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Interplay between research and teaching practice in mathematics education

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Preface

The Tenth Nordic Conference on Research in Mathematics Education (NORMA24) took place in Denmark from the 4th to the 7th of June 2024 at Aarhus University, Campus Emdrup in Copenhagen. It was hosted by the National Centre for Developing Mathematics Teaching (NCUM), the Danish School of Education, Aarhus University, and University College Absalon. NORMA24 is the first NORMA conference after the pandemic in which researchers in the Nordic countries could come together and talk with each other in person. It seemed that such opportunities were much missed as 166 researchers, teachers and others from 12 different countries participated in this NORMA conference. The atmosphere was nice, with engaged, focused and friendly communication among the participants during both the presentations and the social gatherings throughout the three and a half days of the conference. In terms of networking, friendly atmosphere, technical issues, wine and food, the conference seems to have been a success. The conference is documented at https://matematikdidaktik.dk/aktuelt/norma-24.

Even though not as large as NORMA20 – which was exceptionally large – NORMA24 can compare with the previous NORMA conferences. During the conference, 54 regular papers, 25 short communications, 13 poster presentations, four keynotes, one panel debate, and three working groups were presented. After the conference, many of these contributions were revised and resubmitted, using both the feedback from peer reviews and the feedback received during the conference. After a second review process, the accepted papers make up the proceedings presented in this volume. It consists of 49 regular papers, two keynotes, one panel debate, and three working groups. This is an impressive high number of various contributions.

The theme of NORMA24 was *Interplay between research and teaching practice in mathematics education*. The theme represents a long-standing and still current challenge in mathematics education as a research field. It is a dual challenge, which concerns both the question of how research can address and contribute to overcoming issues that are central and pressing for the development of teaching practices and the question of how development of mathematics teaching can be based on research findings. The theme reflects a strong tradition in mathematics education research in the Nordic countries, but it is revitalized with the new technological innovations omnipresent in mathematics education at all levels in our countries such as programming, and there is no firm research ground to support its use in teaching yet. During the conference, the conference theme was addressed in insightful and various ways, which we now briefly outline.

All plenary sessions addressed the theme albeit in different ways. The first plenary *What is the ‘work’ of mathematics teaching, and how can research contribute to it?* by Deborah Loewenberg Ball, University of Michigan, USA, emphasized teachers’ crucial role in the practice of mathematics teaching. Deborah pointed to the fact that important aspects and conditions for what teachers do and can do to help students to flourish mathematically are left unnoticed by research. Important aspects of teaching are not theorized in ways that help the development of practice. Ball illustrated her points with several examples of the important decisions teachers must take when teaching and discussed what knowledge base teachers need to be able to draw on to succeed. Unfortunately, it was not possible for Ball to contribute to the NORMA24 proceedings. However, as is the case for all the plenary sessions, slides and video from the session are available at the NCUM website: https://matematikdidaktik.dk/aktuelt/norma-24.

In addition, there is a webinar (video lecture) by Ball, Supporting Learners to Flourish Mathematically, which is closely related to the plenary, also available on NCUM's website: https://matematikdidaktik.dk/webinarer.

The second plenary *Teacher education and professional development as practicing core teaching practices* by Janne Fauskanger, University of Stavanger, Norway, addressed the related challenge of how to organize and practice mathematics teacher education and professional development in ways that places the (student) teachers (coming) practice in focus for teacher learning. Fauskanger shared findings from projects on practice-based teacher education, highlighting examples of pedagogies that put teaching practice at the core and that have the potential to foster teacher learning.

The third joint plenary *Math mastery in the digital age: The confluence of digital tools and mathematical competencies* by Uffe Thomas Jankvist and Cecilie Carlsen Bach, DPU, University of Aarhus, Denmark, addressed a comprehensive Danish research project on the confluence of digital tools and mathematical competencies. The project investigates the potentials and pitfalls of digital technologies in relation to the mathematical competencies in the Danish KOM-project in lower and upper secondary mathematics teaching. The project connects theoretical constructs developed in relation to the use of digital tools in mathematics teaching to the KOM framework.

The fourth plenary, *Researching the implementation of programming in school mathematics,* by Cecilia Kilhamn, Göteborgs Universitet NCM, addressed an example of interplay between curriculum development and the implementation hereof in mathematics teaching. Programming was introduced in the Swedish mathematics curriculum in 2018, and the plenary reported on a four-year research project studying this implementation of programming in the mathematics curriculum. The potentials and challenges of using programming as a tool for learning mathematics were discussed. Kilhamn illustrated how research may inform curriculum development and support teachers' professional development. Issues, which still need to be dealt with, such as helping teachers to integrate programming in their mathematics teaching, were pinpointed. The project is and will be reported in journal articles. Kilhamn's plenary is not reported in the proceedings but slides and video from the session at conference can be found at the NCUM website: https://matematikdidaktik.dk/aktuelt/norma-24.

As a special event intended to illuminate the theme of the conference, the IPC decided to invite a panel of Ph.D.-students and early career researchers to present and discuss their view of and experiences with the relationship between the development of teaching practices and research in mathematics education. The panel consisted of five early career researchers from four different countries: Denmark, Iceland, Norway and Sweden. The panel debate was organised and moderated by Tomas Højgaard, DPU, Aarhus University. The presentation from the panellists and the subsequent discussions were structured by these questions: How is this interplay in your own research and in mathematics education in general – and how should it be? Reflections and thoughts on these questions were structured around three perspectives on the quality of research: Relevance and importance: Why should one care? Trustworthiness: Why should one believe what the author says? Generality: What situations or contexts does the research really apply to?

The papers from the above-mentioned plenary sessions are included in the proceedings in the same order as they were presented at the NORMA24 conference.

The 49 regular papers address and investigate in various ways the interplay between research and practice at different educational levels from preschool (kindergarten) to mathematics teacher education or other tertiary level mathematics education. We invited a variety of different types of research projects for the conference ranging from smaller or larger projects that seek to develop teaching practice based on methods and theories from research to research that uses and develops theories or methods based on teaching practice – for example, design-based research approaches.

The regular papers are presented in the proceedings in alphabetic order according to the surname of the first author of the paper.

Three working groups took place at NORMA24. The reports from these working groups are presented in the last section of the proceedings. Two of the groups address issues related to the integration of digital technology in mathematics teaching. The first group addressed *The future of research in programming in mathematics education.* This theme is closely related to the plenary by Cecilia Kilhamn. Several Nordic countries have recently included programming in their mathematics curricula. Therefore, this group initiated a discussion regarding how mathematics education research should investigate the role and function of programming in mathematics teaching. In particular, the group focused on which theoretical frameworks can be used for data analysis and what research questions and designs can advance our understanding. The report from this working group is elaborated into to a peer-reviewed paper following the same process as for the regular papers.

The second working group focused on *Comparative analysis of computational thinking in mathematics education: Invitation to a Nordic discussion* was concerned with relationships between computational thinking and mathematics education. The current state of affairs in Denmark, Sweden, Norway, Finland, Iceland, and Estonia were presented and discussed. A praxeological framework to scaffold the comparison between countries was presented with the comparison between Denmark and Sweden as the case. The group pinpointed issues which needed to be further researched for supporting successful integration of computational thinking in mathematics curricula across diverse educational contexts.

The third working group *What is equal participation in mathematics education research?* addressed an important aspect of the interplay between research and mathematics teaching; namely the collaboration between researchers and practitioners in research and developmental projects. The group discussed the ideal of equal participation based of experiences from two studies, the ethical aspects of such collaboration and the conditions necessary for equal participation. Since it is important for research in mathematics education to be able to establish close collaboration between researcher and teachers or teacher students, such discussions are of importance for our field and for the theme of the conference. The discussion may give raise to consideration on how to develop formats at the NORMA conferences where teachers taking part in research or development projects could be given a voice about the collaboration with researchers.

The IPC warmly thanks all the participants, authors, and reviewers who contributed to the conference and especially to the conference proceedings.

Copenhagen, December 2024

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Table of contents

Working groups

"I know multiplication, but what do I do?"

Teacher education and professional development by practicing core mathematics teaching practices

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The keynote presentation from NORMA 24 entitled: "I know multiplication, but what do I do? Teacher education and professional development by practicing core teaching practices" builds the basis for this paper which focuses on the shift in mathematics teacher education and professional development towards making teaching practice a central source for teacher learning. The paper gives a glimpse into the history of practice-based teacher education, followed by digging into the concept of core teaching practices and examples where practicing teachers are provided with opportunities to learn core practices in a practice-based approach to professional development, the Mastering Ambitious Mathematics teaching project. The paper concludes by presenting questions that scholars are currently grappling with and future directions for research on practice-based approaches to professional development.

Keywords: Professional development, mathematics teachers, core teaching practices.

Introduction

This paper builds on the keynote presentation entitled "I know multiplication, but what do I do? Teacher education and professional development by practicing core teaching practices." I will return to the first part of this title, but the last part relates to the shift witnessed in the field of mathematics teacher education and professional development – a shift towards making teaching practice a central source for teacher learning (e.g., Grossman & Fraefel, 2024a). Critiques that novice teachers are not well-enough prepared for the challenges of the mathematics classroom, has led to a worldwide redesign of teacher education programs where teaching practice is prioritised (e.g., Ball & Cohen, 1999; Grossman & Fraefel, 2024b; Korthagen & Kessels, 1999; Zeichner, 2012). Such redesigns have included both changing and lengthening field experiences in schools and bringing more opportunities to engage in teaching practice into universities (Grossman & Fraefel, 2024b). In the Nordic countries, teacher educators have been experimenting with how to reorganize their programs around a set of core practices (e.g., Hovtun et al., 2021; Klette et al., 2024; Wæge & Fauskanger, 2021, 2023; Wæge et al., 2024), or around things mathematics teachers do, that are central to the work of teaching mathematics (Grossman & Fraefel, 2024b) – for example eliciting student strategies for finding the number of dots in a quick image and representing the strategies on the board as a mathematics teacher has done in Figure 1.

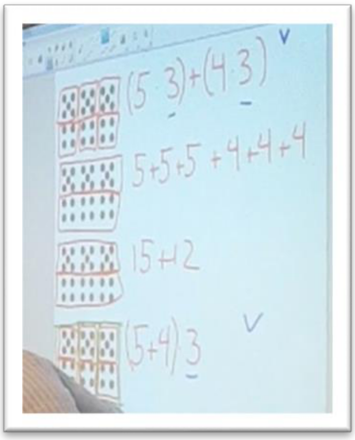

Figure 1: A teachers' representation of student strategies for finding the number of dots in a quick image

This turn towards practice-based teacher education is better documented in the United States (e.g., Ball & Forzani, 2009; Grossman et al., 2009; McDonald et al., 2013; Zeichner, 2012) than in the Nordic countries. With the recently published book edited by Grossman and Fraefel (2024a) – "Core practices in teacher education. A global perspective" – as one exception, what has been less well documented are the global efforts to develop more practice-based teacher education programs, in particular programs for practicing teachers (Wæge et al., 2024). Following this, my examples of documentation will in this paper be from the Norwegian context and from professional development for practicing teachers. The work is, however, inspired by the TeachingWorks project at the University of Michigan[1], the Teacher Education by Design project (TEDD) at the University of Washington[2] and the work done to support excellence in teaching at Stanford University[3].

The first part of the title "I know multiplication, but what do I do?" is more personal. 15 years ago, I was supervising novice teachers their first year in school. In one of these supervision meetings, a third-grade teacher started to cry saying: "I know multiplication, but what do I do?" (Fauskanger et al., 2010). As she was educated at the University of Stavanger – my university – and her grades were high, I realized that the way we prepared teachers did not always help them *do* the work of teaching mathematics. Following this, our pre-service teacher education (Hovtun et al., 2021) and professional development for practicing teachers (e.g., Wæge & Fauskanger, 2021, 2023: Wæge et al., 2024) have slowly turned towards focusing on core mathematics teaching practices. This is also supported by our national guidelines for teacher education stating that teacher education should qualify teachers to *do* the challenging and complex work of teaching (NRLU, 2024).

Before continuing, I would like to invite you as a reader into a situation from a mathematics classroom – to pause and *do* some work of teaching mathematics. The instructional activity in this class was a string starting with the task 100 : 4. The context in the string was a relay race. The total run was one

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[1] https://www.teachingworks.org/

[2] https://tedd.org/

[3] https://cset.stanford.edu/

hundred metres, and four students were going to run equally long. One of the students in the class said that “they have to run 25 metres each” because “four times 25 equals one hundred”. If you were the teacher in this classroom, how would you represent this student’s strategy on the board?

The teacher in this classroom represents the student’s idea this way: First, she divides a line segment into two equal parts. Then then she divides each of these two parts into two equal parts. Since the context was professional development for practicing teachers, they discussed the difference between representing (100 : 2) : 2 and 4 × 25. After a short discussion the group agreed that the teacher should represent 4 × 25 as four jumps of the length 25 and in addition write 25 above each arch as illustrated in Figure 2. You probably had similar ideas?

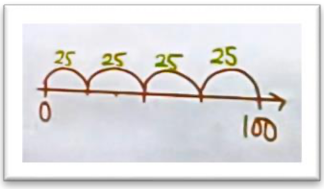

Figure 2: “They have to run 25 metres each” because “four times 25 equals one hundred” represented on the board

This is an example of one core practice of the work of teaching mathematics, namely representing students’ strategies or their ideas on the bord. The situation is also an example of the necessity to practice representing students’ strategies also for practicing mathematics teachers. Later in the paper, I will return to this example.

With this as a background, this paper will continue by providing a short glimpse into the history of practice-based teacher education. This will be followed by diving into the concept of core practices and a presentation of findings from studies of practice-based professional development for practicing teachers. I will conclude by identifying issues and challenges for future research and for future professional development.

So, what can be learned from the history of practice-based teacher education?

A glimpse into the history of practice-based teacher education

Teaching used to be viewed as a relatively simple practice and according to Forzani (2014), many Americans in the 19th century held the belief that teaching did not require much in the way of special preparation or knowledge. Teacher education became more academic in the first half of the 20th century. In addition, this part of the century was the starting point for a focus on teaching practices. One example is “The commonwealth study” (Charters & Waples, 1929) which resulted in a list with more than 1000 practices important for teachers. In more recent work, teaching is conceived of as a combination of technique, analysis, interpretation, and judgment: The teacher uses skills to elicit and interpret student thinking and decides based on that interpretation what to do next (Forzani, 2014) – for example how to represent the students’ strategies on the board (Figures 1 and 2). This is very different work from what has been common throughout much of the history of teaching. The focus on student thinking is at the core and researchers aim at identifying the teaching practices most important for students' learning. Practice is important, but 1000 practices as in "The commonwealth study" might be too many. In addition to the number of practices, the view of the teacher also differs. The first half of the 20th century, the view was a teacher practicing discrete skills, often divorced from judgement, whereas today, the teacher is seen as being engaged in complex practices that require complex thought, judgement and continual reflection. However, the concept of practice is of importance, but what is practice?

Lampert (2010) attempted to "provoke" clarity (p. 21) by investigating four conceptions of practice. Conceptions of practice are important, because when we talk about practice, we might not talk about the same thing. One conception of practice relates to the commonly made distinction between theory and practice, the distinction that implementing an idea is different from having the idea. A second conception is that practice is something that is done repeatedly to improve performance in it, in the sense of how one might practice playing the piano. Third, the practice of teaching includes people who have adopted "the identity of a teacher" and who have taken on "the common values, language, and tools of teaching" (p. 29). A fourth meaning of practice relates to routines that are done "constantly and habitually" (p. 25) in the classroom. An example of such a practice is representing student strategies on the board (Figures 1 and 2). These practices done constantly and habitually are often described as core practices and they are often understood as identifiable components fundamental to teaching (Grossman & Fraefel, 2024ab).

Learning core teaching practices

According to Grossman and Fraefel (2024b, p. 139), core practices are "based in research on teaching and learning, and on interaction." The research area on core practices aims at providing a common language to describe the work of teaching. Core practices are central to the daily work of teaching and to supporting student learning, and they are fundamental to developing other, more complex practices. The goal for a core practices approach to teacher education is development of adaptive expertise (Grossman & Fraefel, 2024b). Examples of core practices across subject areas are, eliciting student thinking, providing instructional explanations, and facilitating productive classroom discourse. For mathematics, launching a problem to provoke mathematical thinking and representing students' strategies (Figure 2) are examples of core practices (e.g., Lampert et al., 2013). But it is easier to talk and write about core practices than to *do* them. This will be illustrated with an example where students work on the following problem (Empson & Jacobs, 2008, p. 258): "Janelle has 21 beads. She wants to make three braids in her hair and put the same number of beads in each braid. How many beads can go in each braid?" In the conversation between one student and the teacher the student starts by saying six, but soon says that she must think a bit more. After a few seconds, thinking while looking at her fingers she says: "There would be seven." The teacher replied: "Tell me how you got seven." And now you as a reader must concentrate because the student continues: "Well, because I knew that seven [putting up seven fingers] threes were 21, and then I took two from each [pairing up her fingers], and that would make six, and then there'd be three others, and then you'd put one from that three into each one. And the two would be six, so plus one more would be seven." And then she smiles. It is challenging to elicit this student's thinking, but Table 1 summarises how Empson and Jacobs (2008) have described the student's mathematical understanding.

Table 1: A student's mathematical understanding (slightly adapted from Empson & Jacobs, 2008, p. 263)

The student's description	Possible equation	Fundamental principle of arithmetic
"I knew that seven threes were 21"	3 + 3 + 3 + 3 + 3 + 3 + 3 = 21	Known fact
"I took two from each"	3 + 3 + 3 + 3 + 3 + 3 + 3 = (3 + 3) + (3 + 3) + (3 + 3) + 3	Associative property of addition
"That would make six, and then there'd be three others"	(3 + 3) + (3 + 3) + (3 + 3) + 3 = 6 + 6 + 6 + 3	Addition
"You'd put one from that three into each one"	6 + 6 + 6 + 3 = 6 + 6 + 6 + 1 + 1 + 1 6 + 6 + 6 + 1 + 1 + 1 = (6 + 1) + (6 + 1) + (6 + 1)	Decomposition and commutative and associative properties of addition
"The two would be six so plus one more would be seven"	(6 + 1) + (6 + 1) + (6 + 1) = 7 + 7 + 7	Addition

When the student says that "I knew that seven threes were 21" (Table 1, first row), this seems to be a known fact, and the associative property of addition is used to combine "two from each" (second row). Addition is used to add them together and "make six" (third row) and decomposition and both the commutative and associative properties of addition are used when the student said: "put one from that three into each one" (Row 4). At the end the student adds six and one to get seven.

