Skrifter från SMDF, Nr. 20

Talking mathematics

Proceedings of Madif 15
The fifteenth research conference of
the Swedish Society for Research in
Mathematics Education
Gothenburg, 2026

Editors:
Mattsson, L., Eckert, A. & Kilhamn, C.

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Preface

On 27–28 January 2026, the Swedish Society for Research in Mathematics Education, SMDF, organized the fifteenth Swedish Mathematics Education Research Conference, MADIF 15, at Chalmers University of Technology in Gothenburg. With the theme *"Talking Mathematics,"* the conference attracted a record number of participants. Twice as many papers and short communications were submitted as in the previous MADIF, and 150 scholars registered for the conference, a 50% increase.

The conference opened with a plenary lecture by Professor Johan Lithner from Umeå University. His presentation introduced the research program *Learning by Imitative and Creative Reasoning*, initiated in the 1990s to investigate students' difficulties in learning mathematics. Lithner described how this work has developed into a research framework that contrasts learning based on imitation of procedures with learning through creative reasoning grounded in mathematical properties. Over the course of the last 30 years, the framework has been applied in studies analyzing learning processes, in quasi-experimental research on task design, and in design research aimed at supporting teachers in fostering students' mathematical problem solving. The lecture provided a stimulating starting point for many of the discussions that continued throughout the conference.

During the second day, the focus shifted to the participants' research contributions. With more accepted submissions than ever before, 19 full papers, 44 short communications, and one workshop, the program featured five parallel sessions, allowing a broad range of current research in mathematics education to be presented and discussed.

Before the closing of the conference, a plenary lecture was given by Professor Jo Boaler from Stanford University, USA. In her presentation, titled *A Mindset Mathematics Approach*, Boaler discussed a growing body of research indicating that mathematics teaching emphasizing conceptual understanding, connections between mathematical ideas, and supportive mindset messages can contribute to both higher and more equitable student achievement. Her presentation offered a stimulating conclusion to the conference and invited further reflection on approaches to teaching and learning mathematics.

A subsequent evaluation indicated high participant satisfaction and confirmed the importance of MADIF as a meeting place for researchers in mathematics education.

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Acknowledgements

The program committee would like to express its sincere appreciation to all participants of MADIF 15 for contributing to a collegial and stimulating atmosphere in which research perspectives and ideas were exchanged openly and constructively. Through your engagement, the conference became a vibrant forum for scholarly dialogue and reflection.

We are very grateful to all authors who submitted full papers, short communications, or workshop proposals to MADIF 15. These contributions enriched the conference program and sparked many of the stimulating discussions reflected in the work presented in this volume.

The nineteen full papers included in this volume report original research that has not been previously published. In addition, forty-four short communication abstracts present ongoing research projects or aspects of work discussed during the conference, offering insights into current investigations in the field. The workshop contribution focuses on research questions under development and provides a collaborative setting for exploring emerging ideas and perspectives.

This volume constitutes the proceedings of MADIF 15 and offers a snapshot of current research in mathematics education in Sweden and the Nordic countries. All full paper presentations were followed by discussions led by designated discussants. The program committee would like to acknowledge the valuable work of these scholars: Anna-Ida Säfström, Ann-Sofi Röj-Lindberg, Catarina Andersson, Daniel Brehmer, Erika Boström, Hanna Fredriksdotter, Hanna Knutsson, Helena Roos, Ida Bergvall, Jorryt van Bommel, Julia Tsygan, Maria Johansson, Martin Purfürst, Olov Viirman, Peter Frejd, Petra Svensson Källberg, Robert Gunnarsson, Sara Julsgård, and Serife Sevinc.

We are also very grateful for the excellent work our reviewers have carried out. Some reviewers presented papers at the conference, while others contributed to the community as external reviewers, i.e. reviewers who did not contribute a paper or short communication themselves. The external reviewers were Anette De Ron, Andreas Bergwall, Anna Pansell, Bjørn Smestad, Dorota Lembré, Eva Fülöp, Heidi Strømskag, Helena Johansson, Hendrik Van Stenbrugge, Håkan Lennerstad, Inger Eriksson, Jake McMullen, Jan-Fredrik Olsen, Jenny Svanteson Wester, Johanna Rylner Kjellgren, Karl de Fine Licht, Kim Berts, Lisa Österling, Magnus Fahlström, Lovisa Sumpter, Magnus Österholm, Malin Norberg, Manya Raman Sundström, Mats Brunström, Monica Johansson, Morten Blomhøj, Ove G Drageset, Patrik Gustafsson, Peter Nyström, Päivi Juvonen, Samuel Sollerman, Serik Sagitor. Tasmin Meaney, Timo Tossavainen, and Yukiko Asami-Johansson.

The program committee would also like to express its sincere gratitude to the local organizers of MADIF 15 for their invaluable work in making the conference possible. Their efforts in arranging the venue, refreshments, and the conference dinner contributed greatly to the smooth running of the event and to creating a welcoming and collegial atmosphere throughout the conference. The local committee consisted of Cecilia

Kilhamn, Samuel Bengmark, Laura Fainsilber, Anna Holmlund, Eva Fülöp, Rimma Nyman, Seif Sharif, and Hoda Ashjari. The members of the local committee, with Ann-Sofi Röj-Lindberg added, also served as moderators for all short communications.

Finally, as chair of the program committee, I would like to express my sincere gratitude to the members for their expertise, dedication, and collegial spirit. Their careful reviews, thoughtful deliberations, and generous investment of time and knowledge were essential to the preparation of the conference and this volume. Throughout the process, the committee demonstrated both professionalism and a strong commitment to supporting the development of the mathematics education research community in Sweden. The program committee for MADIF 15 consisted of the following members: Andreas Eckert (Linnaeus University), Anna Teledahl (Örebro University), Ann-Sofi Röj-Lindberg (Åbo Akademi University), Cecilia Kilhamn (University of Gothenburg), Hanna Palmér (Linnaeus University), Jonas Jäder (Mid Sweden University), Kicki Skog (Stockholm University), Samuel Bengmark (Chalmers University of Technology), and Ulf Ryberg (University of Gothenburg).

Review process and reference to the proceeding

All full papers published in this volume were subject to a rigorous two-step review process for both conference presentation and publication. Each submission was peer-reviewed by three researchers. Contributions submitted as short communications or workshop proposals were reviewed by members of the program committee. Nineteen out of forty submitted papers were accepted after minor revisions, and seven of the rejected papers were instead presented as short communications. The acceptance rate for short communications was 85 %.

The MADIF Proceedings are published by SMDF (*skriftserie från Svensk Förening för MatematikDidaktisk Forskning*). Since 2010, this publication series has been classified as a scientific publication channel at Level 1 in the Norwegian Register for Scientific Journals, Series and Publishers (kanalregister.hkdir.no).

In line with environmental considerations, SMDF has decided to publish the proceedings exclusively in digital format. When citing this volume, please use the following reference:

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The MADIF conference proceedings are published in the SMDF publication series. Since MADIF 9, the volumes have also been made available electronically and can be accessed through the SMDF website (matematikdidaktik.org).

The next conference, MADIF 16, will take place at Mid Sweden University in Sundsvall in 2028. We look forward to welcoming both new and returning participants and to continuing the shared exploration of research in mathematics education.

MADIF 1, 1999, Stockholm
MADIF 2, 2000, Göteborg
MADIF 3, 2002, Norrköping
MADIF 4, 2004, Malmö
MADIF 5, 2006, Malmö
MADIF 6, 2008, Stockholm
MADIF 7, 2010, Stockholm
MADIF 8, 2012, Umeå
MADIF 9, 2014, Umeå
MADIF 10, 2016, Karlstad
MADIF 11, 2018, Karlstad
MADIF 12, 2020, Växjö
MADIF 13, 2022, Växjö
MADIF 14, 2024, Örebro
MADIF 15, 2026 Göteborg

The program committee for MADIF 15
by chair Linda Mattsson

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Mellanstadieelevers generalisering genom identifikation och tillämpning av regelbundenheter

Hanna Fredriksdotter
Uppsala universitet

I studien analyseras mellanstadieelevers formulering och användning av generella argument under gemensamt arbete med matematisk problemlösning. ”Generalisering” *operationaliseras som handlingar som innebär* Identifikation *och/eller* Tillämpning *av regelbundenheter, där relationen mellan specifika exempel och ”för alla”-argument betraktas som central. Studiens resultat understryker vikten av att elevers generella argument stämmer överens med vad konkreta exempel visar – samt antyder att elevers förmåga att generalisera stöds av att undersökningar av specifika exempel upprepas.*

Bakgrund

Att *generalisera* kan beskrivas som förmågan att se samband mellan det specifika och det allmängiltiga. Denna förmåga, menar Mason (1996, s. 65), är att betrakta som ”the heartbeat of mathematics” – vilket motiverar att aktiviteter som kan bidra till att utveckla elevers generaliseringsförmåga bör vara en del av matematikundervisningen på alla utbildningsnivåer. Att uppmuntra yngre elever att formulera generella argument när lösningar förklaras och motiveras i samband med matematisk problemlösning ger även en grund för framtida arbete med formell bevisföring (Harel & Sowder, 2007).

Forskning om generalisering har vanligen handlat om algebraundervisning då denna förmåga är en central del av ett algebraiskt tänkande, vilket bör introduceras i tidig ålder (se Blanton m.fl., 2019). Studier har till exempel visat att mellanstadieelevers identifiering av generella mönster i talserier underlättas av att uppgifterna knyter an till elevernas förkunskaper, samt att lösningarna representeras på ett strukturerat sätt, såsom i tabellform (Yeap & Kaur, 2008). Yngre elevers förmåga att hantera och formulera algebraiska uttryck stöttas även av att elevernas ”naturliga” språk används i både uppgiftsformulering och lösningsförslag (Ayala-Altamirano m.fl., 2022). Vidare stärks förmågan att formulera och använda generella matematiska argument av att få dela med sig av sina idéer då detta ger möjlighet att både öva på att motivera sina egna resonemang och bygga vidare på andras förslag (Strachota, 2020). Att delta i samtal med jämnåriga leder dock inte alltid till att elever kan dra generella slutsatser eller upptäcka samband mellan matematiska fenomen; de flesta elever, oavsett ålder, behöver lärarens stöd i att identifiera vad som är generellt giltigt för olika situationer (Jurow, 2004). Det finns även ett flertal studier som tyder på att många elever uppfattar generalisering som svårt, samtidigt som det saknas kunskap om hur ordinarie undervisning bör organiseras för att bidra till att elevers förmåga att generalisera utvecklas och befästs (Ellis m.fl., 2024).

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Tidigare studier om elevers generalisering har ofta byggt på intervjuer med enskilda elever, eller utformats som undervisningsexperiment, medan det finns färre studier om elevers förmåga att generalisera i samband med lektionsarbete i matematikklassrummet (Ellis m.fl., 2024). Syftet med den här artikeln är därför att bidra till att synliggöra yngre elevers generalisering under grupparbete kring uppgifter inom andra områden än algebra, när detta sker som del av det ordinarie lektionsarbetet. Följande fråga fokuseras:

> Vilka strategier använder elever när de formulerar generella argument på ett korrekt sätt under gemensamt arbete med matematisk problemlösning?

Teoretiskt ramverk

Den här artikeln utgår från Masons (1996, s. 65) definition av generalisering som ”*seeing a generality in the particular* and *seeing the particular in the general*”. Att betrakta generalisering som förmågan att *se det generella genom det specifika* respektive att *se det specifika i det generella* utgör således en grundläggande teoretisk referensram för analysen av artikelns empiriska material. I analysen betraktas även förmågan att formulera och hantera korrekta *”för alla”-argument* – det vill säga, argument som varken accepterar undantag eller begränsas till specifika fall (Harel & Sowder, 2007) – som en central aspekt av generalisering. Vidare har resultat av tidigare forskning bidragit till att operationalisera begreppet generalisering som handlingar som innebär att elever *identifierar* och/eller *tillämpar regelbundenheter* (se Ellis m.fl., 2024; Jurow, 2004; Strachota, 2020; Yeap & Kaur, 2008).

Analysen av det empiriska materialet utgår även från ett *dialogiskt perspektiv på kommunikation*, vilket innebär att människor anses ha förmåga att ”*tillsammans med andra* skapa mening och ordning” (Linell, 2022, s. 373). Detta perspektiv innebär bland annat att samtalsdeltagares bidrag till samtal inte kan ses som ”en serie atomära yttranden” utan måste förstås i relation till de samtalssekvenser inom vilka de yttras (ibid.). I analysen av det empiriska materialet uppmärksammas därför inte enbart det matematiska innehållet i elevernas argumentation utan även elevernas användning av olika språkliga resurser. Till exempel noteras användningen av pronomenet *man* som har en generaliserande funktion då det kan likställas med *vem som helst* (Teleman m.fl., 1999). Elevers användning av modalpartikeln *ju* noteras också, då denna modalpartikel kan antyda att det som sägs hanteras som allmänt känd information (Lindström, 2008).

Metod

Datainsamlingen gjordes i två klassrum i årskurs sex i en F-6 skola belägen i en större kommun i Mälardalen. Klasslärarna hade undervisat sina respektive klasser sedan årskurs fyra och tillämpade regelbundet olika former av samarbete. Att arbeta i par och grupp kan således betraktas som en del av elevernas ordinarie klassrumsarbete.

Det empiriska material som analyseras i denna studie bestod av observationer av 17 lektioner (som varade i genomsnitt 55 minuter). I enlighet med klasslärarnas önskemål handlade uppgifterna om kombinatorik respektive bråk och procent.

Kombinatorik var ett nytt arbetsområde för eleverna och valdes då lärarna önskade introducera detta område på ett annat sätt än via mer traditionell helklassundervisning. Bråk och procent hade eleverna däremot träffat på under tidigare läsår och lärarna ansåg att grupparbete kunde vara en lämplig arbetsform för repetition av arbetsområdet. Kombinatorik behandlades under nio lektioner medan åtta lektioner handlade om bråk och procent. Eleverna tilldelades en uppgift per lektion och utöver att lösa uppgiften instruerades de att förbereda en presentation av gruppens lösning inför resten av klassen i samband med en avslutande helklassdiskussion. Det var också explicit uttalat att alla gruppmedlemmar förväntades att självständigt kunna förklara sin grupps lösning.

Under varje lektion videofilmades två grupper; således ingick 34 videoinspelningar i det empiriska materialet. Efter lektionerna samlades även elevernas anteckningar in. De sekvenser som presenteras i den här artikeln handlar om tre uppgifter som presenteras i Tabell 1:

Kläder	Lejon	Tärningar
Lisa har tre jeans och fyra tröjor. Hon använder alltid jeans och tröja. På hur många olika sätt kan hon klä sig? En dag är det kallt och Lisa vill också ha en jacka på sig. Hon har två jackor. På hur många sätt kan hon klä sig nu? Försök finna en regel för antalet olika sätt att klä sig.	En djurpark har fler än 15 och färre än 25 lejon. Hur många lejon kan det vara om: a) exakt 20% är hanar och resten är honor? b) exakt 25% är hanar och resten är honor? c) exakt 50% är honor, exakt en tredjedel är unga hanar och resten är vuxna hanar? Förklara varför det inte kan vara vilket antal lejon som helst mellan 15 och 25.	Några klasskamrater kastar två likadana tärningar och adderar de två talen som kommer upp. Vilka är de vanligaste summorna de får om de använder: a) två sexsidiga tärningar? b) två tiosidiga tärningar? c) två tjugosidiga tärningar? Försök finna en regel för vilken summa som är vanligast när man kastar två tärningar, beroende på antalet sidor på tärningen.

Tabell 1. Uppgifter som behandlas i exemplen på elevers generalisering.

Videoinspelningarna transkriberades ordagrant och analyserades i enlighet med principer för *deduktiv innehållsanalys* (Mayring, 2015). Detta innebär att sekvenser där elever formulerade generella argument identifierades enligt operationaliseringen av generalisering som presenterades i avsnittet Teoretiskt ramverk. I de fall elever explicit tillämpade ”för alla”-argument (se Harel & Sowder, 2007), till exempel genom att formulera vad som gäller i ”varje” eller ”alla” fall eller vilken procedur som ”alltid” ska utföras, noterades dessa som indikationer på att elever *identifierade* regelbundenheter mellan specifika exempel. Även elevers skiftningar i användningen av olika pronomen, till exempel från ”jag” till det generaliserande pronomenet ”man” (se Teleman, 1999), noterades som en form av indirekta ”för alla”-argument.

Elevers *tillämpning* av regelbundenheter noterades däremot på ett mer implicit sätt. I de fall en elev använde modalpartikeln ”ju” i samband med att ett argument presenterades analyserades detta som en indikation på att eleven hänvisade till – och tillämpade – tidigare kunskaper, samt att dessa kunskaper hanterades som kända av alla inom gruppen som samarbetade (se Lindström, 2008).

Ett delsteg i deduktiv innehållsanalys är att söka efter ”anchor samples” (Mayring, 2015, s. 377), vilket i den här studien innebar att finna utdrag som tydligt exemplifierar hur elever *identifierar* och/eller *tillämpar* regelbundenheter. Tabell 2 presenterar tre anchor samples, som i sin tur representerar olika ”typer” av generalisering:

Generalisering	Anchor samples
Typ 1 – *Identifikation* Elever undersöker specifika exempel och *identifierar* en regelbundenhet. Exemplifieras av samtal om uppgiften ”Kläder”.	Elever undersöker kombinationer av ett visst antal klädesplagg – och identifierar en ”regel”: **man tar alltid dom olika sakerna gånger varandra, som tröjerna gånger byxerna gånger jackerna**.
Typ 2 – ***Tillämpning*** Elever redogör för en regelbundenhet och *tillämpar* den i specifika beräkningar. Exemplifieras av samtal om uppgiften ”Lejon”.	Elever redogör för sambandet mellan 20% och 1/5, vilket tillämpas vid beräkning: **tjugi procent är ju samma sak som en femtedel […] det betyder att […] det måste va i femmans tabell för att det inte ska bli halva lejon**.
Typ 3 – *Tillämpning-och-identifikation* Typ 1 och Typ 2 kombineras då elever först *tillämpar* en regelbundenhet vid formulering av ett lösningsförslag och därefter *identifierar* densamma genom undersökning av specifika exempel. Exemplifieras av samtal om uppgiften ”Tärningar”.	Elever tillämpar en regelbundenhet vid formulering av ett förslag på den vanligaste siffersumman när två sexsidiga tärningar kastas: **en tärning har ju sex sider [och] det talet bakom den siffran blir summan sju**. Siffersummorna undersöks praktiskt, vilket innebär att en regelbundenhet identifieras – och i sin tur bekräftar det inledande förslaget: **nie funkar inte, inte åtta heller, det är sju som är rätt svar**.

Tabell 2. Presentation av tre typer av generalisering samt ”anchor samples”.

Nedan presenteras analys av tre utdrag ur videoinspelningarna. Utdragen har i huvudsak formulerats enligt normer för skrivet språk där vissa ”talspråkliga” inslag har behållits. Även några symboler används i enlighet med de konventioner som har sammanfattats av Jefferson (2004): understreck innebär betoning, hakparentes anger samtidigt tal och text inom dubbla parenteser utgör mina kommentarer.

Resultat

I 15 av de 34 videoinspelningarna identifierades sekvenser där elever formulerade korrekta generella argument, enligt operationaliseringen som presenteras i Teoretiskt ramverk och exemplifieras i Tabell 2. Elevers generalisering kategoriserades i sex sekvenser som *Identifikation* (”Typ 1”), i åtta sekvenser som *Tillämpning* (”Typ 2”) och i en sekvens som *Tillämpning-och-identifikation* (”Typ 3”).

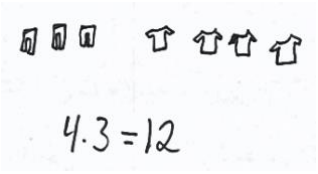
Identifikation: elever identifierar en regelbundenhet

Elevers generalisering i form av *Identifikation* exemplifieras genom utdrag ur samtal mellan Anna, Bibi, Carl och Darin om uppgiften ”Kläder” (Tabell 1). Eleverna arbetade först i par och därefter i grupp bestående av fyra elever. Samtalet mellan Anna och Bibi inleddes genom att Bibi presenterade en illustration (Figur 1) som hon menade visar att ”alla tröjerna kan vara med alla jeans” på tolv olika sätt.

Figur 1. Kopia av Bibis bild, ritad under enskilt arbete.

Anna ansåg dock att Bibis illustration behövde förtydligas och tog initiativ till att gemensamt rita en ny bild. Under det gemensamma ritandet enades eleverna också om att antalet kombinationer kan beräknas via multiplikationen 4 · 3 (Figur 2).

Anna: Jag kan rita fyra tröjer. ((Börjar rita))
Bibi: Ja. ((Ritar tillsammans med Anna))
Anna: Kan man tänka tolv [gånger tre?
Bibi: [Nej vänta.
Anna: Eller?
Bibi: Tolv gånger, nej inte tolv gånger tre men fyra gånger [tre.
Anna: [Nej, jag menade fyra [gånger tre.
Bibi: [Ja, fyra gånger tre kan man [tänka.
Anna: [Ja vänta. ((Skriver 4 · 3, se Figur 2)) Fyra gånger tre, det blir ju tolv.



Figur 2. Kopia av Anna och Bibis gemensamma anteckningar.

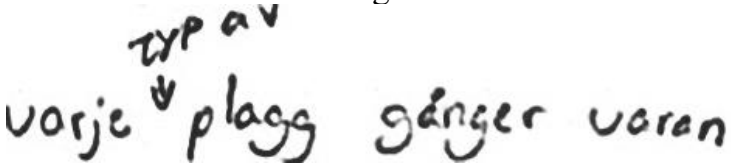
Förslagen att ”tänka tolv gånger tre”, respektive ”fyra gånger tre”, tyder på att eleverna rörde sig bort från den konkreta representationen av lösningen. Användningen av det generaliserande pronomenet *man* (Teleman m.fl., 1999) kan även ses som ett tecken på att multiplikationen hanterades som en generellt tillämpbar strategi. I tillägget ”det blir ju tolv” visar också modalpartikeln *ju* att antalet kombinationer hanterades som känd information (Lindström, 2008).

Efter en stunds pararbete bildade Anna och Bibi grupp med Carl och Darin som även hade löst deluppgiften där två jackor lades till (se ”Kläder” i Tabell 1). Anna och Bibi höll med Carl och Darin om att antalet kombinationer då är 24 – och gruppens diskussion handlade därför huvudsakligen om att formulera en ”regel”.

Carl: Okej, men en regel det kan vara att man tar alltid dom olika sakerna gånger varandra. Som tröjerna gånger byxerna gånger jackerna.
Anna: Kan vi inte bara lägga till ett till [klädesplagg?
Darin: [Kolla kolla kolla, man kan ha, man tar två gånger kläderna som är, det är tre, fyra tröjer. Tre gånger fyra, det är tolv.
Carl: Mm, man tar alltid klädesplaggen gånger varann.
Darin: Exakt.
Carl: Mm, ska vi skriva det? Alltid klädesplaggen gånger varandra.
Anna: Alla klädesplaggen gånger varann.
Carl: Alltid, man ska alltid multiplicera klädesplaggen med varann.

Carls exemplifiering av ”dom olika sakerna” i form av ”tröjerna gånger byxerna gånger jackerna”, tillika med yttrandet ”man tar alltid klädesplaggen gånger varann” (vilket sammanfattade Darins förslag att beräkna ”två gånger kläderna som är”) tyder på en förmåga att se samband mellan det generella och det specifika, i enlighet med Masons (1996) definition av generalisering. Vidare tyder Carls användning av både det generaliserande pronomenet *man* och ordet ”alltid” på att hans förslag till ”regel” inte enbart gällde gruppens lösning av den specifika uppgiften. Att Carl sedan höll fast vid att ”alltid multiplicera klädesplaggen med varann”, i stället för att följa Annas förslag om att multiplicera ”alla klädesplaggen”, tyder också på att han försökte formulera en strategi som gäller för alla uppgifter som handlar om att kombinera klädesplagg (se Harel & Sowder, 2007), snarare än att fokusera på den specifika uppgiften ”Kläder”.

Gruppen formulerade sedan lösningen i skrift, så som Figur 3 visar. I den skriftliga formuleringen av gruppens ”regel” framgår generaliseringen ännu tydligare; eftersom det tydliggörs att det är ”varje typ” som ska multipliceras kan ”regeln” tillämpas även vid beräkning av andra uppgifter som handlar om att finna antalet kombinationer när element som hör till olika kategorier kombineras.



Figur 3. Kopia av gruppens skriftliga formulering av sin ”regel”.

Tillämpning: elever tillämpar en regelbundenhet

Elevers *tillämpning* av en generellt giltig regelbundenhet vid beräkning av specifika uppgifter representeras av en diskussion mellan eleverna Erik, Fia och Gösta om uppgiften ”Lejon” (Tabell 1). Efter en stunds enskilt arbete inledde Erik den gemensamma diskussionen genom att explicit hänvisa till sambandet mellan tal i procent- respektive bråkform. Därefter tillämpade Erik denna generella regelbundenhet i argumentationen för sitt resultat till den specifika uppgiften.

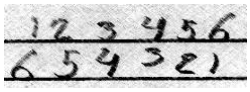
Erik:	Tjugi procent, det är ju samma sak som en femtedel.
Fia:	Ja det skrev jag också. ((Pekar på sina anteckningar))
Erik:	Och det betyder att på fem, eller det liksom, det måste va i femmans tabell för att det inte ska bli halva lejon. Alltså, med sex lejon, om man delar det på fem, det går ju inte.
Gösta:	[Varför då?
Erik:	[Men om man har fem [lejon…
Gösta:	[Varför går inte det?
Erik:	För att det blir ett halvt lejon, eller…
Gösta:	Bra!
Erik:	…det blir liksom noll komma två lejon. Och det går inte.
Gösta:	Om man tar tjugi då?
Erik:	Men om har fem, liksom alla dom i femmans tabell går ju.

Användningen av modalpartikeln *ju* antyder att Erik hanterade sambandet mellan 20% och 1/5 som ett känt faktum (Lindström, 2008), vilket Fia bekräftade. Erik förde sedan fram argumentet att resultatet måste finnas ”i femmans tabell” för att undvika ”halva lejon”. Detta innebär indirekt en hänvisning till generella regler för delbarhet, vilket i sin tur kan tyda på en förmåga att identifiera något generellt i den specifika uppgiften (se Mason 1996). Därefter exemplifierade Erik sitt påstående med argumentet att ”sex lejon” dividerat med fem ”går ju inte”, där Eriks sätt att respondera på Göstas ”varför”-frågor tyder på att de hanterades som ett behov av förtydligande, snarare än som ett ifrågasättande av slutsatsen i sig; Göstas utrop ”bra” när Erik sa att ”det blir ett halvt lejon” kan möjligen också tyda på att invändningen hade karaktären av en retorisk fråga. Frågorna bidrog även till att Erik specificerade sitt exempel då han ändrade exemplet om ”ett halvt lejon” till att en femtedel av sex lejon ger resultatet ”noll komma två lejon”. Gösta föreslog därefter antalet ”tjugi”, vilket är ett fungerande antal för den specifika uppgiften, medan Erik formulerade ett ”för alla”-argument genom att understryka att ”alla dom i femmans tabell går ju” (se Harel & Sowder, 2007), där *ju* åter antyder att Erik hanterade delbarhetsreglerna som allmänt kända (se Lindström, 2008).

Tillämpning-och-identifikation: elever tillämpar och identifierar en regelbundenhet

Vid generalisering av typen *Tillämpning-och-identifikation* kombineras ”Typ 1” och ”Typ 2” (Tabell 2). Detta exemplifieras av diskussioner kring uppgiften ”Tärningar” (Tabell 1) mellan Heidi, Ida, Jim och Kalle. Heidi inledde sitt pararbete med Ida genom att föreslå att 7 är den vanligaste siffersumman när två sexsidiga tärningar kastas.

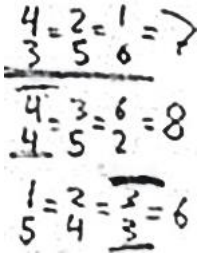
Heidi:	Kolla, jag tänkte sju.
Ida:	Mm.
Heidi:	För jag visste inte hur jag skulle rita så jag skrev. ((Pekar på anteckningar, Figur 4)) För en tärning, den har ju sex sider.
Ida:	((Nickar))
Heidi:	Och se’n, det talet bakom den siffran blir summan sju.



Figur 4. Kopia av Heidis anteckningar.

Heidis betoning av ”jag” tyder på att förslaget ”sju” enbart var hennes egen specifika idé; däremot kan hänvisningen till strukturen hos ”en tärning” ses som ett generellt argument då den obestämda artikeln *en* även kan representera *alla* (Mason, 1996). Vidare antyder användningen av modalpartikeln *ju* att den sexsidiga tärningens struktur hanterades som allmänt känd (Lindström, 2008).

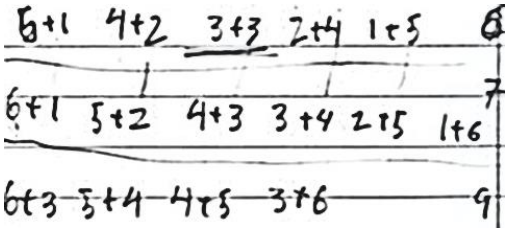
Idas ”mm” och nickande antyder att hon höll med Heidi, men kort därefter föreslog Ida att summorna 6 och 8 skulle kunna vara lika vanliga som summan 7, då hennes alternativa tabell (Figur 5) antyder att alla dessa siffersummor ges av tre par av termer. En stund senare avfärdade dock Ida sitt eget förslag, med argumentet att additionerna 4 + 4 respektive 3 + 3 ”blir samma siffra” när termerna ”byter plats”. Med andra ord ges siffersumman 7 av sex par av termer medan summorna 6 och 8 ges av endast fem par.



Figur 5. Kopia av Idas anteckningar.

När Jim och Kalle bildade grupp med Heidi och Ida inledde de gruppdiskussionen med att föreslå att summorna 7, 8 och 9 är lika vanliga. Efter undersökning av ytterligare exempel (Figur 6) enades dock gruppen om att endast 7 är den vanligaste summan.

Jim:	Nej förlåt, nie funkar inte. Inte åtta heller, det är sju som är rätt svar.
Ida:	Och sex.
Heidi:	Och sex, [sju och sex.
Ida:	[Funkar inte sex?
Jim:	Men sex är bara fem kombinationer. ((Pekar på Figur 6)).
Ida:	Men kan, kolla kolla, kan inte dom härа byta plats? ((Pekar på 3+3, därefter på 4+2 och 2+4)) Alltså, [fyra två, två fyra.
Jim:	[Men det blir ju inte en gång till, det är ju samma. Fem.
Ida:	Det var därför vi sa sju från början också.



Figur 6. Kopia av Jims anteckningar.

Jims användning av modalpartikeln *ju* när han underströk att treornas platsbyte i additionen ”3 + 3” inte ger en ny kombination antyder att detta förhållande hanterades som redan känt (se Lindström, 2008). Genom att säga att det var ”därför vi sa sju från början också” visade Ida därefter att hon inte bara accepterade resultatet av Jims undersökning utan även använde det konkreta exemplet (Figur 6) för att bekräfta Heidis initiala förslag att 7 är den vanligaste siffersumman när två sexsidiga tärningar kastas – vilket i sin tur byggde på tillämpning av ett generellt argument gällande strukturen hos ”en tärning”.

Diskussion

Syftet med den här artikeln var att synliggöra mellanstadieelevers strategier för formulering och användning av generella matematiska argument under gemensam problemlösning. Detta gjordes via observation av klassrumsarbete i årskurs 6, vilket visade exempel på elevers *identifikation* och/eller *tillämpning* av regelbundenheter.

Artikeln bygger på analys av 34 videoinspelningar av elevers samtal kring matematiska problem, där sekvenser som innebar att elever formulerade korrekta generella argument förekom i 15 inspelningar. Detta tyder på att korrekt generalisering var ganska vanligt förekommande, vilket i viss mån kontrasterar tidigare studier som har påtalat att elevers svårigheter att generalisera är väldokumenterade (Ellis m.fl., 2024).

Utdraget som exemplifierar *Identifikation* av en regelbundenhet visar att beräkning (i form av multiplikation) föreslogs under det att en ny figur ritades. Initiativet att rita en ny bild kan därför både ses som en *gemensam granskning* av Bibis lösningsförslag i form av en bild (Figur 1) och som en indirekt utmaning av att använda ritande som metod för att lösa uppgiften. På ett liknande sätt visar exemplifieringen av *Tillämpning-och-identifikation* hur den gemensamma granskningen av olika tabeller (Figur 5 & 6) bidrog till att identifiera (och därigenom bekräfta) den regelbundenhet som tillämpades vid formuleringen av det inledande förslaget gällande den vanligaste siffersumman när två sexsidiga tärningar kastas. Vidare kan Göstas ”varför”-frågor i exemplet som illustrerar generalisering i form av *Tillämpning* av en regelbundenhet också ses som ett bidrag till en gemensam granskning av Eriks argument. Tidigare studier har uppmärksammat vikten av att elever uppmuntras att förklara och motivera sina lösningar (Strachota, 2020), vilket underlättas av möjligheten att använda sitt ”naturliga” språk (Ayala-Altamirano m.fl., 2022). Även om den här artikeln bygger på ett begränsat empiriskt underlag bidrar den med konkreta exempel på hur elevers ”naturliga” samtal kring förklaringar och motiveringar kan utformas, där en gemensam granskning av lösningsförslag i sin tur kan bidra till generalisering.

Analysen i denna artikel visade även att elevers gemensamma granskning byggde på konkretisering av lösningsförslagen. Att strukturerade undersökningar av konkreta exempel stöttar elevers förmåga att formulera generella argument är i linje med tidigare forskning (se Yeap & Kaur, 2008). De elevsamtal som presenteras i den här artikeln visar dock att eleverna också utförde *upprepade undersökningar* av specifika exempel; Anna och Bibi ritade en ny bild, Erik utförde flera beräkningar, Ida och Jim skrev mer än en tabell. Elevsamtalen visar dessutom att eleverna inte enbart repeterade sina argument när de upprepades utan varje upprepning bidrog till lösningar som var både exaktare och mer generella. I exemplet som visar *Tillämpning* ledde den upprepade beräkningen till ökad exakthet, då Erik först angav att en femtedel av ”sex lejon” resulterar i ”ett halvt lejon” och sedan ändrade till ”noll komma två lejon”, medan elevernas anteckningar i exemplifieringen av *Tillämpning-och-identifikation* visar en viss progression – från att enbart ange tärningens siffror (Figur 4) till att redovisa alla kombinationer av termer som ger en specifik siffersumma (Figur 6) – vilket också ökade lösningens exakthet. Vidare ledde upprepningen av argumenten kring en ”regel” i exemplet som visar *Identifikation* till en mer generellt formulerad slutsats (Figur 3).

Resultaten antyder således hur lärare skulle kunna arbeta för att bidra till att elever utvecklar sin förmåga att generalisera under gemensamt arbete med problemlösning. Elever bör uppmuntras att tillämpa strategin *gemensam granskning* av lösningar genom att uppmärksamma värdet av olika sätt att lösa uppgifter (såsom ritande och räknande).

Lärare bör även stötta elever i att ställa och hantera frågor som ett led i att öka lösningarnas exakthet, snarare än att se frågorna som en negativ kritik mot själva lösningsförslaget. Vidare bör lärare uppmuntra elever att konkretisera sina lösningar med hjälp av olika representationsformer, inklusive beräkningar, då strategin *upprepade undersökningar* av specifika exempel också tycks kunna bidra till att elever formulerar exaktare och mer generella lösningar.

References

Ayala-Altamirano, C., Molina, M., & Ambrose, R. (2022). Fourth graders' expression of the general case. *ZDM*, *54*. 1377–1392. https://doi.org/10.1007/s11858-022-01398-8

Blanton, M., Isler-Baykal, I., Stroud, R., Stephens, A., Knuth, E., & Murphy Gardiner, A. (2019). Growth in children's understanding of generalizing and representing structure and relationships. *Educational Studies in Mathematics, 102,* 193–219. https://doi.org/10.1007/s10649-019-09894-7

Ellis, A. B., Waswa, A., Tasova, H. I., Hamilton, M., & Moore, K. C. (2024). Classroom supports for generalizing. *Journal for Research in Mathematics Education*, *55*, (1), 7–30. https://doi.org/10.5951/jresematheduc-2022-0140

Harel, G., & Sowder, L. (2007). Toward comprehensive perspsectives on the learning and teaching of proof. I F. K. Lester (Red.), *Second Handbook of Research on Mathematics Teaching and Learning* (ss. 805–842). National Council of Teachers of Mathematics.

Jefferson, G. (2004). Glossary of transcript symbols with an introduction. I G. Lerner (Red.), *Conversation Analysis: Studies from the first generation* (ss. 13–31). John Benjamins Publising.

Jurow, A. S. (2004). Generalizing in interaction: Middle school mathematics students making mathematical generalizations in a population-modeling project. *Mind, Culture, and Activity*, *11*(4), 279–300. https://doi.org/1020/s15327884mca1104_4

Lindström, J. (2008). *Tur och ordning*. Norstedts Akademiska Förlag.

Linell, P. (2022). *Med språket genom tillvaron: En introduktion till dialogiska perspektiv på språkande, tänkande och kommunikation.* Linköpings universitet.

Mason, J. (1996). Expressing generality and roots of algebra. I N. Bednarz, C. Kieran & L. Lee (Red.), *Approaches to Algebra: Perspectives for Research and Teaching* (ss. 65–86). Kluwer Academic Publishers.

Mayring, P. (2015). Qualitative content analysis: Theoretical background and procedures. I A. Bickner-Ahsbahs, C. Knipping & N. C. Presmeg (Red.), *Approaches to qualitative research in mathematics education: Examples of methodology and methods* (ss. 365–380). Springer.

Strachota, S. (2020). Generalizing in the context of an early algebra intervention. *Journal for the Study of Education and Development*, *43*(2), 347–394. https://doi.org/10.1080/02103702.2020.1732611

Teleman, U., Hellberg, S., & Andersson, E. (1999). *Svenska Akademiens grammatik*. Norstedts.

Yeap, B-H., & Kaur, B. (2008). Elementary school students engaging in making generalisation: a glimpse from a Singapore classroom. *ZDM*, *40*, 55–64. https://doi.org/10.1007/s11858-007-0072-x

Rethinking and theorising inclusion and equity - The mathematics is MInE project

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This paper aims to display theoretical and conceptual contributions from the MInE (Mathematics for Inclusion and Equity) project, thereby rethinking inclusion and equity in mathematics education. The project has investigated and defined how inclusion and equity are conditioned and can be supported in and through mathematics teaching. For this purpose, mathematics teaching is understood as a system of principles which are displayed in the lived classroom. A new conceptual construct of inclusion and equity is presented: moments of inclusion and equity. We finally elaborate on how these contributions relate to the overall governance of the school system. The project was carried out in close collaboration with local schools.

Introduction

There is an ongoing critique of the Swedish school system for its lack of inclusion and equity (see, for example, Ding et al., 2024). In research, there is a general gap in knowledge about how inclusion and equity emerge and can be sustained in mathematics teaching (Kollosche et al., 2019). This situation is linked to ambiguity in how the concepts are theorised and used in research and practice, as for example Enchikova et al., (2024) have demonstrated regarding educational equity. This indicates a need to rethink and offer an alternative conceptualisation and theorisation. Namely, one that is closely aligned with the context in which the empirical data have been generated. Hence, the overall purpose of this paper is to rethink and reposition inclusion and equity both theoretically and conceptually, and by that contribute to knowledge in how to sustain these in practice. Furthermore, this rethinking and repositioning is deeply situated in, derives from and has been explored in close collaboration with mathematics teachers, representing mathematics teaching from more than 12 classrooms over a period of 3 years. In extension, the project offers policy, practice and research the opportunity to rethink mathematics teaching and education, by making it possible to explore and develop strategies that support the development of increased inclusive and equitable learning environments (see Bagger & Roos, 2024; Roos & Bagger, 2024).

Subsequently, the specific aim of this paper is to rethink and theorise inclusion and equity by using examples from a project in Swedish primary school: the MInE project (Mathematics for Inclusion and Equity). The aim is twofold: 1) to rethink inclusion and equity by presenting and elaborating on *moments of inclusion and equity* as a concept. 2) to elaborate on the theoretical construct of *inclusion and equity as principles in mathematics teaching as a system*. Drawing on the work of Ainscow (2020), we

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understand *principles* as fundamental ideas or beliefs that guide understanding and action, particularly in the context of teaching.

Inclusion and equity in mathematics education

The notions *of inclusion and equity* are handled in this paper as interrelated and as highly systemic, contextual and lived phenomena. Consequently, when occurring in mathematics teaching, we understand these as principles on a system level that emerge and are recreated in the moment of teaching. By inclusion and equity in teaching, we refer to inclusive teaching that affords all students equal opportunities to learn, feel empowered, and have agency in mathematics. This definition builds on prior scholars' exploration of inclusion and equity, such as Athew (2011), Askew (2015), Lambert (2015) and Kollosche et al (2019), and has ethical aspects embedded (see Roos & Bagger, 2024). Evaluations of Swedish schools reveal a lack of equity in mathematics outcomes, but these evaluations do not offer guidance on how to adapt teaching to promote inclusion and equity (Bagger & Vennberg, 2024). This is worrying since the increasing differences that exist between groups of students in Sweden indicate that disadvantages are reproduced (Muench et al., 2023). In Sweden, the lack of a passing grade is more common for students in need of special support, students with low socio-economic status, students with a foreign background and boys (see Swedish National Agency for Education, 2023). Although, with an exception from the school-year 2024, in which boys slightly over performed girls regarding gender, an opposite trend in terms of equity of outcome is visible in Europe (European Commission, 2022), implying that overarching national and societal principles of (in)equity and in(ex)clusion are manifested in the lived classrooms (see Ainscow, 2020; Bagger & Roos, 2024; Roos & Bagger, 2021; 2024).

In the under-researched area of inclusive mathematics teaching, it is essential to take teachers' experiences as a starting point when developing inclusive practices. These experiences are closely tied to the challenges teachers face in their classrooms (Lindeskov & Lindhardt, 2021), allowing theory to be informed by teachers' professional knowledge. Furthermore, research has repeatedly shown that inclusion in mathematics teaching seeks to address inequity (e.g., Kolloshe et al., 2019), which relates to challenges faced in the classroom. This, in turn, signals a tight relationship between inclusion and (in)equity. In relation to inclusion and equity, we join scholars as Abthai and Planas (2024) who aim to work "towards equity, diversity and inclusion" (p.307) by addressing practices to challenge marginalisation. These practices are influenced by an economic agenda on a societal level where inclusion and exclusion are intertwined, giving rise to processes of in(ex)clusion. This implies that inclusion and exclusion depend on societal norms, where mathematical knowledge is tied to human value (Valero, 2017).

Equity is often under-defined and addressed in a rather general way in research, resulting in blunt instruments that make it difficult to determine for whom this specific equity would work and how to know if it has increased. Consequently, little is known about how teaching strengthens different kinds of equity and their interrelatedness (Enchinova et al., 2024). Despite these conceptual confusions, it is well known that lack of equity is often associated with structural factors that lead to differences in students'

equality of outcome in mathematics and to participate meaningfully in the teaching (e.g., Gutierrez, 2022). Enchinova et al., (2024) have developed a conceptualisation of equity which includes and points towards contextual and individual aspects of equity. This consists of four different constructs, separating *equity of outcome* from *equity of opportunity*, *segregation* and *resilience*.

In mathematics education research, studies on inclusion and counteracting the risks of exclusion often take on the character of reasoning on a theoretical and philosophical level. This highlights a knowledge gap in research about how inclusion and equity are conditioned, shaped, and can be practically implemented (Roos, 2019). How inclusion and equity can be achieved in a didactic sense within the everyday teaching of mathematics remains under-researched. However, as inclusion and equity create conditions for sustainable teaching and learning in the subject (Atweh, 2011; Kollosche et al., 2019), these values must be placed at the centre of mathematics education. The *Mathematics is MInE* project has contributed to this by conducting a practice-oriented and longitudinal investigation into how inclusion and equity are manifested in teaching, what conditions shape them, how they can be mapped, and how they can be sustained across different mathematics primary school classrooms in Sweden (Bagger & Roos, 2024; Roos & Bagger, 2024). Consequently, concepts, methodology and knowledge on moments of inclusion and equity have been generated. In this paper, we further present the context as well as the theoretical and conceptual outcomes of the MInE project. To do so, we first need to provide background on how the project began and the challenges it aimed to address. This is shortly outlined in the following section.

To research and learn from challenges in a challenged practice

The first steps in the project were taken at the Mathematics Education and Society Conference (MES11) in 2021, where we posed the question: *"Developing mathematics education that promotes equity and inclusion—Is it possible?"* (Roos & Bagger, 2021 p. 223). Our motivation for undertaking this work stems from our individual and joint dedication in doing research on inclusive mathematics education, and the reports of students struggling in mathematics and a decline in equity within the subject (for example, OECD, 2014). We recognised the challenge of understanding and developing inclusive mathematics teaching in primary schools and concluded that this work must be grounded in the lived experiences of teachers and students, particularly given the varied and often unhelpful interpretations of inclusion in public and political discourse that risk undermining both inclusion and equity (Bernau, 2011; DePauw & Doll-Tepper, 2000). In response, we initiated the project, *The Mathematics is MInE* (Mathematics for Inclusion and Equity) in 2022. The project is ongoing, ethically vetted (Dnr 2022-02989-01; Dnr 2024-04418-01; Dnr 2025-01378-01), and partly funded by the National FOU-Network for schools with major challenges.

In the project, two lower primary schools participate, both grappling with issues related to inclusion and equity, while also being actively engaged in school development initiatives addressing these themes. Both are situated in working-class communities with structural and socioeconomic challenges. Both schools have experienced frequent changes in school leadership, which have negatively impacted continuity, organisation, resource allocation, support structures, and overall school development. These

circumstances have also placed significant strain on individual teachers. At one school, this resulted in four out of five participating teachers taking long-term sick leave, while at the other school, it led to sense of resignation. Naturally, these challenges also influenced the research process, requiring us as researchers to remain highly flexible while maintaining consistency in research objectives and theoretical framework.

The theoretical framework and methodology of MInE

A well-established and used theoretical framework for working practice-based, but at the same time recognising inclusion and equity as societal and highly discursive phenomena is Ainscows (2020) principles for inclusion and equity in school systems. This theoretical model puts school development at the core of investigation and as the site on which the history, legislation, culture, values and norms of a school system are manifested. This way of understanding and relating to inclusion and equity has guided data collection and analysis in the MInE project and aligns well with prior research on in(ex)clusion and disadvantage in mathematics education (for example Lambert, 2015).

Ainscow's (2020) theoretical model offers a comprehensive framework for examining inclusion and equity at a systemic level, positioning school development as a central mechanism for driving change. It conceptualises inclusion and equity not only as outcomes but also as dynamic forces that shape and are shaped by teaching practices. In doing so, the model highlights the systemic nature of educational challenges while emphasising the importance of school-level strategies and classroom methods that support all learners. Five interrelated components form the foundation of the model: *Inclusion and equity are guiding and governing principles* (1) on a system level. These principles impact and are manifested in three of the other components: *school development* (2), which is placed in the centre and interacting with *administration* (3) and *community involvement* (4). Finally, the fifth component is the *use of evidence* (5), which means using data from multiple sources to guide reflection and support inclusive educational practices. This builds up and informs school development administration and community involvement. A core tenet is the creation of inclusive learning environments that address the diverse needs of all students, moving beyond a narrow focus on supporting only disadvantaged students. The model underscores the influence of societal and administrative contexts on teachers' practices, emphasising the interconnectedness between classrooms and educational systems.

The project is methodologically inspired by an Inclusive Inquiry Approach (IIA), fostering teacher collaboration to try out and evaluate inclusive strategies. IIA can be seen as a way of responding to learner diversity and means that teachers explore and investigate their practices and understanding of inclusion and equity (Messiou & Ainscow, 2020) and use it in their development of inclusive practices. Hence, it can be connected to Ainscow's (2020) *use of evidence*. This method is well-tested in 30 schools across five countries over 30 years (see Messiou & Ainscow, 2020). Thus, IIA can be considered a stable and reliable method for exploring these types of issues. In the MInE project, we have advocated IIA, holding inquiry-based workshops with one team of teachers in each school. The central questions in all the workshops are drawn from IIA; How can inclusion and equity be made visible? What can we learn from diversity in the classroom? How can we adjust teaching practices?

We collaborate with two teams of teachers from the two schools, consisting of five respectively eight teachers teaching in grade F-3 in each team. Since the project started, we have had 11 exploratory workshops for approximately 100 minutes/workshop with each team of teachers (22 workshops altogether). During the workshops, our interactions with the teachers were guided by the IIA (Messiou & Ainscow, 2020), through which the teachers examined the learning environment with a specific focus on inclusion and equity. Together, the workshops and this methodological stance formed a collaborative effort in the investigation. After each workshop (i.e two workshops, one with each team of teachers), the researchers listened to the collected audio and transcribed it. Then followed a reflexive thematic analysis (Braun & Clarke, 2021) where both researchers discussed the content in the workshops and constructed themes. The themes described different aspects of inclusion and equity based on the broad definition: how teaching affords all students equal opportunities to learn, feel empowered, and have agency in mathematics (Roos & Bagger, 2024). This was returned to the teachers in the following exploratory workshop, signifying that the *use of evidence* formed the foundation of the project's progression, and processes of school development primarily focused on mathematics teaching. Consequently, in the workshops with teachers, the researchers and teachers had a joint focus on the evidence.

Theoretical and conceptual contribution

In this section, we elaborate on the theoretical and conceptual contributions that have evolved over the course of the project. The first elaboration is about *moments of inclusion and equity*, and the second is about *mathematics teaching as the system*.

Moments of inclusion and equity

What we add to Ainscow's (2020) theoretical model and to the IIA (Messiou & Ainscow, 2020), and what marks a theoretical contribution and outcome of our use of evidence during school development of mathematics teaching in primary school, is how inclusion and equity are conceptualised (see Roos & Bagger, 2024). We label the object of inquiry in the IIA as *moments of inclusion and equity.* From this way of viewing inclusion and equity, they are momentary and highly lived, contextual phenomena. Examples of how this is illustrated in our data are when the teachers talk of moments when everything just flows and all students are" a part, talk and participate in the [mathematics] education". Here, they refer to moments that are filled with activity and joy. In addition, we conceptualise inclusion and equity as distinct yet interdependent constructs that, at times, operate in tandem and may even be generated interchangeably (Bagger & Roos, 2023, 2024; Roos & Bagger, 2024). This implies that what is perceived or enacted as inclusion in one context may be perceived as equity in another context. To capture this dynamic relationship, we draw on the metaphor of a Möbius strip, a non-orientable surface, to illustrate the continuous, intertwined nature of inclusion and equity, symbolising their mutual dependence and fluid interchangeability. An illustrative example of this from our data shows how teachers reflect on the importance of being aware of their own assumptions to effectively explain mathematical terms and concepts to all students. This awareness is framed as a means of fostering inclusion in the classroom. However, the discussion also extends to equity, as the teachers also

reflect on how inclusive practices can serve as a foundation for equity. One teacher articulated this interchangeability:" Without inclusion, there is no equity. Not everyone reaches [the goal]. They don't reach the goal that is set, if you don't have inclusion, you don't get equity". This statement encapsulates that the development of inclusive teaching practices is seen not only as a matter of access but as essential to achieving equity in educational outcomes. Hence, moments of inclusion and equity are intertwined, momentary, lived and contextual phenomena.

Mathematics teaching as the system

Drawing on Ainscow's (2020) theory, we have demarcated "the system" to mathematics teaching. Accordingly, we collaborated with teachers in workshops to collect and utilise evidence, focusing on moments where inclusion and equity emerged as central to the dilemmas they encountered and the pedagogical choices they made in their mathematics teaching. These collaborative inquiries also extended to discussions about the broader conditions necessary to sustain and amplify such moments, particularly the roles of the community, administration, and the organisational structures that govern school development. Our findings indicate that the principles underpinning these moments of inclusion and equity emphasise the need for a supportive and adaptable educational environment across multiple levels of governance. Three such principles were identified: *teachers' opportunities and capacity to meet diversity*, *diversified material and teaching strategies,* and *governance and school culture*. These are elaborated in the following sections, followed by a synthesis of key principles for fostering inclusion and equity in mathematics teaching as a systemic endeavour. In this synthesis, we critically examine the roles of administrative structures and community engagement (from Ainscow's model) in shaping and sustaining inclusive school development.

Teachers' opportunities and capacity to meet diversity

How to support a diversity of students emerged as a common theme, in which a principle concerning *the teachers' opportunities and capacity to meet diversity* in the classroom. Examples of this are statements like: "We are different and we must be allowed to be different. [...] You do things differently, so they [the students] need different representations." The teachers also talked about inclusion as "carrying a vision, then you, as a teacher, in your decision-making must constantly embrace diversity among students." One teacher said, "The fascinating thing about diversity is that you are never finished, never fully learned. There is always something more [to learn]." Also, the teachers talked about being able to distribute justice by "hacking the system". This was made at the south school where the teachers felt that they were always alone with the class in mathematics and were begging to have more resources, or at least some time to have half of the group. When this was not recognised by the organisation, they arranged it so that students from grades 8 and 9, who had a free period, came to grade 2 and helped. The teacher in grade 2 said; "this was the best!" The teachers also talked about dilemmas in relation to the need of a supportive and reflexive organization, and the frustration when it was not that; "I can have knowledge, but I don't have the opportunity. […] It reinforces that maybe I don't know how to do it. The ability to do it is what gets hindered". Hence, the principle *teachers' opportunities and capacity to meet diversity*

focus on specific teacher knowledge and possible pathways to practice the knowledge to support inclusive and equitable mathematics teaching.

Diversified material and teaching strategies

In relation to this principle, teachers reflected on how to navigate diversity in ways that allowed all students to engage in the same activities, while simultaneously meeting individual learning needs. Tensions or challenges in this area often surfaced as dilemmas. This is exemplified in the following excerpts: "If you [the student] don't get any accommodations, then this dilemma becomes even bigger. Then the teacher will struggle a lot with diversity. "We talk about time and just finding time for them [the students who struggle] because if you take 4 [students], then there are 4 who get nothing. What should you spend the time on?" This dilemma exemplifies struggles when planning teaching strategies. Another example of planning teaching strategies is when teachers talked about rearranging the classroom to help "the students to find their own way so it works". This was articulated as finding dynamic ways of student grouping and having different ways of explaining. One teacher said, "It's about finding the right tools that will help", highlighting the need for diversified material.

Governance and school culture

This principle played a significant role in sustaining moments of inclusion and equity. This was evident in how teachers discussed governance-related issues in relation to teaching mathematics in inclusive and equitable ways. In connection with this principle, the school's routines, visions, and values emerged as important factors. These were reflected in teachers' statements about the values and visions they personally held, or were expected to conform to, regarding inclusion and equity. Such dynamics were closely tied to underlying issues within the school organisation and its teaching culture. Teachers, for instance, talked about municipal and national tests in relation to students ' not being able to take them from an equity perspective. This principle was also talked about in terms of meeting every student by discussing different approaches on a school level. For instance, a teacher said, "I modelled quite a lot [to reach the student]. So, we talked to each other [about how to approach the students]." Another example is when teachers talked about allocating time to promote equity, "then you have to redistribute [time]".

Administration, School Development and Community Involvement

Drawing on Ainscow's model (2020), we now critically examine the role of administration and community involvement in shaping and sustaining inclusive and equitable school development in relation to the principles. The findings underscore that inclusive and equitable mathematics teaching is not only a matter of individual classroom practice but also deeply influenced by broader systemic conditions. One key insight is the importance of collaborative professional learning among teachers. Such collaboration proved central to enhancing teachers' opportunities and capacity to address diverse student needs. This emerged in discussions about collegial support, the use of special education resources, and teachers expressed a need for further competence development. As one teacher reflected: *"*It takes more, I think, if you are to have everyone in the classroom. It requires that you are truly prepared and well-read.*"* Regarding the distribution of special education, the teachers talked about the need to

constantly reflect on who, how, what and when to support. An illustrative example is when a teacher said:" I really believe that even more support should be implemented from preschool class up to third grade, when it comes to special [education] to include even more students at an earlier stage". In relation, another teacher said, "I don't have any special education in my second grade, but I have [an Arabic speaking person as a resource], [and] my boy, he's learned to count in sequence up to 100 now. He's steadily progressing, but [the person] is paid for by the Employment Agency, [implying the person might disappear], she's invaluable, I'd say, for a boy who speaks Arabic and has very significant difficulties".

Hence, how the administration caters for a sustainable structure and development at the organisational level for collegial support, and the use of special educational resources influenced the school development. Regarding community involvement, there were struggles for the teachers to get involvement from the parents. This negatively affected the school's development work regarding inclusion and equity in mathematics teaching, as it was perceived that teachers lacked access to students' full experiences and to the community-based factors influencing their learning.

Conclusion and contributions

The principles developed in collaboration with teachers within the system of mathematics teaching, teachers' opportunities and capacities to address diversity, the use of diversified materials and teaching strategies, and the influence of governance and school culture, respond to Roos's (2019) call for research and practice to "connect and interrelate the operationalisation and meanings of inclusion in both society and in mathematics classrooms" (p. 25). Accordingly, this study offers both theoretical and practical contributions to the research field of inclusive mathematics education, and has, as proposed by Lindeskov and Lindhardt (2021), taken teachers' experiences as a point of departure.

The principles serve as theoretical constructs that articulate how inclusion and equity are conditioned within mathematics education, while also functioning as practical tools that guide supporting these values across individual, group, and organisational levels, potentially extending beyond the Swedish context. Hence, this study addresses the practice to be able to challenge marginalisation (see Abthai & Planas, 2024). A central concept in our analysis is the notion of "moments," which captures how inclusion and equity are experienced as momentary, fluid and contextual phenomena within the lived realities of mathematics classrooms. These moments illuminate the nuanced and often shifting conditions under which inclusion and equity are either enabled or constrained in everyday teaching practice.

When connecting the principles within the system of mathematics teaching to moments of inclusion in mathematics, the opportunity to promote moments of inclusion in the teaching is influenced by teachers' opportunities and capacities to address diversity. Also, to what extent teachers have the possibility to use diverse materials and teaching strategies influences the opportunity to promote moments of inclusion and equity. Findings from the MInE project illustrate that this, in turn, is conditioned by the governance and school structures within which teachers operate. Our interpretation is that when teachers' professional judgements, needs, and requests are acknowledged and

supported by the school organisation, the likelihood of cultivating moments of inclusion and equity in mathematics teaching is substantially enhanced.

In conclusion, this study offers a nuanced understanding of inclusion and equity in mathematics education and proposes principles aimed at bridging the persistent gap between societal discourses and classroom realities. By foregrounding both actionable and conceptual insights and by emphasising the dynamic and interchangeable nature of inclusion and equity, it seeks to theorise, rethink and advance inclusive mathematics education and support the cultivation of equitable, inclusive, and empowering learning environments.

References

Askew, M. (2015). Diversity, inclusion and equity in mathematics classrooms: From individual problems to collective possibility. In A. Bishop, H. Tan & T. N. Barkatas (Eds.), *Diversity in mathematics education* (pp. 129–145). Springer.

Abtahi, Y., & Planas, N. (2024). Mathematics teaching and teacher education against marginalisation, or towards equity, diversity and inclusion. *ZDM*, *56*(3), 307–318. https://doi.org/10.1007/s11858-024-01602-x

Atweh, B. (2011). Quality and equity in mathematics education as ethical issues. In B. Atweh, M. Graven, W. Secada, & P. Valero (Eds.), *Mapping equity and quality in mathematics education* (pp. 63–77). Springer. https://doi.org/10.1007/978-90-481-9803-0_5

Ainscow, M. (2020). Promoting inclusion and equity in education: lessons from international experiences, *Nordic Journal of Studies in Educational Policy, 6*(1), 7–16. https://doi.org/10.1080/20020317.2020.1729587

Bagger, A., & Roos, H. (2023). Moments of Inclusion and Equity in the Mathematics Classroom. Abstract presented at ECER 2023 in Glasgow.

Bagger, A. & Roos, H. (2024). Inkludering och likvärdighet i och genom matematikundervisning, MONA - Tidsskrift for undervisere, formidlere og forskere inden for matematik- og naturfagsdidaktik, 2024, saernummer, 50–65.

Bagger, A., & Vennberg, H. (2024). Care for the learning of mathematics–put to the test. *Research in Mathematics Education,* 1–19. https://doi.org/10.1080/14794802.2024.2339802

Berhanu, G. (2011). Inclusive Education in Sweden: Responses, Challenges and Prospects.*International journal of special education, 26*(2), 128–148.

Braun, V., & Clarke, V. (2021). One size fits all? What counts as quality practice in (reflexive) thematic analysis? *Qualitative Research in Psychology, 18*(3), 328–352. https://doi.org/10.1080/14780887.2020.1769238

DePauw, K. P., & Doll-Tepper, G. (2000). Toward Progressive Inclusion and Acceptance: Myth or Reality? The Inclusion Debate and Bandwagon Discourse. *Adapted Physical Activity Quarterly, 17*(2), 135–143. https://doi.org/10.1123/apaq.17.2.135

Ding, Y., Klapp, A., & Yang Hansen, K. (2024). The importance of mathematics self-concept and self-efficacy for mathematics achievement: A comparison between public and independent schools in Sweden. *Educational Psychology*, *44*(8), 872–892.

Enchikova, E., Neves, T., Toledo, C., & Nata, G. (2024). Change in socioeconomic educational equity after 20 years of PISA: A systematic literature review. *International Journal of Educational Research Open*, *7*, 100359-. https://doi.org/10.1016/j.ijedro.2024.100359

European Commission: European Education and Culture Executive Agency. (2022). Baïdak, N., Horváth, A., Motiejūnaitė-Schulmeister, A. and Noorani, S., *Increasing achievement and motivation in mathematics and science learning in schools*, Horváth, A.(editor),

Publications Office of the European Union, 2022, https://data.europa.eu/doi/10.2797/031821

Gutiérrez, R. (2022). A spiritual turn: Toward desire-based research and indigenous futurity in mathematics education. *Journal for Research in Mathematics Education*, *53*(5), 379–388. https://doi. org/10.5951/jresematheduc-2022-0005

Kollosche, D., Marcone, R., Knigge, M., Penteado, M. G., & Skovsmose, O. (Eds.). (2019). *Inclusive mathematics education: State-of-the-art research from Brazil and Germany*. Springer. https://doi.org/10.1007/978-3-030-11518-0

Lambert, R. (2015). Constructing and resisting disability in mathematics classrooms: A case study exploring the impact of different pedagogies. *Educational Studies in Mathematics*, *89*, 1-18. https://doi.org/10.1007/s10649-014-9587-6

Lindenskov, L., & Lindhardt, B. (2021). Exploring approaches for inclusive mathematics teaching in Danish public schools. *Mathematics Education Research Journal*, *32*(1), 57–75. https://doi.org/10.1007/s13394-019-00303-z

Messiou, & Ainscow, M. (2020). Inclusive Inquiry: Student–teacher dialogue as a means of promoting inclusion in schools. *British Educational Research Journal, 46*(3), 670–687. https://doi.org/10.1002/berj.3602

Muench, R., Wieczorek, O., & Dressler, J. (2023). Equity lost: Sweden and Finland in the struggle for PISA scores. *European Educational Research Journal, 22*(3), 413–432.

OECD. (2014). *PISA 2012 results: What students know and can do* (Volume I, Revised edition, February 2014). OECD Publishing. https://www.oecd.org/en/publications/2014/02/pisa-2012-results-what-students-know-and-can-do-volume-i-revised-edition-february-2014_g1g3e8ef.html

Roos, H., & Bagger, A. (2024). Ethical dilemmas and professional judgment as a pathway to inclusion and equity in mathematics teaching. *ZDM Mathematics Education*, *56,* 435-446. https://doi.org/10.1007/s11858-023-01540-0

Roos, H., & Bagger, A. (2021). Developing mathematics education promoting equity and inclusion: Is it possible? In D. Kollosche (Ed.), *Exploring new ways to connect: Proceedings of the Eleventh International Mathematics Education and Society Conference* (pp. 223–226). Tredition. https:/doi.org/10.5281/zenodo.5392881

Roos, H. (2019). Inclusion in mathematics education: An ideology, a way of teaching, or both?. *Educational Studies in Mathematics, 100*(1), 25–41.

Swedish National Agency for Education. (2023). *PISA 2022 15-åringars kunskaper i matematik, läsförståelse och naturvetenskap.* Rapporter, internationella studier. Skolverket.

Valero, P. (2017). Mathematics for all, economic growth, and the making of the citizen-worker. In T.S. Popkewitz, J. Diaz, & C. Kirchgasler (Eds.), *A political sociology of educational knowledge: Studies of exclusions and difference* (pp. 117–132). Routledge.

Promoting student engagement with feedback through formative assessment in mathematics

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For teacher feedback to be most beneficial for student learning, students would need to proactively seek and actively engage with it. Embedding feedback within a formative assessment practice has been proposed as a means to support such student engagement. In this study, we investigate a mathematics teacher's development of a formative assessment practice and the associated changes in students' engagement with feedback during individual task solving. Inductive thematic coding of classroom audio-recordings was used to identify characteristics of changes in student-teacher interactions, and Orthogonal Projections to Latent Structures Discriminant Analysis was employed to examine multivariate differences between lessons. Results indicate that changes in the teacher's practice were accompanied by increased student engagement with feedback.

Background

Several meta-analyses and systematic reviews have demonstrated that teacher feedback can yield substantial positive effects on student outcomes, including achievement both in general (Wisniewski et al., 2020) and specifically in mathematics (Palm et al., 2017; Van der Kleij & Lipnevich, 2021). However, effect sizes vary substantially, and Van der Kleij and Lipnevich (2021, p. 373) concluded that: "the relation between feedback and learning is still poorly understood, and past research sheds limited light on the specific mechanisms of feedback and its links to meaningful educational outcomes". One proposed partial explanation for this knowledge gap is the limited body of research examining how students interpret and respond to feedback (Winstone et al., 2017).

In general, students' intentional, proactive and constructive participation in learning situations has been shown to support positive outcomes such as enhanced learning (Patall, 2024). In line with these general results, scholars have suggested that to realize the full benefits of teacher feedback, students need to proactively seek and actively engage with the feedback, rather than remaining passive recipients (Winstone et al., 2017). In the context of mathematics task solving, such engagement involves articulating difficulties and specific support needs rather than leaving interpretation to the teacher, and constructing mathematical reasoning grounded in intrinsic properties of the task rather than relying on superficial recall of procedures (Lithner, 2008).

Mattsson, L., Eckert, A., & Kilhamn, C. (Eds.). (2026). *Talking Mathematics. Proceedings of MADIF 15*. Gothenburg University and SMDF.

It may be noted that engagement is not a fixed student attribute; rather it is shaped by contextual factors such as teaching. Yet, practical guidance on how to support students in shifting from passive to active engagement with feedback remains limited (Winstone et al., 2017). Most empirical research on the topic concerns higher education students and focuses on engagement with feedback for the purpose of resubmission of assignments (Winstone et al., 2017). Research on younger students' engagement with teacher feedback during task solving—particularly in the mathematics classroom—is scarce. Bishop (2021) and Ing et al. (2015) are two exceptions. Bishop (2021) found that in whole-class instruction, "the intellectual work requested by the teacher was related to the level of intellectual work students provided, acting as an upper bound on students' mathematical activity" (p. 466). Analysing whole-class and small-group settings, Ing et al. (2015) found that teacher elicitation of student thinking was positively related to students' production of fully articulated mathematical explanations. However, neither study examined changes in teacher practice or corresponding shifts in student engagement with feedback.

Embedding feedback in a formative assessment (FA) practice has been suggested as a means to support students' active engagement with the feedback (Palmberg et al., 2024). FA is an instructional practice in which teachers and/or students elicit evidence of learning needs and adapt feedback, learning activities, or learning strategies accordingly (Black & Wiliam, 2009). Meta analyses and systematic reviews across subjects and grade levels have consistently shown that such practices can substantially improve achievement, regardless of whether teachers or students act as the primary agents (e.g., Yao et al., 2024), including in mathematics (Palm et al., 2017). However, despite widespread promotion of FA in many countries, its high-quality implementation is uncommon in educational practice, including in Sweden (Andersson et al., 2017). Implementing FA in mathematics education entails specific challenges and changes that are not easy to accomplish. Changes may, for example, involve moving away from intuitive responses such as re-teaching or guiding students toward specific solutions, towards more responsive practices that take up and build on students' ideas (e.g. Jacobs et al., 2022). Such practices hold strong potential but are complex to enact, often requiring sustained professional support to develop both teachers' beliefs and their instructional repertoires.

Palmberg et al. (2024) documented increased student engagement in mathematics classrooms where FA was successfully implemented. However, the most common mathematics learning format internationally involves students working individually on tasks (Hiebert et al., 2003), including in Sweden (Boesen et al., 2014). Palmberg et al. (2024) did not focus on such settings, but the present study does. Here, we focus on FA practices where the teacher, as the primary agent, (1) elicits evidence of student knowledge and skills, (2) interprets this evidence to infer students' learning needs, and (3) provides feedback tailored to those needs. Such FA practices are intricate and challenging (Black & Wiliam, 2009), with weaknesses at any stage risk reducing the overall quality and effectiveness of FA (Palm et al., 2023).

Research questions

In the present study, we investigate an elementary school teacher's development of a FA practice in mathematics and the accompanying changes in students' engagement with feedback during teacher-student interactions when students sought help during individual task solving. We address the following research questions:

1. What changes occur in the teacher's FA practice?
2. What accompanying changes occur in students' engagement with the feedback process?

Methods

Procedure and data collection

Together with all mathematics teachers in two schools, the teacher participating in this study attended a three-year long professional development program (PDP) in FA. The PDP addressed a broad range of FA aspects, including the teacher as a proactive agent who elicits evidence of student learning and adapts feedback or subsequent learning activities to identified learning needs. It also addressed the teachers' role in supporting students to take a more active role in core FA processes through peer assessment and peer feedback, as well as self-assessment followed by learner-initiated actions to reach learning goals

During the second year the participating teacher volunteered for an additional individualized PDP focusing specifically on providing feedback to students when they seek help during individual task solving. We investigated changes in the teacher's practice and students' engagement with the teacher's feedback before, during, and after this individualised support. Data were collected through audio recordings of teacher–student interactions made by the teacher, and these interactions were inductively coded. The coded data were then analysed statistically to identify differences in both teacher and student actions from the beginning to the end of the individualised support period.

Participants

An experienced and highly engaged mathematics teacher, Lisa (pseudonym), volunteered to participate in the study. Twenty of her 25 students (11 female, 9 male; ages 11–12) also consented to participate, with informed consent provided by their legal guardians. The students represented diverse socio-economic and cultural backgrounds.

Professional development

Research has identified key features of effective professional development (PD), including access to instructional resources, opportunities for practice, interactive feedback, time for implementation, and external expertise (Heitink et al., 2016), along with the value of a formative orientation to the PD (Andersson & Palm, 2018). The individualised PD support in this study was designed to align with these principles.

The teacher made classroom audio recordings of her interactions with individual students when they sought help with task solving in March, May, October, and

November 2022, and in January and March 2023, typically two recordings per month. During the individualised PD meetings, initiated in October 2022, the last author provided both oral and written feedback to Lisa based on these recordings. Over the weeks that followed, Lisa used the feedback to further develop her FA practices at her own discretion. New recordings were then collected, followed by a new cycle of feedback. This iterative process continued for 6 months, with meetings held in October and November 2022, and February, March, and April 2023. Each meeting—including discussions of the researcher's feedback and Lisa's feedback to her students—lasted between 60 and 80 minutes.

The feedback provided during the meetings included successful examples of implementations of Lisa's FA practices, and guidance on how to further strengthen her FA. The support focused on the elicitation, interpretation, and feedback processes. It included strategies for eliciting information about students' thinking through questioning, encouraging students to articulate their reasoning, and helping them proactively share information about their understanding. Question types that could be used by both the teacher and students to make students' thinking explicit were discussed. Suggestions regarding feedback practices were also provided, emphasising feedback that is tailored to identified learning needs, supports students' construction of mathematical reasoning, and recognises progress while attributing success to internal, unstable, and controllable causes. Challenges in implementing desired FA practices were also examined, along with possible underlying causes, in order to identify feasible approaches for improvement. These approaches included introducing changes incrementally, starting with students more receptive to these changes, creating a supportive environment for students to practice new behaviours, and providing feedback that reinforces the benefits of, for example, sharing their thought processes. An example of feedback to the teacher was: "Is it clear [to students] how you are actually helping them thanks to what they told you they could and could not do? Can they see the positive consequences of you being better informed?"

Data Analysis

A comprehensive analysis of Lisa's FA practices was conducted through inductive thematic coding (Braun & Clarke, 2006) of classroom recordings collected between March 2022 and March 2023. This period encompassed time both before, during, and after Lisa began receiving individual feedback. In total, 351 minutes of audio recordings were encoded directly in MaxQDA (i.e., without prior transcription). Inductive codes were generated through close examination of the data capturing recurring actions, or patterns in the lessons. Through repeated reading and comparison, these initial codes were refined, grouped, and differentiated until they represented meaningful themes grounded in the data. Code development was carried out collaboratively by five researchers, resulting in high agreement regarding how each code should be applied.

The coding framework captured both concrete aspects of the feedback practices (e.g., telling a student an answer was incorrect) and broader qualities of practice, such as whether the teacher had elicited sufficient information about students' underlying

learning needs, as determined by the coder. Codes were also developed to describe students' engagement with the feedback process. In total, 61 codes were constructed. Using MaxQDA, segments of the audio recordings associated with each code were marked, and the time allocated to each form of FA practice and each type of student engagement was summarized for each lesson. Any uncertainties regarding code application were discussed collectively until consensus was reached, after which the final coding was conducted by one researcher to ensure consistency.

To examine multivariate differences between lessons conducted before and after Lisa began to receive individual feedback (RQ1), Orthogonal Projections to Latent Structures Discriminant Analysis (OPLS-DA; Bylesjö et al., 2006) was employed. OPLS-DA was selected due to the $p >> n$ structure of the dataset, with 61 coded variables (p = 61) derived from only 12 lessons (n = 12), a context in which approaches commonly used in educational research—even more advanced regression-based models—typically require more observations than predictors and rely on assumptions rarely met by coded qualitative data. Because OPLS-DA is designed for high-dimensional, collinear data (Trygg & Wold, 2002; Bylesjö et al., 2006), it allows patterns of co-occurring behaviours to be modelled by separating predictive from non-predictive variance and estimating the contribution of individual codes.

Preliminary analyses suggested no immediate effect of the researcher feedback, so lessons from spring 2023 were compared with lessons from 2022 to maximise contrast.

Results

The OPLS-DA analysis of qualitative codes revealed significant changes in Lisa's FA practices (RQ1) and in student engagement with the feedback process (RQ2) following the period of individualized professional development (Figure 1 and Table 1). The OPLS model demonstrated strong discriminatory power and produced no misclassifications. The OPLS-DA model demonstrated moderate-to-strong predictive validity ($Q^2 = 0.48$), a level considered substantial for qualitative classroom process data. The model accounted for 29% of the variation in coded lesson characteristics ($R^2X = 0.29$) and explained 97% of the variance in time-period classification ($R^2Y = 0.97$), with the Q^2 value indicating that the model generalised well to withheld data. Model reliability was further supported by permutation testing, where the original Q^2 value significantly exceeded those of 100 permuted models, $t(99) = 9.666$, $p < .001$, and the negative intercept (–0.17) confirmed a low risk of overfitting.

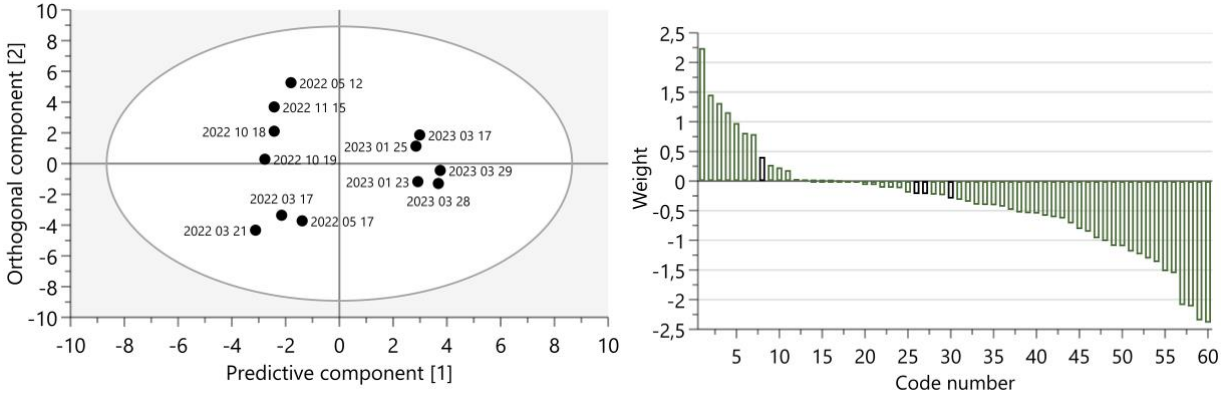
The resulting score plot (Figure 1, left) depicts the relative positioning of individual lessons within a reduced "feature space" derived from the qualitative codes and regression of temporal group membership onto these dimensions. The horizontal axis discriminates between lessons from 2022 and 2023, while the vertical axis captures variation unrelated to this temporal distinction.

The contribution plot (Figure 1, right) displays the extent to which each code contributes to the group separation in the OPLS model (Figure 1, left). In this case, it illustrates how the 2023 lessons differ from the 2022 lessons with respect to FA practices

and student engagement with the feedback process. These variables were measured as the total duration (in seconds) devoted to practices and engagement represented by the codes. Positive values indicate a higher presence of a given code in 2023, whereas negative values indicate a lower presence, compared to 2022. The length of each bar reflects the relative importance of the variable for distinguishing between the two groups.

Table 1 describes the codes in Figure 1 on the right. Due to space limitations, only codes with a substantial variation between 2022 and 2023 are described in the table.

Figure 1. Left: Score plot showing temporal group affiliation of the individual lessons and similarities and differences between individual lessons in terms of code occurrences. Right: Contribution plot illustrating relative contribution of codes to the separation of 2022 and 2023 lessons. Codes with weights > 0.5 are listed in Table 1.



Changes in the teacher's FA practice (RQ 1)

The OPLS-DA data analysis (Figure 1 and Table 1) furnishes specific insights into the shifts within Lisa's classroom practices. Notably, towards the end of the study (2023) Lisa increasingly elicited information concerning students' needs, instead of merely checking that they had understood what the task was about (Codes 4, 50, 54 in Table 1). Her feedback also became more prominently grounded in addressing students' learning needs (Codes 1, 59, 60), and more often requiring the students to construct reasoning that is mathematically founded (Codes 1, 5). Additionally, it was evident that Lisa, towards the end of the study, incorporated more feedback in the form of hints and prompts (Codes 1, 2, 5).

Changes in student engagement with the feedback process (RQ 2)

In 2023, students were more engaged in elaborate reasoning during their interactions with the teacher and described specific gaps in their understanding (Codes 3, 7). In contrast, in 2022 students' engagement with the feedback process was simpler in character, often consisting of short answers—such as "five"— (Codes 55, 58). Students were also more likely to seek affirmation that they were ´doing´ the mathematics

correctly (Code 57), and describing the task and what they have done on the task so far (Codes 40, 43), rather than explaining their reasoning. Instances where students did not understand the teacher's feedback were also more widespread (Code 53).

Table 1. Descriptions of codes of teacher practice and student engagement. Only codes with substantial influence on group separation are displayed. Variables with absolute weights below 0.5 are omitted for brevity. TE denotes teacher elicitation, TF teacher feedback, and S student engagement with the feedback process.

No.	Descriptions	No.	Descriptions
Codes with higher representation in the 2023 lessons (falling order of significance)			
1	TF: Hint that requires student to construct mathematically founded reasoning. Difficulty addressed.	2	TF: Suggests a next step in a solution. Unclear relation to student difficulty. Does not require student to construct mathematically founded reasoning.
3	S: Elaborate partially correct answer	4	TE: Asks about student difficulty
5	TF: Hint that requires student to construct mathematically founded reasoning. Difficulty not addressed.	6	TF: Indicates/states the answer is wrong. Difficulty not addressed.
7	S: Describes their difficulty		
Codes with higher representation in the 2022 lessons (falling order of significance)			
60	TF: Confirms correct response. Difficulty not addressed.	59	TF: Funnelling. Difficulty not addressed
58	S: Short incorrect answer	57	S: Asks if solution is correct
56	S: Silence	55	S: Short correct answer
54	TE: Checks task understanding	53	S: Does not understand the feedback
52	TF: Praise strategy choice/implementation	51	TF: Describes in what way the answer is wrong. Difficulty addressed.
50	TE: Ignores/does not notice student's need description	49	TF: Indicates/states the answer is wrong. Unclear relation to student difficulty.
48	TF: Funnelling connected to student difficulty.	47	TF: Describes solution algorithm. Difficulty not addressed.
46	TF: Describes in what way the answer is wrong. Unclear relation to student difficulty.	45	TF: Describes solution algorithm. Difficulty addressed.
44	TF: Praises thoroughness of work	43	S: Describes the task
42	S: Recalls solution algorithm	41	TF: Confirms correct response. Difficulty addressed.
40	S: Describes prior work on task	39	S: Short partially correct answer

Discussion

In this study, Lisa received individualised professional development support incorporating features known to be critical to generate changes in teachers' FA practices (Andersson & Palm, 2018; Heitink, et al., 2016). The results show that she used this support to make substantive shifts in her FA practice, reshaping the interactional dynamics of the classroom. In 2022, her feedback was predominantly corrective, guiding students toward a "correct" problem-solving procedure without systematically drawing on evidence of their learning needs. Over time, however, her feedback became increasingly attuned to those needs, more frequently taking the form of hints, prompts, and questions that required mathematically founded reasoning. This shift provided

students with greater opportunities for deep cognitive engagement and encouraged them to articulate the difficulties they encountered and to make their reasoning explicit.

The instructional changes were mirrored in students' corresponding shifts in students' engagement with feedback. In 2022 students tended to offer brief answers, seek confirmation of correctness (of their answer or method), or remain silent in response to feedback. By 2023, however, they more frequently produced elaborated explanations and explicitly described gaps in their understanding. Such behaviour suggests a shift from viewing mathematics primarily as "task completion" toward treating it as a process of reasoning and sense-making.

The reciprocal alignment between teacher practices and student engagement with the feedback process highlights a core principle of FA described by Black and Wiliam (2009): effective FA depends not only on the teacher's ability to generate evidence of learning but also on students' capacity to interpret and act on that evidence. Lisa's increasing emphasis on eliciting students' reasoning—and providing feedback in the form of hints that enabled students to build on their existing thinking—appears to have supported the dialogic nature of FA, allowing students to participate more productively in the feedback process.

The relationship observed here resonates with findings from Bishop (2021), Ing et al. (2015), and Palmberg et al. (2024), although our study focussed specifically on individual teacher-student interactions during task solving and its reciprocal nature, a context less emphasised in previous work. Bishop (2021), studying whole-class instruction, found that the intellectual work demanded by the teacher was associated with the level of intellectual work students contributed. Similarly, Ing et al. (2015) reported that when teachers elicited students' thinking in whole-class and small-group settings, students provided more detailed explanations, and engaged more productively with peers' reasoning. Finally, Palmberg et al. (2024), investigating whole-class FA, showed that when FA was developed in ways that positioned both teacher and students as proactive agents, the students to a larger extent than before initiated and maintained learning activities without the teacher telling them to, self-assessed and acted on those assessments, asked the teacher to provide more instruction, and asked fewer but more elaborate questions.

The present study extends this body of work by demonstrating that FA can positively influence compulsory school students' engagement with feedback specifically in one-to-one task solving situations in mathematics. Also, it illustrates the kinds of shifts in student engagement that may occur when FA practices develop along certain dimensions and shows how these changes align with theoretical expectations and findings from other instructional settings in mathematics education.

Limitations and future research

This study offers novel insights into how changes in teaching practice may relate to student engagement with the feedback process, but several limitations should be noted. Our approach is limited in that teacher practice at a specific point in time during a lesson is not directly linked to student engagement with that particular feedback process.

Instead, how much time each agent spends on certain behaviours during each lesson are aggregated at the lesson level. In addition, the use of only inductive coding of student engagement limits the specificity of connections to previous research, and the study design (with changes of one teacher and her class) naturally constrains the generalizability of the conclusions.

Future research could extend this work in several ways. The use of OPLS-DA proved valuable for analysing complex interaction data, and future studies might apply this statistical method to data with codes where the teacher practice and student engagement is connected on a more detailed level than the lesson. This could for example be achieved through an analysis producing codes for larger chunks of the interaction, where teacher feedback and student engagement are directly connected. Including more teachers and students would provide clearer evidence of generalizability. Finally, although embedding feedback in FA appears promising, comparative studies of alternative approaches to promoting engagement with the feedback process could help shed further light on this relatively new area of research.

References

Andersson, C., Boström, E., and Palm, T. (2017) Formative Assessment in Swedish Mathematics Classroom Practice. *Nordic Studies in Mathematics Education 22*(1), 5–20. https://doi.org/10.7146/nomad.v22i1.148754

Andersson, C., & Palm, T. (2018). Reasons for teachers' successful development of a formative assessment practice through professional development—a motivation perspective. *Assessment in Education: Principles, Policy & Practice, 25*(6), 576–597. https://doi.org/10.1080/0969594X.2018.1430685

Bishop, J. P. (2021). Responsiveness and intellectual work: Features of mathematics classroom discourse related to student achievement. *Journal of the Learning Sciences, 30*(3), 466–508. https://doi.org/10.1080/10508406.2021.1922413

Black, P., & Wiliam, D. (2009). Developing the theory of formative assessment. *Educational Assessment, Evaluation and Accountability, 21,* 5–31. https://doi.org/bsqbf2

Boesen, J., Helenius, O., Bergqvist, E., Bergqvist, T., Lithner, J., Palm, T., Palmberg, B. (2014), Developing Mathematical Competence: From the Intended to the Enacted Curriculum. *Journal of Mathematical Behavior,* 33, 72–87. https://doi.org/10.1016/j.jmathb.2013.10.001

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative research in psychology, 3*(2), 77–101. https://doi.org/10.1191/1478088706qp063oa

Bylesjö, M., Rantalainen, M., Cloarec, O., Nicholson, J. K., Holmes, E., & Trygg, J. (2006). OPLS discriminant analysis: combining the strengths of PLS-DA and SIMCA classification. *Journal of Chemometrics: A Journal of the Chemometrics Society, 20*(8–10), 341–351. https://doi.org/10.1002/cem.1006

Heitink, M.C., Van der Kleij, F.M., Veldkamp, B.P., Schildkamp, K., & Kippers, W. B. (2016). A systematic review of prerequisites for implementing assessment for learning in classroom practice. *Educational Research Review, 17,* 50–62. https://doi.org/gf5z3k

Hiebert, J., Gallimore, R., Granier, H., Bogard Givvin, K., Hollingsworth, H., Jacobs, J., Miu-Ying Chui, A., Wearne, D., Smith, M., Kersting, N., Manaster, A., Tseng, E., Etterbeek,

W., Manaster, C., Gonzales, P., Stigler, J. (2003), *Teaching Mathematics in Seven Countries: Results from the Timss 1999 Video Study.* Washington, DC: DIaNe Publishing.

Ing, M., Webb, N. M., Franke, M. L., Turrou, A. C., Wong, J., Shin, N., & Fernandez, C. H. (2015). Student participation in elementary mathematics classrooms: the missing link between teacher practices and student achievement? *Educational studies in mathematics, 90*(3), 341–356. https://doi.org/10.1007/s10649-015-9625-z

Jacobs, V.R., Empson, S.B., Jessup, N.A., Dunning, A., Pynes, D.A., Krause, G., & Franke, T.M. (2022). Profiles of teachers' expertise in professional noticing of children's mathematical thinking. *Journal of Mathematics Teacher Education,* 1–30. https://doi.org/10.1007/s10857-022-09558-z

Lithner, J. (2008), A research framework for creative and imitative reasoning, *Educational Studies in Mathematics, 67*(3), 255–276. https://doi.org/10.1007/s10649-007-9104-2

Palm, T., Andersson, C., Boström, E. & Vingsle, L. (2017). A review of the impact of formative assessment on student achievement in mathematics. *Nordic Studies in Mathematics Education, 22*(3), 25–50. https://doi.org/10.7146/nomad.v22i3.148889

Palm, T., Andersson, C., Palmberg, B., & Winberg, M. (2023). Mechanisms underlying effects of formative assessment on student achievement: A proposed framework to ignite future research. *CADMO, 2*, 9–20. https://doi.org/10.3280/CAD2023-002002

Palmberg, B., Granberg, C., Andersson, C., & Palm, T. (2024). A multi-approach formative assessment practice and its potential for enhancing student motivation: a case study. *Cogent Education*, *11*(1). https://doi.org/10.1080/2331186X.2024.2371169

Patall, E. A. (2024). Agentic engagement: Transcending passive motivation. *Motivation Science*, *10*(3), 222. https://doi.org/10.1037/mot0000332

Van der Kleij, F. M., & Lipnevich, A. A. (2021). Student perceptions of assessment feedback: A critical scoping review and call for research. *Educational assessment, evaluation and accountability*, *33*(2), 345–373. https://doi.org/10.1007/s11092-020-09331-x

Winstone, N. E., Nash, R. A., Parker, M., & Rowntree, J. (2017). Supporting learners' agentic engagement with feedback: A systematic review and a taxonomy of recipience processes. *Educational psychologist*, *52*(1), 17–37. https://doi.org/10.1080/00461520.2016.1207538

Wisniewski, B., Zierer, K., & Hattie, J. (2020). The power of feedback revisited: A meta-analysis of educational feedback research. *Frontiers in psychology*, *10*, 487662. https://doi.org/10.3389/fpsyg.2019.03087

Yao, Y., Amos, M., Snider, K., & Brown, T. (2024). The impact of formative assessment on K-12 learning: a meta-analysis. *Educational Research and Evaluation*, *29*(7–8), 452–475. https://doi.org/10.1080/13803611.2024.2363831

Two teachers' experiences of a large-scale scripted mathematics teaching program intervention

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Through phenomenological analysis, this extreme case study identifies conflicts as key phenomena when two teachers engage in a large-scale, scripted mathematics teaching program. The conflicts relate to the teachers' professional expertise, societal and policy expectations and demands, and didactical considerations. We conclude that scaling should not only be understood as scaling up, but rather as scaling down - adapting programs to specific local contexts, and that implementation of scaled research findings ought to consider teachers' professionalism as a matter of ethics.

The issue of bringing research evidence into teachers' practices has long been debated. Already in 1975, Stenhouse suggested that researchers and teachers work closely together to successfully connect research findings and teaching practice (Elliot, 2024). Stenhouse's suggestion for the implementation of research is small-scale, highly contextualised, and requires teachers' conceptual engagement with research findings. In contrast, instrumental use of research findings leaves teachers to implement a particular teaching model in their classrooms without any deeper conceptual engagement with the research findings that support the model (Sjölund et al., 2022). An extreme version of instrumental implementation of research findings is the so-called scripted teaching programs (Ede, 2006), which have recently gained popularity in Sweden, particularly for mathematics education in the early school years. Several municipalities have enforced sweeping implementation of scripted teaching mathematics curriculum reforms targeting thousands of teachers and their students (see Helenius, 2022), hoping to improve students' mathematics learning. Scripted teaching programs leave little or no room for teachers to adjust the curriculum to their own teaching practice and context (Ede, 2006; Zhang & Sweller, 2023). In other words, research findings have been implemented at scale with no possibilities for adjustments to local context or personal teaching practices. This is in stark contrast to Swedish educational culture, where, traditionally, a policy of empowerment that gives teachers a great deal of autonomy in their teaching has prevailed (Ryve & Hemmi, 2019). This raises questions about the effects of enforced sweeping implementation of scripted teaching mathematics curriculum reforms, not only in terms of its potential efficiency, but also about teachers' lived experiences with the program, which is the focus of the present study.

Hence, the present case study investigates two teachers' experiences of participation in a large-scale scripted mathematics teaching program in Sweden concerning the

Mattsson, L., Eckert, A., & Kilhamn, C. (Eds.). (2026). *Talking Mathematics. Proceedings of MADIF 15*. Gothenburg University and SMDF.

implementation process, the content, the teaching methods, and outcomes. This aligns with Ahl et al.'s (2022) conclusion that small-scale studies are needed to complement large-scale evaluations, as they are essential for gaining an in-depth understanding of what works and why. This study adopts a phenomenological approach (Sohn et al., 2017) by focusing on the meanings that two teachers, Anna and Hanna (pseudonyms), ascribe to their experiences. The study provides empirical insights about the implementation of a scripted teaching program at the "grassroots level". The guiding questions are: How do teachers experience a scripted teaching program? What phenomena emerge from the teachers' experiences?

Background

Scripted teaching programs prescribe, in detail, instructions for how the teaching shall unfold in the classroom. This may include what the teacher must say, how to treat expected student responses, what activities to perform for how long, and so forth. The idea of prescriptive teaching is to strongly regulate teachers' actions to ensure that all students receive the same quality of teaching (Ede, 2006; Zhang & Sweller, 2023). The intention of scripted teaching is to target inequity in instruction quality and the achievements of underperforming students. There is uncertainty about the efficiency of scripted programs. For example, in the US, where scripted teaching programs were launched broadly at the beginning of the millennium, inequity in students' performance remains (Vaughn et al., 2022). Scripted teaching offers a fixed program with little or no possibilities to make pedagogical adaptations. This means, for example, that considerations based on the didactic triad of teacher, students, and content become a matter for the designer of the script rather than for the teacher to consider when planning, conducting, and reflecting on their teaching.

In the context of scaling up, Palmér et al. (2025) discuss the tension between maintaining fidelity and allowing adaptation during a large-scale implementation of a preschool mathematics innovation in Sweden. Their findings indicate that when teachers adhered strictly to the scripted teaching, their teaching became artificial and disconnected from children's experiences, needs, and interests. In contrast, productive adaptations were associated with greater spread, sustainability, and teacher ownership, hence key indicators of successful implementation. Ryve and Hemmi (2019) drew similar conclusions in their study on the role of implicit contextual factors, such as the positionings of teachers in the classroom and in the educational system, and visible and invisible pedagogical traditions, for implementing large-scale projects in Sweden. They emphasised the importance of addressing implicit contextual factors when scaling-up.

The scripted teaching program that the two teachers participated in targets half of the mathematics lessons in each school year from Grades 1 to 3. The program has been scaled pedagogically to cover three years of mathematics education based on findings from a 12-week-long randomized control experiment with 62 pre-school class students in the experiment group, and 62 in the control group (Sterner et al., 2020). Furthermore, the program has been scaled to cover an intervention involving at least 14 municipalities

and more than 1,000 teachers and their students (Ahl et al., 2022). To support implementation of the presumed highly complex scripted teaching program, an advanced implementation model that comprised stakeholders at different levels was used (Helenius, 2022). The model included the design of a stakeholder group of teachers, so-called process leaders. While implementation strategies and teachers' fidelity to the scripted teaching program (Ahl et al., 2022) have been investigated, less is known about teachers' experiences of participating in the program.

Theory – a phenomenological approach

We use hermeneutic phenomenology, drawing primarily on Sohn et al.'s (2017) interpretation of Merleau-Ponty's philosophy of perception and meaning to focus on Anna and Hanna's lived experiences of the scripted teaching program. Lived experiences refer to how people perceive and make sense of their experiences with an emphasis on the subjective and interpretive dimension of human consciousness. Humans are not passive; we take intentional stances towards things and occurrences. Hence, phenomena appear as meaningful wholes against a contextual ground. Lived experiences include subjective perceptions, emotions, and social interactions (Sohn et al., 2017). The aim of phenomenological analysis is to identify core meanings of the phenomenon that reveals itself through analysis of lived experiences (Sohn et al., 2017). In phenomenological analysis, bracketing or epoché is required, meaning that researchers set aside preconceived understandings and biases to understand the phenomena at stake (Sohn et al., 2017). Skovsmose (2014) argued that epoché requires researchers' "pure" and transparent noticing, which is not possible since our consciousness can be "dubious, distorted, mischievous, or simply wrong" (p. 44). Relatedly, Sohn et al. (2017) suggested that researchers use a bracketing procedure to heighten self-awareness of pre-understandings that could influence data collection and interpretation. We recognise that opaqueness influenced what the teachers said and what we noticed in the analysis, which has implications for the findings of this study.

Methodology – researching teacher experiences

To gain an understanding of teachers' experiences of a scripted teaching program, we focused on the meaning they ascribed to those experiences. Hence, a case study (Flyvbjerg, 2006) with a phenomenological approach was conducted. Data was generated through individual semi-structured in-depth interviews with two teachers working in schools in a municipality characterized by high socioeconomic status. This qualifies as an extreme case (Flyvbjerg, 2006) as the study is situated in an unusual Swedish context marked by high student performance in mathematics and a large proportion of parents with academic backgrounds. The teachers were selected due to their lived experiences with participating in and working with the program and their willingness to participate. Informed consent for participation was obtained according to The Swedish Research Council's guidelines (2017). Hanna and Anna have great experiences of working with elementary school mathematics. Hanna has worked as a

teacher for 15 years and Anna for three decades. They have extensive experiences of professional development and leadership. Anna has been involved in the two national professional development initiatives: *Matematikutvecklare* and *Matematiklyftet*. In these initiatives, her role was to lead professional development processes. Hanna was appointed process leader by the municipality and had the role of leading a professional development group of colleagues in the process of implementing the scripted teaching program. Hence, Hanna has experience in both supporting her colleagues' work with implementing the program and in her own implementation in a second-grade elementary mathematics classroom. At the time of the implementation, Anna taught mathematics to first-grade elementary students. The program is designed to target all kinds of teachers and students in all kinds of contexts, including "extreme" contexts like the one in this study, where mathematics teaching generates high student performance compared to the national average. This suggests that the teachers have experiences of successful teaching practices established before the program's implementation, which they can relate to their experiences with the program. With externally recognised expertise and a proven record of producing high-achieving students, they were assumed to be more confident in sharing nuanced perspectives on the program. Hence, contributing with important knowledge from a professional "expert" perspective at grassroot level that may not be possible to receive from cases in "regular" contexts.

The interview guide comprised the following themes: teachers' backgrounds; program impact; experiences of implementation; mathematical content and progression; pedagogical design; influence on students' mathematics outcomes; and parents' and colleagues' perceptions of the program. Interview questions were asked in terms of "What are your experiences of….?", "How did you experience….?" or "Could you describe….?" as suggested by Sohn et al. (2017). Follow-up questions were sometimes used for clarification purposes. Both interviews lasted about one hour and were audio-recorded and transcribed verbatim.

Data processing and analysis

The analysis was conducted in four steps that were inspired by, but did not fully follow, the suggestions for phenomenological analysis by Sohn et al. (2017). *First*, we processed the data by reading the interview transcripts several times, individually paying attention to and marking small units of meaning. *Second*, jointly we discussed our readings and coded data by being mindful of recurring patterns (see Sohn et al., 2017), resulting in the following codes: uncertainty, resistance, loyalty, student performance, and the teacher role. When there were overlaps between codes, meaning that an excerpt could be categorised as belonging to two codes, it was most often categorised as belonging to both codes. *Third*, since phenomena appear as meaningful wholes against a contextual ground, we sought to identify shared phenomena as they emerged through the teachers' lived experiences. We identified *conflicts* as a shared phenomenon, more specifically, conflicts that the teachers needed to handle when participating in the program. *Fourth*, we thematised the coded turns as meaningful wholes in terms of conflicts. We jointly discussed the thematisation of conflicts. It resulted in conflicts related to a) teachers'

professional expertise, b) societal and policy expectations and demands, and c) didactical considerations. The excerpts included and explored in the findings below are selected to illustrate variation within the themes.

Findings

Anna and Hanna initially had a positive approach towards the implementation of the program. But gradually their approach changed. Anna, in agreement with her headmaster, began to leave the program during the first year and left completely during the second year. "I followed the program less and less, and in grade 2, I did what I wanted because I felt that my students did not learn what my professional judgment says that students in Grade 1 need to know". Hanna, in her role as process leader, felt obligated to support the implementation, but she left after two years:

> Hanna: I told those who made the decision to implement the program that I will not jeopardize my students' mathematics development for the sake of following the program. If I feel that I lose my students, I will leave the program, and so I did. I could not be the teacher that I kind of stand for.

Evidently, both Anna and Hanna experienced conflicts with implementing the program. Below, we unpack their experiences of conflicts in three overarching themes.

Conflicts related to teachers' professional expertise

Anna and Hanna wanted to fulfil the program, but the implementation caused conflicts with their professional teaching expertise. Hanna considered herself competent in pedagogy and didactics, but when following the program, she said she had to relearn, which made her feel less competent, particularly in mathematics. This caused a conflict or "a clash with my professional teacher self," as Hanna said. Anna, who said that "the biggest mistake was to remove the teachers' professionalism and personalities," had similar experiences:

> Anna: But why should someone else design lessons for me? There are many good things in the program that I can use in my way because I am a professional. But there was a manuscript to follow. I was stressed about my students not learning what I expected them to learn. This, despite the fact that it was a good group of students who were very willing to learn.

Anna said the program made teaching mathematics less enjoyable: "It was not so much fun to teach mathematics. I did not lose my interest in mathematics, but I was less engaged and felt frustrated". Hanna shared that following the lesson manuscript word for word made the teaching impersonal and rigid, making it difficult to engage the students in the tasks and foster a sense of joy and excitement in learning mathematics, something she considered an essential part of being a good teacher.

Hanna also seemed to experience a conflict between her professional teaching expertise and the research findings that the program draws on. She acknowledged the scientific expertise of the program's designers and their competence in mathematics and

wanted to trust their competence; however, based on her own professional experience, she could not see how the program could be successfully implemented in her classroom:

> Hanna: They [the program designers] say that it doesn't matter if the students do not understand. "It is ok. They will understand. Have patience. It is like this". I have understood it as they rely on their solid research findings and are way more competent in mathematics than me. I am sure that is true. I am humbled that I am a professional in my classroom with my students, but I am absolutely not a professional in mathematics at the same level as the designers. So, during the whole period, I was humble towards the idea that someone else knows better than I.

Anna seemed to experience a conflict between professional development that recognises her professional teaching expertise and knowledge about her students, and the implementation and collegial work of the program:

> Anna: It was a big difference from Matematiklyftet which was very good because there you could adapt the tasks to your own context. It was not possible in this program, so there was not much collegial learning. It was more about practical stuff like "How long will it take to make paper copies for next week?" As teachers, we must critically scrutinize and think critically all the time about what we do. We cannot just "buy" things simply because somebody else decided. Usually, you have the freedom to adapt stuff to your own context and to your professional teacher self. Here I felt like my professional teacher self disappeared.

Conflicts related to societal and policy expectations and demands

The teachers seemed to experience conflicts that refer to obligations and perceived expectations that society and policy may put on teachers. On the one hand, they wanted to comply with the municipality's decision that all teachers should follow the program without making any adjustments. On the other hand, they wanted to fulfil the expectations of delivering students who demonstrate high levels of mathematics performance and have a profound interest in mathematics.

Anna seemed unsure why the specific program would be more efficient than some other mathematics teaching approaches:

> Anna: I did not get a background on why this should be implemented. "Why should this way be so much better than some other way [of teaching mathematics]?" The background was that this would increase the students' interest and result in mathematics. This would be a very good way. I don't know who made the decision at the municipality level.

Hanna shared experiences of uncertainty that referred to students' achievements in relation to the purpose of implementing the program: "From what I understand, it was about that we saw decreasing results in Grade 9, but we did not have decreasing results in Grade 3". Hanna's experiences refer to conflicting issues about the progression of learning and policy requirements. Hanna experienced that Grade 1 to 3 teachers in her municipality delivered according to policy requirements and thus fulfilled their teacher

obligations. Conflicts at this level also include concerns about students' achievements and teachers' loyalty to the program versus the national mathematics syllabi for grades 1 to 3. For instance, Anna said that the program failed to meet the syllabus requirements and learning objectives, and that it caused stress for her and many colleagues.

The conflict that Anna perceived between being loyal to the program and the fact that the program did not provide conditions for Anna's students to meet the learning objectives was so strong that Anna decided to stay loyal to the national policy requirements rather than to the program. Hanna shared that Anna's students scored highest in the municipality at the national test in Grade 3. Anna said that teachers at her school who had been more loyal to the program in Grades 1 and 2 struggled in Grade 3 to cover content that was not part of the program but was required to pass the national tests.

Conflicts related to didactical considerations

The didactical conflicts that the teachers seemed to experience refer to issues that relate to didactical questions such as What? How? For whom? When? and Why?.

Anna and Hanna emphasised the need for varied instruction tailored to students' diverse needs, something the program did not allow. This and the lack of variation in teaching instructions created a conflict with the didactical structure of the program:

> Anna: It was repetitive to work only with the number line or with patterns for four weeks. I thought that it was quite boring, and the children did not find it fun either. They also wanted some variation, and they need to learn in many different ways. Many years ago, I completed a Montessori pedagogy course. This program is similar to that in the sense that there is only one way, which is the right way. That makes things very black or white, and that is not how it works in reality.

According to Anna, "Most often the program suited middle level students. You totally lose the weak students, and the strong students often lost interest since they solved the tasks straight away". Hanna shared the same experience, which seemed to cause a conflict because not all students had equal opportunities to learn the intended material when teaching according to the didactical structure of the program:

> Hanna: Those who are very strong and very weak kind of fell between the chairs. The weak ones just copied from their peer and could not move on by themselves. Not all in Grade 1 can easily express their thinking verbally. It is first in Grades 2 and 3 that all students manage to do that. Since not all students can express themselves mathematically, the conversations did not turn out in the way that I think was the intention of the program. Also, the strong students lose in this program. They finish the tasks superfast. It was stressful to keep them challenged so that they would develop their mathematical thinking, too, because there were no appropriate tasks for these students in the program.

Another didactical conflict concerns the very long introduction phases that the program required that made students lose focus on the mathematics tasks:

> Hanna: Not all students manage that. Other teachers reacted strongly to the long introductions. The students completely zoomed out. They focused on everything except for mathematics. Usually, I have a pretty fast introduction, and then the students work in teams. It depends on what kind of students I have. I need to know my students and know what they know before I can start this kind of teaching, so I know how to place them in teams.

Hanna stresses the importance to know the students' prerequisites. Similarly, Anna said that she could never take her specific group into account when teaching according to the program, which she experienced as impeding.

Both teachers experienced didactical conflicts when they had to progress according to the manuscript even if the students had not learned the intended. Anna said that "the students did not have time to consolidate stuff, you just had to move on to new stuff, and that made many stressed. The answer you got from the program leaders was "Do not worry that they have not understood. This content will also appear in Grades 2 and 3".

Further, Anna felt frustrated because the didactical structure of the program did not allow her to adapt and help the students, which caused a conflict with her social obligations towards her students:

> Anna: For example, the empty number line with the instruction to mark all the numbers that they knew. The students asked questions about what to do, and the only answer that I was allowed to give was that they had to decide for themselves. But they are small children. You need to show them different models and strategies so that they can learn to think mathematically on their own. This was very frustrating. When I brought it up at the meetings, I again got the answer that it is ok that they do not know, they will understand later, this will be good for them when they do lower secondary school mathematics. But there is a long time before lower secondary school, we cannot lose them or their interest in mathematics already in elementary school.

Concluding remarks

The conclusions, based on this small study, contribute to an understanding of how research findings are transferred and scaled from one site to another, which has been discussed previously (e.g., Ahl et al., 2022; Palmér et al., 2025; Ryve & Hemmi, 2019). Given the fact that teacher autonomy is part of Swedish educational culture (Ryve & Hemmi, 2019), the conflicts that Anna, Hanna, and their colleagues experienced when they implemented the scripted teaching program may be expected. The conflicts that the teachers experienced appeared to be multidimensional, comprising conflicts at professional, societal, policy, and didactical levels. Nevertheless, the conflicts raise questions about how to handle results from successful small-scale intervention studies. This particular program seems to have been scaled based on results from a small-scale intervention study in at least two different ways. First, it was scaled didactically. The

original intervention study, on which the scripted program in this study is based, was conducted in preschool classrooms for 12 weeks (Sterner et al., 2020). In the scripted teaching program, the findings were scaled to cover arithmetic and algebra content for Grades 1 to 3 of the compulsory school. This raises questions about scaling small-scale research findings across grades. Can we infer what works for 10-year-olds from what works for 6-year-olds?

Second, the small-scale findings were scaled to include 1000 teachers and their approximately 20 000 students. This means that the small-scale findings were scaled across different contexts. The present study is an extreme case study (Flyvbjerg, 2006) because the interviewees were very experienced teachers teaching in a municipality that is extreme in terms of students' high mathematics performance, high socio-economic standards, and high levels of education among residents. It is possible to imagine other (extreme) cases too. For instance, the opposite of the context of the present study. Anna pointed out that the scripted teaching program may work well for inexperienced teachers, which may very well be the case. That is not our point. Our point is that successful small-scale interventions must clearly document the context and conditions of their success to enable scalability.

Based on the two mentioned points above, and by drawing on the findings of Palmér et al.'s (2025) and Ryve and Hemmi's (2019) studies that emphasise the need for contextual adaptations to the implementation when scaling up research findings, we conclude that scaling may not be a matter of scaling up but of scaling down. By scaling down, we refer to making adaptations to specific contexts so that findings of small-scale studies can be meaningful and useful in different contexts.

Anna and Hanna seemed to experience conflicts between the program's demand not to help students out and seeing their students lose interest in mathematics without having the opportunity to fulfill their social obligation towards their students. Nor were they allowed to "justify themselves" for having given a problem that was too difficult and make adaptations of the program tasks that better suited their students. The social obligations that teachers have towards students raise ethical questions about professional teacher responsibility – the same professional responsibility that Anna and Hanna experienced being stripped of when they followed the program. In addition to issues about conceptual and/or instrumental use of research findings (Sjölund et al., 2022) or tensions between fidelity and adaptation (Palmér et al., 2025), this study points towards ethical dimensions in relation to teaching programs with no room for adaptations that need to be considered when scaling findings. Hence, the implementation of scaled research findings ought to consider teachers' professionalism as a matter of ethics.

This is a small-scale, extreme case (Flyvbjerg, 2006) interpretative study (Sohn et al., 2017). Our pre-understandings of teacher autonomy and vast experiences as mathematics teachers have influenced our analysis. To enhance trustworthiness, we decided to share several of Anna's and Hanna's quotes in this study.

References

Ahl, L. M., Helenius, O., & Koljonen, T. (2022). Gauging fidelity to an implemented teaching model through the lens of the documentational approach to didactics. *Implementation and Replication Studies in Mathematics Education*, *2*(1), 45–75. https://doi.org/10.1163/26670127-bja10003

Ahl, L. M., Helenius, O., Aguilar, M. S., Jankvist, U. T., Misfeldt, M., & Prytz, J. (2023). Implementation research in mathematics education: A systematic mapping review. *Implementation and Replication Studies in Mathematics Education, 3*(2), 135–199. https://doi.org/10.1163/26670127-bja10015

Ede, A. (2006). Scripted Curriculum: Is it a Prescription for Success?. *Childhood Education, 83*:1, 29–32. https://doi.org/10.1080/00094056.2006.10522871

Elliott, J. (2024). The Stenhouse legacy and the development of an applied research in education tradition. *The Curriculum Journal*, *35*(4), 706–721. https://doi.org/10.1002/curj.291

Flyvbjerg, B. (2006). Five Misunderstandings About Case-Study Research. *Qualitative Inquiry*, *12*(2), 219–245. https://doi.org/10.1177/1077800405284363

Helenius, O. (2022). Designing stakeholder scaffolding for implementing a teaching model at scale. In *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education* (CERME12), Feb 2022, Bozen-Bolzano, Italy. hal-03765106.

Palmér, H., Elofsson, J., & Björklund, C. (2025). Success in implementation in early mathematics education- Fidelity and/or adaptation?. In *Proceedings of the Fourteenth Congress of the European Society for Research in Mathematics Education* (CERME 14), Feb 2025, Free University of Bozen-Bolzano, Italy. Hal-05300180.

Ryve, A., & Hemmi, K. (2019). Educational policy to improve mathematics instruction at scale: conceptualizing contextual factors. *Educational Studies in Mathematics, 102,* 379–394. https://doi.org/10.1007/s10649-019-09887-6

Sjölund, S., Lindvall, J., Larsson, M., & Ryve, A. (2022). Using research to inform practice through research-practice partnerships: A systematic literature review. *Review of Education*, 10, e3337. https://doi-org.proxy.mau.se/10.1002/rev3.3337

Skovsmose, O. (2014). *Foregrounds: Opaque stories about learning.* Springer.

Sohn, B. K., Thomas, S. P., Greenberg, K. H., & Pollio, H. R. (2017). Hearing the voices of students and teachers: A phenomenological approach to educational research. *Qualitative Research in Education*, *6*(2), 121–148. https://doi.org/10.17583./qre.2017.2374

Sterner, G., Wolff, U., & Helenius, O. (2020). Reasoning about representations: Effects of an early math intervention. *Scandinavian Journal of Educational Research*, *64*(5), 782–800. https://doi.org/10.1080/00313831.2019.1600579

Swedish Research Council's guidelines. (2017). Good research practice. https://www.vr.se/download/18.5639980c162791bbfe697882/1555334908942/GoodResearchPractice_VR_2017.pdf.

Vaughn, M., Parsons, S. A., & Gallagher, M. A. (2022). Challenging scripted curricula with adaptive teaching. *Educational Researcher*, *51*(3), 186–196. https://doi.org/10.3102/0013189X211065752

Zhang, L., & Sweller, J. (2024). Instructional sequences in science teaching: Considering element interactivity when sequencing inquiry-based investigation activities and explicit instruction. *European Journal of Psychology of Education*, *39*(4), 3791–3801. https://doi.org/10.1007/s10212-024-00799-5

Sarah's construction of a method for division with multi-digit dividends and single-digit divisors

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Division is regarded as the most difficult of the four basic operations for students to learn. Yet, there is little research on how teaching can support students learning of methods for multi-digit division. Supporting students' own construction of methods, rather than letting students imitate given methods, has been shown to result in more sustainable learning for other mathematical topics. However, designing teaching that is contingent to students' own construction requires knowledge of students' construction processes for the method in question. In this case study, we provide a detailed example of such a process regarding a method for multi-digit division. The results show that students can construct increasingly sophisticated methods for multi-digit division based on additive partitions of the dividend into multiples of the divisor.

Division is complex to teach and learn (Hickendorff et al., 2019; Neuman, 1999). Although studies show that different types of teaching lead to different student learning of division methods (Anghileri et al., 2002; van Putten et al., 2005), still too little is known about how multi-digit division teaching should be designed to best promote learning for all students (Hickendorff et al., 2019). Suggestions point to the importance of students building their knowledge from informal methods and inventing their own methods, and that students' development of division methods is influenced by students' multiplicative and relational thinking (Ambrose et al., 2003; Anghileri et al., 2002; Schulz, 2024). In Swedish textbooks, multi-digit division is usually explicitly taught in Grades 4–6, restricted to single-digit divisors and up to four-digit dividends. The most common method taught is short division, first introduced for divisions for which no sub-divisions result in any remainders, and then for divisions with increasing number of sub-remainders and increasing number of digits in the dividend.

For other mathematical topics, studies have demonstrated learning advantages of letting students develop and construct solution methods, rather that imitating solution methods when solving tasks (Jonsson et al., 2014; Lester & Cai, 2016). However, designing teaching that promotes students' own construction requires knowledge of students' construction processes for the method in question and how such processes can be promoted. There are pre- and post-test accounts of changes in students' division methods (e.g., van Putten et al., 2005; Schulz & Leuders, 2018), but, to our knowledge, no accounts of the actual learning process during training. In this case study we therefore investigate one Grade 4 student's process of constructing a method for division of multi-

Mattsson, L., Eckert, A., & Kilhamn, C. (Eds.). (2026). *Talking Mathematics. Proceedings of MADIF 15*. Gothenburg University and SMDF.

digit dividends and one-digit divisors in a teaching design aimed at promoting students' independent construction of such a method.

Difficulties with methods for division

Compared to the other operations, division is conceptually harder since it embraces both quotative and partitive situations (Sellers, 2010). Understanding multi-digit division also requires sufficient knowledge and skill in addition and subtraction, as well as fluency in the 1–10 multiplication tables (Hickendorff et al., 2019). Division algorithms, such as short division, are *digit-based* methods as they allow one to handle each digit separately, ignoring the value of the numbers the digits represent, often leading to mistakes (Ambrose et al., 2003; Hickendorff et al., 2019). Besides general difficulties with algorithms, such as knowing when to apply them (Verschaffel et al., 2007), a specific difficulty of division algorithms is that they go from left to right, while the algorithms for addition, subtraction, and multiplication commonly go from right to left (Hickendorff et al., 2019).

In *number-based* methods, both the numerical value and the position are respected. Therefore, they require students to have a sufficient conceptual understanding of the place value system (Hickendorff et al., 2019). One example of such a method is *Split and add* (Schulz, 2024), where the dividend is additively partitioned into numbers for which the quotient with the divisor is known to the student and the sub-quotients are added to obtain the result, for example $126/3 = 120/3 + 6/3 = 40 + 2 = 42$. When students are allowed to develop number-based multi-digit division methods, they usually start from repeated adding or subtracting for quotative problems and equal sharing of small parts of the dividend using visual representations for partitive problems (Verschaffel et al., 2007). Such methods become cumbersome and error-prone as dividends grow larger (Neuman, 1999; Anghileri et al., 2002). Over time, students may progress to using larger multiples, or 'chunks' of the divisor in successively more efficient ways, as for Split and add (Hickendorff et al., 2019; Verschaffel et al., 2007). However, students' own development of number-based methods can also sometimes result in 'buggy procedures' (Hickendorff et al., 2019), indicating that design of teaching that supports successful and accurate development of division methods may require careful consideration.

Teaching designs for development of methods

A large number of studies have suggested that students are given greater opportunity to develop mathematical understanding when they construct their own solution methods, than when they are given methods to imitate (Boaler, 1998, 2014; Schoenfeld, 1985, 2018). That students better remember methods that they have constructed themselves has been verified in several recent studies using control groups (e.g. Jonsson et al., 2014; Norqvist et al., 2025), but these studies are limited to short interventions and methods for single tasks. For the learning of division, van Putten et al. (2005) studied how Grade 4 students developed long division for multi-digit divisors over five months. The students were supported by teaching based on students' informal methods and own construction of subsequently more advanced methods. Schulz and Leuders (2018), in a multifaceted intervention with some explicit instruction of informal methods and a focus on development of methods, studied students' progression to more sophisticated

methods for solving divisions. Both of these studies leverage pre- and post-tests to infer probable development during instruction, but they do not actually study what happens during training. To our knowledge, there is yet few studies describing how students, without at least in part explicit instruction, may construct general methods over sequences of tasks, and no study describing how this can be done for division with single-digit divisors and multi-digit dividends.

Aim and Research Question

As described above, several previous studies suggest that there are learning advantages to constructing a solution method, even though this is difficult for students. Yet, it is not known how teaching can facilitate this in the case of division with multi-digit dividends and single-digit divisors. A first step would be to investigate if, and if so how, students can construct a reasonably efficient general method for multi-digit division (e.g., one that does not rely on equal sharing of unit objects) without explicit instruction (e.g., without the teacher demonstrating methods). This study aims to contribute to such investigations by considering one student's work during a teaching experiment with constructive task and teacher support design. The research question is: How may a Grade 4 student's process of constructing a method for division with multi-digit dividends and one-digit divisors manifest in a constructive task and teacher support design without explicit instruction?

Method

This study is a sub-study within a larger design research project aiming to develop and test constructive teaching design for learning central mathematical methods. The data was collected from three lessons and two post-tests during an experiment aiming to test and compare teaching designs for learning a general method for division with multi-digit dividends and one-digit divisors. Before the experiment, the students had been taught and practiced on inverse multiplication tasks, that is, division tasks that can be solved by inversing the multiplication facts included in the 1–10 multiplication tables.

The teaching design

The teaching design was based on a sequence of tasks hypothesised to support students' construction of a general method by successively increasing the difficulty of the tasks. Since it is not possible to predetermine the methods students will develop in constructive designs, we weighed in four factors in the task sequence: 1) context: the presence of context could support students' initial attempts at constructing a method. The first five tasks in the sequence were therefore set in a common context of Max's class setting up a school play (Table 1). 2) Size of dividend: larger dividends could be harder to think about and represent, making tasks with two-digit dividend easier than tasks with three-digit dividend. 3) Size of divisor: students in Grade 4 were assumed to be more fluent in the 2–5 multiplication tables than the 6–9 multiplication tables, making tasks with the former as divisors easier. 4) Whether the sub-divisions produced one or two remainders. Though the fourth factor was based on the standard algorithm, we expected that this property would translate to the methods the students developed, as this property is based on the relationship between the dividend and the divisor. We expected the five context

tasks to take up most of the lesson time but also prepared worksheets (WS) with non-context tasks (Table 1). On the post-tests, tasks 1, 3 and 5 were context-tasks, and task 2 and 4 non-context tasks.

Task type	Context task (CT)	WS
Two-digit dividend, one remainder	CT1. 78 of the students go by bus to the theatre where they practise. It takes 3 buses to take all students. How many students are in each bus, if there are equally many in each bus?	WS1
Three-digit dividend, one remainder: all tens	CT2. For all 615 students to see the play they are divided equally to three shows. How many students are there in each show?	WS2
Three-digit dividend, one remainder: all hundreds	CT3. The class has baked cookies to sell at the shows. Max suggests that they sell equally many cookies at each show. They have baked 156 cookies. How many cookies should they sell at each show?	
Three-digit dividend, one remainder: part of tens	No CT.	WS3
Three-digit dividend, one remainder: part of hundreds	CT4. The class has sold tickets to the three shows for 756 SEK. If they have sold equally many tickets to each show, how much money did they get for each show?	
Three-digit dividend, two remainders	CT5. Another school has heard about the play and also want to see it. At that school, there are 765 students. Max's class gives another three shows. If the students are divided equally, how many students are coming to each show?	WS4

Table 1. Task types for context tasks and worksheets included in the task sequence

In addition to the tasks, students were encouraged to use concrete base-10 materials and ask the teacher for help whenever they needed. The teacher was instructed to help students by supporting their creative mathematical reasoning (Säfström et al., 2025). As part of the teacher support, supporting tasks could be given to the context tasks, asking the same question but for a simpler dividend: 60, 600, 150 or 750, and following this up by asking if the answer to the simpler task could be used for solving the original task. Lesson 1 and 3, as well as the two post-tests, each ended with an explanation task that asked the student to describe their solution to one of the division tasks they had worked with earlier that session.

Data and analysis

To answer the research question, the work of one student, Sarah (a pseudonym), was selected as a case. Before the teaching sequence, Sarah could solve simple division tasks but had not worked with tasks where the dividend was not found in the multiplication table and the dividend's digits were not evenly divisible by the divisor. The data consisted of Sarah's written work during lessons and post-tests, and audio recordings of her interactions with the teacher. We selected this case since it illustrates how a student can develop a successively more general method for division with multi-digit dividends and single-digit divisors.

To answer our research question, we adopted qualitative level-raising analysis (Simon, 2019), meaning that each step of the analysis used the result from the previous step as data, forming a recursive process that established a chain of inferences that stayed close to the data in each step. The idea with level-raising analysis is to avoid inferential leaps from data to conclusions and make the overarching conclusion less demanding to

make and easier to trace back to the data. It consisted of three levels, starting with analysis of Sarah's work on each task, then analysing it in relation to her work on previous tasks and finally analysing her process over the whole task sequence. The analysis was first conducted separately by the first and second author, who then discussed their analysis and reached a common, refined analysis, incorporating details identified by each researcher. At Level 1, Sarah's written work on the task was described with as much detail as possible. If Sarah had explained her reasoning for the division task as her response to an explanation task, this was also included as data. If Sarah and the teacher had interacted about the division task, the transcription of the interaction was partitioned into segments where the teacher supported Sarah with the same thing in similar ways, the teacher support in each segment was described, and the teacher support regarding the task was summarised based on the descriptions of the segments. The results of the Level 1 analysis were therefore twofold: a characterisation of the method Sarah used for the task (similarly to characterisations by Schulz (2024) and others) and a characterisation of any support the teacher provided to her. At Level 2, the results from Level 1 for the current and all previous tasks were used to identify which (correct and incorrect) steps in the solution for the task Sarah constructed herself, and which steps she already knew or was given by the teacher. At Level 3, the results of Level 2 for all the tasks Sarah had worked with was used to construct a narrative of when which steps of a general method for division with multi-digit dividends and single-digit divisors were constructed by her. The results consist of the narrative that was the result of the Level 3 analysis, which naturally includes results from the analysis on Level 1–2.

Results

Through working with eleven tasks during three lessons, Sarah developed the Split and add method (Schulz, 2018) for three-digit dividends and one-digit divisors, first with concrete materials and then in writing. She interacted with the teacher regarding three tasks, restricted to the teacher asking questions about Sarah's reasoning, supporting interpretation of the context and a few questions asking for evaluation of what she had done. Her effort and success in applying this method depended on whether an additive partitioning was directly "visible" in the dividend or not, that is, whether the partitioning could be made in between place values, either as one part with the hundreds and another with the tens and ones or as one part with ones and the other with the hundreds and tens. She was able to use her method successfully for dividends with a visible bipartition, for example 615 = 600 + 15, or a partly hidden tripartition, for example 492 = 400 + 80 + 12, where 400 is a visible part whereas 80 and 12 are hidden parts, and extended the method to hidden bi- or tripartition on Post-test 1 (Table 2).

Lesson 1: Constructing the visible bipartition method

Sarah first engaged with CT1 (78/3). In her attempt to solve the task she drew buses on a paper and then shared concrete base-10 materials between the buses until there was nothing left. As seen in Figure 1, she drew three buses and first put twenty students in each bus, sharing out 60. She then added another five to each bus, sharing out 15. However, the materials she had added up to 75, not 78. Therefore, she believed she was finished. At this point, the teacher asked Sarah to explain her solution, and the teacher

highlighted that Sarah seemed to be missing some pieces to represent all 78 students: "Are you not missing some ones? Did you have 78 from the beginning? I think you are missing ones; they are over there, on the table". When the teacher left, Sarah wrote "20 + 5 +" but then crossed over the plus sign. This indicates that Sarah realized that the teacher indicated that something more should be added to each of the buses, but she did not finish the task and instead moved on to the next task.

	Lesson 1			Lesson 2		Lesson 3					Post-test 1					Post- test 2				
Task / Type	78	615	156	756	765	156	492	651	876	165	24	48	126	565	732	22	39	142	575	882
	3	3	3	3	3	3	4	3	4	3	2	4	2	5	6	2	3	2	5	9
Visible bipartition		R	S			R					R	R	R			R	R	R		
Partly hidden tripartition							R	R	R					R					R	
Hidden tripartition				W	W										R					
Hidden bipartition	W									W										NA

Note. R = right answer, S = started but got stuck, W = wrong answer, NA = not attempted.

Table 2. Overview of Sarah's methods and success on tasks she worked with

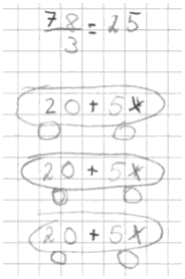
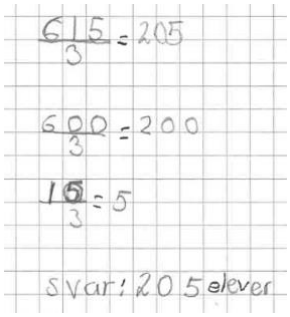


Figure 1. Sarah's solution of 78 divided by 3

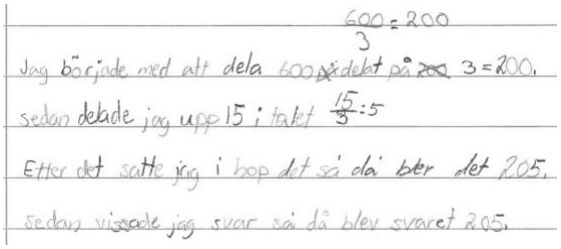
Next, Sarah worked on CT2 (615/3). For this task, she did not use concrete base-10 materials but partitioned the dividend in hundreds and the rest (a visible bipartition of 615) and recorded the sub-divisions in writing (Figure 2A). She then added the quotients, obtaining the correct answer. In Figure 2B we see her own explanation of how she solved the task.

A



English translation. "Answer: 205 pupils"

B



English translation. "I started by dividing 600 divided by 3 = 200. Then I divided 15 in the number 15/3 = 5. After that I put it together, so it became 205. Then I showed the answer, so it became the answer 205."

Figure 2. Sarah's solution and explanation of 615 divided by 3

In her attempt to solve CT3 (156/3) —also a task that can be solved using a visible bipartition—Sarah continued to use the solution method from CT2. Starting with division of the hundreds she wrote 100/3, but got stuck, seemingly because 100 is not

divisible by three. When Sarah asked the teacher for help, she was encouraged to look at the previous task and see if her solution there could help her come up with an idea. Since Sarah had started with dividing the hundreds in the task before, she had a hard time to come up with something new and when she thereafter tried to use manipulatives, she chose a square representing one hundred, five rods representing tens, and six small cubes, which did not help her overcome her difficulty. The lesson ended without Sarah solving the task.

Lesson 2: Beginning to construct the hidden tripartition method

During Lesson 2 Sarah worked with CT4 (756/3) and CT5 (765/3), both requiring hidden partitions. She seemed to realize that if she chose to start with the hundreds, she needed to first find a part of the hundreds (600) that is divisible by three. She worked with these two tasks the whole lesson. At the end of the lesson, she wanted to explain what she had done, and it was evident that Sarah understood that there was one hundred left to divide by three and that she had tried to use a similar approach to the hundred seeing it as ten tens (i.e., to divide as many tens as possible). "They are 200 [600 divided by 3], and then I took those /…/ then I took the hundred and made it into ten [i.e., ten tens] and divided that. /…/ Then there was one piece left. Then I took from there [the tens and ones]" However, as can be seen in Figure 3, Sarah struggled with keeping track of her reasoning, i.e., what she had divided and what remained to be divided of the hundreds, tens and ones. Although she had correctly determined the sub-divisions of a hidden tripartition of 756—600/3, 9/3 (i.e., nine tens divided by three), and 66/3—she has also made many other attempts, and when trying to find the quotient she failed to identify which attempts were successful. When the teacher, after listening to Sarah's explanation leading up to the suggested answer 250, asked her to verify that this was correct, Sarah did not manage to do this.

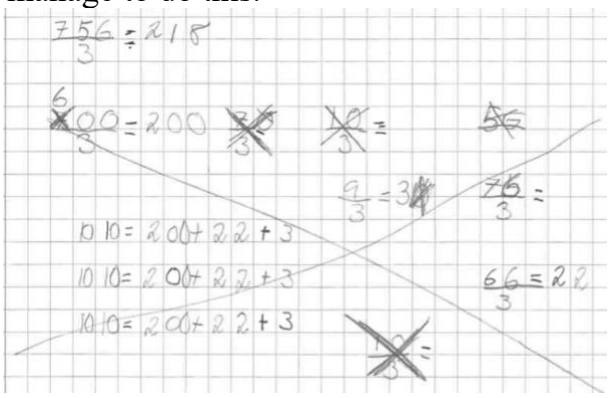


Figure 3. Sarah's solution of 756/3

Lesson 3: Constructing the partly hidden tripartition method

In the third lesson Sarah was first given the supporting task for CT3 (150/3) and then CT3 (156/3) again. Sarah solved the supporting task by realizing that 50+50+50=150 and then used this fact to solve CT3 by a visible bipartition: 150 + 6. Sarah was then given Worksheet 3 on which she successfully solved three tasks (492/4, 651/3, 876/4) by partly hidden tripartition in all hundreds, a part of the tens divisible by the divisor and the rest of the tens together with the ones (Figure 4A). However, Sarah was still

unable to solve the fourth task on the worksheet, which had no visible part divisible by the divisor (see Figure 4B).

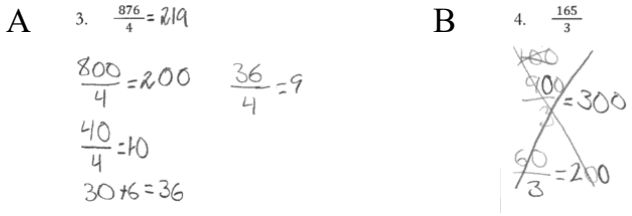


Figure 4. Sarah's written work for 876/4 and 165/3

Post-tests 1 and 2: Extending to a hidden partition method

On the first post-test, a week after the training, Sarah successfully used her visible bipartition and partly hidden tripartition methods to solve 126/2 and 565/5, respectively. She also managed to solve a task with no visible parts (732/6). Her notes of how she solved this task are blurry. She has written the correct answer (122) and several other numbers that do not clearly indicate a solution method. In the explanation task, however, she described her method as taking 600/6 = 100, 30/6 = 5 and that she intended to calculate 42/6, indicating that her solution entails identifying at least three hidden parts of the dividend, but Sarah did not explain how or if she handled the last 60 of the dividend.

On the second post-test, five weeks after training, Sarah still successfully used visible partition and partly hidden tripartition (142/2, 575/5), but she was unable to solve the task requiring hidden bipartition: 882/9. This may indicate that the method she used on Post-test 1 was partly hidden tripartition.

In sum, we answer our research question in the following way: In this context, a Grade 4 student can construct Split and add for division with multidigit dividends and one digit divisors through a process entailing the construction of three key ideas in the following order: 1) partitioning dividends consisting wholly of visible, known divisible parts (e.g., for divisor 3, partitioning 615 into 600 and 15) and adding the results of sub-quotients, 2) after partitioning a dividend with one visible, known divisible part and a not-known part (e.g., for divisor 4, partitioning 492 into 400 and 92), partitioning the not-known part into two known parts (80 and 12), and 3) partitioning a dividend with no visible, known divisible parts into known parts (for divisor 6, possibly partitioning 732 into, 600, 30, 42 and 60). Sarah's wrong answers to CT1, 4 and 5 indicates that for her, it was not possible to construct Idea 2 and 3 without first constructing Idea 1.

Discussion

The study shows that it is possible for students to learn methods for solving multi-digit division with single-digit divisors without being presented with the standard algorithm (as is common in Swedish textbooks and traditional instruction), explicit instruction with discussions of number-based methods or much individual support. It furthermore shows that progress in development of a method can be made over a relatively short period of time, similarly to Schulz and Leuders (2018), complementing studies with a longer time frame (e.g., van Putten et al., 2005). Foremost, it shows the actual process

that students can experience from task to task over a series of lessons. These findings give insights into the feasibility of allowing students to construct their own solution methods and indicates how tasks may promote or not promote such development. The study furthermore exemplifies how students' learning processes can be studied over lessons through Simon's (2019) notion of level-raising analysis.

It has been shown that knowledge of the other operations and their relations can be associated with greater progress in development of general methods for division (Schulz, 2024). Sarah's quick transition from use of concrete materials to written methods, and a lack of examples of where the context supported her, may indicate that her previous mathematical knowledge was strong. For the design of the tasks, we conjectured that dividends with fewer digits would be easier. However, for Sarah, hidden bipartitions of two-digit dividends (as in 78/3) seem to be more difficult than hidden tripartitions of three-digit dividends. This may be because the partitioning of the hundreds only, which you can always start with for hidden tripartitions, may be easier to invent than the partitioning of the hundreds and the tens together, which is required in hidden bipartitions. For the standard short division method, one remainder is likely to be equally difficult no matter where it appears, but for the Split and add method, it can be a significant difference if a partly hidden tripartition starts with the visible or the hidden part. This suggests that students' success in developing their own methods may depend on how tasks are sequenced in terms of which parts of the dividend are visible or hidden.

Some limitations of the study should be noted. The nature of our research question and the detailed analysis of one student afford insights into one student's learning process. However, this focus and scope naturally limit generalizability, both regarding conclusions about whether the process studied is common or idiosyncratic and how the teaching design, as compared to other teaching designs, influences the student's development of a method. Future studies could leverage the affordances of our approach of a detailed study of the student's process over lessons and overcome limitations by, for example, including analysis of more students with different characteristics and compare students' development of a method for solving division problems under different conditions.

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References

Anghileri, J., Beishuizen, M., & Van Putten, K. (2002). From informal strategies to structured procedures: Mind the gap!. *Educational Studies in Mathematics*, *49*(2), 149-170. https://doi.org/10.1023/A:1016273328213

Ambrose, R., Baek, J.-M., & Carpenter, T. P. (2003). Children's invention of multidigit multiplication and division algorithms. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills* (pp. 305–336). Erlbaum.

Boaler, J. (1998). Open and closed mathematics: student experiences and understandings. *Journal for Research in Mathematics Education, 29*(1), 41-62. https://doi.org/c745vv

Boaler, J. (2014). The road to reasoning. In K. Brodie (Ed.), *Teaching mathematical reasoning in secondary school classrooms* (p. v-vii): Springer Nature. https://doi.org/c745vv

Hickendorff, M., Torbeyns, J., & Verschaffel, L. (2019). Multi-digit addition, subtraction, multiplication, and division strategies. In A. Fritz, V.G. Haase, P. Räsänen (Eds.), *International handbook of mathematical learning difficulties: From the laboratory to the classroom* (pp. 543–560). Springer International Publishing. https://doi.org/p4fx

Jonsson, B., Norqvist, M., Liljekvist, Y., & Lithner, J. (2014). Learning mathematics through algorithmic and creative reasoning. *Journal of Mathematical Behavior, 36*, 20–32. https://doi.org/10.1016/j. jmathb.2014.08.003

Lester Jr., F. K., & Cai, J. (2016). Can mathematical problem solving be taught? Preliminary answers from 30 years of research. In Felmer, P., Pehkonen, E., Kilpatrick, J. (Eds.), *Posing and solving mathematical problems: Advances and new perspectives* (pp. 117–136). Springer. https://doi.org/10.1007/978-3-319-28023-3_8

Neuman, D. (1999). Early learning and awareness of division: A phenomenographic approach. *Educational Studies in Mathematics*, *40*(2), 101-128. https://doi.org/d7brs8

Norqvist, M., Jonsson, B., & Lithner, J. (2025). Shifts in student attention on algorithmic and creative practice tasks. *Educational Studies in Mathematics*, *118*(3), 409–428. https://doi.org/gsmkgb

Schoenfeld, A. H. (1985). *Mathematical problem solving.* Academic Press.

Schoenfeld, A. H. (2018). Video analyses for research and professional development: The teaching for robust understanding (TRU) framework. *ZDM Mathematics Education, 50*(3), 491-506. https://doi.org/10.1007/s11858-017-0908-y

Schulz, A. (2018). Relational reasoning about numbers and operations–foundation for calculation strategy use in multi-digit multiplication and division. *Mathematical Thinking and Learning, 20*(2), 108–141. https://doi.org/10.1080/10986065.2018.1442641

Schulz, A. (2024). How Do Multiplicative and Relational Thinking Contribute to Multidigit Division? *Journal für Mathematik-Didaktik*, *45*(2), 21. https://doi.org/p4fw

Schulz, A., & Leuders, T. (2018). Learning trajectories towards strategy proficiency in multi-digit division – A latent transition analysis of strategy and error profiles. *Learning and Individual Differences, 66*, 54–69. https://doi.org/10.1016/j.lindif.2018.04.014

Sellers, P. A. (2010). The trouble with long division. *Teaching Children Mathematics, 16*(9), 516–520. http://dx.doi.org/10.5951/TCM.16.9.0516

Simon, M. A. (2019). Analyzing qualitative data in mathematics education. In K. R. Leatham (Ed.), *Designing, conducting, and publishing quality research in mathematics education* (pp. 111–122). Springer. https://doi.org/10.1007/978-3-030-23505-5_8

Säfström, A. I., Palmberg, B., Anderson, C., Lithner, J., Boström, E., Granberg, C., Palm, T., & Sidenvall, J. (2025). Developing a formative assessment framework for supporting students' reasoning during problem solving [Manuscript submitted for publication]. Department of Science and Mathematics Education, Umeå University.

van Putten, C. M., van den Brom-Snijders, P. A., & Beishuizen, M. (2005). Progressive mathematization of long division strategies in Dutch primary schools. *Journal for Research in Mathematics Education, 36*(1), 44–73. https://doi.org/10.2307/30034920

Verschaffel, L., Greer, B., & De Corte, E. (2007). Whole number concepts and operations. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 557–628). Information Age Publishing.

Cohesion in Static Versus Dynamic Text – Exploring Textual Logic

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This paper presents how static and dynamic text elements in digital teaching materials can express textual logical cohesion in mathematics text. We analyse five items in both static and dynamic versions to explore how these formats expand the text logically. The results reveal that in the versions with dynamic functions, the reader is offered a logic that differs from the static presentation, which more often describes mathematics as given facts without allowing the reader to investigate logical connection themselves. At the same time, the dynamic functions place new demands on the reader.

Introduction

A previous study of digital teaching platforms (Dyrvold & Bergvall, 2023a) identifies rich opportunities to present relations in mathematics when dynamic and interactive functions are utilised. For instance, dynamic functions can demonstrate crucial aspects of mathematics step by step, spatially, or temporally, by allowing the student to click through information or using a video. Mathematics education does however have a long history of using the static media, and the introduction of dynamic–interactive functions does also put new demands on a reader. When dynamic–interactive functions are utilised by a reader, crucial information may be missed because such functions can demand more of a back-and-forth reading (Bergvall & Dyrvold, 2025). In this study we contrast static text to dynamic–interactive text focusing on textual logical expansions to seek a deepened understanding of how a mathematics message can be expanded using different functions. A *textual logical expansion* (defined in Theory) is present when the second of two instances in a text is logically expanded by textual means such as the use of the word "thus" or visually. Logical expansions is a crucial part of text cohesion and therefore also of importance for reading. In the paper logical expansions refer to such *textual* expansions, not to mathematics logic.

Digital modalities in mathematics education

Digital modalities have introduced new possibilities for mathematics instruction, offering a range of representational forms such as text, images, videos, tables, figures, dynamic visualizations, and interactive functions. This section outlines research on

Mattsson, L., Eckert, A., & Kilhamn, C. (Eds.). (2026). *Talking Mathematics. Proceedings of MADIF 15*. Gothenburg University and SMDF.

various digital resources and their potential in mathematics teaching, while also addressing limitations and challenges associated with their use.

Analyses of digital mathematics textbooks highlight how digital resources enable the integration of multimodal elements and the reorganization of mathematical information (O'Halloran, Beezer, & Farmer, 2018). Further examples of digital affordances include dynamic visualizations of mathematical relationships (Demir, 2018; Poon & Wong, 2017; Çeziktürk, 2020), drill-and-practice games (Beserra et al., 2019), automated feedback (Van der Kleij et al., 2015; Stevenson, 2017), and options to access explanations or definitions (Stevenson, 2017; Arroyo et al., 2013).

These affordances are however not used extensively in contemporary materials. Dyrvold (2022) examined the prevalence of four types of dynamic elements in Swedish digital teaching platforms, finding that theoretical sections and examples often feature low interactivity (e.g., hint boxes, text-to-speech), whereas task sections more frequently include highly interactive functions such as adaptive feedback and drag-and-drop activities. A socio-semiotic analysis of seven teaching platforms (Dyrvold & Bergvall, 2023a) similarly revealed a tendency to present theoretical content and examples as written text or dynamic lectures. Interactive activities are mostly allocated to the tasks. Pepin et al. (2017) also note that task sections of teaching materials often revert to traditional formats and emphasize procedural fluency and method acquisition while videos/animations are commonly used to introduce new mathematics.

Students' encounter with digital modalities in mathematics education

The multimodal functions embedded in digital learning resources for mathematics offer both advantages and disadvantages for students' engagement and learning processes. A meta-analysis about dynamic representations of mathematical objects (Moyer-Packenham & Westenskow, 2013) identified enhanced creative thinking, improved ability to direct students' attention to relevant content, and more effective visualization of relationships between mathematical objects in response to student interaction. The analyses also suggest that such resources can foster increased student motivation and persistence when engaging with mathematical tasks, which also is revealed in analyses of students' interaction with dynamic resources (see Dyrvold & Bergvall, 2023b). Benefits of using dynamic modes is also found in an analysis of how use of a digital mathematics textbook with integrated dynamic visualisations and feedback mechanisms impacts students' achievement, in comparison to the use of a traditional printed textbook (Brnic et al., 2024). The results indicate that both boys and girls performed better when using digital materials. The inclusion of dynamic visualisations was critical to students' successful performance.

Another study exploring how students perform when mathematical content is presented through digital and interactive functions, as opposed to static text reveal that students with lower prior achievement in mathematics, but who were familiar with using GeoGebra, outperformed their higher-achieving peers who lacked experience with the tool (Baccaglini-Frank, 2021). Moreover, dynamic mathematics software facilitated a shift in students' engagement from a ritualistic mode of discourse to a more exploratory

form of participation. This transition supported the construction of abstract mathematical objects, as students engaged more deeply with the underlying concepts through interactive and investigative practices. Similar findings are reported by Çeziktürk (2020), about how GeoGebra is an effective tool for fostering mathematical thinking and supporting students in engaging with mathematics in ways that resemble the practices of mathematicians, including formulating and testing hypotheses through dynamic representations. Analyses of students' reasoning indicate that the ability to make connections across multiple representations within such software plays a critical role in the development of mathematical understanding.

Digital environment offers possibilities to display mathematical objects and graphs and thereby support students' development of covariational reasoning skills in a way not possible in print (Kertil et al., 2019). Covariation in calculus generally refers to the way two variables change together continuously or discretely. Covariational reasoning is fundamental also in elementary grades, for example in relation to rate and proportions (Gantt et al., 2023).

To understand the contributions of different modalities and thereby enable informed choices in teaching, research is needed that investigates what is conveyed when the mathematical content is presented in various ways, by means of different semiotic recourses and modalities. The impact of dynamic and interactive tools on students' learning and engagement with mathematics remains only partially understood. Contemporary research does not provide a complete account of whether these functions consistently support students' learning. Moreover, it remains unclear what specific contributions such functions make to mathematical understanding. For instance, while dynamic representations and interactive elements are often assumed to support conceptual development, there is limited empirical insight into *how* they do so, or *what kinds* of understanding they facilitate. Questions persist about whether these tools promote deeper understanding or simply increase surface-level engagement and temporary ability to solve a task. Further research is needed to gain understanding of how dynamic and interactive functions influence students' mathematical thinking and learning. In the present study, we extend the previous work by exploring one specific aspect in depth: logical cohesive relations in static versus dynamic–interactive texts.

Aim

In this study, mathematics texts designed to represent two endpoints on a continuum, from entirely static to texts incorporating both dynamic and interactive functions, are analysed. The aim is *not* to identify a 'best' version. Rather, the study explores what the introduction of dynamic and interactive functions implies from a textual perspective. We examine what the text offers, and the potential demands it places on the reader. Accordingly, the aim of this study is to enhance understanding of the learning opportunities offered to students, by logical expansions in mathematics texts, when static versus dynamic–interactive elements are utilised. Two research questions are addressed.

RQI: How can logical expansions be offered through static and dynamic elements in mathematics text?
RQII: What kinds of demands can logical expansions in dynamic elements place on the reader?

Theory and method

This study is grounded in social semiotic theory (see Kress & van Leeuwen, 2006) highlighting how various semiotic resources can be used to express meaning in different ways. Semiotic resources include for example natural language, mathematical notation, and images. In addition to the variation in semiotic resources, digital media enables the use of these resources in different modes, the channel through which meaning is conveyed. By examining how semiotic recourses and various modalities are used to express the content, we gain understanding of the meaning offered to the reader.

All texts are characterized by cohesive relations (see e.g., Halliday & Matthiessen, 2014). In mathematical texts a facet of cohesion is networks of information between the various semiotic resources. A network is anchored by a *central aspect of a concept* introducing a theme relevant throughout an entire section of text. This anchoring expression can be followed by other central aspects of the same concept, constituting expansions of the concept, creating a cohesive network throughout the text. For example, the concept "per cent" can further be expanded in the text by the central aspects: "%", an explanation using "hundredth", and the fraction "1/100". The theoretical understanding of expansions, which underpins this study, has been developed from van Leeuwen's (2005) (see also Bergvall & Dyrvold, 2021). Three distinctly different types of expansions are, additive expansions (introducing new information), extending expansions (repeating information with new means or slightly reformulated) and logical expansions. We investigate logical expansions because the initial analyses revealed the most differences between static and dynamic–interactive text for logical expansions.

A logical expansion is expressed when a central aspect in the second of two instances of this aspect is justified by a reason, condition, or comparison. For example, a causal logic expansion of the concept premutation is offered when the student drags a fruit to the casket (Figure 1), because the arrangement of fruits appearing below offers the reader the message that *if* the number of selected fruits increases with 1, *then* the number of ways to arrange the fruits increases with … (visualised groups of fruits). Mathematics is a logically constructed science, and logical expansions of the text are central. However, mathematics logic and logical expansions in texts are similar but not the same; there can be both implicit mathematics logic that is not captured in the analysis and logical (textual) expansions that from a mathematics perspective is not considered as logical mathematical reasoning.

Data

The texts analysed in this study are fact boxes in five items developed in a larger research project, specifically designed to introduce new mathematics to Swedish ninth-grade students. The five items include three other parts: introduction, task, and answer options (Figure 1). For each item, there are two versions, one with a static fact box like a PDF version of a mathematical text and one with a dynamic–interactive fact box. The example shows the dynamic–interactive version of facts; depending on students dragging, the numerals changes and arrangements of fruits appear.

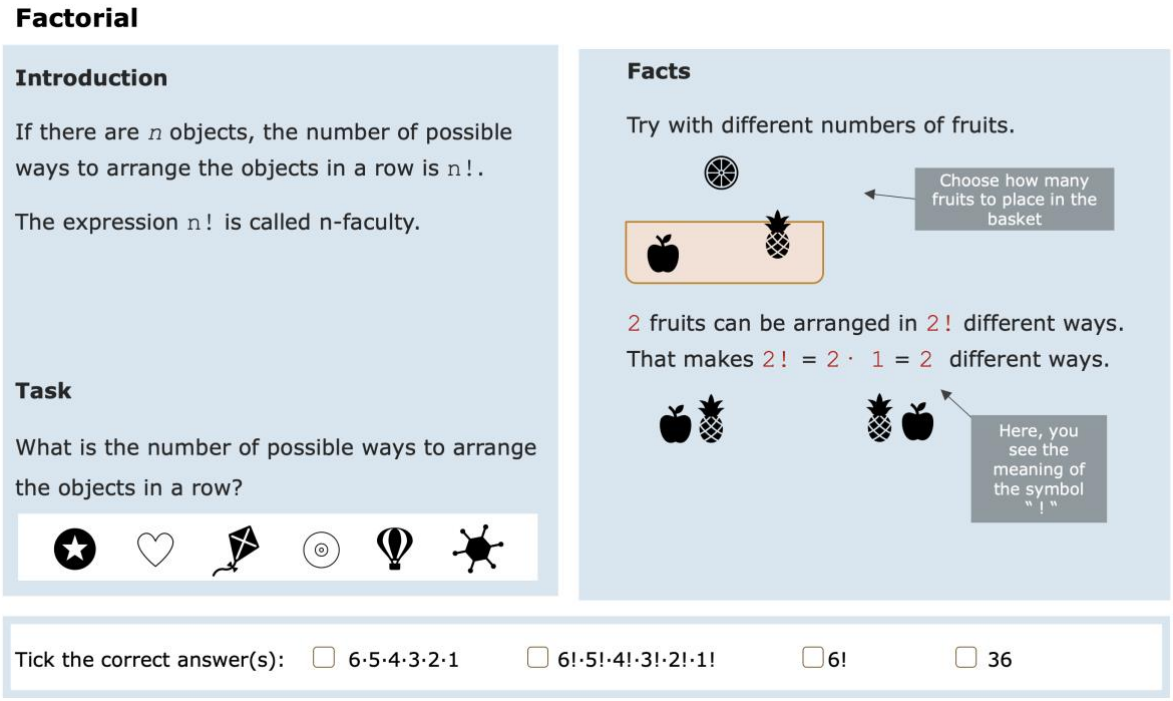

Figure 1. Arrangement of Introduction, Facts, Task, and answers in the items.

Two researchers and two textbook writers developed the items together, ensuring that the fact boxes correspond to each other as much as possible. Each fact box is designed to provide necessary information for solving tasks on five different topics (Table 1).

Item	Topic
1	Inscribed angle theorem
2	The sign of factor k in $y = kx^2$ related to the shape of the parabola
3	Intersection in set theory
4	Roots and their relation to powers
5	Objects and permutations, introducing factorials "!"

Table 1. Content areas in items.

Implementation

In the coding process, central aspects of the mathematical concepts in each item were first identified and all instances of these concepts were marked. Thereafter it was noted if the central aspects express an expansion to a previously presented central aspects and

what kind of expansion. Two researchers independently coded, compared the results, and discussed some questionable cases. Finally, each pair of facts (static and dynamic–interactive) were contrasted regarding how concepts were expanded logically, with a particular focus on if and how the logical expansions were dependent on dynamism and on actions (interactive functions). A challenge in the coding process was identifying which instances expanded on others, as multimodal mathematics texts do not necessarily follow the traditional reading order. The conclusion that formed the basis for the coding, was that also multimodal texts have a structure where new information is introduced from left to right and from top to bottom (cf. Kress & van Leeuwen, 2006).

Result

The analysis reveals that in static versions of the investigated items, logical expansions are less common, but if present, directly accessible without requiring any active actions from the learner. In contrast, dynamic or interactive media make logical expansions explicit through user interaction. When the learner performs an action such as manipulating an object, this not only affects the object itself but also triggers changes in related elements. In cases where these resulting changes occur on another instance of a central aspect; they have been identified in the analysis as a logical expansion that emerge through engagement with the material. As the visual representation shifts, multiple logical expansions may become apparent. However, it is the learner's responsibility to initiate the interaction and subsequently interpret the relationships between central aspects in the elements. Consequently, logical expansions in dynamic media are often offered implicitly, placing cognitive and interpretative demands on the learner.

An answer to RQI–II is thus that a difference between how logical expansions occur in static versus dynamic–interactive texts is that in the latter, the occurrence of logical expansions is often contingent on the reader's actions. The analysis revealed that only two of the five analysed items contained logical expansions in both the static and dynamic–interactive versions of the facts (2 & 5, Table 1). In the other three facts (item 1, 3, & 4, Table 1), there are no logical expansions present in the static version of the facts. Instead, the mathematics is in these facts presented as statements and other expansions than logical ones are used to make the text cohere. We present one example of an item with logical expansion exclusively in the dynamic version, and one example of an item with logical expansions in both versions to allow for a comparative discussion of how such expansions are manifested across static and dynamic texts.

Logic expansion dynamic/interactive text

The difference between static facts with no logical expansion and dynamic/interactive facts with several logical expansions is demonstrated by the two versions of facts about set theory and the concept intersection. In the static version, there is an intricate path of information, but no logical expansion. Several logical expansions of the concept intersection are however offered in the dynamic/interactive version of the facts when a change in the intersection displayed in the image also leads to changes in the symbolic

expressions. These expansions are dependent on the learner dragging, and thereby changing set B, while noticing the change of objects in B and simultaneously the change in the intersection (see the red letters in Figure 2).

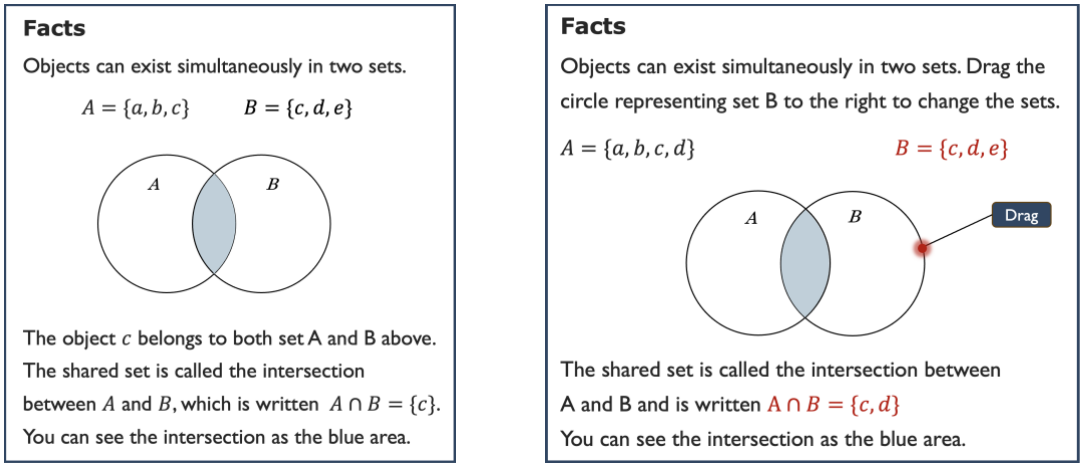

Figure 2. Facts on set theory in static and dynamic/interactive versions.

Logic expansion expressed also in static text

In the second case, both versions of the facts include one or more logical expansions. Recall from the summary that in the dynamic–interactive versions, the facts consistently offer logical expansions, and almost all these expansions are dependent on the readers' acts. The following section presents an examination of the logical expansions in item 2. The facts in this item offer information about the relationship between the shape of the graph of a quadratic function and the coefficient of the x² term (Figure 3).

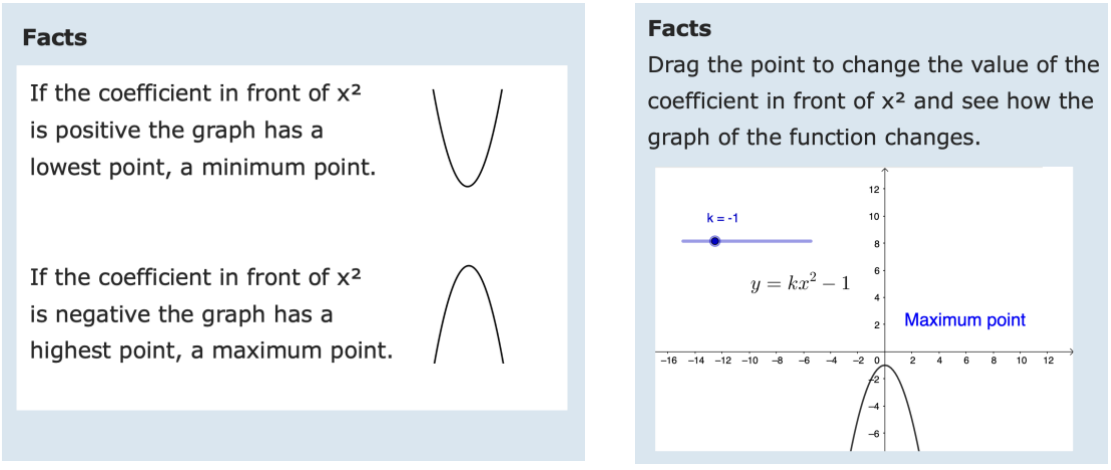

Figure 3. Facts on quadratic functions in static and dynamic versions.

In the static version, a logical expansion of the concept quadratic function is expressed via the central aspects "minimum point" and "maximum point" and conveyed through a conditional expression: "if the coefficient preceding x² is…". This condition constitutes a logical expansion of the central aspects, as it explicates the relationship between the sign of the coefficient and the resulting shape of the graph; specifically, whether the graph has a minimum or a maximum point. Thus, in the static version,

logical expansions are provided in two distinct instances, explicitly and without necessitating active engagement from the reader. In contrast, the dynamic version invites the reader to interact by dragging a point on a slider. This interactive element enables the disclosure of up to five distinct instances of logical expansion related to the concepts of minimum and maximum points. However, the realisation of these expanded concepts is contingent upon the reader's active participation. Without such engagement, the additional layers of logical expansions remain inaccessible.

Conclusion and discussion

The results of the current study highlight how the use of dynamic/interactive features in the presentation of mathematical facts entails increased possibilities for the reader to access a coherent message via logical expansions. By means of the in-depth analysis of logical expansions that makes the text cohere, the results enrich understanding from previous research showing the value of dynamic visualisation of mathematical relations (e.g., Çeziktürk, 2020; Demir, 2018; Poon & Wong, 2017).

The results point out that the potential benefits of the dynamic and interactive means are dependent on the readers active use of them. Thus, dynamic/interactive elements provide opportunities to support learning, but they do also introduce new demands on the reader. Furthermore, the reader must grasp how central aspects in elements emerging dynamically in the text are related, to benefit from different logical expansions offered. If a concept is expanded logically due to actions taken, but the reader does not make the connection in reading, the dynamism does not add much value. On the other hand, readers of static texts must also make connections to benefit from logical expansions of central aspects. Consider, the static version of the facts about maximum and minimum point (Figure 3). Here, the logical relationship between the conditional "if" statement and the corresponding shape of the graph must be followed and interpreted. In contrast, when engaging with the dynamic version (Figure 3), the reader can interact with the content by dragging a slider to observe how the graph flips between a maximum and a minimum as the parameter k shifts from negative to positive values. While dynamic formats can offer richer opportunities for exploring logical relationships, static representations provide immediate access to the core facts. The choice of mode may depend on the student's position within the learning trajectory and whether an exploratory activity is pedagogically appropriate at that stage.

In several of the analysed items, covariational aspects are present. The development of covariational skills is fundamental (Gnatt et al., 2023) and recent research has highlighted the potential of digital environments to support students' covariational reasoning through dynamic visualizations of mathematical objects and graphs (Kertil et al., 2019). The current study provides a possible explanation by pointing out the importance of logical expansions present in the dynamic and interactive texts.

When students engage deeply with mathematical ideas and construct understanding by exploring logical expansions of core concepts, they move beyond passive reception. Previous research has shown how dynamic visualisation of mathematical relations can

direct students' attention to relevant content and enhance their willingness to engage in mathematics (Moyer-Packenham & Westenskow, 2013). Even Çeziktürk (2020) emphasizes how dynamic software can transform students into active participants, engaging in mathematical reasoning akin to that of mathematicians, which is also of importance regarding the view on *mathematics and self* that is conveyed to students. However, mathematics curriculum resources often present new content in a linear fashion, while opportunities to explore mathematical relationships and patterns are typically found in tasks (e.g., Dyrvold, 2022; Pepin et al., 2017). This structure implicitly positions the student as a passive receiver of mathematics posed by a mathematician, suggesting that meaningful exploration by the student is only possible after a formal presentation of the mathematical foundations.

As dynamic–interactive digital resources have potential to counteract this passivity, we suggest that by providing students with opportunities to use such resources to investigate relationships as they emerge, they can be repositioned as active practitioners and thereby develop their conceptions of mathematics and selves. We further propose that dynamic–interactive features may be advantageously employed during teacher-led activities, where their potential to illustrate, among other things, logical relationships can be fully realized, while the demands placed on students to use and interpret the information can be reduced through the support of an experienced teacher.

References

Arroyo, I., Burleson, W., Tai, M., Muldner, K., & Woolf, B. P. (2013). Gender differences in the use and benefit of advanced learning technologies for mathematics. *Journal of Educational Psychology, 105*(4), 957–969. https://doi.org/10.1037/a0032748

Baccaglini-Frank, A. (2021). To tell a story, you need a protagonist: how dynamic interactive mediators can fulfil this role and foster explorative participation to mathematical discourse. *Educational Studies in Mathematics, 106*(2), 291–312. https://doi-org.ezproxy.its.uu.se/10.1007/s10649-020-10009-w

Bergvall, I., & Dyrvold. A. (2021). A model for analysing digital mathematics teaching material from a social semiotic perspective. *Designs for Learning, 13*(1), 1–7. https://doi.org/10.16993/dfl.167

Bergvall, I., & Dyrvold, A. (2025). Timeline analysis of student reading – exploring eye tracking data. In Cornejo, C., Felmer, P., Gómez, D. M., Dartnell, P., Araya, P., Peri, A., & Randolph, V. (Eds.) *Proceedings of the 48th Conference of the International Group for the Psychology of Mathematics Education: Research Reports.* PME.

Beserra, V., Nussbaum, M., & Oteo, M. (2019). On-task and off-task behavior in the classroom: A study on mathematics learning with educational video games. *Journal of Educational Computing Research, 56*(8), 1361–1383. https://doi.org/10.1177/0735633117744346

Brnic, M., Greefrath, G., & Reinhold, F. (2024). Working with digital textbooks or printed materials: A study with boys and girls on conditional probability. *ZDM, 56*(4), 559–572. https://doi.org/10.1007/s11858-023-01543-x

Çeziktürk, O. (2020). Understanding functional dependency on dynamic geometry systems (DGS): Napoleon and von aubel theorems on geogebra. *International technology and education journal, 4*(1), 15–21.

Demir, M. (2018). Effects of virtual manipulatives with different approaches on students' knowledge of slope. *Journal of Interactive Learning Research, 29*(1), 25–50.

Dyrvold, A. (2022). Missed opportunities in digital teaching platforms: Under-use of interactive and dynamic elements. *The Journal of Computers in Mathematics and Science Teaching*, *41*(2), 135. https://doi.org/10.70725/440859vgdrsh

Dyrvold, A., & Bergvall, I. (2023a). Digital teaching platforms: the use of dynamic functions to express mathematical content. *Scandinavian Journal of Educational Research, 68*(5), 861–877. https://doi.org/10.1080/00313831.2023.2196555

Dyrvold, A., & Bergvall, I. (2023b). Static, dynamic and interactive elements in digital teaching materials in mathematics. *LUMAT, 11*(3). https://doi.org/10.31129/lumat.11.3.1941

Gantt, A. L., Paoletti, T., & Corven, J. (2023). Exploring the Prevalence of Covariational Reasoning Across Mathematics and Science Using TIMSS 2011 Assessment Items. *International Journal of Science and Mathematics Education*, *21*(8), 2349–2373. https://doi.org/10.1007/s10763-023-10353-2

Gracin, D. G., & Krišto, A. (2022). Differences in the Requirements of Digital and Printed Mathematics Textbooks: Focus on Geometry Chapters. *Centre for Educational Policy Studies Journal, 12*(2), 95–117. https://doi.org/10.26529/cepsj.1285

Halliday, M A K., & Matthiessen, C. M. (2014). *An introduction to functional grammar* (4 ed.). Arnold. https://doi.org/10.4324/9780203431269

Jacinto, H., & Carreira, S. (2021). Digital tools and paper-and-pencil in solving-and-expressing: how technology expands a student's conceptual model of a covariation problem. *Journal on Mathematics Education*, *12*(1), 113–132. https://doi.org/10.22342/jme.12.1.12940.113-132

Kertil, M., Erbas, A. K., & Cetinkaya, B. (2019). Developing prospective teachers' covariational reasoning through a model development sequence. *Mathematical Thinking and Learning, 21*(3), 207–233. https://doi.org/10.1080/10986065.2019.1576001

Kress, G., & van Leeuwen, T. (2006). *Reading images: The grammar of visual design* (2nd ed.). London: Routledge. https://doi.org/10.4324/9780203619728

Moyer-Packenham, P. S., & Westenskow, A. (2013). Effects of virtual manipulatives on student achievement and mathematics learning. *International Journal of Virtual and Personal Learning Environments, 4*(3), 35–50.

O'Halloran, K. L., Beezer, R. A., & Farmer, D. W. (2018). A new generation of mathematics textbook research and development. *ZDM, 50*(5), 863–879. https://doi.org/10.1007/s11858-018-0959-8

Pepin, B., Gueudet, G., & Trouche, L. (2017). Refining teacher design capacity: Mathematics teachers' interactions with digital curriculum resources. *ZDM, 49*(5), 799–812. https://doi.org/10.1007/s11858-017-0870-8

Poon, K. K., & Wong, K. L. (2017). Pre-constructed dynamic geometry materials in the classroom-how do they facilitate the learning of "similar triangles"? *International Journal of Mathematical Education in Science and Technology, 48*(5), 735–755. https://doi.org/10.1080/0020739X.2016.1264636

Stevenson, C. E. (2017). Role of working memory and strategy-use in feedback effects on children's progression in analogy solving: An explanatory item response theory account. *International Journal of Artificial Intelligence in Education, 27*(3), 393–418. https://doi.org/10.1007/s40593-016-0129-5

Van der Kleij, F. M., Feskens, R. C. W., & Eggen, T. J. H. M. (2015). Effects of Feedback in a Computer-Based Learning Environment on Students' Learning Outcomes: A Meta-Analysis. *Review of Educational Research*, *85*(4), 475–511. https://doi.org/10.3102/0034654314564881

van Leeuwen, T. (2005). *Introducing social semiotics*. Routledge.

Developing a model of meaning for the concept of logarithms based on the content of university mathematics textbooks

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This paper explores the concept of logarithms as presented in university mathematics textbooks. By means of subject matter analysis, the content was analysed for ways to define and give meaning to logarithms. Seven variants of logarithm definition and ten different facets of meaning were identified, reflecting formal rigor alongside intuition. The findings demonstrate mathematically equivalent yet conceptually diverse ways of defining logarithms, as well as various semantic descriptions highlighting certain properties of the concept. The resulting model of meaning provides a structured framework for understanding how logarithms are conceptualised in higher education, and for studying how logarithms are or could be conceptualised in secondary school.

Introduction

The concept of logarithms has been shown to pose challenges for students to comprehend (e.g., Berezovski & Zazkis, 2006; Chua & Wood, 2005; Kenney, 2005). It is a non-trivial function as it cannot be expressed using a finite combination of numbers, variables, and the standard operations addition, subtraction, multiplication, and division. Despite the basic idea of logarithm being a connection between multiplicative and additive structures, providing a tool for converting burdensome products into easily calculated sums, the notation $log_a(x)$ does not clearly convey the transformation applied to the input x. The abstract nature, shared with trigonometric and exponential functions, might place significant cognitive demands on students' interpretation of mathematical definitions. To begin unpacking logarithms as a mathematics subject matter to be taught and learnt, this paper explores how logarithms are defined and given meaning in university mathematics textbooks.

Among students' challenges with logarithm comprehension identified in previous research, there are difficulties related to grasping the concept and its notation. For instance, common errors are related to the treatment of "lg" as a product of variables, so in "$\lg(x + 100)$", lg is treated as one factor and $(x + 100)$ as another (Chua & Wood, 2005; Kenney, 2005). Hence, the combination of the two letters l and g is interpreted in a mathematical, yet improper, way. This suggests a need of guidance for instructors as to how the logarithm can be conceptualised in teaching to minimise misconceptions.

Ball et al. (2008) developed a theory of mathematical knowledge for teaching (MKT) based on the notion of pedagogical content knowledge (PCK) and observations

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and analyses of mathematics teaching. They identified specialized content knowledge (SCK) as the mathematical knowledge and skills that are unique to teaching. These include the expertise required to be able to explain procedures, interpret student errors, evaluate alternative algorithms, choose effective analogies, and understand different possible interpretations of one and the same topic (Ball et al., 2008).

In the case of logarithms, they could be conceptualised differently when introduced than in later subsequent teaching, just as sine, cosine, and tangent could be introduced as relations between the sides of a triangle, while later defined as functions. But the meaningful selection of ways to define and give meaning to concepts demands SCK (Ball et al., 2008). Within the course of a PhD project, I seek to unpack the mathematical content of logarithms and make it accessible to pre-service and in-service teachers as well as mathematics education researchers. The current paper reports on a substudy of a project aimed at developing a comprehensive model of meaning for the concept of logarithms, grounded in mathematical and educational perspectives. While the full project incorporates literature, interviews, and teacher workshops, this paper focuses on university textbooks, constituting one quarter of the full study.

Aim and research questions

This study explores manifested ways in which the concept of logarithms is defined and given meaning in university mathematics textbooks on real analysis. The aim is to develop a model of meaning containing diverse but normative definitions and meaning facets of the concept of logarithms. The following research questions guide the study:

RQ1: Which definitions and meaning facets of the concept of logarithms are manifested in university mathematics textbooks as the concept is introduced?

RQ2: How can these definitions and meaning facets be organised into a model of meaning for the concept of logarithms?

Background

Mathematical approaches to mathematics education research take mathematics as the main reference point for investigating how mathematical content is recontextualised in educational settings (Bergsten, 2020). Examples include the Anthropological Theory of Didactics (ATD), MKT and the German traditions of Stoffdidaktik (Bergsten, 2020). While ATD emphasizes the relativity of mathematical knowledge with respect to institutions (Bosch & Gascón, 2006), MKT and Stoffdidaktik attach great importance to didactical analyses of mathematical content as a key activity in preparing the subject for teaching (Ball et al., 2008; Hußmann et al., 2016). A key tool is subject matter analysis, which aims to make mathematics accessible and comprehensible for learners through analysis of the subject matter, while preserving mathematical rigor (Hußmann et al., 2016). Such analyses have previously resulted in theoretical foundations for empirical research and scholarly discussions (Kolloshe, 2024; Lankeit & Biehler, 2024). Felix Klein (1908/2016) was a pioneer of this approach. Through mathematical, didactical

and historical analysis, he aimed to make the concept of logarithms teachable for school mathematics. He concluded by recommending the integral definition of logarithms (see below) as the most suitable way to introduce the concept in school.

Subject matter analysis

This study applies subject matter analysis informed by Lankeit and Biehler (2024), who developed a conceptual framework for modelling the meaning of higher-level mathematical concepts. The framework distinguishes between a *formal level*, comprising definitions, theorems and proofs, and a *semantic level*, encompassing sense, meaning and interpretation. Within this model, concept definitions appear as *definition variants* (DVs), which are "mathematically equivalent but conceptually different enough to evoke different associations" (Lankeit & Biehler, 2024, p. 3). This equivalence means that one concept definition can be used to derive theorems which in turn can be used to define the same concept. The DVs may be *constructive*, specifying how to generate an object, or *relational-descriptive*, defining it through its abstract properties (Lankeit & Biehler, 2024). For example, odd numbers can be constructively defined as $2n+1, n \in \mathbb{Z}$, or relational-descriptively defined as integers indivisible by 2.

The semantic level of the framework addresses *facets of meaning* of the concept, which articulate different interpretive perspectives. In Lankeit and Biehler (2024), facets of meaning were organised according to contexts of interpretation (analytic-algebraic, geometric and approximation context), derived from mathematics education literature on a nearby mathematics concept.