This example shows that it is challenging to elicit students' thinking. To use the words of Ma (1999), as a teacher you need profound understanding of fundamental mathematics, but you also need to practice the challenging practice of eliciting student's understanding and of how to respond. This can be done in practice-based teacher education where instruction should be aimed at ambitious learning goals that are grounded in the expectation that all students will develop high-level thinking, reasoning, and problem-solving skills (e.g., Lampert et al., 2013). There are lots of challenges to efforts aiming at such ambitious goals. As an example, in Norway – almost 19 % of the teachers are not educated as teachers (Utdanningsforbundet, n.d.), and sadly, this number is increasing. So, there is a need for a practice-based approach to professional development and for that we need practice-based pedagogies.

Grossman and colleagues (2009, 2018) argue that to learn core teaching practices, both seeing, breaking apart and trying out is important. They identified a framework for teaching practice that includes representations, decompositions, and approximations of practice. *Representations of practice* focus on how practices of teaching are made visible, such as videos of teaching, classroom transcripts, lesson plans and student artifacts. *Decomposition of practice* refers to partitioning the complex work of teaching and its teaching practices into identifiable parts for teachers to learn. *Approximations of practice* refer to creating safe environments for teachers to practice teaching. Approximations include role plays, rehearsals, and co-teaching. Based on these practice-based pedagogies, McDonald and colleagues (2013) developed learning cycles of enactment and investigation (Figure 3) and in our research, we build on this work.

The Mastering Ambitious Mathematics teaching project

Moving on to examples from our own research, the examples will be grounded in the "Mastering Ambitious Mathematics teaching project" (MAM). This professional development and research project takes a core-practices approach to supporting teachers' development of core mathematics teaching practices. It is situated at the Norwegian Centre for Mathematics Education[4] at the Norwegian University of Science and Technology in Trondheim and is led by the leader of the centre Kjersti Wæge. My colleagues Reidar Mosvold and Raymond Bjuland and I have collaborated closely with Kjersti Wæge and her colleagues when exploring the data material from the project. MAM involved 30 mathematics teachers from 10 Norwegian elementary schools who participated in a two-year project that was organised around learning cycles of enactment and investigation (learning cycles). The teachers were divided into four groups. Each group was supervised by a teacher educator. The learning cycles involved six settings as shown in Figure 3 and each cycle was videotaped and observed by a researcher.

Figure 3: Learning cycle of enactment and investigation (Wæge & Fauskanger, 2023, p. 508)

A given instructional activity was used across all the settings. The string starting by the task 100 : 4 presented earlier in this paper is one example of an instructional activity used across all six settings. In the first part of the cycle, preparation, the teachers prepared by reading articles about experiences from enactment of the cycle's activity and by watching videos of the enactment. Some of the teachers tried out the activity in their own class. This preparation, thus, includes *representations* of practice and in addition to videos, the articles often included examples of student artifacts. This preparation was followed by a collective analysis where the teachers discussed what they had prepared. *Decomposing* practices were included in this part of the cycle as well as in the next part, because after this discussion the teachers co-planned to teach and in co-planning discussions, practices were often decomposed into distinct parts. After co-planning, the teachers rehearsed. In the rehearsal, one or two teachers acted as lead teacher whereas the others played the role as students. The rehearsal is thus an example of an *approximation* of practice. This is also true for the classroom co-enactment where the

[4] For information about the center, see https://www.matematikksenteret.no/.

instructional activity was co-enacted with students (aged 11–12 years) and where teachers discussed along the way. The learning cycle ended by a collective analysis of the whole cycle and an introduction to the next cycle's instructional activity.

Different settings of the learning cycle have been explored aiming at understanding how the teachers' engagement in the different parts of the cycle provides opportunities for them to learn the principles and core practices of ambitious mathematics teaching. Among the principles are treating students as sense-makers and providing them with equitable access to learning (e.g., Lampert et al., 2013). During the two years of MAM, we videotaped 12 sessions (nine full learning cycles), and we conducted focus group interviews with teachers at the start and in the end of the project (see Wæge & Svingen, 2023). When analysing the data, we have used several frameworks to understand what teachers might learn from participating in this practice-based approach to professional development. I will present a few examples from our findings, starting with teachers' co-planning.

Co-planning

Teachers' planning is found to be an important part of the work of teaching mathematics but is still an "under-appreciated phase of teaching" (Superfine, 2022, p. 263). Within MAM, we have explored co-planning sessions with different foci such as what teachers plan to notice (Bjuland & Fauskanger, 2023; Fauskanger & Bjuland, 2021, 2022), their anticipating practices (Wæge & Mosvold, in press) and the questions they plan to ask, including the purposes for asking particular questions (Mosvold & Wæge, 2022; Wæge & Mosvold, in process). I will present some findings from my colleagues' explorations on questions and questioning practices. Their explorations of co-planning sessions illustrate the complexity of questions and questioning practices, where teachers considered multiple rationales. Their findings indicate that the teachers most frequently discussed questioning with an emphasis of orienting students towards the learning goals of the lesson parallel to stimulating discussions, but their discussions also focused on questioning aiming at eliciting the details of students' mathematical thinking and engaging students in explanations and arguments. Such considerations on the underlying rationales of questions and questioning practices were entangled in discussions on what type of questions they should ask, as well as the order and timing of questions.

According to Wæge and Mosvold (in process) almost a fifth of the segments addressing questions or questioning practices in the co-planning sessions addressed the purpose of eliciting the details of students' mathematical thinking. I will illustrate this with their example from co-planning of a quick image activity (Figure 4).

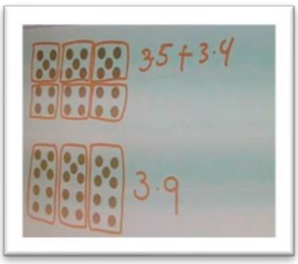

Figure 4: Board work from co-planning to teach a quick image activity

In this quick image activity, the students are shown the quick image in Figure 4 for a few seconds. They are then asked to explain their thinking for finding the total number of dots. The learning goal for the lesson was planned to be the distributive property of multiplication. The following example extract comes after the teacher educator represented an expected student strategy, 3 × 9, on the board at the bottom of the picture in Figure 4 and asked what they might be interested in knowing if the students use the strategy three times nine, seeing a group of nine dots (first line). A teacher suggested questions to uncover the details of students' thinking to show that nine equals 5 + 4 and thus that 3 × 9 equals 3 × (5 + 4) (second line).

1	Teacher educator:	(…) What's interesting for us to know when they see the entire nine group?
2	Teacher:	That it's five plus four. Three times five plus four. The teacher needs to ask "Nine, what do you think then? Nine, how did you get this here?" (…)

This example illustrates how the participants identified questions the teacher could ask to elicit the details of students' thinking. The example also illustrates how the purpose of eliciting students' thinking is closely related to the purpose of drawing their attention towards the learning goal for the lesson, or the distributive property of multiplication in this case.

Rehearsals and co-enactments

Moving on to rehearsals and co-enactments, in these parts of the learning cycle, all participants could pause instruction by initiating a teacher time-out (TTO, see Figure 3). In the TTOs they were discussing how the lead teacher might choose to respond to or represent student contributions and they decided the instruction's next steps (Wæge & Fauskanger, 2021, 2023). Studies have previously explored how teacher educators work with pre-service teachers in rehearsals and findings indicate that rehearsals provide pre-service teachers with opportunities to learn core teaching practices (e.g., Lampert et al., 2013) parallel to developing mathematical knowledge for teaching (Ghousseini, 2021). There are also some studies of practicing teachers rehearsing mathematics teaching. One important finding from these studies is that teacher educators could better support teachers' learning of adaptive instruction if what teachers approximate was reduced (Kavanagh et al., 2021). There are few studies on TTOs in co-enactments. One example is Gibbons and colleagues (2017). They found that the participants in TTO discussions had opportunities to make changes in-the-moment of teaching and to shift the focus of interactions from evaluation to collective considerations. Although the body of research on rehearsals and co-enactments in the context of professional development for practicing teachers is growing, there is more to learn. The aim of our studies has been to shed light on how TTOs in these two different approximations provide teachers with opportunities to learn. So, by analysing TTOs (Wæge & Fauskanger, 2021, 2023) we have explored the teachers' collective engagement and reflections in-the-moment of teaching. As Table 2 illustrates, TTOs in the rehearsals focused mostly on the core practices of using representations, aiming towards goals, launching a problem, organising the board, and facilitating student talk. In the co-enactments, the participants addressed which questions the teachers could ask to elicit the details of students' thinking and how to respond. They also considered how to represent students' thinking and how to aim towards learning goals for the lesson. I will continue by exemplifying aiming towards goals (more examples in Wæge & Fauskanger, 2023, p. 513–516).

Table 2: Core practices in teacher time outs (Wæge & Fauskanger, 2023, p. 511)

Teaching practice	% of all TTOs in rehearsals (n = 175)	% of all TTOs in co-enactments (n = 166)
Elicit and respond		39
Use of representations	29	21
Aiming towards goals	25	24
Launching a problem	23	
Organising the board	18	
Facilitating student talk	16	

Several TTOs addressed the practice of orienting the students towards the mathematical learning goal for the lesson ("aiming towards goals" in Table 2). The goal for students in the following example taken from a rehearsal was to learn the distributive property of multiplication (the example is taken from Wæge & Fauskanger, 2023, p. 513–514). The lead teacher has before this TTO just represented different expected student strategies on the board, both by writing numerical expressions and by using the quick image. Note that the two strategies 3 × 5 + 3 × 4 and 3 × 9 are written as the second and forth strategies on the board in Figure 5.

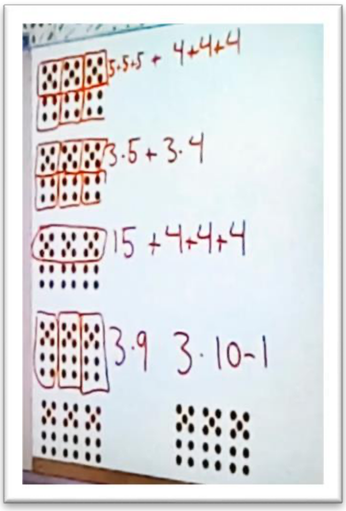

Figure 5: Student strategies from a rehearsal written on the board

The lead teacher has just represented the strategy "three times ten minus one" on the board (Figure 5, at the bottom of the picture) when the teacher educator asked for a TTO and started to talk about the use of parenthesis (first line in the extract below). One of the teachers pointed to the strategy 3 × 9 as something to address instead of parenthesis (second line), and he continued by explaining how starting by 3 × 9 could be used to show that 3 × (5 + 4) = 3 × 5 + 3 × 4. In addition, he suggested follow-up questions that could help students focus on the distributive property of multiplication.

1 Teacher educator: (…) the idea of using parentheses is interesting.
2 Teacher: (…) I was thinking that this [the expression] with 3 × 9 is an excellent opportunity to get them to see 3 × (5 + 4). You [lead teacher] can ask them "Does anyone see 9 in another way?" Then we have 5 and 4, and then you [lead teacher] write down 3 × (5 + 4).
3 Lead teacher: Yes. (…)

(…)

25 Teacher: (…) Most likely few see 4 plus 5 in the parenthesis. That's maybe difficult.
26 Lead teacher: How would you draw it?

This led to a discussion on how the lead teacher could use the students' strategies help them understand the distributive property of multiplication more in-depth (Lines 4–24). Later in the TTO (Line 25), the teacher said that it might be difficult for students to see the connection between 9 and (5 + 4) and suggested that the lead teacher should show the students where they find 4 and 5 in the quick image representing 3 × 9 (Figure 5, at the bottom of the picture). The lead teacher responded by asking how to draw that on the quick image (Line 26). This question led to a discussion on how the lead teacher could use the quick image to connect the two expressions and they worked hard on seeing the connection between the expressions 3 × 9, 3 × 5 + 3 × 4 and 3 × (5 + 4) and the representations in the quick image. Using quick images to support students' understanding of the distributive property of multiplication and helping them see connections between different expressions and representations was highlighted in this TTO discussion from a rehearsal. In the co-enactment following this rehearsal, one of the students explained the connection between the two expressions 3 × 5 + 3 × 4 and 3 × (5 + 4) written on the board (Figure 1) when the lead teacher initiated a TTO and asked for help in responding to this student's explanation. The teacher educator suggested that the lead teacher should focus on why the two expressions had the same answer (first line):

1 Teacher educator: (…), how do I know it'll be the same answer?
2 Teacher: Can you use the images?
3 Teacher educator: Can the images be used to explain why we get the same answer in the two expressions? (…)

One of the observing teachers suggested using the quick images to support the students' thinking (Line 2). This was also discussed in the rehearsal. This suggestion was followed by the teacher educator turning to the students, asking them to use the quick images to explain this (Line 3). From this we can see how "TTOs in rehearsals and co-enactments worked in complementary ways to provide opportunities for teachers to learn the practice of aiming towards goals" (Wæge & Fauskanger, 2023, p. 515).

Wæge and Fauskanger (2021, 2023) also found that the MAM participants in many TTOs discussed representing students' mathematical ideas on the board (Table 2). They discussed both how to make connections between students' thinking and representations and between different kinds of representations. As shown in the next example taken from Wæge and Fauskanger (2023, pp. 516–517), accurate representation of the students' thinking was emphasized in the TTOs in rehearsals. The participants in this example rehearse teaching a string starting with the task 100 : 4 – the one you have already represented yourself when reading the introduction of this paper.

1 Teacher educator: What is interesting here is that you [the student] have thought, you have explained that you were thinking four times 25. You just knew that, right?

	But, when you [the lead teacher] were making the number line now [dividing the number line into two etc.], what did she [the lead teacher] do [represent on the board], do you recognize that?
2 Teacher:	She divided into two and then divided into two again.
3 Teacher educator:	Yes, you [the lead teacher] halved and then halved again.
4 Lead teacher:	Ok.
5 Teacher educator:	It's important that we're aware of this [that this represents (100 : 2) : 2 and not 4 × 25].

And you know that they agreed on the representation in Figure 2.

In the co-enactment, the teacher educator reminded the lead teacher to accurately represent the student's idea. When one of the students said that "I took the first half of 100, and then I took half of 50" and the lead teacher represents the student's idea by drawing "jumps" of 25 on the number line, the teacher educator asks for a TTO (Line 1):

1 Teacher educator:	Can I ask for a time out?
2 Lead teacher:	Yes.
3 Teacher educator:	Can you draw it like the student said [(100 : 2) : 2]?
4 Lead teacher:	Yeah, I can, yeah.

The teacher educator suggested that the lead teacher should represent the student's idea exact as the student said it (100 : 4 as (100 : 2) : 2 instead of 4 × 25) (third line). The lead teacher agreed (last line) and when continuing instruction, she divided the number line into two equal parts and then divided each of these two parts into halves again. By doing this, the lead teacher understood that the teacher educator suggested (Line 3), namely that this student's idea could not be represented as in Figure 2. The lead teacher thus realized that this was the opposite situation from what they discussed in the rehearsal. Representing and connecting student strategies in-the-moment of teaching is challenging and our analyses (Wæge & Fauskanger, 2021, 2023) show that the TTOs provided opportunities for teachers to learn this core practice collectively.

Summary of findings

This was a little glimpse into the findings from exploring data material from MAM. A main finding from our research is that the core practices emphasized in the different settings differed and as a reason, all parts of the learning cycle (Figure 3) are needed to provide teachers with opportunities to learn these core mathematics teaching practices. As examples, co-planning is needed to learn questioning practices (Wæge & Mosvold, 2022), rehearsals to discuss aiming towards the mathematical learning goal for the lesson in-depth (Wæge & Fauskanger, 2021) and co-enactments are needed to learn how to elicit and respond to students' mathematical thinking (Wæge & Fauskanger, 2023). Our analyses also show that rich collective discussions about possible teacher strategies and moves characterises the TTOs in the rehearsals. The TTOs in the co-enactments complemented the TTOs in the rehearsals by supplying "specific suggestions about what the teacher might say or do in a particular situation" (Wæge & Fauskanger, 2023, p. 518). The TTOs in the two settings thus supported teachers' learning in complementary ways. Our results also show that the core practices of mathematics teaching are more than what teachers *do*; they involve thinking, decision-making, and sensemaking. Practice-based teacher education is more than introducing new actions; it involves supporting teachers' sensemaking and decision-making about the work of teaching mathematics.

Core practices into the future

Zooming out, a look across the work happening in different national contexts and not in MAM only, reveals several questions that scholars are currently grappling with. But, in all contexts the need to focus on core practices parallel to content is highlighted (e.g., Grossman & Fraefel, 2024a). As teacher educators we need to help teachers "know what to do" based on thinking, decision-making, and sensemaking. Forzani (2014) and Fraefel and Grossman (2024) provide some issues and challenges for future research. First, they highlight that more emphasis needs to be given to how practice-based pedagogies can support teachers working together in communities of practice in different countries. Second, future studies ought to test if and how these ideas can be applied and transferred into more countries and into different educational systems. Third, so far mostly small-scale studies are published, so investigations of larger-scale attempts on practice-based approaches to teacher learning are needed. Forth, given the emphasis placed on pre-service teachers, exploring the role of a practice-based approach to supporting practicing teachers' learning is important. MAM is one such attempt. Fifth, future research must aim at shared language and what is meant by practice (e.g., Lampert, 2010) is one example. This is highlighted by Fraefel and Grossman (2024, p. 143) underlining that future work in our field must include "definitional work concerning what 'counts' as core practices". 1000 practices such as in the Commonwealth study (Charters & Waples, 1929) are too many and not all of them would be equally important. Sixth, we need agreed-upon methodologies to explore the potential of practice-based approaches to teacher education. Finally, an area of importance for future research is how to best teach core practices in teacher education. This includes how to sequence representations, decompositions, and approximations of practice and how to create opportunities to try out practice at universities (Fraefel & Grossman, 2024). Learning cycles of enactment and investigation (McDonald et al., 2013) as used in MAM (Figure 3) might be one such approach.

Research is of course important. Equally important is providing teacher educators with research-based "help" in doing their work. Within MAM such resources are developed[5], and similar resources are developed within the previous mentioned projects at the universities of Michigan and Washington. The MAM professional development program for practicing mathematics teachers is recommended as a two-year process involving 11 modules. As an example, the third module focuses on representation, and here representation is defined, the goal and instructional activity for the module is presented, and resources for the different parts of the learning cycle are made available for teacher educators to use in professional development for practicing teachers. For preparation for example, there is an article about quick images and a video clip showing enactment of the activity available for teacher educators to use[6].

Teacher educators are of outmost importance for a practice-based approach to teacher education and worldwide, they are not necessarily educated to focus on core mathematics teaching practices in their

[5] See https://www.matematikksenteret.no/kompetanseutvikling/mam/modulene-i-mam for more information about the modules in the MAM professional development program.

[6] For more about the MAM module on representations, see https://www.matematikksenteret.no/kompetanseutvikling/mam/modulene-i-mam/modul-3-representasjon.

work. In Norway, we have developed a course for teacher educators, where practicing core practices of mathematics teacher education (e.g., leading learning cycles) as well as core practices of mathematics teaching in schools (e.g., those presented in Table 2) are focuses of attention[7]. Similar resources are also developed at the University of Michigan, where also online courses for teacher educators are available.