Categorisations of logarithm definitions and interpretations

Previous research on different definitions and interpretive aspects of the logarithm that may be held by students or communicated through teaching suggests various categorisations, summarised in Table 1. Ricardo (2022) outlined four common ways to define the natural logarithm and proved their equivalence. $\ln x$ was defined as the *limit* for $n(\sqrt[n]{x-1})$ as $n \to \infty$, as the *integral* $\int_1^x \frac{1}{t} dt$ for $x > 0$, as the unique continuous *solution of the functional equation* $f(xy) = f(x) + f(y), f(e) = 1$, and as the *inverse* of e^x. Conducting a historical review, Riley (2023) identified five various ways in which

Definitions	**Interpretations**		
Ricardo (2022)	Riley (2023)	Weber (2016)	Gruver (2018)
Limit	Area	Inverse exponent	Factor interpretation
Integral	Exponent	Multiplicative reasoning	Exponent
Unique solution to functional equation	A way to simplify calculations	Decreasing the hierarchy level of operations	
Inverse to the natural exponential	Comparison between two progressions	Counting the number of digits	
	Arithmetic sequence		

Table 1. Definitions and interpretations of logarithms from previous research

the logarithm has been presented. These are an *area*, an *exponent*, a *way to simplify computations*, a *comparison between two progressions*, and an *arithmetic sequence* juxtaposed to a geometric sequence. Weber (2016), on the other hand, present four 'basic models' of the logarithm, namely logarithms as *inverse exponents*, *multiplicative reasoning* (including repeated division), *decreasing the hierarchy level* (of operations), and *counting the number of digits*. Weber (2016) also theoretically evaluated how well basic models help students solve tasks and avoid errors. For example, digit-counting works only in base ten, while multiplicative reasoning supports solving specific logarithmic expressions effectively (Weber, 2016).

Unlike above desk-based studies, Gruver (2018) observed logarithm interpretations emerging in classrooms, accepted by a majority of participating students. Two of the normative ways of reasoning were the *factor interpretation of log* (i.e., log asks for the number of multiplications) and *logarithms as exponents* (Gruver, 2018). Although overlaps exist, such as the "exponent" highlighted in several studies, the general view of logarithms appears scattered. Furthermore, Riley (2023) showed that in addition to definitions such as those in Table 1, Canadian, British and American secondary school textbooks also conceptualised the logarithm as a value and as a button on the calculator. Extending this previous work to Sweden, including university level textbooks, and adding multiple data sources, the PhD project aims to systematise logarithm meaning facets, developing a comprehensive model that could support empirical research and scholarly discussion of different approaches to the teaching of logarithms.

Method

13 university mathematics textbooks were sampled to represent official texts on mathematics subject matter, based on the assumption that the authors of textbooks are authoritative participants in mathematical discourse, and that university teachers purposefully select course literature after careful review. In strive for a diverse sample, I included course literature used in introductory analysis courses at the ten largest universities in Sweden (counted by full-time students, UKÄ, 2025). This resulted in a sample of 11 books and two course compendiums, marked with * in the reference list.

The study employed subject matter analysis following Lankeit and Biehler's (2024) framework. First, the table of content of each textbook was scanned for the terms "logarithm" and "logarithmic function". In response to RQ1, only introductory sections were included, while advanced treatments (e.g., derivatives of logarithms) were excluded. Each identified section was then analysed separately using inductive content analysis (Elo & Kyngäs, 2008). Aligning with the purpose of this study, the method of content analysis seeks to construct a conceptual map by categories describing the phenomenon in a condensed, still broad, manner (Elo & Kyngäs, 2008). Table 2 outlines the different phases of inductive qualitative content analysis as proposed by Elo and Kyngäs (2008), along with a description of how the present analysis was carried out.

Contexts of interpretation were identified using the framework by Berezovski and Zazkis (2006), which is based upon prior work on exponential functions. Using this, I

distinguished between the context of *number* (logarithms as exact or approximated values), *operations* (relating to procedures or to the relation between multiplicative and additive structures), and *function* (relating to functions or function properties).

Phases and steps		Description	
Preparation phase			
Phases and steps of inductive content analysis from Elo & Kyngäs (2008)	Selecting the unit of analysis	The unit of analysis was selected as statements declaring what logarithms are, expressed with mathematical and/or natural language.	
	Making sense of the data	All textbook sections included in the sample were read through several times.	
Organising phase			
	Open coding	Statements about what logarithms are were identified, notes were taken in the margins as interpretations of what logarithms are declared to be in this statement. See example below, extract from Ekstig et al. (2019). Solution to exponential equation Lösningen $x = 3$ till ekvationen $2^x = 8$ kallas *2-logaritmen* av 8. Vi skriver $^2\log 8 = 3$.	
		Definition variants	*Facets of meaning*
	Coding sheets	Identified statements belonging to the formal level were adopted as pre-DVs.	Notes from the margins were used to construct the coding sheet, producing preliminary categories.
	Grouping and categorisation	Pre-DVs were grouped by considering similarities within and differences across the emerging categories.	Preliminary categories were grouped into higher order categories considering similarities within and differences across categories.
	Abstraction	Each category was assigned a number (DV1-7) and a prototypical DV was formulated for each category.	Each category was named using content-characteristic words, and a general description of the data was formulated using typical examples from the data.
Reporting the analysing process and the results			
	Model, conceptual map or categories	DVs were assigned to the constructive or relational-descriptive category and facets of meaning to one of the contexts of interpretation. The model of meaning for the concept of logarithms was constructed.	

Table 2. Phases and steps of inductive analysis from Elo & Kyngäs (2008) juxtaposed with a description of this paper's analysis procedure

In order to get a more reliable description of the data, the open coding process and the identification of statements at the formal level were discussed with a group of five colleagues. These discussions led to the replacement of one facet of meaning with two, the addition of one DV and a refined description of the analysis process.

Results

Figure 1 presents a model of meaning for the concept of logarithms, developed using subject matter analysis of 13 university mathematics textbooks. The model identifies seven DVs – three constructive and four relational-descriptive – alongside ten facets of meaning across three contexts: number (3), operational (4) and function (3).

The constructive DVs show how to mathematically obtain logarithms. DV1 defines the natural logarithm as an integral. In Stewart et al. (2020), this is followed by an acknowledgement that $\ln x$ also can be interpreted geometrically as an area. This conceptualisation is used as a proper definition in other books (DV2), constructively defining the natural logarithm as an area. DV3 defines a logarithm as the exponent y satisfying $a^y = x$, explicitly instructing where in the expression to find the logarithm.

The relational-descriptive DVs define logarithms through abstract properties rather than construction. DV4 specifies the equivalence $y = \log_a x \Leftrightarrow a^y = x$ describing the required relation between logarithms and exponentials. It resembles DV3, however, while DV3 instructs where to find the logarithm (in the exponent), DV4 specifies a relation that logarithms must satisfy. DV5 defines the logarithm as a function with certain properties, namely the domain ($x > 0$) and range ($y \in \mathbb{R}$) along with the equality $\log_a(mn) = \log_a m + \log_a n$ for $x > 0$ and $y > 0$. DV6 defines a logarithm as the inverse of the exponential function with the same base. This DV is relational-descriptive as it prescribes a certain relation to be fulfilled for a function to belong to the class of logarithm functions. DV7 defines logarithms as unique solutions to exponential equations, their existence relying upon the relation with exponentials.

The definition variants are considered to form the mathematical basis for facets of meaning of the concept of logarithms. Being facets, neither comprises the full meaning of the concept, they rather highlight different properties. In the number context, logarithms are interpreted as a value, a solution or (implicitly) an exponent. The exponent facet is exemplified in Johansson and Öhman (2017, p. 174, my translation) calling it a rhetorical definition when stating that "$\ln x$ is what one should take e to the power of in order to receive x". The operational context includes tool-referring facets, namely using logarithms to solve exponential equations or to rewrite numbers. The transformational facet of meaning, turning multiplication into addition, is presented in a historical remark in Olsen (2023, p. 67), stating that "[t]he basic idea of logarithms was to find an invertible function that turns multiplication into addition". The inverse facet also belongs here, reflecting the isomorphic relation between multiplicative and additive structures, as in Rodhe and Sollervall (1998, p. 26, my translation), "the logarithm function is inverse to corresponding exponential function". Although related to DV6, this statement lacks sufficient mathematical rigor to constitute a DV.

Within the function context, two facets capture the general function perspective, either as a single function or as a class of functions. The third gives logarithms the ordered-pair facet of meaning, which is related to functional thinking, and is exemplified by Forsling and Neymark (2011, p. 77, my translation) stating that "for every number $x > 0$, a real number $\ln x$ is ordered".

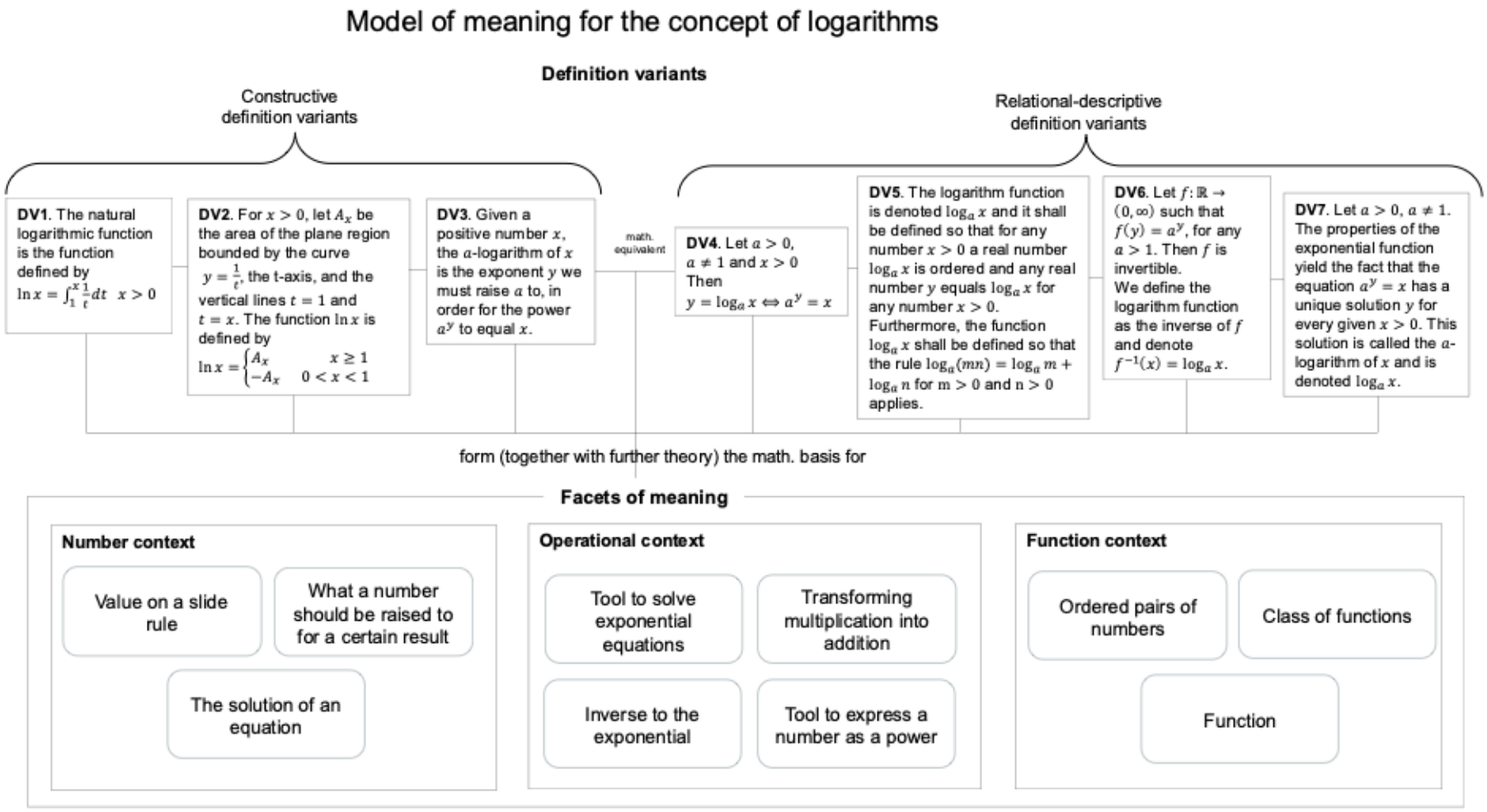

Figure 1. Model of meaning for the concept of logarithms, displaying definition variants and facets of meaning

Discussion

This study analysed 13 university mathematics textbooks to explore how the concept of logarithms is defined and given meaning, resulting in the model of meaning presented in Figure 1. In response to the research questions, seven definition variants (DVs) and ten facets of meaning were identified, the latter spanning three contexts of interpretation. Together, these capture the diverse ways in which logarithms are presented in the sampled textbooks, as they differ in conceptual emphasis and related procedures.

A central theme is the relation between logarithms and exponential functions. Several DVs and meaning facets rely on this inverse link, yet others, namely DV1 (integral), DV2 (area), and DV5 (rule) define logarithms independently of exponentials. Both approaches are mathematically valid, but their didactical implications might differ. These must be explored by the teacher themself or by further research.

Notably, DV1 and DV2 define only the natural logarithm, requiring specification of the base to e when using other DVs to derive them. This adds a condition to the mathematical equivalence that characterises the relations among the DVs.

The identified facets of meaning resonate with prior research. Exponent, inverse, and function meaning facets align with findings by Gruver (2018), Riley (2023) and Weber (2016). The function meaning facet also appears in the framework by Berezovski and Zazkis (2006), while the transformational, value and equation solution meaning facets refine their operational and number contexts. Two of the definitions presented by Ricardo (2022) (integral and inverse to exponential) correspond to DVs in the current result. This study extends earlier work by combining the formal level of definition variants and the semantic level of meaning facets into a unified and detailed model. While the DVs could all serve to derive key properties of logarithms, the facets of meaning each seem to highlight one or a few features, which suggests that we need several of them to actually grasp the concept. Notably, the repeated division/factor interpretation, recognised by Weber (2016) and Gruver (2018), are absent. These views would connect logarithms to the standard operations multiplication and division, rather than to the more advanced concepts of exponential, function, inverse and integral which the current findings emphasise. This study's focus on university textbooks hence delimits the findings by relating to curricula on a certain educational level. Extending the data sources to include mathematics education research literature and insights from mathematicians and teachers could further enrich the model.

The model could also be strengthened by further validation. Lankeit and Biehler (2024) view different DVs as conceptually distinct as they evoke different associations. While the current analysis distinguished DV1 (integral) and DV2 (area), readers might argue that a $\int_a^b f(x)\,dx$ and an area represent the same. Unsurprisingly, the DVs resemble each other mathematically to various degrees, as all aim to grasp logarithms. This study sought to trace conceptual differences among definitions and meaning facets manifested in textbooks. Future research should examine whether these distinctions extend beyond the author's interpretations, explore associations evoked by different DVs, and examine their relation to various meaning facets of the concept of logarithms.

Methodological and theoretical contributions of this study include the productive integration of the framework subject matter analysis with the method content analysis, alongside the initial development of a model of meaning for the concept of logarithms. The emerging model could guide future empirical and theoretical research on teaching and learning logarithms. For instance, research on MKT requires specification of the mathematics that inform teachers' SCK (Ball et al., 2008). Hence, researchers using an MKT framework to study secondary school pre-service or in-service teachers' knowledge of logarithms could make use of this model of meaning, likewise researchers studying definitions and meanings that come to use in classrooms. Additionally, this study empirically contributes with a characterisation of logarithm definitions and facets of meaning brought forth, and not, by a sample of university mathematics textbooks.

The emerging model has didactical implications for subject-specific professional development, encouraging teachers to reflect on productive ways to define and give meaning to logarithms in instruction. It should be noted, though, that the distinction made between DVs and facets of meaning is analytical; in secondary school classrooms, a facet of meaning identified in this analysis could serve as a definition. Which interpretation to foreground depends on learning goals, context, and students' prior knowledge (Lankeit & Biehler, 2024). Notably, the integral definition proposed by Klein (1908/2016) is still just one of many definitions used in university mathematics textbooks. This diversity enables connections between logarithms and several other mathematical concepts. For instance, Riley (2023) proposes introducing logarithms through the relation between arithmetic and geometric sequences, which may be more accessible than an inverse exponential definition. In other classrooms, logarithms could serve to extend students' understanding of functions, particularly non-trivial such, in which the domain, range and ordered-pair idea are significant concepts. Such informed decisions taken by teachers on how to prepare mathematical content for instruction presume an extensive knowledge base, to which the emerging model can contribute.

References

*Adams, R. A., & Essex, C. (2021). *Calculus: A complete course* (10th ed.). Pearson.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching. *Journal of Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Berezovski, T., & Zazkis, R. (2006). Logarithms: Snapshots from two tasks. In J. Novotná, H. Moraová, M. Krátká, & N. a. Stehlíková (Eds.), *Proceedings of the 30th Conference of the Internatioinal Group for the Psychology of Mathematics Education* (Vol. 2, pp. 145–152).

Bergsten, C. (2020). Mathematical approaches. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (2 ed., pp. 498–505). Springer Cham. https://doi.org/https://doi.org/10.1007/978-3-030-15789-0

Bosch, M., & Gascón, J. (2006). Twenty-five years of the didactic transposition. *ICMI Bulletin*, *58*, 51–65.

*Brandell, G., & Sjöstrand, S. (2016). *Envariabelsanalys med datorstöd*. Studentlitteratur.

Chua, B. L., & Wood, E. (2005). Working with logarithms: Students' misconceptions and errors. *The Mathematics Educator*, *8*(2), 53–70.

*Ekholm, T. (2022). *Envariabelsanalys för teknisk fysik*. [Unpublished course material]. KTH Royal Institute of Technology.

*Ekstig, K., Hellström, L. & Sollervall, H. (2019). *Matematik startbok – för ingenjörer och naturvetare*. Studentlitteratur.

Elo, S., & Kyngäs, H. (2008). The qualitative content analysis process. *Journal of Advanced Nursing*, *62*(1), 107–115. https://doi.org/10.1111/j.1365-2648.2007.04569.x

*Forsling, G., & Neymark, M. (2011). *Matematisk analys - En variabel* (2nd ed.). Liber.

Gruver, J. (2018). A trajectory for developing conceptual understanding of logarithmic relationships. *The Journal of Mathematical Behavior*, *50*, 1–22. https://doi.org/10.1016/j.jmathb.2017.12.003

Hußmann, S., Rezat, S., & Sträßer, R. (2016). Subject matter didactics in mathematics education. *Journal für Mathematik-Didaktik*, *37*(S1), 1–9. https://doi.org/10.1007/s13138-016-0105-5

*Johansson, R., & Öhman, L.-D. (2017). *Introduktion till högre studier i matematik*. Liber.

Kenney, R. (2005). Students' understanding of logarithmic function notation. In G. Lloyd, M. WIlson, J. Wilkins, & S. Behm (Eds.), *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1–8). PME-NA.

Klein, F. (2016). *Elementary mathematics from a higher standpoint* (G. Schubring Trans.). Springer. (Original work published 1908)

Lankeit, E., & Biehler, R. (2024). The meaning landscape of the concept of the total derivative in multivariable real analysis textbooks: An analysis based on a new model of meaning. *ZDM – Mathematics Education*, *56*, 1361–1373. https://doi.org/10.1007/s11858-024-01584-w

*Månsson, J., & Nordbeck, P. (2011). *Endimensionell analys*. Studentlitteratur.

*Olsen, J.-F. (2023). *Don't Panic Part 1: A guide to MATA21 analysis in one variable*. [Unpublished course material]. Lund University.

*Persson, A., & Böiers, L.-C. (2010). *Analys i en variabel*. Studentlitteratur.

Ricardo, H. (2022). The equivalence of definitions of the natural logarithm function. *The College Mathematics Journal*, *53*(3), 190–196. https://doi.org/10.1080/07468342.2022.2039553

Riley, S. (2023). *An investigation into historic and contemporary presentations of logarithms in textbooks: Can historic presentations enhance the contemporary presentations of logarithms?* [Doctoral dissertation, Simon Fraser university].

*Rodhe, S., & Sigstam, I. (2006). *Naturlig matematik*. Studentlitteratur.

*Rodhe, S., & Sollervall, H. (1998). *Matematik för ingenjörer*. Bokförlaget KUB.

*Rule, D. (2021). *Ge svar på tal – En introduktion till matematisk analys*. Studentlitteratur.

*Stewart, J., Clegg, D., & Watson, S. (2020). *Calculus: Early transcendentals* (9th ed.). Cengage Learning.

UKÄ. (2025). *Antal helårsstudenter 2023/2024*. https://www.uka.se/integrationer/hogskolan-i-siffror/statistik?statq=https://statistik-api.uka.se/api/totals/34

Weber, C. (2016). Making logarithms accessible – operational and structural basic models for logarithms. *Journal für Mathematik-Didaktik*, *37*, 69–98. https://doi.org/10.1007/s13138-016-0104-6

Developing methods for two-digit subtraction by means of the thieving game

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Multi-digit subtraction is a central part of Grade 1–3 mathematics that many students struggle with and is challenging to teach. We used design research to investigate how and to what extent game-based activities could support Grade 1 students' development of methods for two-digit subtraction. Our results show that a short sequence of activities based in the game led to 85 % of students being able to document at least one method for two-digit subtraction, and 42 % being able to document at least two different methods. However, the way the teacher picked up students' methods for discussion and how the teacher documented them affected what methods students developed.

Multi-digit addition and subtraction is the only area for which Swedish Grade 3 students are required to know written methods (Skolverket, 2022). Since 2016, one subtest of the national mathematics test for Grade 3 has been devoted to such methods. During these years, this subtest has had the lowest student attainment—about 80 % compared to 86–95 % on the other subtests (Skolverket, 2025). Research studies have also found that students struggle with, particularly, methods for multi-digit subtraction (Heinze et al., 2009; Skolverket, 2025). For example, Heinze et al. (2009) and Kullberg et al. (2024b) both report that the students in their samples solved less than half of the tasks correctly. However, the success rate depends on the methods students use, and what methods students use may depend on the design of teaching.

Methods for multi-digit subtraction

In Sweden, and internationally, students are usually taught the *standard algorithm* for subtraction, writing the minuend above the subtrahend and subtracting from right to left digit by digit. If a subtraction cannot be performed due to the minuend digit being smaller than the subtrahend digit, one is taken from the next digit and exchanged for a ten in the current digit. While understanding how and why this algorithm works require making use of properties of our number system, it is possible to *use* the algorithm without considering the place values of the digits, seeing them as separate one-digit numbers. Therefore, this method can be seen as digit-based (Hickendorff et al., 2019). While seldom explicitly taught, it is known that some students use *One-by-one* counting methods (Hopkins et al., 2022). This can be done in several ways—e.g., counting down from the minuend—but all involve double counting that heavily taxes working memory. When terms and differences get larger, this makes the method prone to mistakes.

In contrast to the standard algorithm and one-by-one counting, most methods for multi-digit subtraction make use of the base-10 structure of our number system (Fuson

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et al., 1997) and can thus be seen as number-based (Hickendorff et al., 2019). Some of these methods are based on a "taking away" view of subtraction (Selter et al., 2012). The *Split* method splits both terms into ones, tens and hundreds and entails one subtraction for each number, as in the standard algorithm. Several studies have reported low success rates for using this method for subtractions bridging over ten (Heinze et al., 2009; Kullberg et al., 2024b). A common mistake for when using this method is to subtract the minuend from the subtrahend when the minuend is smaller. The *Stepwise* method splits only the subtrahend, usually into parts that make each sub-subtraction simpler, such as 32 – 18 = 32 – 2 – 10 – 6 = 30 – 10 – 6 = 20 – 6 = 14. This method has been reported to be substantially more successful when used by students (Heinze et al., 2009; Kullberg et al., 2024b; Torbeyns et al., 2009). The *Compensation* method first takes away more than the subtrahend, normally the nearest greater ten, and then adds the difference: 32 – 18 = 32 – 20 + 2 = 12 + 2 = 14. Other methods are based on a "determining the difference" view of subtraction (Selter et al., 2012). *Indirect addition* involves adding parts to the subtrahend up to the minuend using tens and hundreds as benchmarks: 18 + $\underline{2}$ = 20, 20 + $\underline{10}$ = 30, 30 + $\underline{2}$ = 32, 2 + 10 + 2 = 14. *Simplification* involves adding or subtracting the same number to both terms, which can be explained as moving the difference along the number line to a place where it is more easily "seen": 32 – 18 = (32+2) – (18+2) = 34 – 20 = 14.

Teaching designs for methods for multi-digit subtraction

To some extent, students may develop their own methods independent of teaching. However, research has shown that when students are introduced to the standard algorithm, they abandon other methods, favouring the algorithm even if they cannot use it correctly and when other methods would be more efficient (Hickendorff et al., 2019). More recent studies have shown that this phenomenon can be partly avoided if teaching focusses students' own development and comparison of methods before introducing the standard algorithm (Heinze et al., 2018). Designing teaching that supports students' own development of methods may, however, be more difficult since the teaching activities must both require students to develop more advanced methods than One-by-one and support them in doing so. For example, the design used in Heinze et al. (2018) failed to support students' development of indirect addition.

Kullberg et al. (2024a) has shown that strengthening students' knowledge of part-whole number relations can increase their ability to solve subtractions bridging over ten. However, the solution frequencies for tasks with multi-digit subtrahends was still below 50 %, and 55 % of students still used error-prone methods (Kullberg et al., 2024b), indicating that additional teaching is needed to support the development of multi-digit subtraction methods.

Aim and research question

The aim of this study is to further understanding of how activities can be designed to support students' development of multi-digit subtraction methods, by answering the question: How and to what extent can the thieving game support Grade 1 students' development of two-digit subtraction methods by means of their own reasoning?

Method

This study is part of a design research project aiming to develop design principles for teaching Grades 1–3 students about numbers and operations. The study took place at the end of Grade 1 in two classes that had been following a teaching design during Grade 1, based on alternating between student independent work, alone or in pairs, and whole-class discussions about students' methods and reasoning. During Grade 1, teaching had covered additive relationships for numbers 0–20, additive situations (part-part-whole, change, and comparisons), double and half, and two-digit numbers, including adding and subtracting even tens to/from two-digit numbers. The students had worked with several representations of numbers, including the number line, the place-value system and wooden base-10 blocks developed within the project consisting of one- (1B), five- (5B) and ten- (10B) blocks with no visible units (Figure 1A).

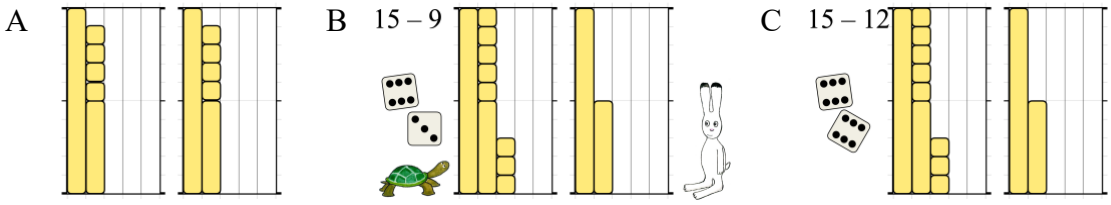

Figure 1. Initial setting for the thieving game and two tricky situations

The study design included developing an initial teaching design for three lessons, which was tested in each of the teachers' classes in two iterations, with revisions of the design in between. The teaching design included instructions and materials for the students' independent work, and suggested examples and questions for the whole-class discussion. The initial idea for the design was to support students' development of methods in the context of *tricky situations* emerging in the thieving game using base-10 blocks played in pairs. In the thieving game, each player starts with an equal amount, in this case 19 represented by one 10B, one 5B and four 1B (Figure 1A). The players take turns in rolling two dice. In each turn, the player rolling the dice is allowed to "thieve" blocks of the same value as the throw of the dice from the other player. A player loses when they have no blocks left. The game forces students to translate from dice to block representations of numbers and thereby practicing number relations such as $3 + 3 = 5 + 1$. It also gives rise to tricky situations where the correct number to thieve cannot be directly taken from the other player (Figure 1B). The game element was hypothesized to make students less likely to solve such situations by taking too much or too little and playing in pairs would make it impossible to avoid tricky situations by choosing to thieve from another player. It was seen as an advantage that tricky situations only start to occur after a few turns, as this would allow all students to get started.

Each lesson started with a whole-class discussion followed by student independent work. The aim of the lesson sequence was to allow students to develop methods in the context of the game and then transfer these methods to subtraction tasks with numbers. The aim of the first lesson was to give students experience of the game and tricky situations. The aim of the second lesson was to develop and practice ways of documenting handling tricky situations with symbolic notation. The aim of the third lesson was to name and practice these ways as methods for subtraction with two-digit

numbers. Through simulated gameplay, we identified three possible methods for handling tricky situations: *Compensate*, and swapping blocks (e.g., one 10B for 1 5B and five 1B) followed by *Stepwise on tens* or *Take from ten* (Table 1).

Code	Description
No documented method	Only an answer or the expression to be evaluated is given.
Not determinable	A method is described or documented, but which one cannot be determined.
One by one	Counting up or down one at a time.
Compensate	Subtracting the nearest larger ten, then adding the difference.
Stepwise on tens	Subtracting stepwise so that the sub-results are even tens.
Stepwise other	Subtracting stepwise so that the sub-results are not even tens.
Split	Subtracting the ones and tens separately
Take from ten	Subtract the subtrahend from one or all tens in the minuend, then adding the ones.

Table 1. Codes for subtraction methods

Data collection and method of analysis

In each class, the teaching design was planned to be implemented over three lessons, each with 40 min devoted to the mathematical activities. During the planned lessons, one researcher observed the lesson and took notes, and audio was recorded by a device worn by the teacher. Student worksheets were also collected. As the study aims to investigate how and to what extent the thieving game can support Grade 1 students' development of two-digit subtraction methods, the analysis focused on identifying students' use of different methods. The first step of the analysis was therefore to identify segments of observation notes, audio recordings and students' written solutions that concerned one solution to one task. Each such segment that concerned a subtraction task was then coded for which method (if any) was described, when and by who. The coding system started out from methods described by previous research and was then developed by adding codes for methods that did not fit any existing code and removing codes that were not present in our data. The final codes are described in Table 1. The codes were then used to create a narrative of the lesson sequence in each iteration, describing when, how and by whom methods emerged.

Results

In this section, we present the results of the two iterations including the revised design.

Iteration 1

The plan for Iteration 1 was: In Lesson 1, introduce the game without discussing tricky situations and let students play in pairs, recording results after each turn, while the teacher identifies and overviews students' handling of tricky situations. In Lesson 2, discuss one non-tricky and one tricky situation, described as occurring when Tortoise and Hare (two figures used since start of Grade 1) are playing. When students suggest ways of handling the tricky situation, the teacher shows ways of documenting these ways using symbolic notation. Each turn in the game includes both a subtraction and an addition, but the teacher was asked to start with and focus on the subtraction. Then, play in pairs, recording how each turn is handled, as done in the discussion. In Lesson 3,

discuss selected examples from the students' documentation, constructing posters for methods used, then individual student work documenting methods for subtractions.

Lesson 1

The teacher started by introducing the game by playing two turns with a student, visibly to the whole class. This seemed to be enough for the students to understand the rules of the game and get started playing in pairs. All pairs seemed engaged and enjoying playing, and most students did not need any support. However, identifying and overviewing students' handling of tricky situations proved difficult for both the observer and the teacher. The teacher was mostly occupied by organising the turn-taking in one group consisting of three students, who also needed support in handling the tricky situations. In this case, the teacher instructed the students how to swap blocks (one 10B for two 5Bs) when they needed to thieve 9 from 14 (one 10B and four 1B). The observer had a hard time capturing the tricky situations, as they were sparse and randomly occurring, but at least one pair was observed to intentionally "forget" what the dice showed to avoid a tricky situation.

Lesson 2

The class first discussed a non-tricky situation, then the tricky situation showed in Figure 1B. This situation necessitates either using Compensate or swapping blocks (one 10B for one 5B and five 1B) followed by either Take from ten or Stepwise on tens. When asked for suggestions, one student demonstrated Compensate without any assistance from the teacher and, when asked, described what he had done in the following way:

Student:	I swapped a 1 for another.
Teacher:	You swapped a 10 for a 1. And why can one do like that?
Student:	You should say that (quietly).
Teacher:	You can say it—why can one do like that?
Student:	Because then you get 1 minus.
Teacher:	So, 10 minus 1?
Student:	Is 9 (emphatically).
Teacher:	Okay, so if we take the 10 and swap it back for a 1, then you take 9 and give back a 1, or take 10 and give back a 1. That means you take 9.

The teacher documented the actions by writing whole sentences on the board, which took considerable time. When documenting the actions using symbols, the teacher first described the addition as Compensate, followed by a student describing an alternative way of adding. The teacher then documented the subtraction as: $15 – 9 = 10 – 9 + 5 = 1 + 5 = 6$, marking that $10 – 9 = 1$, and used the blocks to illustrate the block swapping. This aligns with Take from ten rather than Compensate, which the student described.

The teacher then asked for alternative methods for solving the task $15 – 9$. One student tried swapping a 5B for five 1B, which proves ineffective in this context. With further assistance with swapping from another student, the student was eventually able to thieve 9. The teacher documented this on the board, starting with addition, describing the use of Stepwise other, and then the subtraction as $15 – 9 = 15 – 5 – 4$, describing Stepwise on tens. However, the teacher linked the steps to the order of the actions on the blocks, not to use of tens as benchmarks. At this point, the students seemed tired.

Next, the students set up and played a few rounds of the game. They were instructed to document how they solved the situations, following the teacher's example on the board. However, the students tended to only record what was thieved or lost, for example, writing 29 − 3 = 26. Three pairs used sentences rather than symbolic notation, which was time-consuming. As on Lesson 1, the sparseness and randomness of tricky situations made it difficult to identify and overview students' methods. One instance of students avoiding a tricky situation by thieving less than the dice showed was observed. This led to problems in the next turn, as their documentation did not match the gameplay. These students got a hint to try to swap some blocks, which was enough for them to see that swapping a 10B for two 5B enabled them to thieve 7 from 15 using Stepwise on tens. The observer and the teacher agreed that students needed to encounter more tricky situations and possibly discover methods to provide the teacher with a stronger basis for the subsequent discussion. Therefore, the plan for Lesson 3 was revised.

Lesson 3

First, students continued playing 2–6 rounds. None of the pairs documented how the thieving was conducted; most only recorded the results after each turn. Again, one pair used sentences rather than mathematical symbols. The observer identified one tricky situation (Figure 1C), where the students collaborated to use Compensate: Student 1 was to thieve 12 from 15 (one 10B, one 5B). Student 1 took 10B and then 5B, paused to think, and then stated: "You get two", handing over two 1B to Student 2. Both students then paused to think, then Student 2 took an additional 1B from Student 1, after which both students appeared satisfied.

Since there was no student documentation from the gameplay, the teacher based the whole-class discussion on fictional examples, creating posters illustrating Stepwise on tens for the addition 23 + 8 and the subtraction 15 – 8. The teacher asked for alternative methods, and one student demonstrated Compensate, taking one 10B and returning two 1B, but this was not acknowledged by the teacher, who remained focused on constructing the poster of Stepwise on tens. Attempts to connect the symbolic notation to actions on blocks were not entirely systematic, especially concerning swapping. Several ways of swapping were shown after each other, making it difficult to follow which swap was actually used in the method discussed. Subsequently, the teacher showed another example of addition and students suggested several other methods, but the teacher had some difficulty following their reasoning. For instance, the teacher initially interpreted a student's explanation of Stepwise on tens as a "jump of ten". Thus, most of the examples selected by the teacher focused addition rather than subtraction. As a result, the intended focus on methods for subtraction was partly lost.

The last activity was conducted at a later time and not observed, but worksheets were collected and analysed. In total, 56 solutions from 16 students were collected, of which 46 included at least an attempt to document a method, distributed over methods in the following way: One-by-one: 3, Stepwise on tens: 5, Stepwise other: 7, Take from ten: 23, Not determinable: 8. Thirteen students (81 %) documented at least one method, and 7 students documented two different methods (44 %).

Summary Iteration 1

The game allowed students to quickly get started. However, the sparseness and randomness of tricky situations made it difficult to know to what extent the students were engaged in methods for two-digit subtraction, and, in some cases, students came up with ways to avoid tricky situations. This made it difficult to base the discussions on students' own gameplay. Instead, the teacher had to pick up on students verbally expressed methods during discussions, which was challenging. In the discussions, the teacher started with addition which limited time and focus left for subtraction. Furthermore, the teacher's focus on the order of the steps in Stepwise might not have made the idea of pausing on tens clear for the students. This might explain why more students used Stepwise other than Stepwise on ten in the last activity.

Iteration 2

Based on the outcomes of Iteration 1, two revisions were made. First, it was emphasised that the discussions in Lesson 2 and 3 were to start with subtraction. Second, the gameplay in Lesson 2 was replaced by worksheets illustrating tricky situations with Hare and Tortoise (Figure 2), in order to make sure that all students worked with such situations and to aid the teacher's selection examples for use in the Lesson 3 discussion.

Lesson 1

Lesson 1 was conducted in half-classes but otherwise proceeded as in Iteration 1. Two instances of students using Compensate were noted, and two additional cases using swapping followed by Stepwise on tens. In one of these cases, the method closely resembled Compensate, as the swap and the thieving occurred simultaneously: thieving four by swapping one 5B for five 1B, immediately taking four 1B. Another student, having realized that swapping could be helpful, made a swap that did not aid the theft: To thieve 9 from 28 (two 10B, eight 1B, Figure 2A), five 1B was swapped with one 5B. The teacher suggested an additional swap, which led them to swap one 10B for one 5B and five 1B, enabling them to proceed. Another student always began by thieving 5B. When set to thieve 9 from 27 (two 10B, one 5B, two 1B) they realized they were stuck after taking 5B and came up with swapping one 10B for one 5B and five 1B. One pair asked the observer for support, who merely redirected the question back to the students: "How could you do it? You got to think.". This was enough for them to come up with using Compensate. As in Iteration 1, some students tried to avoid tricky situations which caused problems later. In one such case, the teacher described how a swap could be done, which allowed the students to continue without further issues.

Lesson 2

The discussion began asking for situations where it is not easy to thieve from a peer. One student explained that he was to thieve 8 from his friend having 17 (one 10B, seven 1B). When asked how they handled it, the student explained that they swapped. This was followed by more examples of swaps suggested by students. The teacher then discussed a non-tricky situation, emphasizing the importance of knowing the value of each block, followed by discussing a tricky situation (Figure 1B), focusing on swaps that would enable Tortoise to thieve 9 from Hare. Here, the teacher illustrated swaps by drawing arrows. Students explained Compensate and Stepwise on tens twice, and the teacher documented all contributions on the board using symbolic notation.

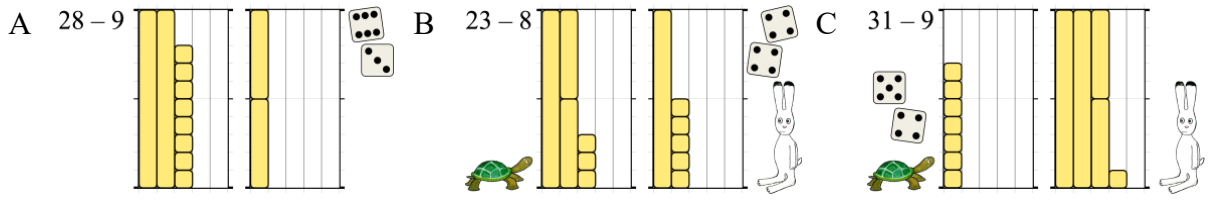

Figure 2. Two tricky situations (A and C) and one non-tricky situation (B)

Next, students worked in pairs with worksheets with tricky situations (such as Figure 2B–C) and the teacher extended the lesson 10 min. The teacher support was mainly in the form of encouragement and prompts to express thinking using "math language". Two pairs were also guided how to document their solutions when applying the Compensate method. Six of eight pairs attempted to document at least one method, four pairs documenting Compensate on at least one task, and three pairs documenting Stepwise on tens.

Lesson 3

The teacher demonstrated how to document a swap (one 10B for one 5B and five 1B), in symbolic notation, holding up blocks to illustrate the situation. Next, the teacher presented an example from the worksheet (Figure 2B) and showed how to document Stepwise on tens for subtraction, both on the board and on a poster. The teacher then created three posters, illustrating Compensate for addition and subtraction based on students' suggestions, starting with taking 9 from 31 (Figure 2C). When discussing Compensate, the teacher showed that nine remains by placing a 1B in front of a 10B.

Next, students worked with subtraction tasks. The teacher gathered six hesitant students at the board and showed Stepwise on tens and Compensate using an example not included in the worksheets. Two of the students were shown an additional example, after which one student was given additional help including number bonds to ten. Most students appeared to write down their answer first, then adding descriptions of their method. The teacher encouraged some students to write down their thinking, stating that it was not enough to write an answer. In total, 80 solutions from 17 students were collected from this activity, of which 56 included at least one attempt to document a method, distributed as: Compensate: 3, Stepwise on tens: 23, Stepwise other: 9, Take from ten: 11, Not determinable: 7. Two solutions used a previously unidentified method starting with increasing the minuend to make the subtraction easier, for example calculating 36 – 8 by the steps: 40 – 8 = 32; 32 – 4 = 28, and one solution mistook a subtraction for an addition. Fifteen students (88 %) documented at least one method, and 7 students documented 2–4 different methods (41 %).

Summary Iteration 2

As in Iteration 1, students were engaged and came up with methods for handling the tricky situations. The change from continued play to worksheets focusing tricky situations resulted in more focus on methods, and a better overview of what methods the students developed and used for the teacher. There was thus a better basis for the discussion, in which the teacher focused more on subtraction. These revisions may have contributed to students documenting more methods for more tasks, compared to Iteration 1.

Discussion

In sum, the answer to our research question is that the thieving game can support Grade 1 students' development of the methods Stepwise, Take from ten and Compensation, by means of their own reasoning by engaging students in an activity that gives rise to situations that require methods based on actions on blocks. One may particularly note the method Take from ten, which to our knowledge, have not been reported in previous studies investigating children's subtraction methods (e.g., Heinze et al., 2018).

Previous studies have investigated long-term effects of different teaching approaches such as investigative, problem-solving, and routine approaches (Heinze et al., 2009), explicit and implicit teaching (Heinze et al., 2018) and structural and non-structural approaches (Kullberg et al., 2024b). The studies of Heinze et al. (2009; 2018) have highlighted the advantages of building on students' own ideas but also noted that students' own methods can be limited. Our study complements these studies by investigating particular activities that support students' development of own methods. In both classes, the game was enough for students to come up with ways of handling tricky situations by means of their own reasoning, and after follow-up discussions of how these ways could be used as methods for subtraction, 28 of 33 students (85 %) could document at least one method for two-digit subtraction and 14 (42 %) could document two or more. The developed methods are all based on a "take away" notion of subtraction, which is expected as the game entails thieving blocks. To support students' development of methods based on a "determining the difference" notion of subtraction (Selter et al., 2012), other activities would be needed.

The game was not in itself sufficient for students to extend their actions on the blocks to methods for symbolically represented subtractions of two-digit numbers. As the tricky situations that require development of methods are sparse in the game, letting students think about selected situations was needed for all students to encounter enough tricky situations. When this was included both in the discussions and as a worksheet (Iteration 2, Lesson 2), the teacher got more information about students' methods, and the class had common examples to discuss. It is however likely that students experience of playing the game was important for their understanding of the worksheet. The development from actions on blocks to symbolic methods may also depend on how well students' descriptions of methods are taken up during discussions, and how well they are connected to block actions and symbolic notation. In Iteration 2, where the teacher focused on using tens as benchmarks, more students used Stepwise on tens than Stepwise other, while the distribution was reversed in Iteration 1, where this idea was less clear. This result aligns with the findings of Heinze et al. (2009), that teacher led discussion and comparison of methods gives better opportunities for students to learn to use at least one method accurately, compared to pure problem solving.

While several students came up with their own ideas during playing, some students may have been introduced to methods by the teacher or students during play or discussions. However, our data show that students documented a variety of methods, including methods never described or discussed in class. This indicates that students' methods were not mere imitation, but that the teaching design supported students' own development and selection of methods. Further research is needed to determine if students can retain and extend the methods to three-digit numbers in later grades.

Besides being supporting the development of methods, an advantage of the game is that it entails translating from dice to block representation, which entail practicing number bonds, which is important for developing subtraction methods (Kullberg et al., 2024a; 2024b). A limitation is that it requires specific materials, which are not available for teachers. While it is possible to play the game with regular base-ten blocks, it reduces the frequency of tricky situations. Teachers can create blocks themselves, but that requires time and effort. Further research would be needed to determine if the function of the game extends to other contexts and outweigh its limitations.

References

Fuson, K. C., Wearne, D., Hiebert, J. C., Murray, H. G., Human, P. G., Olivier, A. I., Carpenter, T. P., & Fennema, E. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education, 28*(2), 130–162.

Heinze, A., Marschick, F., & Lipowsky, F. (2009). Addition and subtraction of three-digit numbers: Adaptive strategy use and the influence of instruction in German third grade. *ZDM–Mathematics Education, 41*(5), 591–604. https://doi.org/10.1007/s11858-009-0205-5

Heinze, A., Arend, J., Gruessing, M., & Lipowsky, F. (2018). Instructional approaches to foster third graders' adaptive use of strategies: An experimental study on the effects of two learning environments on multi-digit addition and subtraction. *Instructional Science, 46*(6), 869–891. https://doi.org/10.1007/s11251-018-9457-1

Hickendorff, M., Torbeyns, J., & Verschaffel, L. (2019). Multi-digit addition, subtraction, multiplication, and division strategies. In A. Fritz et al. (Eds.), *International handbook of mathematical learning* (pp 543–560). Springer.

Hopkins, S., Russo, J., & Siegler, R. (2022). Is counting hindering learning? An investigation into children's proficiency with simple addition and their flexibility with mental computation strategies. *Mathematical Thinking and Learning, 24*(1), 52–69. https://doi.org/10.1080/10986065.2020.1842968

Kullberg, A., Björklund, C., & Runesson Kempe, U. (2024a). Seeing number relations when solving a three-digit subtraction task. *Educational Studies in Mathematics, 115*(2), 271–287. https://doi.org/10.1007/s10649-023-10287-0

Kullberg, A., Björklund, C., Runesson Kempe, U., Brkovic, I., Nord, M., & Maunula, T. (2024b). Improvements in learning addition and subtraction when using a structural approach in first grade. *Educational Studies in Mathematics, 117*(3), 399–417. https://doi.org/10.1007/s10649-024-10339-z

Selter, C., Prediger, S., Nührenbörger, M., & Hußmann, S. (2012). Taking away and determining the difference—A longitudinal perspective on two models of subtraction and the inverse relation to addition. *Educational Studies in Mathematics, 79*(3), 389–408. https://doi.org/10.1007/s10649-011-9305-6

Skolverket. (2024). *Läroplan för grundskolan, förskoleklassen och fritidshemmet. Lgr22 (2nd ed)*. tinyurl.se/aNL

Skolverket. (2025). *Sök statistik*. https://tinyurl.se/cdv

Torbeyns, J., Smedt, B., Ghesquière, P., & Verschaffel, L. (2009). Jump or compensate? Strategy flexibility in the number domain up to 100. *ZDM – Mathematics Education, 41*(5), 581–590. https://doi.org/10.1007/s11858-009-0187-3

Method use and success on the national test for grade 3 after being taught simplification methods

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This study explores which methods Swedish students use for multi-digit addition and subtraction after being taught the simplification methods regroup and move the difference. The data consist of 30 students' solutions to one subtest of the Grade 3 national mathematics test. Results show that students used a variety of methods, with splitting numbers by place value being the most common, despite having lower success rates. The simplification methods were used less frequently, but more successfully. The findings suggest that conceptually focused teaching of different methods can support students in selecting and using efficient methods, although additional support may be needed to increase actual use. The study also confirms earlier research showing that Stepwise methods outperform Split methods, especially in subtraction.

Methods for solving types of tasks is a key part of all mathematical topics at all levels. One of the first type of multistep methods that students encounter in school are methods for multi-digit addition and subtraction. While there are many methods for calculating sums and differences of multi-digit numbers, teaching in Sweden often focus mostly on the standard written algorithms (in Sweden referred to as *uppställning*). However, research has shown that the introduction of the written algorithm often results in students abandoning their own methods even when they would have been more efficient, and even when students use the written algorithm in erroneous ways (Hickendorff et al., 2019). In this study, we investigate to what extent a group of Grade 3 students, who were taught alternative methods for multi-digit addition and subtraction, used varied methods and to what extent these uses were successful on the Grade 3 national test.

Multi-digit addition and subtraction in Swedish schools

In Swedish Grades 1–3, the only written methods included in the curriculum are methods for multi-digit addition and subtraction. Based on the national tests for Grade 3 (Primgruppen, 2025), this should be interpreted to mean addition and subtraction of numbers with up to three digits. Since 2016, three-digit addition and subtraction have been the focus of one or two of the subtests of the national Grade 3 test, usually including six tasks of which 2–3 are addition tasks and 3–4 are subtraction tasks. For each task, students are given one point for documenting a correct method (*strategy correctness*) and one point for obtaining the right answer (*strategy accuracy*). This subtest has consistently been the subtest with the lowest achievement nationwide, with below 80 %

Mattsson, L., Eckert, A., & Kilhamn, C. (Eds.). (2026). *Talking Mathematics. Proceedings of MADIF 15*. Gothenburg University and SMDF.

of students reaching the required level compared to 86–95 % on the other subtests (Skolverket, 2025).

Besides being able to use methods correctly and accurately, researchers and curricula also suggest *strategy flexibility* is an important goal for students (Verschaffel, 2024). Though strategy flexibility is defined in various ways, the core of most definitions is the ability to choose appropriate and efficient methods. This ability implies knowledge of and skill in multiple methods, as well as knowledge of when these methods are efficient. By the end of Grade 3, the Swedish curriculum states that students should be able to choose and use written calculation methods for addition and subtraction with satisfactory confidence (Skolverket, 2024). Though not specified—neither in the curriculum nor the commentary material (Skolverket, 2022; 2024)—being able to choose can be argued to imply that students are supposed to know several methods and have some knowledge of when they are appropriate and efficient, indicating that flexibility is indeed a state mandated goal for Grade 3 students. However, as described above, the scoring of the sub-test of the national Grade 3 test in mathematics that assesses written methods for addition and subtraction does not take flexibility into account. This may encourage both teachers and students to keep to a single method that works the same for every task, such as the standard written algorithm.

Methods for multi-digit addition and subtraction

Previous research has used different terms and classifications of methods for multi-digit addition and subtraction. At the most general level, Hickendorff et al. (2019) and Heinze et al. (2022) differentiate between number-based methods and the digit-based written algorithm, to emphasise that the written algorithm can be used without considering the place value of each digit, treating them as single-digit numbers. A summary of different number-based methods classified by previous research (Fuson et al., 1997; Heinze et al., 2009; 2018; Hickendorff et al., 2019; Kullberg et al., 2024) is given in Table 1. Furthermore, students may use combinations of these methods or record them in different ways than is shown in the examples in Table 1.

The *Split* (also called Decompose) method involves the same steps as the standard algorithm, but records the numbers the different place values represent, rather than just recording digits. This method is rather straightforward for addition bridging over ten but is more complicated for subtraction bridging over ten as it: 1) results in negative results of sub-subtractions, and 2) involves both subtraction (for each place value) and addition (of the results of all sub-subtractions). Several studies have also shown that students have rather low success rate for this method on subtraction tasks; Heinze et al. (2009) report a success rate of 39.0 % in a sample of German third graders and Kullberg et al. (2024) report a success rate of 17.6 % (or 13.3 % if cases where students decomposed into digits are included) in a sample of Swedish second graders. In both studies, students performed substantially better when using the *Stepwise* (also called Jump or Sequential) method on subtraction tasks: 69.6 % and 100 %, respectively.

Method	Description	Addition (45 + 18)	Subtraction (45 – 18)
Split	Decompose both terms in ones, tens etc, do sub-calculations for each place-value and add the results	40 + 10 = 50 5 + 8 = 13 50 + 13 = 63	40 – 10 = 30 5 – 8 = -3 30 + (-3) = 27
Stepwise	Decompose one term (the subtrahend) into numbers that are easily added/subtracted stepwise	45 + 5 = 50 50 + 10 = 60 60 + 3 = 63	45 – 5 = 40 40 – 10 = 30 30 – 3 = 27
Compensation	Add/subtract more and then subtract/add the difference to compensate	45 + 20 = 65 65 – 2 = 63	45 – 20 = 25 25 + 2 = 27
Simplification	Regroup an addition (move part of one term to the other) or move a subtraction (add or subtract equally to both terms) to reach an easier task with the same answer	45 + 18 = 43 + 20 = 63	45 – 18 = 47 – 20 = 27
Indirect addition	Add smaller numbers to the subtrahend until you reach the minuend and then add up all the smaller numbers	–	18 + 2 = 20 20 + 20 = 40 40 + 5 = 45 2 + 20 + 5 = 27
Indirect subtraction	Subtract smaller numbers from the minuend until you reach the subtrahend and then add up all the smaller numbers	–	45 – 5 = 40 40 – 20 = 20 20 – 2 = 18 5 + 20 + 2 = 27

Table 1. Number-based methods for addition and subtraction

Compared to Split and Stepwise, Heinze et al. (2018; 2022) describes *Simplification* (also called Change both numbers) as a more advanced method that students are unlikely to invent by themselves. It is also reported to be used rather seldom by students, especially for subtraction (Heinze et al., 2009). If seen as a method that acts on symbolic expressions—which the name *Simplification* implies—it involves the same steps for both operations: adding a number to one term while subtracting the same number to the other, which will keep the value of the expression the same. This general idea is a property of equalities that is important for students' later development of algebraic thinking (Wilkie & Hopkins, 2024). However, simplifying a symbolic expression is an abstract idea that might be hard to teach directly to Grade 1–3 students. Instead, teaching might be more successful if it is based in the conceptual roots of simplification, which are different for addition and subtraction.

For addition, Simplification can be seen as a regrouping of an additive partitioning by moving one part of one of the terms to the other (Figure 1A). Children are usually able to conceptualise addition as additive partitioning (part-part-whole relationships) even before starting school (Nunes & Bryant, 1996), and the idea of regrouping can be linked to seeing numbers as collections of objects, which is often the first representation of numbers children encounter. This could explain the high success rate (88.1 %) for Simplification for addition reported by Heinze et al. (2009).

For subtraction, on the other hand, Simplification is grounded in the idea of subtraction as difference, which is usually not children's first conceptualisation of subtraction (Nunes & Bryant, 1996). Children usually first encounter subtraction in

"take away" situations with collection of objects, for example, having some cookies, eating some of them, and finding out how many that remain. In such situations, it is not obvious that having more cookies but also eating equally many more will result in equally many cookies left. If, however, one thinks about subtraction as the difference between subtrahend and minuend on the number line, Simplification can be conceptualised as moving the subtrahend and the minuend the same distance, which will keep their difference the same (Figure 1B). Considering these links to concrete models of numbers and operations, it is evident that Simplification might better be seen as two different methods: *Regroup* for addition and *Move the difference* for subtraction, where the second has more complex conceptual roots, explaining the comparatively lower success rate (41 %) for Simplification for subtraction reported by Heinze et al. (2009).

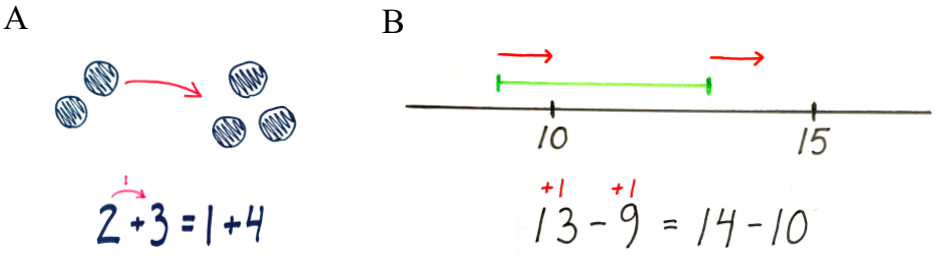


Figure 1. Conceptualisations of simplification for addition and subtraction

Aim and research question

Current teaching can make students overly reliant on digit-based algorithms for multi-digit addition and subtraction, resulting in low achievement. When students are allowed to invent their own methods, it can result in better accuracy, but they may not invent advanced methods that are important for continued learning, such as Simplification. In this study, our aim was to investigate whether teaching students Simplification methods with a focus on their conceptual basis, would result in better achievement for multi-digit addition and subtraction. Our research questions are:

1. What methods did students who were taught the simplification methods Regroup and Move the difference use on the Grade 3 national sub-test?
2. To what extent were students' methods successful, and what errors did students make?

Method

This study is part of a larger design research project that aims to better understand and develop the teaching of whole-number arithmetic in Grades 1–3. In the larger project, teachers and researchers collaborate to design, test and revise a teaching material for Grades 1–3 that aims to promote students' reasoning about number, and to study how teaching affects students' work with and knowledge of number and the four basic operations.

This study focuses on the solutions produced on one of the subtests of the national Grade 3 mathematics test in two classes whose teachers used a first version of the teaching material. The teachers worked at the same school, a school where the socioeconomic background of students is close to an average school in Sweden

regarding foreign background and parent's educational level. They started using the material in Spring in Grade 1. Informed written consent for participation was collected from students' guardians, resulting in a total of 30 participating students.

Teaching design

The design of the teaching material emphasizes the development of mathematical understanding, reasoning, and problem-solving abilities. The lesson plans for each week follows a structure developed by Sterner (2015) including the following parts: introduction, pair work, whole class discussion, individual work, and follow-up discussion. In the material, teaching is viewed not merely as the transmission of procedures, but as a process of supporting students in actively constructing knowledge through exploration, discussion, and reflection. The underlying idea of the material is that learning occurs when students engage meaningfully with mathematical ideas, test their hypotheses, and learn from both success and failure. The aim of the material is therefore to support students in not only applying mathematical methods, but also in understanding the underlying principles and structures.

The teaching material used by the teachers in this study was divided into 153 "cycles" each consisting of three lessons, of which five focused on multi-digit addition or subtraction. The aim was not for the students to develop methods themselves. Two methods were introduced to the students: Regroup for addition and Move the difference for subtraction. Regrouping was introduced during Spring in Grade 1 as a way to generate equalities and was later used as a method for two-digit addition in the middle of Grade 2 and for three-digit addition during Spring in Grade 3. Move the difference was introduced for two-digit subtraction early in Grade 2 and extended to three-digit subtraction in the middle of Grade 3. Both methods were introduced conceptually by concrete use of blocks (Figure 2). These representations were then the basis for the documenting the methods with symbols. Below, we describe the teaching of each method in more detail.

Regroup

In Grade 1, the cycle introducing Regroup addressed the method by focusing regrouping of additive partitions of numbers 10–20. Students learnt ways to represent regrouping of terms in a sum and to use regrouping to explain similarities. In Grade 2, the cycle focused on regrouping to simplify additions. The students explored different ways to regroup additions and how regrouping can be used to calculate sums, including using the number bonds to 10 in regrouping. In Grade 3, the cycle focused on regrouping for addition with three-digit numbers. In this cycle students returned to the same method as in Grade 2, but now for additions when one number had three-digits and the other one-, two- or three-digits.

Move the difference

In Grade 2, the cycle introducing Move the difference addressed the method as a way to simplify subtraction. The students explored different ways to reformulate subtractions by adding or removing equal amounts from both the minuend and the subtrahend and got to use ways to document this. Students also discovered the use of number bonds when adding or removing equal numbers. In Grade 3, the cycle focused on Move the

difference for subtraction with three-digit numbers. This cycle focused on Move the difference, but students could also use, explain and argue for other methods of their choice.

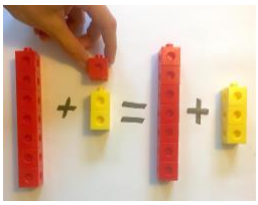
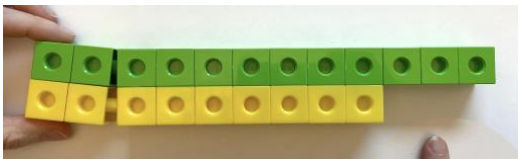
A  B 

Figure 2. Illustrating Regroup and Move the difference using blocks

Data collection

Data consisted of students' solution to subtest F from the national test in Grade 3. This test focused written calculation methods for addition and subtraction. For the test, the written instruction to students was: *use a written calculation method and write the answer.* The year the data was collected, subtest F consisted of six tasks: two addition tasks and four subtraction tasks. One addition task and one subtraction task were word problems while the other four tasks were stated symbolically. Paper copies of the participating student solutions were collected resulting in 30 tests including solutions to a total of 180 tasks. As the national test is still classified and the particular tasks thus not allowed to be reproduced here, we use examples illustrating students' solutions and errors using other tasks with similar properties as the ones on the test, below.

Coding of students' methods

Each student solution to each task was coded for whether or not a correct answer was reached (simply yes/no), which method the student used, and if any, what type of errors were made. The analysis of methods was carried out using content analysis combining both deductive and inductive elements. We started with predetermined codes based on previous research on methods for addition and subtraction (Table 1), with the exception that we differentiated between the two simplify methods *Regroup* and *Move the difference.* These codes formed the basis for an initial coding of the students' solutions and answers. During the analysis, we identified student responses that could not be fully classified within the existing codes. To preserve the nuances of the data, new codes were created based on the students' actual solutions. Three new codes were introduced: *Take from next place value*, *Counting by ones*, and *No information*. In Take from next place value, students divide the minuend into 10 or 100 and the rest, and the subtrahend or part of the subtrahend is then subtracted from the 10 or 100 part. Writing out all numbers between one term and the result or making as many tick marks was coded as Counting by ones and only writing the expression and/or the answer was coded as No information.

The analysis also resulted in dividing the code *Split* into *Split by place value* and *Split other* and the code *Stepwise* into *Stepwise by place value* and *Stepwise other*. In *Split by place value* both terms are decomposed into ones, tens, and hundreds, while in *Split other* both terms are decomposed into parts that *not* exactly made up by place-values. In *Stepwise by place value* one term is decomposed into ones, tens and hundreds, while in *Stepwise other* one term is decomposed into parts that are not exactly by place-

values. As we found no examples of the methods *Compensation*, *Indirect addition* and *Indirect subtraction*, these codes were excluded. Once both the deductively and inductively generated codes had been identified, the codes were reformulated, merged, or clarified to better reflect the variation in students' methods. The coding was then revised and difficult cases discussed between the authors to reach a final coding of each method used.

The analysis of errors was based on a first round of analysis of the solutions with a focus on finding and describing all the different error types that occurred (Table 2), conducted by one author. The types were then used to categorize all errors that occurred in each solution to each task, conducted by both authors. In this way, the errors in 57 % of the tasks were analysed independently by both authors. This resulted in no new error types, and thus no revision of the error types. The error types were grouped into four themes *Insufficient number-bond knowledge*, *Insecure method*, *Insufficient systematics* and *Wrong interpretation* (Table 2).

Theme	Types
Insufficient number-bond knowledge	The student gets the wrong answer in: ● an addition where the sum is ≤18, not 10 ● an addition or subtraction with the number bonds to 10 ● an addition or subtraction with the number bonds to 100 ● a subtraction where the minuend is ≤18, not 10 ● a subtraction where the minuend is >18, not a whole ten
Insecure method	The student: ● makes a mistake while counting up or down by ones ● mixes up units and tens in one or more steps ● switched places of the subtrahend and minuend when the difference is less than 0 ● adds something that should be subtracted or vice versa. ● makes a mistake when in bridging over ten in two sub-calculations ● mixes up place-values
Insufficient systematics	The student: ● misses some part ● forgets that a ten has been bridged and counts it twice
Wrong interpretation	The student draws a wrong conclusion about the operation to use (primarily in a word problem)

Table 2. Error types in each theme

Results

In this section, we present the answer to each of our research questions.

Research question 1: Frequencies of methods used

The students used a total of six different methods on the addition tasks, and eight different methods on the subtraction tasks. The frequencies of solutions using each method is found in Table 3. One can note that the two simplification methods were rather seldomly used, and that some version of Split was still the most common method for both addition and subtraction. Some uses of Split were documented as the standard

written algorithm, suggesting that students had somehow been taught this method, by the teacher or outside of school.

	Addition tasks		Subtraction tasks	
Regroup	9	15 %	–	–
Move the difference	–	–	6	3 %
Split by place value	18	30 %	41	40 %
Split other	25	42 %	21	17 %
Stepwise by place value	1	2 %	26	18 %
Stepwise other	6	10 %	8	7 %
Take from next place value	0	0 %	6	5 %
Counting by ones	1	2 %	2	2 %
No information	0	0 %	10	6 %

Note. There were twice as many subtraction tasks, and thus subtraction solutions, as addition tasks.

Table 3. Frequencies and percentages of use of different methods for addition and subtraction

Research question 2: success rates and errors

Students' success rates varied over methods, especially for subtractions tasks. The success rates for each of the methods used are found in Table 4. One can note that for addition, several methods were highly successful. The two most successful methods were Regroup and Stepwise other. Regroup was typically used on a task where one addend was close to a whole hundred, in the following way: 203 + 298 = 203(-2) + 298(+2) = 201 + 300 = 501. Stepwise other was typically used to get sub-results of even tens and hundreds, for example: 135 + 68 = (135 + 5) + 60 + 3 = (140 + 60) + 3 = 200 + 3 = 203. Both these methods can thus reduce bridging over ten to number bonds to ten, requiring fewer and easier known number facts. Only seven errors were found in students' solutions to addition tasks, of which four were insufficient number-bond knowledge.

For subtraction, variants of Split were the least successful methods, while Move the difference and Stepwise other were the most successful. Of the 57 errors identified in students' solutions to subtraction tasks, 36 were due to insecure methods, with the most common error type (17 cases) being subtracting the minuend from the subtrahend in a sub-subtraction where the minuend was smaller than the subtrahend when using a variant of Split. An additional nine errors were due to unsecure uses of Split variants. Ten errors were due to insufficient number-bond knowledge, seven due to insufficient systematics and four due to wrong interpretation.

	Addition tasks	Subtraction tasks
Regroup	100 %	–
Move the difference	–	100 %
Split by place value	94 %	58 %
Split other	84 %	48 %
Stepwise by place value	0 %	73 %
Stepwise other	100 %	75 %
Take from next place value	–	67 %
Counting by ones	0 %	0 %
No information	–	50 %

Table 4. Success rate of different methods for addition and subtraction

Discussion

In this study, we investigated what methods a group of Swedish Grade 3 students, who had been taught Regroup and Move the difference, used and how successful their methods were. Our results show that students used a variety of methods, and that the most common were variants of Split, while Simplification methods were rather uncommon. These findings are in line previous research in other contexts, such as Swedish Grade 2 (Kullberg et al., 2024) and German Grade 3 (Heinze et al., 2009). However, in our study, students were substantially more successful when using simplification methods than in previous research, especially for subtraction. The high success rate in our study might be a result of students learning these methods separately, based on their differing conceptual bases. Heinze et al. (2009) describe Simplification as one method but does not describe how it was taught to the students in their study.

Simplification of symbolic expressions is an important part of algebraic reasoning (Wilkie & Hopkins, 2024). While some students in our study successfully used of Regroup and Move the difference, there was no wide use of these methods. While teaching these methods conceptually and separately seems like a promising approach, other studies are needed to understand how teaching can support more students in using Simplification methods successfully. Another limitation is that the study design does not allow us to say whether method use affected success rate. One can suspect that use of Simplification methods and success might both be affected by general proficiency, and thus that the higher success rate of Regroup and Move the difference may not be wholly due to the methods in themselves. Further research should investigate the success rate of different methods when used by the same student, and the success rates of different methods when used by different groups of students.

Together with other studies (Heinze et al., 2009; Kullberg et al., 2024), our results support the claim that Split is a comparatively difficult method for subtraction tasks, and not the most successful method for addition. Still, it is the most common method used, and it is the basis for the standard written algorithm, which is commonly the focus of Grade 1–3 teaching. Research, including this study, has consistently shown that Stepwise is at least as successful for addition and substantially more successful for

subtraction (Heinze et al., 2009; Kullberg et al., 2024). Based on this, one might suggest Stepwise to be a better choice of main focus in Grade 1–3 teaching than the algorithm.

References

Fuson, K. C., Wearne, D., Hiebert, J. C., Murray, H. G., Human, P. G., Olivier, A. I., Carpenter, T. P., & Fennema, E. (1997). Children's Conceptual Structures for Multidigit Numbers and Methods of Multidigit Addition and Subtraction. *Journal for Research in Mathematics Education, 28*(2), 130–162.

Heinze, A., Marschick, F., & Lipowsky, F. (2009). Addition and subtraction of three-digit numbers: Adaptive strategy use and the influence of instruction in German third grade. *ZDM – Mathematics Education, 41*(5), 591–604. https://doi.org/10.1007/s11858-009-0205-5

Heinze, A., Arend, J., Gruessing, M., & Lipowsky, F. (2018). Instructional approaches to foster third graders' adaptive use of strategies: An experimental study on the effects of two learning environments on multi-digit addition and subtraction. *Instructional Science, 46*(6), 869–891. https://doi.org/10.1007/s11251-018-9457-1

Heinze, A., Grüßing, M., Schwabe, J., & Lipowsky, F. (2022). The algorithms take it all? Strategy use by german third graders before and after the introduction of written algorithms. In C. Fernández, S. Llinares, A. Guitérrez, & N. Planas (Eds), *Proceedings of the 45th conference of the international group for the psychology of mathematics education* (Vol. 2, pp. 363–370). PME.

Hickendorff, M., Torbeyns, J., & Verschaffel, L. (2019). Multi-digit addition, subtraction, multiplication, and division strategies. In A. Fritz, V.G. Haase, P. Räsänen (Eds.), *International handbook of mathematical learning difficulties: From the laboratory to the classroom* (pp. 543–560). Springer. https://doi.org/10.1007/978-3-319-97148-3_32

Kullberg, A., Björklund, C., & Runesson Kempe, U. (2024). Seeing number relations when solving a three-digit subtraction task. *Educational Studies in Mathematics, 115*(2), 271–287. https://doi.org/10.1007/s10649-023-10287-0

Nunes, T., & Bryant, P. (1996). *Children doing mathematics.* Blackwell.

Primgruppen. (2025, July 1). *Årskurs 3*. https://tinyurl.se/cMe

Skolverket. (2022). *Kommentarmaterial till kursplanen i matematik. Grundskolan.* https://tinyurl.se/cMd

Skolverket. (2024). *Läroplan för grundskolan, förskoleklassen och fritidshemmet. Lgr22 (2nd ed)*. tinyurl.se/aNL

Skolverket. (2025). *Sök statistik*. https://tinyurl.se/cdv

Sterner, G. (2015). *Tal, resonemang och representationer - en interventionsstudie i matematik* https://doi.org/10.13140/RG.2.1.2052.4969

Verschaffel, L. (2024). Strategy flexibility in mathematics. *ZDM – Mathematics Education, 56*(1), 115–126. https://doi.org/10.1007/s11858-023-01491-6

Wilkie, K. J., & Hopkins, S. (2024). Primary students' relational thinking and computation strategies with concrete-to-symbolic representations of subtraction as difference. *The Journal of Mathematical Behavior, 73*, 101121. https://doi.org/10.1016/j.jmathb.2023.101121

YouTube as a Platform for Mathematics Education

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This study investigates mathematics teaching on Youtube. Drawing on Bernstein's code theory and Maton's Legitimation Code Theory, the analysis focuses on classification, framing, epistemic and social relations, and semantic gravity and density. The educational practices in fifty-one videos from ten profiles were analysed. The results show a strong emphasis on the subject (C+) and teacher-led instruction (F+), with limited student agency (SR–). Specialized mathematical language generally dominated (SD+), while contextualization was not common (SG–). The findings suggest that Youtube-based instruction reinforces monologic teaching rather than promoting dialogic engagement. The study contributes to understanding how mathematics teaching on Youtube mediates subject content and in what form.

Introduction

Online tutorials and lessons are common teaching practices today (Isaeva et al. 2025; Vidergor & Ben-Amram, 2020) for which various digital arenas such as Instagram, TikTok and YouTube are used. Some videos aim to support the teachers' own students they regularly meet in class and are used in a flipped classroom practice, as rehearsal or in another way. Other videos are public and can be used by other students, searching for a tutorial online (Adler et al., 2025; Xia et al., 2022). The videos are offered in a range of subjects and a student survey showed that YouTube videos in mathematics dominate amongst students (age 12-19) when asked for their use of online materials (Randahl et al., 2024). In a large-scale survey involving nearly 5,000 Italian students from both lower and upper secondary schools, participants reported that online videos are used more frequently than traditional textbooks to enhance their mathematics performance (Gil-Quintana et al., 2020). These online videos have become a valuable resource for schoolwork (Adler et al., 2025; Kohler & Dietrich, 2021; Xia et al., 2022), with students reporting that they use them to rehearse content, prepare for assessments, and deepen their understanding of various subjects.

Several studies in mathematics education have concluded that a predominant form of instruction and working form in mathematics lessons is individual seat work varied with instructions in whole class by the teacher (e.g., Tengberg et al., 2021). During individual seat work the students work on excersises from their textbook. In the whole class instructions specific constructs are explained or solutions of specific exercises are demonstrated. During such instruction time, the collective learning of the subject is in focus, whereas the individual seatwork aims to provide students an individual learning

Mattsson, L., Eckert, A., & Kilhamn, C. (Eds.). (2026). *Talking Mathematics. Proceedings of MADIF 15*. Gothenburg University and SMDF.

opportunity. With online instructions now are available, a question that arises is if the collective learning is replaced by individual learning moments. If so, these videos change the educational scene and challenge the didactical contract (Brousseau, 1997). When part of the education moves from only including the physical classroom activity to also including the Youtube-classroom, the teacher and subject is not bounded to the physical classroom anymore, and students are part of different didactical contracts.

The aim of this study is to illustrate what schoolsubject is constituted in the online videos. Videos from ten Youtube profiles are analyzed with regard to the content, form and type of instruction to answer our research question: what educational practices are observable in online videos produced by teachers?

The phenomenon

In Sweden there are about 44 000 mathematics teachers at compulsory school (age 6–16-year-olds (ref skolverket statistik, data 2023/2024), teaching around 1,1 million students. For upper secondary school (age 16-19) there are 4 300 mathematics teachers and 370 000 students enrolled during a schoolyear. To give a perspective of how widespread the phenomenon of Youtube videos is, some of these teachers have 50 000 followers on Youtube and some videos are viewed up to 320 000 times. The teachers provide videos from 30 seconds up to several hours and some profiles have uploaded over 60 000 videos. Given the fact that the videos are in Swedish and given the specific target group of the videos, we can assume that most of the followers and views are within Sweden and by Swedish students. For the subject mathematics, Youtube is the predominant form and thus the focus of this paper.

Depending on the target group and the aim, the content, structure and form of the videos varies. For mathematics, single tasks are in focus where the teacher simply demonstrates how to solve the task. In some cases, this can result in thousands and thousands of videos where every single task from a textbook series is calculated. Other videos show an introduction or summary of a specific concept or construct closely connected to the content of a mathematics course. Specific tests result in specific videos. For instance, after the Swedish Scholastic Aptitude Test (Swedish: Högskoleprovet) is conducted and released an hour later, several livestreams and videos appear where all exercises are calculated and demonstrated.

Previous research

Teachers increasingly use YouTube as an instructional medium for a variety of pedagogical purposes. One prominent example is the flipped classroom model, which gained popularity as video-sharing platforms like YouTube became widely accessible to both educators and students (Fulton, 2012; Tucker, 2012; Herreid & Schiller, 2013). In this approach instructional videos (typically lectures or demonstrations) are assigned as preparatory homework, allowing classroom time to be devoted to discussion, collaboration, and problem-solving activities (Bishop & Verleger, 2013). The availability of such platforms has significantly reshaped teaching practices by enabling

instruction to occur outside the traditional classroom setting and flipped classroom evolved into other types of videos. The use of educational videos increased dramatically during the pandemic, with studies indicating that social media platforms like YouTube played a crucial role in supporting remote teaching and learning (Pozo et al., 2024).

Research on science and knowledge communication via social media indicates that video creators must navigate a delicate balance between delivering information and capturing viewer engagement (Xia et al., 2022; Prindle et al., 2024). This fusion of educational content with entertainment, commonly referred to as edutainment (Aksakal, 2015), is believed to enhance learners' motivation and make complex topics more accessible. Among the various formats of educational videos, tutorials stand out as one of the most prevalent and widely used.

Theoretical framework

Bernstein's concepts of classification and framing have been used to analyse educational practice (Barrett, 2024). The nature of knowledge transmission, the methods of instruction, and the expectations placed on students to demonstrate learning can vary significantly across educational contexts. Bernstein (1975) argues that such variations are shaped by underlying principles of power and control.

Power is conceptualized through the notion of classification, which defines the boundaries between different subjects or disciplines. Strong classification (C+) implies sharp distinctions between subjects, while weak classification (C–) allows for more integrated or interdisciplinary approaches. Control, on the other hand, is expressed through framing, which governs who controls the pacing, sequencing, and evaluation of the content being taught. Strong framing (F+) suggests that the teacher tightly regulates the learning process, whereas weak framing (F–) allows for greater student autonomy and negotiation. Both classification and framing can be either strong (C+, F+) or weak (C–, F–), and they may vary independently (e.g., C+, F– or C–, F+).

Bernstein's code theory was further developed by Maton (2014) into Legitimation Code Theory (LCT). Where Bernstein's classification and framing concern the regulation of content and the transmission of this content, LCT focus on what content is valued (legitimated) within a practice and how the learner is positioned visavi this valued content. This is captured by two constructs: Epistemic relations, focussing on what constitutes legitimate knowledge. and Social relations, capturing who is recognized as a legitimate knower. Practices may exhibit stronger relations (ER+, SR+) or weaker relations (ER–, SR–). Practices with strong Epistemic relations, place high value on the specialized knowledge defined within the subject, while giving little importance to the individual's personal knowledge. Practices with strong Social relations place high value on the knower and personal attributes such as experience, opinion, or identity.

Further, the constructs Semantic gravity and Semantic density offer an opportunity to focus on how meaning is constructed and mediated within discourse. Semantic gravity refers to the degree to which meaning is tied to the context. Strong Semantic gravity

(SG+) indicates more concrete, situated meaning, while weak Semantic gravity (SG–) reflects more abstract, generalized meaning. Semantic density captures the complexity and condensation of meaning; stronger density (SD+) is associated with the use of specialized or technical language, whereas weaker density (SD–) corresponds to everyday or less condensed language.

In summary: Classification and Framing highlight how educational knowledge is shaped and transmitted, Epistemic and Social relations focus on who and what is legitimated and thus considered valid. Finally, Semantic gravity and density capture complexity and context dependence in language use.

Data

Ten profiles within the subject mathematics were selected based on the results of a survey (Randahl et al., 2024) and mentioned by the students as learning resources. Eight of these were (Swedish) teachers providing educational videos intended for a general student audience. One profile belonged to a university civil engineering student, a so-called non-teacher profile, providing videos to a broad viewership. Finally, one profile was a non-Swedish edutainer, providing infotainment-style videos targeting a wider public audience.

To capture videos likely perceived as relevant by students, the three most viewed clips from each profile were selected, as these may reflect topics of high urgency or interest. In addition, three further clips were chosen based on predetermined dates distributed across a semester, ensuring the inclusion of recent material. Since not all profiles had published new content on the selected dates, the total number of videos per profile ranged from four to six (Table 1), in total 51 videos and almost 19 hours of video data.

Profile	School level	# videos	total time	shortest	longest
Ma1	Lower Secondary	6	36:21	01:02	15:28
Ma2	Upper Secondary	6	24:17	00:47	05:54
Ma3	Upper Secondary	4	57:00	09:20	28:04
Ma4	Upper Secondary	4	1:01:06	11:01	21:39
Ma5	Upper Secondary	5	3:13:14	18:28	1:14:37
Ma6	Upper Secondary	6	4:12:05	11:16	1:31:14
Ma7	All	6	1:34:22	01:15	24:46
Ma8	Lower Secondary	4	43:18	08:13	12:32
Ma9	Upper Secondary	6	5:05:49	00:29	2:41:01
Ma10	Upper Secondary	4	48:52	04:50	19:00
Total	-	51	18:36:24	-	-

Table 1. Overview dataset

The videos were retrieved using the most recent version of the downloading utility yt-dlp available at the time of each data collection session. Clips were downloaded in batches and organized via a custom Bash script, with output logs from yt-dlp preserved

for documentation. The tool captured each video in the highest available quality, utilizing container formats such as WebM and Matroska. The specific video encoding varied depending on the technical specifics during the original publication of each clip.

Data analysis

To familiarize ourselves with the dataset, we began by jointly viewing a selection of videos from various school subjects. The form of instruction in the videos was denoted: lecture of new content (Swedish: genomgång), rehearsal, solving tasks, edutainment & curiosa, study tips.

Based on the initial observations, a coding manual was developed using the six analytical concepts outlined earlier: Classification, Framing, Epistemic relations, Social relations, Semantic gravity, Semantic density. We decided to code the most prominent feature in each video, a decision that proved especially useful when analysing the dimension of *Semantics*, where shifts between stronger and weaker semantic gravity and semantic density were frequently observed.The subsequent phase involved an iterative process of individual coding and analysis, followed by collaborative discussions to resolve ambiguities and refinement of the manual.

In the present study, classification refers to the extent to which the content is confined within the boundaries of the school subject mathematics, while framing denotes the degree to which the teacher explicitly structures and guides the learning process. In relation to framing, we reflected on the extent to which the teacher must explicitly instruct or address the viewer for the teaching practice to be coded as strongly framed. We reached a consensus that strong framing is characterized by a high degree of teacher control over the transmission of knowledge. For instance, videos in which teachers demonstrate how to solve mathematical problems were coded as examples of strongly framed pedagogic practices.

A key point of discussion centred on whether students should be considered to have agency within the video-based medium itself, particularly given their ability to choose whether or not to engage with the content. We agreed that videos in which teachers explicitly invite students to make choices or reflect on their own learning should be interpreted as exhibiting stronger social relations (SR+).

In our dataset, semantic density is considered stronger when subject-specific terminology is employed, and weaker when explanations rely on informal, everyday language.

Results

Form of instruction

Most profiles offered different forms of instruction, with lecture being the most prominent one (profile 3, 4, 5, 6, 8, 9, 10). For profile 4 this was the only form of instruction used in the collected videos. Profile 6 differed from the others regarding the lectures, as live lectures were used. The profile used live recordings of actual lessons, meaning that the students of this teacher were present in class (though not visible in the

recording). Three profiles focused on national tests in the collected videos (profile 1, 6, 9). Similarly, rehearsal was observable in videos of three profiles (profile 3, 5, 9). Profile 7 only offered edutainment videos. (see Table 2)

Profile	
Ma1	Study tips National test
Ma2	Solving tasks
Ma3	Lecture Rehearsal
Ma4	Lecture
Ma5	Lecture Rehearsal
Ma6	Lecture (live) National test
Ma7	Edutainment
Ma8	Lecture
Ma9	Lecture National test Rehearsal
Ma10	Lecture Curiosa

Table 2. Form of instruction in the videos of each profile

Educational practice

When coding the videos according to the features describing the educational practice observed in the videos, specific patterns occur (Table 3).

There is a clear focus on the subject. A strong classification (C+) dominates in 43 of the 51 videos, meaning that the videos comprise subject specific content and thus show a strong connection to the subject mathematics. The eight videos with a weak classification were detected in the two non-teacher profiles (profile 1 and 7). As for framing, most videos were coded as F+, meaning that the teacher explicitly structures and guides the learning process and thus controls the transmission of knowledge. Also here, the non-teacher profiles (profile 1 and 7) stand out, with a weak framing of their instruction.

For the epistemic relations (ER), most videos show a strong epistemic relation (ER+), which means that the videos showed strong relations to subject specific knowledge, specific principles or procedures within the subject. Weak epistemic relations (ER–), are observed in the videos by Profile 1. For instance, in a video on how to perform well on oral tests, the profile states “Be active, use concepts, give overly clear answers”. This instruction is more of a general advice and not directly connected to the subject mathematics.

As for social relations, all but one video were coded SR–, meaning that students' experiences or views regarding the content are not requested in the video (which gives the student limited influence on the learning process).

Regarding Semantics, the density is high in most videos, profile 1 being an exception, pointing at frequent use of subject specific, specialized language. The semantic gravity differs between videos, with 18 videos contextualizing the content to a high degree and 33 videos only offering generalized content, not tying the content to a context. Notable is that contextualisation is done through visualisations and illustrations, in many cases maintaining a high semantic density (SD+).

Profile	C+	C–	F+	F–	ER+	ER–	SR+	SR–	SD+	SD–	SG+	SG–
Ma1	3	3	2	4	2	4	1	5	2	4	4	2
Ma2	6	0	6	0	6	0	0	6	6	0	2	4
Ma3	4	0	4	0	4	0	0	4	4	0	3	1
Ma4	4	0	4	0	4	0	0	4	4	0	1	3
Ma5	5	0	5	0	5	0	0	5	5	0	0	5
Ma6	6	0	6	0	6	0	0	6	6	0	0	6
Ma7	1	5	0	6	6	0	0	6	6	0	6	0
Ma8	4	0	4	0	4	0	0	4	4	0	1	3
Ma9	6	0	6	0	5	1	0	6	5	1	1	5
Ma10	4	0	4	0	4	0	0	4	4	0	0	4
Total	43	8	41	10	46	5	1	50	46	5	18	33

Table 3. Occurance of the six analytical concepts (Classification, Framing, Epistemic relations, Social relations, Semantic gravity, Semantic density) in the videos of each profile

A closer look at the similarities and differences between the profiles gives us four distinct groups: (I) Profiles 5, 6 and 10 all offer videos with strong Classification, Framing, Epistemic relations, Semantic density and low Social relations and Semantic gravity; (II) Profiles 2, 3, 4 and 8 are very similar to (I) but the Semantic gravity varies over the videos showing both weak and strong Semantic gravity; III) Profile 9, in this profiles videos the Epistemic as well as the Semantic relations varies; (IV) Profile 1 and 7 offer videos that differ in all aspects from the other profiles as pointed out above.

Discussion

This study set out to investigate the schoolsubject constituted in YouTube videos produced by mathematics teachers. The findings reveal that the educational practices embedded in these videos are shaped by a strong classification (C+) and a strong framing (F+), with a clear emphasis on subject-specific content and teacher-led instruction. These results resonate with previous research, indicating that online tutorials and lectures have become a widespread pedagogical tool (Gil-Quintana et al., 2020; Randahl et al., 2024), particularly in mathematics education.

The dominance of strong classification (C+) across the majority of videos suggests that the subject boundaries are clearly maintained, reinforcing the idea that these videos are tightly aligned with formal mathematics curricula. This aligns with earlier studies (e.g., Kohler & Dietrich, 2021; Xia et al., 2022) that highlight how students use online videos to rehearse content, prepare for assessments, and deepen their understanding, activities that presuppose a structured and bounded subject domain. The strong classification (C+) suggest a clear focus on the subject, that is, the videos are easily identified as a practice in mathematics education. Using LCT (Maton, 2014) we can show that this solid subject focus also exhibits strong epistemic relations (ER+), emphasizing specialized knowledge and procedural competence.