As teacher educators and researchers, we also must convince our politicians not to look for "quick fix" solutions (as done by Wæge et al., 2023). We need to convince them to support long term, school-based professional development for practicing teachers focusing on the principles and core practices of ambitious mathematics teaching. With this recommendation, I end my paper.

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[7] For more about the course, see https://www.ntnu.no/videre/gen/-/courses/nv19025.

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Math Mastery in the Digital Age: The Confluence of Digital Technologies and Mathematical Competencies

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In recent decades, there has been an increased focus on implementing digital technologies (DT) and developing mathematical competencies (MC). Yet, the focus on DT and MC has largely been pursued separately. This plenary presents findings from a project involving three PhD studies that investigated the relationship between MC—such as thinking, reasoning, representation, and communication—and DT, employing design research and networking of theories. The studies examined tools like Maple and GeoGebra, using data from both lower and upper secondary schools. The paper presents key findings from the project and suggests directions for future research.

Keywords: digital technologies, mathematical competencies, networking of theories.

Introduction

In 2002, the Danish KOM framework was introduced, emphasizing mathematical competencies (MC) as a core component of mathematical proficiency. This framework has significantly influenced mathematics education, especially within the Nordic countries.

The Danish Mathematics Competency Framework (KOM) was first published in Danish in 2002 (Niss & Jensen, 2002) and was subsequently translated into English (Niss & Højgaard, 2011). Later, an article further detailing the framework appeared in *Educational Studies in Mathematics* (Niss & Højgaard, 2019). Since its inception, the KOM framework has been integrated into various levels of the Danish education system, including primary, lower secondary, and upper-secondary schools, as well as in teacher education for mathematics. Furthermore, the KOM framework's influence extended internationally and, for several years, was included in the PISA framework for assessing MC (e.g., Lindenskov & Jankvist, 2013).

Niss and Højgaard (2011) define a mathematical competency as an individual's "…well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge" (p. 49). The KOM framework comprises eight MC, three types of overview and judgment, and six didactico-pedagogical competencies specifically tailored for teaching. Still, despite its comprehensive approach, the original KOM framework paid limited attention to the role of digital technologies (DT) in the teaching and learning of mathematics. DT are mentioned along with more analog technologies as part of the aids and tools competency. One part of this competency concerns students "*having knowledge of* the existence and properties of the diverse forms of relevant tools used in mathematics and having an insight into their *possibilities and limitations* in different sorts of contexts" and the other part concerns students "being able to reflectively *use* such aids" (Niss & Højgaard, 2011, pp. 68-69, italics in original). Niss and Højgaard (2011) elaborate:

> Mathematics has always made use of diverse technical aids, both to represent and maintain mathematical entities and phenomena, and to deal with them, e.g. in relation to measurements and calculations. This is not just a reference to ICT, i.e. calculators and computers (including arithmetic

programmes, graphic programmes, computer algebra and spreadsheets), but also to tables, slide rules, abacuses, rulers, compasses, protractors, logarithmic and normal distribution paper, etc. The competency is about being able to deal with and relate to such aids. […] Since each of these aids involves one or more types of mathematical representation, the aids and tools competency is closely linked to the representing competency. Furthermore, since using certain aids often involves submitting to rather definite "rules" and rests on particular mathematical assumptions, the aids and tools competency is also linked to the symbol and formalism competency. (p. 69)

The quote sums up the comments related to the use of DT in mathematics teaching and learning in the KOM framework. In fact, discussions about using DT and developing MC have generally occurred separately, both in practice and in research.

Today, some twenty years after the introduction of the KOM framework, Denmark is one of the few countries in the world that have gone "all in" on using DT in the teaching and learning of mathematics. This in particular in upper secondary school, where Computer Algebra Systems (CAS) for some years have permeated the mathematics programs as well as having become a part of the exams, in particular the national written assessments (e.g., see Jankvist, Misfeldt & Marcussen, 2016)—an approach that have also caused both teaching and learning challenges (Jankvist & Misfeldt, 2015; Jankvist, Misfeldt & Aguilar, 2019). While the upper secondary mathematics programs rely on CAS such as e.g., Maple or TI Nspire, primary and secondary mathematics programs in Denmark mainly use Dynamic Geometry Environments (DGE), in particular GeoGebra.

Since 2019, a Danish research project has been investigating the interplay between students' development of MC and the use of DT in lower and upper secondary school. This project has explored the potential for connecting—or "networking"—elements of the KOM framework to other theoretical constructs in mathematics education (Niss & Jankvist, 2022), particularly those related to the use of DT. The two guiding research questions for the project have been: (1) In what ways may the MC approach be "networked" with existing theoretical constructs, results and findings from the field of mathematics education research in relation to the use of ICT in the teaching and learning of mathematics? (2) In what ways may a "networked" frame consisting of the mathematics competencies approach and selected well-established theoretical constructs from mathematics education research inform and improve the use of DT in current and future mathematics teaching and learning? Examples of well-established theoretical constructs applied in such connections with the KOM framework include those of the instrumental approach (Guin & Trouche, 1998), semiotic representations (Duval, 2017), and elements of the theory of conceptual fields (Vergnaud, 2009).

In this keynote presentation, we provide an overview of this research project and its current results. We emphasize the connection between two MC: representation and communication, in the context of using dynamic geometry environments in lower secondary school. First, however, we describe some of the other results of the research project, which besides a number of conference and journal publications also gave rise to both an international anthology (Jankvist & Geraniou, 2022) as well as three doctoral theses (Bach, 2022a; Gregersen, 2024; Pedersen, 2024).

The anthology: “Mathematical competencies in the digital era”

The anthology is organized in such a way that, when possible, younger researchers have co-authored chapters together with more experienced researchers in the fields of MC and DT. In structuring the book, we also attempted to have a somewhat large diversity of different nationalities contribute to the anthology. Besides an introduction, the anthology is structured into four main sections, each addressing distinct facets of the KOM framework.

The first section, titled “Setting the Scene”, comprises two introductory chapters that lay the groundwork by discussing the KOM framework itself and examining the potential of integrating it with various theoretical perspectives from mathematics education research. This integration is explored both generally and with a specific focus on the role of DT. In the first chapter of this section, Niss and Jankvist (2022) introduce and explore the KOM framework broadly, discussing its potential connections with various theoretical perspectives. As part of this exploration, the authors examine terms such as ‘theory,’ ‘theoretical framework,’ and ‘theoretical perspectives,’ aiming to position KOM within this conceptual landscape. The chapter also illustrates possible connections between the KOM framework and other theoretical approaches, offering examples across different areas: for instance, through the modelling competency, in overview and judgment with the historical progression of mathematics, and in didactico-pedagotical competenciew through the exploration of learning processes. Based on this analysis, the authors suggest that, at its current stage, the KOM framework may benefit more from a “mutual fertilization” with related theories rather than an attempt to fully integrate these elements into a unified framework.

The second section of the anthology, “The Eight Mathematical Competencies”, contains eight chapters, each dedicated to one of the competencies identified in KOM. These competencies are as follows: mathematical thinking competency, which involves engaging in mathematical inquiry; mathematical problem handling competency, which focuses on posing and solving mathematical problems; and mathematical modelling competency, centered on analyzing and constructing mathematical models. Additionally, the framework outlines mathematical reasoning competency, which involves assessing and justifying mathematical claims; mathematical representation competency, which relates to managing different representations of mathematical entities; and mathematical symbols and formalism competency, which entails working with mathematical symbols and formal structures. The section also addresses mathematical communication competency, which includes communication within and about mathematics, and mathematical aids and tools competency, which involves using tools and resources in mathematical activities.

In this section, Ahl and Helenius (2022) contribute a notable case study on programming, illustrated through Scratch, within the chapter titled “New demands on the symbols and formalism competency in the digital era.” Here, they integrate Vergnaud’s (2009) theory of conceptual fields with the KOM framework to enhance the understanding of the symbols and formalism competency. These two theoretical lenses offer distinct levels of detail in analyzing the programming scenario. They argue that the KOM’s symbolism and formalism competency serves as a valuable tool for considering essential aspects of mathematical knowledge. By aligning this competency with the more detailed, Vergnaud-based theories, they highlight the complexities of developing symbolic and formal reasoning skills. This coordination underscores the benefits of linking the KOM framework with Vergnaud's theories for a more nuanced insight into the challenges and potential of this competency.

The third section of the anthology, "Overview and Judgement", consists of three chapters, each devoted to one of the KOM framework's types of overview and judgment. Niss and Højgaard (2011) describe these as enabling individuals to "have a set of views allowing him or her overview and judgment of the relations between mathematics and in conditions and chances in nature, society, and culture" (p. 74). The three types of overview and judgment include: the application of mathematics within various other disciplines and practical fields; the historical evolution of mathematics from internal and sociocultural perspectives; and the nature of mathematics as a discipline.

In this section, Thomsen and Clark (2022) examine the second type of overview and judgment in the KOM framework, which concerns the historical development of mathematics, viewed from both internal and sociocultural perspectives. In the chapter "Perspectives on embedding the historical development of mathematics in mathematical tasks," they explore how engaging with the interaction between historical mathematical sources and digital technology (DT) can aid students in developing this form of overview and judgment. This approach emphasizes the dialectical relationship between the praxis and logos blocks, supported by theoretical frameworks such as the Anthropological Theory of the Didactic (ATD). The authors draw upon two empirical examples from Denmark to illustrate these ideas: one involving a 7th-grade classroom, and the other an in-service teacher education course.

The fourth section broadens the focus, with four chapters that examine the relevance of the KOM framework in the context of the digital era. Among other topics, these chapters cover the assessment of MC, teachers' didactico-pedagogical competencies in digital environments, and the relationship between MC, computational thinking, and programming in educational contexts. In the final chapter, Turner, Tout, and Spithill (2022), in "A Rich View of Mathematics Education and Assessment: Mathematical Competencies," discuss the impact of the KOM framework on the OECD's Programme for International Student Assessment (PISA) in terms of shaping both the assessment of mathematical literacy and the reporting of results. They further explore how KOM can serve as a foundation for creating PISA assessment items inspired by its competencies. The authors provide a research-backed argument for incorporating MC into assessment practices in the digital era.

Research on mathematical thinking and DT

One PhD project focused on the KOM framework's mathematical thinking competency (Pedersen, 2024). This competency addresses the thought processes involved in mathematical inquiry, such as examining and posing questions about mathematical concepts, relationships, statements, and logical structures (Niss & Højgaard, 2019). The PhD project investigated the potential advantages and challenges of utilizing DT to facilitate students' expression of mathematical thinking competency, specifically in exploring the concept of differentiability in upper secondary education. The objective of the project was to determine how digital mathematical tools could be effectively employed to support mathematical thinking competency. This involved characterizing the ways mathematical thinking competency operated within digital contexts, detailing its components in terms of mathematical actions and considerations. Consequently, the project emphasized identifying circumstances that enabled or inhibited students' ability to engage in mathematical thinking competency, rather than directly focusing on the competency's development.

A design research approach provided the methodological foundation for the project, structuring the research's data collection, analysis, and conclusions. Additionally, it guided the design of activities that integrated mathematical thinking competency, digital technology, and differentiability. As part of this approach, three task sequences were crafted to encourage students to express mathematical thinking competency through digital investigations of differentiability.

Beyond the KOM framework's mathematical thinking competency, other theoretical lenses informed various stages of design research. Theories such as instrumental genesis (e.g., Drijvers et al., 2013; Guin & Trouche, 1998) and the notion of a scheme (Vergnaud, 2009) underpinned the design of tasks and the analysis of data, offering insights into students' interactions with DT and their mathematical thinking. Complementary perspectives, such as semiotic mediation (Bartolini Bussi & Mariotti, 2008), contextual abstraction (Hershkowitz et al., 2001), and the Anthropological Theory of the Didactic (ATD) (Chevallard, 2008) supported the analysis focusing on context and student cognition.

The study yielded comprehensive insights through its exploration of mathematical thinking competency using DT. The initial phase of the research identified three primary ways in which DT could support elements of mathematical thinking competency, which subsequently guided the design of task sequences. This foundation informed empirical investigations, which documented students' engagement with these tasks and their interactions with DT in various contexts. The project's results showed that students engaged broadly with mathematical thinking competency, demonstrating specific elements of the competency for both the task content and the DT used. Further analysis revealed that digital environments significantly shaped students' ability to engage in mathematical thinking, with certain features enhancing or constraining their engagement. In particular, students' application of logical quantifiers—key to understanding the concept of differentiability—was highlighted as a crucial aspect of their mathematical thinking competency, reflecting their operational approach to definitions and logical structures in a digital context. These findings contribute to an understanding of how DT can enhance or limit mathematical thinking competency, offering guidance for integrating DT in mathematics education (Pedersen, 2022, 2024).

The combined findings provided insights into the characteristics of students' mathematical thinking competency and working methods. Results emphasized the role of students' hypothesis formation in expressing mathematical thinking competency. Additionally, the research identified both the benefits and challenges of employing DT, resulting in the proposal of five design principles to guide the use of digital mathematical tools in supporting mathematical thinking competency (Pedersen, 2024).

Research on the reasoning competency and DT

One PhD project focused on the KOM framework's reasoning competency (Gregersen, 2024b), which involves analyzing or constructing arguments to substantiate mathematical assertions (Niss & Højgaard, 2019). The expanding capabilities of DT, such as DGE and CAS, increasingly combine features from both geometry and algebra (Freiman, 2014; Sutherland & Rojano, 2014), thus offering enhanced educational potential but also complexities that often exceed the common understanding.

The PhD project had both practical and theoretical aims. Practically, it examined how DT in geometry and algebra could support students' reasoning processes in lower secondary mathematics education, focusing on facilitating students' engagement in mathematical reasoning rather than directly developing the competency itself. Theoretically, the study sought to advance sustainable theoretical development by connecting the KOM framework with international research in mathematics education. This integration was framed through a networking perspective on theory development (Prediger, Bikner-Ahsbahs & Arzarello, 2008).

As the other PhD studies in the research project, it employed design research (Bakker, 2018; Gravemeijer & Prediger, 2019) as its primary methodology. This guided data collection and analysis and supported the creation of learning environments that integrated mathematical reasoning competency, DT, and the variable as a generalized number. To achieve this, a digital microworld featuring variable points and related task sequences was created, encouraging students to explore algebraic expressions and their structural impacts on the behavior of variable points.

In addition to the KOM framework, the project incorporated other theoretical frameworks, such as the instrumental approach to mathematics education (Guin & Trouche, 1998), the scheme-technique duality and scheme elaboration (Drijvers et al., 2013; Vergnaud 2009), and Toulmin's (2003) model of argumentation. These perspectives were used to analyze, interpret, and explain empirical data, especially concerning students' use of DT and engagement in mathematical reasoning.

This study presented key findings on the use of DT to support students' mathematical reasoning competency, specifically in the context of variables as generalized numbers. Through a literature review, the study identified useful functionalities within GeoGebra, and its "algebra view", that could aid students in constructing justifications related to variable behavior. These insights informed the design of tasks aimed at fostering mathematical reasoning. Analysis of student interactions with the tasks led to the development of an analytical framework for understanding *instrumented justification*, a concept combining elements from the KOM framework and the instrumental approach through Toulmin's model of argumentation. This framework supported a deeper understanding of how students use DT to justify mathematical claims, refining the notion of instrumented justification by examining how students engaged with mathematical artifacts in task-based contexts (Gregersen, 2024a; Gregersen & Baccaglini-Frank, 2022).

The study's contributions included three principles for task design to enhance students' reasoning competency, a digital microworld focused on exploring and justifying variable point behavior, and task sequences to support these learning objectives. Additionally, the research elaborated on reasoning competency in relation to instrumented justifications, clarified the scheme-technique duality, and provided recommendations for supporting these reasoning processes through classroom activities and task design. Findings also revealed a blended conception of continuous and discrete variable understandings that aided in predicting variable behavior in dynamic geometry and algebra environments. Furthermore, the study suggested connections between the KOM framework and the instrumental approach, laying groundwork for further theoretical integration.

Research on mathematical communication competency and DT

In this section, the PhD project by Bach (2022a) is presented in more depth. This PhD project focused on students' mathematical communication competency and their use of DGE. The tool utilized in the project was GeoGebra.

Niss and Højgaard (2019) characterized mathematical communication competency as the ability to express oneself mathematically and to comprehend and interpret the mathematical expressions of others. Mathematical communication, as defined by Niss and Højgaard (2019), entails the exchange of "notions and concepts, terms, results and theories, or other features of mathematics". (p. 18) Mathematical communication may appear in various media, genres, and discourses and communication is also with diverse audiences, such as peers, teachers, and parents.

Relating unrelated theoretical perspectives: 'Tool-based mathematical communication'

The PhD project aimed to comprehend students' mathematical communication while using DGE. No theoretical perspective in mathematics education research was applicable. Thus, the project related design research and networking of theories to achieve empirical and theoretical results (Bach, 2022a).

Following Prediger et al. (2008), networking of theories is a research practice that aims to bridge previously unrelated theoretical perspectives in mathematics education. By applying more than one theoretical perspective to the data, the research seeks to establish connections between different theoretical and methodological approaches (Tabach et al., 2020). Eight strategies for networking theories are presented in Figure 2.

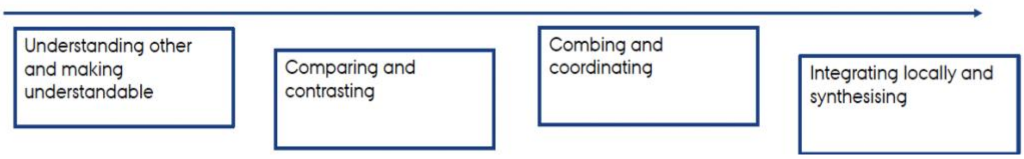

Figure 2: Strategies for networking theories (Prediger et al., 2008, p. 170)

In the first iteration, the relationship between mathematical communication and the use of DGE was explored to investigate possible theoretical perspectives to relate (Bach & Bikner-Ahsbahs, 2020). However, the focus was not on networking of theories. In the second iteration, the relationship was elaborated further. The notion of 'tool-based mathematical communication' was the result of a process of networking two theoretical perspectives as described below (Bach & Bikner-Ahsbahs, 2022). The third iteration tested the notion in a new setting (Bach, 2022a).

The PhD project focused on students' mathematical communication in pairs, while working on a task concerning functions as co-variation. The tasks built on a previously designed task by Johnson and McClintock (2018). In the project, students were given a GeoGebra template as depicted in Figure. In the template, the representations were interrelated. The students could drag Point A in the dragging up and down. Point P's x-coordinate was the height of AB and the area of $ABCD$ was related to Point P's y-coordinate.