Similarly, the prevalence of strong framing (F+) indicates that teachers retain control over the transmission of knowledge, often through direct instruction and demonstration. This reflects the traditional model of mathematics teaching described by Tengberg et al. (2021), where whole-class instruction and individual seatwork dominate. However, when such instruction is relocated to the digital space, it raises questions about the nature of collective learning. When the flipped classroom was introduced, the videos were only available for the specific teacher's students (see Bishop & Verleger, 2013; Fulton, 2012; Herreid & Schiller, 2013; Tucker 2012), but this has changed as videos now are accessible for all. The shift from classroom-based interaction to individualized video consumption may reinforce monologic pedagogical practices rather than dialogic pedagogical practices, as suggested by the absence of strong social relations (SR+) in nearly all videos.

The frequent occurance of weak social relations (SR–) implies that students' personal experiences, interpretations, or agency are rarely acknowledged or valued. This absence of dialogic engagement contrasts with the edutainment model described by Aksakal (2015), where entertainment and personalization are used to enhance motivation and accessibility. Only a few videos, primarily from non-teacher profiles, demonstrated features of SR+, where students' personal experiences and interpretation of the content is addressed. This lack of student agency suggests that teacher-produced content remains largely focused on transmission rather than interaction.

The semantic analysis further supports this interpretation. Most videos displayed strong semantic density (SD+), using specialized mathematical language, while semantic gravity varied. Videos with strong semantic gravity (SG+) contextualized content through illustrations or real-world examples, yet this was not consistently paired with weaker semantic density. Instead, many videos maintained a specialized and dense language even when contextualized, indicating a pedagogical strategy aimed at reinforcing formal knowledge structures.

Our findings have implications for teachers using YouTube videos as part of their practice. While the strong subject focus (C+) can support students' rehearsal and preparation, the limited student agency (SR–) suggests that teachers should consider complementing the use of videos with classroom activities acknowledging students as knowers (SR+). Further, connections between abstract mathematics and everyday

applications may need to be elaborated on (SG+). In this way, YouTube can serve not only as a tool for transmitting knowledge but also as a catalyst for more interactive and student-centred learning.

Taken together, these findings suggest that the Youtube-based mathematics instruction analysed in this study mirrors traditional classroom practices in its emphasis on the subject and on teacher authority. While the medium allows for broader access and flexible engagement, it does not inherently promote dialogic or student-centred learning. The educational scene is indeed shifting, as Brousseau (1997) predicted, but the didactical contract remains largely intact, reproduced in digital form rather than reimagined.

References

Adler, M. V., Madsen, J., Hedberg, J., Steinberg, R., & Parra, L. C. (2025). Effect of explanation videos on learning: The role of attention and academic performance. *Education and Information Technologies*, *30*, 11797–11825. https://doi.org/10.1007/s10639-024-13292-9

Aksakal, N. (2015). Theoretical view to the approach of the edutainment. *Procedia - Social and Behavioral Sciences, 186*, 1232–1239. https://doi.org/10.1016/j.sbspro.2015.04.081

Barrett, B. (2024). *Basil Bernstein: Code theory and beyond*. Springer. https://doi.org/10.1007/978-3-031-50745-8

Bernstein, B. (1975). On the classification and framing of educational knowledge. In B. Bernstein, *Class, codes and control Vol. 3: Towards a theory of educational transmissions* (pp. 85–115). Routledge.

Bishop, J., & Verleger, M. A. (2013, June). *The flipped classroom: A survey of the research*. Paper presented at 2013 ASEE Annual Conference & Exposition, Atlanta, Georgia. https://doi.org/10.18260/1-2--22585

Brousseau, G. (1997). *Theory of didactical situations in mathematics: Didactique des mathématiques, 1970–1990*. Kluwer Academic Publishers.

Fulton, K. (2012). Upside down and inside out: Flip your classroom to improve student learning. *Learning & Leading with Technology, 39*(8), 12–17.

Gil-Quintana, J., Malvasi, V., Castillo-Abdul, B., & Romero-Rodríguez, L. M. (2020). Learning leaders: Teachers or Youtubers? Participatory culture and STEM competencies in Italian secondary school students. *Sustainability, 12*(18), 7466. https://doi.org/10.3390/su12187466

Herreid, C. F., & Schiller, N. A. (2013). Case studies and the flipped classroom. *Journal of College Science Teaching, 42*(5), 62–66.

Isaeva, R., Karasartova, N., Dznunusnalieva, K., Mirzoeva, K., & Mokliuk, M. (2025). Enhancing learning effectiveness through adaptive learning platforms and emerging computer technologies in education. *Jurnal Ilmiah Ilmu Terapan Universitas Jambi, 9*(1), 144–160. https://doi.org/10.22437/jiituj.v9i1.37967

Kohler, S., & Dietrich, T. C. (2021). Potentials and limitations of educational videos on YouTube for science communication. *Frontiers in Communication*, *6*. https://doi.org/10.3389/fcomm.2021.581302

Maton, K. (2014). *Knowledge and knowers: Towards a realist sociology of education*. Routledge.

Pozo, J.-I., Cabellos, B., & Pérez Echeverría, M. d. P. (2024). Has the educational use of digital technologies changed after the pandemic? A longitudinal study. *Plos One, 19*(12), e0311695. https://doi.org/10.1371/journal.pone.0311695

Prindle, A., Kuntz, A., & Breslyn, W. (2024). Short-form videos as an emerging social media tool for STEM edutainment. *Journal of Chemical Education, 101*(3), 1234–1242. https://doi.org/10.1021/acs.jchemed.3c00789

Randahl, A., Nilsberth, M., Olin-Scheller, C., Tarander, E., & Appelgren, J. (2024). *När läraren finns på Youtube: En enkätstudie om elevers användning av material som lärare publicerar på sociala medier.* [When the teacher is on YouTube: A survey study on students' use of material published by teachers on social media]. Karlstad University. https://doi.org/10.59217/mqxs7931

Skolverket. (2024). *Statistik över gymnasieskolans elever – läsåret 2023/24*. [Statistics on upper secondary school students – academic year 2023/24]. https://www.skolverket.se/publikationer?id=12565

Tengberg, M., van Bommel, J., Nilsberth, M., Walkert, M., & Nissen, A. (2021). The quality of instruction in Swedish lower secondary language arts and mathematics. Scandinavian *Journal of Educational Research, 66*(5), 760–777. https://doi.org/10.1080/00313831.2021.1910564

Tucker, B. (2012). The flipped classroom. *Education Next, 12*(1), 82–83. https://www.educationnext.org/the-flipped-classroom/

Vidergor, H. E., & Ben-Amram, P. (2020). Khan Academy effectiveness: The case of math secondary students' perceptions. *Computers & Education, 157*, 103985. https://doi.org/10.1016/j.compedu.2020.103985

Xia, H., Xin Ng, H., Zhu-Tian, C., & Hollan, J. (2022). *Millions and billions of views: Understanding popular science and knowledge communication on video-sharing platforms*. In *Proceedings of the Ninth ACM Conference on Learning @ Scale (L@S'22*) (pp. 163–174). Association for Computing Machinery. https://doi.org/10.1145/3491140.3528279

Exploring Student Teachers' Use of the History of Mathematics in Mathematics Education

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This paper examines how secondary school student teachers engage with the history of mathematics in their first university mathematics course. The analysis draws on lesson plans and questionnaire responses to explore how historical perspectives are integrated into instructional design and pedagogy. Lesson plans mainly use history through teacher presentations focused on well-known mathematicians or ancient cultures. History is typically employed as a tool to contextualize mathematics and motivate learners, though it is sometimes treated as a goal, fostering reflection and cultural literacy. Questionnaire data show generally positive attitudes, noting benefits for understanding, interest, and relevance. Overall, student teachers recognize the potential of history but need further support for meaningful and authentic integration.

Introduction

The integration of the history of mathematics into mathematics education has gained increasing attention in recent decades (Chorlay et al., 2022; Clark et al., 2016). Advocates argue that historical perspectives provide significant pedagogical value, enriching both the teaching and learning of mathematics. This approach is based on the idea that presenting mathematical developments within their historical context can deepen understanding, humanize the subject, and promote more meaningful engagement. Rather than presenting mathematics as a static and decontextualized set of rules and procedures, historical approaches reveal its dynamic and evolving nature, shaped by diverse cultures, real-world problems, and human creativity (Fauvel, 1991). Empirical studies show that historical materials foster conceptual understanding and positive attitudes. Jahnke et al. (2000) and Tzanakis et al. (2000) found that students exposed to history-infused curricula often demonstrate increased motivation and deeper engagement with mathematical ideas. Additionally, historical perspectives are thought to bridge educational levels and strengthen coherent, reflective teacher knowledge (Vinerean et al., 2023). Despite these benefits, challenges remain in teacher preparation, curricular integration, and access to suitable resources.

In practice, the history of mathematics can be integrated into teaching through historical problems, original sources, narratives of discovery, biographies of mathematicians, or by reconstructing historical methods and proofs. These approaches enrich learning by revealing the contingent, non-linear, and often contested development of mathematical concepts, challenging the view of mathematics as fixed and immutable. However, Fried (2001) and Jahnke et al. (2000) stress that these methods require careful handling: without attention to context, there is a risk of trivializing or distorting history, while a more sensitive use of sources can encourage deeper reflection on mathematics

Mattsson, L., Eckert, A., & Kilhamn, C. (Eds.). (2026). *Talking Mathematics. Proceedings of MADIF 15*. Gothenburg University and SMDF.

as a human, cultural, and evolving activity. A meaningful integration therefore requires balancing curricular demands with a historically informed approach that preserves the integrity of the past (Fried, 2001).

This paper aims to examine how secondary school student teachers engage with the history of mathematics in a first-year university mathematics course and how this engagement is reflected in their pedagogical thinking and instructional design. Specifically, it investigates how student teachers incorporate historical perspectives in lesson planning and how they perceive the relevance and impact of the history of mathematics in teacher education and in school.

History of mathematics in the Swedish curricula

The history of mathematics is addressed in both compulsory and upper secondary school curricula, though with different emphasis. In compulsory school, Lgr22 highlights the long, multicultural development of mathematics and its roots in practical needs and human curiosity (Skolverket, 2022). The aims state that teaching should help pupils understand historical contexts of mathematical ideas and reflect on mathematics in everyday life and history. In the core content, history appears in years 1–6 through number symbols and systems from different cultures but is absent in years 7–9. In upper secondary school, Gy25 repeats Lgr22's introductory wording (Skolverket, 2025) but expands the aims to include "professional, societal, and historical contexts." History of mathematics also appears in the core content as an orientation on aspects such as concepts, events, mathematicians, or historical problems. Overall, the curricula differ in emphasis: Lgr22 introduces historical perspectives broadly and concretely in the early years but omits them later, while Gy25 integrates historical content throughout, specifying examples of mathematical concepts, events, persons, or problems.

History as a tool and history as a goal

Jankvist (2009) provides a useful framework for understanding the roles that history can play in mathematics education, distinguishing between *history as a tool* and *history as a goal*. When used as a tool, historical material supports the learning of specific mathematical content by enhancing conceptual understanding and providing motivation. It involves leveraging historical context to deepen students' engagement with mathematical ideas. By examining the original problems that prompted key discoveries, students can gain insight into the reasoning behind the methods. For instance, introducing irrational numbers through the historical shock experienced by the Pythagoreans can make the concept both more meaningful and memorable. History can also help address misconceptions, as the challenges faced by mathematicians in the past sometimes mirror those encountered by students today. By retracing the development of mathematical ideas, students are encouraged to think critically and develop a more flexible understanding of concepts.

As a goal, on the other hand, history promotes broader educational objectives, including critical thinking, cultural literacy, and a reflective perspective on mathematics as a discipline. While mathematics is often presented as a set of fixed truths, examining its historical development reveals it as a human endeavour characterized by trial and error, creativity, and cultural diversity. Through the study of historical developments,

students can understand that concepts such as zero, negative numbers, or functions did not emerge instantaneously – they evolved through struggle, debate, and discovery. In this way, history humanizes mathematics and fosters a sense of connection to the individuals behind the ideas, positioning mathematics as part of our intellectual heritage rather than merely a set of rules to be memorized.

This dual perspective of history as a tool and history as a goal is especially valuable in teacher education, where the development of both mathematical competence and pedagogical awareness is essential.

Methodology

The participants were secondary school mathematics student teachers enrolled in their first mathematics course in a teacher education program at a Swedish university. A total of 63 students participated in the course, which focused partly on geometry. Throughout the course, the history of mathematics was integrated in various ways. Lectures included a chronological overview of key mathematical developments across cultures and periods, such as geometry in ancient Egypt, Babylonia, China, and Greece. The introductory lecture also addressed the rationale for using history in teaching, framed through the concepts of "history as a goal" and "history as a tool." In addition, students worked with original sources and historical problems in lectures and exercise classes, individually, in small groups, and in whole-class discussions. A core component allowed students to write a secondary-school geometry lesson plan incorporating historical content; those who completed this task received bonus points on the course exam.

At the end of the course, a voluntary questionnaire was distributed to collect students' reflections on their experiences with the history of mathematics. The questionnaire contained seven items, two of which addressed another course and were excluded to maintain focus on the course studied here. Two items asked whether specific components – lectures on the history of geometry and writing lesson plans – were perceived as interesting. These were answered using a five-point Likert scale (from completely agree to completely disagree) and included optional free-text comments. Two further items invited reflections on the role of history in mathematics education and on how historical perspectives might be used in future teaching. Another item asked student teachers to rate their agreement with general statements about the history of mathematics using the same scale. Of the 63 students enrolled, 44 submitted a geometry lesson plan and 20 completed the questionnaire. This paper analyses both the questionnaire responses and the 44 lesson plans.

Analysis of Lesson Plans and Questionnaire Responses

The historical components of the geometry lesson plans were analysed qualitatively using the didactical questions *Why?*, *What?*, and *How?* to examine the stated purposes, historical content, and the teaching approaches. This triadic structure aligns conceptually with Shulman's (1986) framework of teacher knowledge – curricular, content, and pedagogical – providing a systematic way to investigate the rationale, substance, and form of teaching. It was therefore well suited to identify the student teachers' reasons for including historical content (*why*), the aspects of the history of mathematics they addressed (*what*), and the ways in which these were incorporated into

the lesson (*how*). The analysis was further guided by Jankvist's (2009) distinction between history as a tool and history as a goal in mathematics education, which informed the interpretation of the function of historical content.

To enhance reliability, the lesson plans were read multiple times and coded iteratively, with the coding scheme refined as the analysis progressed. As the researcher was also the course instructor, reflexivity was emphasized to ensure transparency and reduce potential bias. Each lesson plan had also been peer reviewed by two fellow students, and these comments were considered in the analysis, providing complementary perspectives that supported the consistency and credibility of the interpretations.

Open-ended responses of four questionnaire items were analysed using reflexive thematic analysis (Braun & Clarke, 2006). Data were coded inductively to identify patterns in students' perceptions, experiences, and beliefs regarding the role of history in mathematics education. Themes were iteratively developed to capture descriptive and interpretive meanings, focusing on students' understanding of mathematics, pedagogical perspectives, and future teaching intentions. In addition, one questionnaire item containing statements about the history of mathematics, to which student teachers indicated their level of agreement, was analysed using descriptive statistics.

By combining the lesson plan analysis with the questionnaire analysis, the study aims to provide a comprehensive understanding of how secondary student teachers engage with the history of mathematics and how this engagement is reflected in their pedagogical thinking and instructional design.

Results

The presentation of results begins with the results of the analysis of the lesson plans and is subsequently followed by the results of the analysis of the questionnaire data.

Lesson plans

Why? Of the 44 lesson plans, 11 were designed for lower secondary school (Grades 7–9) and 33 for upper secondary school (Grades 10–12). In all but six lesson plans, references were made to the curriculum to justify the inclusion of history in the lessons (Skolverket, 2022; 2025). Seven plans explicitly referred to both the aims and the core content related to the history of mathematics. Nineteen referred only to the aims, emphasizing that pupils should develop knowledge of historical concepts and methods to understand and reflect on the role of mathematics in everyday life, as well as its influence on society throughout history. Moreover, twelve lesson plans referred only to the core content, focusing on some aspect of the history of mathematics, such as a historical mathematician or the development of a concept. Among the six lesson plans that did not reference the curriculum, three provided no motivation for including historical material. The last three lesson plans refer more generally to using the history of mathematics to spark students' interest, provide context that illustrates the practical applications of mathematics throughout history, and develop students' understanding of mathematics.

What? The topics addressed in the mathematics lesson plans varied considerably. Twenty-nine plans focused on the Pythagorean theorem; five examined the

measurement of circumference, area, or volume in various geometric figures; three dealt with similarity; three with the inscribed angle theorem; two with coordinate systems; and two with geometric proofs and constructions using ruler and compass. All the lesson plans incorporated some element of the history of mathematics. Most commonly, they included biographical or contextual information about significant historical mathematicians: thirty-two plans presented accounts of Pythagoras's life and work, four referred to Archimedes, three to Euclid, and three to Thales. In addition, Descartes and Eratosthenes were each mentioned by two students, while one student included Newton's contributions and another centred an entire lesson on the work of Aristarchus. The use of mathematics in historical civilizations such as Ancient Egypt, Babylonia, Ancient Greece, and Ancient China appeared in 21 of the lesson plans. Some students incorporated only general information about the lives or work of historical mathematicians, whereas others integrated their mathematical contributions more substantively into the lesson content. Notably, none of the students referenced female mathematicians or their contributions.

Even though the lesson plans are short and concise, some of them include wrong historical facts. One common misunderstanding is that Pythagoras came up with Pythagoras theorem, and this misconception is presented as a fact in four of the 44 lesson plans. Other inaccuracies included a misleading description of Archimedes' method of exhaustion and the claim that a Babylonian proof of the Pythagorean theorem exists, although no such proof is known.

How? The lesson plans share a broadly similar structure, beginning with an introduction, followed by descriptions of student activities and, with eight exceptions, ending with student reflections, a joint discussion, or a teacher-led summary. All but two lesson plans incorporated some form of group work, Think–Pair–Share activities, or hands-on tasks, such as measuring the circumference, area, and volume of concrete objects, cutting and assembling the squares on the sides of a right-angled triangle, or searching for and examining right-angled triangles in the classroom. In all but two lesson plans, the history of mathematics is not explicitly included in the students' own work; it is either placed at the beginning or end of the lesson when the teacher gives a presentation or shows a short film. Typically, a significant historical mathematician or an ancient culture is introduced. These presentations provide biographical background and describe the mathematician's work and discoveries, offering historical context and linking the history of mathematics to the lesson's mathematical content. In one of the two remaining lesson plans, students explore, in one activity, how the Pythagorean theorem appeared in different cultures, how mathematics may have developed before modern methods, and why right-angled triangles were culturally important. The final lesson plan centres on a historical problem on similarity, with the teacher and students reconstructing the solution together.

The function of historical content. All lesson plans offer both a historical context and a connection to the mathematical content, though some emphasize one more than the other. Most lesson plans on Pythagoras' theorem focus on context, with teachers lecturing on Pythagoras' life or showing a film about him. One plan describes his belief

that the universe was structured by whole numbers and ratios, the Pythagoreans' strict dietary rules, and the punishment of a member after discovering that $\sqrt{2}$ is irrational. Here, history functions as a tool: engaging narratives make the theorem more meaningful and memorable for students. This represents the most common use of history observed in the analysed lesson plans.

Another lesson plan, by contrast, focuses on mathematics in history and its connection to the lesson's mathematical content (area of the circle). In this lesson plan, the teacher introduces the topic by lecturing on how mathematicians in ancient Egypt and Babylonia estimated the area of a circle, explains how Archimedes approximated π using polygons, and describes how later mathematicians, such as Newton, calculated decimal expansions of π. This lesson plan presents a timeline of the historical development leading to the conceptualization of π. In this context, history is employed as a tool to support students' understanding of the concept of π. At the same time, it serves as a goal by fostering cultural literacy and humanizing mathematics.

One of the two lesson plans on proofs and constructions links historical context to the mathematical content and highlights the influence of Euclid's *Elements* on the later development of mathematics and mathematics education. It focuses on Euclid's axiomatic method, deductive proofs, and ruler-and-compass constructions. The introduction explains how the *Elements* shaped mathematics education for over two millennia and established the modern standards for proof. This is one of only three lesson plans that include an image of an original source, in this case a historical edition of the *Elements*. The teacher also demonstrates a classical ruler-and-compass proof, followed by hands-on group work in which students construct geometric figures such as equilateral triangles. In this lesson, history serves as a goal by fostering critical thinking and a reflective view of mathematics as a human endeavour, while also functioning as a tool to motivate students, support learning, and encourage active engagement.

Finally, the only lesson plan entirely centred on the history of mathematics integrates history both as a learning tool and as a goal. Focusing on similarity, the lesson uses Aristarchus of Samos's third-century BCE method for estimating the distances between the Earth, Moon, and Sun, making historical mathematics central to the activity. Rather than serving merely as motivation, the historical material forms the lesson's structure, guiding students step by step through Aristarchus's reasoning. Active engagement is encouraged through questioning and group and whole-class discussions. The plan also highlights the cultural and scientific context of mathematics, showing how geometric reasoning was used to understand humanity's place in the universe. Overall, the lesson uses history to support understanding of similarity while also promoting reflective awareness of mathematics as a human, historically situated discipline.

Questionnaire

Of the 20 student teachers who answered the questionnaire, 19 chose to write the lesson plan. Most agreed that including the history of mathematics in the lesson plan was meaningful, one neither agreed nor disagreed, and only two disagreed. Analysis of their reflections on this question revealed five distinct categories of responses:

Added value for understanding and motivation: Several student teachers emphasize that the history of mathematics adds value to teaching by making the subject

more vivid and interesting. One notes that history can give the teaching "a bit more flair" and contextualize abstract ideas. Others highlight that exploring historical elements, such as well-known mathematicians and discoveries, is rewarding and enjoyable.

Relevance for students and engagement: Some student teachers discuss the perceived relevance and engagement of the history of mathematics for students. They note that interest varies across learner groups and question whether students aged 12–16 find historical aspects meaningful. In one case, the historical component was perceived as less valuable, as the lesson could have been carried out without it.

Planning and integration in teaching: Student teachers also reflect on the didactic challenges of integrating history. Several view lesson planning with historical elements as valuable and stress the need for inspiring and natural integration. At the same time, they note difficulties in deciding how much history to include, especially given the lack of clear curricular guidance.

Time and practical constraints: A few student teachers mention time as a limitation. One notes that upper secondary mathematics courses are already time-constrained, making it difficult to allocate space for history, while another argues that historical content may reduce time for more complex topics of the course.

Personal experience of the assignment: Several student teachers focus on their own experience of completing the task rather than on the pedagogical role of history. Their comments emphasize enjoyment, interest, or a positive attitude toward the exercise itself, without explicit links to broader pedagogical issues.

In one questionnaire item, the student teachers were asked to indicate their level of agreement with statements concerning the history of mathematics. Table 1 presents a summary of their responses, which show that the student teachers are generally positive towards the inclusion of the history of mathematics in teaching.

	Completely agree	Agree to some extent	Neither agree nor disagree	Disagree	Completely disagree
I think the history of mathematics is interesting.	10 (50%)	8 (40%)	1 (5%)	0	1 (5%)
I think it is important to gain knowledge of the history of mathematics.	6 (30%)	6 (30%)	7 (35%)	1 (5%)	0
I feel motivated to learn more mathematics when I gain insight into the history of mathematics.	7 (35%)	4 (20%)	4 (20%)	3 (15%)	2 (10%)
Historical elements in the course deepened my understanding of its mathematics.	5 (25%)	6 (30%)	2 (10%)	7 (35%)	0

Table 1. Summary of student teachers' responses.

Reflections on the last three questionnaire items, addressing various aspects of the history of mathematics in education, were analysed collectively, resulting in four categories:

History as motivation and interest: Student teachers often found the historical perspective motivating and engaging. It made mathematics feel more exciting and connected to the broader story of the subject. As one student noted, "You feel a bit cool

when you connect your own calculations to those of the great mathematicians" showing how historical links can boost both interest and personal engagement.

Understanding and deeper meaning*:* Several student teachers felt that historical insights deepened their understanding of mathematics. Learning why formulas were developed and how past mathematicians reasoned enhanced both comprehension and appreciation. One noted that historical connections give mathematics a new perspective and purpose, while another emphasized how history reveals the intellectual effort behind mathematical discoveries.

Aim and relevance in education: Student teachers reflected on the importance of linking historical content to the curriculum and to practical classroom application. One observed, "I think it is both broadly educational and helps us place mathematics in context, which I perceive students seek," emphasizing how history can clarify the purpose of mathematical concepts and support meaningful and relevant learning.

Mathematics as human and cultural activity: Several student teachers emphasized that mathematics is not only abstract, but a human and cultural activity shaped by history. They noted how mathematical ideas developed across societies and over time and that formulas are not simply "thrown out" but explained through their origins. These reflections show that student teachers view mathematics as a dynamic, culturally embedded human activity shaped by historical and societal processes.

The reflections from the student teachers complement the earlier findings, showing how their engagement with the history of mathematics is not only pedagogically relevant but also personally meaningful, emphasizing how historical content shapes student teachers' own interest, understanding, and sense of the human and cultural dimensions of mathematics. In summary, the results illustrate that integrating history into mathematics education affects both the teacher's professional thinking and their personal appreciation of the subject.

Discussion

Overall, the findings illustrate that integrating the history of mathematics into teacher education has several interconnected benefits. Most student teachers perceived historical content as motivating and engaging, providing interest and personal connection. Some described how historical explanations helped them better understand formulas and methods, suggesting that history can support conceptual understanding in certain contexts. The analysed lesson plans also show that history can be aligned with curriculum aims, though sometimes only superficially. Additionally, some student teachers expressed that working with historical material helped them see mathematics as a human and cultural activity, though this varied across individuals and lesson plans. Together, these findings highlight the potential of historical content to enrich teacher education by supporting motivation, understanding, and a broader view of mathematics, while also indicating areas for further development.

Analysis of the lesson plans reveals a clear pattern: the history of mathematics is mainly introduced through teacher-led presentations, usually at the beginning or end of lessons, and is rarely integrated into students' own work. In most plans, history functions as a tool (Jankvist, 2009) to contextualize mathematical content through motivation or illustrative examples, such as accounts of Pythagoras or timelines of π approximations.

These uses can enhance engagement and understanding. Some plans also reflect history as a goal (Jankvist, 2009) by incorporating cultural and philosophical perspectives, such as the influence of Euclid's *Elements* or Aristarchus's impact on views of the cosmos. In these cases, history promotes cultural literacy and critical reflection, presenting mathematics as a human enterprise shaped by creativity and continuity. The Euclid lesson on proofs and constructions stands out by integrating historical context with student activity through hands-on geometric constructions linked to historical practices of proof and axiomatic reasoning.

Nevertheless, limitations remain. The reliance on teacher presentations, often focused on biographical or anecdotal details, risks reducing history to a decorative supplement rather than a meaningful part of mathematical inquiry. Another concern is that none of the lesson plans mentioned any women in their historical accounts, reinforcing a "great men" narrative despite women's historical contributions to mathematics. Including women and diverse societal uses of mathematics could help challenge this view. Moreover, students were rarely engaged with original texts or historical problems, and historical sources were seldom shown. One exception is the Aristarchus lesson plan on estimating distances between the Earth, Moon, and Sun, which used a historical method as the basis of the entire lesson. This approach moved beyond illustration toward genuine mathematical exploration, demonstrating how historical reconstruction can support both conceptual understanding and reflection. This aligns with Jahnke et al.'s (2000) argument that well-contextualized historical material can foster deeper reflection on mathematics as a dynamic and evolving discipline.

Students' questionnaire responses highlight both the potential and challenges of integrating the history of mathematics into teaching. On the one hand, historical approaches can enrich learning, enhance motivation, and deepen understanding of mathematics as a human endeavour. For many student teachers, engaging with history stimulates interest and provides purpose by situating mathematics within a broader cultural and intellectual context, supporting earlier research on engagement and positive affect (Fauvel, 1991; Tzanakis et al., 2000). On the other hand, practical constraints, such as limited time, varying student interest, and lack of curricular guidance, pose barriers to implementation. Some students reported limited benefits, echoing Fried's (2001) concern that poorly integrated history may be trivialized or distorted. These tensions suggest that effective inclusion of the history of mathematics requires careful planning and a clear pedagogical role.

In summary, the findings suggest that although student teachers recognize the potential of history as both a tool and a goal, their lesson designs mainly use history for motivation and context, with less emphasis on its role in fostering critical and historically informed mathematical thinking. This points to a need in teacher education not only to emphasize pedagogical benefits but also to develop strategies for more authentic integration, such as working with historical problems, primary sources, or discussing the epistemological implications of historical developments. Future teachers may benefit from training that supports the selection and adaptation of meaningful historical examples while balancing curricular demands and broader educational aims. Such approaches may help reduce trivialization and better realize the educational potential of history.

References

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*(2), 77–101. https://doi.org/10.1191/1478088706qp063oa

Chorlay, R., Clark, K. M., & Tzanakis, C. (2022). History of mathematics in mathematics education: Recent developments in the field. *ZDM Mathematics Education, 54,* 1407–1420. https://doi.org/10.1007/s11858-022-01442-7

Clark, K. M., Kjeldsen, T. H., Schorcht, S., Tzanakis, C., & Wang, X. (2016). History of mathematics in mathematics education: Recent developments. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *Proceedings of the 2016 ICME Satellite Meeting – HPM 2016* (pp. 135–179). IREM de Montpellier.

Fauvel, J. (1991). Using history in mathematics education. *For the Learning of Mathematics*, *11*(2), 3–6.

Fried, M. N. (2001). Can mathematics education and history of mathematics coexist? *Science & Education*, *10*(4), 391–408. https://doi.org/10.1023/A:1011205014608

Jahnke, H. N., Arcavi, A., Barbin, E., Bekken, O., Furinghetti, F., El Idrissi, A., Silva da Silva, C. M., & Weeks, C. (2000). The use of original sources in the mathematics classroom. In J. Fauvel & J. van Maanen (Eds.), *History in mathematics education: The ICMI study, New ICMI Study Series, vol. 6* (pp. 291–328). Dordrecht: Kluwer. https://doi.org/10.1007/0-306-47220-1_9

Jankvist, U. T. (2009). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics*, *71*(3), 235–261. https://doi.org/10.1007/s10649-008-9174-9

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

Skolverket. (2022). *Kursplan för ämnet matematik i grundskolan: Läroplan för grundskolan, förskoleklassen och fritidshemmet (Lgr22)*. Skolverket. https://www.skolverket.se/undervisning/grundskolan/kursplaner-for-grundskolan/kursplan

Skolverket. (2025). *Ämne – Matematik: Ämnesplan för gymnasieskolan (Gy25)*. https://www.skolverket.se/undervisning/gymnasieskolan/program-och-amnen-i-gymnasieskolan/hitta-program-och-amnen-i-gymnasieskolan-gy25/amne-gy2025

Tzanakis, C., Arcavi, A., de Sa, C. C., Isoda, M., Lit, C.-K., Niss, M., de Carvalho, J. P., Rodriguez, M., & Siu, M.-K. (2000). Integrating history of mathematics in the classroom: an analytic survey. In J. Fauvel & J. van Maanen (Eds.), *History in mathematics education: The ICMI study, New ICMI Study Series, vol. 6* (pp. 201–240). Dordrecht Kluwer. https://doi.org/10.1007/0-306-47220-1_7

Vinerean, M., Liljekvist, Y., Brandl, M., & Przybilla, J. (2023). Didactical usefulness of interactive mathematical maps: Designing activities supporting prospective teachers' learning. *Nordic Studies in Mathematics Education, 28*(3–4), 77–102.

Mapping shadow education in mathematics in an upper secondary school in Sweden

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Despite what many mathematics teachers may hope, students in secondary school do not seek help in their mathematics studies because they are curious about mathematics, but rather because they want to pass their tests. This is one of the conclusions of our survey of 229 students. Our study shed light on contemporary shadow education by exploring the types of support students use outside school and what motivates them to seek support in their mathematical studies. We find that regarding digital support types, students choose support where there is little or no interaction. Moreover, we find that almost half of the students claim to have used paid private tutoring, indicating a growing presence of shadow education in Sweden.

Introduction

Mathematics is a challenging subject for many secondary school students all around the world, including in Sweden. In this context, private supplementary tutoring, often described as *shadow education*, has become increasingly important (Bray, 2009; Mikhaylova, 2022; Smyth, 2009). Actually, in many studies of private supplementary tutoring, support specifically in the mathematics subject is often requested (e.g., Ireson & Rushforth, 2009; Zhang & Liu, 2022; Özdere, 2021).

The term shadow education covers supplementary educational support provided outside the formal school system, including private tutoring, online platforms (such as instructional videos or interactive exercises), and support offered by non-profit organizations or homework help-centers (Bray, 2009; Kobakhidze & Suter, 2020). While shadow education has a long history in many countries, particularly in East Asia (Bray, 1999), it is a relatively recent phenomenon in the Nordic countries, including Sweden (Christensen & Zhang, 2021; Hallsén & Karlsson, 2019). The use of shadow education has implications for educational equity, since access to and the quality of supplementary support often vary according to socioeconomic resources (Entrich & Lauterbach, 2019; Ku et al., 2024). This raises questions about how similar patterns might appear in Sweden, where both free and paid support are available.

Until now, there have been only a few studies on Swedish students' use of support in mathematics, with research focusing either on social, institutional, and interactional aspects of private tutoring (Hallsén & Karlsson, 2019; Svahn & Melander Bowden, 2019) or on digital tutoring and tutor-student interaction (Hrastinski et al., 2021). Hence, so far, no study has been devoted to how students in Swedish upper secondary schools

Mattsson, L., Eckert, A., & Kilhamn, C. (Eds.). (2026). *Talking Mathematics. Proceedings of MADIF 15*. Gothenburg University and SMDF.

use existing support structures. Therefore, this present study, which is part of a larger research project on shadow education in Sweden, aims to explore how students in an upper secondary school consult different types of support. We map the different types of support students use within the subject of mathematics, and describe what motivates them to seek such support outside of school.

Previous research

The technological advancement has changed the landscape of private supplementary tutoring in recent years. Although support through one-to-one private tutoring seems to be the most frequent form of support, new forms and types of support are emerging and are increasingly asked for (Karakus et al., 2024). Particularly, the COVID-19 pandemic seems to have accelerated the search for new digital tools to support student learning (see, e.g., Akin, 2022). However, this does not come without complications. Melander Bowden and Svahn (2020) point out that digital support is formed by the limitations and affordances of the online platform it is distributed by. In addition, other studies have pointed out challenges related to communication with online tutoring compared to non-digital tutoring (Hrastinski et al., 2021; Zang et al., 2023).

Students' motives for going beyond the support their school offers for learning mathematics have been described for more than two decades (see, e.g., Bray, 1999). The motives include the desire to achieve higher grades, boost self-confidence, or compensate for (what the student perceives as) inadequate teaching (Hajar, 2018; Wigfield & Eccles, 2000). Actually, students with clear goals and lower confidence in their skills are more willing to seek support outside the school (Wigfield & Eccles, 2000). Furthermore, the initiative to seek support outside school may not only improve academic performance but also positively influence students' motivation and overall satisfaction with school and life (Guill et al., 2020).

However, improving mathematical confidence or general mathematics skills is not the only reason for students to seek support. Studies also indicate that students often seek support to meet performance requirements or pass exams rather than to deepen their understanding of the subject (e.g., Prendergast et al., 2022). Parents often play an active role in this process, especially in contexts where such support is viewed as a strategy to secure educational success and social status (Entrich & Lauterbach, 2019; Ireson & Rushforth, 2009; Mikkelsen & Gravesen, 2021).

Aim and research questions

There is a wide range of types of support that students could use outside of the offer from the school system. There is also a variety of motives for using shadow education identified in previous research. Therefore, this study aims to explore the shadow education used for mathematics education by students in a Swedish upper secondary school. We do this by answering the following research questions: (RQ1) *What types of support are students using?* and (RQ2) *What are the students' motives for using such out-of-school support resources?* We primarily answer these questions quantitatively.

Method

To address the research questions, a digital survey focusing on various aspects of shadow education was constructed. The development of the survey was carried out through repeated discussions and collaborative review of the content. A pilot was conducted in an upper secondary school class (grade 12) at a different school from the one included in this study. Based on the feedback from the pilot and from subsequent focus group interviews, the survey was revised to improve clarity and structure, including providing additional examples for all categories and offering a QR code so that students could complete the survey on their mobile phones rather than via a computer link. In the final design, the survey included multiple types of questions: Likert-scale questions, multiple-choice questions, and open questions, allowing students to elaborate on their experiences in their own words. Open questions such as "What other types of support do you use?" and "Other motives to use support?" were included to ensure that significant alternative answers were not missed.

Participants and data collection

The survey was distributed to students in one (a single) upper secondary school in Sweden, out of which 245 students opened and started answering the questions. Not all students completed the survey. The number of answers was slightly different for the different questions. At most 229 students' answers could be analyzed for a set of questions. For some sets of questions, it was lower. The participants were not selected based on any prior criteria; all students in the participating classes were invited to take part. The students were either currently studying mathematics or had studied mathematics as part of their program curriculum.

All participating students were enrolled in one of four university preparatory programs. The students in the Natural Science Program (16% of the participants) attended grade 3, while those in the Social Science Program (25%), the Economic Science Program (27%), and the Arts Program (27%) attended grade 2; 5% of the participants did not specify which program they were enrolled in. Of the participants, 76% identified themselves as female, 21% as male, and 3% did not disclose their gender. The survey was conducted over two weeks in late May 2025, after students had completed both the national tests and other final assessments in mathematics.

Data handling and analysis

The data were anonymised directly at the point of collection, as students did not provide any names or personally identifying information in the survey. The survey described in this report is part of a larger study of students' use of shadow education that has been approved by the Swedish Ethical Review Authority. Hence, all collected data are stored securely and in accordance with ethical guidelines and relevant data protection guidelines.

The authors analysed the data separately and in joint sessions. In the current study, only quantitative data were included. Other questions included background questions (these are accounted for above), and questions with Likert-scale answers (e.g., "To what

extent have you benefited from support?"), which are not accounted for in this report. A comparative analysis was conducted to examine patterns in students' use of digital and non-digital support. The survey questions asked for students' use of several different types of support, both types that could be counted as shadow education, and types that could be considered outside of shadow education (like support from school and from parents). In some cases, we have included those for comparison. Moreover, to make the data more convenient to handle, we divided the different types into different categories, as described below.

Results

In this section, we present answers to the most relevant questions from the survey, which together provide answers to our research questions.

What types of support are used?

Questions about the types of support students use were divided into several categories, as each type was asked for separately. It could be, e.g., "Do you use this digital/non-digital support in mathematics outside of lesson-time? Private tutoring [Yes/No]". Digital and non-digital support types were asked for in separate clusters.

Digital support types

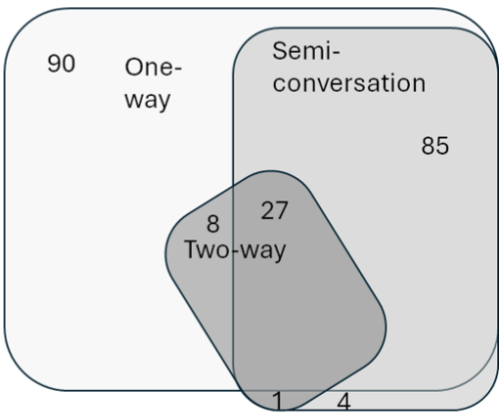
Figure 1(a) shows a summary of the use of various digital support types. We observe that platforms, being either general video-on-demand platforms like YouTube and TikTok, or specific mathematics platforms (with a mix of text and video), like matteboken.se or eddler.se, are commonly used. The content of the videos and on the mathematics platforms is fixed, and communication is directed *from* the platform *to* the student. This is a type of support with one-way communication.

Figure 1. Use of digital support in shadow education, by type (a) and by category (b)

(a) Number of respondents who use each type of support ($N = 215$)

Type of support	Number of "yes, I use..."	
Video-platforms (e.g., YouTube, TikTok,...)	195	One-way support
Digital platforms for self-study (e.g., eddler.se, matteboken.se,...)	144	
Apps/sites for retrieving solutions (e.g., Photomath, ...)	99	
AI-platforms (e.g., chatGPT, copilot,...)	100	Semi-conversation support
Online chat-forums (e.g., pluggakuten.se, social media groups, ...)	42	
Open online support (e.g., mattecentrum livehjälpen,...)	29	Two-way support
Private online tutoring	20	
School-supported platforms (Teams)	86	School's support

(b) Number of students who use support from each category



Also frequently used are forums and platforms where the students can have either a semi-conversation with a chatbot (like, e.g., AI text generators) or an asynchronous chat with a real person. A much smaller number of students are using support that involves an actual, real-time two-way conversation with another (living) person. As a comparison, the school offered a chat function in their digital learning management system (they used Microsoft Teams), and the number of students using this type of support is included at the bottom of Figure 1(a). The number of students using that type of non-shadow education exceeded the number using support by real-time two-way conversations, but was smaller than the number of students seeking support by shadow education tools of the type that did not involve a "live" person-to-person discussion. However, students could use many types of support in parallel. To get an idea of the use of multiple types of support in shadow education, we defined three main categories of support: *one-way support*, *semi-conversation support,* and real-time *two-way support*. In the category *one-way support*, we include video platforms (like YouTube and TikTok), digital platforms for self-studies (two common platforms in Sweden are eddler.se and matematikboken.se), and apps/websites for retrieving mathematical solutions (the one most frequently mentioned by the students was Photomath, but there are many more on the market, like, e.g., Mathway, Quanda). In the category *semi-conversation support*, which comprises semi-two-way support types, we include AI tools and online chat forums. In the category of *two-way support*, we include such types of shadow education that involve real-time person-to-person conversations.

Figure 1(b) shows the number of respondents in each of these categories. Almost all respondents use some kind of one-way support, half of the respondents also use semi-conversation support functions, and a small percentage of the respondents use two-way support functions–most of them as a complement to both one-way support and semi-conversation support. No student used any kind of two-way support only, without using one-way or semi-conversational support. In addition, we can compare the number of students who use some kind of semi-conversational support (117 respondents marked at least one of these support types, see "semi-conversation support" in Figure 1b) to the number of students who used the school's support function (86 respondents, see bottom row in Figure 1a). These two alternatives are comparable since they both rely on asynchronous, chat-like functions. Hence, it is safe to say that outside of mathematics lesson hours, more students are using online semi-conversational chat-support functions (including AI) to communicate with an unknown responder than students who turn to their teachers with similar chat tools.

We did not observe a major difference between females and males regarding the use of different categories of support (at most a difference of a couple of percentage points). However, there could be a difference regarding which study program they attended. Students in the Natural Science Program and those in the Social Science Program utilized two-way support functions to a greater extent (20% and 31%, respectively) than

students in the Arts Program and the Economic Science Program (6% and 10%, respectively).

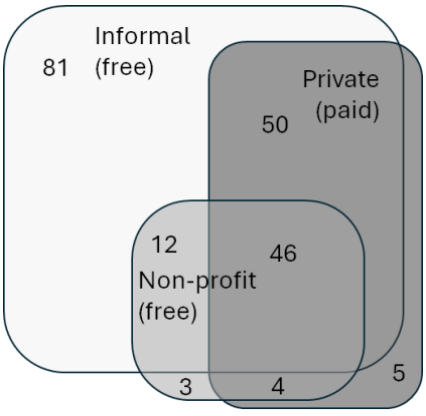
Non-digital support types

Naturally, a student could be asking a friend or a parent for advice and support in their mathematics studies. Indeed, we see that a majority say that they use family and/or friends as support in their mathematics studies (189 of 229 students). To separate this type of support from the organized forms of shadow education, we summarize these under the name *Informal support (free)*, since we assume that the friends or the family do not get paid. Although this does not traditionally count as shadow education, we include it here for comparison.

However, there are types of non-digital shadow education support that are either free or for payment and given by organizations or private persons. For lack of better words, we call these categories *Non-profit support (free)* and *Private support (paid)*, respectively. In the category of Non-profit support (free), we find group tutoring (in Swedish: "räknestugor") from organisations like, e.g., "Mattecentrum" and other non-profit organizations. In the category of Private support (paid), we include all types of paid private tutoring with real-time physical meetings with a person (which could be hired privately, or through a professional study support service).

Figure 2 shows a summary of the number of students that has marked that they use any of the types of support that we divide into these three categories: *Informal support (free)*, *Private support (paid),* and *Non-profit support (free)*. In total, 189 students (i.e., $81 + 50 + 12 + 46$, in Figure 2) marked that they used family and/or friends, i.e., Informal support (free), 65 students marked that they used organizations, Non-profit support (free), and almost half of the students ($105 = 50 + 46 + 4 + 5$, in the figure) marked that they used alternatives that involved some kind of paid study support, Private support (paid).

Figure 2. Number of students who are using other (non-digital) types of support, sorted in three categories.



Note: Total number of respondents for this question: $N = 229$

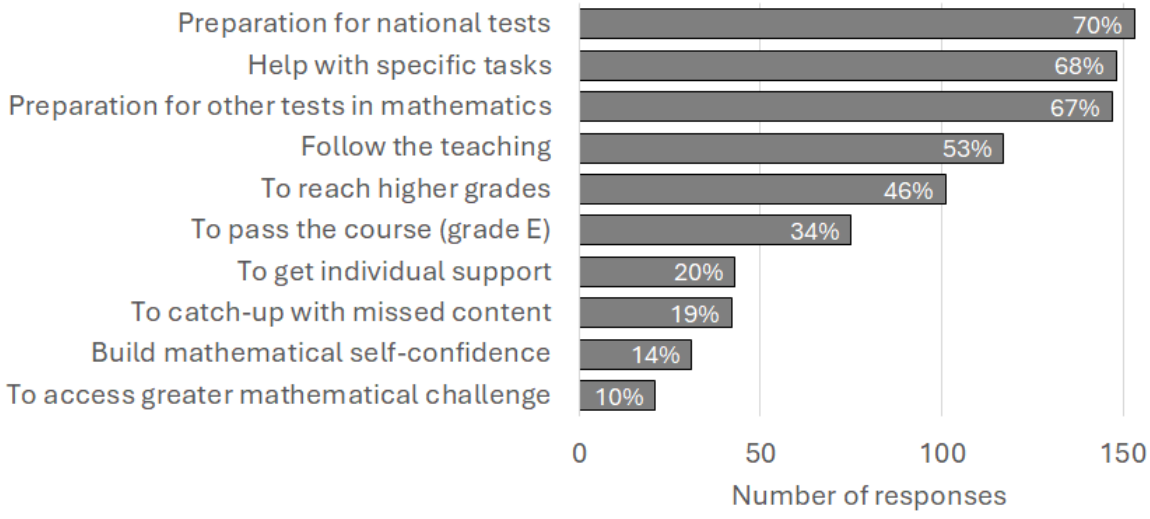
We are aware that some types of support are quite similar. For instance, the (non-digital) support type "Organized (free)" has many similarities to Open online support (from Figure 1a). One can note that the Swedish organization "Mattecentrum" acts in both categories, with both digital and non-digital mathematics support.

Why are students seeking support?

Students' reasons for seeking support are primarily to manage the mathematics course, see Figure 3 for a summary of the different answers. The reasons "as preparation for national/other tests" and "to get help with specific tasks" were each marked by more than two-thirds of the students. Almost half of the students (117 of 219) marked "to be able to follow the teaching in school". Only a small number of students marked reasons indicating other self-improvements within the subject of mathematics.

The answers from female students and male students are almost identical (details not shown here), with only small differences between the groups. The only major difference is that a larger proportion of male students marked that they seek support to access greater mathematical challenge compared to their female peers (15% of the male students compared to 8% of the female students), although the number of students that marked this alternative was small (21 in total, see Figure 3). However, we observe that the female students, on average, selected slightly more alternatives (mean value of 4,1 "reasons" for females compared to 3,7 "reasons" for male students).

Figure 3. Students' reasons for seeking support in mathematics (multiple answers possible) ($N = 219$)



Hence, the main reason for students to seek support in mathematics of the types mentioned above seems to be to prepare for tests, either national tests or other (i.e., teacher-made) tests. Also, the alternatives "to get help with specific tasks" and "to be able to follow the course" were quite frequently marked. The commonality between these options is that they all reflect some type of external factors. The least marked reasons instead reflect internal factors, such as improving their own mathematical skills.

We observe that even though it is the least marked reason, "to access greater mathematical challenge" is marked by almost 10% of all students (21 answers out of 219).

Discussion

Although there was a limited number of participants in this study (*N=229*), we can make some statements based on our quantitative observations.

First, we observe that the largest group providing support is friends and parents, the category we call Informal (free) support (Figure 2). This informal support does not fall under the definition of shadow education (Bray, 2009), but it remains relevant due to the data. We believe we can safely assume the student already has a relationship with friends and parents. Hence, a "safe-haven" for asking questions appears to be important for these students, similar to what was indicated by Melander Bowden & Svahn (2020). They describe that face-to-face support relies on a supportive relationship where the student does not feel judged. Therefore, we believe this is what we see too in our data.

However, we find it interesting that almost half of the students (105 out of 229 in Figure 2) are using paid private tutoring. These findings align with a growing interest in the phenomenon of private paid tutoring in Sweden. Previous research has shown a recent expansion of the organized sector of shadow education in the Swedish context, particularly in paid forms of support (Christensen & Zhang, 2021; Hallsén & Karlsson, 2019). Hence, almost half of the students are paying for face-to-face support, although it is not within the "safe-haven" of family or friends.

Second, we observe that students seem to prefer support types that can be categorized as "one-way", and a much smaller number of students use real-time "two-way" conversation support with an actual person (Figure 1). The use of one-way, typically video-based support types has been described before (Akin, 2022; Ireson & Rushforth, 2009; Mikkelsen & Gravesen, 2021; Entrich & Lauterbach, 2019). We are aware that the study by Karakus et al. (2024) suggests that asynchronous video tutoring (which we include in the category of "one-way") is the only type of support that shows statistically significant improvement in students' mathematical performance. However, other studies point out that online support presents different challenges in the communication between student and tutor as the absence of gestures, facial expressions, and tone of voice can make interaction more difficult (Hrastinski et al., 2021; Zang et al., 2023). Based on our data presented here, we can not say anything about the objective effectiveness of the support, but only comment on the prevalence of students using digital one-way support. We find that only a small number of students report engaging in synchronous (two-way) support (Figure 1b). It seems they instead prefer asynchronous forms that do not require real-time interaction. Without further and more detailed data, we can not fully corroborate Wigfield and Eccles' (2000) observation that students with low academic confidence tend to avoid more interactive support.

Third, we observe that the majority of students seek tutoring or support for their studies to manage the course, pass the test(s), or get answers to specific questions.

Almost two-thirds of the students mark that this is the reason for seeking support (Figure 3). This is a much larger number than the ones seeking mathematics support because they are genuinely interested in the subject (the two alternatives in Figure 3 with the lowest number of respondents). We could only speculate on the reason for this.

The results of this study can, similar to Entrich & Lauterbach (2019) and Ku et al. (2024), be interpreted in the light of research indicating that shadow education may reinforce socioeconomic differences in education. In Sweden, where students can access both free and paid support, it remains an open question how these choices will influence equity in the future. This study represents the initial phase of a larger research project about shadow education; future work will build on the insights gained by exploring qualitative dimensions of students' learning experiences when using out-of-school support.

References

Akin, A. (2022). The effectiveness of web-based mathematics instruction (WBMI) on K-16 students' mathematics learning: A meta-analytic research. *Education and Information Technologies*, *27*(6), 8015–8040. https://doi.org/10.1007/s10639-022-10931-x

Bray, M. (1999). *The shadow education system: Private tutoring and its implications for planners.* United Nations Educational, Scientific and Cultural Organization (UNESCO).

Bray, M. (2009). *Confronting the shadow education system: What government policies for what private tutoring?* UNESCO. International Institute for Educational Planning (IIEP). Paris, France (2009).

Christensen, S., & Zhang, W. (2021). Shadow Education in the Nordic Countries: An Emerging Phenomenon in Comparative Perspective. *ECNU Review of Education*, *4*(3), 431–441. https://doi.org/10.1177/20965311211037925

Entrich, S. R., & Lauterbach, W. (2019). Shadow Education in Germany: Compensatory or Status Attainment Strategy? Findings from the German LifE Study. *International Journal for Research on Extended Education*, *7*(2), 143–159.

Guill, K., Ömeroğulları, M., & Köller, O. (2022). Intensity and content of private tutoring lessons during German secondary schooling: effects on students' grades and test achievement. *European Journal of Psychology of Education*, *37*(4), 1093–1114. https://doi.org/10.1007/s10212-021-00581-x

Hajar, A. (2018). Exploring Year 6 pupils' perceptions of private tutoring: evidence from three mainstream schools in England. *Oxford Review of Education*, *44*(4), 514–531. https://doi.org/10.1080/03054985.2018.1430563

Hallsén, S., & Karlsson, M. (2019). Teacher or friend? – consumer narratives on private supplementary tutoring in Sweden as policy enactment. *Journal of Education Policy, 34*(5), 631–646. https://doi.org/10.1080/02680939.2018.1458995

Hrastinski, S., Stenbom, S., Benjaminsson, S., & Jansson, M. (2021). Identifying and exploring the effects of different types of tutor questions in individual online synchronous tutoring in mathematics. *Interactive Learning Environments, 29*(3), 510–522. https://doi.org/10.1080/10494820.2019.1583674

Ireson, J., & Rushforth, K. (2009). Private tutoring at transition points in the English education system: its nature, extent and purpose. *Research Papers in Education, 26*(1), 1–19. https://doi.org/10.1080/02671520903191170

Karakus, M., Tlessov, A., Hajar, A., & Courtney, M. (2024). Illuminating the shadows: the role of private supplementary tutoring on student math performance in PISA 2022. *Large-scale Assessment in Education, 12*, 42 (2024). https://doi.org/10.1186/s40536-024-00228-5

Kobakhidze, M. N., & Suter, L. (2020). The Global Diversity of Shadow Education. *European Journal of Education*. https://doi.org/10.1111/ejed.12411

Ku, I., Lee, H., & Kim, J. E. (2024). Does shadow education contribute to inequality? *Asia Pacific Journal of Education, 44*(4), 773–791. https://doi.org/10.1080/02188791.2022.2114424

Melander Bowden, H., & Svahn, J. (2020). Collaborative work on an online platform in video-mediated homework support. *Social Interaction. Video-Based Studies of Human Sociality*, *3*. https://doi.org/10.7146/si.v3i3.122600

Mikhaylova, T. (2022). *Shifting shadows: Private tutoring and the formation of education in imperial, Soviet and post-Soviet Russia*. [doctoral thesis] Uppsala University.

Mikkelsen, S. H., & Gravesen, D. T. (2021). Shadow Education in Denmark: In the Light of the Danish History of Pedagogy and the Skepticism toward Competition. *ECNU Review of Education*, *4*(3), 546–565. https://doi.org/https://doi.org/10.1177/2096531120983335

Prendergast, M., O'Meara, N., Duffy, L., & O'Rourke, I. (2022). Delving into the shadow: Mathematics teachers' perspectives on the impact of private supplementary tutoring in Ireland. *Asian Journal for Mathematics Education*,*1*(3), 323–337. https://doi.org/10.1177/27527263221129637

Smyth, E. (2009). Buying Your Way into College? Private Tuition and the Transition to Higher Education in Ireland. *Oxford Review of Education*, *35*(1), 1–22.

Svahn, J. & Melander Bowden, H. (2019). Interactional and epistemic challenges in students' help-seeking in sessions of mathematical homework support: Presenting the problem. *Classroom Discourse, 12*(3), 193–213. https://doi.org/10.1080/19463014.2019.1686998

Wigfield, A., & Eccles, J. S. (2000). Expectancy–Value Theory of Achievement Motivation. *Contemporary educational psychology*, *25*(1), 68–81. https://doi.org/https://doi.org/10.1006/ceps.1999.1015

Zhang, E., & Liu, Y. (2022). Effects of private tutoring intervention on students' academic achievement: A systematic review based on a three-level meta-analysis model and robust variance estimation method. *International Journal of Educational Research*, *112*, Article 101949. https://doi.org/10.1016/j.ijer.2022.101949

Zhang, L., Pan, M., Yu, S., Chen, L., & Zhang, J. (2023). Evaluation of a student-centered online one-to-one tutoring system. *Interactive Learning Environments*, *31*(7), 4251–4269. https://doi.org/10.1080/10494820.2021.1958234

Özdere, M. (2021). The demand for private tutoring in Turkey: An analysis of private tutoring participation and spending. *Journal of Education and Learning, 10*(3), 96–111. https://doi.org/10.5539/jel.v10n3p96

Critical aspects of mathematically rich construction-work tasks

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This paper examines characteristic features of vocational mathematics knowing, as it is expressed in authentic vocational task-solving. Vocational teachers' descriptions of students' difficulties and required understanding were analysed alongside student dialogues while solving vocational mathematics tasks. Using a phenomenography-variation theory framework, the study identifies critical aspects that students need to discern to engage with mathematically rich construction-work tasks in proficient ways. An illustrative case exemplifies how such tasks involve a complex interplay of formal mathematics, vocation specific practices, physical constraints, and interdisciplinary knowledge. The findings highlight the need to address both context specific elements and broader knowledge structures in vocational education.

Introduction

Affording opportunities for students to develop the ability to use mathematics efficiently in vocational activities is a significant challenge in vocational education. Although the utility of mathematics in vocational settings is widely recognized as a key argument for its place within vocational education, research indicates a discrepancy between students' mathematical knowing and workplace demands (Muhrman, 2016; Hoyles et al., 2010). Previous research show that mathematical aims, applications and activities may differ significantly between the mathematics classroom and workplace situations (FitzSimons, 2014; La Croix, 2014, Sundtjønn, 2021). Vocational mathematics is described as a multifaceted knowledge area, shaped by an interplay between mathematical and vocational dimensions, each comprising both theoretical and practical components (FitzSimons, 2014; FitzSimons & Boistrup, 2017; Johansson et al., 2024). In vocational practice, mathematics appears in diverse forms, sometimes explicit, but often implicitly embedded in vocational practice (Hoyles et al., 2010; FitzSimons & Boistrup, 2017). This diversity contributes to the field being under-researched, with mathematical competencies for vocational students, requested by policy makers, frequently redefined (Bakker, 2014; Dalby & Noyes, 2022).The question of how mathematics is conceptualised within vocational activities, and how relations between school mathematics and vocational practices can be understood and reflected in teaching, thus remains a key area for further research (FitzSimons & Björklund, 2017; Sundtjønn, 2021). Discussing "numeracy at work," Straesser (2015) problematizes the common assumption that mathematical knowledge in vocational practice is confined to basic arithmetic. In contrast, research across various vocational fields reveal a much broader

Mattsson, L., Eckert, A., & Kilhamn, C. (Eds.). (2026). *Talking Mathematics. Proceedings of MADIF 15*. Gothenburg University and SMDF.

range of mathematical competences (e.g., Hoyles et al., 2010; Muhrman, 2016; LaCroix, 2014; Johansson et al., 2024; Straesser, 2015). However, Straesser (2015) argue that, apart from basic arithmetic skills, required mathematical qualifications vary widely between workplaces. Research must therefore be conducted for each vocation specifically, with close attention to the characteristics of the respective workplace. Analyses of specific workplace tasks confirm that such activities may involve mathematical calculations of varying difficulty as well as vocation-specific demands such as interpreting drawings, performing measurements, and choosing appropriate strategies. (e.g. Martin & LaCroix, 2008; LaCroix, 2014; Johansson et al., 2024). Considering problem solving in a construction work practice, Martin and LaCroix (2008) suggest that students focus on three interfaces. They start by forming an image of the task as a whole and then move between images of visible mathematics and images of the workplace. Martin and LaCroix (2008) further propose that, for vocational aims, it is more appropriate to talk about the *learning of a mathematically rich workplace tasks*, than simply the learning of mathematical topics. This idea is reflected in the present paper, which takes authentic workplace tasks as a point of departure. By analysing students' and teachers' reasoning about a particular task type, the study provides a concrete example of the knowledge demands and potential learning objectives associated with the kind of integrated knowledge that emerges from the interplay between mathematical and vocational elements within workplace practice, as described in previous research (e.g., FitzSimons, 2014). The contribution of this study lies in its fine-grained analysis, highlighting *critical aspects*, i.e. the components that must be discerned to engage with mathematically rich vocational tasks in proficient ways (Marton, 2015). Articulating these aspects provides a detailed account of such activities focusing on what students need to learn. In recent decades, phenomenography-variation theory (Ph-VTL) have made significant contributions to mathematics education research by providing conceptual tools that enable teachers and researchers to identify essential aspects of specific objects of learning (e.g. Marton, 2015). This study extends previous research within the tradition by applying the framework to authentic, vocational situations, a context seldom considered in this line of research. In a phenomenographic interview study, Knutson & Kullberg (2025) highlight qualitatively different ways of experiencing vocational mathematics knowing in the context of construction work practice and education. They conclude that teachers, in various ways, acknowledge the complexity of integrating various aspects during vocational problem solving. This paper aims to further illustrate how this kind of knowing is explicitly expressed in vocational mathematics task-solving by addressing the research question: *Which critical aspects do students need to discern in order to understand and engage with mathematically rich, construction work, vocational tasks, in proficient ways?*

Theoretical framework

The analysis conducted in this study draws on the conceptual framework developed within the phenomenography and variation theory research tradition, (from now on

denoted Ph-VTL). Ph-VTL frames learning as gaining more powerful ways of seeing the world. It is assumed that the way a person may act in a situation, depends on how the situation is perceived. Consequently, certain ways of seeing, or experiencing, could be regarded as more powerful than others (Pang & Ki, 2016). Marton and Booth (1997) suggest that differences in ways of experiencing, could be traced to the discernment of *critical aspects*. The way something is experienced depends on the aspects that are discerned and simultaneously brought into focal awareness. The aspects that are focused on, and internally and externally related, thus constitutes a way of experiencing, and the aspects that are critical for experiencing something in a certain way are called *critical aspects* (Marton & Booth, 1997; Pang & Ki, 2016). That is, learning is, according to Ph-VTL, essentially a matter of discerning critical aspects. Ph-VTL is grounded in the principle of intentionality, which means that learning is understood as inherently directed toward something. The verb "to learn" must have an object, i.e. *the object of learning* (Marton & Booth, 1997). The object of learning could be conceptualised as the particular way of experiencing a situation which is regarded the most powerful (Pang & Ki, 2016) and thus corresponds to the critical aspects that are necessary to discern to become able to do, or understand, something in a desirable way. These aspects constitute the essential differences between ways of experiencing and are thus critical components in the transition from inability to proficient action (Pang & Ki, 2016; Marton, 2015). Intentionality pertains to experience as well; every experience is directed to something. In each moment, that what is in the focus of attention tends to be perceived as a meaningful entity (Marton & Booth, 1997). Such intentional objects are in this paper referred to as *phenomena*. When we focus on and delimit a phenomenon, other elements are necessarily perceived as a background, an *external horizon*. Those elements, which are present in awareness in an inarticulate yet significant way, constitute a *structure of relevance* and are also central to the experienced phenomenon. (Marton & Booth, 1997). Booth (1992) suggests that problem solving could be described in terms of a shifting focus, a stream of consciousness, which is guided by references to the external horizon. In that way elements from the external horizon enter focal awareness and emerge into new phenomena. When solving vocational tasks, students can thus be assumed to shift between making sense of the task as a whole, and "zooming in" on other phenomena. According to Ph-VTL, human mind is delimited to consider just a few aspects at a time (Marton & Booth, 1997). However, during the process of task solving, different phenomena, and consequently several different aspects, might be considered. *Critical aspects* of the task are thus the totality of aspects, which are necessary to discern to understand the task, and its' component parts, in a powerful and proficient way.

Method

The empirical data includes teacher interviews and a task-based student dialog. Eight construction work teachers (denoted V1-V8) were interviewed about their experiences of students' challenges when solving authentic vocational mathematics tasks. In addition, a discussion between two students, (denoted S1 and S2) solving corresponding

tasks was recorded. The interviews (1h each) and the dialogue (1h) were transcribed verbatim. During the teacher interviews, participants were asked to give examples of vocational tasks and to highlight aspects that they considered particularly important and challenging for students to discern. The interview transcripts were read several times to identify recurring examples of workplace tasks (i.e. tasks sharing a similar practical purpose and underlying structure). Based on this reading, five task-types were selected for further analysis, one of which is presented in this paper. To identify suitable tasks for the student dialogue, the task types were sought in mathematics textbooks aimed at construction-related vocations (e.g., Schmale, 1966). Since older textbooks provide a wide range of such tasks, a selection of older mathematics textbooks, chosen based on their availability in library collections, was examined. The task, forming the basis for the dialogue presented in this paper is described in further detail below.

The transcribed teacher interviews and student dialogue were analysed through the conceptual lens of Ph-VTL, focusing on *critical aspects*. In the teacher interviews, critical aspects were identified as elements that teachers emphasised as central or challenging when describing the task and explaining how they would support students in approaching it. In the student dialogue, critical aspects were identified as aspects at the edge of students' understanding, manifested as focal points of discussion, where differences or shifts in understanding became apparent. The two students were selected to provide suitable variation in understanding: one of them attending the first year of the construction work programme, while the other had recently completed the technology programme. Both students contributed constructively to the task solution and ultimately reached a shared understanding. Using different data sources contributed to analytical robustness as well as a more comprehensive account. The identified aspects were compared across interviews and dialogue, both to confirm overlap and to capture a broader range of aspects. The analysis also distinguished between aspects that are typical of the task type (i.e., even distribution of items) and aspects that are specific for the vocational situation (e.g. roof tiling or wall panelling). Including both types of aspects enabled a more comprehensive picture, showing how more general principles interact with situation specific demands.

The selected task type

The task type selected for analysis in this paper concerns constructing even spacing. It is described, in several interviews and appears in one of the textbooks (Schmale, 1966). Given a total length and a maximum spacing, the task is to calculate the number of items and the spacing that result in even distribution of items along the full length. Situations, such as panelling, roofing, and fencing are exemplified contexts. There are a number of such tasks in the textbook. The task below was selected for student dialogue:

> **250**. The c/c distance between the roof trusses in a building must not exceed 120 cm. Calculate the number of roof trusses needed if the c/c distance between the two outermost trusses is 24.36 meters. What will the actual distance between the trusses be if they are evenly distributed? (Schmale, 1966, authors' translation)

According to the authors' own investigation, the core procedure for solving the task-type typically involves the following steps:

- Calculate the minimum number of spacings by dividing the total length by the maximum spacing and then round the obtained value up to nearest integer.
- Calculate the proper spacing by dividing the total length by this number.
- Determine the number of items (typically one above the number of spacings).

Result

For students to complete vocational mathematics tasks, certain critical aspects must be discerned. Examples from the interviews, along with the identified critical aspects, are presented below. The *critical aspects* are italicised in the text and summarised at the end of the section. Significant passages in the interview excerpts are bolded.

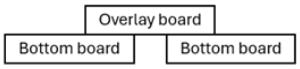
Making sense of the task as a whole -aim and conditions

As previously described (see Martin & LaCroix, 2009; Booth, 1992), students initial focus is typically on the task as a whole, including discerning its framing: the desired outcome, the preconditions and the constraints. In this case this entails understanding that *the spacing must be even across the entire distance,* that the *total distance is fixed*, that the *spacing is adjustable*, and that the *spacing is constrained*. These aspects can be traced in the excerpts below:

> V8: Something the students really struggle with is dividing up a panel surface. A distribution of boards, with a **certain spacing in between**. (…) You have a surface, and a **panel of a certain width,** and it must **divide evenly at the end.**

> V5: This very thing of dividing things up. It's really hard for them. Distributing panelling on a wall. You have a panel [figure 1], bottom board and overlay board. (…) **it's that distance** [between the bottom boards] **that we can adjust** (…) We must **end up with a board here** [beside window] so that the **overlay board covers any gap**.

Figure 1: *Panel cross-section*



It is pinpointed that it is the *distance between the boards that can be adjusted*, that the *panel-width comprises a constraint*, and that the *spacing must remain invariant until the last item*. This approach is also reflected in students' reasoning when solving the task described in the methods section. They initially focus on understanding the task, observing that the items should be evenly distributed, identifying a maximum allowable spacing, noting that the number of spacings must be calculated and suggesting that the actual spacing will likely become slightly less than the maximum value:

> S1: You need to have them **evenly spaced** (…) It can be a **maximum of 120** cm (…) and then you're supposed to **see how many will fit** in 24.36 m. So, it'll probably **end up being a bit less than 120** cm then.

That is, a strategy is suggested and fixed, and adjustable quantities are distinguished.

Initial measurements and practical conditions, dependence on contextual factors

In authentic vocational situations, such as those described in the teacher interviews, the required measurements are typically not given in advance. It is therefore critical to discern which measurements that are essential, how they should be obtained, and how they may be influenced by physical constraints, such as material dimensions. For example, aspects related to proper measuring practice are emphasised by V6:

> V6: [Concerning tile battens] You must take the **centre-to-centre measurement** [between the upper and lower tile batten] However, (…) students might take the **measurement between the edges** of the tiles. It's common. Then, they **end up with the wrong distance** at the last item (…) and must start all over again.

Here V6 points to the significance of distinguishing the *correct centre-to-centre measurement* from the commonly mis-used measurement between adjacent edges of the items. Similarly, the excerpt below underscores the need to discern how physical conditions such as wall configurations impose context-specific constraints:

> V5: [Concerning panelling] We get a certain measurement under the window, and we get a **completely different measurement over there**, and then, possibly, above this window there is another window, but with an offset of 15 cm. There they get stuck!

In particular VM5 points to the important challenge of considering how *physical features of the wall generate various fixed distances across the wall*. In a similar manner, the excerpt below shows that concerning the task of distributing roof tiles, it is necessary to discern that the *maximum spacing depends on roof slope*:

> V6: Let's say this roof slopes at 27 degrees. I use the tile instructions (…) If the slope is 27 degrees, the spacing between the battens should be, say, 340 mm to 375 mm (…). **If the slope is less, the tiles need to overlap more**, to protect against wind and water.

In this specific context, it is essential to discern *that the degree of overlap, and thus the maximum allowable spacing, depends on the slope of the roof* ("if the slope is less, the tiles need to overlap more") *why this is the case* ("to protect against wind and water") and *how this is expressed in numerical terms* (e.g., 340–375 mm). These aspects relate to general principles of the physical environment as well as vocation-specific demands. More implicit here is the concept of slope itself. The task might include finding the roof angle, which might require seeing *slope as rice over run* and understanding *how this is related to the roof angle* through the *tangent function*. Other contexts may involve different types of constraints, such as safety regulations, material dimensions, or specific workplace standards, which are then critical aspects in these specific situations.

Recognizing and applying a general strategy

Once the practical conditions are accounted for, and proper measures are obtained, the task solving process proceeds by calculating the spacing. Here, recognizing an *over-all task solving strategy* is a critical aspect, with the strategy, as such, as a dimension of

variation. Understanding the procedure, in turn, involves a number of interrelated aspects. A central aspect in this regard is *the multiplicative relation between the spacing, the number of spacings, and the total distance,* an aspect commonly encountered within school mathematics curricula. However, solving the task also demands awareness of how numerical results must be adapted to practical constraints. For example, understanding that the *number of spacings must be an integer*, and more specifically, that the proper value is obtained by *rounding up* the calculated result. This insight is in turn based on the understanding that, starting from *maximum allowable spacing yields minimum number of required items*. These aspects are evident in the student dialogue:[1]

> S2: All right… I know exactly how to do it! [calculates: 24.36 / 1.2 = 20.3 i.e. total length divided by maximum spacing]
>
> S1: Should it be 20.3 items? (…) well, should it be 20 then? No, it must be 21, because **you can't have more than 120 cm** between, **so you have to round up**. [calculates: 24.36 / 21] We get 1.16. So, it's less than 120 cm then!

The critical point of discussion here concerns the fact that the *number of items must be an integer* (i.e., not 20.3) and that *rounding up is necessary to ensure compliance with the maximum spacing constraint* (i.e., 21 rather than 20). As a contrast, it is concluded that *rounding down would result in a spacing that exceeds the allowed maximum*. Finally, using multiplicative reasoning, the appropriate spacing is calculated as the total length divided by the number of intervals, and it is confirmed that the resulting spacing satisfies the constraint. Another critical aspect emphasised is the *distinction between the number of items and the number of spacings*. Dividing the total distance by the spacing gives the number of spacings, not the number of items. The students first suggestion is that the number of items should be 21. They later reconsider this answer:

> S1: If you have 1.16 m between each and multiply by 21, you get 24.46 m…but… **then you've only counted the spaces between**, and you're **missing the final truss**.

Applying the result, accumulation of errors

Having calculated the proper spacing and the number of items, the authentic task proceeds with practically marking out their placements. A critical aspect at this stage is the *accumulation of errors*. In other words, students need to understand that *simply adding rounded values will cause a deviation at the final item*, and that this must be accounted for. V6 emphasizes this issue and suggests a method to add non-round values:

> V6: I use the first measurement and mark it out. I use the [+] [+] [=] method [using the calculator in a specific manner which implies adding a non-rounded value] to add the same spacing again. By the end, I'll reach 4332 mm exactly. (…) **If you use the plus-plus method, you can go twelve miles with millimetre precision.**

[1] The students are solving the task exemplified in the method section

Summary:
As illustrated above, the vocational task-type of distributing items evenly, require a variety of aspects to be discerned. Some of them concerns making sense of the specific task at hand and understanding how relevant features of the physical environment affect the initial values. Such aspects might be general or specific to the situation. Furthermore, solving the task involves aspects related to the solution procedure, and when applying the calculated values in the context, yet other aspects need to be discerned.

Table 1:
Summary of critical aspects at different stages of the task solving process

Conditions	Initial values	Solution	Application
-The spacing should become **even** across the entire distance **-The total distance** is **fixed** **-The spacing** is **adjustable** **-The spacing** is **constrained**	-The significance of taking the **centre-to-centre measurement** in contrast to the measurement between adjacent edges	**-Multiplicative relation** between the spacing, the number of spacings, and the total distance **-Number of spacings** must be an **integer** -The **maximum allowable spacing determines the minimum number of items.** (Which in turn implies that:) -The **number of spacings** must be **rounded up** -Distinction between **number of items vs. number of spacings**	**-Accumulation of errors** (in contrast adding rounded values will cause a deviation at the final item)
	Aspects, specific to a particular context		
	-Maximum spacing depends on roof slope/panel width etc. (why and how) **-Total distance depends on** e.g. **wall-configuration**		

Note: Most of the aspects are typical of the task type, while a few are specific to a particular context, as indicated by the shaded area in the table above.

Discussion

To address the research question: "Which critical aspects do students need to discern in order to understand and engage with mathematically rich vocational construction tasks in proficient ways?" a detailed analysis of learning demands was conducted using a Ph-VTL framework. Confirming earlier findings (e.g., LaCroix, 2014; Martin & LaCroix, 2008; Hoyles, 2010; Johansson et al., 2024) the result demonstrate that workplace activities may involve aspects from various domains, and their interrelationships. The present study adds to previous research through its fine-grained analysis of critical aspects, which clearly exemplifies and illustrates this multifaceted nature of vocational mathematics reasoning. As the result show, the aspects derive from various stages of the problem-solving process. There are critical aspects involved in the initial sense making of the task as a whole (e.g distinguishing fixed and adjustable quantities), in the process of finding appropriate initial values, in the solution procedure and in the final application of the results within the context. Furthermore, the aspects belong to various mathematical, contextual and vocational domains. Some of them align closely with a formal mathematics curriculum (e.g., discerning multiplicative relations), while others represent a kind of mathematical thinking which is intertwined with logical reasoning

within the vocational context (e.g. principles for appropriate rounding, and effects of cumulative error). Additionally, the task incorporates vocation-specific aspects related to tools, norms, and practices particular to the profession, such as measuring techniques and construction regulations (e.g. correct c-c measurement and maximum spacing values). Beyond these, there are aspects of the physical environment, related to mathematics but also requiring knowledge that transcends mathematics (e.g. understanding how, and why, the appropriate tile spacing is affected by the roof slope). Moreover, in addition to aspects of general strategies, the task involves situation-specific aspects, including unique conditions of the specific situation (e.g effects of wall-configuration and material dimensions). The results thus suggest that vocational mathematics proficiency is not merely a matter of transferring school mathematics to vocational practice but rather involves the ability to discern and integrate critical aspects across domains, and to understand how mathematical principles interact with contextual and physical constraints.

These findings have implications for teaching. If mathematics in vocational education aims to prepare students for authentic workplace situations, the objects of learning need to reflect such interactions (FitzSimons & Boistrup, 2017). In doing so, however, objects of learning will extend beyond a formal mathematics curriculum. The present study provides a detailed account of the critical aspects involved in solving one such task. If the educational goal is primarily to enable students to handle specific vocational situations, a particular set of aspects such as those identified above may constitute a significant and sufficient object of learning. Yet there are also potential objects of learning that are more general and disciplinarily coherent, while still connected to those aspects (e.g., an in-depth, generalized understanding of the concept of proportionality, of the concept of slope, and of the relationship between these concepts). FitzSimons (2014) argues that besides contextual knowledge, such disciplinary coherent knowledge structures, are important components of vocational mathematics knowing. Booth (1992) suggests that problem-solving processes might be described as a stream of consciousness guided by references to relevance structures, surrounding the phenomena. That is, besides the directly discerned critical aspects, the perceived phenomena are potentially related to structures including more peripheral yet connected aspects. It is reasonable to assume that such structures also affect the task solving process, not least by affording opportunities for creativity and adaptability. Focusing mainly on the learning of specific tasks, might run the risk of ignoring coherent structures within disciplines, which, beyond their intrinsic value, might be essential for creative vocational problem solving. However, focusing only on objects of learning within a formal mathematics curriculum may run the risk of overlooking contextual and interdisciplinary aspects, which are equally essential to discern when solving mathematically rich workplace tasks. How such different knowledge structures affect vocational task solving, and how the corresponding objects of learning might be connected and mutually enhancing, remains a question for future research and indicates a direction for further development of Ph-VTL.

References

Bakker, A. (2014). Characterising and developing vocational mathematical knowledge. *Educational Studies in Mathematics, 86*(2), 151–156. doi.org/10.1007/s10649-014-9560-4

Booth, S. (1992). *Learning to program: A phenomenographic perspective.* Göteborgs Universitet,

Dalby, D., & Noyes, A. (2022). The waxing and waning of functional skills mathematics. *Journal of Vocational Education & Training, 74*(3), 434–453. https://doi.org/10.1080/13636820.2020.1772856

FitzSimons, G. E. (2014). Commentary on vocational mathematics education: where mathematics education confronts the realities of people's work. *Educational Studies in Mathematics 86*(2), 291–304. https://doi.org/10.1007/s10649-014-9556-0

FitzSimons, G. E., & Björklund Boistrup, L. (2017). In the workplace mathematics does not announce itself: towards overcoming the hiatus between mathematics education and work. *Educational Studies in Mathematics 95*(3), 329–349. https://doi.org/10.1007/s10649-017-9752-9

Hoyles, C., Noss, R., Kent, P., & Bakker, A. (2010). *The need for techno-mathematical literacies*: Routledge. https://doi.org/10.4324/9780203854655

Johansson, T., Björklund Boistrup, L., & Dahl, J. (2024). Patterns of connections between mathematics and electrician knowledge: Praxeology in vocational education. *Quaderni di Ricerca in Didactica, 13*, 641–650.

Knutson, H., & Kullberg, A. (2025). *Teachers' perceptions of educationally significant vocational mathematics knowing.* Proceedings of the Fourteenth Congress of the European Society for Research in Mathematics Education (CERME14).

LaCroix, L. (2014). Learning to see pipes mathematically: preapprentices' mathematical activity in pipe trades training. *Educational Studies in Mathematics, 86*(2), 157–176. https://doi.org/10.1007/s10649-014-9534-6

Martin, L., & LaCroix, L. (2008). Images and the growth of understanding of mathematics-for-Working. *Canadian Journal of Science, Mathematics and Technology Education, 8*(2), 121–139. https://doi.org/10.1080/14926150802169263

Marton, F. (2014). *Necessary conditions of learning*. Routhledge. https://doi.org/10.4324/9781315816876

Marton, F., & Booth, S. (1997). *Learning and awareness*. Routledge. https://doi.org/10.4324/9780203053690

Muhrman, K. (2016). *Inget klöver utan matematik: En studie av matematik i yrkesutbildning och yrkesliv. [No clover without mathematics: A study of mathematics in vocational education and professional life]* [Doctoral thesis, Linköping University] https://doi.org/10.3384/diss.diva-12456

Pang, M., & Ki, W. (2016). Revisiting the idea of "critical aspects. *Scandinavian Journal of Educational Research, 60*(3), 323–336. https://doi.org/10.1080/00313831.2015.1119724

Schmale, W. (1966). *Yrkesräkning för byggnadsyrken*. Berlingska Boktryckeriet.

Straesser, R. (2015). Numeracy at work: a discussion of terms and results from empirical studies. *ZDM Mathematics Education, 47*, 665–674. https://doi.org/10.1007/s11858-015-0689-0

Sundtjønn, T. (2021). *Opportunities and challenges when students work with vocationally connected mathematics tasks.* [Doctoral thesis, Agder University]

From Prescription to Participation: Teacher Agency in a Long-Term Mathematics PD

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This article explores teacher agency and the interplay of prescriptive and self-directed professional development (PD) approaches within a long-term mathematics initiative in Sweden. Thirteen teacher reflections were analysed to examine how agency and ambit are expressed in practice and how they relate to student learning and instructional change. Drawing on Boylan et al.'s (2018) conceptual model, the findings reveal a shift from traditional, top-down teaching to more exploratory, personalized, and collaborative mathematics education. The article contributes to ongoing discussions about how PD can be designed to support sustained teacher engagement, reflection, and autonomy.

Introduction

In recent years, professional development (PD) in mathematics education has shifted from a predominantly theoretical orientation toward models that integrate practical resources and classroom strategies. Historically, PD programs emphasized theoretical foundations, with the expectation that teachers would independently translate these into practice (Desimone, 2009; Boylan et al., 2018). Previous research highlights that the design of professional development materials should not only address content and methods, but also the relational dimensions between developers, teachers, and students (Helenius et al., 2014). Swedish research has also shown how teacher agency develops in the interplay between centrally designed PD materials and collegial discussions, highlighting the dual influence of structure and teacher interpretation (Insulander et al, 2019). In Sweden, the Mathematics boost exemplifies this model, providing structured opportunities for theory-based professional development (Helenius et al., 2021).

Furthermore, shifts in classroom authority are central to understanding teacher agency. Wagner and Herbel-Eisenmann (2014) identified how authority structures in mathematics classrooms, including personal authority, discourse as authority, discursive inevitability, and personal latitude, influence both teachers' and students' opportunities to participate. This perspective provides an important background for analysing how teachers in long-term PD initiatives negotiate authority and participation. More recent initiatives, however, combine explicit teaching strategies and curriculum materials with theoretical insight (Maass et al., 2019; Koichu et al., 2021). These hybrid approaches do

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not fit neatly into the PD models analysed by Boylan et al. (2018). Instead, they are reflecting a growing international focus on implementation, how innovations are enacted, sustained, and adapted in practice (Coburn, 2003; Century & Cassata, 2016; Ahl et al., 2023). This shift raises important questions about the design of PD programs that balance innovation, sustainability, and teacher autonomy. This development aligns with the systematic mapping review by Ahl et al. (2023), which showed that implementation research in mathematics education has increasingly focused on how innovations are enacted, sustained, and adapted in practice, rather than solely on outcomes. This broader perspective situates the present study within an international research trend.

In Sweden, national initiatives such as Mathematics Boost and TRR (Tänka, Räkna, Resonera) illustrate attempts to align pedagogical development with both policy reforms and the everyday realities of classroom practice (Helenius, 2021). This aligns with research showing how teacher agency develops within national PD initiatives (Insulander et al, 2019). Yet the challenge remains to develop models that simultaneously support innovation and foster teacher agency. As Helenius (2022) shows in his analysis of stakeholder scaffolding in the TRR program, large-scale implementation requires deliberate coordination across multiple organizational levels. Such scaffolding plays a crucial role in supporting teachers' enactment of new teaching models, highlighting the institutional dimension of agency.

This paper examines a multi-year PD initiative in Sweden, designed to promote professional growth "on teachers' terms." Grounded in research on self-directed and collaborative learning (Boylan et al., 2018), the project supported early years and lower elementary teachers in refining mathematics teaching. Rather than offering a standardized program, it emphasized adaptability, teacher choice, and sustained engagement. To analyse the design and impact of this initiative, the study draws on Boylan et al.'s (2018) framework, with particular attention to agency, teachers' control over their professional learning and ambit. The research investigates how the program's features, and the connections among them, shaped teachers' professional learning, how practical and theoretical components interacted, and how teacher agency was expressed in participants' written reflections.

Theoretical Background

According to Boylan et al. (2018), agency in professional learning refers to an individual's ability to set goals, make decisions, and influence the course of their learning. In their study of the Swedish national mathematics modules, Helenius et al. (2014) identified three key components for sustainable professional development, content, tasks, and relationships, that interact rather than follow a linear sequence. In addition, studies of Swedish PD programmes demonstrate how professional development materials and structured discussions can serve as mediating tools for teacher agency (Insulander et al., 2019). High levels of agency support autonomy,

motivation, creativity, and ownership of both learning and teaching practices. In contrast, low agency can result in passivity and compliance with top-down directives.

Ambit describes the range of experiences and practices considered legitimate within a PD framework, a dimension that has recently been examined in depth to understand how various innovation carriers, curriculum reforms, professional development, and curriculum resources, interact with enactment and outcomes (Ahl et al., 2023). A narrow ambit implies prescriptive content and delivery, while a broader ambit allows for adaptation, experimentation, and inclusion of local needs. The balance between agency and ambit influences not only teachers' professional growth but also students' opportunities to learn. To summarise, teacher agency refers to teachers' capacity to make intentional choices, act with professional judgment, and shape their learning trajectories. In contrast, ambit delineates the boundaries of what is considered legitimate or possible within a professional development framework. While agency emphasizes teachers' capacity for action, ambit refers to the scope and structure that either constrain or enable such action. The balance between them can therefore be understood as the dynamic interplay between professional autonomy and institutional prescription. Recent reviews show that these two elements interact in multiple ways depending on the implementation context and the types of innovations being introduced (Ahl et al., 2023).

In addition, research on highly prescriptive curriculum programs shows that even in tightly structured settings, teachers adapt materials in ways that reflect their professional experience. Ahl et al. (2022), using the Documentational Approach to Didactics, demonstrated how teacher adaptations in the TRR program shaped alignment and implementation outcomes. This highlights the importance of analysing not only whether teachers follow prescribed guides but also how their adaptations contribute to or challenge intended program goals. Research in mathematics education highlights that prescriptive PD can ensure alignment with standards but may restrict creativity and responsiveness, a finding echoed in a systematic mapping review of over 100 studies on implementation research in mathematics education (Ahl et al., 2023). Self-directed PD supports personalization and innovation but may suffer from a lack of structure. Thus, finding a productive balance is critical, especially when designing long-term initiatives aimed at sustainable change.

Methodology

The study involved thirteen teachers from early years and lower elementary grades who have taken part in an ongoing professional development (PD) initiative in Sweden, which began in 2019. Data for this study consist of written reflections collected in spring 2025. All participants provided consent for their reflections to be used in research.