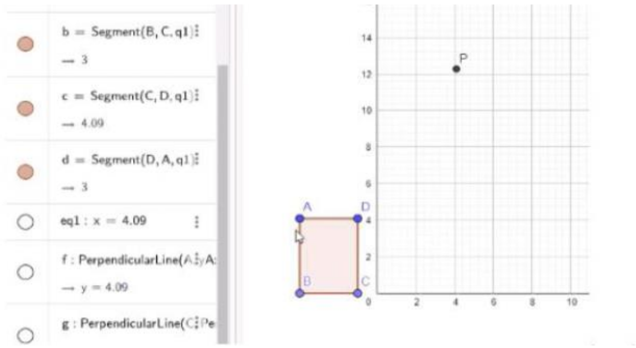

Figure 4: The GeoGebra template

Tool-based mathematical communication

As presented in Bach and Bikner-Ahsbahs (2022), tool-based mathematical communication locally integrated two theoretical perspectives. The notion drew on psycholinguistic concepts, i.e., conversation and empractical talk (O'Connell & Kowal, 2012), and the instrumental approach to mathematics education (Guin & Trouche, 1998). Both theoretical perspectives were compatible with KOM, meaning that their basic assumptions were not too far apart.

O'Connell and Kowal (2012) distinguished between two ways of verbal mathematical communication: conversation and empractical talk. In Bach and Bikner-Ahsbahs (2022), these two notions were related to mathematics. Mathematical conversation was considered having interactions centered around mathematics, including discussions in, with, and about mathematical concepts. These conversations required active speaking and listening (listening is active in opposition to hearing, which is passive), promoting equal participation and turn-taking among participants. The goal was to collaboratively move the conversation forward by building on each other's ideas and contributions. On the other hand, empractical mathematical communication was characterized by non-linguistic elements, as practical activities were in focus. This communication often involved frequent pauses and a lack of continuous, coherent dialogue. It tended to rely on gestures or brief, directive phrases like "there" or "look," emphasizing action over explicit verbalization of mathematical concepts. Communication competency was exercised during a mathematical conversation.

For the perspective on DT, we used the instrumental approach to mathematics education in which artefact and instrument are distinguished as well as instrumentation and instrumentalization. Furthermore, Guin and Trouche (1998) characterized five profiles for the use of CAS: random, mechanical, resourceful, rational, and theoretical (which can be adapted to DGE) (Bach, 2023a).

Based on the following excerpt 1, tool-based mathematical communication is explained.

Frida	Yes, so if we change…
Emma	So, *P* moves askew to the right [grabs the mouse].
Frida	And…
Frida	The rectangle's … area [she drags point P so the figure and point P change].
Frida	Okay, then we change…
Emma	Well, if it moves askew to the right, more and more.

Excerpt 1 (Bach & Bikner-Ahsbahs, 2022, p. 190, lines 35-40).

In Excerpt 1, the students showed a mechanical instrumentation profile due to the students' difficulties of using the tool, their references to the DGE (e.g., dragging) and their exploration of dynamic commands. Their communication was 'empractical' as the students were both speaking, but it is questionable if they were listening actively to one another, and they were focused on the practical activity of using GeoGebra and solving the task.

Tool-based mathematical communication referred to "a particular type of communication linking cognitive-linguistic resources with cognitive action schemes developed during instrumentation, which emerge in connection with communicating and acting" (Bach & Bikner-Ahsbahs, 2022, p. 194). This form of communication was context-dependent and strongly influenced by the specific tools being used, i.e., the use of GeoGebra played a significant role in facilitating mathematical communication. Four kinds of tool-based mathematical communication arose when networking mathematical communication and the instrumental approach. See Table 1.

	Embedded digital tool use (related to dynamic natures of represent-tations)	Idled digital tool use (related to static representations or template given
Mathematical conversation	*Tool-embedded* mathematical conversation	*Tool-idled* mathematical conversation
Empractical mathematical communication	*Tool-embedded* empractical mathematical communication	*Tool-idled* empractical mathematical communication

Table 1: Overview of the four profiles of tool-based mathematical communication. Tool-idle mathematical conversation was not empirically found (after Bach & Bikner-Ahsbahs, 2022, p. 193)

Summarizing the PhD project's (Bach, 2022a) results in general, exercising mathematical communication competency in the context of DGE presented notable challenges. The use of DGE introduced new forms of mathematical communication, such as dynamic communication (Bach, 2022a; Bach et al., 2024). However, when students focused too heavily on the technical aspects of DGE, their communication could shift toward empractical forms, where practical activities dominated the communication (Bach & Bikner-Ahsbahs, 2022).

The dynamic linking of mathematical representations in DGE offered significant potential for fostering investigation and conversation (Bach, 2023b; Bach & Bikner-Ahsbahs, 2020). However, constraints tied to the process of instrumental genesis, where the integration of tools into learning processes was identified to sometimes hinder the students' possibilities to exercise of communication competency (Bach & Bikner-Ahsbahs, 2020, 2022; Bach, 2023a; Bach et al., 2024).

Research on mathematical representation competency and DT

A shared research interest across PhD projects (Bach, 2022a; Gregersen, 2024; Pedersen, 2024) was mathematical representation competency. Representation competency involves the ability to interpret, translate, and move between different forms such as graphs and equations. It also includes choosing appropriate representations and understanding their limitations (Niss & Højgaard, 2019).

In a literature review (Pedersen et al., 2021), we identified five recommendations for teaching involving DT and aimed to exercise mathematical representation competency. Subsequently, we aimed to test and revise our recommendations as part of design research. In Bach et al. (2022b), we aimed to explore how practices from the networking of theories could support design research, while also refining our recommendations into a cohesive design principle. We focused on refining recommendations 1, 3, and 4 from Pedersen et al. (2021).

- Recommendation 1: Include the discursive multifunctional register (linguistic), both at the beginning and at the end of each task […].
- Recommendation 3: Break objects/representations into smaller units or gradually introduce new windows and features within a given MDBO.
- Recommendation 4: Use sliders, dragging and tracing as they hold potentials for students' ability to move and translate (using the tools) as well as interpreting and understanding the representations and their reciprocal relations. […]. (Pedersen et al., 2021, p. 18)

The study showed that the processes of instrumental genesis and students' use of representations and transformations are closely intertwined. The theory of semiotic registers (Duval, 2017) and the instrumental approach (Trouche, 2005) proved to be compatible, particularly from a cognitive perspective, and could be applied effectively within the KOM framework (Bach et al., 2022b). A design principle with three additional characteristics emerged:

> If you want to design teaching with a digital tool for exercising and developing the representation competency in schools, then you are best advised to *gradually relate all four registers (linguistic, symbolic, figurative and graphic) in a task sequence when using DT.* Collaboration between students is essential for all three characteristics and procedures. (Bach et al., 2022b, p. 2927).

This principle highlighted the importance of linking various representations and foster collaboration.

Research on 'networking of theories' in relation to the KOM framework

In Bach (2022), design research and networking of theories were related to enhance the development of theoretical perspectives. The relationship between the two acted as in Figure 5.

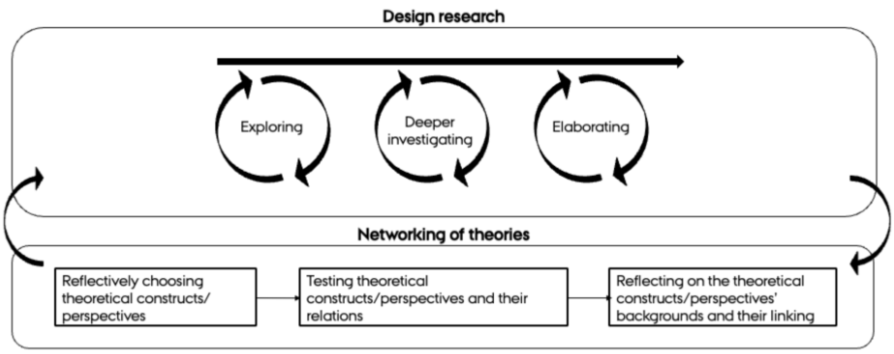

Figure 5: Networking of theories and design research (Bach, 2022b)

The examples presented in our plenary lecture highlighted the need for relatively unrelated theoretical perspectives from mathematics education research to gain a deeper understanding of the relationship between MC and DT, as well as to better understand empirical cases. Common for the projects was that networking theories was carried out around an empirical case of data to coordinate these theoretical perspectives. The theoretical perspectives related to mathematical representations and communication were chosen because they are compatible with the KOM framework (Niss & Højgaard, 2019). Thus, KOM functioned as a ‘framing theory’—i.e., a background theory grounded in mathematics education research (Bach et al., 2021).

Based on the three PhD projects, potentials and limitations of using KOM appeared. KOM offered a broad lens, focusing on the mastery of mathematics (Niss & Højgaard, 2019). However, we also recognized the need for more granular analytical tools to study communication and the use of DGE in detail. KOM’s aids and tools competency did not provide explicit guidance on how tools should be used. Yet, the strengths of KOM were in its ability to offer direction and contextualize results within a broader perspective (Bach, 2022; Gregersen, 2024; Pedersen, 2024).

To summarize the relationship between design research and the networking of theories, this work demonstrated that both research approaches aim to contribute to theoretical development. We identified potential in using networking practices within design research in the following situations: 1) when needing to relate previously unrelated theoretical perspectives; 2) when aiming to strengthen theoretical development by refining and connecting theoretical perspectives through additional steps of theorizing; 3) when employing networking strategies such as coordinating, combining, and locally integrating perspectives, as these strategies align particularly well with the goals of design research, focusing on both theory and practice; 4) when attempting to triangulate theoretical perspectives; and 5) when exploring the foundational assumptions of the theoretical perspectives in use. Looking in the opposite direction, design research can support the networking of theories in the following situations: A) when exploring theoretical perspectives, and B) when testing networked theoretical frameworks.

Future research

As part of future research perspectives building on the results of the project, we mention two: One related to students’ Mathematical Digital Competency (MDC) and another related to the teaching of this, Mathematical Digital Competency for Teaching (MDCT).

Geraniou and Jankvist (2019), building on the KOM framework (Niss & Højgaard, 2019), the instrumental approach (Trouche, 2005), and the theory of conceptual fields (Vergnaud, 2009), proposed that students with MDC exhibit specific attributes. The first characteristic, referred to as MDC1, entails the ability to participate in techno-mathematical discourse. This competency involves aspects of the artifact-instrument duality, where the process of instrumentation enables students to develop techno-mathematical fluency. Secondly, MDC2 requires students to recognize appropriate DT for various mathematical contexts and understand each tool’s capabilities and limitations, highlighting the duality between instrumentation and instrumentalization. Lastly, MDC3 involves reflective use of digital technology in problem-solving and mathematical learning, where students are mindful of how DT serves both practical and knowledge-building purposes. This reflective use incorporates the scheme-technique duality for both predictive and operative forms of knowledge. The mentioned dualities are due to Drijvers et al. (2013), who connected Vergnaud’s (2009) concept of schemes with techniques from instrumental genesis, thus providing a clearer framework for understanding the interplay between student techniques and their evolving digital mathematical skills.

In further expanding on the KOM framework with a focus on teachers, Geraniou et al. (2024) introduced KOM's six didactico-pedagogical competencies (Niss & Højgaard, 2011, 2019) as a primary perspective for examining teaching practices. These competencies provide a detailed description of the practices involved in mathematics instruction. However, like the development of MDC for students, the KOM framework alone is not entirely sufficient for addressing technologically enhanced classroom settings. To address this gap, two additional theoretical frameworks connected to the instrumental approach were integrated to support KOM's didactico-pedagogical competencies.

The first framework is the Theory of Instrumental Orchestration (TIO), initially developed to address the pedagogical complexities of organizing and managing students' use of DT and resources in classrooms. TIO examines the challenges and possibilities that arise in technology-rich classrooms, using a language of 'orchestrations' to describe the teacher's role in facilitating digital resources. TIO is highly relevant for understanding how teachers design and manage orchestrations that involve various DT in mathematics. This framework should be interpreted considering the three dualities in the Theory of Instrumental Genesis (TIG), as outlined by Drijvers et al. (2013), since teachers need to consider how to support students in achieving these dualities through their orchestrations.

The second framework, the Documentational Approach to Didactics (DAD), focuses on how teachers incorporate DT to support their teaching processes. DAD proposes that teachers' pedagogical work relies on instrumented techniques like students' engagement with DT in mathematics (Trouche et al., 2018). In this approach, the resources supporting instruction, e.g., lesson plans, online platforms, and handouts, are referred to as 'documents' to differentiate them from the mathematical tools used by students. DAD examines the process of documentational genesis, where these teaching artifacts are shaped by and adapted to instructional practices. DAD provides insights into teachers' strategies for integrating DT, particularly as they work toward fostering students' MDC. As the emphasis on MDC grows, DAD allows for a nuanced view of how teachers' preparation, planning, presentation, and documentation processes are influenced by both technological infrastructures (documentational systems) and new learning objectives, ultimately shaping how MDC is embedded in teaching.

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Panel discussion on the interplay between research and teachingpractice in mathematics education

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This paper provides reflections and thoughts from five early career researchers in mathematics education on the interplay between research and teaching practice. How is this interplay in one's own research and in mathematics education in general – and how should it be? Reflections and thoughts on these questions are structured around three perspectives on the quality of research: Relevance and importance: Why should one care? Trustworthiness: Why should one believe what the author says? Generality: What situations or contexts does the research really apply to?

Keywords: Mathematics education, early career researchers, research versus teaching practice, relevance and importance, trustworthiness, generality.

Introduction

As described thoroughly elsewhere in these proceedings, the interplay between research and teaching practice in mathematics education was the theme of the NORMA 24 conference. One consequence of this choice was that a plenary panel debate with the same theme was part of the program. In this debate, five of us, Anna, Lóa, Maria, Alexander, and Jóhann, all early career researchers in mathematics education, formed a panel discussing the theme, moderated by the sixth of us, Tomas.

The present paper is an attempt to reflect this discussion. Not by putting what was said in writing, but rather, we have structured the paper the same way we structured the oral debate, and then allowed ourselves to write down our thoughts and opinions under the different headings.

In what follows, we first introduce the five contexts for reflecting on the research/practice relationship spanned by the members of the panel, accompanied by a short description of what each member finds to be among the most pressing issues regarding the research/practice relationship. Then, the remainder of the paper is spent on sharing thoughts and opinions about this relationship, structured by a three-dimensional quality of research perspective with the headings relevance and importance, trustworthiness, and generality.

Five contexts for and general thoughts on the research/practice relationship

Anna Holmlund

The research-wise youngest of us (with two years until the PhD defense) is me, Anna Holmlund, a teacher in mathematics and physics in upper secondary school in Sweden. After working as a teacher for a few years, I was accepted to a research school for teachers, CUL – a graduate school in educational studies for teachers at the University of Gothenburg. The aim of the research school is to develop the scientific base for the teaching profession (Bergviken Rensfeldt & Säljö, 2012), which directed my research interest towards finding a topic that teachers easily could relate to. In my licentiate thesis I examine how coefficients other than natural numbers (e.g., decimal and negative numbers) influence students' solving of linear equations (Holmlund, 2024). The continued work with my thesis concerns how teachers can support students to recognize familiar structures of equations. This work includes a learning study where I collaborated with three mathematics teachers in upper secondary school. This project was supported by the governmental funding called ULF (education–learning–research), that is specifically aimed at enabling collaborations between teachers and researchers (Regeringsbeslut U2017/01129/UH).

The link between research and practice is, in my view, a relation that is built on mutual trust and respect. I believe collaboration between practice and research needs to be rewarded more in the research community. Looking at my experience, the research school of CUL and the ULF funding are examples of such initiatives that steer research to be closer to practice. Furthermore, when researchers collaborate with teachers, I believe they strengthen the relation between research and practice, which in itself can increase teachers' interest in research. In my work I found that the possibility to collaborate digitally opened up to meet teachers at different schools and in other cities. However, collaboration in research is only one of many settings for interactions between teachers and researchers (e.g., social media, teacher conferences, teacher education) and it is important to use all of these venues.

Lóa Björk Jóelsdóttir

For several years now, including my time as a PhD student in 2020–2023, the main focus in my work has been research in mathematics education. By definition I am a new researcher but with my background as a schoolteacher in music and mathematics in Iceland for 10 years (1997–2007) and a teacher educator in Denmark since 2009, I have been in the teaching practice for over 25 years and my professional identity is closely related to my background and experience.

My research focus has developed through my experience, first, as a mathematics teacher in Iceland, where me and my colleagues experienced an important development of the teaching practice, including a new set of teaching materials for all grade levels. Second, as a pre-service teacher educator, developing my ideas on teaching practice from the teacher educator point of view. Third, as a very important factor, working with in-service teachers in different courses, workshops and meetings in practice has brought a variety of topics, which today's teachers deal with in their everyday practice. And fourth, the experience gained in several research projects which I attended before my PhD.

My choice of research topic for my PhD was flexibility and adaptivity in arithmetic. From the Danish curriculum guidelines it is expected that all students' strategy development is based on their understanding of numbers but not training a standard method (Børne- og Undervisningsministeriet, 2019). At the same time, my experience from practice indicated that the reality may differ from the curriculum guidelines. In my study, I had the opportunity to test 2,298 students from grades 3, 6, and 8 in arithmetic. The test was designed to encourage careful examination of numbers that were selected to elicit the use of shortcut strategies. However, the findings showed that only few students compute using shortcut strategies, such as solving 199 + 323 by adapting it to 200 + 323 – 1. Use of standard algorithms is by far the most common strategy (Jóelsdóttir & Andrews, 2023, 2024). Further, it was more common that students in grade 6 used shortcuts compared to grade 8 students. This evokes questions about practice, culture, and socio-mathematical norms in the classrooms – how students and teachers think about and appreciate mathematics. The focus of the research is on what occurs in practice, and the research findings bring us back to practice and new issues to be studied.

Maria Møller

I have just completed my PhD and can now call myself a researcher, but I have almost always been interested in research and development projects. Years ago, when I taught mathematics and science in primary and lower secondary schools, I participated in several interdisciplinary development projects. The same has been the case during my time as a teacher at a teacher training college. I am sure that my experiences with developing projects have impacted my research into STEM teaching and STEM didactics in my PhD.

I base my work on the belief that research and practice are linked, and I accept that they will often interact dynamically. Conducting research in practice and with practice will be natural in my future work. But I know that it will also bring some thoughts. Throughout my PhD study, I have had challenges finding my identity as a researcher in a teaching practice I know so well. On the one hand, I feel like a practitioner and a part of the practice because of my experience with developing, implementing, and reflecting on teaching. On the other hand, my task as a researcher regards various pedagogical and didactical issues that arise in practice. I intend to point out or draw attention to the opportunities and challenges the problems give rise to. This is not always easy to do.

Alexander Jonas Viktor Selling

While recently completing my PhD, my role within the field of education has been multifaceted. From initially teaching at the upper secondary level as a mathematics and science teacher I was offered a position as a doctoral research fellow at the Quality in Nordic Teaching (QUINT) centre, where I was to research teaching quality in Nordic lower secondary mathematics classrooms. Following my position as a PhD student, I remained at QUINT as a researcher, paired with a role as a mathematics teacher educator. As a former teacher, and at present a researcher and teacher educator, I have fulfilled different roles and been exposed to different perspectives on the interplay between research and teaching practice.

In light of these different perspectives, I believe there are a few tensions that seem to arise in many cases. Firstly, the relevance of educational research that is being conducted seemingly fails at proving its relevance to the field of practice. This can be a problem based on the lack of exchanges of ideas between research and practice. That is, research is often not presented in schools, meaning that findings are not made readily available. Secondly, teachers' ideas of relevance are rarely integral to what research is conducted, thus the relevance of the research could be diminished in relation to the needs of teachers (e.g., Broekkamp & van Hout-Wolters, 2007; Rust, 2009). Thus, there seems to be a need for more collaboration between researchers and teachers. I have observed examples of this, for instance in applying observation manuals, commonly utilized as tools for conducting research, as a tool for teachers to use for professional development (e.g., Magnusson et al., 2023), but in a further stage, also by the teachers themselves as a tool for reflections among colleagues in schools.

Jóhann Örn Sigurjónsson

As a recent doctoral graduate, I have navigated many roles within education. I have taught at the university level since 2013, and I also have experience teaching at the lower secondary level. After a degree in computer science, I dove into PhD studies in educational science, focusing on teaching quality in mathematics across the Nordic countries as a fellow within QUINT centre, just as Alexander. After defending my thesis, I continued as a post-doc within the centre for one year. I currently serve as a specialist in mathematics education at the Icelandic Directorate of Education and School Services where we are developing a new national assessment system with an enhanced formative role in assessing learning progress and growth.

The link between research and practice in mathematics education will likely be a perennial topic and area of tension in my view. The reason is that not all researchers are teachers, and not all teachers are researchers – nor should we expect them to be. But some are in this intersection! And it is precisely in this intersection (between the set of researchers and the set of teachers within mathematics education) that the link can be strengthened by moving (some) research closer to practice, and practice closer to research. An example of this comes from my experience with using lesson video data within QUINT. Some colleagues have initiated collaborations between them as researchers with in-service teachers to watch and notice enactment of some core practices of teaching using lesson video libraries (Grossman & Fraefel, 2024; Magnusson et al., 2023). Although some of this work has been within language arts education, these avenues may be one pathway forward for reflecting on the research/practice relationship within mathematics education as well.

A three-dimensional quality of research perspective

Having introduced five contexts for reflecting on the research/practice relationship as spanned by the members of the plenary panel at the NORMA 24 conference, we now turn to sharing our thoughts and opinions about this relationship in a thematically structured manner. The structure comes from a discussion of the quality of research in general by Alan Schoenfeld (2007), with the implication that in this paper, we – as researchers in mathematics education contributing to a research conference – quite naturally look at the research/practice relationship from a research perspective. This is not to indicate that we find that more important than looking at the relationship from a practice perspective. In fact, we don't, as will be evident when reading the thoughts shared below.

In his analysis, Schoenfeld introduces three dimensions along which the quality of research can and – according to him – should be discussed, pinpointed by posing three guiding questions:

- *Importance*: Why should one care?
- *Trustworthiness*: Why should one believe what the author says?
- *Generality*: What situations or contexts does the research really apply to?

Schoenfeld then suggests the visual representation of the three dimensions depicted in Figure 1. Among other things, this is to indicate that the three quality concerns regarding research are mutually interdependent, and that each of them spans a continuum, not an either/or-dichotomy.

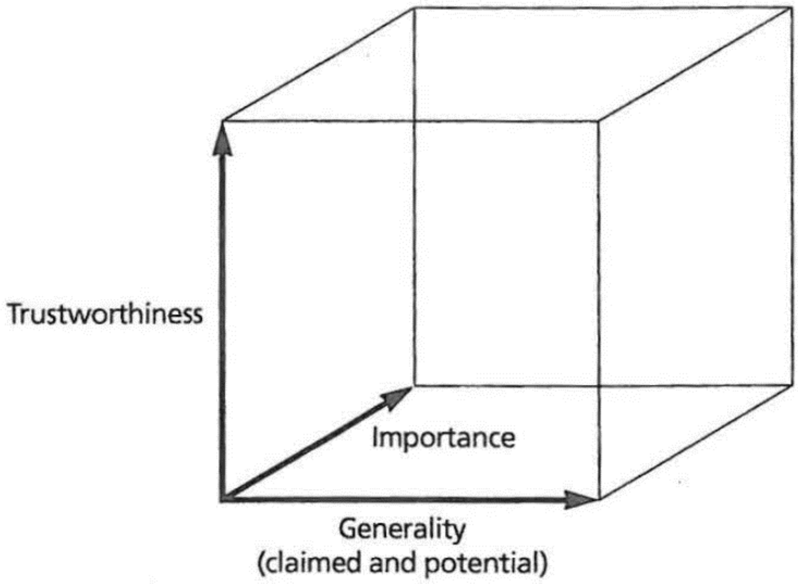

Figure 1: Three important dimensions along which research studies can be discussed and assessed (Schoenfeld, 2007, p. 82)

A reflection matrix

When discussing the research/practice relationship, we provide and have made use of the reflection matrix depicted in Figure 2. The matrix combines two suggested distinctions: Between reality and ideal when it comes to the research/practice relationship, and between how one sees the world when it comes to one's own research on the one hand, and when it comes to (mathematics education) research in general on the other hand.

Research versus practice	Your own research	Mathematics education research in general
What is the reality?		
How would you like it to be?		

Figure 2: A 2 × 2 reflection matrix, distinguishing between reality and ideal on the one hand and between one's own research and (mathematics education) research in general on the other hand

When combining Schoenfeld's three perspectives on the quality of research with our reflection matrix, we end up with the following questions to be addressed in the remainder of the paper:

How is/should your own research – and how should mathematics education research in general – relate to the research/practice relationship when it comes to the relevance and importance of research, the trustworthiness of the research and the generality of the research?

Relevance and importance: Why should one care?

This thematic part is guided by the following question: How is/should your own research – and how should mathematics education research in general – relate to the research/practice relationship when it comes to the *relevance and importance* of the research?

Anna: The answers to the question in the first column of Figure 2 will likely display a dilemma for many researchers as it is natural to wish for one's research to be useful even though it is very hard to measure what difference it makes. A lot of research is very specific in its inquiries, as my own for example, that investigate how teachers can support students when generalizing their equation solving skills to other numerical domains (Holmlund, 2024). Teachers can use my research directly only in a few lessons. But looking at my results, not on their own, but in relation to the gathered research of mathematics education, it contributes to the gathered knowledge on how students learn about numbers and algebra, and how we should teach mathematics. That is why the second column of the reflection matrix is so important, as my (and many others') research results gain importance and relevance when combined with other research results.

Now that I have done some research, I have learned that it is essential to promote it, both to teachers and researchers, so that it can increase in importance. I believe this might be even more important when you work alone on your project, as I did in the beginning, and there are no other project researchers to advertise it for you. One way to improve the relevance is to collaborate with teachers. If the teachers in the project find the research useful, it will most likely strengthen other teachers' interest. Da Ponte et al. (2023) suggest that the process of communicating the research to teachers in general is simplified by such collaborations as the research will already be partly adapted for practice.

Lóa: Even though the common aim of research in mathematics education must be to improve the practice of teaching, there are different perspectives which all play an important role as pieces in the big puzzle. When choosing the research focus for my PhD, I would have loved to work in the mathematics classroom, developing teaching practice aiming for flexibility, but there were a lot of unanswered questions. First of all, is there a problem to develop solutions for? In the case of Danish students' strategy choices and the relation to the curricular aims, there was knowledge to be gained about the actual situation. Both national tests in different grade levels and the final examination after grade 9 (age 15–16) are digital without showing the actual strategies used, but with possibilities to analyze the students' errors, which indicate differences between the intended curriculum and the received curriculum (Andrews, 2007). For me, and for the relevance of my study, it was important to first investigate the actual situation before studying possible interventions to improve an unknown but experienced situation.

The findings showed both limited flexibility/adaptivity and a strong relationship between students' level of adaptivity and level of flexibility and their general achievement in mathematics (Jóelsdóttir et al., 2023). These findings provide grounding for the importance of the next study: an intervention study aiming to answer new questions arising from the previous study. For example, how can students' flexibility/adaptivity be advanced, knowing that more flexibility/adaptivity is related to improved achievement.

Maria: It is essential for me that my research is relevant for practice, and I am aware that it also plays a role in mathematics and STEM education research in general. In the first part of my PhD (Møller, 2024), I sought knowledge about how STEM competence can be defined and characterized. I developed and described seven STEM competencies (Møller, 2022). This result is relevant for mathematics and science educators and researchers because it provides a framework for understanding STEM education. This relates to the second column in Figure 2. It mainly contributes to mathematics education research because it is a framework for how mathematics can contribute to the international political STEM agenda. In the second part, I worked with a team of mathematics and science teachers to develop, implement, and evaluate teaching to achieve one of the described STEM competencies: STEM data handling competence. The knowledge from this part shows how such teaching can take place and what opportunities and challenges there may be. This knowledge is relevant to practice, for example, in schools and teams of teachers interested in implementing competence-oriented STEM education.

For me, the knowledge I produce in my research must reach the stakeholders for whom it is relevant. Therefore, the dissemination of research plays a central role. Consequently, I believe that it is essential that research is communicated in different ways. For example, it is relevant to communicate through research articles that have undergone a formal review process. However, it is also essential to communicate through other channels, such as popular articles, podcasts, and conference presentations for researchers and practitioners. By doing so, we as researchers contribute to our research becoming and being justly perceived as relevant to more stakeholders.

Alexander: Overall, I believe that research should prioritise relevance to the field of practice, as educational research projects often require teachers and students to use time to engage and participate in the data collection. Some research that is important from a research perspective, like examining smaller aspects of a larger concept (e.g., instructional quality), might in an article seem dedicated to a very specific classroom practice, such as goal communication in my case, and thereby less relevant to a practitioner, as it does not consider teaching in a broader sense. Still, issues of overly focusing on practices that may not hold the same level of relevance to the participating practitioner has been described previously as a methodological issue of magnification (see e.g., Blikstad-Balas, 2017) and is thus something to be considered when selecting an area of research. Echoing Anna's point, promoting and presenting the research could be part in underscoring the relevance of the research, providing an opportunity to explain the relevance, as well as discussing it with practitioners. Similarly, collaborative research could increase relevance through allowing teachers to take part in deciding what is being studied beforehand, thus utilizing the perspectives of both the teachers and the researchers.

In my own research, I focused on goal enactment. While relatively rarely studied, some previous studies have indicated that goal communication is a valuable practice. Therefore, I found goal enactment to be an important topic both to teaching practice and teacher education. In addition, it adds to research on instructional quality, as a part of instructional clarity, thus I intended it to act as a part of the larger concept. Even so, access to this research might be limited to many teachers, as it is published in international journals, something many schools may not have access to, nor teachers time to read. This underscores the previous point of the importance of presenting research to teachers and schools, both in terms of bringing the research back to schools, as well as allowing for different perspectives on the findings and discussions.

Jóhann: A central question for relevance and importance in relation to my research is: Do we care about the quality of mathematics teaching? Most people involved in mathematics education would probably say that they do, but then some might question the exact meaning of quality in this context. Whose quality (cf. Elf, 2022) – quality as experienced, as prescribed, or as documented? More fundamentally, one could ask: Do we care about what mathematics is understood to be? In the concluding remarks of my dissertation (Sigurjónsson, 2023), I asserted that we still have ground to cover in fully understanding the way mathematics is taught and what teaching and learning mathematics is understood to be in a Nordic context. I would expect that those who care about these issues should find this work both relevant and important.

The topic of my dissertation was cognitive activation in mathematics teaching. Cognitive activation is defined as one of three basic dimensions of teaching quality (Praetorius et al., 2018). I explored it through both video observation of lessons using a systematic analysis of intellectual challenge and classroom discourse and through student perceptions. In the most recent paper based on this work, I focused on teacher-student interactions in the lessons with the highest levels across four Nordic countries (Sigurjónsson, 2024). Returning to Schoenfeld (2007), he mentions Bauersfeld, a pioneer in the interactionist point of view (i.e., seeing human interaction as intrinsic to learning mathematics), who defined teaching as “the attempt to organize an interactive and reflexive process” and “the establishing and maintaining of a classroom culture, rather than the transmission, introduction, or even rediscovery of pregiven and objectively codified knowledge” (1994, p. 139). Again, I would expect that those who care about these issues should find my research both relevant and important.

Response from Anna: In our panel discussion someone made a distinction between the relevance *before* the research and the relevance *after* the research. The initial arguments motivating certain research need to adapt to the results of the research as they bring new information on the topic. The relevance is then extended to involve how the actual knowledge can reach practice, as Maria’s comment above that it “must reach the stakeholders for whom it is relevant”, and new arguments for what is relevant research in the future is generated, as for Lóa and I that started by examining student’s knowledge which generated questions on how to improve practice.

Summing up

The above discussion serves to underscore that relevance and importance are indeed two main factors when conducting research. The discussion also highlights some key issues pertaining to the relevance and importance of the research. One such issue that is reported by several members of the discussion group is that research in education often tends to focus on very specific aspects of teaching or classroom situations, which can seemingly limit its direct applicability to practice. From an outside view, individual studies can thus be perceived as relevant only in narrow contexts, making it difficult for practitioners, e.g., teachers, to see their immediate value. However, as is underscored above, such studies are often a necessary starting point for a larger study and when various studies are conjoined, they can form a more cohesive and comprehensive picture, which can be more easily understood as important and relevant to the field of practice.

Another issue that is brought up several times, is that even when the larger picture has been produced, it is not always accessible to the field of practice. As such, it is pointed out by many in the group that communicating the research is essential, including targeting different audiences, both within the field of practice and the research community, through different channels. This can provide broader access to the research and help highlight its relevance and importance. Collaborations between researchers and teachers are also mentioned as one way of strengthening the relevance and importance. Such collaboration can serve to make research more relevant by being more closely aligned with the needs in the field of practice and by including perspectives from practitioners, but also make the research more easily communicated to teachers, as it becomes more closely aligned with practical needs (see e.g., da Ponte et al., 2023). As such, the above discussion presents some key issues in terms of showing the importance and relevance of mathematics education research, as well as in what ways this has been remedied, such as in presenting the research as a whole and could be further enhanced through communication and collaborations.

Trustworthiness: Why should one believe what the author says?

This part of the paper is guided by the following question: How is/should your own research – and how should mathematics education research in general – relate to the research/practice relationship when it comes to the *trustworthiness* of the research?

Jóhann: A central question for the trustworthiness in relation to my research is: Do we trust coding procedures of established observation systems and peer-review processes? First, regarding coding procedures, there have been some critical discussions on both interpretations of observation scores (White & Klette, 2023) and the benefits and challenges in using rubric-based observations for teacher development (White & Maher, 2024). Interpretation of scores is critical both for research work and for observation work in collaboration with teachers. For the instructionally-focused dimension that was the topic of my dissertation, cognitive activation, studies across many different countries have typically found it to be of low quality. One large-scale example is the Global Teaching Insights study, that used the term cognitive engagement (OECD, 2020). Across classrooms in eight countries/economies, all with observed lessons on the topic of quadratic equations, seven of them had a mean cognitive engagement score below 2.0 out of 4.0, with these seven countries having over 80% of classrooms with a score below 2.5 out of 4.0 (Japan was the outlier with a higher mean score in this dimension). Therefore, it may not come as a surprise that the Nordic countries show a similar distribution as England and Germany in cognitive activation.

Second, regarding peer-review processes, I can share a story of the second paper I wrote for my doctoral dissertation. In the paper, I explored the connection between observation scores and student perceptions of cognitive activation and generally found a rather weak connection (Sigurjónsson et al., 2022). This involved mostly statistical interpretation of observation scores in relation to classroom-level averages in a student survey. However, the initial draft of results included a sub-chapter where I provided qualitative descriptions illustrating the different groups of classrooms used for the analysis according to observation evidence. I viewed this as a way to enhance the trustworthiness of the results. In Schoenfeld's (2007) terms, I viewed the study's strength being in its descriptive power, rather than explanatory power. But when it came to finalizing the paper in peer-review, this sub-chapter of roughly 1100 words needed to be cut due to word limits. The moral of the story is not that peer-review processes are not to be trusted, but rather that sometimes limitations such as journal word limits bring on the need to sacrifice – in this case my ambition to increase trustworthiness. The silver lining here is that in my dissertation, I had the freedom to keep the approach of illustrative descriptions intact.

Alexander: Like Jóhann, I encountered issues of trustworthiness in relation to coding and observation manuals in my research. While observation manuals often are seen as valuable tools in video studies (Bell et al., 2019), there are many inherent limitations and critiques. For instance, the issue of rater error has been widely discussed (White & Klette, 2024). Such issues have few easy remedies and presenting scores generated by observation manuals to practitioners can easily make the analytical procedures seem like a "black box", thus possibly reducing trustworthiness. Moreover, observation manuals reduce the complexity of practices in order to make them observable in large scale studies, creating a risk of nuances being lost.

To alleviate such issues, I focused on providing rich descriptions of the data in conjunction with the scores in my own studies (e.g., Selling et al., 2024). In addition to providing an insight into the analyses of my studies, I believe such practices could raise relevance, by providing "inspirational" examples of both valuable practices and practices best avoided, which could be useful to teachers and as examples in teacher education.

While not a possibility in my project, allowing the participants access to the analyses and an opportunity to comment and discuss these, to ensure a joint understanding of what has been observed, could be a way of utilizing the researcher/practice relationship to raise trustworthiness. This again highlights the affordances provided by a greater focus on collaborations when conducting research. On the other hand, this also limits the amount of data which can be included in a study, due to time constraints and feasibility of allowing large numbers of participants to comment on data and analyses. Thus, it might be necessary to consider whether the purpose of the research requires more data or greater attendance to the perspective of the practitioners.

Maria: It is essential that research is taken seriously and appears trustworthy. Throughout my PhD, I have worked with this dimension in several ways. Firstly, I have taken an autoethnographic approach to collecting and analyzing data. This means that I have chosen to involve myself actively. For example, I am aware that my experiences as a teacher in primary school and teacher education, and the knowledge I have gained through my previous work in research and development, influences the methodological and analytical choices I make throughout my research process.