The PD initiative has been running continuously since 2019 and is still ongoing. It is organized around meetings, each lasting 3.5 hours, 4-6 times per year. Teachers gather either in grade-level groups or as a full school staff. Sessions typically include:

- Structured discussions of classroom challenges,

- Follow-up on topics from previous meetings, and
- A teacher-selected mini-lecture on a theme of shared interest.

The project leader, first author, also maintains ongoing dialogue with school leaders and special education staff to ensure alignment with school priorities and student needs.

In spring 2025, participating teachers were invited to submit written or recorded reflections as part of the project's evaluative process. The reflections were guided by four open-ended prompts designed to capture their experiences of instructional change, student learning, and professional growth:

1. Changes in teaching practice – How do you feel your teaching has changed during the course of the project? Reflect on the causes of these changes and what you think they depend on.
2. Students' behaviour in the classroom – How have the students' strengths changed over time? Reflect on their behaviour now compared to earlier.
3. Reflections on the journey – Share your reflections on the whole process.
4. The role and selection of teaching materials – How do you choose teaching materials, and what role do they play in your teaching?

These reflections provided insight into how teachers understood the long-term influence of the PD initiative on their practice and how they exercised agency in relation to classroom teaching and curriculum resources. Thematic analysis was employed, guided by Boylan et al.'s (2018) concepts of agency (teachers' control over their professional learning) and ambit (the scope of experiences considered valuable). Coding combined inductive and deductive approaches: codes were generated to capture emergent themes while being informed by the conceptual framework. The analysis focused on how teachers described changes in teaching practice, their perceptions of student engagement, the role of teaching materials, and their experiences of professional learning.

Participation in the study was voluntary, and teachers provided informed consent before submitting their reflections. Data were anonymized during transcription and analysis to ensure confidentiality, and all identifying details were removed. The study followed ethical standards for educational research (Vetenskapsrådet, 2024), ensuring that participation had no consequences for teachers' professional roles and that they retained the right to withdraw their contributions at any stage.

Results

This analysis summarizes teacher reflections from a mathematics professional development (PD) project. Thirteen written responses were analysed and categorized into key themes including; changes in teaching practice, teacher's perceptions of student engagement, the role of teaching materials, and their experiences of professional

learning. These findings are discussed in relation to the concepts of agency and ambit in professional learning, as outlined by Boylan et al. (2018).

These findings resonate with Wagner and Herbel-Eisenmann's (2014) framework on authority structures. In particular, the increase in student participation and collaborative reasoning observed in this study can be linked to shifts toward greater personal latitude in classroom discourse, where students are positioned as active contributors to mathematical meaning-making.

Changes in Teaching Practice

Many teachers described a shift toward more intentional and reflective teaching. Rather than strictly following a textbook, teachers reported selecting tasks that encourage reasoning and mathematical language development. One teacher noted: "I spend more time on tasks that focus on students' reasoning. I want to see and hear that they understand the mathematics rather than just producing the correct answer." This change can mean that teachers dared to move away from a prescriptive model and instead allowed for both teacher and student agency, aligning with Boylan et al.'s (2018) framework for self-directed professional learning. Several participants also described adopting "mathematical glasses" as a metaphor for emphasizing the conceptual and linguistic components of mathematics. "Now the math lesson has become like we put on our math glasses and step into a role". Another teacher stressed: "I am more careful to use precise mathematical terms, like saying 'equals' instead of 'becomes', and I follow up students' answers with open questions such as 'How do you know that?' or 'How did you think?". Teachers also highlighted that this shift required more preparation but gave deeper rewards. As one teacher explained: "Earlier, I just followed the book with little preparation. With the new book, and the activities we've tested, I prepare more, both theoretically and practically, but it's worth it. I feel calmer now, and I can really hear the students' understanding when they talk to each other." Another wrote: "I have challenged both myself and the students much more. I have dared to move away from the book and use other tasks, like those we received in workshops. That has given the students more variation and better understanding."

A recurring theme was the value of open-ended and exploratory tasks, where there was not always one "correct" solution. One teacher described: "When we work with broader problems where there isn't just one right answer, more students dare to try. Earlier some never said anything for fear of being wrong, but now they contribute." Another wrote: "Students have become more relaxed and want to reflect and experiment with everyday math problems, like figuring out how many students are absent and setting up different equations to show it."

Several teachers pointed out that the project helped them to see mathematics everywhere and to use everyday contexts for learning. One explained how she used her students' curiosity about her newborn lambs as a way of exploring number concepts: "When the children ask if more lambs are born, we can investigate together." Another wrote about the importance of collective learning: "Instead of individual tasks, I now

choose activities where everyone can join in, regardless of whether they are strong or struggling in mathematics."

The reflections also show that teachers felt less stressed and more confident in prioritizing: "I have reflected more over children's mathematical learning. I feel less stress over math lessons now that I know how to target specific areas." In summary, the reported changes in teaching practice shows a movement toward more reflective, language-rich, and student-centered mathematics teaching. Teachers reported increased confidence, greater awareness of mathematical language, and more varied and open tasks that foster reasoning and collaboration.

Perceptions of Student Engagement in the Classroom

Teachers reported notable changes in students' classroom behaviour. A prominent theme was increased student participation in mathematical discussions. As one teacher stated: "They talk about mathematics with me and, more importantly, with each other!" Teachers described that students who previously remained silent began to contribute: "When we work with broader problems where there isn't just one right answer, more students dare to try. Earlier some never said anything for fear of being wrong, but now they contribute." Another teacher reflected: "Students who once struggled with confidence now participate in discussions and show their reasoning openly."

A recurring theme in the written reflections was that collaborative tasks not only improved mathematical reasoning but also strengthened social interaction. One teacher noted: "Working in groups has made students better at listening to each other and helping each other. They are more used to reasoning together and it creates a more open climate in the classroom." Several teachers emphasized that the shift toward dialogue and reasoning helped reduce students' anxiety around mathematics. As one explained: "Before, some students panicked when they didn't understand immediately. Now they take more time, analyse the problem, and feel safer as mathematicians."

In summary, teachers reported that students' engagement in mathematics lessons grew stronger as they gained confidence, learned to express their reasoning, and discovered the value of collaboration. They expressed that open-ended tasks and mathematical dialogue gave space for more voices to be heard, contributing to a richer and more inclusive classroom culture. This aligns with research showing that student agency and open-ended tasks foster deeper mathematical understanding (Ball, 2003; Boylan et al., 2018).

Professional Learning: Ambit and Agency

Many teachers expressed that they had felt encouraged to experiment with new teaching methods, often due to concrete examples and collegial discussions. Helenius et al. (2014) argue that relational aspects, between teachers, between teachers and students, and between teachers and designers, must be integrated into PD design to create meaningful learning opportunities. Similar to findings from earlier Swedish case studies, teachers in this project exercised agency by appropriating and reshaping PD tasks in ways that reflected their local needs and collegial practices (Insulander et al., 2019).

As one participant wrote, receiving concrete tips and engaging in meaningful dialogue with colleagues expanded their willingness to try new approaches. Similarly, another teacher wrote that they had challenged both themselves and their students through more varied teaching, illustrating how collaboration fostered the confidence to depart from routine practices, another teacher emphasized that collaboration provided courage: "I have challenged both myself and the students with more varied teaching. It feels easier to test something new when I know my colleagues are also trying it."

Teachers frequently highlighted that this type of peer collaboration expanded their view of what counts as legitimate and valuable practice. What once might have felt like stepping outside the norm gradually became part of a broader ambit of professional learning. Several wrote that they now reflect more on why certain mathematical concepts are difficult for students and share strategies to make them more accessible. This peer collaboration appeared to expand the perceived ambit of their professional learning, what counts as legitimate and valuable practice. This broader ambit, as defined by Boylan et al. (2018), reflects greater professional agency and ownership of instructional methods. The reflections in this study reveal how teachers gradually expanded the ambit of legitimate practice, from following textbooks to designing tasks collaboratively. This expansion was made possible through increased agency: teachers took ownership of instructional decisions and justified them through shared professional reasoning.

The Role of Teaching Materials

A number of teachers described a more flexible relationship to teaching materials. While previously constrained by textbooks, they now felt more empowered to supplement with investigative and hands-on activities. One teacher shared: "I used to see teacher guides as something to copy from, but now I reflect more on how I explain mathematical relationships." Another reflected: "No material can provide everything a group needs. I use the textbook as a foundation, but I adapt and add activities so all students can be included."

Still, some tension remains around the comprehensive scope of the new textbook, which was described as both beneficial and overwhelming. On the one hand, it provided a structured sequence, ready-made activities, and research-based content. On the other hand, several described it as overwhelming: "If you try to do everything in the new book, there is little time left to pause, reflect, or practice more." This highlights the need for a balance between structured support and teacher autonomy, a key interaction between ambit and agency. Some teachers concluded that the most productive approach was to see textbooks and curriculum packages as valuable tools, but not as strict scripts that must be followed line by line.

Summary of Key Themes

A thematic analysis of the thirteen teacher responses revealed a number of interrelated developments. Many teachers reported that their teaching had become more intentional,

with greater awareness of planning and prioritization. The teachers reported that this shift was accompanied by students' improved reasoning and collaboration. Teachers also described a more flexible use of teaching materials, no longer constrained by textbooks but instead combining structured resources with exploratory and hands-on activities.

A central theme was the role of collegial learning. Through professional dialogue, teachers not only gained concrete strategies but also expanded their sense of what counts as legitimate practice. This reflects a broadening ambit of professional learning, where experimentation and shared reflection became valued aspects of teaching. At the same time, teachers reported an increased sense of agency, greater confidence and ownership of their instructional choices, even when moving beyond routine practices.

Several teachers also pointed to challenges, particularly with group work for students with neurodivergent needs, highlighting the need for careful scaffolding and differentiated approaches. Nonetheless, the overall findings align with Wagner's (2014) argument that professional development is most transformative when it expands teachers' repertoires and strengthens their professional identity. In this case, the project not only changed classroom practice but also contributed to a culture of agency, expanded ambit, and a more exploratory, student-centred approach to mathematics instruction. The findings indicate that the PD project contributed to an enhanced sense of professional agency, a broadened ambit of teaching methods, and a more exploratory and student-centred approach to mathematics instruction.

In this study, teachers' reflections suggest that agency was exercised within an expanding ambit. Initially, their professional choices were confined by prescriptive curriculum materials, but over time collegial discussions and reflective practice broadened the ambit of what counted as valid teaching. Thus, professional growth emerged not through the absence of structure, but through teachers' negotiation of structure and freedom.

Discussion

These results demonstrate that structured, yet reflective professional development can support meaningful shifts in both teaching and learning. The project encouraged teachers to move from a prescriptive model of instruction, where outcomes and methods are externally set, towards a more self-directed approach that values experimentation, dialogue, and conceptual understanding. These results resonate with research showing that teacher agency in PD programmes often emerges through negotiation between prescribed materials and collective sense-making processes in collegial discussions (Insulander et al., 2019). This is consistent with findings from a systematic review showing that effective mathematics PD often combines structured input with opportunities for inquiry and collaboration (Alsina et al., 2021).

The expanded ambit of acceptable instructional practices, fostered by collegial sharing and practical examples, appears to have played a key role in promoting teacher agency. This supports Boylan et al.'s (2018) assertion that professional learning is most

effective when it balances structure with autonomy. Moreover, the reflections suggest that increasing student agency, through collaborative work, open-ended questions, and mathematical discourse, had a positive impact on classroom culture and learning outcomes. Boylan et al. (2018) emphasize that the ambit of a professional learning model defines the scope of legitimate practices, while agency reflects how actors operate within—or seek to expand—that scope. In this study, the teachers' agency was enacted through the very act of stretching the ambit of acceptable mathematics teaching.

However, challenges were also identified. Some teachers noted that certain students, particularly those with NPF, struggled with group dynamics, and others felt time-pressured by the comprehensive curriculum. These tensions underscore the importance of maintaining flexibility within professional development frameworks and adapting instructional materials to local contexts. This is consistent with Koljonen et al, (2023), who found that even in tightly scripted teaching models such as TRR, teachers adapted lessons in ways that reflected their professional judgement. Their findings illustrate that fidelity and adaptation are intertwined, reinforcing the central role of teacher agency in large-scale PD.

Overall, the findings reinforce the need for professional learning that empowers teachers to navigate between prescriptive curricula and student-centred practices, expanding both their sense of agency and the ambit of what is possible in mathematics education.

Conclusion

This study contributes to a growing body of implementation research in mathematics education by highlighting how long-term, dialogic PD initiatives can cultivate teacher agency and broaden the ambit of acceptable instructional practice. Teachers in the project reported a shift from compliance with fixed curricula toward more reflective, student-centred approaches. Through collaboration, self-direction, and support for local adaptation, teachers engaged deeply with both theoretical underpinnings and practical enactment.

Importantly, this work shows that implementation success is not solely a matter of alignment to a predefined model but also depends on the space afforded for professional judgement, experimentation, and the cultivation of trust among the teachers. While some tensions remained, particularly around time constraints and differentiated student needs, the reflections collected offer compelling evidence that sustainable instructional change can be achieved when PD is responsive, participatory, and grounded in both research and practice.

Further research could continue to investigate the dynamic interplay between top-down reform goals and bottom-up adaptation, especially in underexplored areas such as the planning and piloting phases of innovation. Future studies may also address how implementation evolves over longer timeframes and under varying organizational conditions, thereby contributing to more robust design and theorization of mathematics PD programs.

References

Alsina, Á., Pincheira, N., & Franco, J. (2025). Mathematics teachers' professional development: A systematic review of training methods. *EURASIA Journal of Mathematics, Science and Technology Education*, *21*(3).

Ahl, L. M., Helenius, O., & Koljonen, T. (2022). Gauging fidelity to an implemented teaching model through the lens of the Documentational Approach to Didactics. *Implementation and Replication Studies in Mathematics Education, 2*(1), 45–75.

Ahl, L. M., Helenius, O., Sánchez Aguilar, M., Jankvist, U. T., & Misfeldt, M. (2023). Implementation research in mathematics education: A systematic mapping review. *Implementation and Replication Studies in Mathematics Education*, *3*(2), 135–199.

Ball, D. L. (2003). Mathematical proficiency for all students: Toward a strategic research and development program in mathematics education. *RAND Corporation*.

Boylan, M., Coldwell, M., Maxwell, B., & Jordan, J. (2018). Rethinking models of professional learning as tools: a conceptual analysis to inform research and practice. *Professional Development in Education*, *44*(1), 120–139.

Century, J., & Cassata, A. (2016). Implementation research: Finding common ground on what, how, why, where, and who. *Review of Research in Education*, *40*(1), 169–215.

Coburn, C. E. (2003). Rethinking scale: Moving beyond numbers to deep and lasting change. *Educational Researcher*, *32*(6), 3–12.

Desimone, L. M. (2009). Improving impact studies of teachers' professional development: Toward better conceptualizations and measures. *Educational Researcher*, *38*(3), 181–199.

Helenius, O. (2021). A stakeholder analysis of the development of a large-scale professional development program. *Implementation and Replication Studies in Mathematics Education*, *1*(2), 227–256.

Helenius, O. (2022). Designing stakeholder scaffolding for implementing a teaching model at scale. In *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12)*. Bozen-Bolzano, Italy. HAL Archive.

Helenius, O., Johansson, M., Lange, T., Meaney, T., Riesbeck, E., & Wernberg, A. (2014). Theorising the design of professional development web modules. In 9th *Swedish mathematics education research seminar*, Umeå, February 4-5, 2014. Svensk förening för matematikdidaktisk forskning (SMDF).

Insulander, E., Brehmer, D., & Ryve, A. (2019). Teacher agency in professional development programmes–A case study of professional development material and collegial discussion. Learning, Culture and Social Interaction, 23, 100330.

Koichu, B., Aguilar, M. S., & Misfeldt, M. (2021). Implementation and implementability of mathematics education research. *ZDM – Mathematics Education*, *53*(5).

Koljonen, T., Helenius, O., & Ahl, L. M. (2023). Evaluating the enactment of a scripted teaching model. In *Proceedings of the Thirteenth* Congress *of the European Society for Research in Mathematics Education (CERME13)*. Budapest, Hungary. HAL Archive.

Maass, K., Cobb, P., Krainer, K., & Potari, D. (2019). Different ways to implement innovative teaching approaches at scale. *Educational Studies in Mathematics*, *102*(3), 303–318.

Vetenskapsrådet (2024). *God forskningssed*. Stockholm: Vetenskapsrådet.

Wagner, D., & Herbel-Eisenmann, B. (2014). Identifying authority structures in mathematics classroom discourse: A case of a teacher's early experience in a new context. *ZDM Mathematics Education, 46*(6), 871–882.

Designing and validating a test for the assessment of Algebra-Monday

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We report on the design and validation of a 34-item algebra test intended as an outcome measure for Algebra Monday, a long-term, structural-algebra intervention. The test design operationalizes algebra as structure, operations on symbols, and functional thinking, aligning the conceptualization of algebra with the intervention focus of Algebra Monday. Administered in regular eighth- and ninth-grade lessons from May to June 2025, the instrument demonstrated good reliability, appropriate difficulty coverage, and minimal construct-irrelevant variance. These findings support its use for evaluating Algebra Monday's effectiveness at the group level and point to opportunities for further research on the nature of students' algebra difficulties and on validation of tests used in intervention studies.

Introduction

Algebra is a major constituent of mathematics. It has historical roots in arithmetic calculations involving unknown quantities. In the modern era, algebra denotes the study of abstract systems in the form of algebraic structures. In school, algebra has traditionally played a significant role in secondary education, often serving as a gatekeeper that marks the point at which students either gain access to advanced mathematical studies or begin to disengage from the subject. For a few decades, research in mathematics education has therefore focused on early algebra as a way of fostering algebraic thinking already in the early grades, hopefully making the transition into formal algebra in secondary education less harsh (Kieran 2004).

Research on school algebra and early algebra has traditionally focused on students' (mis)understandings of algebraic concepts. Additionally, complex frameworks are presented to help understand early algebra and algebraic thinking. Some of these frameworks have also served as the basis for systematic intervention work. Notably, the work of Blanton and colleagues, who built upon an algebra framework developed by Kaput (2008), resulted in a new framework. Blanton et al. included the components of generalized arithmetic, equivalence, expressions, equations, and inequalities, and functional thinking (Blanton et al., 2015). The group then moved on to conduct a series of interventions from a K-6 perspective, with the randomized controlled trial of the group's 3-5 intervention, involving approximately 3,000 students, being the most notable effort (Blanton et al., 2019). Unfortunately, a follow-up study in grade 6

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revealed that the advantage gained by the experimental group in grade 5 had vanished one year later, when a retention study was conducted (Stephens et al., 2021).

In an effort to enhance the long-term positive effects of an increased algebra focus, we have designed the project Algebra Monday. On Algebra Mondays, all teachers from K (6-year-olds) to 9 (16-year-olds) teach specifically designed algebra lessons once a week, not only for a specified number of weeks, but indefinitely. In each participating municipality, we first work with pilot teachers for a minimum of 20 weeks. These pilot teachers later act as facilitators in collegial collaborations (Roesken-Winter et al., 2021).

In Algebra Monday, we are particularly interested in promoting structural algebraic or arithmetic reasoning. By **algebra as a structure,** we mean ways of discovering, explicating, symbolizing, and using general relationships among, primarily arithmetic, operations (e.g., Section 4.2.2 in Kieran, 2018a). Examples include knowledge that $a \cdot b = c$, $a = c/b$, and $b = c/a$ will all hold simultaneously (unless zeroes are involved). Since this category of algebra cuts across several of, for example, Blanton's categories, we constructed an algebra framework of our own that we will briefly mention here, with three categories in addition to the main category algebra as structure.

Algebra as operations on non-numeric symbols concerns operations on symbols that represent numerical entities but are not themselves numerical symbols. This involves typical solutions of linear equations. We also consider tasks like $8 + _ = 3$ to belong in this category, as well as tasks such as simplifying $4a^3 + 4a + 2a^3 + 3a$.

Algebra as variables and functional thinking. Functional thinking is centered around using a symbol that can stand for any number (for which the function is defined), i.e., function as a variable. Functional thinking also includes graphical representation of covariation between variables. We also include modeling, i.e., when a function is created to represent some phenomenon, in the functional thinking category.

Algebra as patterns. Working with figural or geometric patterns has been proposed as a way to get acquainted with certain forms of algebraic thinking. Tasks in this class typically involve continuing and representing growing or repeating patterns (section 4.2.3 in Kieran, 2018b). This category is not a main priority in Algebra Monday because we do not see proficiency with pattern work as generative for other algebraic proficiency in the same way as we see algebra as structure as generative. This is a design hypothesis we will not delve deeper into here. For more information on the four categories above, including how they are currently represented in Swedish textbooks, see Helenius & Ahl (2024).

The functioning of Algebra Monday will be evaluated mainly by its effects on students' algebra knowledge development. For this we need a test. There are mandatory national tests in Sweden in grades 3, 6, and 9, which will also help assess the effectiveness of Algebra Monday. These tests are, however, not algebra specific, and individual tasks cannot be shared openly, meaning they are of limited use for sharing research results. In targeted interventions such as that of Blanton et al. (2019), researchers may design their own assessments based on pre-existing assessments and may put them through various validation processes such as establishing reliability and

ascertaining item difficulties. To our knowledge, there is little or no research on or recommendations for how such validation of test use for intervention research are done. In this paper, we report on the design of an algebra test for grade 8 and 9, intended for evaluating the Algebra Monday intervention. Beyond this specific purpose, however, we hope to use it as an illustration through which to initiate a conversation with the wider mathematics education research community about how we design instruments for ascertaining the efficacy of our innovations, and how the interpretation and use of such instruments may be validated. Additionally, the test is made available upon request[1] in the hope that other researchers and teachers may find it useful.

Design

We had two design purposes. First, we wanted the test to distinguish between algebra abilities in a population of Swedish 8th- and 9th-grade students who had not received any special treatment. Second, we wanted the test to measure aspects of algebra teaching that we were focusing on in our intervention, Algebra Monday but we were *not* trying to presuppose cognitive subconstructs that can form the basis for latent variables in an assessment, as Zhang et al. (2018) do. In addition, we included items that had either been proven in previous tests or were otherwise well-known in the mathematics education community. For example, the task $14 - 5 = _ + 7$ is a well-known type of task in the mathematics education research literature, which tests a relational understanding of the equal sign. There are known typical errors that are helpful when designing multiple-choice items. The proportion of pupils of different ages that can correctly handle this task is well known both internationally and in the Nordic perspective (see Opsal & Topphol, 2023, for an overview). Using established tasks like this also makes it easier to build a future discussion based on previous research. In this limited format, we cannot describe the origins of all items, but several tasks are related to the following references (Hackenberg & Sevinc, 2022; Hilton et al., 2013; Niss & Jankvist, 2013). A few tasks also come from older national tests for grade 9 in Sweden.

The test was trialed in two schools during March and April 2025, with 68 students ranging from grade 7 to the third course in upper secondary school. Based on this pilot trial, 11 questions were removed to make the test shorter, based on low point-biserial correlation with the overall test result, inappropriate difficulty level, and technical difficulties in the testing platform. Moreover, test questions were reordered by difficulty. While the initial test consisted of 45 questions, the final test contained 34 questions.

We based the test construction loosely on the three focused categories in Algebra Monday: algebra as structure, as operations on non-numerical symbols, and as variables and functional thinking. Table 1 shows the distribution of tasks between the three categories. 26 of the 34 items were multiple-choice and had one option *I don't know* which was included to avoid students guessing. A reason to have an open answer format on 8 items was that we wanted to avoid trial and error type solutions for some equations type items. A test blueprint is presented in Table 1.

[1] By email to ola.helenius@ncm.gu.se

Category	Target skills	Examples	Number of items
Structure	General arithmetic facts, such as $\frac{m}{m} = 1$	What should be on the empty line? $a + b + _ = a + c$	15
Functional thinking	Using algebra to model worldly situations	The blue flower's height (B) is $\frac{2}{5}$ of the red flower's height (R). Which equality is true?	8
Non-numeric symbols	Operations on variables	Solve the equation $x - 1 = 3x + 5 - 8x$	11

Table 1. Test Blueprint.

The test contains groups of items within which items will likely appear quite differently difficult to typical students. According to the design hypothesis of Algebra Monday, students who have experienced teaching focused on structural relationships should instead see many of these items as variants of the same task, in the sense that a unified principle can be used to deal with them. For example, we strongly advocate that operations often described as simplifying or expanding fractions should be understood via the neutral element in multiplication, in the form of $m/m = 1$: simplifying is always done by factorising, identifying a common factor m in numerator and denominator, and then removing $1 = m/m$. So, the tasks in Figure 1 should differ markedly in difficulty for standard students but be more similar in difficulty for Algebra Monday students, according to the hypotheses behind the Algebra Monday intervention.

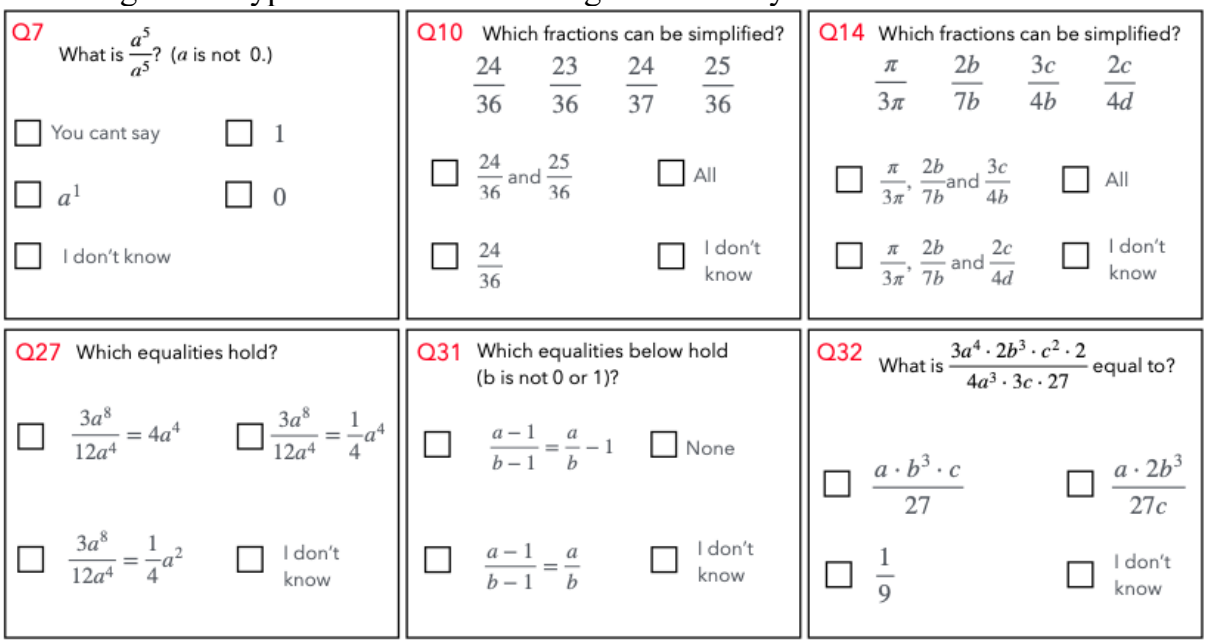

Figure 1. Five items that can be solved by repeated usage of $m/m = 1$ and one (Q31) designed to capture erroneous usage of $m/m = 1$. Items here are presented in a compact form, slightly different from the actual test layout.

Validation

We approach validation loosely based on Kane's (2013) *Interpretation/Use Argument* (IUA) framework, which posits that validity is a property of an argument concerning the interpretation and use of test scores, and that validation is the scrutiny of this argument. Such an argument will use evidence that is pertinent to the specific interpretation and use of the test scores sought in each specific case.

Our test is intended to be used for evaluating the effectiveness of Algebra Monday in improving students' algebra knowledge. The proposed interpretation of test scores is that they reflect relevant algebra knowledge. Validation therefore required evidence of the spread of item difficulty, sufficient item discrimination power, reliability, and potential presence of construct-irrelevant factors in the test administration. Additional information is included in Supplementary materials.

First, we needed to decide what to do about anomalous and missing data. We tested 471 students (May–June 2025). We excluded anomalies: 7 students spent <5 minutes (which we interpreted as not a serious attempt), and 79 had recorded times >60 minutes despite the 60-minute limit (sessions where students continued the test at a later date, which allows for discussing items between testing sessions and other algebra-irrelevant variables). For the remaining data, non-responses were scored 0. Because nearly all students finished within 50 of the 60 allotted minutes, even blanks at the end of the test are better interpreted as uncertainty than as a sign of time expiry. While we expected students to choose "I don't know" for any question they did not have an answer for, many students left at least one missing answer. A possible explanation is that students may have skipped a question, intending to return or preferring not to choose "I don't know" simply because of the mouse maneuvering required. We therefore code missing answers as incorrect and note this as a minor limitation to monitor in the future.

Spread of difficulty and discrimination. For our test to capture a suitable range of student algebra ability, items should have a spread of difficulty which reflects the range of algebra knowledge of the students the test will be used with. Additionally, correct answers should be contingent on algebra ability, meaning that the items should have sufficient discriminating potential. We also wanted to later be able to examine the relative difficulty of the various items for standard students compared to Algebra Monday students, investigating the hypothesis that some item difficulties would change substantially when students have been instructed in algebraic structures. We therefore chose an item response theory (IRT) approach using the two-parameter logistic (2-PL) model, which estimates both item difficulty (β) (i.e., at what ability a student has a .5 probability of answering the item correctly) and item discrimination (α) (i.e., how well the item score distinguishes between students with lower and higher abilities). A 3-PL model which takes guessing into account was not appropriate because we had the "I don't know option", which changes the effect of guessing.

To choose which 2-PL IRT model to fit, we needed first to decide on the dimensionality of the data. While the test design included multiple categories of questions, these were meant to reflect teaching foci rather than ability subdomains. We

therefore have no theoretical grounding either way for a uni- or multi-dimensional structure. We opted first to try a unidimensional model because of parsimony and ease of interpretation of the IRT statistics. We fit a one-factor Confirmatory Factor Analysis (CFA) using Weighted Least Squares Mean and Variance adjusted (WLSMV).

Given the near-identical content and format of Q04 and Q05, and a high tetrachoric correlation between them (r ≈ .900), we included a residual covariance between these two items in our model. The one-factor model showed a good fit and provided support for a one-dimensional model: scaled Comparative Fit Index (CFI) = .962 scaled Tucker-Lewis Index (TLI) = .959; Root Mean Square Error of Approximation (RMSEA) =.017 90% CI [0, .025]; Weighted Root Mean Square Residual =.983.

Based on the solid fit of the one-factor CFA, we then fit a unidimensional 2-PL IRT model. The evaluation of this model likewise fit well: the average correlation between residuals (denoted Q3, not to be confused with question Q03) was –.02, 99.5% of pairs $|Q3| \leq .20$). Two pairs of items (Q04 and Q05, Q11 and Q17) showed notable local dependence (Q3 > .20) and may be revised in future forms of the test.

Discrimination parameters ranged from .192 to 1.82, with 24 of 34 items having discrimination above 0.64, which is considered moderate (Baker, 2001), and difficulties ranging from -1.54 to 3.56 logits, with a 6.51 logit outlier (Q27, see Figure 1). Most items thus successfully distinguish between students with lower versus higher abilities.

As the average difficulty of the items, we used the weighted mean of difficulties (β-values), multiplying each item's β-value by the discrimination α-value of each item, so that the items that discriminated best were given higher relative importance in the calculation. The weighted mean of β-values was .580, meaning that the test was towards the difficult side for this sample of students. The spread of item difficulties in comparison with the students' abilities is shown in Figure 2.

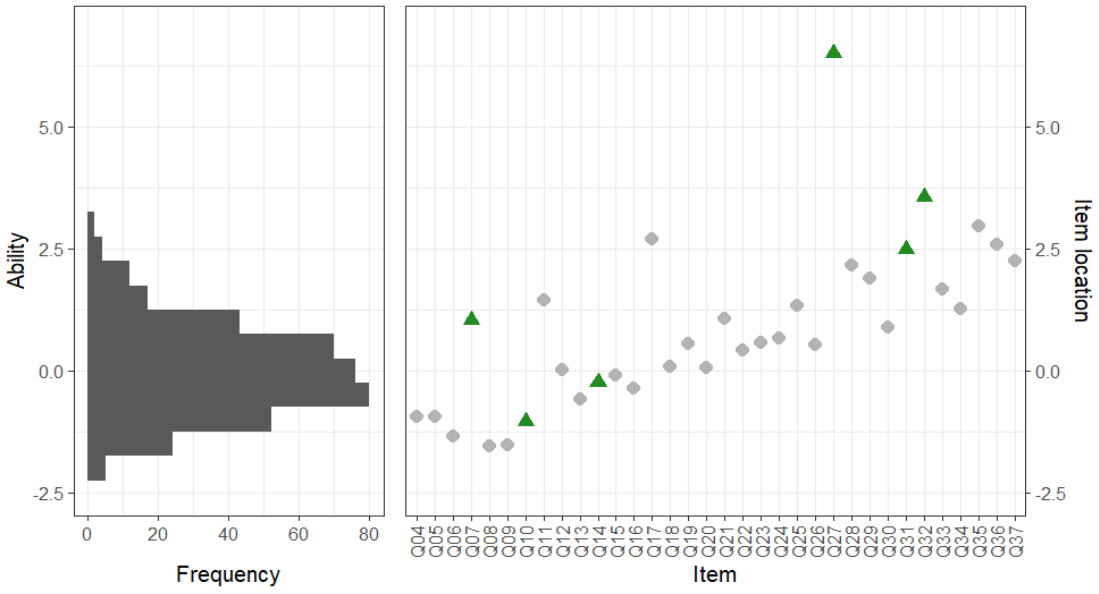

Figure 2. Respondent Algebra Ability and Item Location. Questions 7, 10, 14, 27, 31, and 32 (see Figure 1) are highlighted as triangles.

In relation to the categories and unified algebraic principles, such as $m/m = 1$, as illustrated in Figure 1, discrimination and difficulty of items in each category vary. For illustration, Table 2 presents the discrimination parameter α and item difficulty β for the sample of items presented earlier in Figure 1. While the two questions that ask about which fractions can be simplified are easier than what the average student in this sample has equal odds of answering correctly (β<0), the other four questions are more difficult. For example, question 7, asking students to simplify a^5/a^5 (a is not 0) has difficulty 1.048, meaning roughly that for students to have equal odds of answering correctly and incorrectly, they need to be one standard deviation above the average ability in this group. Questions 27 and 32 are about the simplification of fractions that involve exponents. These questions have low discrimination power (below 0.65). Keeping in mind that the poor discrimination means that the estimate of difficulty is uncertain, they are the two most difficult items.

Item:	**Q07**	**Q10**	**Q14**	**Q27**	**Q31**	**Q32**
α	0.80	0.61	0.36	0.23	0.56	0.38
β	1.05	-1.05	-0.23	6.51	2.48	3.56

Table 2. Item parameters for questions related to $\frac{m}{m} = 1$.

Reliability. Reliability here means the absence of random or algebra-irrelevant fluctuations in students' scores and is a pre-condition for our interpretation of scores as reflecting algebra knowledge. Because we conducted an IRT analysis, we chose to use the expected a posteriori (EAP) marginal reliability, ρ, which is the average reliability across the range of different abilities, weighted by the proportion of students of each ability. It is considered good when it is above .800 and very good when it is above .900. The EAP marginal reliability ρ was .847, indicating good reliability, which in turn means that there is one clear signal, or construct, captured by the scores on this test.

Construct-irrelevant variance. Construct-irrelevant factors, in this case factors not related to algebra knowledge which systematically affect scores, can be detrimental to validity. Potential construct-irrelevant variance from timing and reading demands appears minimal. Under a 60-minute limit, 95% of students finished within 50 minutes. Regarding reading, some items necessarily used brief real-world scenarios to elicit algebraic modeling. While this introduces a possible reading-comprehension component, wording was kept straightforward, and automatic translation into multiple languages was available, reducing unintended language demands. Because the post-intervention cohort will have similar demographics and the same language supports, we tentatively believe that any remaining reading demand should be stable across groups and thus unlikely to affect estimates of intervention effects.

Discussion

We have presented an algebra test whose primary purpose is in the evaluation of the Algebra Monday intervention. The algebra test was designed with prior research on

students' algebra difficulties in mind. The questions were chosen, whenever possible, among questions that had previously been studied in assessments of students' algebra skills, used on previously released Swedish National Tests, and, similarly to Blanton et al. (2019) aligned with the intervention that the test is meant to help evaluate. In this way, the test is simultaneously fit for the intended purpose and in parts similar to realistic assessments students may encounter during their schooling. A test designed to evaluate a different intervention should align with the narrow concerns of that intervention, as well as include items that more broadly capture the way the construct of interest is often represented.

The test showed good reliability with our sample of students, indicating that *something* has been measured. Because the items are designed as described above, that something is very likely algebra ability, as construed within the Algebra Monday intervention. A unidimensional structure was likewise observed, indicating that algebra knowledge, in this sample of students, is well-described as one thing, rather than multiple. The items are also spread out in difficulty and target the range of abilities in the sample of students, so it is suitable for the students we intend to use it with. For use with another group of students, reliability, dimensionality and appropriateness of difficulty should first be established for that group.

A limitation of our interpretation and use argument is that we have not yet measured students' response processes and, therefore, cannot say what knowledge or reasoning students used to solve test questions. While we designed the questions with the implicit belief that students would use knowledge of general algebraic structures to solve the structure questions, for example, it would be helpful in the future to assess whether some students may instead have used knowledge of exponent rules. Response processes should likewise be part of test validation for other algebra tests, because algebra problems often lend themselves to multiple solution methods.

A challenge for tests used in intervention studies is that they must be sensitive enough to capture the gains researchers hope to see from the intervention, yet not so difficult that control-group students become discouraged and stop trying. In this study, we cannot rule out that the overall difficulty of the test may have influenced some students' effort on later items. Further research on students' response processes could help clarify the extent to which test difficulty affects engagement and, in turn, score interpretation.

Additionally, because we believe that the algebra understanding and skills assessed by our test could be useful for learning various mathematics topics, such as coordinate geometry, the results should correlate with national test scores. They should, however, correlate less well with topics such as elementary probability or geometry, in which algebra plays a more minor role, at least in school mathematics. Future directions for validating our stated interpretation and use of this test could, therefore, include comparison with such external measures.

Some auxiliary findings also emerge from our analysis and may serve as inspiration for future research.

Questions which involve rational expressions with exponents appear especially difficult for our students, despite being eminently solvable by reasoning about exponents as repeated multiplication and the identity $m/m = 1$. Also difficult were those questions that ask for scaling by fractions, where the correct answer is given as a fraction. Based on student notes in the testing platform, some students seem to have calculated a correct answer but then chosen an answer option that has an (incorrect) answer in decimal form. Additionally, it is noteworthy that so few students in our sample, in eighth grade, answered that $a^5/a^5 = 1$.

Despite designing our test based on three didactic approaches to teaching algebra, a one-factor model fit the data well. This is in contradiction to Zhang's (2018) multi-factor approach to algebra. To our knowledge, Blanton et al. (2019) did not conduct a dimensionality analysis. Further research on dimensionality of algebra knowledge could explore dimensionality in different populations, with different conceptual frameworks, and based on exposures to different algebra curricula.

Conclusion

We have presented a 34-item algebra test that is closely aligned with the structural focus of Algebra Monday and validated its use as a group-level outcome measure for students in grade 8. The test shows good reliability, a unidimensional structure, and a reasonable spread of item difficulties, while potential construct-irrelevant influences from timing, reading load, and anomalous test-taking behaviors appear limited. At the same time, our analyses highlight areas of algebra that are particularly challenging for students, such as rational expressions involving exponents and scaling by fractions.

As Algebra Monday is implemented over a longer period, this test and subsequent IRT analyses will allow us to examine whether instruction with a strong emphasis on algebraic structure changes the relative difficulty of structurally related items, and how change in this type of algebra knowledge relates to performance on broader national assessments. Researchers or educators who wish to use this test should be mindful that dimensionality and difficulty levels may differ based on students' prior algebra experience. More generally, we hope this study contributes to wider discussion about how researcher-designed tests for intervention studies are constructed and how the interpretation and use of such instruments can be systematically validated.

Acknowledgement

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References

Baker, F. B. (2001). *The basics of item response theory* (2nd ed). ERIC Clearinghouse on Assessment and Evaluation.

Blanton, M., Stephens, A., Knuth, E., Gardiner, A. M., Isler, I., & Kim, J. S. (2015). The development of children's algebraic thinking: The impact of a comprehensive early algebra

intervention in third grade. *Journal for Research in Mathematics Education*, *46*(1), 39–87. https://doi.org/10.5951/jresematheduc.46.1.0039

Blanton, M., Stroud, R., Stephens, A., Gardiner, A. M., Stylianou, D. A., Knuth, E., Isler-Baykal, I., & Strachota, S. (2019). Does early algebra matter? The effectiveness of an early algebra intervention in grades 3 to 5. *American Educational Research Journal*, *56*(5), 1930–1972. https://doi.org/10.3102/0002831219832301

Hackenberg, A. J., & Sevinc, S. (2022). Middle school students' construction of reciprocal reasoning with unknowns. *The Journal of Mathematical Behavior*, *65*, 100929. https://doi.org/10.1016/j.jmathb.2021.100929

Helenius, O., & Ahl, L. M. (2024). A framework for analyzing long-term early algebra progression in textbook series. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.), *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education, Vol. 3* (pp. 9–16). IGPME.

Hilton, A., Hilton, G., Dole, S., & Goos, M. (2013). Development and application of a two-tier diagnostic instrument to assess middle-years students' proportional reasoning. *Mathematics Education Research Journal*, *25*(4), 523–545. https://doi.org/10.1007/s13394-013-0083-6

Kane, M. (2013). The Argument-Based Approach to Validation. *School Psychology Review*, *42*(4), 448–457. https://doi.org/10.1080/02796015.2013.12087465

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5–18). Lawrence Erlbaum Associate.

Kieran, C. (2004). Algebraic thinking in the early grades: What is it? *The Mathematics Educator*, *8*(1), 139–151.

Kieran, C. (2018a). Seeking, using, and expressing structure in numbers and numerical operations: A fundamental path to developing early algebraic thinking. In C. Kieran (Ed.), *Teaching and Learning Algebraic Thinking with 5- to 12-Year-Olds* (pp. 79–105). Springer International Publishing. https://doi.org/10.1007/978-3-319-68351-5_4

Kieran, C. (Ed.). (2018b). *Teaching and Learning Algebraic Thinking with 5- to 12-Year-Olds*. Springer International Publishing. https://doi.org/10.1007/978-3-319-68351-5

Niss, M., & Jankvist, U. T. (2013). *13 Spørgsmål fra Professoren (detektionstest 3). Materiale udleveret til matematikvejlederuddannelsen [13 Questions from the Professor (detection test 3). Material handed out at the maths counsellor programme]*.

Opsal, H., & Topphol, A. K. (2023). The problematic equal sign. *NOMAD Nordic Studies in Mathematics Education*, *28*(1–2). https://doi.org/10.7146/nomad.v28i1-2.149263

Roesken-Winter, B., Stahnke, R., Prediger, S., & Gasteiger, H. (2021). Towards a research base for implementation strategies addressing mathematics teachers and facilitators. *ZDM – Mathematics Education*, *53*(6), 1193–1206. https://doi.org/10.1007/s11858-021-01220-x

Stephens, A., Stroud, R., Strachota, S., Stylianou, D., Blanton, M., Knuth, E., & Gardiner, A. (2021). What early algebra knowledge persists 1 year after an elementary grades intervention? *Journal for Research in Mathematics Education*, *52*(3), 332–348. https://doi.org/10.5951/jresematheduc-2020-0304

Zhang, Z. (2018). Designing cognitively diagnostic assessment for algebraic content knowledge and thinking skills. *International Education Studies*, *11*(2), 106–117. https://doi.org/10.5539/ies.v11n2p106

Learning for work: A comparative study of vocational mathematics in three contexts

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This paper explores and compares vocational students' experiences of mathematics across three teaching approaches: vocationally integrated, classroom-integrated, and textbook-based. Based on 22 group interviews with 91 students in their final year, the study investigates how these approaches influence students' perceptions of effectiveness and motivation. Students in vocationally embedded settings report stronger connections to workplace practice and higher motivation, while those in traditional settings often view mathematics as disconnected. The classroom-integrated context shows mixed results. Findings suggest that students' attitudes toward mathematics remain stable over time and are shaped by the teaching approach, highlighting the value of contextualized and collaborative methods in vocational education.

Introduction

There is broad consensus among educational researchers on the need to develop teaching strategies that connect mathematics learning to students' future professions (e.g., FitzSimons, 2001; Frejd & Muhrman, 2022). Vocational mathematics education in Sweden is designed to prepare learners for employment, yet many students struggle to see its relevance and often lack motivation (Muhrman, 2016). Research suggests that embedding mathematics in realistic workplace scenarios can enhance student engagement and make the subject more meaningful (Dalby & Noyes, 2016). However, integrating authentic scenarios into vocational mathematics education can be done in several different ways (Frejd & Muhrman, 2022). Students might, for example, explore workplace mathematics by interviewing professionals, shadowing them on the job, or engaging with realistic problems in the classroom.

This paper presents new findings from a longitudinal research project investigating how mathematics teaching approaches in vocational education can better prepare students for their future careers. The project involves nine upper secondary vocational schools across Sweden, with over 1 000 students and 20 teachers. Data is collected through pre- and post-tests, surveys, interviews, and follow-ups 18 months after the interventions. This paper shares preliminary insights from *the first follow-up interviews* that took place right before the students were leaving upper secondary school for going into work or further education. The aim is to compare students' attitudes, feelings and

Mattsson, L., Eckert, A., & Kilhamn, C. (Eds.). (2026). *Talking Mathematics. Proceedings of MADIF 15*. Gothenburg University and SMDF.

experiences of working with different teaching approaches at the end of their schooling, to explore their view of the effectiveness of the teaching approaches to prepare them for their future occupation. These approaches are: Context 1 (C1), a subject-integrated approach with a connection to students' professional settings; Context 2 (C2), a subject-integrated approach in the mathematics classroom; and Context 3 (C3) a traditional textbook-based approach. The analysis is guided by the following research questions:

RQ1: *To what extent do students' experiences differ regarding the perceived effectiveness of the teaching approaches in preparing them for their future professional occupation?*

RQ2: *To what extent differ students' experiences about how the teaching approaches they have encountered motivated them to learn?*

Vocational mathematics education and teaching approaches

Vocational mathematics education is shaped by diverse educational systems and apprenticeship models, resulting in varied integrations of mathematics across formal and informal settings (Wedege, 2000). Despite these differences, research suggests that effective vocational mathematics should support three interconnected forms of knowledge: *formal mathematical understanding*, *practical application in vocational contexts*, and *reflective awareness of mathematics as a tool* (Wedege, 2000). The balance between academic and workplace-oriented mathematics often reflects broader political and social priorities, but the mainstream teaching in upper secondary education focus on formal mathematics (Muhrman, 2016; Wake, 2014). To enhance relevance and quality, scholars advocate for interdisciplinary collaboration, where mathematics teachers work closely with vocational educators and professionals (Frejd & Muhrman, 2022). This cooperation helps design authentic tasks rooted in workplace practices and tools (Sundtjønn, 2021).

Authenticity is central to vocational mathematics. It involves connecting mathematical tasks to real vocational tools, traditions, and environments (Mouwitz, 2013). Studies show that learning in vocational settings, such as training halls, can boost student motivation and engagement, and reshape classroom roles and perceptions of competence (Frejd & Muhrman, 2022; Sundtjønn, 2021).

Moreover, vocational mathematics should support identity development and self-esteem by linking mathematical learning to students' professional roles and responsibilities (Mouwitz, 2013). Finally, communication and teamwork are essential workplace skills that should be embedded in vocational mathematics through collaborative tasks that reflect real-world demands (Wake, 2014).

There exist many different types of teaching approaches for developing students' mathematical skills that the need for their future occupation, which also depend on what forms of knowledge is in focus. Teachers that focus on formal mathematics might use a *Textbook-Based-Learning* (TBL) approach, similar to ones that are used in upper secondary courses in preparing for further education. Whereas as teacher focusing on mathematical applications in vocational contexts use other teaching methods. One such

a teaching approach is *Subject-Integrated Team-Teaching* (SITT) (Frejd & Muhrman 2022), which is an approach where a mathematics teacher and a vocational subject teacher co-design and co-deliver instruction, often situated within or closely connected to students' training hall environments. This joint effort aims to bridge the gap between abstract mathematical concepts and their practical applications in vocational contexts.

Another integrated teaching approach is also based on interdisciplinary collaboration between mathematics and vocational teachers but takes place in the mathematics classroom. In this approach, the mathematics teacher is responsible for teaching, but the underlying ideas of the workplace situation to be discussed are developed in collaboration (Muhrman, 2016).

The extent to which these different approaches are effective in the short and long term is a matter of research. A previous study by Muhrman and Frejd (2025) found, by the end of students first year, a notable difference in students' attitudes to the teaching approaches of SITT and TBL. The SITT-group rated their experience more positively (mean score: 5.92 out of 10), appreciating the vocational relevance. In contrast, the TBL-group gave a lower rating (3.98), feeling the mathematics was less applicable.

Motivation to learn also differed, the SITT-group scored higher (4.93) than the TBL-group (3.87). Still, both groups identified positive aspects of their respective approaches. The TBL students valued small groups and clear lesson structure, while SITT students highlighted engaging lessons, varied workspaces, and practical tasks. Across both groups, the teacher played a key motivational role, alongside other factors like grades.

This paper will expand upon and contrast these findings by investigating students' experiences at the end of their final year of education, while also incorporating an integrated approach within the classroom.

The theoretical framework

This longitudinal research project is grounded in Self-Determination Theory (SDT) (Ryan & Deci, 2000), a framework for understanding human motivation in educational contexts. SDT distinguishes two main types of motivation: *intrinsic* and *extrinsic*. Intrinsic motivation is always internal, driven by interest or enjoyment. Extrinsic motivation, however, ranges along a continuum from external to internalized forms: *External*, *Introjected*, *Identified*, and *Integrated regulation*. External regulation refers to actions driven by rewards or the avoidance of punishment, while Introjected regulation involves internal pressures to maintain self-worth. Identified regulation occurs when individuals recognize the personal value of an activity, such as learning mathematics because it supports future career goals. Integrated regulation represents the most autonomous form of extrinsic motivation, where the behaviour is fully aligned with one's values and identity, such as professional identity. Intrinsic and internal motivation is associated with deeper engagement, persistence, and long-term learning outcomes (Taylor et al., 2014). By designing learning activities that support intrinsic and internal motivation, teachers can enhance both mathematical development and positive attitudes toward learning (Ryan & Deci, 2000).

Background information about the research study and data collection

As described in the introduction, the research project involves nine vocational schools across Sweden and over 1 000 students have participated in the study. This paper is based on a sub-study which included 22 group interviews with 91 students across three contexts: C1 (2 schools, 7 groups from 6 programmes, 29 students), C2 (4 schools, 7 groups from 4 programmes, 26 students), and C3 (3 schools, 8 groups from 7 programmes, 36 students). One school in the larger research project linked to C2 is not a part of this sample due to a scheduling conflict during work practice period.

Two of the schools have adopted the teaching approach connected to C1, which includes six vocational programmes. These schools allocate three one-hour lessons per week to Mathematics 1a. One of these lessons is designated as ”vocational mathematics”, which often takes place in the vocational classroom. These lessons are designed collaboratively by mathematics and vocational teachers and relate directly to students’ ongoing vocational tasks. These lessons may also require students to adhere to workplace dress codes. The remaining two lessons follow a more traditional, textbook-based format in the mathematics classroom.

The teaching approach related to C2 is applied at five schools and include seven programs. About three hours per week is spent on mathematics education and there is a difference between the schools how much time is used for vocational education. Some of the schools work more projects based with a few vocational tasks during the semester, whereas other have more regular teaching which they also call vocational mathematics. The mathematics teacher in C2 is responsible for the teaching and planning, but have discussions and collaboration with vocational education teachers.

The sample C3 includes groups of students from three schools (whereas one school also is found in C2) covering seven vocational programmes. The mathematics instruction is delivered over three to four hours per week, primarily through textbook-based lessons. There is no collaboration between mathematics and vocational teachers in lesson planning or delivery.

To explore our research questions, semi-structured group interviews were conducted with students in their final year of upper secondary school (Year 3, 18-19 years of age). These students had been selected in their year-1 from a pool of volunteers and earlier been interviewed twice in the project. The interview guide was organized around three topics in relation to the research questions.

The first topic focused on students’ motivation to learn mathematics. Students were asked to reflect on their overall experiences with mathematics education, particularly considering that many had not studied mathematics since completing Mathematics 1a. Questions addressed their initial associations with mathematics lessons, their attitudes toward learning mathematics, and the factors that had influenced their motivation.

The second topic explored the role of mathematics in vocational work. As students had gained more practical experience since earlier stages of the study, they were also invited to discuss whether mathematics is important in their vocational area, and to provide concrete examples of how mathematics is used in workplace tasks.

The third topic addressed how mathematics education can support vocational competence. Students were asked to evaluate their teaching approach if it had prepared them for their future profession. They also reflected on the content and structure of their mathematics education, suggesting improvements that could better align with vocational needs. Finally, they were asked whether the mathematics course in Year 1 primarily served as preparation for their profession, further studies, or other goals.

The empirical material for this study consists of transcribed interviews. The interviews were analysed by a *theoretical thematic analysis* as outlined by Braun and Clarke (2006). This approach involves identifying and interpreting patterns within the data based on predefined theoretical categories, rather than generating themes inductively. We applied *nine thematic categories*. These categories reflect both practical and psychological dimensions of learning, connected to Self-Determination Theory.

For effectiveness of the teaching approaches in preparing them for their future professional occupation, the analysis focused on four categories on how students perceived the relevance of mathematics to their vocational future. This included whether they saw a *clear connection to workplace tasks*, *felt mathematically competent*, *could provide concrete examples of workplace mathematics*, and how they viewed the balance between *formal and vocational content*.

For motivation, the analysis examined students' sense of purpose in learning mathematics, including five categories about whether they *felt it was necessary* for their future profession. It also explored their *attitudes* toward the subject itself, the *variation* in teaching methods they experienced, the role of *the teacher* in supporting engagement, and the influence of *grades and assessments*.

Results and analysis

We describe the results from each context in detail below, and summarise them in a table at the end.

Context 1, vocational mathematics tasks in professional settings

The students in C1 recalled weekly lessons referred to as 'vocational mathematics', which they experienced throughout their mathematics course. These lessons were widely seen as more engaging and meaningful than traditional instruction, primarily due to their *explicit connection* to future workplace tasks. As student **S43** noted, vocational mathematics allowed them "to test everything," referring to hands-on experiences in areas like painting, housebuilding, and tinsmithing. This integration of practical tasks with mathematical concepts contributed to a sense of authenticity and relevance.

Students expressed a strong *feeling of competence*, describing themselves as mathematically qualified for workplace situations. **S54** reflected on bakery work, stating, "I think we have won a lot… it has been good for us," linking math to success in vocational exams. Students provided *concrete examples* of workplace mathematics, such as load securing (**S46**), cable dimensions (**S50**), recipe scaling (**S53**), and sight

distance calculations (**S65**). These examples illustrated the *usability* of mathematics in vocational contexts, and all students agreed on its *necessity*.

Students **S59** and **S62**, suggested extending the course to two or three years, arguing that math is better learned alongside vocational practice. **S66** noted that supervisors during working practice were "not educated mathematics teachers," highlighting the need for more teaching. **S65** echoed this, saying, "Now that we have the answer, we should have wanted more vocational mathematics, even though it had not been that fun."

Students generally viewed the teaching approach as effective. **S53** stated, "this approach is good," though others suggested improvements in timing as previously discussed. Traditional lessons were often described as "boring" and more suited to academic study than vocational preparation.

Motivation was shaped by both internal and external factors. *Teachers* were frequently mentioned as central motivators, alongside *grades* and *attendance*. **S59** explained, vocational mathematics was motivating because "you will use it in the future." Attitudes toward mathematics varied: some found it "not fun" (**S47**), "not exciting" (**S43**), or difficult (**S52**), while others described it as "fun" (**S39**), "easy" (**S48**), and expressed interest in continuing mathematics studies (**S54**). Regardless of their views, all students recognized its importance for their future occupation. **S43** emphasized, "Yes, it [math] is important in construction and all craft occupations."

Finally, *variation* in teaching and learning activities was seen as a motivating factor. The combination of a supportive teacher and diverse instructional methods was highlighted by **S53** as particularly effective.

Context 2, vocational mathematics tasks in the mathematics classroom

In Context 2 (C2), students' experiences with vocational mathematics were less consistent and less explicitly tied to their future professions than in C1. Several students did not recall dedicated lessons on workplace mathematics. For instance, **S69** noted, "no specific mathematics focusing on our occupation," highlighting a lack of structured integration. Some students remembered isolated examples, such as estimating hair length (**S31**) or coloring (**S30**), but emphasized that vocational math differed from classroom instruction. One student explained, "we do not use multiplication when mixing colors… we just add," and another mentioned using "cheating notes" (**S27**), suggesting a disconnect between formal math and practical use.

Despite this, many C2 students felt *confident* in their mathematics skills, often describing vocational mathematics (e.g., hairdressing, animal care, IT-support) as simple. Students gave *concrete examples* like pricing and coloring (**S27**), feeding rations and vitamins (**S80**), and medication calculations (**S90**). However, IT-support students struggled to identify specific applications, referring vaguely to programming.

Students often viewed the curriculum as geared more *toward academic* progression than vocational relevance. Many felt the mathematics taught was unnecessarily complex. **S24** remarked, "It has not much to do with what we do… we could have skipped some content." Others believed that math from grades 7–9 would have sufficed. In contrast, **S81** valued mathematics 1a, especially for those who had struggled earlier.

She stressed the importance of refreshing prior knowledge and connecting it to vocational contexts: "If I start my own business, even algebra can be useful."

Suggestions for improvement included stronger links to vocational content and individualized difficulty. **S72** proposed "more tasks with specific focus to our occupation," while **S82** suggested tailoring tasks to students' skill levels.

Motivation in C2 was driven more by external factors, such as grades, attendance, and teacher support, than internal ones. A few students, like **S68**, acknowledged practical relevance of mathematics ("you need to use it"), but this view was not widespread. Attitudes varied: some enjoyed math ("I like it", **S68**), while others found it "boring" (**S28**) or "difficult" (**S83**). A recurring theme was that mathematics became enjoyable "when you understand," underscoring the importance of clarity.

Students generally agreed mathematics had some *relevance to their future* jobs, but often limited. **S27** noted tools like calculators could replace mental math, and **S28** saw mathematics as useful "such as in pricing, percent, but not mathematics as a subject, more as in economics." Unlike C1, *variation* in teaching methods was not mentioned, suggesting a more uniform and possibly less engaging experience.

Context 3, working with traditional textbook-based instruction

In C3, students primarily encountered mathematics through traditional, textbook-based instruction, with little to no integration of vocational content. None recalled consistent exposure to workplace-related mathematics. One group mentioned a single exception, a seed rate task, which they found more engaging. As **S16** noted, "[the teacher] gave us one task about seed and then it became more interesting," suggesting that even isolated vocational links could increase interest.

For most students, motivation was driven by external factors like *grades* and attendance. Comments such as "If I had not been forced to be there I would not have been there" (**S19**) and "when we were young, we learned because we wanted, but now we learn to get grades" (**S21**) reflected a lack of intrinsic engagement.

Students were generally sceptical about the *relevance* of mathematics to their future careers. Rather than citing workplace examples, they often dismissed the content as irrelevant. **S19** stated, "driving a harvester does not require knowledge about mathematics," and **S20** added, "Pythagoras' theorem and functions are not useful in the stable." Many felt the curriculum was disconnected from their occupational realities.

Several students questioned the *necessity* of the mathematics course itself, describing it as repetitive or redundant. **S19** claimed it was "just repetition of grade 9," while others argued that "grade 6 mathematics is enough to become a baker" (**S35**). **S3** summarized, "I feel that 7–9 mathematics was enough. I have never used everything we learn in upper secondary school… most things [in the workplace] are in forms where you type them in and they make all the calculations."

One student, **S9**, training to become a truck driver, offered a more nuanced view. He acknowledged that mathematics was more important than expected, noting many calculations in vocational training. However, he emphasized that these skills were

learned in vocational courses, not in the mathematics classroom, which remained focused on textbook exercises.

Students consistently described the teaching approach as *monotonous and uninspiring*. **S13** remarked, "you sit and solve tasks in a textbook, tasks that you do not understand," while others used stronger language: "F… boring" (**S17**), "a complete hell" (**S20**), and "difficult and tiring" (**S24**).

Teacher quality was seen as a key factor in shaping attitudes. **S22** noted, "if the teacher is not good, you get a negative attitude towards mathematics." When asked how to improve the course, students unanimously called for more workplace-relevant content. As **S16** put it, "You get more motivated if you know that you can use it in your working life, especially if you also get to use it in practice."

Summary of analysis

The results of our analysis of the interviews are summarized in Table 1 below.

Table 1. Summary of the analysis in relation to RQ1 and RQ2, split by context (C1, C2 and C3). Y = Yes, N = No, F =Formal, V= Vocational.

RQ1,"Effectiveness"	**C1**	**C2**	**C3**	**RQ2, "Motivation"**	**C1**	**C2**	**C3**
Clear connection to workplace tasks	Y	Y/N	N	Felt it was necessary	Y	Y/N	N
A feeling of competence	Y	Y	Y/N	Attitude towards math	Mix	Mix	Neg
Provide examples of workplace math	Y	Y/N	N	Variation	Y	N	N
Formal/ Vocational focus	F/V	F/V	F	Teacher	Y	Y	Y
				Grade and assessment	Y	Y	Y

Table 1 indicates whether most students in the different groups within each context experienced the thematic categories (Yes), did not experience them (No), or if the analysis showed no clear distinction (Yes/No).

Discussion and conclusion

This study explored how different teaching approaches in vocational mathematics, (C1) subject-integrated teaching in vocational settings, (C2) subject-integrated teaching in the mathematics classroom, and (C3) traditional textbook-based instruction, influence students' perceptions of effectiveness and motivation. The findings, as displayed in table 1, reveal differences between the contexts.

Students in C1 consistently described their mathematics education as explicitly connected to their future professions. They provided concrete examples of workplace mathematics and expressed a strong sense of competence, suggesting that the integration of mathematics into vocational settings enhances both relevance and confidence. In contrast, students in C3 struggled to identify any vocational relevance, often viewing the mathematics as disconnected and unnecessary. C2 students fell somewhere in between, with some vocational connections but less consistency and clarity.

These results align with previous research (e.g., Frejd & Muhrman, 2022; FitzSimons, 2001; Dalby & Noyes, 2016), which emphasizes the importance of authenticity and contextualization in vocational mathematics. The findings suggest that students' perceptions of mathematics as a vocational tool are shaped not only by the content but also by the context in which it is taught.

The results of this study are in line with those found in Muhrman and Frejd (2025), where students filled in questionnaires at the end of their mathematics course in Year 1. In both studies, students exposed to the C1 approach reported more positive attitudes and stronger perceptions of relevance compared to those taught through textbook-based instruction. The similarity in findings suggests that students' attitudes and perceptions of mathematics tend to remain stable throughout their education. This implies that the teaching approach adopted early in their studies may have a lasting impact on how students engage with and value mathematics in vocational contexts.

The inclusion of a third context (C2) in this study adds nuance to the discussion. While not as clearly vocationally embedded as C1, C2 still offered some integration and collaboration between mathematics and vocational teachers. Students in C2 expressed mixed views, some saw relevance and usability, while others felt the mathematics was too academic or disconnected. This intermediate position highlights the importance of not only integrating vocational content but also ensuring that the integration is explicit, consistent, and meaningful to students.

There were also pronounced motivational differences. Students in C1 reported both intrinsic and extrinsic motivators. Intrinsic, internal motivation was characterised by an interest in mathematics itself. Extrinsic, internal motivation was described as driven by the relevance of mathematics to their future work. In C2, there was no intrinsic motivation, but there was extrinsic motivation driven by both internal and external factors. In C3, motivation was largely extrinsic and externally driven, with students citing grades and attendance as the main reasons for their engagement.

Across all groups, the results show that internal motivation is affected by variations in teaching methods and supportive teachers (cf. Ryan & Deci, 2000). In addition, students from all groups expressed a desire for stronger connections between mathematics and vocational practice. This shared sentiment suggests that vocational relevance is not only beneficial but necessary for meaningful mathematics education in vocational programmes. Based on these findings, it may be worth considering the development of a dedicated course in vocational mathematics, tailored to the specific needs and practices of different professions.

However, the study has limitations. One school in C2 was unable to participate in the follow-up interviews, which may have affected the representativeness of the data. Additionally, the analysis is based on qualitative coding without frequency counts, as the aim was to explore perceptions rather than quantify responses. The sample, while diverse, is still a subset of the larger intervention and should be interpreted accordingly.

Looking ahead, the next phase of the project will include a second round of follow-up interviews and further analysis of long-term outcomes.

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References

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*, 77–101.

Dalby, D., & Noyes, A. (2016). Locating mathematics within post-16 vocational education in England. *Journal of Vocational Education & Training*, *68*(1), 70–86. https://doi.org/10.1080/13636820.2015.1110828

FitzSimons, G. E. (2001). Is there a role for mathematical disciplinarity in productive learning for the workplace? *Studies in Continuing Education*, *23*(2), 261–276. https://doi.org/10.1080/01580370120102000

Frejd, P., & Muhrman, K. (2022). Is the mathematics classroom a suitable learning space for making workplace mathematics visible? – An analysis of a subject integrated team-teaching approach applied in different learning spaces. *Journal of Vocational Education & Training*, *74*(2), 333–351. https://doi.org/10.1080/13636820.2020.1760337

Mouwitz, L. (2013). Rationality of practice and mathematical modelling: Connections, conflicts, and codifications, In G. A. Stillman, G. Kaiser, W. Blum, & J. P. Brown (Eds.), *Teaching mathematical modelling: Connecting to research and practice* (pp. 563–572). Springer.

Muhrman, K. (2016). *Inget klöver utan matematik: En studie av matematik i yrkesutbildning och yrkesliv* (Publication No. 194) [Doctoral dissertation, Linköpings universitet]. Diva. https://www.diva-portal.org/smash/get/diva2:900159/FULLTEXT01.pdf

Muhrman, K., & Frejd, P. (2025). Subject integrated team teaching vs. textbook learning: A comparative study on students' experiences in vocational mathematics education. *To appear in the Proceedings of the Fourteen Congress of the European Society for Research in Mathematics Education February, 4-8, 2025. Bozen-Bolzano, Italy*.

Ryan, R. M., & Deci, E. L. (2000). Self-determination theory and the facilitation of intrinsic motivation, social development, and well-being. *American Psychologist*, *55*(1), 68–78. https://doi.org/10.1037/0003-066X.55.1.68

Sundtjønn, T. P. (2021). *Opportunities and challenges when students work with vocationally connected mathematics tasks* (Publication No. 309) [Doctoral thesis, University of Agder]. CRIStin. https://hdl.handle.net/11250/2727570

Taylor, G., Jungert, T., Mageau, G. A., Schattke, K., Dedic, H., Rosenfield, S., & Koestner, R. (2014). A self-determination theory approach to predicting school achievement over time: The unique role of intrinsic motivation. *Contemporary Educational Psychology*, *39*(4), 342–358. https://doi.org/10.1016/j.cedpsych.2014.08.002

Wake, G. (2014). Making sense of and with mathematics: The interface between academic mathematics and mathematics in practice. *Educational Studies in Mathematics*, *86*(2), 271–290. https://doi.org/10.1007/s10649-014-9540-8

Wedege, T. (2000). Mathematics knowledge as a vocational qualification. In A. Bessot & J. Ridgway (Eds.), *Education for mathematics in the workplace* (pp. 127–136). Kluwer.

Eliciting cultural awareness of Catalan, Swedish, and Turkish prospective teachers using representative solutions to a Fermi problem

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This study aims to elicit the cultural aspects that 18 Catalan, 16 Swedish, and 19 Turkish prospective teachers (PTs) noticed when examining their counterparts' written solutions to a Fermi problem. The paper focuses on the PTs' identification of so-called representative solutions to a Fermi problem. The results revealed that PTs could readily notice the cultural markers from their own culture and, to a lesser extent, from other cultures, thus demonstrating basic and advanced cultural awareness. Furthermore, the PTs noticed culturally induced behaviours from the solutions, which have implications in mathematics education regarding achieving curricula goals related to "international perspectives". This study also illustrates the merit of analysing peers' solutions, which carry rich potential for modelling and problem solving in mathematics classrooms.

Introduction

Cultural aspects in education are important, and as a result, many national curricula incorporate learning goals focusing on fostering and raising students' cultural awareness (Baker, 2011). In the Swedish context for example, both Lgr22 and Gy25 emphasise that "*An international perspective* is important in order to see one's own reality in a global context and to create international solidarity…" (Lgr22, 2022; Gy25, italics in original and our translation), and further that "[T]he school shall be responsible for ensuring that, after completing primary school, every pupil … is able to interact with other people based on knowledge of similarities and differences in living conditions, culture, language, religion and history" (Lgr22, 2022, our translation), and to "prepare pupils for a society with increasingly close contacts across national and cultural boundaries…." (Gy25, our translation) so that "every pupil… is able to interact with other people based on respect for differences in living conditions, culture, language, religion and history." (Gy25, our translation).

It can be challenging to realise these general ambitions in the curricula in the mathematics classroom, however, engaging in mathematical modelling is a fruitful opportunity for problem solvers at all ages that connect and extend the links not only between mathematics and the real world but also between mathematics and culture. The connection between modelling and culture is emphasized in the ethnomodelling perspective, which stresses the recognition of cultural aspects in a local context and establishes links to related global perspectives through the process of *glocalization* (Orey & Rosa, 2021; Rosa & Orey, 2013). Building on this perspective, we recognized

Mattsson, L., Eckert, A., & Kilhamn, C. (Eds.). (2026). *Talking Mathematics. Proceedings of MADIF 15*. Gothenburg University and SMDF.

the potential of modelling to reveal cultural elements and launched a collaborative research project called *FerMICA – Fermi Modelling for Intercultural Awareness* (Ärlebäck et al., 2024a) involving PTs from four different universities in four different cities around the world: Ankara (Turkiye), Barcelona (Spain), Linköping (Sweden) and Valencia (Spain). As the project's name suggests, we use Fermi problems as compact yet potentially rich modelling situations (Ärlebäck & Albarracín, 2019) with the aim to investigate and develop intercultural awareness by exploring the cultural aspects involved in solutions to Fermi problems, as well as the cultural similarities and differences arose from the comparison of solutions that originated from different cultural settings. Through this work, we aim to explore whether FerMICA project can serve as a means to help realise the overarching curricular ambitions (cf. Lgr22 and Gy25) within the mathematics classroom.

As part of the FerMICA project, this study explored this idea in the context of teacher education and focused on the stage of a teaching sequence where prospective teachers (PTs) examined so-called *representative solutions* (cf. Albarracín et al., 2025). The PTs read and discussed generic solutions that represented the approaches of PTs from four different cultural contexts (i.e. representative solutions) to a Fermi Problem. Thus, the objective of this study is to investigate and identify the cultural aspects noticed by PTs in terms of *cultural markers* when examining the representative solutions of their counterparts in Ankara, Barcelona, Linköping, and Valencia. Specifically, the following research questions were addressed:

1. To what extent could Catalan, Swedish, and Turkish prospective teachers identify a solution to a Fermi problem representative of their own culture?
2. To what extent could Catalan, Swedish, and Turkish prospective teachers identify other cultures' representative solutions to the Fermi problem?
3. What cultural markers did the prospective teachers notice in their counterparts' Fermi problem solutions in four different sites?

Theoretical Considerations

Mathematical Modelling with Fermi Problems

According to the perspective of Lesh and Doerr (2003), models are conceptual systems used to describe, explain, organize, or predict real-life situations. The modelling process involves making sense of the real-world context and meaning making about the context, which therefore demands the formulation of contextual assumptions and interpretations. Their perspective introduces so-called Model-Eliciting Activities (MEAs) based on six design principles: reality (meaningfulness) principle, model-development principle, model-documentation principle, self-assessment principle, model-shareability and reusability principle, and effective prototype (Lesh et al., 2000). While these principles demonstrate the cyclical and iterative nature of the modelling process, developing qualified MEAs is challenging (Lesh & Doerr, 2003). Nevertheless, in this context, Fermi problems are considered relatively simple, compact (Sriraman & Lesh, 2006) and well-aligned with MEAs (Ärlebäck, 2009; Ärlebäck & Albarracín, 2019).

Fermi problems are "open, non-standard problems requiring the students to make assumptions about the problem situation and estimate relevant quantities before

engaging in, often, simple calculations" (Ärlebäck, 2009, pp. 331–332). Traditionally, solving a Fermi problem involves making sense of a real-life or realistic situation and developing a strategy for generating reasonable estimates based on legitimate assumptions. Thus, solutions to Fermi problems rely on estimation-based thinking, or what has been termed *model-based estimation* (Ärlebäck et al., 2024b). In other words, estimation is not merely based on assumptions, but rather on a model that strategically uses those assumptions to mathematise the situation at hand (Ärlebäck et al., 2024b). Solving Fermi problems therefore involves interpretation, mathematisation, and reinterpretation to approach real-world situations strategically. As a result, Fermi problems provide opportunities to consider cultural and intercultural aspects, conditions, and constraints (Ärlebäck et al., 2024a). Building on this potential, the FerMICA project uses Fermi problems to elicit and investigate the cultural perspectives that become visible and are incorporated into PTs' solutions and models.

Cultural Awareness

One view of culture in education encompasses shared human activities involving interaction based on social norms. Therefore, communication plays a crucial role in developing a shared understanding within a community (Stenhouse, 1967). Another view is that culture is defined as the surrounding environment, such as school or classroom culture, where shared norms and interaction practices are established among pupils (Cobb & Yackel, 1996; Nickson, 1994).

In the FerMICA project, we primarily adopt the first view and attend to the problem solvers' shared beliefs, attitudes, values, goals, and practices within their country, city, and university. Although culture is a shared understanding, it is not static; rather, it gradually evolves. We further conjectured that cultural aspects could be elicited from the work of problem solvers because "a conscious understanding of the role culturally based forms, practices, and frames of understanding can have in intercultural communication, and [*demands*] an ability to put these conceptions into practice in a flexible and context specific manner in real time communication" (Baker, 2011, p. 66). Furthermore, Baker scrutinised cultural awareness in three hierarchical levels: *Level 1–Basic cultural awareness; Level 2–Advanced cultural awareness; and Level 3–intercultural awareness*. Although Baker's conceptualization of the levels of cultural awareness originated in English language education research, it effectively frames the layers of shared understanding about one's own and others' cultures. Level 1 entails an understanding of the role of one's own culturally shared beliefs, values, and attitudes in interpreting situations, as well as recognizing others' culturally induced behaviours compared to one's own culture. Level 2 involves appreciating one's own and other cultures, understanding the relative and provisional nature of cultural norms, and developing respect for alternative perspectives. Lastly, level 3 involves interacting with another culture and developing intercultural communication that transcends cultural stereotypes and biases. We used Baker's (2011) levels of cultural awareness as one of the theoretical standpoints of our research.

Building on these perspectives, we propose the notion of *cultural markers* to refer to observable features, practices, or references that signal underlying cultural norms, values, or assumptions within educational settings. These markers may include language

use, interaction styles, problem-solving strategies, or contextual references that reflect the cultural background of learners and teachers. Introducing this concept allows us to analytically trace how culture becomes visible and influential in classroom practices and modelling activities.

Methods

In this paper, we focused only on the written data from three sites of the FerMICA project (see Table 1). Those groups of PTs were relatively similar in the sense that they were in either the third or last year of their teacher education program and so completed most of their mathematics and/or mathematics education studies. However, they were different in the sense that completing their teacher education programs, the PTs would qualify to teach mathematics at different grade levels (see Table 1).

Table 1. Overview and description of participants in the study

City, Country	Description of the participants	Number of participants
Ankara, Türkiye	Middle school (Grades 5-8) PTs	19
Linköping, Sweden	Secondary school (Grades 7-12) PTs	16
Barcelona, Spain	Primary school (Grades 7-12) PTs	18

In data collection, we used a Fermi problem relatable and relevant to all of these participants' cultural settings (see Figure 1).

The Tea & Coffee Problem: In a middle or primary school in Spain/Sweden/Turkiye, teachers often have their refreshments including tea and coffee in the teacher's room. Make a realistic recommendation to the school administrator who has to plan the budget about how much tea and coffee would be sufficient to last until the end of the school year?

Figure 1. The Fermi problem *The Tea and Coffee Problem* used in this study

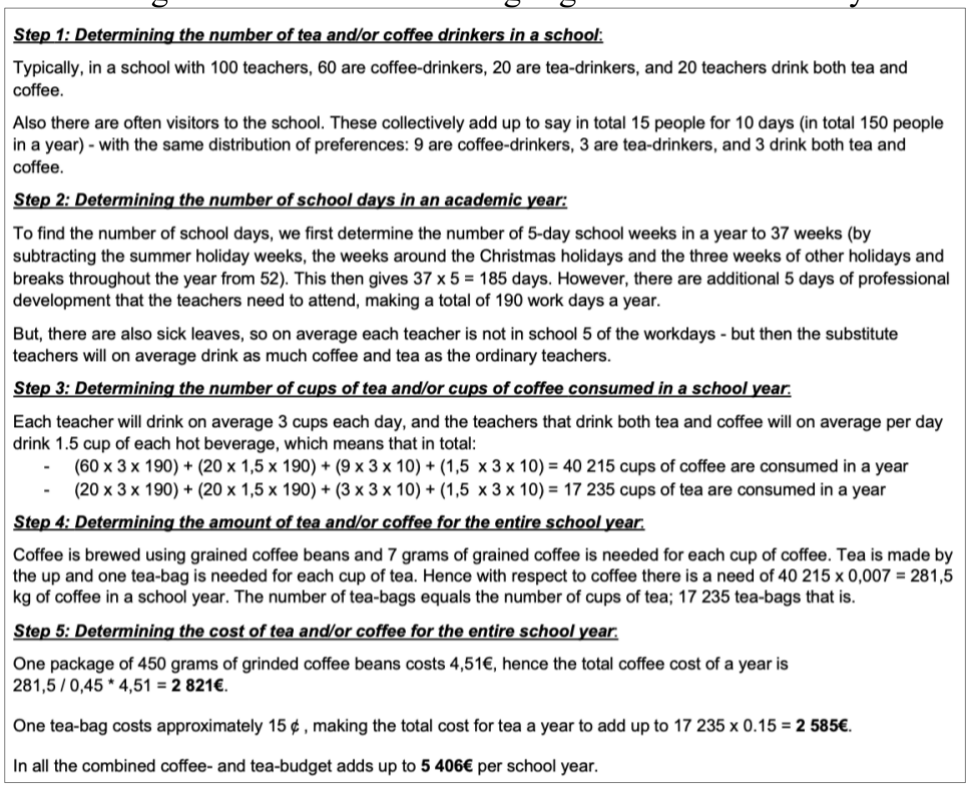
Throughout the FerMICA project, the PTs engaged in a teaching sequence entailing various activities based on The Tea and Coffee Problem. Table 2 provides an overview of the activities in the teaching sequence.

Table 2. Sequences of the activities implemented as part of the FerMICA project

Main Activities	Sub-activities
1. Problem Solving	1.1. Working individually on developing a solution plan (not a solution) for the Tea and Coffee Problem 1.2. Working in pairs to discuss the solution plans 1.3. Working in pairs to develop a solution and write a solution report
2. Problem Posing	2.1. Working individually on posing structurally similar problem but in a different context
3. Analysis of Representative Solutions	3.1. Working in pairs to examine four representative solutions, each of which demonstrates a generic solution of the prospective teachers in four different cities of the world 3.2. Working either individually or in pairs to report on the similarities and differences noticed in four representative solutions

3.3. Working either individually or in pairs to match the representative solutions with the locations they demonstrate and write an explanation for the reasons of their matching trend

The present study focused solely on the third activity: PTs' examination of the representative solutions from four sites (i.e., Ankara, Barcelona, Linköping, and Valencia) and matching representative solutions with the cultural sites based on the cultural markers they identified. To develop the representative solutions, we analysed the data from the problem-solving activity and identified commonly used ways of thinking and cultural assumptions made to solve the Fermi Problem (for more details see Albarracín et al., 2025). Figure 2 shows the sample representative solution of the Linköping site, noting that euros was used as a currency in all representative solutions not to reveal the origins of the sites from the get-go in the third activity.



Step 1: Determining the number of tea and/or coffee drinkers in a school:

Typically, in a school with 100 teachers, 60 are coffee-drinkers, 20 are tea-drinkers, and 20 teachers drink both tea and coffee.

Also there are often visitors to the school. These collectively add up to say in total 15 people for 10 days (in total 150 people in a year) - with the same distribution of preferences: 9 are coffee-drinkers, 3 are tea-drinkers, and 3 drink both tea and coffee.

Step 2: Determining the number of school days in an academic year:

To find the number of school days, we first determine the number of 5-day school weeks in a year to 37 weeks (by subtracting the summer holiday weeks, the weeks around the Christmas holidays and the three weeks of other holidays and breaks throughout the year from 52). This then gives 37 x 5 = 185 days. However, there are additional 5 days of professional development that the teachers need to attend, making a total of 190 work days a year.

But, there are also sick leaves, so on average each teacher is not in school 5 of the workdays - but then the substitute teachers will on average drink as much coffee and tea as the ordinary teachers.

Step 3: Determining the number of cups of tea and/or cups of coffee consumed in a school year:

Each teacher will drink on average 3 cups each day, and the teachers that drink both tea and coffee will on average per day drink 1.5 cup of each hot beverage, which means that in total:

- (60 x 3 x 190) + (20 x 1,5 x 190) + (9 x 3 x 10) + (1,5 x 3 x 10) = 40 215 cups of coffee are consumed in a year
- (20 x 3 x 190) + (20 x 1,5 x 190) + (3 x 3 x 10) + (1,5 x 3 x 10) = 17 235 cups of tea are consumed in a year

Step 4: Determining the amount of tea and/or coffee for the entire school year:

Coffee is brewed using grained coffee beans and 7 grams of grained coffee is needed for each cup of coffee. Tea is made by the up and one tea-bag is needed for each cup of tea. Hence with respect to coffee there is a need of 40 215 x 0,007 = 281,5 kg of coffee in a school year. The number of tea-bags equals the number of cups of tea; 17 235 tea-bags that is.

Step 5: Determining the cost of tea and/or coffee for the entire school year:

One package of 450 grams of grinded coffee beans costs 4,51€, hence the total coffee cost of a year is 281,5 / 0,45 * 4,51 = **2 821€**.

One tea-bag costs approximately 15 ¢ , making the total cost for tea a year to add up to 17 235 x 0.15 = **2 585€**.

In all the combined coffee- and tea-budget adds up to **5 406€** per school year.

Figure 2. An example of a representative solution

The data collection procedure was parallel at the three sites, except a minor difference in data recording. In Linköping and Barcelona, the PTs examined, matched, and reported on their rationale for the matching in pairs. In contrast, the PTs in Ankara also examined the representative solutions in pairs but reported their matching and explanations individually.

For data analysis, we quantified correct and incorrect matches of representative solutions, employing open coding (Lincoln & Guba, 1985) to identify cultural aspects the PTs notice in the representative solutions. We triangulated the qualitative interpretations within the group of co-authors and recorded the audit trails of the direct

links between the codes and the raw data, which were made available to other researchers to verify the credibility of the findings (Lincoln & Guba, 1985).

Findings

We present our findings in two subsections. While the first one answers the first research question, the second addresses the second and third research questions.

Prospective teachers' noticing the markers of their own culture: Indicators of basic cultural awareness

First, we explored the cases in which Catalan, Swedish, and Turkish PTs correctly or incorrectly identified their own culture's solution to the Tea and Coffee Problem. Figure 3a presents this exploration, only focusing on groups' identification of their own culture, no matter if the rest of the matching is correct or incorrect; therefore, the percentages add up to 100% for each group. As seen in the figure, approximately 15% of the PTs in Barcelona and Linköping had difficulty in identifying their own culture's solution, as did 10% of the PTs in Ankara. Yet, at all three sites, the majority of PTs recognized the cultural elements of their respective solutions.

When examining their explanations, we found that the PTs had noticed several cultural markers in the representative solutions (see the coloured diagonal cells in Table 3). For example, Turkish PTs noticed the "1 kg package of tea", which reflects a culturally specific way of brewing tea in Türkiye, where loose-tea leaf is commonly used instead of teabags. Similarly, Turkish coffee is prepared in a special pot, which differs from the methods used for filter coffee, and the PTs noted that the solution did not involve filtered coffee. They also noticed that the administrators and guests were included in the solution, which is another cultural marker, as it is customary for administrators or teachers to offer tea to guests at school. Likewise, the Swedish PTs noticed familiar prices, a reasonable number of cups of coffee per person per day, and the use of ground coffee instead of coffee capsules. These elements also indicate cultural markers related to Swedish schools and broader societal norms. Finally, the Catalan PTs noticed the number of teachers and the length of internships in Barcelona schools, which are specific to their school culture.

Prospective teachers' noticing the markers of other cultures: Indicators of advanced cultural awareness

When the PTs' matching patterns of the representative solutions were explored in more detail, we found that approximately 40% of Swedish and Turkish PTs identified the representative solutions of all other cultures correctly, while 83% of Catalan PTs identified them correctly (see Figure 3b). The percentages in Figure 3b do not necessarily add up to 100% because the focus was only the correct matching cases.

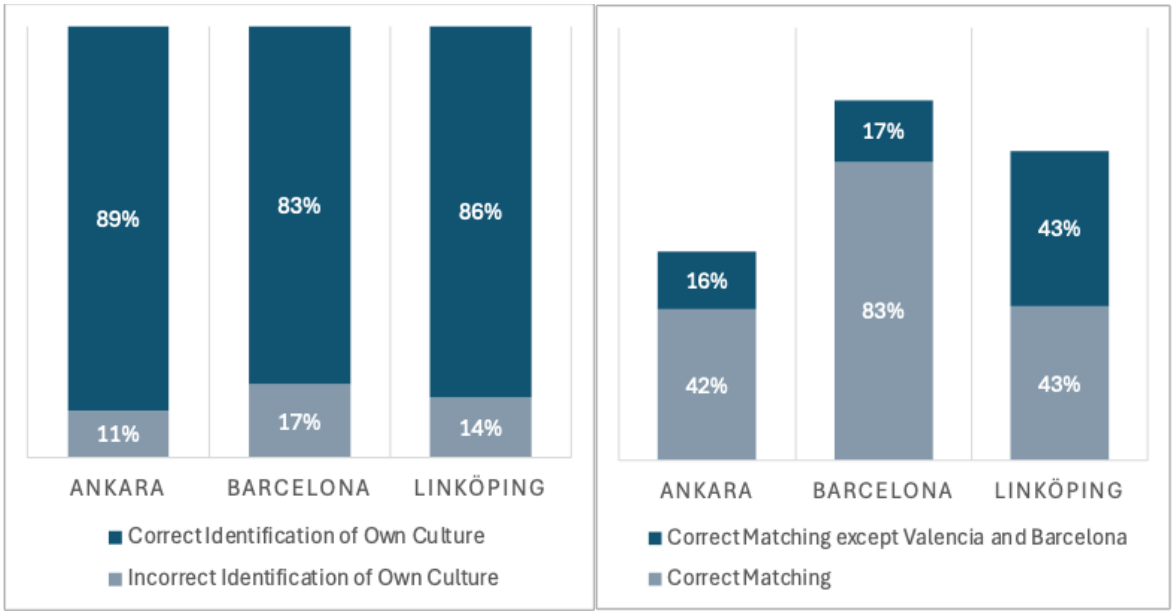


Figure 3. PTs' representative solution identification of (a) their own culture and (b) other cultures

As seen in Figure 3b, there are also partially correct matching in the sense that the representative solutions from Ankara and Linköping were identified correctly but the solutions from Barcelona and Valencia were mismatched although recognized to be from the same country (Spain). 16% of Turkish PTs and 17% of Catalan PTs mismatched the representative solutions from Valencia and Barcelona. However, 43% of Swedish PTs did so, although they noticed and reported that both solutions took the same solution approach. The Catalan, Swedish, and Turkish PTs who mismatched the representative solutions from Valencia and Barcelona noticed that both solutions had a similar school structure. While Turkish PTs both noticed teabags and coffee capsules as cultural markers, Swedish PTs mentioned only coffee capsules as a common feature of these solutions.