Secondly, I pay particular attention to being transparent throughout the research process. This applies to how I collect my empirical data and subsequently document and analyze it. For example, I make visible representations of how and in what order texts in a review have been included. Likewise, I am thorough in how qualitative interviews and observations are conducted and documented. In my analysis, I describe how I have gotten from the empirical data to concrete situation descriptions. To be transparent, I explain in detail how I analyze the situation descriptions to get to the didactic issues. I do this, for example, by stating that in my analytical approach to the empirical data, I was particularly interested in situations that did not work the way we (the teachers and I) had expected. This was expressed by the fact that I experienced surprise or wonder. This kind of transparency is quite different from what Jóhann describes when he talks about coding procedures, observation scores, and interpretation of the scores across countries. The trustworthiness of my research result depends on whether the reader trusts me and how I handle my empirical data. One concrete way to address that is to create visual representations to make the analysis process as transparent as possible. For example, I have made a table of which articles I have included in the four iterations of my hermeneutic review (Boell & Cecez-Kecmanovic, 2014). Another example is making a table of the format and scope of the empirical data I have collected in and with practice, and yet another table of the empirical situation descriptions that contribute to the didactic issues. These visual representations contribute to increased transparency and thus strengthen the credibility of the research.

Lóa: When communicating and discussing the findings of your research, everyone you communicate with has a different background, both regarding experience and knowledge. I had the opportunity to do my PhD in a research environment that is separate from the teacher education environment I have been employed in since 2009. I did my PhD at *Trygfondens' Center for Child Research* working with professors with a background in educational economics and political science, as part of the School of Business and Social Science, Aarhus University. At the same time, I continued working with my research group from teacher education and my network from practice.

What I experienced was that there were different questions about trustworthiness depending on which group I was communicating with. First, when meeting the teachers from practice, discussing my research and findings, the questions often focused on how representative the choice of participants is, and if data was new (as the situation might have changed). In this case it helped that the participants came from 121 different classes, in 20 different schools from 5 different municipalities, both bigger and smaller, rural and urban.

Second, when meeting colleagues from mathematics education the main questions were about definitions of the concepts and the clear protocol of the coding. The core concepts as flexibility and adaptivity, which are central for interpreting the findings of the study, are found in the literature with variety in the definitions (Verschaffel et al., 2009). A well-defined conceptual framework, and a coding process reflecting the concept definition, became an important element in the discussion of trustworthiness.

Third, when presenting the findings with the group at the research centre, the questions often related to the methodology, the statistical analysis, control variables and similar elements from the study. The N of 2298 students, which often is seen as a big sample in educational research, was not so big in their mind, and different but equally important questions about classroom effects, school effects, gender and so on played a central role in the question of trustworthiness.

Anna: I have also addressed both theoretical and methodological issues of trustworthiness in my research, such as "why I use a certain definition of the term structure" or "how students were selected for interviews". My personal experience is that these questions often come from other researchers and not from practitioners.

What has surprised me as a young researcher is how *the writing* of research articles affects how your research is interpreted and how decisions about its presentation are left to the researcher. For example, to some extent researchers are free to choose what previous research to include in an article, what aspects of the results to display and what conclusions to draw and not to draw – while still being true to the collected material. The results are trustworthy in a methodological way, but many choices have been made about how to present it.

It is also an art to describe the essence of the research results, not describing too much and making it irrelevant, or too little and enabling miscomprehensions. Schoenfeld (2007, p. 93) states that a "major challenge in the conduct of educational research is the tension between the desire to make progress and the dangers of positivism and reductivism". These difficulties of presenting research do not increase the trustworthiness of our research community when we approach practice, but these are the conditions we have. Therefore, opportunities to discuss these decisions, including the peer-review processes, are crucial.

Summing up

The discussion on trustworthiness in mathematics education research highlights concerns about the reliability of methods such as coding systems, observation manuals, and the peer-review process. Jóhann, for instance, observed low cognitive activation scores across countries, including the Nordics, and faced peer-review constraints when he tried adding qualitative data for transparency. Observation manuals, as Alexander notes, can simplify classroom practices, risking loss of nuance; he suggests using detailed descriptions and involving participants in data analysis, while acknowledging practical challenges. Maria emphasizes transparency by using tables and visual aids to clarify her data analysis, enhancing credibility.

Trustworthiness perspectives also vary with audience: Lóa found that practitioners value participant diversity, mathematics education peers prioritize conceptual clarity, and researchers in educational economics focus on sample size and control variables. Finally, Anna addresses the impact of researchers' choices in presenting findings, noting the tension between depth and simplification, and underscoring transparency in peer-review processes.

This multidimensional view on trustworthiness highlights that in mathematics education research, trustworthiness relies on clear communication and transparency in methodology.

Generality: What situations or contexts does the research really apply to?

This part of the paper is guided by the following question: How is/should your own research – and how should mathematics education research in general – relate to the research/practice relationship when it comes to the *generality* of the research?

Maria: The analytical themes and the didactic recommendations that emerged in my PhD have the character of an existence proof, and therefore, they cannot be interpreted as general. However, they can be seen as having potential. It may turn out that over time, they acquire "potential significance" and point to aspects or phenomena that are potentially interesting and that can be refined over time through a growing body of research. Therefore, there is naturalistic generalizability in the results (Kvale & Brinkmann, 2015) that builds on deep explanations and moves tacit knowledge to more explicitly assumed knowledge.

My PhD also includes descriptions of concrete situations from practice followed by analytic arguments that justify and relate why and how the partial conclusions can be transferred to other situations and contexts. In this way, specifying the supporting evidence and making the argumentation explicit supports another kind of generalization, the analytical (Kvale & Brinkmann, 2015), which focuses on allowing the reader to assess to what extent the conclusions are generalizable to other situations.

Jóhann: Since my PhD study was based on classroom observations from Iceland and other Nordic countries, a central question in relation to my research is: Are the findings meaningful outside of the Icelandic or Nordic context? The general database does provide rich qualitative descriptions along with a quantitative overview of all (~160) mathematics lessons. However, a quantitative representation may sometimes give an "illusion of total generality" that is worth being mindful of.

In my paper on the high-level lessons in cognitive activation, I provided descriptions of instructional formats and teacher-student interactions (Sigurjónsson, 2024). One could argue that this may have served (in Schoenfeld's terms) as an *existence proof*. That is, lessons with high cognitive activation *do* exist in the Icelandic and Nordic context, and this is what they look like. They are quite different from the "typical" norms and patterns of teaching described in multiple previous studies. Given the evidence (from other contexts) of educational benefits for students receiving instruction with high cognitive activation, this suggests pathways to develop teaching in Iceland, and possibly, in a Nordic context. In that way, I would argue that given its context of viewing multiple classrooms from multiple neighboring countries, it may specifically apply to that context, even if they may be informative for similar classroom contexts.

Anna: Looking at the generality of my research, I have found it harder to gather and draw conclusions from the quantitative data than the qualitative data. In a learning study with vocational students solving equations, it was difficult to collect the data as some of the students rarely attended all the lessons of a school week, and thereby missed either the pre-test or post-test or the research lessons. As these sorts of difficulties result in small datasets, it is naturally harder to draw conclusions of generality in a quantitative way.

Regarding the generality of the qualitative conclusions, I find the topic of my thesis – how some students change their way of thinking when solving equations with decimal or negative coefficients involved (Holmlund, 2024) – relevant for teachers and students at different levels and in different countries. Hence, it has the potential to be generalised to many settings. In line with previous mentions of the existence proof, I agree with Bakker (2018, p. 41) that writes about research findings (in design research) as "generalizable if they are transferable to other situations, where they have to be adjusted to local circumstances".

Alexander: As my research utilizes video data from the Nordic countries, one primary issue is one mentioned by Jóhann: is the research valuable beyond the Nordic context? Certainly, there can be different understandings of what constitutes high quality teaching (Praetorius et al., 2019), meaning that practices could differ across contexts. Therefore, I found it important to rationalize why the study, in this case goal communication, could be important beyond the context of the research, and to situate it in the international context. -Similarly, I addressed whether the goals in the Nordic curricula could be comparable to goals in other contexts, and mathematics education internationally does target many of the same overarching goals. As such, while the study is by no means generalizable, it could provide insights valuable to other contexts and situations.

Thus, I argue that even when studies are situated in a specific context, like those described in this manuscript, they can provide information that is important in general, provided that the general importance is well described.

Lóa: Cultural differences, different curricula (intended, implemented and received) and even different opinions about the aims of mathematics teaching and learning in school practice will all affect the generality of studies like my PhD project, where the aim was to investigate Danish students' strategy use in different grade levels.

To reflect on this topic, an important part of my research process was to compare and contrast different findings from international studies on similar topics. The findings indicate that Danish students' strategy flexibility is limited, yet similar to Dutch students that share a similar reform-based curriculum (Hickendorff, 2018). As previously mentioned, Danish curriculum guidelines do recommend developing number-based strategies, while in other countries a standard algorithm often is the main focus in the mathematics teaching practice, even when aiming for adaptive expertise.

Comparing results across cultures is not the only challenge. Another limitation is different measurement tools which might affect the results. In the case of flexibility/adaptivity, several studies rely on the Choice/No Choice method which needs individual settings and focus on time and accuracy (Luwel et al., 2009). In my study, I use a classroom setting and do not focus on time, but only the task-related adaptivity, and an assessment adapted from Xu et al. (2017), which studied flexibility/adaptivity of students solving linear equations.

When generalising the results, all the limitations mentioned above should be considered. At the same time, results from different cultures give opportunities for extended understanding of the topic. For example, research findings show a strong relationship between developing adaptivity and general achievement in mathematics. These findings are seen across different cultures and thereby provide a deeper understanding about student's learning of mathematics. The opportunities to generalise are in the findings but must be carefully discussed in relation to differences in methodology and mathematical culture.

Summing up

In educational research, generality is complex and requires careful consideration. Naturalistic generalization moves from tacit knowledge to explicit assumptions, building on deep explanations. Both Anna and Maria state that transparent arguments allow readers to evaluate the generalizability of conclusions. When empirical data is collected from various contexts, such as different countries, thorough descriptions of those contexts are essential. Research findings, especially in design research, are considered generalizable if they can be applied to other situations, provided they are adjusted to local circumstances.

We agree on a naturalistic and analytic understanding of generality for qualitative studies, and Jóhann comments that in quantitative studies, significant contextual differences can exist, and researchers must be cautious of the "illusion of total generality." Lóa points out that research from different cultural contexts offers opportunities for deeper understanding and highlights the significance of context in educational research. By acknowledging these nuances, researchers can better navigate the complexities of generalization.

A bird's eye recapitulation

As has been evident from the above, this is not a report of a research study, like most conference papers in this and other proceedings are. Hence, there have been no sections about methods, results, or other headings from a classic structuring of a research paper, and we will not finish off with an answer to a research question. Instead, we will recapitulate a main impression from our discussions at the NORMA plenary panel and the elaborations of it in this paper.

As described above, we have structured our discussions according to a three-dimensional quality of research model from Schoenfeld (2007). We have used the model to pose and address the following questions: How is/should your own research – and how should mathematics education research in general – relate to the research/practice relationship when it comes to the

- *relevance and importance* of the research? (" Why should one care?")
- *trustworthiness* of the research? (" Why should one believe what the author says?")
- *generality* of the research? (" What situations or contexts does the research really apply to?")

Our impression is that for a couple of reasons we believe this approach to a discussion of the research/practice relationship in mathematics education makes good sense. Firstly, because the three questions from Schoenfeld's model (in brackets above) seem quite fundamental to educational research in general, and therefore worth addressing as part of any mathematics education research study. At least, we have enjoyed and benefitted from giving ourselves the challenge and opportunity to discuss them among us.

Secondly, because each of the three fundamental questions has proven to be a relevant and thought-provoking perspective on the research/practice relationship. When discussing them, it has been very clear to us that research in mathematics education can raise both its relevance and importance, trustworthiness, and generality. Those conducting future research would be well to consider how to involve and attend to the practice of mathematics education, and the ways suggested in this panel discussion have often been in favor of quantitatively and qualitatively more involvement, not less.

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Pre-service teachers' proving praxeologies

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The paper aims to raise awareness of possible constraints in mathematics teacher education regarding the topic of proof. Reasoning and proving is emphasised in school mathematics and is thus a focus in teacher education. In this paper, I analyse 80 pre-service teachers' (PSTs') exam answers to three proving tasks on a compulsory national exam for primary teacher education in Norway. The results emerge from using the Anthropological Theory of the Didactic (ATD) as a theoretical lens. Whilst some findings are recognised from previous research on PSTs' struggles with proving, other findings indicate that challenges exceed PSTs' lack of knowledge.

Keywords: Proof, pre-service teachers, teacher education, ATD

Introduction

Reasoning and proving is an important aspect of mathematics in most national curricula (e.g., NCTM, 2000; Kunnskapsdepartementet, 2019), as encouraged by researchers in mathematics education (e.g., Hanna, 2000; Stylianides, 2007). However, the idea of teaching reasoning and proving to all students, at all levels, is relatively novel (Balacheff, 2024). And it does not come without a cost: Numerous studies describe students' challenges with reasoning and proving. It is the responsibility of teacher education institutions to prepare mathematics teachers to meet the demands of the curricula they will teach in school, so also for the topic of reasoning and proving. However, research shows that learning to teach reasoning and proving in teacher education is also demanding (e.g., Stylianides et al., 2017). In this paper, I focus on Norwegian primary teacher education (for grades 1–7), where content and didactics are treated simultaneously. The empirical data that informs the study comes from a compulsory exam that is given to all Norwegian primary pre-service teachers (PSTs) during their undergraduate years. I analyse the proving techniques the PSTs used on the proving tasks in the exam, as well as seek to uncover how the proving techniques were explained and justified.

As a theoretical framework, I employ the Anthropological Theory of the Didactic (ATD) (Chevallard, 2019). The ATD comes with assumptions and considerations that enable a holistic perspective on teaching and learning. Moreover, it provides a functional model for knowledge, praxeologies, that is used in the analysis. The research question guiding the study is *What praxeologies related to proof are visible in the PSTs' answers to the exam questions on proving?* Within the ATD, knowledge is seen relative to a larger social infrastructure. Thus, the findings are further used to identify and discuss potential constraints regarding the study of proof in Norwegian primary teacher education.

Theoretical background

A central focus of the ATD is how knowledge is transformed between institutions, in particular institutions of teaching and learning (Chevallard, 2019). An *institution* in the ATD is "any created reality of which people can be members" (Chevallard & Bosch, 2020, p. xxxi). In this study, the knowledge under scrutiny is proof and proving in mathematics, which has originated and evolved among mathematicians. Here, however, I consider how this knowledge "lives" inside the institution of primary teacher education in Norway. Knowledge that has moved from one institution to another is "transformed, deconstructed, and reconstructed in order to adapt to their new institutional settings" (Chevallard & Bosch, 2014, p. 215). This process is called *institutional transposition* (Chevallard & Bosch, 2014). In teacher education, proof is studied in ways that are considered relevant for PSTs' future teaching profession, which is inherently different to, e.g., how it is studied by undergraduate mathematics students at university.

To operationalise knowledge, I use the ATD's notion of *praxeologies*. Any human action can be modelled as a praxeology, which consists of four elements: the task (what did the action seek to achieve?), the technique (how was the action carried out?), the technology (to explain and justify the technique), and the theory (more global explanations). These four elements are sorted into two pairs: The *praxis* (task and technique) and the *logos* (technology and theory). It is an underlying assumption of the ATD that both praxis and logos are always present in any human action, although they are not necessarily detectable to the observer. "Tasks" are interpreted in a wide sense: To open a drawer is a task, and so is to prove a theorem. In mathematical praxeologies, proofs are often associated with the technology element, being part of what justifies and explains a given mathematical procedure (technique) (Chevallard, 2019, p. 93). For this study, however, I am interested in praxeologies where the actual actions of proving constitute the technique, as the tasks under scrutiny are proving tasks.

Another type of praxeologies that is of interest in mathematics education research is *didactic praxeologies*. They consist of the tasks of teaching, the techniques teachers use, and their associated logos (García et al., 2020). "Teacher knowledge" can be modelled in the ATD as teachers' mathematical and didactic praxeologies (García et al., 2020). Since teachers' mathematical and didactic praxeologies are intertwined (Pansell, 2023), I assume that a teacher's or PST's praxeology is not necessarily either mathematical or didactic, it can be a combination. Thus, the PSTs' answers to the exam tasks on proving may include elements of their didactic praxeologies as well as their mathematical praxeologies.

Methodology

This qualitative study takes a phenomenological approach, where the phenomenon under scrutiny is how PSTs engage with proving tasks. ATD's praxeological analysis fits well with this qualitative approach because praxeologies are situated within institutions. Thus, it makes sense to assume that there are common features in the participants' praxeologies related to the phenomenon. Data comes from answers to the Spring 2023 national assessment examination in teacher education for the grades 1–7 teacher programmes. This 5 ECTS credits exam is mandatory for all 1–7 pre-service teachers, which is given either in the first or second year, according to the university's programme schedule. The Norwegian Agency for Quality Assurance in Education (NOKUT) facilitates the exam, which is compiled by a group of five mathematics teacher educators representing different universities. The author of this paper has been a member of the group since 2022. The recurring topic of the exam is *algebraic thinking* from a didactic perspective, including reasoning and proving. Since the institutions are free to choose readings for their teaching leading up to the exam, the exam tasks can not require knowledge of any specific curricular resources. Moreover, admission to primary teacher education in

Norway does not require mathematics education from upper secondary school exceeding the mandatory minimum, thus the exam does not contain mathematical content beyond the lower secondary curriculum. An emphasis on reasoning and proving in the Norwegian curriculum (grades 1-13) was not added until 2020 (Kunnskapsdepartementet, 2019), i.e., after the PSTs in the study finished their primary and lower secondary education. Thus, it is safe to assume that in most cases, the PSTs' demonstrated knowledge in answering the proving tasks was acquired in teacher education.

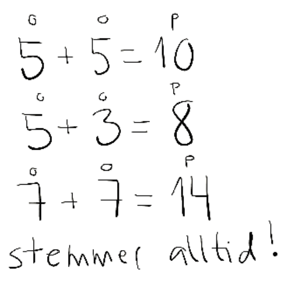
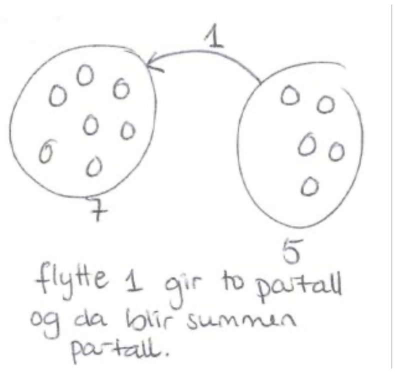
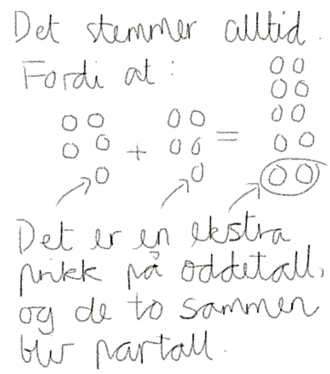
The data for the study is 80 exam answers from across universities. I have analysed the answers to all tasks related to proving in the NOKUT Spring 2023 exam (Figure 1). The process of analysis was carried out by first searching for the praxis, and then for the logos. The *tasks* are explicitly given as the three exam items. All the tasks can be considered mathematical tasks. However, the two last tasks also have a didactic flavour, as Task 5a relates to the didactic task type of assessing students' work, and Task 5b relates to the didactic task type of building on students' work. Yet, one could argue that mathematical praxeologies only, and not didactic, are sufficient to solve the tasks. In the task, Student 1 proposes an exclusively empirical argument, but the two other arguments build on some structural idea, yet can be criticised for a lack of generality or assumptions. Thus, the task was quite open.

Task 1a. Decide and justify whether the claim is always true, always false or sometimes true: *If we take an even number and add half of the number, the answer is always divisible by 3. For example, 2+1=3, and 3 is divisible by 3.*

Task 5. Middle-grade students are asked the following question:

Is it *always, never* or *sometimes* true that the sum of two odd numbers is an even number?

Below you see three student answers:

<table>
<tr><th>Student 1</th><th>Student 2</th><th>Student 3</th></tr>
<tr>
<td>
</td>
<td>
</td>
<td>
</td>
</tr>
<tr><td>always true!</td><td>moving 1 gives two even numbers so the sum is an even number.</td><td>It's always true. Because that:
It's one extra dot on the odd numbers, and those two together make an even number</td></tr>
</table>

a) Choose two student answers that you consider *not* to be a valid proof that the sum of two odd numbers *always* is an even number. For each of the two student answers, give one reason for why it is not a valid proof.

b) Choose one of the three student answers that you think builds on a correct idea. Complete the student's answer so that it becomes a valid proof that the sum of two odd numbers always is an even number.

Figure 1. The three tasks (1a, 5a and 5b) for which answers were analysed (NOKUT, 2023)

To identify *techniques*, I have chosen to compare PSTs' proving techniques with those identified by Stylianides (2008). Stylianides' framework consists of four types of arguments, here called techniques: Demonstrations and generic examples (proofs) and empirical arguments and rationale (non-proofs). A priori constructed examples of how the first three techniques, which are well-described in the literature, can be used for Task 1a are given in Figure 2. The role of the rationale in the framework is to "capture arguments for or against mathematical claims that are neither proofs nor empirical" (Stylianides 2008, p. 12), which I interpret as incomplete arguments that point to some potentially relevant mathematical properties. I used the same analytical framework to classify proving techniques in Task 5b, although here the PSTs' freedom was restricted because they were starting from one of the student's answers. In addition, I have paid attention to how the PSTs used examples in their arguments.

$\tau_{d,a}$ (demonstration, algebraic): Any even number is of the form $2n$ for some n, and thus an even number added with its half is $2n + n = 3n$ which is divisible by 3.

$\tau_{d,v}$ (demonstration, verbal): Any even number consists of two equal parts. Half of an even number is the same as one such part, and if we add it we get three equal parts, so the sum is divisible by 3.

$\tau_{d,f}$ (demonstration, figure): An even number can be divided in two equal parts, so if this is an even number: □□ we get the sum □□+□=□□□ which is three times □.

τ_g (generic example): 8 is an even number, and half of 8 is 4. So $8 = 4 + 4$. If we add half of 8 we add another 4 so that gives $12 = 3 \cdot 4$. This will be similar for any even number, since you get three times the half of it. So the claim is always true.

τ_e (empirical argument): $8 + 4 = 12 = 3 \cdot 4,\ 10 + 5 = 15 = 3 \cdot 5,\ 20 + 10\ = 30 = 3 \cdot 10$, it always works.

Figure 2. Possible techniques for solving Task 1a. All but τ_e are mathematically valid

To identify and reconstruct the logos, I have chosen an inductive approach, partly because the tasks were not designed to elicit particular technologies and theories and partly because little is known about PSTs' justifications for their proving techniques. Here, a *technology* is related to justifying or explaining a particular proving technique, while a *theory* is a more global conception or property about proofs and proving. Whilst the PSTs' choice of proving techniques on tasks 1a and 5b can only occasionally indicate elements of the logos, their choices and justifications on tasks 5a and 5b are likely to reveal for example what arguments the PSTs consider valid and why/why not.

Findings

Praxis: The PSTs' proving techniques

I give an overview of the proving techniques used by the PSTs on Task 1a. The proving techniques used on Task 5b are of a similar nature and are only briefly summarised at the end.

Less than half (36 out of 80) of the PSTs presented valid proofs on Task 1a. All except one were *demonstrations*: 18 were algebraic, 9 were verbal arguments, and 8 used figures together with a verbal explanation. Only one PST provided what I have recognised as a *generic example*. However, 12 out of the 35 PSTs who gave valid demonstrations also included examples. The function of most examples seemed to be to illustrate that the claim was true. Yet, some (5) used examples to highlight the key idea. The border between such use of examples in an argument and generic examples is not watertight, but there is a subtle difference between using the example to establish the key idea and then generalizing, and establishing the key idea in general terms and then exemplifying it.

The remaining 44 PSTs presented *non-proofs*. Half of them were *empirical arguments*, mostly in the form of some examples that the claim holds, and a conclusion. However, the number of examples varied from one to 11, and some used different examples, e.g., big numbers, small numbers, and some used subsequent numbers. Some also noted a pattern, e.g., sums increasing by 3, or that you get half of the even number when you divide by 3, without attempting to justify it. Further, there were four *rationales*. The remaining 18 non-proofs were attempts at demonstrations which failed due to the wrong application of algebraic rules or that the PSTs used or added wrong definitions or assumptions, often causing them to asses the claim as "not true", and some missing an argument altogether.

The proving techniques used on Task 5b have an overweight of (attempts at) algebraic representations of the sum of two odd numbers, intending to generalise Student 3's argument. As on Task 1a, many of the PSTs fail at using algebraic notation.

Logos: The PSTs' technology and theory related to proving

The large number of (attempts at) algebraic demonstrations on Tasks 1a and 5b suggests that PSTs strongly associate this technique with proof. However, for Task 5b, it could also be that the PSTs see the use of algebraic notation as a natural next step in Student 3's learning trajectory, as indicated by a PST's comment: "next step for the student could be to make a formula." The many empirical arguments could indicate that some PSTs consider this a valid technique, but this will be further discussed. However, I consider the awareness of using suitable examples, as shown in many empirical arguments on Task 1a, to be an element of the technology for this (yet invalid) proving technique. Next, I turn to the PSTs' reasons for their choices on Tasks 5a and 5b.

Almost all PSTs (74) chose Student 1 on Task 5a, and most of them (70) justified their choice by saying that the student only gave examples. Others pointed to the lack of explanation, for instance: "Student 1 doesn't really give a proof, the student just shows 3 different calculations with odd numbers, and points to that these give even numbers. The student doesn't explain how s/he found out that it is always correct by pointing to 3 quite small calculations. Miss an explanation from the student." Another PST missed the use of manipulatives (likely to mean a visual representation), and one said that the student does not show his thinking. All these objections to Student 1's argument are plausible, but whilst the objection pointing to the insufficient use of examples is of a mathematical nature, the others reflect didactic justifications for turning down Student 1's argument.

Student 2 was also chosen on Task 5a by a majority (72) of the PSTs. Of them, 19 pointed to what can be characterised as deductive mistakes or the use of unaccepted statements in the student's argument. Examples are that Student 2 ends up not adding two odd numbers, but two even numbers; and that Student 2 would have to show that the sum of two even numbers is even. Further, six PSTs replied that Student 2's drawing is not structured so that it is clear that you get two even numbers, for example: "The argument could have been strengthened by saying like Student 3 that there is always an extra dot on even numbers, and they can make a pair together, but to say that you just move one dot and get two even numbers is not a general proof that it holds for any odd numbers." Two PSTs pointed to a lack of generalization in the argument. I regard all the above comments as elements of the PSTs' technologies explaining the missing steps in Student 2's technique.

Notably, 37 of the PSTs who chose Student 2 on Task 5a, gave the same justification for turning down Student 2's answer that dominated for Student 1: That the argument is example-based and thus not valid. Some of them added to their justification, e.g., issues described in the previous paragraph, that Student 2 makes a better attempt than Student 1, or that Student 2 has a "nice strategy". Still, many did not add more. Moreover, of the twelve PSTs who chose Student 3 on Task 5a, seven gave the same reason "the student only shows the claim for one example" as justification for it not being valid. The common use of this justification strongly indicates that the PSTs *are* aware of the insufficiency of empirical arguments. I relate this to the theory part of the logos because this is seen across techniques and seems to be a "global belief" about proving.

Finally, I consider the PSTs' justification for their choice in Task 5b. Only about half of the PSTs were explicit about what is lacking in the students' argument. 20 of them pointed to a lack of generalization, which is dominantly attempted solved with algebra and/or generalizing the figure, and we can guess that this is the underlying reason for many other PSTs as well who used similar techniques to complete the proof. This is a strong indicator that the PSTs see "generality" as a part of what a proof should include, which I also relate to the theory part of their praxeologies.

Discussion and concluding remarks

Some of the findings strengthen what is known about PSTs' proving praxeologies: Many PSTs struggle with relatively simple proving tasks (like Task 1a), and demonstrations—often algebraic—seem to be a preferred proving technique (Doğan, 2020; Kempen, 2018). Moreover, the PSTs frequently used empirical argumentation, and some labelled Student 2's argument "empirical" without recognizing that the argument held something more. The findings suggest that the PSTs' knowledge of proof can be developed further. Within the theoretical framework of the ATD, however, we should also address the possible conditions and constraints of teacher education at a higher level and not reduce challenges to a matter of the PSTs' incompetencies. In particular, we can consider what characterises the body of knowledge (i.e., praxeologies) concerning proof and proving in Norwegian primary teacher education institutions, and the consequences of the transposition of knowledge that has taken place, in light of the PSTs' future positions as primary mathematics teachers.

Indeed, the PSTs' answers to the exam tasks analysed in this paper (mainly) reflect their mathematical *and* didactic praxeologies acquired in teacher education. Many of the PSTs included logos elements that are better described as didactic than mathematical, like emphasizing explanations, students' learning trajectories, and labelling students' strategies as "nice". I hypothesise that to call Student 1 and Student 2's arguments "empirical", *may* be a didactic rather than mathematical justification—that is, the PSTs know that they are not supposed to accept empirical argumentation from their students, a point likely to have been stressed in their teacher education courses. The fact that many PSTs do not know the mathematical difference between an empirical argument and a (however incomplete) generic example supports this claim. I suggest that this observation exemplifies how, in the case of proving, the PSTs' mathematical and didactic praxeologies are intertwined (Pansell, 2023), even to the extent that the PSTs might not know the difference.

Many of the PSTs involved in this study will work as primary school teachers, probably teaching aspects of proof to their students. In that situation, another institutional transposition has occurred: The praxeologies of proof and proving are likely to behave differently in a primary school classroom than in teacher education, because the conditions under which they live are different. Any praxeology has its *raisons d'être* ("reasons for being") that justify why this piece of knowledge appeared, and to what cause; yet "it often occurs that the raisons d'être at the origin of most praxeologies disappear with time" (Bosch & Gascón, 2014, p. 70). Proof plays several roles in mathematics and thus has a complex raison d'être, which will also be manifold when transposed into education, where verification and explanation have been emphasised (Hanna, 2009). However, from the previously described confusion of mathematical and didactic proving praxeologies, I suspect that many reasons for (the teaching of) proof and proving in mathematics could be obscured to the PSTs, and thus represent a constraint for their teaching of proof.

I end this paper with some methodological remarks. First, the data for the study is limited, because as written exam answers to a very situated examination, the data do not provide first-hand insight into what goes on in teacher education. Yet, NOKUT exam data provides interesting insights across universities, and is thus a way of accessing "the institution of Norwegian primary teacher education". However, it is also possible to define an institution based on the NOKUT exam itself, because its six years' lifespan has, arguably, generated its own discourse. Taking this view could indicate that the limitations of empirical arguments is also something PSTs acquire through working on previous NOKUT exams. Furthermore, identifying and analyzing logos is a difficult task, and for this study, I have taken the pragmatic decision to ignore that PSTs' justifications on task 5a can be seen as a technique and not a technology. In particular, if we consider the institution under scrutiny to be that generated by the NOKUT exam, rejecting a student argument based on its empirical features could be seen as a technique, with associated logos "that is how we are supposed to answer on this exam". Further studies is needed to investigate such possible, yet unintended effects of the NOKUT exam.

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“I drew myself.” The Interplay Between a Reasoning-based Discourse Strategy and Primary Students' Mathematical Identity

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Mathematically rich learning environments often use discourse as a means of promoting learning. However, when students do not see themselves as someone who can be successful in mathematics, that is, when they have a negative mathematical identity, meaningful engagement in exploring mathematical relationships is not likely to happen. Knowing how to support, develop, and foster a positive mathematical identity in students is thus important in students' current and future academic achievement. The purpose of this study was to understand the extent to which a discourse strategy that focuses on reasoning helped students understand what it means to be a mathematician and do mathematics as reflected in their drawings of a mathematician. Initial data indicates students' perceptions of themselves as student mathematicians positively changed and were more reflective of their own gender, age, and appearance or setting.

Keywords: Discourse, Identity, Primary Grades.

Introduction

Developing students' mathematical identity is central to their success in doing mathematics (Aguirre et al., 2013). Part of one's mathematical identity is how they see themselves and how others view them in relation to doing mathematics; it is a crossroads of beliefs dealing with several different constructs and variables. Lock et al. (2013) indicate that some of these variables include being recognized as one who can do mathematics, one's own belief with respect to their performance when faced with challenging mathematics, having a curiosity and inquisitiveness about exploring mathematical relationships, a belief in one's competencies to arrive at a solution and the interplay of these beliefs with current and past cultural experiences (Gutiérrez, 2009).

However, building up and supporting this identity also relies on meaningful mathematical interactions that allow students to experience what it means to do mathematics and be a mathematician (Boaler & Greeno, 2000). The nature of the content and the ways in which students interact with mathematics can take many forms but one of these ways is through whole-class discourse. Mathematical discourse includes the various ways students talk, listen, and interact with each other as well as how they feel and think about what they are studying (Grifenhagen & Barnes, 2022). Thus, learning to be a student mathematician is more than being proficient on exams. Students must also see themselves as an equal member of the learning environment. In doing so, they build stronger connections to the learning and see themselves as being more capable. All of which help develop a strong mathematical identity.

With that said, some students face additional challenges that inhibit their willingness to talk about mathematics. For example, some students who academically struggle may not believe they are capable of succeeding in mathematics, some students come from immigrant or migrant families and are still learning the language of the host nation, and some have learning disabilities that require more time to process information. As a result, the equal participation in mathematical discourse is not the same for all students (Bennett, 2010). So, while exploring complex mathematical relationships in a collaborative and discourse-rich environment represents an ideal learning interaction, not all students equitably engage in such interactions.

Theoretical Framework

This study is theoretically framed within two key constructs. First, engaging in discourse is central to developing students' mathematical habits of mind and interaction (Stentoft & Valero, 2009). Second, when students engage in mathematical discourse, as a normative classroom process, their identities of being a mathematician change (Cobb et al., 2009). Furthermore, the change in these identities can be seen not just in the ways in which students verbally or visually communicate their thinking and understanding of mathematics in relation to others in the classroom (Radovic et al., 2017). The changes can also be seen in the ways in which they illustrate their perceptions of one who does mathematics; namely, a mathematician. Building on the work of Ganesh (2011), who highlighted how students' drawings of engineers and scientists can be used to understand how students view themselves with respect to these professions, students' personal mathematical identities can be captured in how they draw mathematicians. Through the process of creating and illustrating another person, and teachers' critical inquiry (Jaworski, 2006) as a means of positively shaping students' mathematical identities, children highlight their own projected interpretations of themselves. Thus, the research question for this project is to determine the extent to which students' drawings of mathematicians change over time to more accurately represent themselves as student mathematicians.

Review of Literature

Mathematical identities are complex, nuanced and depend on many different personal, socio-cultural, and socio-linguistic elements (Goldin, et al., 2016) all of which are deeply entwined in learning mathematics (Aguirre et al., 2013). Additionally, students' mathematical identity builds on their past interactions in doing mathematics and as Crossley and colleagues (2018) indicate, students with a strong mathematical identity persist when faced with challenging mathematics; ideal traits to being successful in learning and doing mathematics in both the short and long-term. Conversely, repeated failure in mathematics may diminish students' belief in being successful (Crossley et al.) and can lead to anxiety, fear, and apprehension when faced with challenging situations. This increased level of anxiety limits working memory (Young et al., 2012) and negatively predicts students' use of advanced problem-solving strategies (Ramirez et al., 2016). This means supporting the development of students' mathematical identities can benefit students' long-term success in mathematics.

Likewise, students' engagement in collaborative learning centres on communicating thinking and ideas, which is often through some form of discourse. The research literature has long highlighted the importance of collaborative and socially supportive learning environments, such as whole-class discourse, in promoting student engagement (Mende et al., 2021; Bennett, 2010). And, the role of the teacher as a participant in co-constructing mathematical meaning in the classroom (Eckert, 2017) and in transforming the discursive interactions to increase more equitable learning environments (Puntambekar, 2022) cannot be understated. This means that teachers also need to be explicit in establishing socially co-constructed mathematical norms and deliberate in equitably facilitating discursive interactions to leverage students' thinking and mitigate power structures. Therefore, it is important that educators foster an environment that promotes equitable collaborative opportunities to listen to, interpret and discuss the mathematical thinking.

A primary component of mathematical discourse is the use of logical reasoning. One strategy that has become more visible in elementary classrooms to support reasoning is number talks (Parish, 2014). Number talks are short, non-instructional moments that require students to reason about and defend relationships by using flexible strategies to mentally calculate solutions. However, number talks are typically designed around computational exercises wherein the accuracy of the solution is also discussed. A modification to this process is to use reasoning talks (Bennett, 2018), which uses contextual problems and centres instead on reasoning about relationships by developing conjectures, formulating a plan, or recognizing key information and then defending the logic and reasoning behind these actions. As such, reasoning talks were used as they create rich discursive interactions.

Methods

This study focuses on the extent to which students' drawings of mathematicians change over time to more accurately represent themselves as student mathematicians. It developed as a result of a larger project centred on understanding how reasoning-centric discourse protocols help students develop mathematical behaviours for constructing and critiquing mathematical arguments. Directly related to students' mathematical behaviours and mathematical ways of thinking, is their sense of self, their perceived mathematical identity, during these whole-class discursive interactions. As such, an exploratory qualitative case study design (Bogdan & Biklen, 2007) was used as it allowed the university researcher and teacher researcher to explore students' discourse-identity relationship in an authentic and natural context and to develop initial understandings of how such interactions influence students' perceptions of themselves and what it means to be a mathematician and do mathematics.