Moreover, we observed that the Catalan and Swedish PTs both noticed the tea culture in Türkiye and correctly identified the representative solution that included more tea than coffee (see Table 3). It is important to note that these cultural markers were elicited from the written explanations of the PTs. For the representative solutions from Linköping, the Catalan and Turkish PTs identified cultural markers such as the use of coffee beans and grounded coffee, high prices, and guest consideration. In the case of Barcelona, both the Swedish and Turkish PTs recognized the use of coffee capsules. Furthermore, the Turkish PTs compared the representative solutions from Barcelona and Valencia, arguing that Valencia has fewer teachers and a lower budget, Barcelona has more teachers and a higher budget. Turkish PTs also noted that the Barcelona solution did not consider guests, indicating another noticed cultural marker.

Table 3. Cultural markers noticed by PTs in the representative solutions

	Cultural markers noticed in the representative solution of			
	Ankara	**Linköping**	**Barcelona**	**Valencia**
Turkish PSTs	● more tea consumption ● variety and familiarity of brands ● 1 kg packages of tea for tea brewing, instead of tea bags ● both tea and coffee consumption ● no filtered coffee ● considering administrators, guests, and other school staff	● more coffee than tea ● grounded coffee and coffee beans ● high budget (high tea and coffee prices) ● guests counted ● last choice left	● tea bags and coffee capsules ● guests not counted ● more diverse community, possibility of less religious courses ● Relative to Valencia: ○ more population ○ more teachers ○ higher budget	● tea bags and coffee capsules ● religious courses ● guests not counted ● emphasis on pedagogies ● Relative to Barcelona: ○ less population ○ fewer teachers ○ lower budget
Swedish PSTs	● more tea consumption (tea culture)	● grounded coffee and coffee beans, not capsules ● 3 cups of coffee per day is the average for Swedish people ● prices are familiar ● holidays are considered	● coffee capsules ● last choice left	● cappucino likers
Catalan PSTs	● more tea consumption (tea culture) ● low prices ● different school schedule	● School structure (higher teacher quantity) ● coffee beans, not capsules ● visiting teachers and other staff ● high prices ● professional development	● More coffee consumption ● number of teachers ● length of internships	● organization of Spanish school system ● familiar school schedule ● holly week considered ● last choice left

Discussion and Conclusion

The findings showed that the Catalan, Swedish, and Turkish PTs could readily identify cultural markers from their own context, although a small percentage failed to do so. This indicates basic cultural awareness, which remains important because the PTs recognised that the solutions to the Fermi problem were embedded with cultural markers. This observation aligns with the ethnomodelling perspective (Orey & Rosa, 2021). Although the PTs did not consciously consider these cultural markers while solving the problem, the mathematical modelling process enabled the interpretation of culturally induced behaviours, such as tea and coffee drinking habits in different countries, and whether tea or coffee are routinely offered to the guests visiting schools. This suggests that a possible approach in the mathematics classroom to address the goals related to "international perspectives" emphasized in curricula (cf. Lgr22 and Gy25), is to engage in comparing and contrasting representative solutions of Fermi problems.

All PTs could identify solutions representative of other cultures to some extent. For the Swedish and Turkish PTs, this percentage was nearly half that of their basic cultural awareness performance. Catalan PTs performed twice as well as their Swedish and Turkish counterparts, perhaps because Barcelona is a more metropolitan city that provides opportunities to interact with people from different cultures. For mathematics education this indicates, that for the local sites that do not have the advantage of encountering various cultures, then mathematical tasks may offer such possibilities.

Although we do not have direct evidence that the PTs acknowledged the provisional nature of culture, their explanations for each representative solution suggested that they considered cultural markers to be relative (cf. Baker, 2011). Furthermore, different solutions to the Fermi problem entailed distinct cultural assumptions and constraints that influenced the solution (Ärlebäck et al., 2024a). This was possible through the interpretation and mathematisation of the situation during the problem-solving stage of the FerMICA project. Moreover, the analysis of the representative solutions stage, the focal point of this paper, required a reinterpreting of the situation. This reinterpretation enabled the identification of the cultural markers involved in both the context and the solution process. Specifically, PTs engaged in hypothetical intercultural communication with their peers about solutions developed by their Catalan, Spanish, Swedish, and Turkish counterparts. This interaction represents an important step in extending advanced cultural awareness to intercultural awareness. Therefore, we find this experience promising for developing the intercultural awareness of future teachers in the global educational context of the 21st century.

We also acknowledge several limitations of this study, such as the small number of participants involved in each research site and the diversity of their educational backgrounds. Although our Fermi problem addressed a global issue, the mathematics and modelling courses that PTs took in their teacher education programs, as well as their familiarity with Fermi problems, were factors that we could not control. For example, the Catalan PTs were more familiar with Fermi problems than the Swedish PTs, while the Turkish PTs had no prior experience with Fermi problems. Additionally, our analysis relied solely on the written responses of the PTs, which were limited by their self-expression skills and the time they devoted to reporting after the peer analysis of the representative solutions. Therefore, we suggest a more thorough data collection session supported by audio and/or video records of peer discussions.

As a final remark, we emphasise that mathematical modelling is not only the mathematisation of real-life situations but also the mathematisation of cultural situations. In this regard, our study shows that the development and peer analysis of representative solutions offer a valuable opportunity to implicit elicit cultural markers that are embedded and manifested in, and come to the fore in, the modelling process. Lastly, in the context of Fermi problems, this study highlighted the value of analysing the solutions of peers alongside the problem-solving experiences. By engaging in the analysis of others' solutions students can be prompted to attend to (inter)cultural-, contextual-, interdisciplinary-, mathematical-, and creative-dimension of the work. This in turn, can support students in developing awareness across these multiple dimensions of modelling and problem solving in mathematics classrooms.

References

Albarracín, L., Bergman Ärlebäck, J., Sevinc, S., Irene, F. (2025). Developing local representative solutions to fermi problems. In Ikeda, T., Saeki, A., Geiger, V., Kaiser, G. (eds) I*nternational Horizons in Mathematics Modelling Education. International Perspectives on the Teaching and Learning of Mathematical Modelling.* Springer, Cham. https://doi.org/10.1007/978-3-031-53533-8_7

Ärlebäck, J. B. (2009). On the use of realistic Fermi problems for introducing mathematical modelling in school. *The Montana Mathematics Enthusiast, 6*(3), 331–364.

Ärlebäck, J. B., & Albarracín, L. (2019). The use and potential of Fermi problems in the STEM disciplines to support the development of 21st century competencies. *ZDM Mathematics Education, 51*(7), 979–990. https://doi.org/10.1007/s11858-019-01075-3

Ärlebäck, J. B., Albarracín, L., Orey, D. C., Rosa, M., & Sevinc, S. (2024a). Exploring the potential of using Fermi problems to elicit and develop cultural aspects in modelling processes. In V. Geiger, G. Kaiser, & H.-S. Siller (Eds.), *Researching mathematical modelling education in disruptive times* (pp. 525–535). Springer.

Ärlebäck, J. B., Albarracín, L., & Sevinc, S. (2024b). Model-based estimation: Stressing the role of estimation in STEM projects. Paper presented at *the 15th International Congress on Mathematical Education,* Sydney.

Baker, W. (2011). From cultural awareness to intercultural awareness: Culture in ELT. *ETL Journal, 66*(1), 62–70. https:doi.org/10.1093/elt/ccr017

Cobb, P. & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31,* 175-90.

Läroplan för grundskolan, förskoleklassen och fritidshemmet [Lgr22]. (2022). Skolverket. https://www.skolverket.se/undervisning/grundskolan/laroplan-lgr22-for-grundskolan-samt-for-forskoleklassen-och-fritidshemmet

Läroplan för gymnasieskolan [Gy25]. (2025). Skolverket. https://www.skolverket.se/undervisning/gymnasieskolan/laroplan-gy25-for-gymnasieskolan

Lesh, R. A., & Doerr, H. M. (Eds.). (2003). *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching*. L. Erlbaum.

Lesh, R. A., Hoover, M., Hole, B., Kelly, A. E., & Post, T. (2000). Principles for developing thought revealing activities for students and teachers. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 591–645). L. Erlbaum.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic Inquiry.* Beverly Hills, CA: Sage Publications.

Nikson, M. (1994). The culture of the mathematics classroom: An unknown quantity? In S. Lerman (Ed.), *Cultural perspectives on the mathematics classroom* (pp. 7-35). Kluwer.

Orey, D. C., & Rosa, M. (2021). Ethnomodelling as a glocalization process of mathematical practices through cultural dynamism. *The Mathematics Enthusiast, 18*(3), 439–468.

Rosa, M., & Orey, D. C. (2013). The mathematics of the curves on the wall of the colégio arquidiocesano and its mathematical models: a case for ethnomodeling. *Journal of Mathematical Modelling and Application, 1*(8), 42–62.

Sriraman, B., & Lesh, R. A. (2006). Modeling conceptions revisited. *ZDM, 38*(3), 247–254.

Stenhouse, L. (1967). *Culture and education.* Weybright & Talley.

Investigating students' activities and approaches when solving Fermi problems using Fermi Graphs

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This paper explores how a relatively new analytical tool called Fermi Graphs can be used to characterize and compare students' approaches to solving Fermi problems. Drawing on data from two groups of Swedish upper secondary students working on two Fermi problems, we examine the distribution and sequencing of students' activities throughout the solution process. The analysis shows that while both groups engaged in a wide range of activities, most frequently proposing assumptions and making guesstimations, their approaches differed structurally. One group worked sequentially through sub-problems, while the other adopted a strategy making assumptions before performing calculations. Fermi Graphs effectively captured these differences, offering a nuanced view of students' reasoning and problem decomposition strategies.

Introduction

In recent years, so-called *Fermi problems* have gained attention in mathematics education research for their potential to foster critical thinking, estimation skills, problem decomposition, and connection to mathematical modelling (Segura et al., 2025). As interest in their didactical potential grows, so does the need for robust analytical frameworks that can capture the nuances of students' problem-solving and modelling processes when working on Fermi problems. Existing frameworks such as *MADs—Modelling Activity Diagrams* (Ärlebäck, 2009) and *FPATs—Fermi Problem Activity Templates* (Albarracín & Ärlebäck, 2019) offer valuable perspectives; MAD emphasizes temporal sequencing of overarching modelling activities, while FPATs focus on task decomposition and data sourcing (Ärlebäck & Albarracín, 2019; 2024).

However, these frameworks do not necessarily fully capture the structural and logical complexity of students' solutions, especially when multiple solution paths and reasoning strategies are involved (Albarracín et al., 2025). In this context, a relatively new analytical tool called Fermi Graphs (Borys & Hartmann, 2022; Hartmann et al., 2020), offer a complementary approach by representing students' solutions as directed graphs that visualize the decomposition, recomposition, and flow of reasoning. This graphical representation allows for a more nuanced analysis of how students engage in various activities (such as proposing assumptions, validating results, and performing estimations) and how these activities are embedded within the overall structure of the solution (Hartmann et al., 2020).

This paper seeks to contribute with an empirical basis to enhance the theoretical discussion on analytical tools for analysing, understanding and mapping students' work

Mattsson, L., Eckert, A., & Kilhamn, C. (Eds.). (2026). *Talking Mathematics. Proceedings of MADIF 15*. Gothenburg University and SMDF.

on Fermi problems by exploring how Fermi Graphs can be used to characterize and compare students' solutions. Drawing on empirical data from two groups of Swedish upper secondary students solving two Fermi problems, we explore and examine the distribution and sequencing of the activities the students engage in. The research questions we address in the paper are: *What are the distributions of students' activities when solving the Fermi problems that can be discerned using a Fermi Graph analysis?* and *What characteristics of the problem solving approaches to the Fermi problems come to the fore using a Fermi Graph analysis?*

Theoretical considerations and selected prior research

Fermi problems, MADs and FPATs

Fermi problems are open-ended estimation tasks that require students to make assumptions, break down complex questions into simpler sub-problems, and use rough estimates to reach a plausible solution. Named after physicist Enrico Fermi, these problems typically lack precise data and encourage logical reasoning and quantitative thinking (Efthimiou & Llewellyn, 2006). In mathematics education research they have been defined as "open, non-standard problems requiring students to make assumptions about the problem situation and estimate relevant quantities before engaging in, often, simple calculations" (Ärlebäck, 2009, pp. 331–332). Their structure (decomposition, estimation, and synthesis) makes them similar to small-scale modelling tasks (Robinson, 2008), offering rich opportunities for mathematical thinking and discussion.

Several analytical tools have been developed to study students' engagement and work with Fermi problems. One such tool is the *Modelling Activity Diagram (MAD)*, used by Ärlebäck (2009) and Albarracín et al. (2019) to link students' work to the so-called modelling cycle (cf. Niss & Blum, 2020). MADs help identify which modelling activities students engage in during different stages of problem solving and how much time is spent on each. Unlike cyclical models, MAD presents the process linearly in terms of the six *modelling activities* Reading, Making Model, Estimating, Calculating, Validating, and Writing (Ärlebäck, 2009).

Another tool, the *Fermi problem Activity Templates (FPATs)*, has been used to analyse FP solutions and support task design (cf. Albarracín & Ärlebäck, 2024). FPATs highlight how students break down a problem, determine necessary data, and build toward a final solution. FPATs are particularly useful for comparing the final chosen solution strategies and examining how students generate numerical data. For the latter, Albarracín and Ärlebäck (2019) identify four methods for this: guesstimation, measurement and experimentation, information retrieval, and statistical data collection. Although MADs and FPATs provide useful information about students' overall working on, and solutions to, Fermi problems, the descriptions of students' work is rather coarse.

Fermi graphs

A less researched but more fine-grained description of the process is given by *Fermi Graphs* (see examples in Figure 2 and the Appendix), which visually represent a solution

to a Fermi Problem, designed to emphasize the structure and complexity of the solution process in the form of a logical tree as a directed mathematical graph (Borys & Hartmann, 2022). Within this graph, numerical assumptions and intermediate results are noted along the edges, while the nodes capture how these data are processed, such as for example arithmetic operations when relevant (Jablonski et al., 2025). The processing of data at each node is also categorized using distinct *Fermi activities*, which describe the type of reasoning or calculation taking place (Hartmann et al., 2020). In some cases, a single node may involve more than one Fermi activity, reflecting the complexity of the students' reasoning. The solution process is read from left to right (giving the directional aspect of the graph), but unlike MADs, does not include an explicit timeline. Instead, it illustrates the structural flow of the solution process and Fermi activities engaged in, allowing for a multi-tiered analysis of how students decompose and reconstruct the problem more in line with a FPAT (Albarracín et al., 2025). In this paper we investigate the potential of Fermi Graphs to get a more detailed and nuanced description of students work as an analytical complement to MADs and FPATs.

Participants, data collection and analysis

We now turn to describe the participants of the study, the data and the data collection methods used, and how the data was analysed in the study [1].

Participants

Seven upper secondary school students from southern Sweden, divided into two groups based on their availability for data collection, participated in this study. Group A consisted of three male students (Alf, Ali, and Ari) and group B consisted of three male students (Bill, Ben, and Bo) and one female student (Bea). These names are pseudonyms to allow the students to remain anonymous. The students are all high performing math majors, but they had no explicit experience of solving Fermi problems prior to this study.

Methods and data collection

This study was conducted as a case study (Yin, 2014) in which two student groups' work on two Fermi problems was observed and analysed. The entire solution process of both groups was video recorded, and the students' utterances were transcribed verbatim. Non-verbal forms of communication, such as gestures or other forms of communication, were noted in the transcription using square brackets ("[]"), and instances when a student's utterance was unintelligible was marked as such using a regular parenthesis. A short break in a student's utterance, or instances when a student was trailing off (e.g. when a student was interrupted mid-sentence, or when a student paused to ponder what to say next), was transcribed using three dots ("…").

Two Fermi problems were given to the students to solve in this study: *The Google and ChatGPT problem*, and *The wrapping paper problem*. These two tasks are presented in Figure 1. The problems were presented in Swedish and were translations of two original Fermi Problems developed within the EU-funded Dim^{2}ension project [2] by Jonas Bergman Ärlebäck and Simon Barlovits respectively.

Many people like to wrap presents. Especially at Christmas, many gifts are wrapped in gift paper to decorate the gift. Wrapped gifts may look beautiful, but wrapping paper has two main problems.

Firstly, wrapping paper is a product that is usually used once and then thrown away after unwrapping. Secondly, wrapping paper is usually plastic-coated paper. This type of wrapping paper looks very decorative, but it cannot be recycled. Only wrapping paper without a plastic coating (i.e. "real" paper) or plastic film can be recycled.

These two problems lead to the following question: How many tonnes of non-recyclable wrapping paper is used in the European Union every year at Christmas?

A single Google search uses approximately 0.0003 kWh of energy. This is enough energy to power a 60-watt light bulb for about 17 seconds, or the equivalent to 0.2 grams of CO2.

Similarly, each ChatGPT query uses approximately 0.0005 kWh of energy.

How much energy is used by Google searches each day? By ChatGPT queries? What are the corresponding CO2 emissions?

Compare your answers with the facts that:

- the average household in the EU uses approximately 10 kWh of electricity per day; and that
- a flight between London and New York emits 1 590 kg of CO_2.

Figure 1. The two Fermi problems used in this study.

Both groups were first given *The Google and ChatGPT problem* to work on. They were instructed to "think aloud" and to provide a written solution accounting for all their assumptions, estimates, and calculations. The written solution was collected after the students were finished with the task. After a brief pause the students were then given *The wrapping paper problem* to work on in the same manner. The first author of this paper was present in the room as the students were working on the problems, but did not partake in the groups' solution processes.

Analysis

Based on the prior research literature on Fermi Graphs, MADs and FPATs, a list of Fermi activities was compiled to be used as an analytical lens (cf. Purfürst, 2025). These activities are presented in Table 1 below:

Table 1. A short description of all eleven Fermi activities used in this study.

Fermi activity	Description
Common Knowledge (CK)	The students make use of information or knowledge that in context can be considered common knowledge.
Propose Assumption (PA)	The students propose an assumption or estimation.
Set Assumption (SA)	Previously proposed assumptions are set/confirmed and used throughout the rest of the solution process.
Guesstimation (Ge)	An assumption or estimate in the form of a *"qualified guess"*.
Collect/Split up (C/Sp)	Previous assumptions or sub-results are collected/split up for further use.

Validate (Va)	The students interpret or critically scrutinize an assumption or (sub-)result.
Unit Conversion (UC)	The students perform a simple unit conversion.
Get from Text (GT)	The students get, and use, information from the task description.
Calculate (Ca)	The students perform an exact calculation without rounding or using approximate values.
Rough Calculation (RC)	The students perform a rough/approximate calculation using rounding or approximations.
Adjust/Simplify (A/S)	The students adjust or simplify a (sub-)result or previously set assumption.

For each student group and task, a Fermi graph was created. An initial draft of each graph was based on the groups' written solution, in which different solutions' steps were identified and categorized as instantiations of distinct Fermi activities. Once the basic structure of each group's Fermi graph was established, the transcriptions of their work on the tasks were used to provide a more detailed representation of the solution process, gradually extending and revising the initial graphs.

The transcriptions not only allowed us to establish the sequential aspect of the solution (i.e. in what order the Fermi activities were carried out), but also provided insight into the assumptions and estimates the students did not account for in their written solutions. Different parts of the transcriptions, in which the groups were considered to progress in the solution, were categorized as different Fermi activities. To illustrate; in the following short excerpt, we see Ben proposing an assumption about how many times a day an average person uses Google based on a guesstimation (PA and Ge; #1). Bea and Bo confirm this assumption (SA; #2–4), then Ben and Bill propose an assumption about the Earth's population based on common knowledge (PA and CK; #5–#6), and the whole group confirms this assumption (SA; #7):

#1, Ben: I think it's at least 10… if you take an average.

#2, Bo: Should we… Average people Google 10 times a day, then we can just strike out a zero here. [Writes something] … (Unintelligible) times the amount of people.

#3, Bea: Yes, 10 feels pretty reasonable, I think.

#4, Bo: Yes, and it… How many people are we here on earth?

#5, Ben: 8–9?

#6, Bill: 8 billion. That was around a year ago so it should…

#7, Everyone: Yes…

From this short excerpt we can identify several Fermi activities and clearly establish in what order they were carried out in the solution process. This type of categorization was done for all instances in the transcribed material where the students could be considered to engage in activities or discussions related to a progression in their solution.

After incorporating the activities categorized in the transcriptions into the Fermi graphs, the distribution of each student group's activities related to each task was summarized in a table. A complete distribution of activities (i.e. adding all activities for each student group) was also created. Finally, the Fermi graphs for each task was compared from a structural perspective to identify qualitative differences in the solution strategies between the two groups. The focus of this part of the analysis was to highlight different approaches the students chose to use to solve the two problems. In particular, we focused on contrasting what type of Fermi activities the students mainly were engaged in at different times of the solution process, as well as identifying the types of sub-problems the students decomposed the problems into.

Results

Next, we present the results from the analysis. We first provide an overview of the two groups' activity distributions. After this we present a qualitative comparison of the groups' solution strategies for each task based on the Fermi graphs created.

The distribution of students enacted Fermi activities

Both student groups employed a wide range of Fermi activities when solving the two Fermi problems. The most frequently observed activity in both groups was *Propose Assumption (PA)*, followed by *Guesstimation (Ge)*. Group A also frequently engaged in the *Set Assumption activity (SA)*, whereas Group B did not to the same extent. However, Group B participated in the *Calculate* activity *(Ca)* more often than Group A. A complete and more detailed summary of the distribution of activities across both student groups and problems is presented in Table 2 below:

Table 2. A summary of the distribution of both student groups' Fermi activities when solving the two Fermi problems.

Task & group	CK	PA	SA	Ge	C/Sp	Va	UC	GT	Ca	RC	A/S
Google & ChatGPT, Group A	1	11	8	10	5	4	0	2	4	1	3
Wrapping paper, Group A	5	21	12	16	6	1	1	0	7	3	2
Total, Group A	**6**	**32**	**20**	**26**	**11**	**5**	**1**	**2**	**11**	**4**	**5**
Google & ChatGPT, Group B	1	10	3	9	3	7	2	7	10	1	1
Wrapping paper, Group B	3	18	6	15	4	2	1	0	5	2	2
Total, Group B	**4**	**28**	**9**	**24**	**7**	**9**	**3**	**7**	**15**	**3**	**3**

Comparing and contrasting the groups' work using Fermi graphs

We now turn to a comparison of the two student groups' work on the tasks, using the constructed Fermi graphs as the basis for our analysis. The primary focus here is on structural differences in how the groups approached the problems. Specifically, we

examine the types of Fermi activities the students engaged in at various stages of the solution process, as well as the kinds of sub-problems they chose to decompose the tasks into. The Fermi graphs representing each group's work on the tasks are presented in Figure 2 below. A larger version of Fermi graph c) is available in the Appendix.

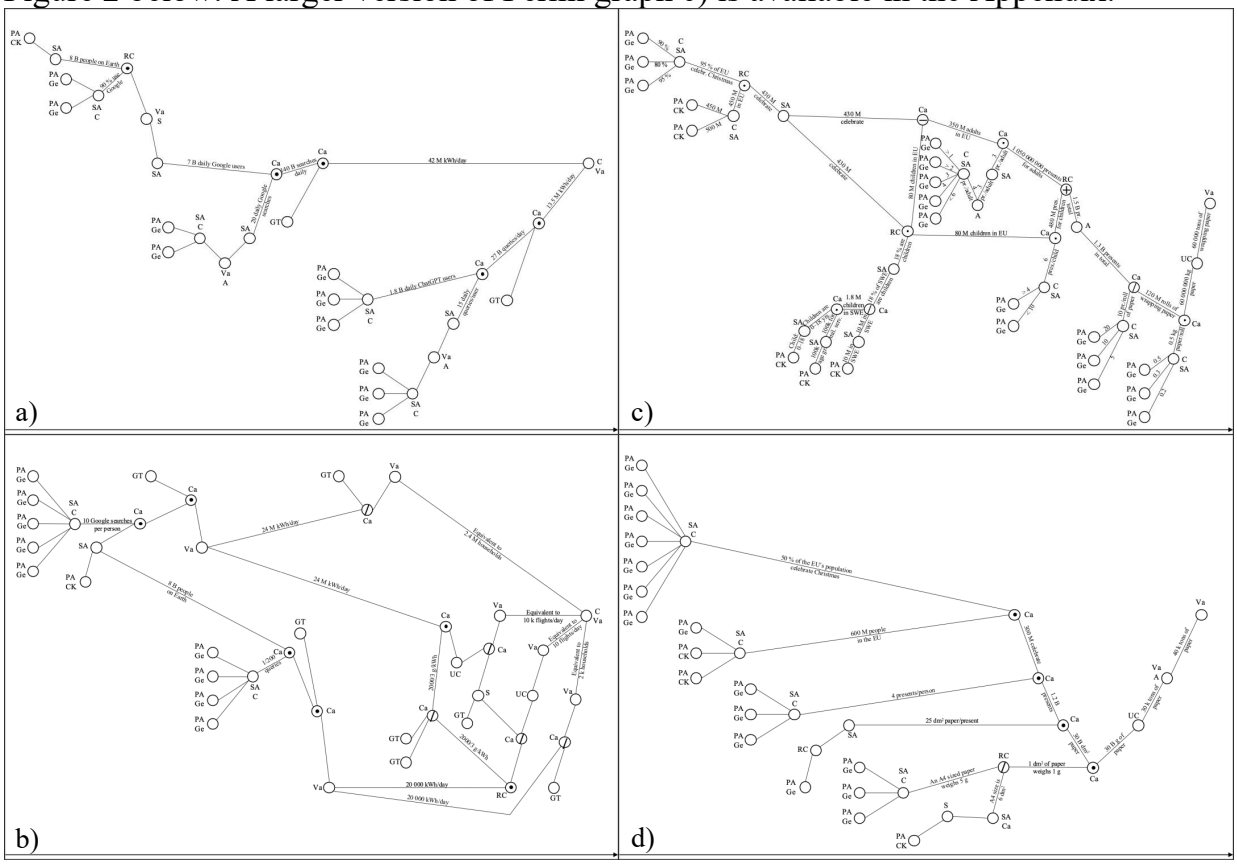


Figure 2. Fermi graphs of the groups' solution processes. a) and b) depicts group A and B's solutions to *The Google and ChatGPT problem* respectively, and c) and d) depicts the groups' solutions to *The wrapping paper problem*.

The overall structure of the Fermi graphs in Figure 2 illustrate that the two groups approached the tasks in distinctly different ways. By comparing the structure of the Fermi graphs, it is evident that group A (graphs a) and c)) generally solved one sub-problem at a time before arriving at a final solution. In contrast, Group B (graphs b) and d)) first proposed and set all assumptions they considered necessary, and then primarily engaged in calculating activities, effectively solving their (implicit) sub-problems.

These contrasting approaches are particularly evident in the groups' solutions to *The Wrapping Paper Problem* (graphs c) and d) in Figure 2). Group A displays a relatively even distribution of Fermi activities throughout the solution process, solving one sub-problem at a time, each involving a range of activities. In contrast, Group B primarily engaged in activities related to proposing and setting assumptions before transitioning to calculating activities, once all necessary assumptions had been established. A similar structural approach can be observed in the groups' work on *The Google and ChatGPT Problem*, although the comparison is less clear in this case, as Group A did not fully answer the posed questions.

Conclusion and discussion

The analysis presented shows that both student groups engaged in all Fermi activities listed in Table 1 when solving the two problems presented in this study, with Propose Assumption (PA) and Guesstimation (Ge) being the most frequently observed. Notably, although the overall distributions of activities were similar across the groups, their approaches to solving the Fermi problems differed structurally. Group A addressed the tasks by working through one sub-problem at a time, engaging in a variety of activities throughout the process. In contrast, Group B adopted *a mise en place* approach, first identifying and setting all necessary assumptions and estimations before proceeding with calculating activities based on those predefined assumptions.

The applied theoretical framework, Fermi graphs, proved effective in highlighting the distinct flows of reasoning within each student group. When considering the sequential aspects of the students' solution processes, the graphs clearly revealed contrasting approaches between the groups. As used in this study, the Fermi graphs provided insight into the overarching structure of each group's solution strategy and effectively illuminated the different types of sub-problems the students chose to address.

Contrasting a Fermi Graph's description of a solution process with representations from MAD or FPAT analysis, Fermi Graphs generalize the *modelling activities* of MADs by introducing more nuanced *Fermi activities*, and offer a clearer overview of problem decomposition strategies. In this sense, Fermi Graphs provide a hybrid representation that captures both the logical structure and flow of reasoning. Meanwhile, MADs and FPATs retain strengths in their accessibility and in representing the key temporal progression and actual solution of a Fermi problem. Together, these three tools offer complementary lenses for analysing students' engagement with Fermi problems: Fermi Graphs excel in mapping reasoning structure, MADs in highlighting activity sequences, and FPATs in clarifying (the final) decomposition of the Fermi problem and data sourcing. Future research could explore hybrid analytical models that integrate these perspectives for a more holistic understanding of students' modelling processes.

In addition, we observed that throughout the solution processes, both groups engaged in various validating activities that could not adequately be represented in the Fermi graphs. To fully capture the logic of a group's solution strategy the students' various validating activities should be taken into consideration. To accomplish this, future researchers could consider introducing additional and more nuanced Fermi activities connected to students' validating activities; or using other analytical tools (i.e. MADs or FPATs) as a complement to the Fermi graphs (cf. Albarracín et al., 2025).

Notes

1. This work originate from, and develops the ideas in, the student thesis by the first author (Purfürst, 2025) carried out under the supervision of the second author.

2. DIM^{2}ENSION–**DI**gital Supported **M**athematical **M**od**E**lli**N**g For **S**ustainable Development Goals **I**n Eur**O**pean Educatio**N**–is an EU-funded research project (2024-1-DE01-KA220-HED-000245297) aimed at integrating digital tools into teaching about (and with) mathematical modelling related to the Sustainable Development Goals. https://dimension-project.eu

References

Albarracín, L., & Ärlebäck, J. (2019). Characterising mathematical activities promoted by Fermi problems. *For the Learning of Mathematics*, *39*(3), 10–13. https://www.jstor.org/stable/26854427

Albarracin, L., Ärlebäck, J., Civil, E., & Gorgorio, N. (2019). Extending modelling activity diagrams as a tool to characterise mathematical modelling processes. *The Mathematics Enthusiast, 16*(1), 211–230. https://doi.org/10.54870/1551-3440.1455

Albarracín, L., Ärlebäck, J. B., Okamoto, H., Hartmann, M., & Borys, T. (2025). *Tools for analysing student behaviour when solving Fermi problems: Insights and future developments* [Manuscript submitted for publication].

Ärlebäck, J. B. (2009). On the use of realistic fermi problems for introducing mathematical modelling in school. *The Mathematics Enthusiast*, *6*(3), 331–364. https://doi.org/10.54870/1551-3440.1157

Ärlebäck, J. B., & Albarracín, L. (2019). The use and potential of Fermi problems in the STEM disciplines to support the development of twenty-first century competencies. *ZDM Mathematics Education, 51*(6), 979–990. https://doi.org/10.1007/s11858-019-01075-3

Ärlebäck, J. B., & Albarracín, L. (2024). Fermi problems as a hub for task design in mathematics and STEM education. *Teaching Mathematics and its Applications: An International Journal of the IMA*, *60*(7), 25–37. https://doi.org/10.1093/teamat/hrad002

Borys, T., & Hartmann, M. (2022). Der „Fermi-Graph“. Illustrative Rekonstruktion der Lösungsprozesse von Fermi-Aufgaben. In T. Borys, M. Brandl, & A. Brinkmann (Eds.), *Mathevernetzt–Anregungen und Materialien für einen vernetzenden Mathematikunterricht* (pp. 26–32). Verlag Bücherbunt im MUED e.V.

Efthimiou, C. J., & Llewellyn, R. A. (2006). Cinema, Fermi problems and general education. *Physics Education*, *42*(3). https://doi.org/10.1088/0031-9120/42/3/003

Hartmann, M., Borys, T., Okamoto, H., & Kawasaki, T. (2020). *Der „Fermi-Tree“ – Prozesse und Strukturen beim Lösen von Fermi-Fragen erfassen*. https://doi-org.e.bibl.liu.se/10.17877/DE290R-21348

Jablonski, S., Ehemann, J., Deng-Xin, K., & Ludwig M. (2025). One step at a time? –An empirical study on the use of graphical representations for solving Fermi problems. *Proceedings of CERME14*. hal-05201444

Niss, M., & Blum, W. (2020). *The teaching and learning of mathematical modelling.* Routledge.

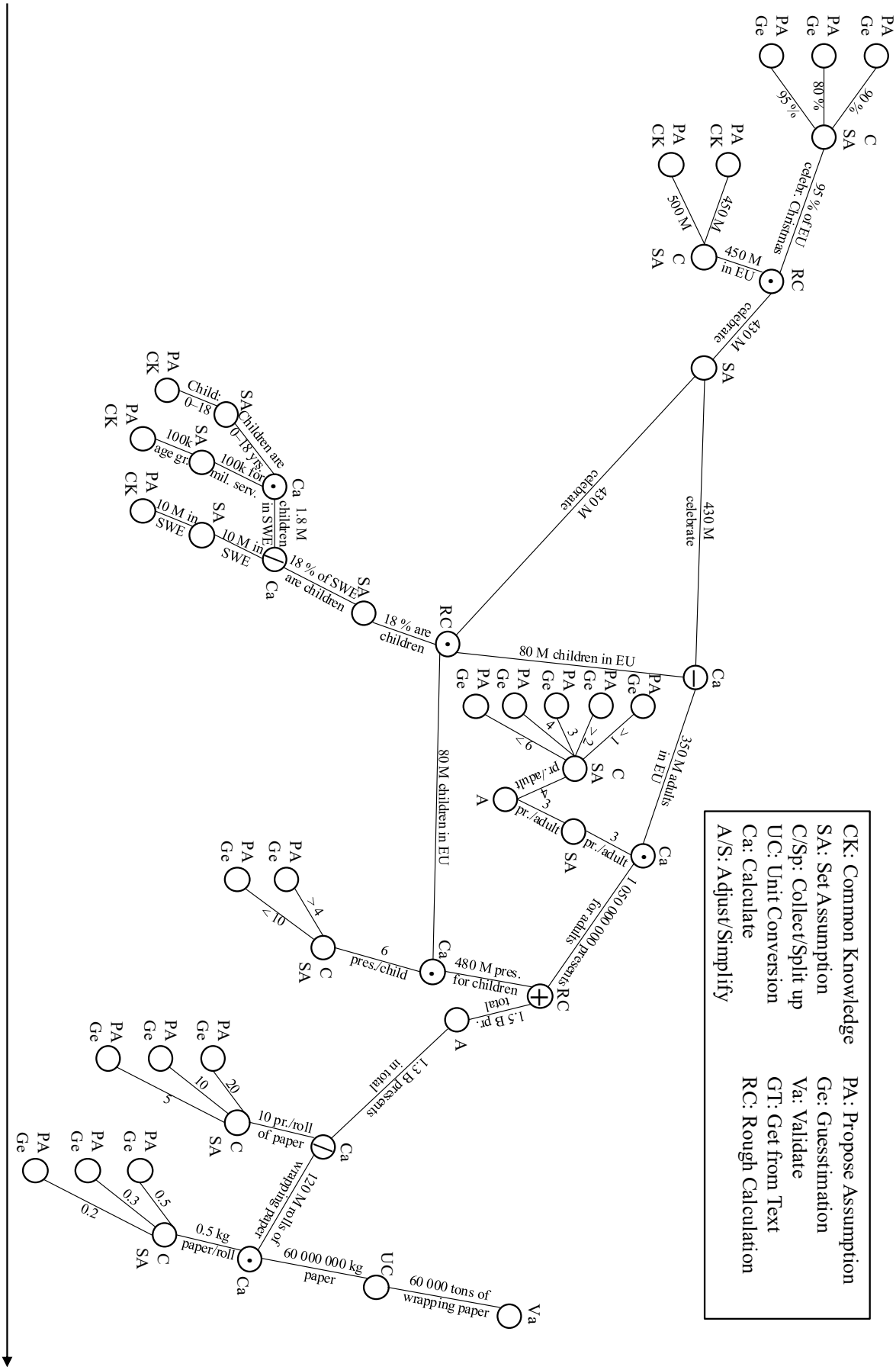
Purfürst, M. (2025). *Fermiproblem för hållbar utveckling: En analys av svenska gymnasieelevers arbete med Fermiproblem kopplade till FN:s globala mål för hållbar utveckling* (Master’s thesis, Linköping University). DiVA Portal. https://liu.diva-portal.org/smash/record.jsf?pid=diva2:1968712

Robinson, A. W. (2008). Don’t just stand there—teach Fermi problems! *Physics Education*, *43*(1), 83-87. https://doi.org/10.1088/0031-9120/43/01/009

Segura, C., Gallart, C., & Ferrando, I. (2025). Influence of pre-service primary school teachers’ prior knowledge of measurement and measurement estimation in solving modelling problems. *Journal of Mathematics Teacher Educ*ation. http://doi.org/10.1007/s10857-025-09685-3

Yin, R. K. (2014). *Case Study Research Design and Methods* (5th ed.). Sage.

Appendix – Fermi graph of group A's work on *The wrapping paper problem*



Developing a model for implementing mathematics curriculum materials in teacher education

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Based on a configurative systematic review of 44 papers, this study proposes a model for implementing mathematics curriculum materials in teacher education. The review identifies three learning objectives — learning from, about, and with the materials. For each objective, the suggested model outlines (a) activities and associated prospective teacher's learning outcomes, (b) applied theoretical constructs, and (c) key research findings. The model offers an actionable and structured framework for integrating Mathematics curriculum materials into teacher education and is discussed in relation to its implications for both practice and research.

Introduction

Mathematics curriculum materials (MaCMs) constitute teachers' major resource for planning and teaching mathematics (e.g., Choppin et al., 2022; Fan et al., 2021), operationalising steering documents (e.g., Choppin et al., 2022) and shaping what teachers teach and students learn (Reys et al., 2004). Despite their importance, teachers are rarely taught how to interpret and use MaCMs effectively (Remillard, 2016), even though research (e.g., Ben-Peretz et al., 1984; Drake et al., 2014) has advocated for the development of such competencies in teacher education (TE). Studies on MaCMs are a growing field of research (Rezat, 2024), focusing on their use by teachers (e.g., Rezat et al., 2021), their design (e.g., Ball & Cohen, 1996, Davis & Krajcik, 2005, Van Steenbrugge & Ryve, 2018) and their potential to support teacher development (e.g., Fan et al., 2021), but recent comprehensive reviews on MaCMs (e.g., Buch et al., 2022; Fan et al., 2018; Lloyd et al., 2017; Rezat, 2024) have not addressed MaCM use in TE.

Aim and research questions

This study aims to develop a model for implementing MaCMs in TE. The aim is operationalised through a configurative systematic review on key components when using MaCMs in TE, based on the research questions:

When using mathematics curriculum materials in teacher education,

1) What activities are implemented, and what do prospective teachers learn from them?
2) What theoretical constructs are used?
3) What are the main research findings and conclusions?

Mattsson, L., Eckert, A., & Kilhamn, C. (Eds.). (2026). *Talking Mathematics. Proceedings of MADIF 15*. Gothenburg University and SMDF.

Terminology

This study includes research on the use of curriculum materials such as textbooks, teacher guides, and their digital equivalents in TE. Remillard and Kim (2020) define curriculum materials as "a set of resources, those designed to support a program of instruction and student learning over time" (p. 3). For the purposes of this paper, the term MaCMs refers to "textbooks, teacher guides, and their digital equivalents designed to support a programme of mathematical teaching and learning over time."

Background

Research has consistently shown that MaCMs have a significant impact on mathematics teaching. They are teachers' primary resource for planning and establishing classroom practice (e.g., Choppin et al., 2022; Fan et al., 2021), operationalise steering documents (e.g., Choppin et al., 2022), shaping teachers' teaching and students' learning (Reys et al., 2004) and thus still have a reach in the system to exert a pervasive influence on teachers' daily practices, as Ball and Cohen (1996) noted already 30 years ago.

Recent reviews on MaCMs have highlighted various perspectives such as the intended, enacted and attained curriculum (Lloyd et al., 2017), teacher interaction with resources, teachers collective work through resources, interaction teacher/pupils through resources and pupils and teachers learning from MaCMs (Fan et al., 2018) and a focus on what teachers' guides should provide for being educative rather than merely instructive manuals (Buch et al., 2022). Rezat (2024) categorizes the field of research on curriculum resources in mathematics education into seven areas: the role of curriculum resources, content analysis, user studies, effects of resources, design of resources, curriculum resources as data and reviews. However, despite research advocating for MaCM inclusion in TE (e.g., Ben-Peretz, 1984; Drake et al., 2014) and the breadth of these reviews, none of them has addressed the use of MaCMs in TE. This supports Remillard's (2016) assertion that teachers are rarely taught how to interpret and effectively use MaCMs, which is contradictory to their widespread use.

Research suggests that teacher knowledge can grow through MaCM use, depending on the material's design, the support provided for how to understand and use them and teachers' capacity to read and interpret it (e.g., Davis et al., 2017; Ku & Lee, 2020; Li & Harfitt, 2018; Remillard & Kim, 2017). Teachers' ability to interpret MaCMs – and thus learn from them – can be enhanced through deliberate implementation in TE (Ben-Peretz, 1984; Drake et al., 2014). In Sweden, a national TE degree learning outcome states that prospective teachers (PTs) should "demonstrate the ability to use and evaluate textbooks, other teaching materials and other learning tools" (SFS 1993:100, author's translation). Given the influence of MaCMs on mathematics teaching and their potential to support continuous teacher learning, this requirement is of particular relevance for mathematics education and universities should be supported by research-based arguments for its relevance and provided with guidance on what to include and how to implement it. This study aims to develop a model for implementing MaCMs in TE through a systematic review on key components of integrating MaCMs into TE. The

research questions are focusing on essential components for actionable implementation of MaCMs in TE – activities and PT learning outcomes from these activities, adopted theoretical constructs and crucial research findings and conclusions.

Methods

A *configurative review* aims at providing an overview and understanding of a research field, identifying its themes and providing new insights into it (Gough et al., 2012). Such approach is applicable to the aim to develop a model for implementing MaCMs in TE and can be described in two main steps: 1) finding and selecting relevant literature, and 2) analysing and synthesising the selected literature according to relevant themes.

Finding and selecting relevant literature

The strategy for identifying relevant literature was conducted in two steps, 1) searching databases and 2) searching reference lists in hits from the first step. The databases used were ERIC, Scopus, Educational Research Complete, Teacher Reference Center and Web of Science, which were searched through as illustrated in table 1.

Search #	Search strings	Hits
1	TI (“curriculum material*” OR textbook* OR ”teacher guide*” OR “mathematic* curricul* material*” OR “mathematic* textbook*”) OR AB (“curriculum material*” OR textbook* OR ”teacher guide*” OR “mathematic* curricul* material*” OR “mathematic* textbook*”) OR KW (“curriculum material*” OR textbook* OR ”teacher guide*” OR “mathematic* curricul* material*” OR “mathematic* textbook*”)	Number of hits
2	TI (“teacher* education” OR “teacher student*” OR “pre-service teacher*” OR PSTs OR “teacher educator*” OR “teacher* learning”) OR AB (“teacher* education” OR “teacher student*” OR “pre-service teacher*” OR PSTs OR “teacher educator*” OR “teacher* learning”) OR KW (“teacher* education” OR “teacher student*” OR “pre-service teacher*” OR PSTs OR “teacher educator*” OR “teacher* learning”)	Number of hits
3	TI mathematic* OR AB mathematic* OR KW mathematic*	Number of hits
4	1 AND 2 AND 3	Number of hits
Inclusion criteria		
Years	2000 - 2023	Number of hits
Language	English	Number of hits
Total		Number of hits

Table 1. The search methodology to identify publications addressing the use of mathematics curriculum materials in teacher education.

Crucial in the database search is step 4, in which a combination of step 1 – 3 is selected. For example, in ERIC step 1 gave 27 591 hits, step 2 gave 63 060 hits and step 3 gave 92 516 hits. When combining them in step 4, only 93 hits remained and when inclusion criteria were applied, 87 hits remained. From the same procedure, Scopus provided 157 hits, Educational Research Complete 126 hits, Teacher Reference Center 33 hits and Web of Science 133 hits. In total, 536 hits from the selected databases were found. Duplicates were removed and papers were excluded based on title and abstract screening, as well as full-text reading if they: focused on in-service teachers rather than PTs, addressed subjects other than mathematics, involved university textbooks or focused on syllabi or teacher educators. A total of 35 records remained for analyses and for the next step in the procedure – searching the reference lists of the 35 hits.

In the search through the reference lists of the 35 included papers from the initial step, 17 hits were found. The same approach regarding exclusion as in the first step was

applied, excluding 8 records and remaining 9 papers for analyses. In total, 44 papers (see Appendix) were selected for analyses and synthesis.

Analysing and synthesising selected papers

This study aims to develop a model for implementing MaCMs in TE. Such a model should provide research-based descriptions of what PTs are supposed to develop by integrating MaCMs in TE – describing its learning objectives. For the model to be actionable for teacher educators, ideas of *what* to do (activities), *how* to do it (what frameworks to depart from) and *crucial aspects to consider* (research findings and conclusions) are relevant to describe and constitute the foundation of the three research questions. Thus, the first step in the analysis was to establish learning objectives for PTs when using MaCMs in TE. Then, for each learning objective, content addressing research questions 1, 2 and 3 were identified.

Learning objectives

The process of establishing learning objectives departed from Lloyd (2006), who identifies three critical areas of PTs learning in relation to MaCMs: 1) Learning mathematics, 2) Learning about curriculum, and 3) Learning mathematics teaching and learning. The critical area *learning mathematics* corresponds to what Nicol and Crespo (2006) term learning *from* MaCMs, where the aim is for PTs to learn mathematical content from the materials. The critical area *learning about curriculum*, focuses on PTs' analyses of MaCMs from various perspectives to understand their structure and purpose – learning *about* the material. The critical area *learning mathematics teaching and learning* encompasses PTs' engagement with curriculum materials as a foundation for lesson planning and teaching. This corresponds to what Nicol and Crespo (2006) term learning *with* MaCMs. As illustrated in Figure 1, three learning objectives for implementing MaCMs in TE are established – learning *from*, *about* and *with* MaCMs.

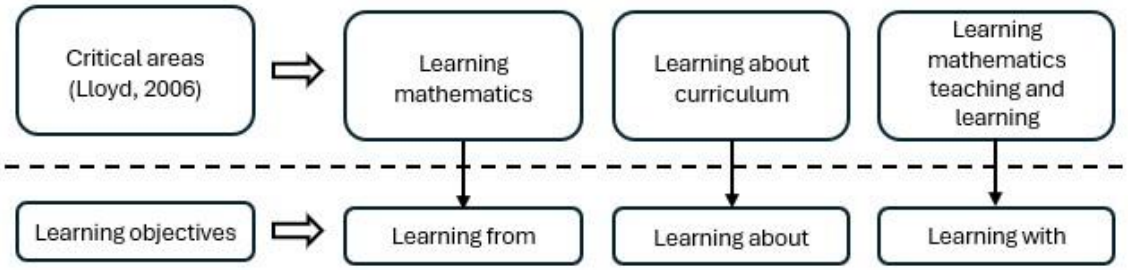

Figure 1. Connection between the three critical areas for PTs learning from MaCMs (Lloyd, 2006) and applied learning objectives when implementing MaCMs in TE.

Activities and prospective teachers' learning, theoretical constructs and research findings and conclusions

For each established learning objective – learning *from, about* and *with* MaCMs – the 44 included papers were searched for crucial actionable aspects when implementing

MaCMs in TE – implemented activities and PTs' learning outcomes, theoretical constructs, and research findings and conclusions.

In learning *from* MaCMs, typical activities found in the analysis involve PTs engaging with materials as pupils (e.g., Frykholm, 2005), working with several mathematics areas, classifying mathematical tasks (Dempsey & O'Shea, 2020) and interpreting mathematical definitions (Chesler, 2012). In searching for applied theoretical constructs, some frameworks for classifying mathematical tasks (e.g., Lithner, 2008) were found and the *MKT framework* (Ball et al., 2008) was found to structure the investigation of PTs' mathematical understanding. Regarding research conclusions, the analysis revealed that mathematics instruction for PTs should be qualitatively different from traditional university mathematics teaching, directly connected to the mathematics they will later teach (Lutz & Berglund, 2007). It was also indicated that PTs learn mathematical content using standards-based materials, sometimes more effectively than with traditional university textbooks (Spielman & Lloyd, 2004).

In learning *about* MaCMs, typical activities found in the analysis include students analysing MaCMs from several perspectives (e.g., Males et al., 2015) and comparing different MaCMs (Lloyd & Behm, 2005). For such analyses, various frameworks for textbook analysis were employed, such as *Documentational genesis* (Gueudet & Trouche, 2009), *Key Pedagogies of Practice* (Grossman et al., 2009), and the *Curriculum Spaces Analysis Tool* (Drake et al., 2015). Examples of found research conclusions are 1) PTs benefit from analysing MaCMs to deepen their understanding of the materials and enhancing their ability to use different resources effectively (Cross & Lloyd, 2011), 2) engaging in critical analysis of MaCMs is more beneficial than focusing solely on their practical use in lesson planning and teaching (ibid) and 3) textbook analyses are most effective when PTs receive support both from teacher educators and using analytical frameworks, guiding their analyses (e.g., Earnest & Amador, 2019).

In learning *with* MaCMs, typical activities found in the analysis include reading, planning, and enacting lessons based on MaCMs (e.g., Davis, 2009). PTs became familiar with MaCMs as tools for both planning and teaching and recognized that effective curriculum use is more complex than initially anticipated (Behm & Lloyd, 2005). Various frameworks are employed to examine how PTs use and make sense of curriculum materials, for example *Types of Curriculum Use* framework (Brown & Edelson, 2003) and *Teachers' Curriculum Use* (Remillard & Bryans, 2004). The analysis noticed research conclusions such as differently designed materials promote different types of learning, supporting either subject matter knowledge or pedagogical content knowledge (Davis, 2009). Also, modifying preexisting lesson suggestions produces better lesson plans and higher efficacy in teaching the lesson, than planning and teaching without support (Lim et al., 2018).

Result

The result of the analysis and synthesis is presented as a suggested model for implementing MaCMs in TE, illustrated in Figure 2.

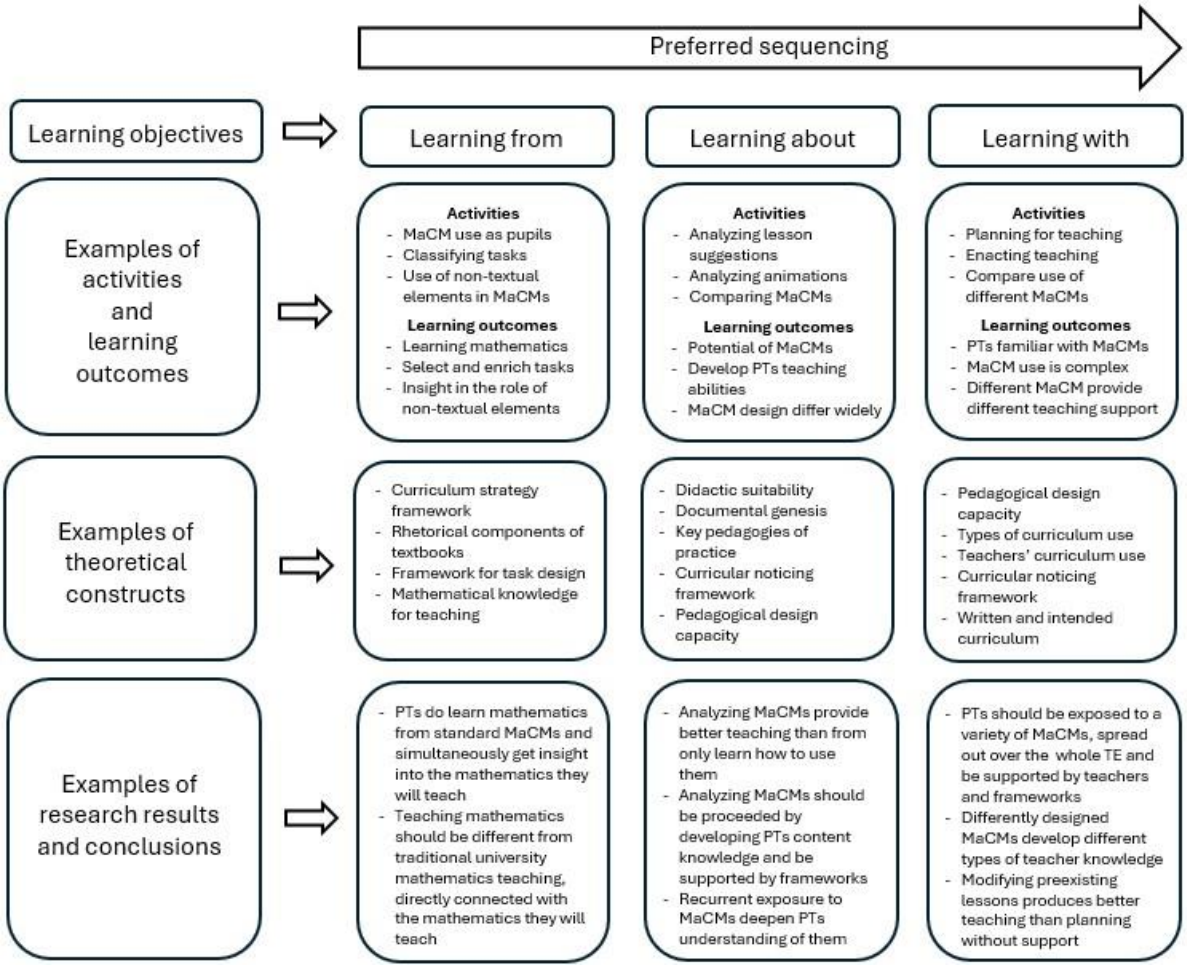

Figure 2. A suggested model for implementing MaCMs in TE.

The suggested model, based on an analysis of 44 papers on the use of MaCMs in TE, is structured around three learning objectives – learning *from, about* and *with* MaCMs. For each learning objective, findings regarding a) activities and student learning outcomes, b) theoretical constructs and c) research findings and conclusions are linked. Altogether, the suggested model is offering a research-based guidance for teacher educators to consider when implementing MaCMs into TE programmes, by simultaneously providing an overview of learning objectives, details on their content and structures and their preferred sequencing.

The preferred sequence of the learning objectives – first learning *from* MaCMs, followed by learning *about* MaCMs and last learning *with* MaCMs – is based on two key findings from the analysed literature. Firstly, Castillo Céspedes and Burgos Navarro (2022) found that PTs with well-developed content knowledge demonstrated deeper insights when analysing MaCMs than those with weaker content knowledge. This suggests that learning the mathematics *from* MaCMs ideally precedes learning *about* MaCMs. Second, Tyminski et al. (2011) observed that PTs who analysed MaCMs before

planning and teaching made more effective use of them. This implies that learning *about* MaCMs should precede learning *with* MaCMs. Together, these findings suggest the proposed sequence: first learning *from* MaCMs, then learning *about* MaCMs and last learning *with* MaCMs.

Discussion

The discussion reflects on limitations of the applied methods and implications of the suggested model for both practice and research.

The systematic review approach is limited by factors such as selected databases, search strings and inclusion criteria. If one or more of these factors is changed, the result and content of the suggested model could be different. To counteract such bias, the applied search strings and criteria for inclusion or exclusion, are a product of tests of several different ones, refined and adjusted to be precise according to the aim and research questions. At the end of this process, it was noted that small changes in words or sequences of words lead to minimal changes in number of hits, indicating the used strings and criteria were satisfying. Further, the completed search gave a large number of duplicates, indicating more databases was probably not necessary. Another factor for result bias is interpretations in the process of analysing and synthesising the papers. To counteract such bias, distinct themes based on the research questions guided the analysis and synthesis procedure. Also, iteratively readings of each paper were conducted, reducing the risk for misinterpretations or placing papers in the wrong category. For example, some papers included more than one learning objective, initially suggesting them to be placed in more than one category. Further reading though revealed one of the objectives to be in focus, placing it in one specific category.

Regarding the result of the study, implementing MaCMs in TE appears highly relevant given the calls from research to address this area (Ben-Peretz, 1984; Drake et al., 2014) and MaCM's significant impact on mathematics teaching (e.g., Choppin et al., 2022). However, recent reviews on the field (e.g., Buch et al., 2022; Fan et al., 2018; Lloyd et al., 2017; Rezat, 2024) have not addressed the use of MaCMs in TE. This review study suggests a model with the aim to provide an actionable structure for teacher educators to consider when implementing MaCMs into TE programmes. Firstly, the model identifies three distinct learning objectives, offering a structured approach for implementation with a suggested, not prescriptive order. Some studies, for example, focused solely on learning *about* MaCMs, but still reported positive outcomes. These studies, however, also indicated that PTs with stronger content knowledge gained more from the analysis, suggesting some activity involving learning *from* should precede learning *about*. Similarly, but the other way around, in many cases multiple learning objective were addressed within one single activity. For example, PTs first analysing a MaCM (learning *about*) and then using it to plan and teach a lesson (learning *with*), describing an internal sequence within one activity. Second, for each learning objective, the model provides examples of applied activities and information on PTs learning outcomes from these activities, along with theoretical constructs to apply. This helps

teacher educators designing research-informed learning experiences from MaCMs in TE. As the model is grounded in research findings and conclusions, it provides arguments for applying specific activities and indications of their effectiveness.

Together, results regarding learning objectives, activities and theoretical constructs to apply and crucial research findings to consider, lead to five overarching conclusions:

- Teaching mathematics to PTs should be qualitatively different from traditional university mathematics teaching, being directly connected with the mathematics they will later teach. This can be achieved through learning mathematics *from* MaCMs (Lutz & Berglund, 2007).
- Exposure to MaCMs should be recurrent throughout TE (Nicol & Crespo, 2006), involving a variety of materials (Block et al., 2023; Males & Setniker, 2019) and be supported by teachers and functional frameworks (e.g., Earnest & Amador, 2019).
- It is more beneficial for PTs to first analyse and then use MaCMs, than to only learn to use them (Cross & Lloyd, 2011).
- PTs should ideally develop content knowledge before analysing MaCMs (Castillo Céspedes, & Burgos Navarro, 2022) and analysing MaCMs before using them for planning and teaching (Tyminski et al., 2011).
- Modifying preexisting lesson suggestions is more effective than planning and teaching without support (Lim et al., 2018).

These conclusions and the suggested model together provide a comprehensive and holistic overview with actionable detailed structures for teacher educators to consider when implementing MaCMs into TE programmes.

From the findings and the suggested model, some interesting avenues for future research emerge. First, all analysed papers could be categorised into at least one of the purposed learning objectives learning *from, with* and *about* MaCMs. While this somewhat strengthen the validity of the suggested model, future research with similar approach might find additional learning objectives, which complement or reshape the suggested model. Second, the initial search identified several publications addressing similar themes within science education. A study similar to this study, on curriculum materials designed for science subjects, would be potentially illuminating regarding the implementation of curriculum materials per se in TE. Third, developing PTs subject matter knowledge *from* MaCMs is projected as successful, but not all forms of knowledge were effectively cultivated. For example, learning about mathematics definitions was found to be difficult from this approach. It thus seems relevant to investigate which content areas are most conducive to PTs' development, and why certain areas may be less suitable in the context of TE. Fourth, it would be valuable to apply this model in a longitudinal study, spanning an entire TE programme. Such a study could evaluate PTs understanding and use of MaCMs during their first year of teaching and compare outcomes with those of teachers who did not follow this structured approach.

References

Ball, D. L., & Cohen, D. K. (1996). Reform by the book: What is—or might be—the role of curriculum materials in teacher learning and instructional reform?. *Educational researcher*, *25*(9), 6-14.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: what makes it special? *Journal of teacher education, 59*(5), 389-408.

Ben-Peretz, M. (1984). Curriculum theory and practice in teacher education programs. In L. Katz & J. Raths (Eds.), *Advances in teacher education* (pp. 9-27). Norwood: NJ: Ablex.

Buch, B., Gissel, S. T., Oksbjerg, M., Kjeldsen, K. & Albrechtsen, T. R. S. (2022). A systematic review of research on teachers' guides. *Learning Tech – Tidsskrift for læremidler, didaktik og teknologi*, (12), 12-40.

Castillo Céspedes, M. J., & Burgos Navarro, M. (2022). Developing Reflective Competence in Prospective Mathematics Teachers by Analyzing Textbooks Lessons. *EURASIA Journal of Mathematics, Science and Technology Education, 18*(6).

Choppin, J., Roth McDuffie, A., Drake, C., & Davis, J. (2022). The role of instructional materials in the relationship between the official curriculum and the enacted curriculum. *Mathematical thinking and learning*, *24*(2), 123-148.

Davis, E. A. & Krajcik, J. S. (2005). Designing educative curriculum materials to promote teacher learning. *Educational Researcher, 34*(3), s. 3–14.

Davis, E. A., Palincsar, A. S., Smith, P. S., Arias, A. M., & Kademian, S. M. (2017). Educative curriculum materials: Uptake, impact, and implications for research and design. *Educational Researcher*, *46*(6), 293-304.

Drake, C., Land, T. J., & Tyminski, A. M. (2014). Using educative curriculum materials to support the development of prospective teachers' knowledge. *Educational Researcher*, *43*(3), 154-162.

Fan, L., Cheng, J., Xie, S., Luo, J., Wang, Y., & Sun, Y. (2021). Are textbooks facilitators or barriers for teachers' teaching and instructional change? An investigation of secondary mathematics teachers in Shanghai, China. *ZDM–Mathematics Education, 53*, 1313-1330.

Fan, L., Trouche, L., Qi, C., Rezat, S., & Visnovska, J. (2018). Research on mathematics textbooks and teachers' resources. *Switzerland: Springer International Publishing*.

Gough, D., Thomas, J., & Oliver, S. (2012). Clarifying differences between review designs and methods. *Systematic reviews*, *1*, 1-9.

Ku, N., & Lee, K. (2020). Middle school teachers' interpretations and use of geometric curriculum materials: Focused on the three aspects analysis. *Journal of Educational Research in Mathematics*, *30*(3), 397–425.

Land, T. J. & Drake, C. (2014). Understanding preservice teachers' curricular knowledge. In J. Lo, K. R. Leatham & L. R. Van Zoest (Eds.), *Research trends in mathematics teacher education* (pp. 3–22). Cham, Switzerland: Springer International Publishing AG.

Li, Z., & Harfitt, G. (2018). Understanding language teachers' enactment of content through the use of centralized curriculum materials. *Asia-Pacific Journal of Teacher Education*, *46*(5), 461–477.

Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in mathematics*, *67*(3), 255-276.

Lloyd, G. M. (2006). Using K-12 mathematics curriculum materials in preservice teacher education: Rationale, strategies, and teachers' experiences. In K. Lynch-Davis, & R. L.

Rider (Eds.), *The work of mathematics teacher educators: Continuing the conversation* (vol. 3, AMTE monograph series, pp. 11-27). San Diego, CA: Association of Mathematics Teacher Educators.

Lloyd, G. M., Cai, J., & Tarr, J. E. (2017). Issues in curriculum studies: Evidence-based insights and future directions. *Compendium for research in mathematics education*, 824-852.

Nicol, C. C., & Crespo, S. M. (2006). Learning to teach with mathematics textbooks: How preservice teachers interpret and use curriculum materials. *Educational studies in mathematics, 62*, 331-355.

Remillard, J. T. (2016). How to Partner with Your Curriculum. *Educational leadership*, *74*(2), 34-38.

Remillard, J. T., & Bryans, M. B. (2004). Teachers' orientations toward mathematics curriculum materials: Implications for teacher learning. *Journal for Research in Mathematics Education, 35,* 352-388.

Remillard, J., & Kim, O. K. (2017). Knowledge of curriculum embedded mathematics: Exploring a critical domain of teaching. *Educational Studies in Mathematics*, *96*(1), 65-81.

Remillard, J. T., & Kim, O. K. (Eds.). (2020). *Elementary mathematics curriculum materials: Designs for student learning and teacher enactment*. Springer.

Reys, B. J., Reys, R. E., & Chavez, O. (2004). Why mathematics textbooks matter. *Educational leadership*, *61*(5), 61-66.

Rezat, S. (2024). Research on curriculum resources in mathematics education: A survey of the field. *ZDM–Mathematics Education*, *56*(2), 223-237.

Rezat, S., Fan, L., & Pepin, B. (2021). Mathematics textbooks and curriculum resources as instruments for change. *ZDM–Mathematics Education, 53*, 1189-1206.

SFS 1993:100, Högskoleförordningen (the Swedish Higher Education Ordinance, author's translation).https://www.riksdagen.se/sv/dokument-och-lagar/dokument/svenskforfattningssamling/hogskoleforordning-1993100_sfs-1993-100/

Tyminski, A. M., Land, T. J., & Drake, C. (2011). Elementary Preservice Teachers' Critiques, Comparisons, and Preferences in Examining Standards-Based Curricular Materials. In Wiest, L. R., & Lamberg, T. (Eds.)*, Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Reno, NV: University of Nevada, Reno.

Van Steenbrugge, H., & Ryve, A. (2018). Developing a reform mathematics curriculum program in Sweden: relating international research and the local context. *ZDM, 50*(5), 801-812.

Large-scale implementation of preschool mathematics education

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[4]Dalarna University

This presentation is about a study on large-scale implementation of preschool mathematics education. Mathematics education for toddlers (1-3-year-olds) was developed in collaboration between researchers and preschool teachers and implemented small scale at three preschools. Research indicated that toddlers in these preschools enhanced their understanding and use of numbers both qualitatively and quantitatively compared to those in the control group (Palmér & Björklund, 2024). Based on these positive results, a new study was initiated with the aim of implementing the mathematics education large scale (financed by the Swedish Institute for Educational Research, 2022-00010). Research has however demonstrated that successful interventions are not easily implemented large scale and that all interventions undergo changes when implemented in new settings. Such changes imply a need to focus on quality rather than quantity in large-scale implementation (Century & Cassata, 2016). This presentation concerns such qualitative aspects found to be significant for implementing the toddler mathematics education large scale.

The study

The first step in the implementation study was to identify the essential and indispensable elements of the preschool mathematics education in the initial study, that is, to identify the core components of mathematics education to be implemented large scale. Based on these core components, a guiding material was developed in collaboration between researchers and preschool teachers. This guiding material was divided into five parts, each focused on one of the core components identified: part 1: mathematizing; part 2: contrast and generalization; part 3: representations; part 4: cardinality; and part 5: part-whole relationship of numbers. Each part consisted of one text to read, one video illustrating the content of the text in authentic teaching situations, questions to discuss with colleagues, scripted activities to be carried out with children, and finally, one activity to be planned and carried out by the preschool teacher.

The implementation study was conducted in line with educational design research, which implies implementation in cycles, where new preschools were successively included in an increasingly broader implementation. In the first cycle, the guiding material was implemented in two preschools. In the second cycle, six new preschools were involved, and in the third cycle all preschools from one municipality (7)

implemented the guiding material. The data generated were individual interviews with the preschool teachers from the first cycle (12), focus group interviews with the teachers from the second and third cycle (28), and video documentations (219) from the teaching at the preschools from all cycles. Also, delayed interviews with preschool teachers (after 6 and 12 months) were conducted. The interviews focused on the guiding material and how it was used and experienced by the preschool teachers. The video documentations made visible how the mathematics education was adopted by the preschool teachers. These empirical data were analysed with a focus on whether the core components were visible in the teaching and, if not, how the guiding material needed to be developed. In line with focusing on quality rather than quantity, a framework placing focus on the nature and quality of the implementation, emphasizing depth (the nature and quality of change), sustainability (the persistence of change over time), spread (within each setting), and ownership (the autonomy of teachers) was used (Coburn, 2003).

Results

The results from the study are still tentative but point to important considerations for large scale implementation of preschool mathematics education. One consideration concerns the motives behind participation. In the cycles, there has been a difference regarding whether the implementation has been initiated by individual preschool teachers or principles of preschools. The results show that teacher-initiated implementation generally generates better results. Another key finding relates to the guiding material. The core components must be expressed explicitly. This is essential, as everything in the mathematics education except the core components needs to be adapted to the local context, otherwise the mathematics education becomes instrumental. Adaptations, when they are productive and do not interfere with the core components, indicate depth, spread, sustainability and ownership (Coburn, 2003) and can therefore be considered empirical evidence of successful implementation.

References

Century, J., & Cassata, A. (2016). Implementation research: Finding common ground on what, how, why, where, and who. *Review of Research in Education, 40*, 169–215. https://10.3102/0091732X16665332

Coburn, C. E. (2003). Rethinking scale: Moving beyond numbers to deep and lasting change. *Educational Researcher, 32*(6), 3–12. https://doi.org/10.3102/0013189X032006

Palmér, H., & Björklund, C. (2024). Extending common preschool activities for mathematics teaching with toddlers. *The 15th International Congress on Mathematical Education*, Sydney, 7–14 July 2024.

Learning mathematics in English

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This presentation describes a comparative study investigating how the language of instruction affects learning in mathematics. The language of mathematics is largely subject-specific, including symbols, a variety of representations such as graphs, abstract concepts and domain-specific vocabulary (e.g., Schleppegrell, 2007). Although the symbol system is the same across languages, natural language is necessary to explain concepts and to make sense of symbols and representations. In Sweden, some schools use a content and language integrated learning approach (CLIL), where some or all subjects are taught in English (Coyle, Hood & Marsh, 2010; Cenoz, Genesee, & Gorter, 2014). When English is the language of instruction in mathematics, students are introduced to new mathematical concepts in a language that is not their first language, which could be more of a challenge (Bruton, 2013). However, for some of the students neither English nor Swedish is their first language.

The study measures mathematics learning in school years 5 and 6 (ages 11–12), comparing student groups where the medium of instruction in mathematics is English (EMI) or Swedish (SMI) respectively. Our two research questions are: *What differences are there, if any, between EMI and SMI groups in students' progression in mathematics from school year 5 to year 6? How does students' language background influence this progression, if at all?*

A total of 370 students from five independent schools participated in the study, of which 182 were EMI students and 188 were SMI students. Drawing on research-based descriptions of mathematical competencies (Boesen et al., 2014; Mullis, 2009; Niss & Højgaard, 2019), three different areas were selected to give a broad measure of competence: 1) arithmetic, a procedural test where only mathematical symbols were used; 2) terminology, where students were asked to name a number of mathematical objects, and 3) problem solving, with word problems chosen from TIMSS released items for mathematics in school year 4, covering different content areas.

Statistical analyses of the test results were conducted, showing some significant differences in progression between the two groups. The EMI students' progression was significantly greater than that of SMI students on the arithmetic test, while the SMI students' progression was greater than that of EMI students on the terminology test. However, on the problem-solving test, there was no significant difference between the groups, indicating that all students had progressed equally in a general sense of mathematical competence overall, although their competencies may be stronger or weaker in different areas. The students' language background did not have a statistically

significant impact on the arithmetic and terminology tests, but on the problem-solving test students with a non-Swedish background performed significantly lower than students with a Swedish background. When looking at only students with a non-Swedish background, the results showed that the EMI students progressed significantly more in terms of arithmetic competence than the SMI students.

In addition to the statistical analyses, qualitative analyses of test results from school year 6 were conducted. Items with poor results in one or both groups were selected for analysis to identify typical or frequent incorrect answers in the two groups. Differences between the groups were highlighted. In the presentation, we discuss our results and possible reasons for the identified differences. We also discuss limitations of the study and the necessity of further studies including interviews and observations, which were not possible in the present study due to the COVID-19 pandemic raging during the time of data collection.

References

Boesen, J., Helenius, O., Bergqvist, E., Bergqvist, T., Lithner, J., Palm, T., & Palmgren, B. (2014). Developing mathematical competence: From the intended to the enacted curriculum. *The Journal of Mathematical Behavior, 33*, 72–87. https://doi.org/10.1016/j.jmathb.2013.10.001

Bruton, A. (2013). CLIL: Some of the reasons why...and why not. *System, 41*(3), 587-597. doi: https://doi.org/10.1016/j.system.2013.07.001

Cenoz, J., Genesee, F., & Gorter, D. (2014). Critical analysis of CLIL: Taking stock and looking forward. *Applied linguistics, 35*(3), 243–262.

Coyle, D., Hood, P., & Marsh, D. (2010). CLIL: *Content and language integrated learning* (Vol. 1). Cambridge university press.

Mullis, V.S., Martin, M.O., Ruddock, G.J., O'Sullivan, C.Y., Preuschoff, C. (2009). TIMSS 2011 Assessment Frameworks. TIMSS & PIRLS International Study Center

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics Education, 102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9

Schleppegrell, M. (2007). The linguistic challenges of mathematics teaching and learning: A research review. *Reading & Writing Quarterly, 23*(2), 139–159. https://doi.org/10.1080/10573560601158461

What properties of a teacher guide can produce support for mathematics teachers?

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This presentation focuses on teacher guides, which together with textbooks play important roles in the work of mathematics teachers. However, mathematics education research on teacher guides is not as developed as research on textbooks (Jukić Matić & Glasnović Gracin, 2021). A review of research on teacher guides in any subject found only 43 studies from 1990-2020 (Buch et al., 2022). Results in the review are primarily at more general levels, for example, about teachers' views on guides, whether guides are used by teachers and how often and when. Few studies make specific links between properties of guides and the work of teachers. Our study contributes by developing design principles for teacher guides, in relation to principles for teaching and learning.

The aim of our ongoing study is to produce a deeper understanding of the role that teacher guides can play in supporting mathematics teachers, specifically addressing the following research question: How do different properties of a teacher guide affect teachers' work when planning and evaluating mathematics lessons?

Framework

We rely on design research methodology, including a focus on design principles, which refers to empirically and/or theoretically grounded descriptions and arguments of how to achieve a goal (van den Akker, 2010). We focus on two types of design principles.

First, we use design principles concerning teaching and learning, based on previous research. These design principles describe how the interaction between different forms of expression, or modes, in mathematics teaching can support students' learning, based on aspects of cohesion (e.g., similarity, continuity) and tension (e.g., contrast, distance) between modes. Our main design principle state that to promote learning, there must be a balance between cohesion and tension in the teaching situation, and tension between modes must be sharp enough to be challenging for the students (Engebretsen, 2012).

Second, through empirical analyses, we intend to develop design principles for a teacher guide. The purpose of our teacher guide is to support teachers to work in line with the principles of teaching and learning. Our study aims to create design principles on how properties of the teacher guide can enable teachers to receive this support.

Method

This study is based on two projects: One project with four teachers in lower primary school, delimited to subtraction, and one project with three teachers, one each from lower primary, upper primary, and lower secondary school, focusing on the concept of mathematical equality. In each project, we first met with the teachers to present and discuss the design principles for teaching and learning. Then followed cycles of testing

and revising a teacher guide of about two pages, where teachers and researchers met before and after lessons to jointly plan and evaluate the teaching. We recorded all meetings, covering 17 lessons, where 7 versions of the teacher guide have been used.

Data analyses clarify how properties of the teacher guide seem to influence teachers, which is done in two ways. First, teachers' explicit descriptions of the impact of properties are included. Second, we analyse how teachers relate to and (do not) use different parts of the guide in their work with planning, implementing and evaluating lessons.

Tentative results from ongoing analyses

So far, we have located some links between properties of the guide and teachers' work. In the project about subtraction, a change was made to the guide that seemed to support the teachers in a more concrete way to reflect on aspects of cohesion and tension to create a contrast between different modes. We changed from an overall focus on *expected challenges for the students*, with respect to using different modes, to *contrasts between modes* that are present in the teaching material. In the project about equality, revisions of the guide continued at a more detailed level, in line with the change of the overall focus. For example, when teachers talked about *contrasts for students* when planning lessons, we changed formulations in the guide, to instead of asking teachers to reflect on how students need to handle a contrast between modes, we asked in what way a *contrast between modes* should exist in the teaching material.

Based on these preliminary results, a tentative design principle can be formulated: A teacher guide for mathematics teachers should focus on supporting the teacher to reflect on properties of the teaching material to be used, since this seems to make the support more concrete and useful. Continued analyses will scrutinize this design principle in relation to other data and develop more design principles for teacher guides.

References

Buch, B., Gissel, S. T., Oksbjerg, M., Kjeldsen, K. & Albrechtsen, T. R. S. (2022). A systematic review of research on teachers' guides. *Learning Tech – Tidsskrift for læremidler, didaktik og teknologi*, *12*, 12-40. https://doi.org/10.7146/lt.v7i12.132330

Engebretsen, M. (2012). Balancing cohesion and tension in multimodal rhetoric: An interdisciplinary approach to the study of semiotic complexity. *Learning, Media and Technology, 37*(2), 145-162. https://doi.org/10.1080/17439884.2012.655745

Jukić Matić, L., & and Glasnović Gracin, D. (2021). How do teacher guides give support to mathematics teachers? Analysis of a teacher guide and exploration of its use in teachers' practices. *Research in Mathematics Education*, *23*(1), 1–20. https://doi.org/10.1080/14794802.2019.1710554

van den Akker, J. (2010). Building bridges: How research may improve curriculum policies and classroom practices. In S. Stoney (Ed.), *Beyond Lisbon 2010: Perspectives from research and development for education policy in Europe (CIDREE Yearbook 2010)*. National Foundation for Educational Research.

An analysis of secondary students' performance on two linear equation tasks

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This presentation extends the work presented in our previous papers (Holmlund, 2024; Tossavainen & Holmlund, 2024), focusing on why some students solve linear equations with natural numbers, but not equations with decimal and negative numbers. Analysis of interviews display that the numbers in the equations yield uncertainty concerning what is possible and allowed in handling them (Holmlund, 2024). The interviews also show that decimal and negative numbers attract students' attention, making them overlook other aspects of the equation, e.g., the operations. As algebra revolves around the analytic *handling* of unknown quantities (Radford, 2018) it is natural to put focus on operations in teaching algebra. Though based on our previous results, students also need to learn about the underlying algebra to be able to generalize their knowledge of equation solving (Tossavainen & Holmlund, 2024). Contemplating the relevance of the underlying algebra for students' solving of equations, it is relevant to explore how students perceive the algebra of an equation with parameters, such as $a = b \cdot x$. In mathematics, a and b are usually specified to a certain domain of numbers, but as students might not be familiar with these domains, there is reason to investigate how a and b are perceived.

Therefore, the aim of this short communication is to study quantitatively how students' ability to solve linear equations with parameters correlate to their solving of other similar equations—but with different operations or, especially, which number sets the contained coefficients in the equation belong to.

Method and results

The data analysed was collected from 110 students in their first year of upper-secondary school in the autumn 2020 (for more details, see Holmlund, 2024). The test, where students had calculator aid, included three purely numerical tasks, and ten equations: A. $819 = 39 \cdot x$, B. $2.01 = 0.434 + x$, C. $42 = 3 \cdot x$, D. $37 = 7 \cdot x + 2$, E. $352 = 63 + x$, F. $0.12 = 0.4 \cdot x$, G. $a = b \cdot x$, H. $-24 = 6 \cdot x$, I. $2.5 = 0.8 \cdot x + 2.1$, and J. $0.657 = 0.045 \cdot x$. In G, it was specifically asked what x is if a and b are two numbers we do not know the value of. Students' performances on each thirteen exercises have been evaluated on the dichotomous scale: 1 = a correct solution; 0 = an incorrect solution. Thereafter we have computed the phi coefficient for each pair of the

performance variables. To fit this analysis into a short communication, we choose to limit our analysis to two equations: $819 = 39 \cdot x$ and $a = b \cdot x$.

The phi coefficient is a measure of the strength of association between two binary variables, in our case the ability to solve different equations, based on independent observations. For dichotomous variables, it coincides the Pearson correlation coefficient. Consequently, it should be interpreted similarly. The values belonging to the interval (–0.3, 0.3) are often interpreted as weak or negligible relationship.

Focusing only on items with $\phi \geq 0.3$, students' solutions to $819 = 39 \cdot x$ are associated with their responses to equations B, C, D, E, H, and I. In contrast, their answers to $a = b \cdot x$ show no association with any other equations. We also note that most students who did not solve $a = b \cdot x$ left the blank answer. Some of these students gave an example of *one* possible solution—often using natural numbers for a and b. Among the students who successfully solved the task, some were able to generalize their solution after initially assigning concrete values to a and b.

Discussion

Our results show that successful solving $819 = 39 \cdot x$ is associated with success on several other tasks, particularly those involving only integers. However, performance on this equation does not correlate with that on equation F and G, both of which are multiplicative but involve decimal numbers. Furthermore, successful solving $a = b \cdot x$ is not significantly associated with success on any other tasks and shows only a weak positive association ($\phi = 0.21$) with $0.657 = 0.045 \cdot x$. Therefore, it seems unlikely that students share a common approach to solving equations with unknown parameters. However, we cannot say for sure, how students who solved the equation perceived the domain of a and b; some may have considered integers, while others another domain. Nevertheless, we can conclude that secondary students feel quite comfortable with integers—especially natural numbers—in linear equations, but they are clearly more insecure with equations that contain rational or real numbers.

Based on our findings, we conclude that secondary students would benefit from more experience from solving linear equations with parameters from different number domains. A topic for further research could be how their conceptions of underlying algebra are influenced when they encounter linear equation with unknown parameters.

References

Holmlund, A. (2024). How numbers influence students when solving linear equations. *Mathematical Thinking and Learning*, *27*(3), 305–322. https://doi.org/10.1080/10986065.2024.2314067

Radford, L. (2018). The emergence of symbolic algebraic thinking in primary school. In C. Kieran (Ed.), *Teaching and Learning Algebraic Thinking with 5- to 12-Year-Olds: The Global Evolution of an Emerging Field of Research and Practice* (pp. 3–25). Springer.

Tossavainen, T. & Holmlund, A. (2024). We can solve linear equations–in some algebras. *For the Learning of Mathematics, 44*(1), 38–40.

Recontextualization of discourses on mathematical reasoning within the curriculum chain

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This presentation focuses on mathematical reasoning, which is a core competence in curriculum documents, both in Sweden and many other countries. However, analyses show that there is great variation in what is meant by mathematical reasoning, for example, in curriculum documents (Reid, 2022) and among teachers (Herbert et al., 2015). It is thus important to analyse different parts of the *curriculum chain* (Stein et al., 2007) to understand different conditions for, and influences on, the teaching and learning of mathematics. Such a chain may, for example, consist of the following discourses: research → policy documents → textbooks → teaching → learning. The discourse on mathematical reasoning may be different in the different steps of the curriculum chain, especially since there is always a recontextualization, which means that a discourse "selectively appropriates, relocates, refocuses and relates other discourses to constitute its own order" (Bernstein, 2000, p. 33).

This study is part of a larger project that aims to understand the recontextualization of discourses on mathematical reasoning in the curriculum chain in Sweden and England. This study is delimited to how Swedish teachers' discourse on reasoning is similar or different to other discourses, focusing on the following research questions:

- RQ1: What different discourses exist in the conditions for teachers' work, regarding three types; research, everyday life, and formal governance?
- RQ2: How does teacher discourse relate to the different discourses in RQ1?
- RQ3: Do the relations in RQ2 differ depending on school level?

The analysis focuses on parts of the discourses where the notion of "reasoning" is explicitly used. We initially exclude textbooks, since they tend not to be explicit about when and how reasoning is relevant for specific tasks or activities.

Method

We use four data sets that correspond to four different discourses. *Teacher discourse* is investigated through interviews with five teachers from each of lower primary, upper primary, and lower secondary schools. They have long experience as teachers and come from different parts of Sweden and different types of schools. The interview had a variety of questions about the teacher's perspective on mathematical reasoning. A *research discourse* is investigated through three international frameworks that focus on school mathematics; Adding it up (Kilpatrick et al., 2001), NCTM Standards (NCTM, 2000),

and the KOM project (Niss & Højgaard, 2011). An *everyday discourse* is investigated through different types of Swedish dictionaries; Svensk Ordbok (https://svenska.se/so) and Wiktionary (https://sv.wiktionary.org). The *discourse of formal governance* of Swedish schools is investigated through the Swedish national curriculum for compulsory school, including the commentary material, and the national tests.

All data sources are analysed in the same way. First, we searched for passages including the word "reasoning" (Sw. "resonemang") or its cognates. Then, each such passage was analysed to find: (1) what processes or objects are identified as types of reasoning, and (2) how students are expected to engage with reasoning, to recognise (understand), to realise (construct), or to evaluate. For example, in "it consists of the ability *to devise* informal **arguments**", the process/object is arguments (bold), and the student engagement is a realisation (italics).

Tentative results from ongoing analysis

Analyses of the frameworks have been conducted and reported (Morgan et al., 2024). The results show a total of 39 processes/objects describing what mathematical reasoning is. Only five of these were common to all three frameworks: argument, (logical) chain, explanation, justification, and proof. Overall, the frameworks emphasise production of reasoning (i.e., realisation) over recognition and evaluation.

Ongoing analyses of the interviews have also produced a variety of different processes/objects, but any comparisons of discourses have not yet been done.

References

Bernstein, B. (2000). *Pedagogy, Symbolic Control and Identity: Theory, Research and Critique* (revised ed.). Rowman and Littlefield.

Herbert, S., Vale, C., Bragg, L. A., Loong, E., & Widjaja, W. (2015). A framework for primary teachers' perceptions of mathematical reasoning. *International Journal of Educational Research, 74*, 26-37. https://doi.org/10.1016/j.ijer.2015.09.005

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. National Academies Press.

Morgan, C., Bergqvist, E., Adler, J., & Österholm, M. (2024). Discourses of mathematical reasoning: analysis of three reform frameworks. In T. Fujita (Ed.) *Proceedings of the British Society for Research into Learning Mathematics 44*(1). BSRLM.

NCTM. (2000). *Principles and standards for school mathematics*. NCTM.

Niss, M., & Højgaard, T. (2011). *Competencies and mathematical learning: ideas and inspiration for the development of mathematics teaching and learning in Denmark* (English ed.). IMFUFA, Roskilde University.

Reid, D. A. (2022). 'Reasoning' in national curricula and standards. Presented at the Twelfth Congress of the European Society for Research in Mathematics Education, Bozen-Bolzano, Italy. https://hal.science/hal-03746833v2

Stein, M. K., Remillard, J., & Smith, M. S. (2007). How curriculum influences student learning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 1, pp. 319-369). IAP.

Two reading behaviours and their effect on reading comprehension of mathematical texts

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This presentation communicates the results of a pilot study involving eight upper secondary and ten lower secondary students reading mathematical texts. Using eye-tracking, we investigated whether excessive focus on symbols or longer gaze shifts impact reading comprehension. The study is part of a larger research project, with data collection planned from 75 students, ranging from lower secondary to tertiary education.

Linking the reading process to comprehension of mathematical texts offers insight into which reading skills are beneficial to develop. Experienced readers have learned to attend to all semiotic resources involved (Fang, 2023) while students tend to prioritize symbolic expressions over written language, which might lead to the neglect of important content and impaired comprehension (Dee-Lucas & Larkin, 1991; Österholm, 2008). Compared to students, mathematics professors also tend to shift more frequently between different parts of proofs, likely in order to grasp more implicit meaning (Inglis & Alcock, 2012).

Methods

We use six different mathematical texts for each educational level. Each participant read six texts with new mathematical content, each primarily presenting either mathematical concepts (2) or procedures (4). The texts combine written language and symbols, while half of the texts also include mathematical visualisations. Eye movements were recorded both while reading the texts and during the reading comprehension test that followed each reading. To distinguish between levels of comprehension, the reading comprehension test included both reproduction and application questions, corresponding to textbase and situation model questions (Kintsch, 1998). To clarify the students' answers, brief interviews were conducted after the eye-tracking session.

While the dataset from upper secondary education (8 students) was complete and suitable for statistical analysis, the dataset from lower secondary school (10 students) was not, and was therefore examined only exploratively to describe the reading behaviours that distinguish the most successful readers from the least successful. In the statistical analysis, reading comprehension was modelled as the dependent variable. As independent variables, we used the grades (A-E) of the last completed mathematics course and reading behaviour (length of gaze shifts/time spent reading). The analysis was conducted using Generalized Estimating Equations (GEE) to account for

correlations within repeated measurements, as each student read six different texts.

Results

The results showed that in the data from lower secondary level—consistent with previous research (Dee-Lucas & Larkin, 1991; Österholm 2008)—students appeared to focus particularly on symbols and formulas, at least in some contexts. For example, in a text presenting some concepts of symmetry, the students scored 44% correct on concept reproduction, 6% on concept application, and 75% on concept application when using a presented formula. Review of the eye-tracking video also showed that students displayed greater interest in the part of the text containing the formula, compared with the visualisations or the words describing the concepts.

The statistical analysis of the data from upper secondary school showed that both grades ($\beta = 2.31$, $p = 0.009$) and length of gaze shifts per time unit ($\beta = 1.32$, $p < 0.001$) were positively associated with reading comprehension. The interaction between these two variables ($\beta = -0.25$, $p = 0.002$) was statistically significant in predicting reading comprehension. That is, students who performed better on the comprehension test in general made longer gaze shifts per time unit, which is consistent with previous findings (Inglis & Alcock, 2012). However, this contrasts with our results for students with lower grades (D and E), who performed worse as their gaze shift lengths increased.

Discussion and implications

Although our results largely align with earlier research, they also raise questions about what constitutes optimal reading behaviour, and whether this can be measured in the same way regardless of students' levels of prior knowledge and skills. Moving forward, we plan to expand the statistical analysis, using more independent variables, including additional measures of reading behaviour—for example, the proportion of time spent on symbols and the number of visits to various text areas of interest. We also plan to analyse the eye-tracking data and test responses using qualitative methods.

References

Dee-Lucas, D., & Larkin, J. H. (1991). Equations in scientific proofs: Effects on comprehension. *American Educational Research Journal, 28*(3), 661-682. https://doi.org/10.3102/00028312028003661

Fang, Z. (2023). *Demystifying academic reading: A disciplinary literacy approach to reading across content areas*. Taylor & Francis. https://doi.org/10.4324/9781003432258

Inglis, M. & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. *Journal for Research in Mathematics Education, 43*, 358-390. https://doi.org/10.5951/jresematheduc.43.4.0358

Kintsch, W. (1998). *Comprehension: A paradigm for cognition*. Cambridge University Press.

Österholm, M. (2008). Do students need to learn how to use their mathematics textbooks? The case of reading comprehension. *Nordic Studies in Mathematics Education, 13*, 53-73. https://doi.org/10.7146/nomad.v13i3.148124

Spänning mellan lek och matematikundervisning i förskolans praktik

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Denna presentation riktar fokus mot matematikundervisning med de yngsta barnen, och hur sådan undervisning kan förstås i relation till förskolans uppdrag och praktik. Med bakgrund i en identifierad brist i tidigare forskning gällande hur matematikundervisning kan iscensättas tar studien sin utgångspunkt i ett praktiknära projekt att utveckla och implementera matematikundervisning som är responsiv på lek, så kallad lekresponsiv undervisning (LRU) (Pramling m.fl., 2019). LRU är i linje med det pedagogiska uppdrag som formuleras i den svenska förskolans läroplan i betydelsen att bevara lekens centrala roll samtidigt som barns uppmärksamhet riktas mot specifika kunskapsinnehåll. Studien syftar till att fördjupa kunskapen om det komplexa samspelet mellan målinriktad matematikundervisning och barns lek inom ramen för förskolans praktik. Studien vägleds av följande forskningsfråga: Vilka utmaningar framträder när matematikundervisning och lek kombineras i förskolans praktik?

Bakgrund

För matematikundervisning av hög kvalitet är lärare den viktigaste faktorn (Clements et al., 2023). Lärares uppfattningar om undervisning i förskolan har dock visat sig varierar stort, från att undervisning inte hör hemma i förskola till att allt på förskola är undervisning (Vallberg Roth, 2020). I linje med denna osäkerhet kring förskolans undervisning visar en nationell granskning av förskolan (Skolinspektionen, 2016) att potentiella undervisningssituationer inte tas till vara samt att målstyrda processer ofta saknas. Forskning visar vidare att lärares uppfattning av undervisning har en praktikbaserad förankring med svag teoretisk anknytning (Vallberg Roth, 2020), vilket kan betraktas som särskilt problematiskt då forskning om matematikundervisning med de yngsta barnen fortfarande är begränsad (Svane et al., 2023).

Förskolans matematikundervisning kan dock inte reduceras till en specifik metod (Clements et al., 2023). Undervisning är mångfacetterad och rymmer en variation av former, från ”fri lek” till lek vägledd av vuxna, till undervisning som är lekfull men lärarledd. Denna studie tar sin utgångspunkt i en form som förenar lek och undervisning i en enhetlig process, så kallad LRU (Pramling m.fl., 2019). LRU benämns som både en teori och didaktik för förskola där responsivitet på barns initiativ står i centrum. Lekresponsiv matematikundervisning formas i interaktionen mellan barn och lärare där matematikinnehållet är relevant för leken och meningsfull för barnen.

Metod

Studie är en interventionsstudie där två arbetslag i samverkan med forskaren under ett läsår planerat, iscensatt och utvärderat Lekresponsiv matematikundervisning i en iterativ

design. Interventionen genererade empiriskt material bestående av videoinspelning av undervisningssituationer samt ljudinspelning från reflektionstillfällen, där arbetslagen analyserade och utvärderade undervisningen. I denna presentation har lärarnas reflektioner analyserats induktivt och tematiskt.

Resultat

Resultatet visar vissa spänningar kring lek. Å ena sidan ses lek som en självklar och central del av förskolans praktik, å andra sidan framträder en osäkerhet gällande vad som är lek och hur den kan definieras. En syn på lek som självklar kan återspegla ett motstånd mot pågående förändring i svensk förskola som av vissa uppfattas som ett hot mot lek. Svårigheten att definiera lek pekar på dess komplexa och mångfacetterade natur där lek till stor del upplevs snarare än verbalt definieras. Vidare framkommer en osäkerhet gällande lärarens roll i lek, där frihet, glädje, spontanitet och barns initiativ lyfts fram som centrala kännetecken i relation till lek.

Lek framhålls som viktig i syfte att göra matematikundervisning rolig, spännande och intressant, vilket i sin tur anses främja barns uppmärksamhet och engagemang. Lärarna anger barns engagemang i aktiviteter som centralt för lärande, att barn ”gör matematik” betonas. Detta tyder på en utbildningsfilosofi som värdesätter lärande genom handling och utforskning snarare än passiv mottagning av information. Dessutom distanserar lärarna förskolan från formaliserad undervisningsmetod i skolan. Enligt lärarna är lek och undervisning i förskolan integrerade på ett sätt som skapar en mer flexibel och glädjefylld lärandemiljö än den i skolan. Denna distansering från skolan kan ses som ett försök att bevara förskolans unika karaktär där lek är en viktig komponent.

Referenser

Clements, D. H., Lizcano, R., & Sarama, J. (2023). Research and Pedagogies for Early Math. *Education Sciences, 13*(8), 839-. https://doi.org/10.3390/educsci13080839

Pramling, N., Wallerstedt, C., Lagerlöf, P., Björklund, C., Kultti, A., Palmér, H., Magnusson, M., Thulin, S., Jonsson, A., & Pramling Samuelsson, I. (2019). *Play-Responsive teaching in early childhood education*. Springer. https://doi.org/10.1007/978-3-030-15958-0

Svane, R. P., Willemsen, M. M., Bleses, D., Krøjgaard, P., Verner, M., & Nielsen, H. S. (2023). A systematic literature review of math interventions across educational settings from early childhood education to high school. *Frontiers in Education (Lausanne), 8*. https://doi.org/10.3389/feduc.2023.1229849

Skolinspektionen. (2016). *Förskolans pedagogiska uppdrag: Om undervisning, lärande och förskollärares ansvar.* Skolinspektionen.

Vallberg Roth, A. (2020). What may characterise teaching in preschool? The written descriptions of Swedish preschool teachers and managers in 2016. *Scandinavian Journal of Educational Research, 64*(1), 1–21. https://doi.org/10.1080/00313831.2018.1479301

Designing inclusive learning environments through talk in co-taught secondary classrooms

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Background

This presentation reports early findings from an educational design research study in secondary mathematics education, framed by two major reform ideals that currently influence policy and practice in Sweden and internationally: the demand for high-quality instruction and the pursuit of inclusive education.

Ambitious mathematics instruction (AMI) has emerged as a response to both reforms, aiming to support *all* students in engaging with mathematical reasoning and classroom discourse. This as, rather than simplifying tasks for struggling students, AMI emphasises instruction that is responsive, dialogic, and deliberately designed to help students take intellectual risks and participate in whole-class reasoning. However, sustainable implementation of AMI requires structural and instructional alignment, particularly in supporting all students to access complex content. Proactive support, planned in advance, is necessary to prepare these students for participation in rich mathematical dialogue (Cobb & Jackson, 2021).

Alongside instructional approaches such as AMI, co-teaching has emerged as a structural strategy to foster inclusive learning environments, bringing together general and special education teachers in shared classroom practice. While co-teaching holds potential for addressing needs for all learners, it often manifests in a "one-teach, one-assist" arrangement (Gardesten, 2023). As a result, the full potential of co-teaching to support both academic and social inclusion remains unrealised.

The overall aim of this study is to add to the knowledge base of teacher practices and learning environments in co-taught secondary mathematics classrooms that promote participation in mathematical reasoning for all students. The study seeks to collaboratively develop, test, and refine design principles for ambitious mathematics instruction and is guided by the following research questions: *How can co-teaching practices be structured in secondary mathematics education when the aim is to support all students' participation in mathematical reasoning?*

Theoretical and methodological framing

The study adopts an educational design research (EDR) approach (Bakker, 2018) and draws on two interwoven theoretical frameworks: First, instructional vision (Munter, 2014) and teacher self-efficacy (Bandura, 1997) are used as meta-design principles, guiding how the interventions support teacher learning. Vision interviews help surface

teachers' criteria for high-quality instruction, while self-efficacy structures support collaborative enactment and reflection. Then, AMI (Jackson et al., 2017) provides the instructional ideal, emphasising students' conceptual engagement and teachers as facilitators. Cobb and Jacksson's (2021) systems framework for AMI grounds the need for proactive support, coherent instructional support, especially for students traditionally marginalised in mathematical reasoning.

Two tentative design principles have been identified to support inclusive learning environments and scaffold teacher practice: Talk moves (Kazemi & Hintz, 2014) may help teachers to reorient from focusing on correctness to valuing student reasoning, and proactive support (Cobb & Jackson, 2021) may challenge deficit views and enable participation for all students. These principles are deliberately modest and introduced early in the intervention to create entry points for teachers. The principles are refined through iterative design cycles involving collaborative planning, classroom enactment, and reflection.

The study involves two co-teaching teams from different secondary schools. Each team will participate in three design cycles during a term. Data include planning/reflection meetings, video-recorded lessons, teaching materials and student work.

At MADIf 15, I will report early insights from the design cycles and engage with colleagues around definitions of inclusive learning environments in secondary mathematics education.

References

Bakker, A. (2018). *Design Research in Education: A Practical Guide for Early Career Researchers*. https://doi.org/10.4324/9780203701010

Bandura, A. (1997). *Self-efficacy : the exercise of control*. W. H. Freeman.

Cobb, P., & Jackson, K. (2021). An Empirically Grounded System of Supports for Improving the Quality of Mathematics Teaching on a Large Scale. *Implementation and Replication Studies in Mathematics Education*, *1*, 77-110. https://doi.org/10.1163/26670127-01010004

Gardesten, M. (2023). How co-teaching may contribute to inclusion in mathematics education: A systematic literature review. *Education Sciences*, *13*(7), 1-17. https://doi.org/https://doi.org/10.3390/educsci13070677

Jackson, K., Gibbons, L., & Sharpe, C. J. (2017). Teachers' Views of Students' Mathematical Capabilities: Challenges and Possibilities for Ambitious Reform. *Teachers College Record*, *119*(7), 1-43. https://doi.org/10.1177/016146811711900708

Kazemi, E. & Hintz, A. (2014). Intentional Talk: How to Structure and Lead Productive Mathematical Discussions. Portland: Stenhouse Publisher.

Munter, C. (2014). Developing Visions of High-Quality Mathematics Instruction. *Journal for Research in Mathematics Education*, *45*(5), 584-635. https://doi.org/10.5951/jresematheduc.45.5.0584

Att använda kollegiala samtal för att förbättra professionell bedömning i matematik

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Denna presentation ger en översikt över ett internationellt samarbetsprojekt som syftar till att stärka professionell bedömning i matematik. Inom utbildning syftar professionell bedömning på användandet av kunskap, färdigheter, erfarenhet eller intuition för att uppfatta och utvärdera kvalitet, styrkor och svagheter i studenters arbeten. Bedömning är den grundläggande basen för lärares pedagogiska beslut och handlingar i klassrummet (Poskitt & Mitchell, 2012). Trots att det finns omfattande forskning om professionell bedömning bland lärare i grund- och gymnasieskolan, saknas det i stor utsträckning empiriska studier som undersöker hur universitetslärare i matematik bedömer studenters matematiska resonemang och prestationer. Med hjälp av teoretiska ramverk för lärares bedömningskompetens (Xu & Brown, 2016) samt bedömnings- och beslutsfattandeprocesser (Borko et al., 2008), adresserar denna studie denna forskningslucka genom: (a) ett deltagarbaserat kompetensutvecklingsinitiativ som främjar tillförlitlig och konsekvent bedömning genom modererade, strukturerade matematiska samtal mellan kollegor, samt (b) en undersökning av vilka utmaningar lärare möter när de bedömer studentarbeten i matematik. Genom att betona samarbete och dialog mellan matematiker främjar projektet kommunikativa praxis som kan stärka kvaliteten på professionell bedömning i matematikundervisningen.

Metod

Studien genomfördes under en period av 18 månader och involverade fyra universitetslärare med specialisering inom tillämpad och teoretisk matematik. Dessa undervisade grundkurser i komplex analys, statistik och talteori vid två högre utbildningsinstitutioner i Norge och Tyskland. Studien genomfördes i två faser. I den första fasen utvecklade lärarna gemensamt bedömningsverktyg, såsom bedömningsmatriser och exempelarbeten, i syfte att stödja en konsekvent och transparent professionell bedömning. De förde samtal för att etablera gemensamma kriterier och standarder för att bedöma studenters matematiska arbeten. I den andra fasen använde lärarna dessa matriser för att analysera studentarbeten tillsammans med

kollegor i strukturerade samtal som främjar reflektion. Data om professionell bedömning och beslutsfattande samlades in från tre källor:
(a) Tänka-högt-protokoll: Lärarna arbetade i par och verbaliserade sina tankar och resonemang medan de bedömde studenters arbeten. De motiverade sina bedömningar med hänvisning till matrisen. Om samsynen mellan två bedömare var otillräcklig, tillkallades en tredje.
(b) Kognitiva kommentarer: En kognitiv kommentar hjälper till att synliggöra lärares tänkande och professionella bedömning, som ofta är implicit och osynlig (Wyatt-Smith, et al., 2024). Lärarna skrev ned sina resonemang och motiverade de poäng de tilldelade.
(c) Fokusgrupp: De deltagande lärarna ombads att reflektera över vilka utmaningar de mött under bedömningspraktiken, utifrån styrande frågor. Fokusgrupp-intervjun (90 minuter) spelades in och transkriberades ordagrant med ett automatiskt transkriptionsverktyg.

Resultat och diskussion

Analys av de olika datakällorna visade att lärarnas professionella bedömningar i hög grad är kontextberoende och ämnesspecifiktat. Deras bedömningar formades genom kontinuerliga kompromisser och förhandlingar mellan personliga övertygelser och ämnesdidaktisk kunskap (t.ex. kännedom om kursplaner, förväntad lärandeprogression och studentprofiler). Dessa resultat överensstämmer med tidigare forskning om att professionell bedömning till sin natur är adaptiv och situationsbaserad, och betonar värdet av strukturerade samtal med fokus på matematik för att öka träffsäkerheten i bedömning. Studien belyser behovet av skräddarsydd, evidensbaserad kompetensutveckling som bygger på kollegiala samtal för att stödja universitetslärare i att vidareutveckla sina professionella bedömningar och beslut (det bör noteras att matematiklärarna också involverade sina studenter i att diskutera kvalitetskriterier i både bedömningsmatris och exempel, men i denna presentation fokuserar vi på lärarnas uppfattningar och utvärdering av kvalitetskriterier genom kollegiala diskussioner).

Referenser

Borko, H., Roberts, S. A., & Shavelson, R. (2008). Teachers' decision making: From Alan J. Bishop to today. In P. Clarkson, & N. Presmeg (Eds.), *Critical issues in mathematics education. Major contributions of Alan Bishop* (pp. 37e67). Springer.

Poskitt, J., & Mitchell, K. (2012). New Zealand teachers' overall teacher judgements (OTJs): Equivocal or unequivocal? *Assessment Matters*, *4*, 53–75.

Wyatt-Smith, C., Adie, L. E., & Harris, L. (2024). Supporting teacher judgement and decision-making: Using focused analysis to help teachers see students, learning, and quality in assessment data. *British Educational Research Journal, 50*(3), 1420–1448. https://doi.org/10.1002/berj.3984.

Xu, Y., & Brown, G. T. L. (2016). Teacher Assessment Literacy in Practice: A Reconceptualization. *Teaching and Teacher Education*, *58*, 149-162. https://doi.org/10.1016/j.tate.2016.05.010

Using questions to develop children's mathematical reasoning and conceptual understanding in preschool

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This presentation concerns an ongoing project about mathematical communication in preschool. The project aims to understand how talk moves (Kazemi & Hintz, 2014), and different types of questions (Cunningham, 1987; Kilhamn & Liljekvist, 2018) can be used in the teaching of mathematics. This presentation will focus on questions and how these can be used to promote children's conceptual understanding and collective (Sumpter & Hedefalk, 2015) and creative mathematical reasoning (Lithner, 2008) in a preschool setting. The ability to reason mathematically has been shown to predict later mathematical performance in school (Nunes et al. 2012), and conceptual understanding is needed to be able to reason and communicate about and in mathematics. Children already reason at preschool age, but the development of mathematical reasoning requires teaching from someone who asks questions and helps children make progress (see e.g. Hedefalk et al., 2024).

The four types of questions used in the project are: factual questions, focusing questions, conceptual questions, and evaluative questions (Cunningham, 1987; Kilhamn & Liljekvist, 2018). *Factual questions* are asked by the preschool teacher when they already know the answer. This type of closed question can be used to introduce a new concept or clarify a concept. For example: "What is this shape called?", "What is the name of...?" *Focusing questions* direct the children's attention to something specific. This type of question can be used to find out the child's understanding of a concept. For example: "Can you see a figure that has the same shape?"—while pointing to a circle. *Conceptual questions* are used by preschool teachers to encourage children to suggest strategies and describe their thinking and reasoning. These questions help move the children's thoughts forward and allow them to verbalise their experiences or solutions. For example: "Could you do it in another way?", "What happens if…?" *Evaluative questions* are used to encourage children to justify and argue, to provide evidence for different thoughts and claims. For example: "How come?", "How do you know that?" The different types of questions have been discussed in research on early childhood mathematics education. However, it appears that preschool teachers primarily use the first two types of questions, that is, closed questions that are more intended to introduce a new concept or direct the children's attention to something than to challenge ideas or develop reasoning (e.g. Björklund & Palmér, 2024; Carlsen, 2013). It can therefore be assumed that asking open-ended questions, such as conceptual and evaluative questions, is something preschool teachers need to practice.

Therefore, one aim of this project is to identify the potential uses of each question type, and to explore and examine how conceptual and evaluative questions can be formulated in relation to specific mathematical domains in preschool. The research question is: 1) How can different types of questions be used as tools to develop children's mathematical reasoning and understanding of mathematical concepts?

This newly started project is conducted as action research in collaboration with teams of preschool educators from one Swedish preschool. Through multiple cycles of planning (including input from researchers and practising different questions), action, observation, and reflection, we aim to jointly identify effective questions in preschool mathematics. The "action" refers to teaching designed to develop children's conceptual understanding and mathematical reasoning.

The presentation will reveal preliminary findings, discuss methodological issues and the planned way forward.

References

Björklund, C., & Palmér, H. (2024). The challenges of mathematizing in Swedish early childhood education. *Journal of Early Childhood Education Research, 13*(2), 167-186.

Carlsen, M. (2013). Engaging with mathematics in the kindergarten. Orchestrating a fairy tale through questioning and use of tools. *European Early Childhood Education Research Journal*, *21*(4), 502–513. https://doi.org/10.1080/1350293X.2013.845439

Cunningham. R, (1987). What kind of question is that? In W. Wilen (Ed.) *Questions, Questioning Techniques, and Effective Teaching*, (s. 67-93). National Education Association of the United States.

Hedefalk, M., Sumpter, L., & Eriksson, H. (2024). Matematiska och etiska resonemang i förskolan: didaktisk modellering som intervention. *Forskning om undervisning och lärande, 12*(2), 68-84.

Kazemi, E. & Hintz, A. (2014). *Intentional talk: How to structure and lead productive mathematical discussions.* Stenhouse Publishers.

Kilhamn, C. & Liljekvist, Y. (2018). Interaktion i matematikklassrummet. In O. Helenius (Ed), *Att bli lärare i matematik.* Liber.

Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics, 67(*3), 255–276. https://doi.org/10.1007/s10649-007-9104-2

Nunes, T., Bryant, P., Barros, R., & Sylva, K. (2012). The relative importance of two different mathematical abilities to mathematical achievement. British Journal of Educational Psychology, 82(1), 136–156.

Sumpter, L., & Hedefalk, M. (2015). Preschool children's collective mathematical reasoning during free outdoor play. *The Journal of Mathematical Behavior, 39,* 1-10.

Coordinating Teacher Noticing with Responsive Teaching to Support Mathematical Dialogue

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This presentation is about a research project in which three teachers and two researchers collaborated to understand more about students' doings and learnings in mathematical dialogue. Using Teacher Noticing (Sherin et al., 2011) together with Responsive Teaching (Dyer & Sherin, 2016), made it possible to understand how two concrete examples of classroom management approaches affected the classroom interaction: the use of image support and the development of open mathematical questioning. The research question guiding this short communication is: How can Teacher Noticing and Responsive Teaching be used together, to inform teachers' actions in and promotion of mathematical dialogue?

Teacher Noticing and Responsive teaching

In this project, three teachers worked with strategies on Responsive Teaching, striving to 1) ask students to develop their thinking, 2) ask students to follow up on each other ideas and 3) follow up students' suggestions (Dyer & Sherin, 2016). This framework was coordinated with Teacher Noticing (Sherin et al., 2011) in which teacher *attend* to certain situations in the classroom, *interpret* the situations (with help of responsive teaching) and thereafter choose how to *act/respond*.

Method

Two problem-solving activities on geometry were designed, one for middle school and one for lower secondary school. The activities were implemented several times with different groups of students. In middle school, we *attended* to the use of mathematical questioning, and in lower secondary school, we *attended* to the use of image support. In the analysis, we identified and *interpreted* differences in implementation between the first and the last lesson in relation to teachers' work with responsive teaching. The *acting/responding* was related to what teachers learnt from the first lesson to the last.

Four video-recorded lessons were analysed regarding how the three strategies in the Responsive Teaching framework affected the mathematical dialogue.

Learnings

The first lessons were not very pre-planned. Rather, teachers gave students problem-solving tasks related to geometry, and then curiously *attended* to what happened in the mathematical dialogue. Teachers *interpreted* the results as if students were eager to find the correct solution to the problem, but that they were less interested in understanding

the mathematics behind the solution or deepening their mathematical thinkings. Students' focus was on doings rather than learnings. Few Responsive Teaching strategies were utilized. Teachers realised that they had an important role in giving students support in their interactions and chose to *act/respond* by developing the lesson to include more open questions when students did not continue their strive for learning, or image support, to scaffold the dialogues and make students explain their thinking to each other. In the last lesson, the teachers were more active in using Responsive Teaching techniques and shifting students' focus to not only want to find the correct answer but also understand and learn the mathematics and follow up on each other's ideas. Teachers actively used questions to make students develop their thinking, they gave students image support and concrete lesson materials to use in group discussions, and they followed up on students' ideas in whole class discussions.

Conclusions on using the two methodological frameworks together to understand mathematical dialogue

The use of Teacher Noticing (Sherin et al., 2011) together with Responsive Teaching (Dyer & Sherin, 2016), made it possible to understand and systematically sort what parts of the mathematical dialogues that teachers wanted to *attend* to, to *interpret* what happened using the three strategies in responsive teaching, and to *act/respond* to better support students' needs. The two models together enabled a deep focus when wanting to understand the dialogues while using students' own ideas and responding to them.

There might however also be problematiques that need to be considered when using the two frameworks together. Hence, our intensions forward are to investigate the possibilities to network the frameworks, for instance by coordinating them (Prediger et al., 2008), looking at similarities in system of principles, methodologies and research questions (Radford, 2008). Also, the strategies included in the Responsive Teaching framework, might not correlate to all actions taken by the teachers. Additional ways of *acting/responding* might have occurred. These two unsolved questions will be further investigated in coming research publications (Sjöblom & Silwer, to be submitted).

References

Dyer, E. B., & Sherin, M. G. (2016). Instructional reasoning about interpretations of student thinking that supports responsive teaching in secondary mathematics. *ZDM 48*(1), 69-82. https://doi.org/10.1007/s11858-015-0740-1

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework. *ZDM, 40*(2), 165-178. https://doi.org/10.1007/s11858-008-0086-z

Radford, L. (2008). Connecting theories in mathematics education: challenges and possibilities. *ZDM 40*(2), 317-327. https://doi.org/10.1007/s11858-008-0090-3

Sjöblom, V. M., & Silwer, M. (to be submitted). *Using teacher noticing to develop responsive teaching in mathematical dialogue.*

Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (2011). *Mathematics teacher noticing: seeing through teachers' eyes.* Taylor and Francis.

A pilot study on using data cards for inferential statistics in primary teacher education

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This presentation examines preservice teachers' perspectives on statistical concepts and ideas as they engage with a teaching sequence on data modelling using biology-related content in teacher education. Inferential reasoning is increasingly recognised as a core statistical competence that learners of all ages should acquire (Makar & Rubin, 2018). Yet, little is known about didactic resources that support students in acquiring such competences through a holistic approach to teaching, where statistical concepts are explored in meaningful and integrated contexts rather than in isolation (Burrill & Pfannkuch, 2024). In primary school, promoting inferential competence through a data modelling approach involves more than supporting pupils in drawing conclusions. It encompasses developing their competence in posing statistical questions, collecting and interpreting data, describing patterns, and reasoning about uncertainty (Lehrer & English, 2018). Despite the recognised importance of these competencies, research remains limited on effective ways of teaching core statistical ideas in real-world contexts, and even less is known about preservice teachers' beliefs and competence in this area.

Data cards as a didactic resource

Technological advancements have greatly simplified the collection and processing of first-hand data and this holds also for downloading, compiling and processing second-hand data sets. Yet, one drawback of generating first-hand data is that it can be time-consuming. Conversely, a limitation of using second-hand data is that it removes the learning opportunity to design the measurement process for the property under investigation. Research suggests strong potential in using data cards containing authentic datasets for students to interpret (Burrill & Pfannkuch, 2024). Hence, our study aims at developing teaching activities that integrate student-relevant ready-made data sources with an explicit focus on core statistical content. This approach streamlines classroom implementation while deepening students' understanding of the underlying concepts in statistical inference. The study is guided by the following research question: *What statistical competence and curricular goals do preservice teachers articulate when working with data cards as a didactic resource in the classroom?*

Methodology

This short presentation reviews the design when implemented in primary school teacher education and discusses its theoretical basis. This pilot study adopts an educational design research approach, which comprises three main phases: analysis and exploration, design and construction, and evaluation and reflection (McKenney & Reeves, 2018). In the first phase, we analyse the opportunities and challenges associated with introducing statistical inference in primary school mathematics. This has resulted in the development of a set of age-appropriate data cards based on real local biological conditions as a tentative didactic resource. Two versions of the data cards were produced – one tailored for early primary school students and one for older primary students.

The analysis is guided by praxeology (Chevallard, 2006) as an overarching theoretical lens, complemented by a framework that integrates positioning structures with data modelling (Blomberg, under review). While praxeology helps distinguish between what is done (praxis) and what is said about what is done (logos), the theory of positioning structures offers a valuable didactic tool for interpreting how mathematical ideas are positioned in relation to human activity. Together, these perspectives provide tools for examining how preservice teachers articulate and position core inferential and statistical concepts when working with data cards in the classroom.

Conclusion

Placing data in authentic contexts highlights the subject-integrative potential central to our design. The age-appropriate data cards are intended to help students to pose meaningful questions, consider alternative interpretations, and draw conclusions from uncertain information. This approach reflects our aim to embed core statistical ideas, especially inferential reasoning within relevant contexts, making them both accessible and conceptually rich. In a world shaped by data and AI, such experiences are essential for developing the statistical literacy needed for informed participation in society.

References

Burrill, G., Pfannkuch, M. (2024). Emerging trends in statistics education. *ZDM Mathematics Education 56*, 19–29. https://doi.org/10.1007/s11858-023-01501-7

Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In *Proceedings of the IV congress of the European society for research in mathematics education* (pp. 21-30).

Lehrer, R., & English, L. (2018). Introducing children to modeling variability. In D. Ben-Zvi, K. Makar, & G. J. (Eds.), *International Handbook of Research in Statistics Education* (pp. 229–260). Springer International Publishing. https://doi.org/10.1007/978-3-319-66195-7_7

Makar, K., & Rubin, A. (2018). Learning about statistical inference. In D. Ben-Zvi, K. Makar, & J. Garfield (Eds.), *International handbook of research in statistics education* (pp. 261–294). Springer International Publishing. https://doi.org/10.1007/978-3-319-66195-7_8

McKenney, S., & Reeves, T. C. (2018). *Conducting educational design research* (2nd ed.). Routledge. https://doi.org/10.4324/9781315105642

Unequal participation in mathematics? Towards a model networking commognition and ATD

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This presentation focuses on theoretical perspectives for understanding students' unequal access to and participation in the canonical mathematical discourse, which are shaped by a wide range of social and educational factors. While some students engage fluently in classroom practices that align with the institutionalised norms, others remain on the periphery; excluded, silent, or disengaged. This disparity raises critical questions about how the learning is organised and who gets to be recognised as a legitimate participant in mathematical activity. Addressing these questions involves critically examining the norms, values, and practices that are often taken for granted in mathematics education, which is essential for understanding how certain mathematical practices are recognised as legitimate, and for contributing to practices that promote social justice. The theoretical networking in this paper is part of a PhD project, where the focus is on mathematics in upper secondary school in Sweden; specifically social justice perspectives on school algebra in vocational education and higher education preparatory programmes. The algebra content in the mathematics courses differs depending on the programme the students attend. These differences raise questions about what mathematical objects are made available to whom, and under what conditions.

To this end, we explore the potential of networking two theoretical frameworks within mathematics education research: the Anthropological Theory of the Didactic (ATD) (Bosch & Gascón, 2006) and commognition (Sfard, 2008). The ATD provides a lens for examining the institutional organisation of mathematical knowledge, enabling analyses of how access to mathematical content is mediated by didactic transpositions and institutional (e.g., discursive) constraints. Commognition, on the other hand, emphasises the discursive nature of learning and the development of mathematical thinking through participation in specific forms of communication. It offers tools for analysing how students' identities and positions are shaped through discourse, which is crucial for understanding why some students may not identify with mathematics or feel excluded from the mathematics classroom. While both frameworks engage with discourse, they focus on different dimensions: commognition on the organisation of participation and identity, and ATD on the organisation of content and practice. Together, they offer complementary perspectives for investigating how mathematics education can reflect and reproduce social inequalities, and how it might be reimagined to support more equitable participation. Therefore, we want to highlight the need for a more integrated framework that can account for both i) the institutional organisation of mathematical content, and ii) the discursive construction of identity.

Commognitive theory defines discourse through four key components: words, visual mediators, routines, and narratives, which all help us describe how mathematical ideas are communicated and negotiated. However, it does not explicitly address how the content of discourse is organised or how it shifts across institutional contexts. This is where ATD offers complementary insights (see Bosch & Gascón, 2006); the notion of *praxeology* allows for a detailed examination of how mathematical knowledge is organised within a given teaching institution, while the concept of *didactic transposition* addresses how scholarly knowledge is transformed into teachable content within educational institutions, a process involving deliberate choices that reshape the mathematical content to fit pedagogical aims and institutional constraints. These can be understood in terms of the *scale of levels of didactic co-determination* that can be used to identify and analyse external influences, ranging from national curricula and institutional policies to broader societal expectations. We argue, that understanding how mathematical content is structured and made accessible within educational settings, is central when analysing students' opportunities to engage meaningfully in mathematics. Yet, ATD is limited when it comes to understanding students' identity and unequal participation.

Commognition and ATD both offer perspectives on participation. However, they differ significantly in how they conceptualise the role of the *individual*. More recent attempts to incorporate individual perspectives within ATD have been made. For instance, the notion of *personal relations* (Chevallard, 2019), aims to account for how individuals relate to didactic systems. However, these efforts remain limited. The structuralist orientation of ATD tends to reduce individual agency to institutional positioning, assuming that interactions are primarily shaped by the roles individuals occupy within educational institutions. This approach offers little insight into *how* identities are construed or *how* they influence learning. In contrast, commognition provides a more nuanced account of *identity* by treating it as discursively constituted. Identity is constructed through narratives of different forms; 1st-person, 2nd-person, and 3rd-person, which reflect and shape individuals' participation in discourse (Sfard, 2008). From this perspective, identity is inherently linked to learning: to learn mathematics is to become a participant in mathematical discourse. This makes it essential to understand the characteristics of such discourse, for example in domains like school algebra.

Together, these frameworks reveal that participation in mathematics is not solely a matter of individual engagement or classroom dynamics, nor merely a consequence of curricular design. Rather, it emerges through the interaction between students' discursive positioning and the institutional regulation of access to mathematical content.

References

Bosch, M. & Gascón, J. (2006). Twenty-five years of the didactic transposition. *ICMI Bulletin 58*, 51–63.

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. *Hiroshima journal of mathematics education, 12,* 71–114.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing.* Cambridge: Cambridge University Press.

Solving Ambiguity in Mathematics Text

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This presentation reports on the results of an in-depth analysis of ambiguity in mathematics task texts, with particular emphasis on the interpretative challenges such ambiguity poses for students. Research on word problems in mathematics education has long emphasized the challenges of comprehension and the importance of precise mathematical terminology for successful problem solving (Verschaffel et al., 2020). Studies have shown that interpreting word problems requires not only mathematical competence but also linguistic and cognitive strategies to navigate the textual and representational complexity. While considerable attention has been given to the role of vocabulary and the structure of mathematical language (Niss & Jablonka, 2020), in fact, many words, symbols, and images do not carry fixed meanings but instead require the reader to infer the intended sense based on surrounding textual and situational cues. There is still much to uncover regarding the interpretive demands posed by ambiguous representations within task texts, particularly in how students navigate multiple possible meanings (Barwell, 2005; Peterson et al., 2020).

The IRIS model (Dyrvold & Ribeck Nyström, 2025) outlines the interpretive path a reader follows when resolving ambiguity, to assign a particular sense to a representation. For each representation, the reader first determines whether it is used in a technical sense. If so, they assess whether it relates to the current subject or another. If not technical, the focus shifts to whether the representation is used in an academic or everyday sense.

In this case study, we use the IRIS model as an analytical lens to explore how ambiguity in mathematical representations affects the reader's discernment process, highlighting cognitive and linguistic demands that may otherwise remain invisible. The analysis focuses on two mathematics tasks: one contextualized in everyday life and one intra-mathematical. In each task, all words, symbols, and images are examined. For each representation, we note both its possible meanings (independent of context) and its intended sense within the specific task.

While almost all representations exhibit a degree of ambiguity (Barwell, 2005), the findings, consistent with expectations, suggest that those with a technical sense tend to be comparatively less ambiguous. An ambiguous representation in the contextualised task is "coffee scoop" which can be interpreted in a technical sense (as an object with an exact volume) or in an everyday sense (as a kitchen utensil). The contextual cues

"decilitre" and a table with volumes of measuring utensils contribute to reveal the intended, technical, sense.

In both analysed tasks the introductory premise consists of one or more *constructions* with a technical sense that must be recognised. Constructions are conventionalized patterns of one or more representations with a specific form and meaning (Goldberg, 2013). An example of such a construction in the data is

$$" A = \frac{B}{B+1} \ where \ B \ is \ a \ positive \ number. "$$

In this context, the construction represents a conventionalized way to clearly relate a *mathematical identity,* specifically, an equation, to a *mathematical statement* using the word "where". It qualifies as a construction because the intended meaning depends on recognising the overall pattern, not just the individual elements. This pattern helps eliminate potential ambiguity in the singular components, as it is the technical function of the construction as a whole that guides its interpretation. In conclusion, disambiguating mathematics task text entails discernments spanning from disambiguating singular representations as words or numbers, to the identification of larger constructions that must be prioritised over the included singular representations.

In our conference presentation, we showcase the two mathematics tasks and walk through our analytical process using the IRIS model, illustrating how ambiguity is navigated and resolved in practice. By highlighting the interpretive work involved, we hope to spark a conversation about the broader cognitive and linguistic dimensions of mathematical meaning-making and the educational relevance of our findings across diverse learning contexts.

References

Barwell, R. (2005). Ambiguity in the mathematics classroom. *Language and Education, 19*(2), 117–125.

Dyrvold, A., & Ribeck Nyström, J. (2025). A subject language model of Interpreting Representations in their Intended Senses. *Designs for Learning, 16*(1), 88–99. https://doi.org/10.16993/dfl.231

Peterson, B. E., Leatham, K. R., Merrill, L. M., Van Zoest, L. R., & Stockero, S. L. (2020). Clarifiable ambiguity in classroom mathematics discourse. *Investigations in Mathematics Learning, 12*(1), 28–37. https://doi.org/10.1080/19477503.2019.1619148

Goldberg, A. E. (2013). Constructionist approaches. In T. Hoffmann & G. Trousdale (Eds.), *The Oxford handbook of construction grammar* (pp. 15–31). Oxford University Press.

Verschaffel, L., Schukajlow, S., Star, J., & Van Dooren, W. (2020). Word problems in mathematics education: A survey. *ZDM – Mathematics Education, 52*(1), 1–16. https://doi.org/10.1007/s11858-020-01130-4

Niss, M. A., & Jablonka, E. (2020). Mathematical literacy. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 548–553). Springer. https://doi.org/10.1007/978-3-030-15789-0_100

Matematik i anpassad grundskola

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Den här presentationen fokuserar på en studie som genomförs i anpassad grundskola med elever med lindrig intellektuell funktionsnedsättning (IF).

Anpassad grundskola är en av Sveriges fyra skolformer. Skolan erbjuder plats till elever som har en intellektuell funktionsnedsättning (IF) och som bedöms att inte kunna uppfylla grundskolans kriterier för bedömning av kunskaper. Diagnosen innebär kognitiva svårigheter som påverkar hur information tas in, bearbetas, lagras och tas fram vid behov. Taluppfattningsutveckling är samma för elever med IF och elever utan IF, men i utvecklingen stannar de ofta vid grundläggande talfakta (Schnepel & Aunio, 2022)

I studien undersöks hur olika faktorer kan samverka och stödja elever med lindrig IF att förstå subtraktion. Faktorer som studeras är hur elevernas lärande kan främjas av undervisningsstruktur, representationsformer och delaktighet.

Tidigare studier visar att subtraktion generellt upplevs som svårare jämfört med addition. Vid en subtraktion krävs flera steg för att få ett svar. Det räcker inte med en uppräkning som vid addition utan eleven kan behöva räkna nedåt Att kunna gå från talens helhet till delar kan underlätta förståelsen av subtraktion. För att underlätta för arbetsminnet kan sambandet mellan addition och subtraktion uppmärksammas. (Löwing, 2016). I studien prövas hur talstrukturer kan främja elevers lärande av subtraktion.

Gällande undervisningsstruktur visar forskning om undervisning för elever med IF på vikten av lärarstyrda metoder som inkluderar instruktionsstrategier (Schnepel & Aunio, 2022). Kritik har riktats mot att dessa strukturer inte alltid ger eleven utrymme för egna initiativ och konstruktioner (Göransson et al., 2016). I studien prövas hur den lärarstyrda metoden explicita instruktioner (Hughes et al., 2017) kan medge ökat elevutrymme och möjligheter till egna initiativ från eleverna när den kombineras med Math-talk learning communites (Hufferd-Ackles et al., 2004). Målet med undervisningen i Math-talk learning communites (Hufferd-Ackles et al., 2004) är att eleven ska bli med-lärare och med-lärande (co-teaching och co-learning). Ramverket kommer att användas vid planering av lektionerna men även vid analys av studien.

Matematiken måste tolkas genom olika representationer (Duval, 2017). Representationer, främst konkret material, har lyfts fram som ett stöd för att få tillgång till matematiken (Uttal et al., 1997). Samtidigt påpekas att svårigheter kan uppstå i att förstå sambandet mellan material och matematik, då material kan uppfattas som matematiken (Sterner, 2015). Eleven behöver bortse från delar i materialen som är

irrelevanta för att förstå matematiken (Duval, 2017). I studien prövas hur en förståelse av matematiken kan uppstå i användandet av representationer.

Metod

Studien genomförs genom en intervention inom designforskning med fokus på lärandeprocesser. I studien avses att skapa och studera nya undervisningssituationer genom att utveckla och förfina tidigare framkomna resultat (Cobb et al., 2003). Studien genomförs i samarbete mellan forskare, lärare och assistenter i anpassad grundskola där lektioner planeras och utvärderas gemensamt. Lektionerna filmas och samtal innan och efter lektionerna spelas in.

Diskussion

Kan explicit undervisning (Hughes et al., 2017) kombineras med Math-talk learning communties (Hufferd-Ackles et al., 2004)? Om explicit undervisning handlar om att ta in elevernas svar och att läraren sedan ger rätt svar (om det inte var rätt.) Går det att utveckla denna del till elevdiskussioner och elevledda samtal? Eller frångår man målen med explicit undervisning (Hughes et al., 2017) om detta görs?

Referenser

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, *32*(1), 9-13. https://doi.org/10.3102/0013189X032001009

Duval, R. (2017). *Understanding the mathematical way of thinking – the registers of semiotic representations* (1st 2017. uppl.). Springer International Publishing. https://doi.org/10.1007/978-3-319-56910-9

Göransson, K., Hellblom-Thibblin, T., & Axdorph, E. (2016). A conceptual approach to teaching mathematics to students with intellectual disability. *Scandinavian journal of educational research*, *60*(2), 182-200. https://doi.org/10.1080/00313831.2015.1017836

Hufferd-Ackles, K., Fuson, K. C., & Sherin, M. G. (2004). Describing levels and components of a math-talk learning community. *Journal for research in mathematics education*, *35*(2), 81-116. https://doi.org/10.2307/30034933

Hughes, C. A., Morris, J. R., Therrien, W. J., & Benson, S. K. (2017). Explicit instruction: Historical and contemporary contexts. *Learning disabilities research and practice*, *32*(3), 140-148. https://doi.org/10.1111/ldrp.12142

Löwing, M. (2016). *Diamant - diagnoser i matematik : Ett kartläggningsmaterial baserat på didaktisk ämnesanalys*. Acta universitatis Gothoburgensis.

Schnepel, S., & Aunio, P. (2022). A systematic review of mathematics interventions for primary school students with intellectual disabilities. *European journal of special needs education*, *37*(4), 663-678. https://doi.org/10.1080/08856257.2021.1943268

Sterner, G. (2015). Tal, resonemang och representationer - en interventionsstudie i matematik i förskoleklass. .

Uttal, D. H., Scudder, K. V., & DeLoache, J. S. (1997). Manipulatives as symbols: A new perspective on the use of concrete objects to teach mathematics. *Journal of applied developmental psychology*, *18*(1), 37-54. https://doi.org/10.1016/S0193-3973(97)90013-7

The experiences of being a mathematics teacher aligning with a translanguaging teaching approach

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This presentation focuses on the empirical results of a study exploring multilingual mathematics teacher's professional identity in primary school in Sweden. Research on the identity of multilingual teachers of mathematics in pre-school education has highlighted that multilingual teachers have to navigate through multiple expectations of parents, colleagues and school administration about multilingualism and teaching in mathematics classrooms (Delacour, 2020). Therefore, this presentation aims to further address aspects of language diversity in relation to the multilingual teachers' professional identity in the mathematics classrooms and the school. The research question posed is, what are the experiences of a multilingual mathematics teacher in relation to a multilingual approach to teaching mathematics in multilingual classrooms and schools?

Conceptual framework

The conceptual framework of Patterns of Participation (Skott, 2018) is followed to explore the multilingual mathematics teacher's professional identity. Particularly, the multilingual teacher's professional identity is viewed as the shifting experiences of being, becoming and belonging in interactions with multilingual students and colleagues in school. The teacher's experiences are interpreted by addressing the teacher's relation with multiple practices and figured worlds. This is accomplished through the theoretical construction of multiple patterns of participation in classroom interactions. A pattern of participation is a presentation and description of the practices and figured worlds significant to a particular classroom interaction, from the teacher's point of view. The concept of practice refers to groups of people and is viewed as a social phenomenon created as the individual teacher participates in an endeavour while interpreting the situation in multiple ways (Skott, 2018). Figured worlds, following the definition of Holland and her colleagues, existing in society without the individual teacher re-negotiating their meaning in broader society (Skott, 2018). An example of a practice is a teacher education program, while an example of a figured world is an educational reform.

Methods

A primary school mathematics teacher, anonymised as Neda, was followed over two academic years. She was a multilingual teacher speaking Persian, Swedish and English,

working in a multilingual school in Sweden. She was a teacher of 20 multilingual students, most of whom spoke Swedish. Eight more languages were part of the students' linguistic repertoires. Swedish was the language of instruction; however, other languages were occasionally used by Neda and students. Lesson observations of the same multilingual class were conducted, constituting background information for interviews. Reflection interviews were conducted, in which Neda reflected on instances from the lessons observed. Moreover, semi-structured interviews were conducted, focused on her educational and professional background and reflections on meetings or discussions with colleagues observed in the school. An inductive thematic analysis (Braun & Clarke, 2006) of the interviews was followed, allowing insights into multiple practices and figured worlds that contributed to interactions with students from Neda's perspective. Neda's experiences of being, becoming and belonging were addressed according to how the teacher relates to different practices or figured worlds, during interviews and in different classroom and school interactions.

Results

Neda's experiences of *being a teacher aligning with a translanguaging approach to teaching* relate to her re-engagement with research and collaboration initiatives in the past. These experiences relate to *being* a multilingual teacher of multilingual mathematics students who was eventually allowed to use students' mother tongue after teacher education on translanguaging took place in the school. During interaction with multilingual students, Neda balances between the theoretically conflicting figured worlds of *Swedish language as the language of instruction* and *students' mother tongue*. She balances between students, pronouncing mathematical concepts according to their mother tongue versus using the Swedish pronunciation of the concepts. During the second academic year of data collection, administration changes happened in the school, and its future was debated due to achievement results. Then the issue of using students' mother tongue or how often students are allowed to use their mother tongue becomes a topic of discussion between teachers. However, Neda's experiences of *being a teacher aligning with a translanguaging approach* remain during discussions with colleagues, supporting students' right to use their mother tongue and participate in multilingual schools and classrooms.

References

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101. https://doi.org/https://doi.org/10.1191/1478088706qp063oa

Delacour, L. (2020). Expectations and discourses in multilingual preschool mathematics: Case study of a teacher of immigrant background. *Philosophy of Mathematics Education Journal*, *36*, 1–13.

Skott, J. (2018). Re-centring the individual in participatory accounts of professional identity. In G. Kaiser, M. Forgasz, A. Graven, E. Simmt, & B. Xu (Eds.), *Invited lectures from the 13th International Congress on Mathematical Education* (pp. 601–618). Springer. https://doi.org/10.1007/978-3-319-72170-5_33

Tasks to identify mathematical competency profiles for high-achieving students

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This presentation is about the first phase of a larger project that aims to develop design principles for how mathematical problems can be designed to develop high-achieving lower secondary students' mathematical competencies. Students' opportunities to develop mathematical competencies depend on the types of tasks they engage in, and since many textbook tasks lack sufficient challenge (Jäder et al., 2020) there is a need to better design tasks that support high-achieving students' mathematical growth. To be able to reach the aim of our project, knowledge of the students' present mathematical competence profiles is important, that is the competencies the student already possess to a high level. The first step in our project is therefore to find a way to identify high-achieving students' mathematical competence profile. This first step of the project strives to answer the questions: 1) What kind of tasks can be used to identify students' competence profile? 2) How can the competence profiles be visualised?

We start from a theoretical approach where mathematical knowledge is described as a set of competencies, and learning in mathematics involves developing these competencies (e.g., Kilpatrick et al., 2001; Lithner et al., 2010; Niss & Højgaard, 2019). Applying procedures, reasoning, communication, representation, connection and problem solving are the six competencies used in the Mathematical Competency Related Framework (MCRF) (Lithner et al., 2010). Each of the six competencies are, according to Lithner et al., assumed to consist of a productive part and an analytical part. To capture the productive and analytical parts and characterize the engagement with respect to these parts, three competence-related activities (CRA) are defined: *Interpret* (analytical), *Do and use* (productive), and *Judgment* (analytical) (Lithner et al., 2010). Since our project is closely related to practice and done in collaboration with a lower secondary school, we will eventually connect the competencies; *concept, method, reasoning, problem solving and communication*, in the Swedish curriculum (Skolverket, 2024) to the competencies in MCRF.

To answer our two questions, we will first, following the analysis procedure in Boesen et al. (2018), analyse tasks from former national tests in mathematics for grade nine. This means that for each CRA and competence we will from a reasonable solution (for the age group) try to show that the competence is not tested through the task. If we cannot show that the competence is not tested, the task is classified as *Testing the competency*. Through the analysis we strive to find tasks that can be used to identify how developed the mathematical competencies are on an individual level. The result will therefore also reveal what competencies a student needs to develop to deepen her/his mathematical knowledge.

Thereafter, we will use the tasks to identify competency profiles among high-achieving lower secondary students to explore what different competency profiles exist among them. We will also elaborate on different ways to visualize students' competency profiles with the goal to give the students' teacher guidance in how to further support student learning.

Our analysis is ongoing; in our presentation we will show how we analyse tasks and show some visualizations of students' competency profiles. In Figure 1, a hypothetical example is given showing strength and weaknesses in mathematical competencies for each CRA. We hope to get a fruitful discussion about our analysis and our visualisations.

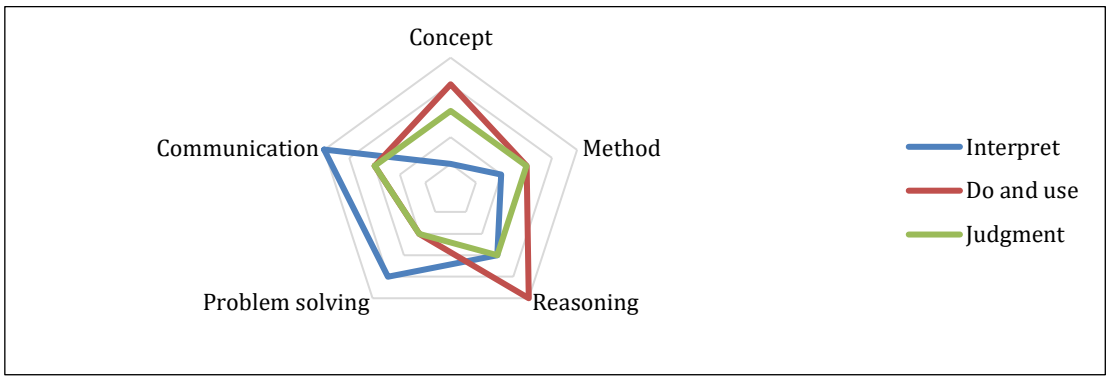

Figure 1. Visualisation of a competence profile

References

Boesen, J., Lithner, J., & Palm, T. (2018). Assessing mathematical competence: An analysis of Swedish national mathematical tests. *Scandinavian Journal of Educational Research, 62*(1), 109-124. https://doi.org/10.1080/00313831.2016.1212256

Jäder, J., Lithner, J., & Sidenvall, J. (2020). Mathematical problem solving in textbooks from twelve countries. *International Journal of Mathematical Education in Science and Technology, 51*(7), 1120-1136. https://doi.org/10.1080/0020739X.2019.1656826

Kilpatrick, J., Swafford, J. O., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. National Academy Press.

Lithner, J., Bergqvist, E., Bergqvist, T., Boesen, J., Palm, T., & Palmberg, B. (2010). Mathematical competencies: A research framework. In C. Bergsten, E. Jablonka & T. Wedege (Eds.), *Mathematics and mathematics education: Cultural and social dimensions. Proceedings of MADIF 7, the Seventh Mathematics Education Research Seminar, Stockholm, January 26-27, 2010*. (pp 157-167). SMDF.

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9

Skolverket. (2024). *Läroplan för grundskolan, förskoleklassen och fritidshemmet: Lgr22* (2 uppl.). https://www.skolverket.se/undervisning/grundskolan/laroplan-lgr22-for-grundskolan-samt-for-forskoleklassen-och-fritidshemmet

Consequences of a new assessment model in Mathematics 1 for engineering students

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This presentation takes its starting point from a major change in the assessment model of the course Mathematics 1, a first semester 7.5 ECTS credits course for engineering students at the University of Agder in Norway. We report on how the new assessment model affected the students' pass rate, but also about how the new assessment later might affect the students' performances in engineering subjects at bachelor's level.

The course Mathematics 1 used to suffer from low pass rates. For example, in the autumn semester 2021, 44 % of the students failed on their first attempt. The course was assessed by a classical written high-stakes exam at the end of the semester. The result on the exam decided the student's course grade. The autumn semester 2022, a new assessment model was implemented, which consisted of four smaller tests taken throughout the semester and a written essay at the end of the semester. The small tests each counted 15 % towards the final grade, and the essay counted 40 %. The tests were taken under supervision, digitally in the computer aided assessment system STACK, which also means they were assessed automatically. Each student had a maximum of four attempts on each test, and the best result counted.

The first semester with the new assessment, the failure rate (grade F) dropped from 44 % to 12 % (Larsen, 2023), while 19 % got the best grade (A). The mathematics teachers observed that after a test, the students gathered to discuss the tasks. A plausible conclusion is that the new assessment model contributed to that the students began to study early in the semester, instead of possibly waiting until a few weeks before the exam.

Although the failure rate has dropped and the average grade now is quite high, one might question the reason for this. One possible explanation is that the high grades depend on that the students starts studying earlier and study more frequently, and thus that their mathematics competency in fact has improved. An alternative explanation is, however, that the students have learnt to anticipate what will come in the tests. This explanation would imply they have improved their knowledge in how to take tests, i.e. cracked the system, rather than enhanced their mathematics knowledge. Moreover, even if the improved results actually depended on enhanced student knowledge in mathematics, there is also a risk that this knowledge is temporary and that the students forget much of it after the test is taken. Hence, we wish to investigate how the teachers in engineering subjects at bachelor's level find the new assessment in Mathematics 1 and what consequences it has had on students' performances in their subjects.

Our research questions are "What reflections do teachers in engineering subjects have regarding the new assessment model in Mathematics 1?" and "How has the new

assessment model in Mathematics 1 affected the students' knowledge in mathematics in relation to engineering subjects at bachelor's level?"

Implementation and results

We employed a convenience sample to find six interviewees, who had taught engineering subjects at bachelor's level for at least three years at the University of Agder. The individual interviews were conducted by the first author, in Norwegian, on Zoom. They were semi-structured, steered by an interview guide, which previously had been tested in a pilot interview with a seventh teacher. The duration of the interviews was 24–40 minutes. They were video recorded and then transcribed.

The transcripts were analysed by a thematic content analysis (Braun & Clarke, 2006). The analysis showed that none of the interviewees had identified any change in the students' mathematics knowledge after the new assessment model was implemented. As interviewee 2 said, "If I had not known there had been a change, I would never have thought about it." The interviewees were all positive about the assessment model, although also some drawbacks were mentioned. A positive consequence was that the students now are more relaxed, since they do not have to struggle with a re-exam of Mathematics 1 after the first semester. As interviewee 3 said, "The students are very pleased with the new assessment. /…/ They can lower their shoulders and be more relaxed." A drawback that was mentioned is that since a subject with regular and frequent testing tends to make the students putting more effort into it, they put less effort into other courses that run parallelly with e.g. Mathematics 1. Other minor complaints expressed were rather connected to the course content and lack of knowledge in different topics, that are essential in engineering courses. For example, in a subject about construction and solid mechanics a lack of knowledge in probability might be an issue, knowledge of vectors is essential in electromagnetism, while deep knowledge of integrals is not always so relevant in programming. Such issues, and weak pre-knowledge when entering the university, were much more identified as problematic than any consequences of the new assessment model in Mathematics 1.

Conclusions

Our results suggest that the new assessment model in Mathematics 1 has not affected the students' mathematics competency in any negative way, regarding their performance in engineering subjects at bachelor's level. The participants in the study were positive about this assessment model, and also expressed that the students find it positive.

References

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101. https://doi.org/10.1191/1478088706qp063oa

Larsen, H. (2023, February 2). Fra 44 til 12 prosent stryk i matematikk. Khrono. www.khrono.no/fra-44-til-12-prosent-stryk-i-matematikk/755711

Play-Responsive Mathematics Teaching in Swedish preschools

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This presentation is based on a study conducted within the theoretical and didactic model Play-Responsive Teaching (PRT), that is, a model integrating play, teaching, and learning (Pramling et al., 2019). PRT aims to promote teaching in play without losing the essential characteristics of play. When applied in the context of preschool mathematics teaching, this study uses the term Play-Responsive Mathematics Teaching (PRMT) to highlight the specific focus on mathematics. Within mathematics, particularly the concept of cardinality was focused on in this study exploring how PRMT can be planned and implemented in preschool settings. The research question addressed is: What characterizes activities where teachers intend to implement play-responsive teaching focusing on cardinality?

Early mathematical knowledge influences children's later academic success, highlighting the need for research on the relationship between early mathematics teaching and young children's learning (Björklund & Palmér, 2019). Cardinality, a fundamental aspect of early mathematics, refers to understanding that a specific number of counted objects represents a particular quantity. This concept goes beyond reaching a numerical answer, encompassing the relationship between object count and the idea of 'how many' (Björklund & Palmér, 2023). To understand how cardinality can be supported in preschool through PRMT, it is essential to consider how teaching is conceptualized within the PRT model. In PRT, teaching is seen as a collaborative process shaped by both teachers and children. Play must be organized without losing its playful nature, allowing children to shift between fantasy (as if) and reality (as is). One key feature of PRT is double responsiveness; that is, teaching is goal-oriented but at the same time adapted to children's initiatives.

Method

The study is conducted in line with educational design research (Bakker, 2018) following a cyclic process of designing, implementing, and revising teaching activities in collaboration between a researcher and three preschool teachers from different preschools. The teachers had prior experience with early number sense, including cardinality, but were new to PRT. The teachers and the researcher collaboratively planned four mathematics activities focusing on PRT and cardinality, which were carried out with 48 children aged three to four years. Each activity went through several iterative cycles of planning, implementation, and evaluation, and was documented through teacher-recorded video (73 recordings). Content analysis, as described by Hsieh and Shannon (2005), was conducted using NVivo. The analysis focused on key PRT

concepts such 'as if' and 'as is', responsiveness, intersubjectivity, and agency (Pramling et al., 2019). The analysis also examined how cardinality appeared in children's actions and expressions, including subitizing, ordinality, and use of gestures (Paliwal & Baroody, 2020).

Results

In the recorded videos of the activities, the analysis revealed three distinct categories: (1) play without mathematics teaching, (2) mathematics teaching without play (MT), and (3) PRMT, where play and mathematics teaching were integrated. The results highlighted the teacher's role as a responsive mediator in PRMT. Rather than merely transmitting knowledge, the teacher co-constructed meaningful learning situations by shifting between 'as if' and 'as is', thereby fostering intersubjectivity and agency. By responding to children's initiatives in the play, the teacher contributed to the development of the learning environment. When teachers engaged with children's initiatives, double responsiveness emerged, enriching the conditions for learning. However, if responsiveness or intersubjectivity was lacking, the teacher risked being excluded from the play, which reduced its educational potential. In conclusion, both the sequences, mathematics teaching without play and PRMT, were found to be important and complementary. By combining structured teaching with playful responsiveness, the teacher supported children's understanding of concepts and mathematical thinking.

References

Bakker, A. (2018). *Design research in education*: *A practical guide for early career researchers.* Routledge. https://doi.org/10.4324/9780203701010

Björklund, C., & Palmér, H. (2019). I mötet mellan lekens frihet och undervisningens målorientering i förskolan. [At the intersection of the openness of play and the goal-orientation of teaching in preschool]. *Forskning om undervisning & lärande, 7*(1), 64–85.

Björklund, C., Palmér, H., 2023. Matematisering i lek på vetenskaplig grund. Nordisk barnehageforskning 20, 29–50. https://doi.org/10.23865/nbf.v20.261

Hsieh, & Shannon, S. E. (2005). Three Approaches to Qualitative Content Analysis. *Qualitative Health Research*, *15*(9), 1277–1288. https://doi.org/10.1177/1049732305276687

Paliwal, V., & Baroody, A. (2020). Cardinality principle understanding: the role of focusing on the subitizing ability. *ZDM Mathematics Education 52*, 649–661, https://doi.org/10.1007/s11858-020-01150-0

Pramling, N., Wallerstedt, C., Lagerlöf, P., Björklund, C., Kultti, A., Palmér, H., Magnusson, M., Thulin, S., Jonsson, A., & Pramling Samuelsson, I. (2019). *Play-Responsive Teaching in Early Childhood Education*. Springer.

The relation between mathematical reasoning and communication: How teachers describe it

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The present study is part of a project studying the role of reasoning in mathematics education in Sweden and England. Here, we focus on how Swedish teachers in different school years describe mathematical reasoning in relation to communication. Reasoning is commonly seen as central to mathematics, both in research-based frameworks and in curriculum documents (see e.g., Niss & Højgaard, 2011; Skolverket, 2024). However, the concept is seldom explicitly defined and, as shown by Herbert et al. (2015), teachers can have different views on what mathematical reasoning actually entails. In addition, the different competencies in frameworks and curriculum documents sometimes overlap (Niss & Højgaard, 2011).

According to the Swedish curriculum (Skolverket, 2024), students are supposed to develop "the ability to carry out and understand mathematical reasoning" (Sw. 'resonemang'). Reasoning is not explicitly defined, although it is exemplified to some extent. Being able to communicate about/with mathematics is included as a separate goal. However, the grading criteria connected to reasoning include actions such as "asking and answering questions" (Year 3) and "putting forward and responding to claims" (Years 6 and 9), which are activities that include communication, at least in an everyday sense. It is therefore possible that Swedish teachers interpret and implement reasoning in different ways, especially regarding its relation to communication.

The purpose of the present study is therefore to increase the understanding of how Swedish teachers at different school years relate mathematical reasoning to communication.

RQ1. In what ways do teachers relate mathematical reasoning to communication?

RQ2. Are there differences in how teachers across different school years relate mathematical reasoning to communication?

We interviewed 15 experienced (>3 years) teachers, five from each of school years 3, 6 and 9, and from different parts of Sweden. The interviews lasted 15–45 minutes and all questions concerned reasoning in mathematics education. In particular, it focused on the teachers' views on, and instructional practices related to, mathematical reasoning. Examples of questions are 'It would be interesting to know what mathematical reasoning is for you?' and 'Can you describe how you as a teacher work with mathematical reasoning?' Preliminary interaction with the data highlighted frequent references to communication when talking about reasoning. We therefore started to focus on these references, identifying instances when teachers describe some aspect of communication,

that is, when addressing the production or reception of information in some external medium. For example, this includes production through talking or writing and reception through listening or reading. The next step of the analysis is to characterize the relation between mathematical reasoning and communication in the teachers' descriptions.

The ongoing analyses indicate that teachers in Year 3 often describe how they use general communication activities when they want to develop students' reasoning. For example, when talking about what mathematical reasoning means for them, one teacher said: "I really want the children to sit and talk and discuss and come up with things, experience things together". Teachers in Year 9 instead tend to describe different and more specific forms of communication (e.g., graphs, algebraic expressions) as a means to express mathematical reasoning. For example, one Year 9 teacher said: "In this case, maybe look at how linear functions would illustrate it [the logical reasoning]. Or if you want, you could set up an equation to illustrate this."

The continued analyses will characterize how the relationship between mathematical reasoning and communication is described by all teachers and then compare teachers from different school years. For example, the Year 3 teacher above seemed to see *reasoning as communication*, while the Year 9 teacher used *communication as a means or a prerequisite for reasoning*. The results from this analysis can give insights into potential cultural differences between different school levels or give information about some type of progression over school years concerning properties of mathematical reasoning.

References

Herbert, S., Vale, C., Bragg, L. A., Loong, E., & Widjaja, W. (2015). A framework for primary teachers' perceptions of mathematical reasoning. *International Journal of Educational Research, 74*, 26–37. https://doi.org/10.1016/j.ijer.2015.09.005

Niss, M., & Højgaard, T. (2011). *Competencies and mathematical learning: ideas and inspiration for the development of mathematics teaching and learning in Denmark* (English ed.). IMFUFA, Roskilde University.

Skolverket (2024). *Läroplan för grundskolan, förskoleklassen och fritidshemmet: Lgr22* (2 uppl.).

Relationship between teaching experience and teachers' use of teacher guide content

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Introduction

This presentation reports on preliminary findings of a research project where Swedish upper-secondary school teachers used a designed topic-centered research-based teacher guide for teaching quadratic equations. The project is situated within the vast field of research on teachers' use of curriculum materials (Remillard, 2005). Even though research has identified that this use relates to contextual factors and teacher variables such as teaching experience, less research has targeted the relationship with specific usages of different features of teacher guides. A first design cycle showed some patterns of use for 6 teachers (Gustafsson, 2025). In the second cycle, 55 teachers of various experience used the guide, allowing for the investigation of usage patterns.

The overarching question addressed in this presentation is: what is the relationship between teachers' use of teacher guide content and teaching experience?

Methods

The teacher guide was designed and distributed to teachers in a second cycle of a design research project in spring 2025. It consisted of 3 expository texts, 5 lesson activities, and 4 types of tasks, representing content designed to be possibly educative (Davis et al., 2017). Expository texts consisted of a research summary, suggestions on possible implications for teaching, and curriculum information, all for quadratic equations. The activities were designed to meet issues raised in the texts, and included rationale for their use. Tasks included old national exam tasks, challenging tasks, exit tickets and a quick reference. Teachers were recruited by mail through principals, and 68 teachers answered a post-use survey, out of which 55 teachers from 49 different schools stated to have used some part of the guide. Self-reported items from the survey on teaching experience and content use constitute the data. Statistical analysis, frequentist and Bayesian, was performed in JASP v0.95.1 (Marsman & Wagenmakers, 2017).

Theoretical framing

Teachers' curriculum use is here seen as teachers perceiving and mobilizing the resources available to them in designing instruction (Brown, 2009). In this work, they distribute their agency between themselves and the resources through different degrees of artifact appropriation, termed *offload*, *adapt* and *improvising*. Perceiving an affordance can result in such forms of mobilization, representing a decreasing spectrum of agency transfer from teacher to resource.

Preliminary findings

Teaching experience (EXP) measured in years taught (M = 14, SD = 10), were used as the independent variable. Dependent variables were describing aspects of use: what texts were read and how thorough, what activities and tasks were used, and in what way.

In terms of relations, statistical analysis indicated negligible correlations between EXP and the thoroughness of expository texts read, and between EXP and the offloading, adapting or improvising behaviour in using activities. Trivial to small correlations were found between EXP and the number of activities and tasks used, positive and negative respectively, cautiously suggesting that more experienced teachers in the sample tended to use activities more and tasks less. However, none of these correlations were statistically significant, and Bayes factors provided moderate evidence for no relationships, and weak evidence regarding task use.

To investigate patterns in use of individual activities and tasks, logistic regressions were conducted for each activity and task. Results indicated that EXP was a significant predictor for only the first activity, with a one-year increase in experience associated with a 7% increase in the odds ratio. Bayes factor also indicated weak evidence for this relation, but weak evidence was also found for no relationships between EXP and the other activities and tasks, providing inconclusive findings.

Separate ordinal logistic regressions were conducted to examine whether EXP predicted the extent to which participants read each text, but it did not significantly predict the odds of teachers reporting greater thoroughness in reading any of the texts.

Discussion

These results, suggesting that teaching experience has weak or no relation to different usages of teacher guide content, require further investigation. They also raise methodological questions in relation to certainty of findings constructed from analysis.

References

Brown, M. W. (2009). The teacher–tool relationship: Theorizing the design and use of curriculum materials. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.), *Mathematics teachers at work* (pp. 17–36). Routledge.

Davis, E. A., Palincsar, A. S., Smith, P. S., Arias, A. M., & Kademian, S. M. (2017). Educative curriculum materials: Uptake, impact, and implications for research and design. *Educational Researcher, 46*(6), 293–304. https://doi.org/10.3102/0013189X17727502

Gustafsson, M. (2025, February 4–8). *Upper-secondary school mathematics teachers' use of a topic-centered research-based teacher guide.* [Paper presentation]. CERME14. Bozen-Bolzano, Italy.

Marsman, M., & Wagenmakers, E. J. (2017). Bayesian benefits with JASP. *European Journal of Developmental Psychology*, *14*(5), 545–555. https://doi.org/10.1080/17405629.2016.1259614

Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research, 75*(2), 211–246. https://doi.org/10.3102/00346543075002211

Co-teaching as a setting promoting mathematics teachers' professional development

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This presentation is about mathematics teachers' professional development in co-teaching settings. The purpose of mathematics teachers' professional development can be described as the continuous learning process through which teachers develop within their profession (Borko, 2004). In mathematics education settings, various models of professional development have been implemented, and diverse perspectives have been employed to evaluate their outcomes. However, the field still struggles with unanswered questions concerning impact on classroom practice and, by extension, student outcomes.

Recent studies highlight the importance of catalysts that support teachers' learning (Brehmer & Ryve, 2024) and the significance of mathematics teachers' commitment to professional development initiatives (Johari et al., 2022). These insights are consistent with a growing recognition of the need to integrate professional development with teachers' everyday practice. In line with this, the present study adopts a bidirectional approach where mathematics teachers' professional development and classroom practice are developed in duality, explored within the context of co-teaching.

Co-teaching, as defined by Cook & Friend (1995), involves two or more teachers collaborate in planning, teaching, and evaluating instruction within a shared learning environment. The purpose of co-teaching is to improve teaching quality, support professional development, and enhance student learning through a collective teaching practice. In this study, co-teaching is viewed as an organizational model for collaboration in mathematics instruction, offering a context where professional development is directly linked to teaching practice, unlike externally driven professional development programs. The purpose of the study is to investigate the opportunities and challenges of co-teaching when the aim is to enhance high-quality mathematics instruction. The study is guided by the following research question: *What opportunities and challenges for mathematics teacher professional development emerge in teachers' talk within a co-teaching setting?*

Method

Data were collected through semi-structured interviews with twenty mathematics teachers from different schools and municipalities, all with experience of co-teaching, but with varying experiences. The interviews were informed by research on teachers' visions (Munter, 2014) and self-efficacy (Bandura, 1997), as aspects that affect the

potential for professional development and the enactment of outcomes in instructional practice. Questions that were asked included, for example: *How do you define high-quality mathematics instruction? How do you perceive that co-teaching has influenced your instructional practice?* and *Describe a situation within co-teaching in mathematics that you experienced as particularly successful?*

All interviews were recorded, transcribed and analysed using content analysis. To capture aspects of teachers' learning in the co-teaching setting Communities of practice (Wenger, 1998) provided a theoretical lens.

Results

At the conference, results will be presented concerning the teachers expressed opportunities and challenges related to roles and relationships, structures for collaboration, the effects on mathematics instruction, and visions of high-quality mathematics instruction. In the next step, these insights will serve as the foundation for developing design principles intended to support high-quality mathematics instruction, which will subsequently be tested in a design study.

References

Bandura, A. (1997). *Self-efficacy: the exercise of control*. W. H. Freeman.

Brehmer, D., & Ryve, A. (2024). Facilities for Mathematics Teachers' Learning from Professional Development Programmes: A Qualitative Systematic Review. *Mathematics Teacher Education & Development*, *26*(2).

Borko, H. (2004). Professional development and teacher learning: Mapping the terrain. *Educational researcher*, *33*(8), 3–15. https://doi.org/10.3102/0013189X033008003

Cook, L., & Friend, M. (1995). Co-teaching: Guidelines for creating effective practices. *Focus on exceptional children*, *28*.

Johari, M. I., Rosli, R., Maat, S. M., Mahmud, M. S., Capraro, M. M., & Capraro, R. M. (2022). Integrated professional development for mathematics teachers: a systematic review. *Pegem Journal of Education and Instruction*, *12*(4), 226–234. https://doi.org/10.47750/pegegog.12.04.23

Munter, C. (2014). Developing visions of high-quality mathematics instruction. *Journal for research in mathematics education*, *45*(5), 58–-635. https://doi.org/10.5951/jresematheduc.45.5.0584

Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. Cambridge University Press.

Learning multiplicative reasoning in preschool class

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In this presentation, we highlight young students' ways of solving a problem characterized by its multiplicative structure. Through an analysis of interview data with 199 students at three occasions, guided by the variation theory of learning (Marton, 2015), we identified four qualitatively different ways of solving the problem that reveal differences in discerned aspects of the numerical content. The data analysed and presented here is part of the research project SATSA (grant no. 2020-03712), aiming to study how students in preschool class (mean age 6 years) develop their number knowledge and arithmetic skills. The question, which the current study will contribute to answer is: What aspects are critical to discern, solving a multiplicative structure problem in preschool class? The analysis will contribute to what teaching should attend by unfolding the complexity and progression of learning multiplicative reasoning.

Our interest lies in how a problem can be interpreted, and what students find possible to do with the numbers given in the problem. Multiplicative structures can be found in situations where two variables are linked by operations of multiplication or division, e.g., "one candy costs 3 crowns, how many candies can you get for 12 crowns?" where the price is invariant and the number of candies and total amount of crowns are related proportionally. In general, it is considered difficult to learn the multiplicative structure of numbers (van Dooren et al., 2010; van den Heuvel-Panhuizen & Elia, 2020), why we need a deeper understanding of students' perspectives to support their arithmetic development.

To extend our knowledge of students' developing understanding of multiplicative structures, we were guided by the variation theory of learning (Marton, 2015), which means we direct our interest to the qualitatively different ways students experience numbers, number relations and operations. Observations of students' strategy use in arithmetic problem solving is used as data for analysing what constitutes observed differences; what aspects of numbers a student discerns is reflected in this student's ways of experiencing and thus handling numbers in the problem solving. This further means that some aspects are necessary to discern to advance one's way of experiencing the numbers. In this analysis, (un)discerned aspects are the object of analysis. The results from this analysis may thereby explain progression in arithmetic skills.

Methods

Three task-based interviews were conducted with students at the start and the end of preschool class, and at the end of grade 1. The problem analysed here is inspired by van den Heuvel-Panhuizen and Elia's study (2020), given orally to the students: *If you have ten mittens, how many children is it enough for*? Students' problem-solving strategies were documented in a protocol, completed by field notes capturing their actions and verbal expressions. The analysis is done in two steps: first, identifying discerned and undiscerned aspects, reflecting the different meanings of the number relations in the multiplicative problem expressed in the students' answers; second, studying the progression through the three interview occasions.

Results and conclusions

The analysis revealed the following four aspects to be critical for the students solving the problem:

A. Discerning the size of the unit 2 (each child in the task needs two gloves, and the size of the unit [2] is not the same as one child "*2 mittens for each*").
B. The unit 2 is repeated several times in the given whole [10] (*"2 mittens, 2 mittens, 2 mittens,* raising one finger at a time, … *I don´t know"*).
C. Keeping track of the iterations of the same composite units [2's] (how many will 'fit' in the whole; *Places 10 fingers on the table, separates them into 2s:* "*Ten children, ten mittens*").
D. Simultaneously discerning units in relation to the whole (Compose two fingers at a time: *"1, 2 to one child, 3, 4, to two, 5, 6 to three, 7, 8 to four, 9, 10 to five, five children"*).

These aspects are considered inclusive, meaning that, being able to successfully solve the problem, the ABCD aspects need to be discerned. If only A is discerned, this limits the student's problem solving, ending up in answering "2" without relating it to the whole set of mittens or number of children. Thus, the more aspects the student discerns, the more advanced understanding of the multiplicative structure. This is confirmed in the analysis of the progression of number of students discerning all (ABCD) aspects: interview 1; 52(199) 26%, interview 2; 90(199) 45%, and interview 3; 133(199) 67%.

References

Marton, F. (2015). *Necessary conditions of learning*. Routledge.

Van den Heuvel-Panhuizen, M., & Elia, I. (2020). Mapping kindergartners' quantitative competence. *ZDM Mathematics Education 52* (4), 805–819. https://doi.org/10.1007/s11858-020-01138-w

Van Dooren, W., De Bock, D., & Verschaffel, L. (2010). From addition to multiplication … and back: The development of students' additive and multiplicative reasoning skills. *Cognition and Instruction, 28*(3), 360–381. https://doi.org/10.1080/07370008.2010.488306

Training pre-service teachers to teach statistics: What can they learn from their degree project?

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This presentation outlines initial results on how pre-service teachers analyse and present quantitative data in degree projects at an advanced level. Degree projects from a period of seven semesters are analysed. The guiding research question was: What statistical elements can be identified in degree projects in science and mathematics education? The results show that statistical engagement was present in only a small proportion of the projects and was confined to basic descriptive forms, rather than analytical ones, which limited opportunities for deeper interpretation. The purpose of this short communication is to raise the question of how teacher education can find ways to foster statistical engagement in degree projects, moving beyond algorithmic approaches and supporting higher-quality student research.

Developing data literacy through data treatment in degree project

In the 21st century, data literacy has developed as an extension of statistical literacy, requiring not only an understanding of statistical concepts but also the ability to critically engage with data across various contexts (Schreiter et al., 2024). There is a need to become both a consumer and a producer of data (Gould, 2017). Pre-service teachers need authentic experiences that foster inference skills and enhance statistical reasoning, thereby moving beyond algorithmic approaches that restrict interpretative abilities. These experiences can be practised as part of a degree project.

The degree project holds a privileged place in higher education, as it is the culmination of students' studies, providing a space to integrate theory, research, methodology, and analysis, and fostering independent and in-depth learning (Hemmings, 2001). One of the learning objectives in the degree project is to conduct a study where the collected data are systematically analysed and the results are reported in a well-structured manner (Stockholm University, 2020). Participating in research projects and designing statistical investigations during their training can provide valuable practical experience, enabling teachers to promote inquiry-based learning in their classrooms (Batanero et al., 2011; Gould, 2013). Expanding statistical literacy in this way better prepares teachers and students to navigate a data-driven world and address the complex challenges of interpretation and decision-making. The preparation of teachers remains central to advancing education. Strengthening teachers' competencies, fostering the development of tailored pedagogical approaches, and sustaining systematic research are essential for shaping education (Schreiter et al., 2024).

Method

The data comprised degree projects completed within teacher education at the Department of Education, Stockholm University, during the past seven semesters. From this total corpus, we identified 185 projects within science and mathematics education. In a second step, we included only those projects that contained elements of statistical content, resulting in a subset of 43 projects. These were analysed using a coding schema aligned with the research questions, in which "statistical elements" were classified as descriptive (data handling, summaries, graphs) or analytic (inferential tests).

Preliminary results and conclusions

Only a small proportion of the examined degree projects included statistical processing of results, amounting to 23% percent of the corpus**.** The statistical elements that did appear were almost exclusively descriptive, typically limited to nominal data, frequency counts, tables, and simple bar charts. In some cases, graphs seemed to be included decoratively, without a clear analytical purpose or alignment with the stated research questions. While the qualitative orientation of many projects was well justified, keeping statistical use at such a low descriptive level restricted the possibility of drawing more nuanced or robust conclusions from the data. Engaging even with basic analytical approaches would allow students to interpret and critically engage with their results more fully, thereby strengthening both the validity of their findings and the educational value of their projects. This raises the question of how teacher education might more fully integrate statistical engagement in degree projects, moving beyond algorithmic approaches that restrict interpretation, and contribute to more robust and higher-quality research experiences (Gould, 2013, 2017; Schreiter et al., 2024).

References

Batanero, C., Burrill, G., & Reading, C. (2011b). Overview: challenges for teaching statistics in school mathematics and preparing mathematics teachers. In: Batanero, C., Burrill, G., & Reading, C. (Eds.), *a ICMI/IASE Study* (Vol. 14, pp. 407–418). Springer. https://doi.org/10.1007/978-94-007-1131-0

Hemmings, S. (2001). The place of the dissertation in learning to research. *Learning to research: resources for learning and teaching in sociology and social policy. Sheffield, SSP2000/Teaching and Learning Network for Sociology and Social Policy*

Gould, R. (2017). Data literacy is statistical literacy. *Statistics Education Research Journal*, *16*(1), 22–25. https://doi.org/10.52041/serj.v16i1.209

Schreiter, S., Friedrich, A., Fuhr, H., Malone, S., Brünken, R., Kuhn, J., & Vogel, M. (2024). Teaching for statistical and data literacy in K-12 STEM education: a systematic review on teacher variables, teacher education, and impacts on classroom practice. *ZDM–Mathematics Education*, *56*(1), 31–45. https://doi.org/10.1007/s11858-023-01531-1

Stockholm University (2020). *Mathematics Education, School Years 4-6, Degree Project II, 15 ECTS credits.* UM9020.

Review of mathematical generalisation

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This presentation focuses on a pilot study for a review of research on mathematical generalisation, MG, in school algebra. MG is an important step when going from the mathematical context through detecting a mathematical pattern, to develop algebraic thinking (Sibgatullin, et. al. 2022). Effects of technology on algebraic thinking – including making generalisations – have been reviewed (Mustaffa et al., 2015) as well as theoretical considerations and processes of generalisation (Malara, 2012). However, reviews focussing on MG seem scarce. Given that the ability to make MGs for a long time has been claimed to be crucial knowledge and of vital importance for developing algebraic thinking, there is a need for a more overarching research review.

In this pilot study we delineate and compare different views on MG in research on school algebra. More specifically, we ask:

- What methods characterise empirical research on MG in school algebra?
- What characterises the relation between MG and school algebra in this empirical research?

In our presentation we would like to discuss these questions and alternative ways to approach our material.

Theoretical and analytical frames, and methods of analysis

Developing awareness of MG has been explained as enhancing "seeing a generality through the particular and seeing the particular in the general" (Mason, 1996, p. 65). MG can thus be seen as a process which may be developed in many directions, and it has been expressed differently by different researchers (Mason, 1996). Choosing an already fixed analytical framework would have included presumptions about how MG can be depicted in relation to school algebra. Therefore, we have chosen an inductive thematic approach. Our focus has been the object of study in the papers and its relation to MG. Three positions were identified: MG as the object of study *in the study's foreground*; MG *in the study's background*, with something else, e.g., algebraic thinking, as the object of study; and MG *in relation to* something else as the object of study.

ERIC and Scopus were used for the review. The search terms were combinations of *generali**, *arithmetic**, *algebra** and *abstract reasoning.* Studies on higher education and teachers were not included. In total, 654 records were identified in Scopus and 902 in ERIC, whereof 186 were selected. A flow chart of the selecting process is lifted in the presentation. For this pilot, 31 papers (every sixth paper, chronologically) of the 186 were selected for full reading. Three papers were then excluded due to wrong population. All researchers coded all papers with respect to educational level, methods

of analysis, and the object of study. Citations of the studies' aims and results were then thematised inductively (Braun & Clarke, 2021).

Results and discussion

Studies that involved middle school were common (11 of 28, ages 11-13). Seven studies involved several educational levels, mostly both lower and upper primary school (ages 6-10). Only a handful of studies involved high school (ages 14-18), none kindergarten (ages 0-5). Content analysis is the main research method (12 studies). Eight papers employ a combination of methods; often content analysis paired with descriptive statistics. Three studies use only quantitative methods. Finally, MG is in the study's foreground in seven papers and studied in relation to something else in six papers; but MG is mainly in the study's background (15 papers). Studies about *Technology, problem solving, and modelling* frequently sought to identify different aspects of mathematical content in tasks, students' solutions, strategies or reasoning, while studies on *Algebraic thinking, reasoning and structure* predominantly concerned students' learning trajectories or mathematical development. *Patterns, functional thinking, and generalized arithmetic* mostly concerned either students' learning trajectories or mathematical development, or students' perceptions of mathematical content. The theme *Representational aspects* was identified in all kinds of studies.

Sibgatullin et al. (2022) seem to conceptualise MG as a backdrop to algebraic thinking, but this pilot points out other possible conceptualisations. The review has an empirical focus and can thus be regarded as a complement to Malara's (2021) theoretical considerations on MG. Compared to Mustaffa et al. (2015), who focus on technology in relation to algebra, we have identified other themes as well.

References

Braun, V. & Clarke, V. (2021). One size fits all? What counts as quality practice in (reflexive) thematic analysis? *Psychology, 18*(3), 328–352. doi.org/10.1080/14780887.2020.1769238

Malara, N. (2012). Generalization Processes In The Teaching/Learning Of Algebra: Students Behaviours And Teacher Role. In B. Maj-Tatsis & K. Tatsis (Eds.), *Generalization In Mathematics At All Educational Levels* (pp. 57–90). Wydawnictwo Uniwersytetu Rzeszowskiego.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Berdnarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 65–86). Kluwer.

Mustaffa, N., Ismail, Z., Tasir, Z. & Said, M. N. H. M. (2015). A Review on the Developing Algebraic Thinking. *Advanced Science Letters, 21* (10), 3381–3383 doi.org/10.1166/asl.2015.6511

Sibgatullin, I. R.,Korzhuev, a. V., Khairullina, E. R., Sadykova, A. R., Baturina, R. V., & Chauzova, V. (2022). A Systematic Review on Algebraic Thinking in Education. *EURASIA Journal of Mathematics, Science and Technology Education, 18*(1), 1–15. doi.org/10.29333/ejmste/11486

Early functional thinking through mathematical talk

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This presentation aims to exemplify and problematize how students in the early grades engage in talking mathematics concerning functional relationships, and how such engagement can foster the development of functional thinking through pattern generalizations. The terminology associated with functional thinking, such as linear functions, coordinates, rate of change, covariation and independent and dependent variables is not only novel to students but frequently unfamiliar and challenging to teachers in lower grades (Wilkie, 2020). For teachers working with younger students, this specialized terminology can seem abstract and disconnected from everyday classroom practice. However, research demonstrates that these concepts can be made accessible and pedagogically meaningful, even for young students, A central question arises: What mathematical concepts might be important for supporting the development of functional thinking in young learners, and in what ways can these concepts be explored through mathematical talk? One approach to engage students in mathematical talk about functional relationships is focusing on pattern generalization and relations through the lens of Smith's (2008) three modes of functional thinking: recursive patterning, covariational thinking and correspondence relationships. These modes provide a conceptual structure for identifying how functional thinking can be introduced and developed in the early years. By situating mathematical talk within these modes, teachers can create opportunities for deepening students' understanding of generalizations and promoting functional thinking from an early age (Sterner, 2022; 2024).

An intervention study involving Grade 1 (seven-year-olds) and Grade 6 (twelve-year-olds) students and their teachers demonstrated that the graph representation, often considered complex, can serve as an accessible tool for supporting students' functional thinking, particularly pattern generalization (Sterner, 2022; 2024). Through engagement with graphs and patterns exploration, both students and teachers were able to conceptualize and articulate the three modes of functional thinking. Classroom activities enable students to identify and interpret functional relationships and formulate generalizations. Notably, the mathematical talk in the classroom shifted from a focus on recursive patterning toward richer conversations about covariational thinking and correspondence relationships. These latter modes became visible when the students used the graph in linear relationships and using discrete values. For instance, students talked about the meaning of proportionality by interpreting the structure and the correspondence relationships within a general formula. In Grade 1, students referred to the slope of the graph using the metaphor of "big and little tents" to talk about rate of change along the y-axis and x-axis, an intuitive way of expressing rate of change despite

not having access to formal variable notation. Such examples illustrate how young students can engage meaningfully with functional relationships through specific pattern generalizations.

It is worth mentioning that the use of graphical representation does not automatically contribute to mathematical talk about covariational thinking and correspondence relationships. Intentional and reflecting teaching is required to ensure that the graphs contribute to the development of students' functional thinking. A challenge identified by participating teachers was the national curriculum. In *Lgr22* (Skolverket, 2022), functional thinking is only implicitly referenced through terms such as *relationships* and *change* (samband och förändring). This lack of explicitness of functional thinking poses a risk that teachers may struggle to grasp or convey the core ideas of functional thinking in the early years. Furthermore, the absence of a shared and accessible terminology complicates the planning and implementation of learning activities that meaningfully support students' development of functional thinking through mathematical talk (Sterner, 2022; 2024). Research (Wilkie, 2020) underscores that instruction related to functional relationships is often absent in the early grades, largely due to teachers' limited experience and support in teaching algebraic ideas grounded in functional thinking.

References

Skolverket. (2022). *Läroplanen för grundskolan, förskoleklassen och fritidshemmet 2022, Lgr22.* Skolverket.

Smith, E. (2008). Representational thinking as a framework for introducing functions in the elementary curriculum. In J. J. Kaput, D. W. Carraher, & M. L. Blanton, (Red.), *Algebra in the early grades,* (s. 133–160). Taylor & Francis Group.

Sterner, H. (2022). Using the graph when talking about functional relations in Grade 1: The importance of terminology. *Proceedings of the twelfth congress of the European Society for Research in Mathematics Education* CERME 12: (s. 612–619). http://erme.site/wp-content/uploads/2023/02/indexPROCEEDINGS.pdf

Sterner, H. (2024). Using graphical representations to develop students' correspondence relationships and covariational thinking in pattern generalizations in primary school. *International Journal of Science and Mathematics Education.* https://doi.org/10.1007/s10763-024-10520-z

Wilkie. K. J. (2020). Investigation students' attention to covariation features of their constructed graphs in a figural pattern generalisation context. International Journal of Science and Mathematics Education, *18*(2), 315–336. https://doi.org/10.1007/s10763-019-09955-6

Measuring interleaving of learning goals through an index for proximity

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Interleaving learning goals as lesson design

This presentation introduces a new – and easy to implement – proximity coefficient P_3 for quantifying the degree of interleaving. Firth et al. (2021) contrasted interleaving and block study to each other where the latter is to focus only one learning goal during a sequence of exercises while interleaving means to intertwine different learning goals. Organising the order of exercises for students according to interleaving or block study has different consequences for students' learning. In a post-test, Noh et al. (2016) found that when students had practiced according to interleaving, they achieved better on tasks that required the students to integrate information. On the other hand, they also found that when students had practiced according to block-study, they achieved better on tasks that required rule-based skills. Interleaving is relevant not only for the mathematical content, but also for how the content is represented, namely, Rau et al. (2013) found that students achieved better when the lesson design was to present learning goals as interleaving and representations as block-study when compared to the opposite design of their classroom experiment.

Measuring interleaving in textbooks

Most research on interleaving seem to be classroom studies. However, Rohrer et al. (2020) compared six mathematics textbooks in USA and found that in each textbook, about 10% of all the tasks were interleaved and 80% blocked while about 10% were ambiguous in this respect. If including demonstrational examples in a textbook, the definition of interleaving used in Rohrer et al. (2020) is the same as the one in Firth et al. (2021) and the proximity index in equation (1) should capture the degree of interleaving in a textbook.

$$P_3(learning\ goal2\ |learning\ goal1) = \frac{A}{A+B} \quad (1)$$

The sum $A + B$ in equation (1) denotes the number of tasks having learning goal 1. In equation (1), letter A denotes the number of tasks with learning goal 1 that also has learning goal 2 in the same, the previous or the following task, that is, in a proximity of three task, therefore the index 3 in equation (1). This definition implies that $0 \leq P_3 \leq 1$, where the case $P_3 = 0$ corresponds to no interleaving and $P_3 = 1$ corresponds to full

interleaving. Though there are methods for how to quantify similarity between different textbooks, see Petersson (2022), this proximity coefficient seems to a new development that also measures interleaving.

Methods

Equation (1) was used for comparing textbooks with respect to interleaving of Foundational Number Sense – FoNS as defined by Andrews and Sayers (2015). The FoNS framework applies to the number range [0; 20] and mathematical knowledge that pupils should learn by their first year in compulsory school. As data for empirically exploring the proximity coefficient P_3, units of analysis were all tasks in three Swedish mathematics textbooks (Eldorado, Favorit and Singma) and these were analysed with respect to the occurrence of FoNS.

Empirical results

Some results are that the textbooks Eldorado and Singma had more interleaving than Favorit for tasks on simple addition and subtraction. Singma had more interleaving through tying together counting pictorial objects and transforming this into an arithmetic situation. This could be connected to the emphasis on concrete-pictorial-abstract in Singapore-based textbooks. Despite that, Singma showed less interleaving for systematic counting (stability in the number track) than Eldorado and Favorit. This presentation gives some more details about these results.

References

Andrews, P., & Sayers, J. (2015). Identifying opportunities for grade one children to acquire foundational number sense: Developing a framework for cross cultural classroom analyses. *Early Childhood Education Journal, 43*(4), 257–267. https://doi.org/10.1007/s10643-014-0653-6

Firth, J., Rivers, I. and Boyle, J. (2021), A systematic review of interleaving as a concept learning strategy. *Review of Education, 9*(2), 642–684. https://doi.org/10.1002/rev3.3266

Noh, S. M., Yan, V. X., Bjork, R. A., & Maddox, W. T. (2016). Optimal sequencing during category learning: Testing a dual-learning systems perspective. *Cognition, 155*, 23–29. https://doi.org/10.1016/j.cognition.2016.06.007

Petersson, J., Sayers, J., & Andrews, P. (2022). Two methods for quantifying similarity between textbooks with respect to content distribution. *International Journal of Research & Method in Education, 46*(2), 161–174. DOI: https://doi.org/10.1080/1743727X.2022.2093846

Rau, M. A., Aleven, V., & Rummel, N. (2013). Interleaved practice in multi-dimensional learning tasks: Which dimension should we interleave?. *Learning and Instruction, 23*, 98–114. https://doi.org/10.1016/j.learninstruc.2012.07.003

Rohrer, D., Dedrick, R. F., & Hartwig, M. K. (2020). The Scarcity of Interleaved Practice in Mathematics Textbooks. Educational Psychology Review, 32(3), 873-883. https://doi.org/10.1007/s10648-020-09516-2

Collaboration between parents and preschool teachers for children's learning of mathematics

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This presentation highlights the tension between parental engagement and professional practice in early childhood mathematics education. Parents and preschool teachers often value different aspects of children's learning due to their roles and experiences (Skwarchuk, 2009). While policy encourages parental involvement (Ministry of Education and Research, 2017) differences in how children engage with mathematics at home versus in ECEC (Early Childhood Education and Care) can make collaboration challenging (Einarsdottir & Jónsdóttir, 2017). Teachers, as professionals, may perceive parental involvement as interfering (Wager & Whyte, 2013), complicating efforts to integrate home-based learning into formal education (Blevins-Knabe et al., 2000; Lee & Ginsburg, 2007). Drawing on Foucault's theory of power and knowledge (Foucault, 1980), I will present a study which explores how dynamics in interactions between parents and preschool teachers shape what is considered valuable math knowledge and how that affects collaboration. For example, Peressini's (1998) research shows how reform documents in mathematics education have created what Foucault calls a "regime of truth"–a system in which certain forms of knowledge are legitimised while others are marginalised. In this regime, teachers are positioned as experts with pedagogical authority, while parents are often seen as outsiders, limiting their role in shaping their children's learning. But power is not fixed, it circulates through everyday interactions. In this presentation, I will illustrate how interactions between parents and teachers offer opportunities to challenge dominant truths and renegotiate what is valuable knowledge about mathematics education for young children.

Methodology

The data were collected through photo-elicited focus group interviews (PEIs). PEIs involve participants bringing and discussing their own photographs, which serve as visual prompts to stimulate conversation and reflections. This presentation will be based on one interaction between two parents and three teachers discussing a photograph of a child playing *Pokémon: Magikarp Jump* on a tablet. This app is a digital game in which players raise and compete with their Pokémon by feeding them, tracking their growth, and making choices about training activities. These actions involve mathematical ideas such as counting items, comparing sizes, recognising numerical increases, and making simple predictions. Such activities reflect early mathematical concepts like number recognition, one-to-one correspondence, and basic measurement. The interaction about this app was chosen as a data source because it integrates mathematics into everyday

activities. This makes it a relevant context for examining how parents and teachers interpret mathematics learning in informal digital environments.

Results and conclusions

This presentation sought to highlight that parent-teacher discussions about digital apps for children's mathematics revealed varied forms of collaboration, shaped by how knowledge about mathematics was accepted, resisted, or negotiated. One form emerged when parents accepted teachers' expertise on suitable apps for learning. A second form of collaboration emerged when parents resisted teachers' views on digital tools, leading to fragmented interactions and a lack of shared understanding, highlighting how collaboration can break down when knowledge is not mutually accepted. A third form of collaboration emerged through negotiation, in which differing perspectives led to the co-construction of shared knowledge about the use of digital apps, not only for mathematics learning but also for broader educational purposes.

References

Blevins-Knabe, B., Austin, A. B., Musun, L., Eddy, A., & Jones, R. M. (2000). Family home care providers' and parents' beliefs and practices concerning mathematics with young children. *Early Child Development and Care*, *165*(1), 41–58. https://doi.org/10.1080/0300443001650104

Einarsdottir, J., & Jónsdóttir, A. H. (2017). Parent-preschool partnership: many levels of power. *Early Years*, *39*(2), 175–189. https://doi.org/10.1080/09575146.2017.1358700

Foucault, M. (1980). *Power/Knowledge: Selected interviews and other writings, 1972-1977* (C. Gordon, Ed.). Harvester Press.

Lee, J. S., & Ginsburg, H. P. (2007). Preschool teachers' beliefs about appropriate early literacy and mathematics education for low- and middle-socioeconomic status children. *Early Education and Development*, *18*(1), 111–143.https://doi.org/10.1080/10409280701274758

Ministry of Education and Research. (2017). *Framework plan for the content and tasks of kindergartens*. Norwegian Directorate for Education and Training. https://www.udir.no/contentassets/7c4387bb50314f33b828789ed767329e/framework plan-for-kindergartens--rammeplan-engelsk-pdf.pdf

Peressini, D. D. (1998). The portrayal of parents in the school mathematics reform literature: Locating the context for parental involvement. *Journal for Research in Mathematics Education*, *29*(5), 555–582. https://doi.org/10.5951/jresematheduc.29.5.0555

Skwarchuk, S.-L. (2009). How do parents support preschoolers' numeracy learning experiences at home? *Early Childhood Education Journal*, *37*(3), 189–197. https://doi.org/10.1007/s10643-009-0340-1

Wager, A. A., & Whyte, K. (2013). Young children's mathematics: Whose home practices are privileged? *Journal of Urban Mathematics Education*, *6*(1), 81–95. https://doi.org/10.21423/jume-v6i1a196

Who owns the mathematics in pre-school?

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In this presentation we will present how mathematical teaching and learning emerge during planned and spontaneous play situations in preschool, and how ownership of the play and the learning process is negotiated between teachers and children, using Helenius' (2018) dimensions of teaching and Stensson's (2022) concept of attention as analytical frameworks.

Background

Mathematical competencies at pre-school are highly relevant for learning outcomes at school. In a meta-analytic regression, maths competencies in kindergarten, i.e. number recognition, number sequence, counting, ordinality, relative size, addition and subtraction were found to be the strongest predictors for later school achievement (Duncan et al. 2007). Whilst there is a growing awareness that children need to be supported in their mathematical learning in pre-school, there is little consensus about the pedagogical approach.

Play-based pedagogy has increased in pre-school during the last decade (se for example Chukwudi Ekeh, 2023). In Sweden, pre-school mathematics has a tradition of being taught through a play-based approach. 1998 the pre-school got the first national curriculum with goals for the pre-school mathematic teaching and learning but with no evaluation of the children´s achievement. Lately there has been a shift towards a pedagogy where pre-school teachers should interact more with the children to challenge them in mathematics. In the curriculum this can be noticed by the word teaching being stated (Skolverket, 2019).

Helenius (2018) writes about the move in Sweden towards a more teacher-driven and formalised pedagogy and present conceptual framework in order not to push Swedish pre-schools in such direction whilst still maintaining a focus on teaching. He calls for visibility of teaching in preschool being more than planned situations. In line with Helenius research, we draw on four short episodes to see how the dimensions change during unplanned session with children in a preschool setting.

Method

Björklund (2008) and Lynch and Stanley (2018) outline the benefits and use of video when examining the actions of very young children. Björklund (2008) states that using video enables a more thorough examination of the actions, often in a more natural or everyday environment. Both Björklund (2008) and Lynch and Stanley (2018) espouse the capacity of video to be viewed and reviewed when being analysed.

Participants and situation

The research was undertaken in a private preschool in a large city in southern Sweden. Filming was undertaken with different groups over several days. In this presentation we report on 4 episodes of play. Although initially the teacher had not nominated it as being a mathematical activity, this focus became evident as the children engaged in the situations. Therefore, they were chosen because they exemplified how the activity was developed from the children's own interests.

In order to analyse the material, we used a table to note what happened during the episodes. Se table 1 for example.

A dimension of pedagogic explication		A dimension of teacher participation		A dimension of situational planning		Attention	
Pedagogical	Instrumental	Participation	No participation	In moment	Formally	Shared	Non shared

Table 1. Analyses of situation

Result

The analysis of the four episodes shows that the didactic dimensions – pedagogical explication, teacher participation, situational planning, and attention – are all present in various forms during spontaneous activities in preschool. These dimensions are not static but shift dynamically throughout the course of the activity. In summary, the results show that mathematics teaching in preschool does not necessarily need to be formally planned to be effective but rather depends on the teacher's ability to recognise and build on the mathematical opportunities that emerge in children's play and exploration.

References

Björklund, C. (2008). Toddlers' opportunities to learn mathematics. *International Journal of Early Childhood, 40*(1), 81–95. https://doi.org/10.1007/BF03168365

Duncan GJ, Dowsett CJ, Claessens A, Magnuson K, Huston AC, Klebanov P, Pagani LS, Feinstein L, Engel M, Brooks-Gunn J, Sexton H, Duckworth K, Japel C. School readiness and later achievement. Dev Psychol. 2007 Nov;43(6):1428-1446. doi: 10.1037/0012-1649.43.6.1428. Erratum in: Dev Psychol. 2008 Jan;44(1):232. PMID: 18020822.

Helenius, O. (2018). Explicating professional modes of action for teaching preschool mathematics. *Research in Mathematics Education, 20*(2), 183-199.

Lynch, H., & Stanley, M. (2018). Beyond words: Using qualitative video methods for researching occupation with young children. *OTJR: occupation, participation and health*, *38*(1): 56–66. http://doi.org/10.1177/1539449217718504

Skolverket (2019). *Curriculum for the pre-school (Lpfö18*).

Stensson, C. (2022). Didaktik attention: Preschool teaching from a didaktik and variation theory perspective using examples of mathematical aspects of programming. *Educare.*

Mathematics teacher students' designs of degree projects

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This presentation concerns degree theses that students in mathematics teacher education write as reports of their degree projects. These theses represent the peak of the students' scientific engagement within the programs and are used as examples for other students and benchmarks in governmental evaluations of teacher educations. This raises the question of what students select to study within their scientifically founded teacher education. Skau et al. (2025) found that qualitative methods along with themes related to learning and instruction are most common in Swedish teacher students' theses. Empirical choices are influenced by various factors, such as difficulties in applying scientific frameworks, as found in a study of mathematics teacher students' use of frameworks (Juter, under review), or challenges in reading academic texts (Yore & Tang, 2022). The presentation builds on the question: What topics, objects of study and methods of analysis do students choose in degree projects in mathematics teacher education programs for secondary school levels?

The study

This study examined 74 published degree theses authored by 98 students from 14 Swedish mathematics teacher education programs aimed at teaching school years 7–12. Only regular programs, whether campus-based or online, were included. Of the 74 theses, 50 are written individually and 24 are written in pairs. A qualitative content analysis was conducted through reviews of the theses to identify characteristics in the students' textual descriptions.

The objects of study in the theses were categorized according to the categories of the didactical tetrahedron (Rezat & Sträßer, 2012) and one thesis may cover one, two or three of the objects *students*, *teachers* and *artifacts* (T: teachers, S: students, and A: artifacts in Table 1) alongside the overarching object *mathematics,* which they all cover. The objects of study were then linked to the mathematics education topics derived from the students' research questions, as well as to the analytical methods and frameworks employed (as defined by Magana, 2022). Table 1 presents the distribution of theses by topics, methods for analysis, and object of study, with topics and methods represented by fewer than four theses omitted. A single thesis may appear in multiple categories.

Topics in research questions	Obj of study				Methods of analysis	Obj of study		
	T	S	A			T	S	A

Digital activities	14	6	3		Thematic analysis	24	11	2
Math. communication	5	6	3		Content analysis	13	11	13
Stud. identity & experience	4	10	0		Statistical analysis	4	11	3
Assessment	7	2	3		Discourse analysis	3	1	2
Problem-solving	4	3	4		Phenomenography	3	1	1
Math. areas, s.a. algebra	2	5	2		Grounded theory	3	0	1
Students' use of textbooks	2	3	3					
Students' math. difficulties	2	3	0					
Inclusion of all students	4	1	0					
Mathematical reasoning	1	0	3					
Modelling	2	1	1					

Table 1. Topics in research questions and methods of analysis used in the theses related to objects of study (numbers of theses).

Digital activities was shown to be the most frequently addressed topic in the theses, in contrast to Skau's (2025) results, particularly from the perspective of teachers. Other popular topics are mathematical communication, student experiences, and assessment. Fewer theses focused on mathematical problem-solving or specific mathematical content areas. Thematic analysis and content analysis were the most employed methods, the former especially in theses with teachers as objects of study, which may be a consequence of trends in the research community. The theses reveal varied combinations and justifications of methodological frameworks, often simplified and insufficiently problematized, with several theses lacking clarity in how frameworks are combined and applied in the analysis.

The results have implications for teacher education regarding content and how it is taught, particularly regarding scientific methods related to didactical practices.

References

Juter, K. (under review). Pre-service mathematics teachers' framework usage in degree theses.

Magana, A. J. (2022). The role of frameworks in engineering education research. *Journal of Engineering Education, 111*(1), 9–13. https://doi.org/10.1002/jee.20443

Rezat, S., & Sträßer, R. (2012). From the didactical triangle to the socio-didactical tetrahedron: artifacts as fundamental constituents of the didactical situation. *ZDM Mathematics Education, 44*, 641–651. https://doi.org/10.1007/s11858-012-0448-4

Skau, S., Börjesson, M., Nyman, R., & Thompson, W. H. (2025). Trendlines in research methodology and topics in teacher education: big data analysis of theses in Swedish teacher education between 2005 and 2023. *Scandinavian Journal of Educational Research*, 1–16. https://doi.org/10.1080/00313831.2025.2506378

Yore, L. D., & Tang, KS. (2022). Foundations, insights, and future considerations of reading in science and mathematics education. *International Journal of Science and Mathematics Education, 20*, 237–260. https://doi.org/10.1007/s10763-022-10321-2

Exploring teacher student interaction to support reasoning in a Thinking Classroom context

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This presentation reports on a study examining teachers' use of questions to facilitate reasoning when interacting with students working in groups according to the teaching approach Thinking Classroom. Teacher-student interaction is recognised as a central component of effective mathematics teaching with opportunities to access students' reasoning, making visible the ways in which learners interpret and construct mathematical ideas (Walshaw & Anthony, 2008). Classroom interaction is also associated with students' cognitive development (Zhang & Cao, 2022), however not all classroom talk is equally productive; it is the quality and structure that determines its educational value (Walshaw & Anthony, 2008).

Reasoning is a central goal of mathematics education, contributing to students' deeper understanding and conceptual development (Hiebert, 2003). Teachers' questions are crucial, as well-designed questions can prompt students to articulate their thinking and make connections (Boaler & Brodie, 2004). Different types of questions vary in their cognitive demands and impact on reasoning (DeJarnette et al., 2020). Ultimately, teacher–student interaction enables questioning that fosters reasoning and can "greatly enhance the development of their [the students] mathematical thinking" (Walshaw & Anthony, 2008, p. 535).

The teaching approach Thinking Classroom (Liljedahl, 2016) consists of 14 practices aimed at creating a student-centred environment with a focus on students' own thinking. The first three practices are considered foundational, as they quickly shift classroom norms and should be implemented together. These are: (1) using tasks that are accessible but not routinely and require thinking rather than imitation; (2) randomly assigning students to groups of up to three that change regularly; and (3) students solve tasks standing at whiteboards, or other erasable vertical surfaces. These first three practices are relatively easy to implement and can bypass the normative behaviours of both students and teachers (Liljedahl, 2016).

The analysis drew on 44 recordings of students working in groups, with durations ranging from 31 to 55 minutes. These recordings yielded more than 150 transcribed teacher–student interactions. The research question, 'How do teachers facilitate the students' mathematical reasoning in the context of Thinking Classroom?' guided the analysis. Every teacher question concerning mathematics was labelled using Boaler and Brodie's (2004) categories. Utterances that functioned as questions were also included, for example "Let's hear," and "Tell me," both urging the students to describe their work. The category in focus was *probing* (to encourage the students to explain their thinking) which I viewed as one way to facilitate reasoning. The preliminary results show that about 15 % of the questions were probing, for example: "Why did you choose thirteen?",

"What was your strategy?" and "Because?". Not every probing question led to reasoning among the students and since an interaction never is one-sided the next step of the analysis will focus on the students' responses.

The analysis of the data revealed an unexpected pattern, many opportunities for reasoning were left unexplored. Instead, the focus often shifted towards obtaining the correct answer with the correct method and moving quickly to the next task, as in:

Teacher: How much [money] did you get?

Student: Twenty-three, thirty-three… forty-one, fifty

Teacher: Good, now [next part of the task].

This raises an important issue: if the students are to develop mathematical reasoning, they must be given the opportunity to practice it.

It is suggested that implementing the first three practices of Thinking Classroom can substantially shift students' thinking and influence teachers' behaviour. In this material, the same patterns regarding teachers' questions as previously reported (e.g. Boaler & Broadie, 2004) can be observed. This raises a critical question: if teachers' approaches to questions remain unchanged, how can the students' long term reasoning practices be expected to evolve?

References

Boaler, J., & Brodie, K. (2004). The importance, nature and impact of teacher questions. In D. E. McDougall & J. A. Ross (Eds.), *Proceedings of the twenty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 774–782). University of Toronto/PME-NA.

DeJarnette, A. F., Wilke, E., & Hord, C. (2020). Categorizing mathematics teachers' questioning: The demands and contributions of teachers' questions. *International Journal of Educational Research, 104,* Article 101690. https://doi.org/10.1016/j.ijer.2020.101690

Hiebert, J. (2003). What research says about the NCTM Standards. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 5–23). National Council of Teachers of Mathematics.

Liljedahl, P. (2016). Building thinking classrooms: Conditions for problem-solving. In P. Felmer, E. Pehkonen, & J. Kilpatrick (Eds.), *Posing and Solving Mathematical Problems* (pp. 361–386). Springer International Publishing. https://doi.org/10.1007/978-3-319-28023-3_21

Walshaw, M., & Anthony, G. (2008). The teacher's role in classroom discourse: A review of recent research into mathematics classrooms. *Review of Educational Research, 78*(3), 516–551. https://doi.org/10.3102/0034654308320292

Zhang, S., & Cao, Y. (2022). Classroom interaction in mathematics: Learning of mathematics and learning to participate. *International Journal of Science and Mathematics Education, 20*(3), 607–626. https://doi.org/10.1007/s10763-021-10167-8

Teacher perspectives on ”Thinking Classrooms”: a longitudinal case study in Norwegian upper secondary mathematics education

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Introduction

In this ongoing study, we explore Norwegian upper secondary mathematics teachers' perspectives on teaching in what is referred to as ”Thinking Classrooms”. At a surface level, this method comprises students' working jointly on vertical surfaces, to solve mathematical problems. The teachers' role in such classrooms is to act as a facilitator, providing necessary prompts or questions if the problem solving comes to a standstill, and at the end of the session, to consolidate the learning through a joint discussion or debrief (Liljedahl, 2016). This teaching method has become more prominent in Norwegian classrooms, aided by a strong support from different municipalities and courses arranged by the Norwegian Centre for Mathematics Education. Similar trends can also be found internationally.

Despite increasing interest, there is still a lack of research from classrooms utilizing the method in secondary education and on teacher perspectives on the implementation of ”Thinking classrooms”. Therefore, we aim to investigate Norwegian upper secondary mathematics teachers' beliefs and narratives (cf. 'perspectives' in Xenofontos, 2019) on implementing ”Thinking classrooms”, working in a school where this method has been adopted as a primary method of mathematics teaching across all classrooms.

Theoretical perspectives

”Thinking classrooms” are focused on students working in groups of three to solve problems on vertical non-permanent surfaces, and comprises a set of teaching practices meant to enhance learning, comprising standards for what tasks are utilized and how they are given, how questions are answered and hints are provided during task work, and how to consolidate learning (Liljedahl, 2016).

As is the case with any sort of reform or introduction of a new teaching practice, this may entail new ways of teaching, thinking, and planning for teachers, and sometimes, a change in the teachers' role in the classroom. Previous research has found that such changes can be challenging for teachers, and that there is a complex relationship between teachers' beliefs and changes in teaching practices (Cobb et al., 1990). Such changes can be challenging, particularly in situations where the teachers' identity shifts from a provider of learning, into a facilitator. Similarly, tensions between introducing a new set of practices, as per ”Thinking classrooms”, without forfeiting practices that have previously worked well can become challenging and thus lead to a strained relationship with the newly adopted teaching method.

Methods

This study is part of the overarching ”Problem solving and inquiry with vertical boards” project at the University of Oslo, which collects video, interview, and survey data from secondary schools. In this study, we utilize a longitudinal case design, where teachers are interviewed three times across three years, utilizing semi-structured interviews asking the teachers about their perspectives on mathematics, mathematics education, ”Thinking classrooms”, and various related practices such as assessment, differentiation, and tasks. The data is analyzed thematically, using inductive and deductive coding. The presentation reports on two cases.

Preliminary results and discussion

Both teachers have previous experiences implementing inquiry-based teaching approaches, and thus, are positive toward the adopted method from the start. However, in the first year they feel restricted by the many practices detailed in Liljedahl’s approach, expressing some worry about not being able to utilize other practices, practices they have used prior to adopting the new method, and having less “control” over what happens in the classroom. Over time, they slowly adapt their teaching approaches to align more to the upper secondary mathematics curriculum, their students, and their own teaching preferences, indicating a greater feeling of agency in applying the ”Thinking classroom” method. They also express a greater sense of control over the course of the years, as they feel more comfortable relying on their observations of the groups and vertical surfaces as a means of both providing insights into students’ learning processes and as a ground for providing feedback in line with assessment for learning practices, showcasing some affordances with the new teaching method.

Despite more experience with the method between interviews, the teachers expressed some concerns in terms of being able to differentiate teaching sufficiently for students with lower achievement, as they might feel less inclined or have less opportunities to participate in joint problem-solving. However, they are both confident the joint problem-solving is beneficial even to these students.

References

Cobb, P., Wood, T., & Yackel, E. (1990). Chapter 9: Classrooms as learning environments for teachers and researchers. *Journal for Research in Mathematics Education. Monograph*, *4*, 125–210.

Liljedahl, P. (2016). Building thinking classrooms: Conditions for problem-solving. In *Posing and solving mathematical problems: Advances and new perspectives* (pp. 361-386). Cham: Springer International Publishing.

Xenofontos, C. (2019). Primary teachers' perspectives on mathematics during curriculum reform: A collective case study from Cyprus. *Issues in Educational Research*, *29*(3), 979–996.

Using contradictions as a didactical tool in early mathematics education

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This presentation explores how contradictions that emerge in student collective actions can be used as a didactical tool to promote theoretical thinking in early primary mathematics. Drawing on a cultural-historical activity theory (CHAT) perspective and the principles of the El'konin–Davydov (ED) curriculum (Davydov, 2008), we present data from two research lessons where students work with tasks designed to support an understanding of the structural properties of the positional number system (Björk, 2023).

From this perspective, contradictions are not treated as individual cognitive errors but as essential drive force in learning activity (Davydov, 2008; Ilyenkov, 2007). A contradiction arises when the students' empirical understanding proves insufficient to resolve a conceptual demand, thereby creating the need for a new way of seeing the object of study. As Radford (2021) emphasise, contradictions surface not only at the conceptual level but also through students embodied and tool-mediated actions.

Theoretical background and analytical perspective

In a learning activity, students must identify and define the problem themselves, assess their existing tools, and engage in collective theoretical exploration (Repkin, 2003; Zuckerman, 2003). Tools used as *learning models* become central – they are not merely aids or illustrations but mediators of meaning that help make visible essential relations and structures within the content (Davydov, 2008). The teacher's role is to design tasks and sustain conditions where contradictions can emerge and be explored jointly.

Contradictions, in this sense, are not methodical additions but intrinsic to a process where habitual ways of seeing are disrupted. They offer developmental potential when they emerge within a meaningful, culturally situated collective activity – especially when mediated by shared tools and models (Davydov, 2008; Radford, 2021).

Data and methodology

The analysis draws on video recordings from two collaboratively – researchers and teachers – designed research lessons (25 in total) in a Swedish lower grade classroom, conducted within a larger design-based study. Each lesson was divided into 13 analytic sequences, three of which contained content-related contradictions identified through verbal and embodied indicators such as hesitation, inscription conflicts, meaning negotiations, and student utterances (Radford, 2021).

Findings

The identified contradictions led to collective exploration that enabled the students to go beyond their existing knowledge and ideas about key properties of the positional system. In their efforts to resolve the contradictions, their modelling work with tools such as number lines and place value tables was of crucial importance. The three content related contradictions surfaced when the students 1) tried to understand how two symbolic tools – the table and the number line – used to model the same counting system, could correspond to each other, 2) were to construct successively larger number units according to a multiplicative principle, and 3) tried to understand that a number unit has a value regardless of its graphical representation.

Conclusion and implications

We argue that contradictions such as these can support the development of theoretical thinking by creating a need to challenge and go beyond existing knowledge among the students. In line with Radford (2021), contradictions become visible *not only* through speech but also through how students interact with tools and each other. Furthermore, contradictions can arise when the teacher asks provocative questions or makes provocative comments.

We argue that contradictions, when recognised and maintained within the classroom activity, can function as powerful didactical tools. They help to shift the focus from surface features to essential relationships within the content. Designing for such contradictions, however, requires both content sensitivity and pedagogical responsiveness. The teacher must be able to identify when a contradiction is about to surface and support its exploration without closing it down too quickly.

References

Björk, M. (2023). *Att främja elevers teoretiska utforskande av bassystemet – en undervisningsutvecklande studie i matematik på mellanstadiet* [Promoting students' theoretical exploration of the base system: A teaching developmental study in mathematics in grade four]. [Licentiate thesis, Stockholm University].

Davydov, V. V. (2008). *Problems of developmental instruction. A theoretical and experimental psychological study.* Nova Science Publishers. (Original work published 1986)

Ilyenkov, E. V. (2007). Our schools must teach how to think! *Journal of Russian & East European Psychology, 45*(4), 9–49. https://doi.org/10.2753/RPO1061-0405450402

Radford, L. (2021). *The theory of objectification. A Vygotskian perspective on knowing and becoming in mathematics teaching and learning*. Brill.

Repkin, V. (2003). Developmental teaching and learning activity. *Journal of Russian and East European Psychology, 41*(5), 10–33. https://doi.org/10.2753/RPO1061-0405410510

Zuckerman, G. (2003). The learning activity in the first years of schooling. In A. Kozulin, B. Gindis, V. S. Ageyev & S. M. Miller (Eds.), *Vygotsky's educational theory in cultural context* (pp. 177–199). Cambridge University Press. https://doi.org/10.1017/CBO9780511840975.011

Developing a Framework for Differentiating Elicitation and Feedback

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Introduction

This presentation introduces a framework for differentiating between teacher utterances intended to elicit information about student knowledge and those intended to provide feedback that supports student learning needs (SLN).

Feedback is a central component of formative assessment (Hattie & Timperley, 2007). It has been shown to support student achievement, but research also reports variation in its effects, ranging from negative to very positive outcomes (Shute, 2008; Wisniewski et al., 2020). Despite these findings, research has not yet provided a clear understanding of the mechanisms that explain when and why feedback is effective. One underexplored mechanism concerns how teachers' elicitation of student thinking contributes to their interpretation of SLN and influences the feedback they provide. This mechanism can, however, be difficult to capture in research. Distinguishing whether a teacher's utterance is intended as an elicitation of information or as feedback towards learning can often be challenging. In some cases, the same utterance may involve elements of both. Currently, there is no analytic tool that distinguishes between teachers' elicitation and feedback, making it more challenging to examine how teachers elicit information about students' thinking and how this information forms the basis for the feedback that follows.

The guiding research question is: How can a framework be formulated that can be used to distinguish between teacher utterances intended to elicit information about student knowledge and those intended to provide feedback that supports SLN? By developing such a framework, the study aims to clarify the processes involved in formative assessment and offers a basis for further research on how teachers' practices can be analysed and supported in mathematics education.

Method

The data consisted of audio-recorded mathematics lessons in grades 6 to 9, transcribed into teacher–student dialogues and interviews with teachers. The analysis focused on individual teacher–student interactions, while whole-class discussions and non-mathematical talk were excluded. Only teacher utterances were coded, focusing on intentions rather than functions, since the study seeks to examine what teachers aim to achieve rather than how students interpret their words.

The development of the framework followed an iterative process. Initial distinctions between elicitation and feedback were informed by existing models of formative assessment and feedback (Hattie & Timperley, 2007; Shaughnessy et al., 2021) as well as observations of the classroom data. These preliminary categories were applied to the transcripts, discussed with supervisors, and revised through several cycles until definitions and indicators became sufficiently clear. The framework was also tested through interviews where teachers commented on their own utterances and described their intentions.

Framework

The framework consists of two main categories of teacher utterances: elicitation and feedback. Within elicitation, four subcategories were identified: (E1) prior reasoning, (E2) student perceived learning need, (E3) specific knowledge needed to continue, and (E4) ideas on how to proceed. Feedback was also divided into three subcategories: (F1) explicit feedback, which entails suggestions of strategies or instructions, (F2) implicit feedback, which guides students towards learning and/or problem solving, and (F3) evaluation of students' work or ideas.

Conclusion

Most utterances could be categorised without difficulty, while a few remained problematic. Certain teacher questions appeared to serve both elicitation and feedback at the same time, which made it challenging to decide how to code them. In the analysis of ambiguous cases, it became clear that context is important. This refers to aspects such as the whole sequence of the interaction, the point at which the utterance occurs, and the character of the task that the student was working on.

During the presentation, the framework will be described in more detail and demonstrated with examples from classroom transcripts, including cases where utterances seem to involve both elicitation and feedback. While no framework can capture teachers' intentions with complete certainty, this framework offers a close approximation that provides a useful basis for studying how the initial stages of formative assessment are carried out in mathematics classrooms.

References

Hattie, J., & Timperley, H. (2007). The Power of Feedback. *Review of Educational Research*, *77*(1), 81–112.

Shaughnessy, M., DeFino, R., Pfaff, E., & Blunk, M. (2021). I think I made a mistake: How do prospective teachers elicit the thinking of a student who has made a mistake? *Journal of Mathematics Teacher Education*, *24*(4), 335–359.

Shute, V. J. (2008). Focus on Formative Feedback. *Review of Educational Research*, *78*(1), 153–189.

Wisniewski, B., Zierer, K., & Hattie, J. (2020). The Power of Feedback Revisited: A Meta-Analysis of Educational Feedback Research. *Frontiers in Psychology*, *10*, 3087.

Students' impasses during joint problem-solving

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Introduction and theoretical background

In recent years, the teaching approach *thinking classrooms* has become increasingly prevalent in mathematics education. In a thinking classroom, students are assigned random groups of three and work on non-permanent vertical surfaces with problem-solving tasks that are intended to promote thinking (Liljedahl, 2016). In this context, problem-solving activities can be understood as students engaging with tasks where the solution strategies are not immediately available to them. During such activities, there are situations wherein students are unable to generate new solution strategies, stop attempting to reach a solution, or repeat previous strategies. Such situations can be termed *impasses* (Ohlsson, 2011). Students can also become fixated on pursuing non-productive strategies, often referred to as a *wild goose chase* (Schoenfeld, 1987), which can be understood as another type of impasse. Aiming to understand these issues, this study examines the occurrences of impasses in thinking classrooms and how students navigate out of them, with or without facilitation. The study is based on data from a larger overarching project at the University of Oslo, "Problem solving and inquiry with vertical boards", and presents preliminary findings from this ongoing research.

Methodology

Video data is collected to observe student groups in upper secondary mathematics classrooms as they work jointly with problem-solving tasks in a thinking classroom. We will present and discuss preliminary findings from the study, using two illustrative examples that showcase what may happen when students reach an impasse and how they navigate out of it. Deductive codes (*unable*, *stopping, repeating,* and *wild goose chase*, cf. Ohlsson, 2011; Schoenfeld, 1987) were used to analyse impasses, whereas both inductive and deductive codes (*asking teacher*, *asking neighbour group*, *looking at neighbouring vertical surface, collaboration,* and *teacher initiating guidance*) were used for attempts to navigate out of impasses. The research project received the necessary approvals from Sikt, and written consent letters was signed by the participating students and teachers.

Preliminary findings and discussion

In the first sequence a group is trying to figure out when a given function is greater than another function. The group aims to determine the y-value for the intersection but is unable to generate a solution strategy, indicating an impasse. Attempting to navigate out of the impasse, the group reaches out to a neighbouring group but fails to understand

their provided suggestions. Therefore, the group turns to their teacher in a new attempt to navigate out of the impasse. The teacher replies by decomposing the solution with several questions. This leads to the group being able to determine the y-value.

The sequence can be categorized as the impasse type *unable*. It can also be understood as a *wild goose chase*, as the solution strategy fails to provide information about when the function is greater and thus is non-productive. The teacher guided them out of the impasse type *unable* by posing questions decomposing the solution strategy. This indicates that the students needed assistance from the teacher in uncovering and structuring the necessary knowledge to get "back on track" in their problem-solving process. However, the group was still on a *wild goose chase*. A possible reason for this, is that the students were unconscious of this impasse. In addition, the teacher did not notice the impasse or chose to leave it out of the guidance. This indicates that a combination of two types of impasses can make it more complex to move beyond.

In the second sequence a group is trying to solve a quadratic inequality. They start formulating the expression as an equation but are unable to generate a solution, indicating an impasse. One of the group members looks at a neighbour group's vertical surface and copies the beginning of their solution, claiming that: "I do not know what I have done now, but it is something here". This leads to a group discussion regarding what the copied beginning means and how to use it to reach a solution. This collaboration leads to them uncovering that the quadratic formula provides the solution.

The sequence can be categorized as the impasse type *unable*. By looking at a neighbouring vertical surface, the group were able to start a discussion and find a solution. It appears that the group needed something to base their discussion on to get "back on track" in their problem-solving process. In addition, the sequence indicates that examining a neighbouring surface can be insufficient for moving beyond an impasse, because the group also relied on collaboration.

Conclusion

To summarize, the sequences have shown that impasses and navigation can occur in various ways. The preliminary findings indicate that it may be more complex for students to overcome combinations of impasses, and that sometimes different types of navigation need to be combined to move beyond an impasse.

References

Liljedahl, P. (2016). Building Thinking Classrooms: Conditions for Problem-Solving. In P. Felmer, E. Pehkonen & J. Kilpatrick (Ed.), *Posing and Solving Mathematical Problems: Advances and New Perspectives* (pp. 361-386). Springer. https://doi.org/10.1007/978-3-319-28023-3_21

Ohlsson, S. (2011). *Deep Learning: How the mind overrides experience.* Cambridge University Press. https://doi.org/10.1017/CBO9780511780295

Schoenfeld, A. H. (1987). What's all the fuss about metacognition? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (1st ed., pp. 189-215). Lawrence Erlbaum Associates. https://doi.org/10.4324/9780203062685-8

Dilemmas using digital textbooks when learning math

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This presentation will present and discuss findings from a project which is based on an extensive video ethnographic research project using cameras and screen recordings in upper secondary classrooms to follow teachers' and students' use of a learning platform (Nilsberth et al., forthcoming). The platform includes different kinds of teaching resources, including the textbook, and becomes a base for subject transformation. The conditions that emerge in its use uncover what mathematics learning is enabled in such kind of digitalized teaching environment. We will present a case from one student's work to display some dilemmas that arise in this learning environment when working with a digitalized textbook. The textbook is nested in the learning platform. Each task in the textbook generates points if you enter the correct answer. If the student is unsure, they can look at a hint, but then they get fewer points. If they look at the proposed solution, they get no points.

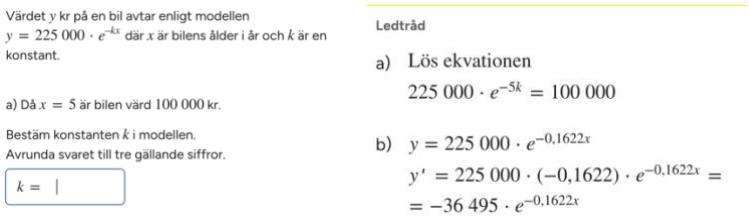
Analysis procedure

In the analysis of the video and screen recordings we first made a transcript of the actions taken by the student. Then we analysed sequences of the student's activities using the theoretical lens of Duval (2017) to describe the learning process.

The situation

The student in focus struggles with a task (see Table 1) and how to write the answer and get it accepted in the digital book.

Table 1. The task, and description of clues the student can use if she chooses



The student writes in her notebook and uses the calculator. She enters k= 0.833 in Noteflex, and it becomes marked in red (wrong). The student looks at her notes and enters 0.83, which is also marked in red. 'No' she says and sighs. She turns the page in her notebook and continues to write and calculate on the calculator. Enters 0.067, which is marked in red. Enters 0.0674, which is marked in red. She mumbles 'I would like to say that it's correct… I feel'. The student looks at her calculations on paper and the

screen. ‘But is it…’, she says, and tests -0.067, which is wrong, then she tests -0.0674, which is also wrong. She sighs heavily. Looks at her notes again and raises her hand.

Analysis and dilemma

According to Duval (2006) the learning process of mathematical thinking involves two types of transformations: treatments and conversions. Treatments are defined as transformations within one register, such as rephrasing or solving an equation using algebraic signs and symbols. Conversion, which is seen as a more challenging process, is defined as a transformation where the register changes, such as going from an algebraic to a graphic representation of a function, while maintaining the same conceptual reference, that is, keeping the mathematical meaning of the object. Registers are a semiotic system that permits the transformation of representations (Duval, 2006, p.111). Duval (2006) divided registers into two groups, monofunctional and multifunctional. Registers that enable mathematical processes such as algorithms (e.g. manipulations of algebraic expressions) are described as monofunctional registers. Semiotic systems that can be used for various cognitive functions, such as awareness and communication which mathematical processes cannot be made into algorithms are called multifunctional registers.

The task is about solving an equation. When the student tries different answers, both positive and negative, she works in the symbolic register. This means she is doing a treatment since she is working in one register only. In this task the clues did not invite the students to make a conversion, as the clue and solution was only written in algebraic form. This means that the student has fewer opportunities to engage in learning processes supporting more challenging mathematical thinking. She does not look at any hints or proposed solutions – she doesn't lose any points.

The analysis indicates that digital tools may shift students’ attention from mathematical reasoning to the procedural demands of interacting with the system. This displacement reflects a semiotic reorientation in which the technological format of the response becomes the locus of activity, while the underlying mathematical structures remain underexplored.

These findings highlight a critical tension within ‘platformisation’ of education: although digital tools promise to expand the range of representations and provide immediate feedback, they also risk narrowing students’ engagement to surface-level interactions. When such challenges remain unnoticed by teachers, opportunities for fostering deeper reflection and conceptual understanding are further constrained.

References

Duval, R. (2017). Understanding the mathematical way of thinking – The registers of semiotic representations. Springer. https://doi.org/10.1007/978-3-319-56910-9

Nilsberth. M., Gustafsson, B., Wendell, J., Liljekvist, Y., Olin-Scheller, C., & Samuelsson, J. (forthcoming). Subject traditions and the role of digital platforms in upper secondary education. A comparative study.

Mathematical Self-Efficacy and Test Anxiety: Mediating Effects of Rehearsal, Metacognition and Effort on Achievement

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This presentation presents ongoing work exploring relationships between mathematical self-efficacy and test anxiety, and how these factors influence mathematical achievement via rehearsal strategies, metacognition and effort. According to Bandura's (1997) social cognitive theory, self-efficacy is associated with more efficient study strategies and effort, whereas mathematics anxiety (Dowker et al., 2016) may lead to avoidance behaviours that hinder mathematics development. Understanding how these factors might mediate through learning behaviours is of importance to understand how these could influence mathematical achievement.

Method

The sample consists of upper secondary students (n = 1005) studying their second mathematics course from social science and science tracks. Students completed a Likert-style questionnaire in their final month of the course prior to their national tests. Measures included; mathematics self-efficacy, test-anxiety, rehearsal strategies, metacognition and perceived effort. Mathematics achievement was measured by the final score on the national test. A partial least square structural equation modelling approach (Hair et al., 2021) was used to test a multiple mediation model.

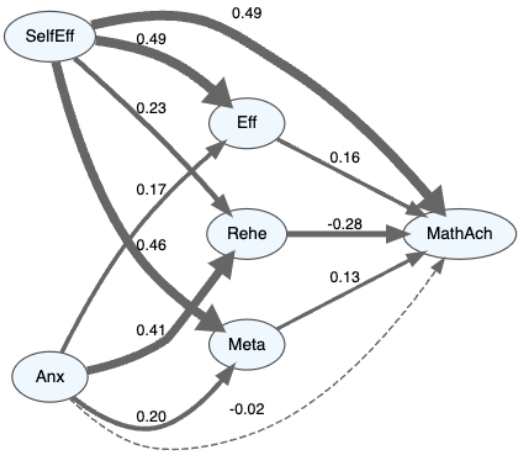
Results

The explored model explained 42% of variance in mathematics achievement and between 15-20 % of rehearsal, metacognition and effort variables. Self-efficacy positively predicts achievement both directly and indirectly via metacognition and effort suggesting a complementary partial mediation (VAF = 0.14). Rehearsal strategies mediated a negative pathway, suggesting a counterproductive relationship with achievement. In contrast, relating to test anxiety, the model suggests a mediation mainly via rehearsal strategy with a negative relationship towards achievement with no significant direct relationship (VAF = 0.838).

Table 1. Direct and indirect relationships between self-efficacy and test-anxiety on mathematical achievement.

Path	β	95% CI
SelfEff → Rehe → MathAch	-0.062	[-0.088, -0.039]
SelfEff → Meta → MathAch	0.061	[0.030, 0.092]
SelfEff → Eff → MathAch	0.080	[0.048, 0.113]
SelfEff (Total indirect)	0.078	[0.044, 0.112]
SelfEff (Direct)	0.488	[0.435, 0.539]
Anx → Rehe → MathAch	-0.113	[-0.144, -0.083]
Anx → Meta → MathAch	0.027	[0.012, 0.045]
Anx → Eff → MathAch	0.027	[0.012, 0.045]
Anx (Total indirect)	-0.058	[-0.088, -0.030]
Anx (Direct)	-0.023	[-0.080, 0.036]

Figure 1. Explored model connecting self-efficacy and test-anxiety via rehearsal, meta-cognition and effort on achievement.



These findings highlight the importance of fostering self-efficacy and metacognitive engagement and the counterproductive use of rote rehearsal strategies for mathematics learning. Further potential implication will be discussed during the presentation.

References

Bandura, A. (1997). *Self-efficacy: The exercise of control*. W. H. Freeman.

Dowker, A., Sarkar, A., & Looi, C. Y. (2016). Mathematics Anxiety: What Have We Learned in 60 Years? *Frontiers in Psychology*, *7*. https://doi.org/10.3389/fpsyg.2016.00508

Hair, J. F., Hult, G. T. M., Ringle, C. M., Sarstedt, M., Danks, N. P., & Ray, S. (2021). *Partial Least Squares Structural Equation Modeling (PLS-SEM) Using R: A Workbook*. Springer International Publishing. https://doi.org/10.1007/978-3-030-80519-7

Ray, S., Danks, N., & Calero Valdez, A. (2021). SEMinR: Domain-Specific Language for Building, Estimating, and Visualizing Structural Equation Models in R. *SSRN Electronic Journal*. https://doi.org/10.2139/ssrn.3900621

Elevgenererade ledtrådar i problemlösning

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Denna presentation handlar om hur elevers skapande och användande av ledtrådar kan utveckla deras problemlösningsförmåga inom matematik och fysik på gymnasienivå. Ledtrådarna kan ge eleverna stöttning i problemlösningsprocessen, utan att den kognitiva utmaningen försvinner, samtidigt som processen att ta fram ledtrådarna i sig kan utveckla problemlösningsförmågan.

Hagvall Svensson et al. (2025) studerade vilken stöttning elevgenererade ledtrådar ger i gymnasiematematik, och fann att de erbjöd både kognitivt och affektivt stöd, klassificerat utifrån van de Pol et als (2010) ramverk för beskrivning av stöttning. Innevarande studie kompletterar dessa resultat, med mer fokus på ledtrådarnas innehåll och effekten på lärande av att generera dem. Studien är genomförd i kursen Fysik 2, där problemen som studeras till stora delar är tillämpad matematik. Att studien genomfördes i en fysikkurs kan öka resultatens generaliserbarhet även inom matematik.

Studiens syfte är dels att skapa mer kunskap om hur arbetet med elevgenererade ledtrådar kan stärka lärande, dels att se om arbetssättet fungerar på samma sätt i fysikämnet som det gjorde i matematik. Tre forskningsfrågor formulerades: Vilken typ av ledtrådar genererar eleverna? Om och i så fall hur förändras de elevgenererade ledtrådarna över tid? Hur upplever eleverna aktiviteten med att skapa och använda ledtrådar i fysikämnet ur ett lärandeperspektiv?

Datainsamling och analys

Generering av ledtrådar sker inför att eleverna ska ha fysikprov. En vecka före provet får de ett övningsprov med kluriga uppgifter liknande dem som kommer på provet. Parvis får eleverna först lösa och sedan generera ledtrådar till var sin av de uppgifter som kräver utförliga svar. När eleverna läser till provet kan de välja att öppna ledtrådarna för att få hjälp att lösa uppgiften om de känner att de behöver stöttning.

Ledtrådar genererandes vid fem tillfällen under kursen Fysik 2 i tre klasser på en gymnasieskola. Vid de två första tillfällena sparades alla ledtrådar osorterat. Vid övriga tillfällen fick två elevpar först var för sig skapa ledtrådar till en uppgift, därefter diskutera dem och slutligen göra ett urval och bestämma i vilken ordning de skulle ges. Syftet med förändringen var att se om kvaliteten på ledtrådarna ändrades om de processades i två steg. Totalt samlades närmare 200 elevgenererade ledtrådar in för analys. Ledtrådarna klassificerades med avseende på om de var procedurella eller konceptuella (Rittle-Johnssen et al., 2015) samt vilken typ av stöttning de utgjorde.

I slutet av terminen genomfördes fyra gruppintervjuer i form av fokussamtal med sammanlagt 12 elever. I intervjun fick eleverna prata om sina upplevelser av att generera ledtrådar och av att använda sina kamraters ledtrådar, samt reflektera över vad de lärt sig av processen. Intervjuerna transkriberades och en tematisk innehållsanalys genomfördes (se Braun & Clarke, 2006).

Resultat

De typer av stöttning som förekom var: a) tips i form av ”tänka på att…” eller ”fundera på…”, b) förslag på metod eller strategi, c) frågor, d) faktapåståenden, e) enstaka ord, f) negationer i form av vad som inte ska göras eller vad något inte är. Vanligast var kategori a, b och d. Andelen metodförslag (kategori b) ökade stadigt under studiens gång från 34 % i början till 41 % i slutet. Att ge en ledtråd i form av en fråga (kategori c) var mycket vanligt i samband med prov 1 och 4 men var för övrigt sällsynt. Enstaka ord och negationer förekom sporadiskt.

I intervjuerna uppgav eleverna att processen med att generera ledtrådar hjälpt dem att identifiera uppgiftstyper och fokusera bärande idéer samt att de lärt sig dela upp uppgifter i mindre delar. Att de fått skapa ledtrådarna tillsammans i par var viktigt eftersom de då fick öva sig på att förklara, och att även en svagare elev på så sätt blev en resurs. De var positiva till att lägga tid på aktiviteten eftersom de menar att de lär sig genom att reflektera och genom att förklara.

Slutsats

Studien visar att idén med elevgenererade ledtrådar kan användas i fysikämnet med positivt resultat. Eleverna upplever att de lär sig mycket på att skapa ledtrådarna, vilket torde gälla även inom matematik. Upplevelserna av att använda ledtrådar överensstämmer med den tidigare studien i matematikämnet, vilket illustreras tydligt av följande citat:

> Ledtrådarna är ju skrivna av folk som liknar oss. Vi har fått samma utbildning inom det här området, lärt oss samma saker. Vi har liknande perspektiv på hur vi ska besvara de här frågorna, så att då ökar chansen ganska mycket att ledtrådarna träffar rätt hos just mig eftersom de är skrivna av folk som baserar det på samma förkunskaper.

Referenser

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101. https://doi.org/10.1191/1478088706qp063oa

Hagvall Svensson, O., Marander, S., & Larsson, D. (2025). “Det är riktat till mig, det borde gå att lösa”: Elevgenererade ledtrådar som stödstruktur vid självständig problemlösning. *Nordic Studies in Mathematics Education, 30* (3), 101-127. https://doi.org/10.7146/nomad.v30i3.160879

Rittle-Johnson, B., Schneider, M. & Star, J.R. (2015). Not a One-Way Street: Bidirectional Relations Between Procedural and Conceptual Knowledge of Mathematics. *Educational Psychology Review, 27*(4), 587–597. https://doi.org/10.1007/s10648-015-9302-x

van de Pol, J., Volman, M., & Beishuizen, J. (2010). Scaffolding in Teacher–Student Interaction: A Decade of Research. *Educational Psychology Review 22*(3), 271–296 https://doi.org/10.1007/s10648-010-9127-6

Teaching ordinality of numbers in preschool counting rhymes with representations

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In this presentation, the starting point is an ambition to design teaching activities for preschoolers to whom numbers are novel concepts. In the tradition of variation theory of learning (Marton, 2015), to learn a new concept such as numbers, the child first has to differentiate the aspects ordinality and cardinality of numbers, before these can be coordinated and constitute a powerful way of experiencing and making use of numbers. Fuson (1988) defines ordinality as ordinal number, sequence and order relations. Cardinality describes the number of elements in a set. How to teach ordinality has shown to be more complicated than teaching cardinality, because ordinality usually appears intertwined with other attributes of numbers (Askew & Venkat, 2020).

Children encounter numbers in daily situations and teaching practices, through representations in different modes. Using different representations when teaching is a way to support conceptual understanding and help children see connections between different ways of expressing a mathematical idea (Lesh, 1981). Representations are complex in themselves because they refer to something that is not present or directly experienced. Representations of numbers to preschoolers can be manipulative materials, pictures, and symbols including finger patterns (see Björklund & Palmér, 2022).

This study investigates specifically what challenges there are in teaching the idea of ordinality of numbers. The research questions are: 1) What is critical for differentiating ordinality of numbers as an object of learning? and 2) In what ways do representations support this differentiation? We will in the analysis reveal how differentiation is made possible in observed teaching situations, regarding ordinality as the object of learning.

Methods

Interventions designed as games and activities familiar to the preschool settings were developed and trialled in collaboration between preschool teachers and researchers. Data for analysis consists of 38 video recorded observations (230 minutes) of five teachers conducting teaching activities with 3–5-yearolds. They were using a counting rhyme of their own choice together with materials, gestures and spoken language to represent mathematical meaning. The analysis, informed by variation theory, reveals significant challenges in offering children opportunities to discern critical aspects of ordinality, which is related to both the nature of the object of learning and what representations are used to support the learning process.

Results

Activities that include the number sequence in the shape of counting rhymes were chosen as means to teach ordinality as a significant aspect of numbers. The counting

rhymes are enacted together with different materials, and consequently, different representations of ordinality are presented to the children. The observations of the enacted teaching shows that there seems to be some fundamental features that are necessary to discern before numerals (symbols) can be introduced: *units of a sequence, stable order of the sequence*, and *positions of units within the sequence*. These are brought fore in the teachers' use of different representations, such as identical figures, that are handled systematically to make units, order and positions discernible. The key point identified is to direct attention to units as separate but ordered (rather than as a collection), and that each unit can be experienced as situated within a stable sequence. To discern units of a sequence, is primarily non-numerical and attention is directed to this by pointing out units (objects) one at a time in an orderly fashion, preferably in a straight line. The latter can be taught through activities (games and play) where in particular number words are used to recite the sequence of words and explore positions and order relations in numbers by contrasting the positions of the numbers in the sequence. These build a foundational way of experiencing sequence, as a prerequisite for discerning ordinality in numbers.

Discussion

This study shows that when trying to bring for ordinality, the choice of representations may in fact hinder this objective and instead emphasize cardinality or trigger enumeration. By identifying fundamental ways of making ordinality discernible, we broaden the idea of what it means to learn about ordinality as an aspect of numbers (c.f. Fuson, 1988). This also implicates a need for further research and intervention studies to confirm or challenge the findings.

References

Askew, M., & Venkat, H. (2020). Deconstructing South African Grade 1 learners' awareness of number in terms of cardinality, ordinality and relational understandings. *ZDM Mathematics Education, 52*(4),793–804. https://doi.org/10.1007/s1185 8-020-01132-2

Björklund, C. & Palmér, H. (2022). Teaching toddlers the meaning of numbers - Connecting modes of mathematical representations in book reading. *Educational Studies in Mathematics, 110*(3), 525–544. https://doi.org/10.1007/s10649-022-10147-3

Fuson, K. C. (1988). Some further clarifications of numerical terminology using results from young children. *Behavioral and Brain Science, 11*(4), 583–585.

Lesh, R. (1981). Applied mathematical problem solving. *Educational Studies in Mathematics, 12*(2), 235–264. https://doi.org/10.1007/BF00305624

Marton, F. (2015). *Necessary condition of learning*. Routledge.

Samverkande undervisning i matematik

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Presentationen fokuserar på ett forskningsprojekt om metoden samverkande undervisning i matematik i ett flerspråkigt klassrum med elever i årskurs tre. Vi undersöker vad metoden kan bidra med för elevers möjlighet att samtala matematiskt. Samverkande undervisning är en matematikdidaktisk metod som innebär att elever löser textbaserade problemlösningsuppgifter tillsammans med matematiklärare, studiehandledare i modersmål och forskare i ett flerspråkigt klassrum. Resultatet visar att samverkande undervisning i matematik påverkar elevers matematiska engagemang, självförtroende och identitet genom att flera resurser samverkar i undervisningen.

Introduktion

Detta forskningsprojekt pågick under 2023/24 inom ramen för ULF (Utbildning, Lärande, Forskning) och syftar till att undersöka metoden samverkande undervisning och vad den kan bidra med för flerspråkiga elevers matematiska samtal. Samverkande undervisning är en matematikdidaktisk metod som innebär att elever löser textbaserade problemlösningsuppgifter tillsammans med matematiklärare, studiehandledare i modersmål och forskare i ett flerspråkigt klassrum. I undervisningen inkluderas flera resurser såsom gester och kroppspråk, talat språk, begrepp, ord och bilder, material och symboler (jfr. McIntosh, 2007, s. 144). de Freitas och Sinclair (2014) beskriver att matematiklärande handlar om att skapa erfarenheter genom kroppsligt engagemang, materialiteter, miljö och sammanhang. Vilka matematiska erfarenheter och kunskaper som kan skapas beror på vad som inkluderas, och hur. Detta teoretiska perspektiv, *inkluderande materialism* (de Freitas & Sinclair, 2014), har varit vägledande genom hela projektet. Matematisk identitet är inte något ”fast eller stabilt utan något som ständigt skiftar. Det går att likna vid ett slags görande” (Palmer, 2020, s. 58). Att inkludera språk, konkret material och bilder i matematikundervisningen, kan bidra till att elever kan bearbeta ett matematiskt innehåll samt koppla innehållet i matematikuppgifter till den egna vardagen (Norén & Caligari, 2020).

Resultat

Intervjuare:	Vad tänker du om att använda konkret material?
Elev 1:	Man kan räkna snabbare och sånt.
Intervjuare:	Vad säger du då? (vänder sig mot den andra eleven i intervjun)
Elev 2:	… också snabbare och sen blir det lite lättare om man använder sina händer… Om det är svårare då tar man kuber. Då är det lite lätt.

Det *eleverna* uttrycker handlar om att det konkreta materialet blir en möjlighet för dem att skapa erfarenheter och lösa matematiska problem. De beskriver också att i den samverkande undervisningen hjälps man åt att ”göra” matematik tillsammans med hjälp av material, men även genom att förklara ord och begrepp på olika språk och med hjälp av gester. Det bidrar till att de förstår orden i uppgifterna, vad de ska göra och hur de kan lösa problemen. *Matematiklärarna* beskriver att samverkande undervisning stöttar eleverna i att våga uttrycka sig om sådant de inte förstår, om ord och begrepp, ”… jag tycker att den [samverkande undervisning] fångar upp mycket av språket”. Det läraren lyfter fram har att göra med matematiskt självförtroende, att våga engagera sig och uttrycka att man inte kan. *Studiehandledarna* i sin tur beskriver att när modersmålet blir en resurs kan eleverna se betydelsen av att använda modersmålet i olika sammanhang. Detta är något som kan sammankopplas med matematisk identitet (jfr. Palmer, 2020). Enligt studiehandledarna kan denna undervisningsmetod stärka upplevelsen av modersmålet som en resurs för elevens matematikutveckling och inte enbart som ett språk som talas i hemmet. Under intervjun återger en av studiehandledarna att den samverkande undervisningen blir något ”mer” och att den möjliggör för inkludering av matematiska ord, begrepp och beskrivningar av kontexter på modersmålet. Även en av matematiklärarna beskriver betydelsen av att undervisa om kontexter. Eleverna behöver också undervisning om ord, begrepp och sammanhang som ingår i textbaserade problemlösningsuppgifter för att kunna lösa uppgifterna (jfr. Norén & Caligari, 2020). Kontexten kan handla om ”…typiska aktiviteter som man kanske gör när man växer upp som barn i Sverige, det är inte de här barnen [flerspråkiga elever] bekanta med… Jag vet redan i mitt bakhuvud att de här orden har inte dom, eller de flesta av dom”.

Diskussion

Elever uttrycker att de ”gör” matematik tillsammans. De kan ”bli” matematiska, vilket synliggörs genom den inkluderande materialismen (de Freitas & Sinclair 2014). Samverkande undervisning påverkar det matematiska engagemanget, elevens matematiska självförtroende och identitet (jfr. Palmer, 2020). Bidraget synliggör betydelsen av att *inte* enbart se språkliga aspekter och konkret material som isolerade enheter i undervisningen. Det är genom samverkan de kan bli reella resurser som möjliggör för elever att utveckla matematiskt engagemang, självförtroende och identitet.

Referenser

de Freitas, E., & Sinclair, N. (2014). *Mathematics and the Body: Material Entanglements in the Classroom*. Cambridge University Press.

McIntosh, A. (2007). *Förstå och använda tal: En handbok*. Göteborgs Universitet NCM.

Norén, E., & Caligari, L. (2020). Practices in multilingual mathematics classrooms: Word problems. In *MADIF 12, The twelfth research seminar of the Swedish Society for Research in Mathematics Education*, Växjö, Sweden, January 14–15, 2020 (Vol. 15, pp. 61-70). Swedish Society for Research in Mathematics Education.

Palmer, A. (2020). *Hur blir man matematisk? Att skapa nya relationer till matematik och genus i arbetet med yngre barn.* Liber.

Learning to teach the relationship between addition and subtraction in grade 3 – the use of visual representation

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In this presentation we will present and discuss how prospective teachers learn to teach the relationship between addition and subtraction in grade 3. Visual representations are powerful tools for supporting arithmetic learning, as diverse visualisations have a significant positive impact (Schoenherr et al., 2024). The aim of this study is to identify how prospective teachers use visual representations when planning a lesson in early arithmetic in a learning study. Teaching the relationship between addition and subtraction is challenging in the early years. Previous research shows that children typically begin with counting-based strategies, using objects or fingers (e.g Neuman, 2013). As understanding deepens, non-counting strategies and a part–whole view of number are developed. Understanding subtraction as the inverse of addition is crucial for early mathematical reasoning (Vilette, 2002). For example, knowing that 4+3=7 supports recognising that 7−3=4. Encouraging pupils to frame subtraction in terms of addition (What must be added to 4 to make 7?) supports deeper mathematical insight (Fuson, 1992). Visual representations enhance conceptual understanding, showing how addition and subtraction relate. As Björklund and Palmér (2022) note, "it is significant to pay attention not only to the fact that different representations are used in an act of teaching, but also to which representations are chosen and how they are presented to the learner" (p.529). From this follows that prospective teachers need to be responsive to different visual representations when it comes to relationship between addition and subtraction.

Method

The study is a part of a larger project conducted within a mathematics course for primary teacher education at two universities. In this paper two groups' planning and revising of lessons taught in grade 3 where analysed, one group from each university. The learning study involved an iterative process in which a lesson was collaboratively planned and revised in a cyclic process with the aim to enhance student learning of the targeted object of learning (e.g., Pang & Runesson Kämpe, 2019). The learning study was conducted during the seventh of eight semesters. The groups worked with the relationship between addition and subtraction: if $a + b = c$, then $c - a = b$ (the complement principle). Over two weeks, four group meetings focused on students' learning difficulties, the mathematics, and teaching. This study draws on over ten hours (600 minutes) audio

recordings from two meetings: planning Lesson 1 and revising L1/planning Lesson 2, written documentation of lesson plans and written analysis of L1 and L2. Analysis based on variation theory was employed, incorporating the notion of external visualization (Schoenherr et al., 2024), including graphs, charts, shapes, number lines, and bar-graph diagrams that illustrate structural relationships. References to representations were coded and analysed in depth.

Results and discussion

The results demonstrate how PTs progressively develop their ability to use visual representations, particularly bar charts and triads, to illustrate part-whole relationships when teaching addition and subtraction. Initially, these visual tools served as central aids for pupils' understanding. After L1, PTs revised their approach by enhancing the visualizations with additional arrows within the triads to clarify relationships among elements, emphasizing the connections between parts and the whole and moving beyond simple counting strategies (Neuman, 2013). In L2, they used arrows to indicate the direction of subtraction as an operation, but also to show how parts and the whole remain constant across both addition and subtraction. Further, while continuing to employ bar charts and triads in subsequent lessons, their approach became nuanced and detailed, reflecting their teaching experiences from L1. They interconnected the models with the corresponding algebraic expressions written alongside, deepening their understanding of how addition and subtraction relationships can be effectively communicated. This process of revision fostered structure in their lesson plan. While PTs improved their capacity to structure visual representations purposefully, they could not fully anticipate potential obstacles or misconceptions prior to implementation. This highlights the importance for teacher educators to guide PTs in designing comprehensible visual models, enhancing their readiness for classroom practice.

References

Björklund, C., Palmér, H. (2022). Teaching toddlers the meaning of numbers: connecting modes of mathematical representations in book reading. *Educational Studies in Mathematics*, 110, 525–544

Fuson, K. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 243–275). Macmillan.

Pang, M. F., & Runesson Kempe, U. (2019). The Learning study: recent trends and developments. *International Journal for Lesson and Learning Studies*, 8(3), 162–169.

Schoenherr, J., Strohmaier, A.R., Schukajlow, S., (2024). Learning with visualizations helps: A meta-analysis of visualization interventions in mathematics education, *Educational Research Review*, 45, https://doi.org/10.1016/j.edurev.2024.100639.

Vilette, B. (2002). Do young children grasp the inverse relationship between addition and subtraction? *Cognitive Development*. 17(3), 1365–1383.

Students in Grades 2 and 3 solving a small difference subtraction task with a number line

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In this presentation, we report on how Grade 2 and 3 students solve a subtraction task with a small difference between the subtrahend and minuend – first without a number line, and later with one. The data emerged from a collaboration project (ULF-project), conducted with teachers and researchers in preschool classes through Grade 3 during one school year. In this project, relations between numbers were emphasized, with particular attention to the number line, its structure, and use as a tool for solving arithmetic problems. Young students' ability to handle various arithmetic problems relies on flexibility, in terms of choosing an adequate strategy for solving a task based on its numerical properties. Also, students' use of strategies focused merely on manipulating digits irrespective of the task at hand has been questioned (Selter, 2001). In contrast with such strategies, recognizing the minuend and subtrahend as being close to each other may underpin strategies based on number relations (Torbeyns et al., 2009). In previous research, there are arguments for the number line functioning as a powerful representational tool that supports students' understanding of number relations. In particular, viewing the number line as a measurement, rather than a counting model, supports students when estimating the position of a given number in relation to numbers already on the line (Diezmann & Lowrie, 2006). However, how the number line may support students' use of number relations when solving subtraction tasks may be further discussed.

This presentation aims to add knowledge about how students in lower grades solve a subtraction task where the difference between the minuend and subtrahend is small, and discuss the role played by the number line as a tool in this task. The study is guided by the following research question: How do the Grade 2-3 students' ways of solving a small difference subtraction task change when a number line is provided?

Methods

22 students in Grades 2 and 3 were selected for interviews. Tasks were presented orally, and students explained their solutions verbally. The interviewer noted both explanations and gestures. The task reported here differs between the grades only by the inclusion of hundreds (Grade 3) and was presented in two steps:

1. *If you have (1)83 kronor and spend (1)77, how much is left?*

2. The interviewer then placed a number line in front of the student and asked: *Can you show it on the number line?*

We were interested in how students explained their solution without a number line and if their focus changed when one was shown. Separately, we analyzed strategies used without and with the number line shown. Then we compared them and identified two underlying approaches: *take-away* and *difference*. Finally, we analyzed each student's solutions based on these approaches and noted any changes.

Results

Seven students changed their way of solving the task from *take-away* to *difference.* For instance, Bea first solved the task by subtracting digit by digit: (100–100=0; 80–70=10; 7–3=4), and answered 14. When the number line was shown, Bea reconsidered, starting at 77 and counting to 83, finding the difference as 6. This shift demonstrates a move from *take-away* to *difference*. Cai first solved the task by consecutively removing hundreds, tens, and units. When shown the number line, Cai focused on 80 as a benchmark, saying: "3 up to 80 and 3 to 83, 3 plus 3 is 6." Like Bea, Cai first adopted a *take-away* approach. After the number line was shown, his strategy shifted towards finding the difference.

One student changed from a *difference* to a *take-away* approach. Mio, focused on the small gap between the numbers and used indirect addition: "177 plus 3 is 180, 180 plus 3 is 183, 3 plus 3 is 6." When the number line was shown, Mio changed to a *take-away* approach, counting 77 single steps backwards from 83.

In total, fourteen students did not change their way of solving. Five students maintained a *difference* approach. Nine students' solutions were based on a *take-away* approach when the task was presented without a number line, as well as with the number line. These results indicate that students' ways of solving subtraction tasks may change when given access to a physical number line, as it may allow them to focus on distance and reference points. But the results also show an example of a student who abandoned an initial *difference* approach and used the number line for single-step counting.

References

Diezmann, C, M & Lowrie, T., (2006) Primary students' knowledge of and errors on number lines. In Grootenboer, P. and Zevenbergen, R. and Chinnapan, M., Eds. *Proceedings 29th Annual Conference of the Mathematics Education Research Group of Australasia.* 1, (171–178). Canberra.

Selter, C. (2001). Addition and Subtraction of Three-digit Numbers: German Elementary Children's Success, Methods and Strategies. *Educational Studies in Mathematics 47*, 145–173. https://doi.org/10.1023/A:1014521221809

Torbeyns, J., De Smedt, B., Stassens, N., Ghesquière, P., & Verschaffel, L. (2009). Solving Subtraction Problems by Means of Indirect Addition. *Mathematical Thinking and Learning*, *11*(1–2), 79–91. https://doi.org/10.1080/10986060802583998

Mathematics Teachers' Adoption of Novel Instructional Practices: Factors Shaping Implementation in Professional Development

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Educational reforms frequently introduce new pedagogical approaches with the expectation that teachers will integrate them into practice. However, adoption is complex and shaped by teachers' beliefs, classroom contexts, and practical constraints. This presentation highlights how five themes that emerged from a thematic analysis affect implementation of new pedagogical approaches within the context of professional development (PD).

Method

Three teachers (2–25 years' experience) who had participated in a project involving physical and digital mathematics resources were interviewed. Semi-structured interviews were transcribed and analysed inductively using Braun and Clarke's (2006) six-step thematic analysis. Codes were generated from teachers' descriptions of their experiences, challenges, and reflections about new teaching methods.

Findings

The five themes found, and explored deeper in Sharif (2026), were as follows: (1) Professional autonomy. Flexibility to adapt innovations to personal teaching styles is essential; rigid mandates face resistance. (2) Student engagement. Materials that spark curiosity and collaboration motivate continued use, though effectiveness varies by student group. (3) Learning outcomes. Preference for methods that enhance conceptual understanding and confidence, especially for struggling learners. (4) Practical and logistical barriers. Barriers like noise, preparation time, unclear instructions, and monitoring challenges can outweigh benefits. (5) Student-material fit. Success hinges on matching difficulty and structure to diverse class needs; adaptability is critical.

Discussion

The themes should not be viewed independently; teachers evaluate innovations holistically. A method that fits students well and improves learning may outweigh logistical challenges, whereas a method that limits autonomy or fails to support learning is unlikely to persist. The findings align with theoretical frameworks such as Rogers' (2003) *Diffusion of Innovations*, Clarke and Hollingsworth's (2002) *Interconnected Model of Teacher Growth*, and Hall and Hord's (2020) *Concerns-Based Adoption Model*

(CBAM), all of which emphasize compatibility, perceived benefits, and manageable complexity.

Implications

A central contribution of the study lies in its implications for how (PD) can more effectively support sustained instructional change in mathematics.

The findings suggest that professional development initiatives should prioritize teacher autonomy by offering adaptable strategies rather than imposing rigid, prescriptive methods, thereby respecting teachers' professional judgment. Training programs need to equip educators with practical techniques for fostering student engagement, including leveraging the material to accommodate diverse learner profiles. Furthermore, PD should emphasize the pedagogical value of concrete tools in enhancing conceptual understanding and confidence, particularly for students who struggle with abstract representations, while guiding teachers to weigh minor drawbacks against demonstrable learning gains. Addressing practical barriers is equally critical; programs should provide classroom management strategies for mitigating noise and time-intensive preparation, alongside guidance for efficient implementation and monitoring of digital tools. Finally, PD should promote flexible design principles by offering teachers the option to adjust difficulty and structure to suit mixed-ability classrooms.

Conclusion

This short communication highlights broader implications from a qualitative study of mathematics teachers' reflections on adopting a new instructional method. Teachers' adoption decisions were shaped by a combination of pedagogical reasoning, practical constraints, and perceived alignment with student needs. These insights underscore the importance of PD that is responsive to both the structural and instructional realities of mathematics classrooms. A forthcoming full article will present the detailed thematic analysis and theoretical integration underlying these implications.

References

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, 3(2), 77–101. https://doi.org/10.1191/1478088706qp063oa

Clarke, D., & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education*, 18(8), 947–967. https://doi.org/10.1016/S0742-051X(02)00053-7

Hall, G. E., & Hord, S. M. (2020). *Implementing change: Patterns, principles, and potholes* (5th ed.). Pearson.

Rogers, E. M. (2003). *Diffusion of innovations* (5th ed.). Free Press.

Sharif, S. (2026). *Mathematics teachers' adoption of novel pedagogical methods* [Conference paper in preparation]

Mathematics as lived experience

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This presentation explores how mathematics, as lived experience, shapes the development of mathematics teacher identity for pre-service and novice teachers with engineering backgrounds. Within a hermeneutic-existential phenomenological perspective, lived experience captures how the past and the anticipated future are interpreted and gathered into the present, shaping how mathematics is encountered and understood in a person's life (Heidegger, 2013; Gadamer, 2015). For mathematics teachers with disciplinary backgrounds in engineering, this involves navigating a shift from mathematics as a stable and objective discipline (Niss, 1996) to a form shaped by students' encounters with the subject, marked by uncertainty and variation (Skovsmose, 2005).

Prior research

Research on mathematics teacher identity highlights tensions between disciplinary knowledge and pedagogical demands (Mitiche, 2025). Boaler (2002) emphasizes that teaching mathematics involves relational pedagogy, focusing not just on abstract procedures but also on students' real-world connections with the subject.

Method

Data were collected through semi-structured interviews with Sarah, a pre-service teacher, and Peter, a novice teacher, both with engineering backgrounds. Mathematics is approached in this study as a phenomenon with existential meaning, influencing how individuals come to see themselves as mathematics teachers. The analysis, framed by a hermeneutic-existential phenomenological perspective (Heidegger, 2013; Gadamer, 2015), draws on data from two interviews originally collected in a previous study (Mitiche, 2024). Gadamer's concept of fusion of horizons guides the analysis, exploring how the disciplinary and educational perspectives of mathematics intersect. This fusion occurs as the participants' horizons, shaped by their individual experiences and disciplinary backgrounds, merge with the perspectives of the educational situation, within the hermeneutic understanding that meaning is always embedded in lived experience, temporal, and shaped by evolving horizons of interpretation.

Findings

The participants' lived experiences of mathematics, shaped by their disciplinary backgrounds, their encounters with students, and the emotional tensions that arose in these situations, continuously informed how mathematics appeared to them as teachers.

Mathematics is stable in disciplinary contexts but uncertain in schools
For both participants, mathematics emerged stable and coherent within their engineering backgrounds, but in teaching it appeared differently. Meeting students with varied understandings and emotions made the subject feel uncertain, revealing how mathematics no longer appeared as it once had.

Continual redefinition of mathematics in teaching
Teaching requires continually redefining the content and purpose of mathematics. Sarah worked to make the subject relevant to students' lives, while Peter adjusted its presentation to students' different ways of understanding.

Mathematics as relational meaning-making
Mathematics appeared through relational encounters. Sarah described how building trust helped ease students' negative attitudes, while Peter saw interest grow when a disengaged student felt personally recognized.

Discussion, conclusion and implications

Disciplinary knowledge is re-anchored in encounters with students, revealing mathematics as a relational, meaning-making practice (Boaler, 2002). For pre-service teachers and mathematics teachers with deep disciplinary mathematics expertise, the study underscores the need to reflect on lived experiences and how these shape classroom meaning-making. Teacher education and schools should support them in adapting disciplinary knowledge to pedagogical demands.

References

Boaler, J. (2002). *Experiencing school mathematics: Traditional and reform approaches to teaching and their impact on student learning* (2nd ed.). Lawrence Erlbaum.

Gadamer, H.-G. (2015). *Truth and method*. Bloomsbury Academic.

Heidegger, M. (2013). *Being and time*. Wiley-Blackwell.

Mitiche, A. (2024). Redefining professional and academic identities: Turning points in pre-service and novice mathematics teachers' development of lived experience in and after short-track teacher education. *Cogent Education, 11*(1). https://doi.org/10.1080/2331186X.2024.2417626

Mitiche, A. (2025). Pre-service mathematics teachers' professional identities: A systematic literature review. *Review of Education, 13*(2), e70078. doi.org/10.1002/rev3.70078

Niss, M. (1996). Goals of mathematics teaching. In A. Bishop et al. (Eds.), *International handbook of mathematics education* (pp. 11–47). Springer.

Commognitive analysis of mathematics textbooks

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This workshop aims at advancing the analysis of mathematics language and discourse in textbooks, with a basis in commognitive analysis. We have developed a framework for analysing mathematics tasks, the mathematical objects and routines or activities. In particular, we want to understand how explanations and tasks together constitute a narrative about mathematical objects in the textbooks. In the workshop, we strive to advance commognitive analysis of textbooks, but also the understanding of how textbook analysis can contribute to teacher education and teacher professional development. The workshop will include textbook excerpts from different educational levels, from the first school years to university, and will include excerpts in both Swedish and English.

Introduction

Given the central role of mathematics textbooks in shaping communication and discourse in mathematics classrooms, we propose an analytic framework based on commognition for researching mathematics discourse in textbooks.

Commognitive tools are now used to investigate mathematics teaching and learning at all levels, up to and including university (Sfard, 2008; Thoma & Nardi, 2016). So far, however, the theory has mainly been used to study interaction happening in classrooms and lecture halls. There are examples of it being used for analysing assessment tasks (Thoma & Nardi, 2016), but analyses of teaching materials, such as textbooks, are scarce. One exception is the work by Park (2016) on US university-level calculus textbooks. We believe that textbook analysis is an area of research where commognition has the potential to offer new insights.

According to a recent review study (Rezat, 2024), current research on mathematics textbooks primarily focuses on five areas: content analysis, user studies, design studies, impact studies, and educational material as data. Hence, few studies have explored mathematics discourse in textbooks. The main aim of this workshop is to explore the potential of a framework based on commognition, for researching the language and discourse in mathematics textbooks. A further aim is to contribute to our understanding of commognitive concepts, and of the mathematics discourses in textbooks for different levels.

Theory and framework

Commognition is a theory looking at mathematics learning from a discursive perspective, that is, it views mathematics and mathematical objects as discursively constituted and learning mathematics as increasing participation in mathematical discourse (Sfard, 2008). Mathematics as a discourse is characterised by its use of *words* and *visual mediators* (e.g., symbols, graphs, etc), the *narratives* concerning the objects

of the discourse deemed endorsable within the discourse, and the repetitive patterns, *routines*, engaged in by the discursants (ibid., p. 133f). More recent work (Lavie, Steiner & Sfard, 2019) has seen further development of the notion of routine, conceptualizing a routine as a task-procedure pair, with a task, as understood by the performer, together with the procedure to solve the task.

We will explore a framework based on a current characterization of mathematics discourse, where Macchioni and Bachlieli-Frank (2025) synergistically combined concepts from recent developments in commognition. Our textbook analysis focuses on the mathematics discourse, and therefore includes the following analytic aspects:

- the task, the task situation and the procedures to solve the task
- the realisations (symbols, graphs, written text) and connections between them

The analysis will enable us to describe the narratives about mathematical objects, and compare tasks for learners to the explanations and worked examples.

Workshop structure

1. Short interactive introduction: jointly operationalising the analytic framework on a brief textbook example. (5 minutes)
2. Group work: groups engage in an analysis of a textbook excerpt, one per group, and one school-level per group. We will also bring English language textbooks, for non-Swedish speaking participants. (20 minutes)
3. The groups present and discuss findings and challenges (15 minutes)

References

Lavie, I., Steiner, A., & Sfard, A. (2019). Routines we live by: From ritual to exploration. *Educational Studies in Mathematics*, *101*, 153-176. https://doi.org/10.1007/s10649-018-9817-4

Macchioni, E. & Baccaglini-Frank, A. (2025). When the whole is greater than the sum of its parts: towards a characterization of students' mathematics discourse participation profiles. *Educational Studies in Mathematics, 120,* 1-31. https://doi.org/10.1007/s10649-025-10410-3

Park, J. (2016). Communicational approach to study textbook discourse on the derivative. *Educational Studies in Mathematics, 91*, 395–421. https://doi.org/10.1007/s10649-015-9655-6

Rezat, S. (2024). Research on curriculum resources in mathematics education: a survey of the field. *ZDM Mathematics Education, 56*, 223–237. https://doi.org/10.1007/s11858-024-01559-x

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge University Press.

Thoma, A., & Nardi, E. (2016). A commognitive analysis of closed-book examination tasks and lecturers' perspectives. *INDRUM 2016 Proceedings*, 411-420. University of Montpellier & INDRUM.