Participants

Participants included one grade four teacher researcher and his 18 students, which include seven girls and eleven boys. The school in which this study takes place is located in the mountain-west of the United States and it is situated in a smaller urban community, but the neighbourhood and micro-community it serves has substantial social needs and financial insecurities. The school has approximately 500 students across all primary grades and less than ten percent are English Language learners but nearly half receive government financial support for meals while at school.

Procedures

The teacher researcher and the university researcher facilitated reasoning talks with students about three times a week for 12 weeks. Again, the purpose was to help students learn how to reason about complex problems through a discourse protocol as a means of learning behaviours mathematics demonstrate in doing mathematics. Before each session, the teacher reminded the students that effective mathematicians can carefully talk about their ideas, questions, and doubts in mathematics. They listen carefully to others and develop mathematical arguments based on their reasoning. Then, the teacher or the university researcher facilitated a reasoning talk focusing on students defending their reasoning and justification for a problem. For example, one question originally stated "Mike is comparing the area of three states. He said Alaska's area is about 10 times greater than Georgia's land area. Is Mike correct?" and had an accompanying table with supporting data. However, the question was modified to focus more on reasoning that calculating by removing the table and then asking the students "what information would you need and why would that be important?" Again, the focus was on considering what information is necessary to begin and not simply calculating a solution. After six weeks students were then asked to draw a second picture of a mathematician.

Data Collection & Analysis

For the purpose of this study, data collected included students' initial drawings, the base line drawing, the ones created six weeks later, and field notes based on students' explanations of their drawings, when shared, as they turned in the drawing. Students were asked to draw mathematicians twice during a six-week period. The first was at the start of the school year before mathematical classroom norms were established and prior to any substantial instruction in mathematics. At this time, some students were unfamiliar with the word and were unsure what to draw. Students were asked to think about what a mathematician looked like, how they saw it in their mind, and to draw that image. Students were not allowed to search for pictures of mathematicians on the internet but were encouraged to add notes on the side of their drawing to highlight anything they were unsure would not be understood, or seen, by someone else. Then, when the students turned in their drawings, the teacher researcher and university researcher asked students clarifying questions to better understand the drawings. This included questions such as, "where is the mathematician?" if there were several people on the paper or "do you know this person?" if it was unclear who the drawing represented. The second occurred six weeks after students were introduced to reasoning talks. For the upcoming year, three additional drawings will be collected each about six to eight weeks apart.

Drawings and/or annotations were then independently and openly coded to identify initial themes (Ganesh, 2011). Building on Ganesh's analysis of drawing an engineer, the broad theoretical categories for these codes included gender, age, tools/equipment and physical appearance (i.e., clothing, hair, and eyeglasses). Based on this coding, initial themes were then discussed amongst the team to determine the central, emergent themes to be used for a second round of analysis looking at holistic classroom shifts by quantifying the frequency of the theoretical categories. While changes for individual students across time were also examined, for the purpose of this paper, only the holistic classroom shifts will be discussed. This was a critical phase of the analysis as the teacher researcher had additional insights relating to how students' mathematical identities were developing. Three emergent themes were then identified and used to analyse changes in students' projected identities of a mathematician.

Limitations

A primary limitation of this study revolves around the fact that it included only one classroom with 18 students. It is possible that these 18 students, and their perspectives on what a mathematician looks like, are a non-representative sample. That is, the representative samples shown below, which include many common themes and nuanced features or traits created by multiple students, may not be the same in other classrooms or cultural contexts. It is also probable that other interactions that were not studied impacted students' perceptions of a mathematician. This would include other conversations and interactions within and beyond school that may have impacted their beliefs, self-efficacy or confidence in doing mathematics. As such, this reduces the richness of the data and thus the interpretations that can be made despite promising findings for future research.

Findings

The findings will be discussed according to when the data were collected, initial and second, and in relation to the three central themes, namely gender, appearance, and the age of the person(s) drawn for each data collection cycle. For representative samples of the initial drawings, see Figure 1 and then see Figure 2 for the second set of drawings collected six weeks later.

Initial Drawings

Of the 18 students in the class, 14 of their initial drawings appeared to be men. Additionally, three drawings appeared to be women (i.e. longer hair, wearing dresses or skirts) and one drawing was of an entire family as the student said they were unsure what the term mathematician meant so just drew a family instead. All of these non-men drawings were done by girls and of the seven girls in the class, four drew men. Furthermore, initial drawings typically included people with eye glasses, who wore button down shirts and ties or other clothing typically worn by adults. Most also had short hair that looked like styles many men might have or hair that was sticking up in a wild an unkept manner, though generally no facial hair. Furthermore, all were of people that were older than the students and appeared to be adults. Even if children were in a drawing, which occurred in only four drawings, when asked who the mathematician was, students all identified the adult. This indicates that none of the students initially perceived that a child could be a mathematician.

Figure 1: Representative samples of initial drawings showing gender, appearance, and age.

A few drawings even appeared to be more like magicians with a top hat, cape, and some object in their hand; one student even stated it was a wand. These appearances suggest that multiple students were unaware of what a mathematician was and thus drew someone who, in English, has a similar sounding title; a magician rather than a mathematician. Some students noted that this object was a marker and one indicated it was a wand. In almost all drawings, students' mathematicians held calculators, books, white board markers, rulers, or sticks to point at a whiteboard. These drawings suggest that being a mathematician may rely on or incorporate external objects.

Second Drawings

For the second drawings, they were more representative of students' gender and age. In particular, and with respect to the seven girls in the class, only one drew a male person and nearly all (n=17) drew someone younger in age; one student drew an older man with wild white hair. One student used both sides of the paper to draw the inside and outside of the school. The inside drawing included desks, a student, and a teacher. When asked who the mathematician was, he pointed to the student. On the outside drawing of the play area there were many children. When asked if there were mathematicians in this drawing, too, he said yes, there was "all of the kids are mathematicians, they are just playing." Otherwise, annotated descriptions indicated they were children and every student drew a person that was their own gender.

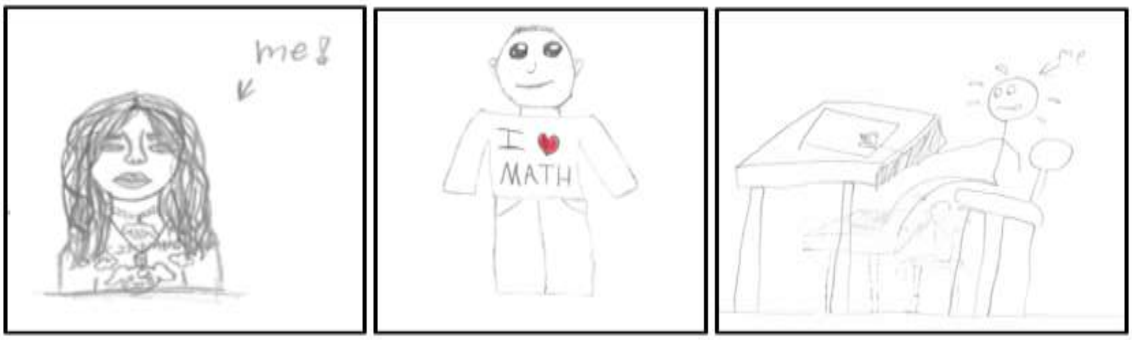

Figure 2: Representative samples of the second drawings showing gender, appearance, and age

With respect to appearances, one drawing still had a top hat, and many had objects that are often used in class, such as books, pencils, and paper, or whiteboards. However, an additional change in the drawings was the inclusion of beliefs and/or behaviours. For example, three drawings included concepts of persistence and increased effort, another included thought bubbles to show thinking, and another included a student justifying their thinking ("I think 9=3x3 because 6=3x2 and that is one more 3"). One student even wrote, "I drew myself because I feel more confident in math than last time. I drew a mathematician." Whereas two others specifically identified themselves as mathematicians. This suggests that students are beginning to see themselves, people their own age and gender, as mathematicians.

Discussion of Results & Implications

Students' mathematical identities are but one part of helping them be successful in learning and doing mathematics (Gutiérrez, 2009). This project highlighted how the mathematical identities of these students, as evident in their drawings of mathematicians and discussed above, positively changed to reflect a more inclusive and favourable belief about themselves as student mathematicians. This is particularly important when considering initial drawings were mainly of older men; a very small subset of the human population. Additionally, using artistic representations provided insight into students' perceptions that may not otherwise have been captured in the classroom. That is, given the complex construction of identities (Goldin, et al., 2016), it is important to recognize the varied ways in which students see themselves or adult professional versions of themselves. This means supporting students' positive mathematical identities is a multifaceted endeavour and should constantly build upon the socio-mathematical norms and on-going interactions within the classroom (Cobb et al., 2009).

With that said, caution is warranted. As Stentoft and Valero (2009) clearly warned that only attending to a few fixed characteristics limits the ways in which identity can be understood. While future research may want to consider how to leverage students' illustrative representations of themselves as doers of mathematics, it is not the characteristics alone that matter. But rather, it is the ways in which these characteristics are shaped and change over time. That is, the value comes not from the drawing itself, or what may or may not be shown in a drawing, but in the rich socio-mathematical interactions that influence one's perception of self as may be evident through the changes in the drawings over the course of a school year.

Lastly, the understandings from this project on developing students' mathematical identities were a direct result of the inquiry, interactions and curiosity between the teacher research and university research (Jaworski, 2006). That is, while analysing the data for themes, the university researcher was able to consider alternate perspectives (i.e. the way in which productive mathematical behaviours were illustrated) because he was not part of the daily learning environment whereas the teacher researcher provided additional insight on the data because he was working with the students daily (i.e. students shift towards drawing friends, not just students). Having both a more ethnographic and content analysis approach created rich and robust discussions on how best to support students' mathematical identities through the reasoning talk protocol. Ultimately, this deliberate collaboration between the teacher researcher and the university researcher created opportunities to help students positively change their personal stories and thus change the beliefs and views students hold about being successful in mathematics.

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The role of rituals and ambiguities in a mathematical discourse

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This study examines two students' reasoning processes when solving a Calculus problem, focusing on their discourse characteristics and factors influencing narrative endorsement. Using video recordings of paired discussions, the results show rituals, social acceptance, ambiguity, inconsistency, and the impact of mathematical representations as characteristics of the narratives. The results also indicate that rituals and ambiguities are key factors in determining whether the narrative becomes endorsed. Additionally, ambiguity provides opportunities for the two students to think critically and creatively, which can support their reasoning process.

Keywords: Ambiguity, derivative, discourse, reasoning.

Introduction

In a typical first-year Calculus course for prospective teachers, students solve problems to enhance their reasoning, which is crucial for their future as educators. They are expected to effectively communicate mathematical concepts using the accepted terminology defined by the discourse of the mathematical community, characterised by endorsed narratives (Sfard, 2008). These concepts will be further explored below in the section "Theoretical framework". Researchers in mathematics education have investigated whether teacher-led instruction benefits students' learning and the differing discourses between students and teachers (e.g., Viirman, 2021; Moschkovich, 2007). Typically, in university settings, students passively listen to lectures without actively engaging in the learning process (e.g., Petropoulou et al., 2020). The findings from these studies highlight the discursive aspect of university teaching even if there is a lack of dialogue during lectures. There is also a lack of knowledge supporting prospective teachers' ability to engage in mathematical reasoning with their students in the classroom. One aspect of teaching that remains underexplored is the student discourse that occurs after lectures, particularly in relation to problem-solving. To gain a deeper understanding of students' reasoning, which can be used to support their explorative participation in mathematical discourse, our goal is to better understand students' narratives and the endorsement process. We address the following question: *What are the characteristics of narratives in a problem-solving discourse involving derivative, and what factors contribute to the endorsement of these narratives?*

Related literature

Understanding derivatives is critical for students at the university level (Biza, 2021; Haghjoo et al., 2020; Lefrida et al., 2021; Zandieh, 2000). Previous studies have identified that many students lack a conceptual understanding of derivatives (e.g., Haghjoo et al., 2020), tend to focus more on symbolic rather than graphical representations (e.g., Biza, 2021), struggle with making logical connections between these representations (e.g., Zandieh, 2000), and switch rapidly within and between different mathematical objects (Berggren et al., 2023). Studies have also shown that students commonly make repeated mistakes in terminology and symbol usage (Lefrida et al., 2021), and exhibit ambiguity in their reasoning about calculus problems (Berggren et al., 2023).

Ambiguity in mathematics has been defined by Byers (2007) as "a single situation or idea that is perceived in two self-consistent but mutually incompatible frames of reference" (p. 28). Foster (2011) further categorised ambiguity into four types: symbolic ambiguity (where the same letter represents different things), multiple-solution ambiguity (where a problem or concept has several possible solutions or interpretations), paradigmatic ambiguity (involving multiple interpretations or perspectives on a concept or object), and definitional ambiguity (where there are multiple ways to interpret the meaning of a mathematical term). Previous studies have explored ambiguity concerning classroom teaching and have identified two additional types of ambiguity: clarifiable (Peterson et al., 2020) and calculus symbolism (Tasara, 2018). However, there is a lack of literature exploring ambiguity as a tool for understanding post-lecture student discourse at the university level.

Theoretical framework

In our analysis, we examine students' discourse on derivatives to identify the characteristics of narratives in problem-solving and the factors that contribute to the endorsement of these narratives. We focus on four aspects of mathematical discourses: word use (e.g., tangent, derivative, intersect), visual mediators (e.g., graphs, symbols), narratives, and routines. Narratives encompass expressions describing mathematical objects, relationships, and processes, including definitions, theorems, and proofs. The adjective endorsed indicates that these narratives are accepted by the mathematical community and guide the development of the discourse (Sfard, 2008). Routines in mathematics are activities that evolve through repetition and adaptation based on previous practices (Sfard, 2008). These routines are characterised by defined and recurring patterns within mathematical discourse, governing various tasks such as calculation and proof. They become evident through an examination of word use, visual mediators, and the analysis of how narratives are formulated and supported. Examples of mathematical routines are calculating the derivative to find the slope of a tangent line or finding a tangent passing through a given point on the graph. In mathematics teaching, teachers often employ specific practices, that can be seen as routines within classroom discourse (Nachlieli & Elbaum-Cohen, 2021). In this study, students utilise earlier practices, which can also be considered routines within students' discourse.

Sfard (2008) categorises mathematical routines into explorations, deeds, and rituals. Exploratory routines aim to develop accepted narratives about mathematical objects, while deeds focus on producing actions or modifying objects. Rituals help maintain connections with others, such as colleagues or teachers, while recognising their authority within the discourse. Explorations are further divided into construction, substantiation, and recall (Sfard, 2008). Construction involves generating entirely new narratives that can be accepted within the mathematical discourse. Substantiation entails evaluating the validity and reliability of previously constructed narratives. The terms and criteria for endorsement can differ considerably from one discourse to another (Sfard, 2008). Recall enables individuals to retrieve narratives endorsed in the past. Lavie et al. (2019) highlight the complexity of analysing routines, as the variation in how students perform the same task is shaped by their past experiences. Therefore, when examining the routines used in post-lecture students' discourse, it is important to delve deeper into analysing the ambiguity that can arise in how students execute the same task. For example, ambiguity can occur between mathematical areas that may seem unrelated to students (e.g., the derivative of a function and the tangent to the function).

Method

This qualitative study analyses the discourse between two students when working on a mathematical problem-solving task (Figure 1, translated from Swedish to English). In this task, students engage in reasoning that involves various discursive elements such as word usage, visual aids, narratives, and routines. Word usage includes the use of mathematical language, such as algebraic, numerical, and geometric terms, equations, and other mathematical expressions when solving problems. Visual aids involve the use of graphs and symbols to support the problem-solving process. The narrative emphasises students' ability to communicate essential mathematical information, clarifying relevant mathematical facts related to the problem.

> At a point on the curve, $y = e^{-x}$, $x \geq 0$ a tangent to the curve is drawn.
>
> The coordinate axes together with the tangent form a triangle.
>
> What is the largest area such a triangle can have?

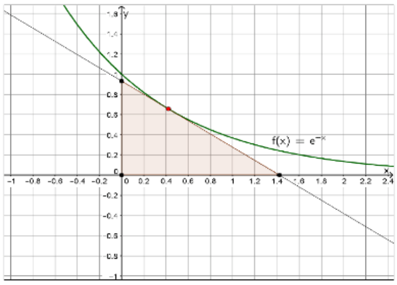

Figure 1: The task, as presented to the students (Berggren et al., 2023)

The data presented in this paper were collected during a mathematics course on Calculus for prospective teachers in the first year of their mathematics teacher education program. The course provides core knowledge for the students' future work in upper secondary school. This paper analyses the discourse between two students in a problem-solving situation. They worked in pairs on a task chosen by the researcher from a set of tasks provided by the course teacher. The students worked without the supervision of the teacher and outside the classroom. Data consists of the students' written answers and a videorecording of their hands and actions on a touchscreen laptop, which they used for writing and drawing during the session. We used the transcription of what the students were saying, as well as their written documents, and analysed every sentence and action in detail.

In our data analysis, we employed a combined inductive and deductive approach. Initially, we used an inductive coding to closely match with the content of the data. As we proceeded with coding and analysing, we changed to deductive coding, guided by codes generated during the inductive phase. This deductive thematic analysis was guided by the research question and the study's conceptual framework. We began by analysing spoken sentences to identify key aspects of the discourse: word usage, visual mediators, and routines, and their interrelationships. Next, we divided episodes based on the constructed or endorsed narratives and studied the factors influencing the students' transitions between different narratives. The review of preliminary narratives was closely aligned with the study's theoretical framework and the research question.

In presenting our results, we include student quotes as references to clarify our interpretations of the identified narratives (Figure 2) related to determining the equation of the tangent line to a given function at a specific point of tangency. It is important to note that narratives have a hierarchical structure, with some being subordinate to others, referred to as sub-narratives.

Results and analysis

The results are presented in four narratives: “Triangle area”, “The tangent line and its equation”, “The slope of the tangent line”, and “The intersection of the tangent line and the y-axis”. Figure 2 illustrates the order in which the narratives are constructed by the students, as they switched between different parts of the task, and they came back to unsolved subtasks several times.

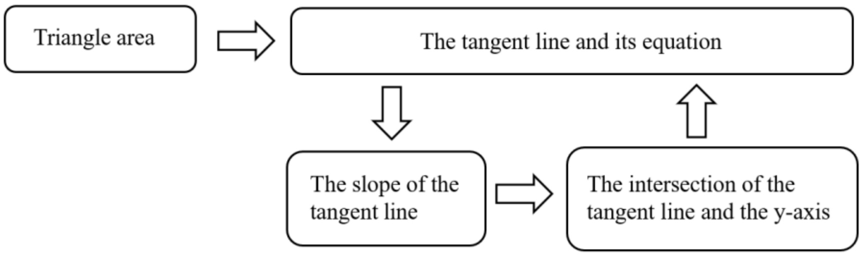

Figure 2: The narratives in chronological order

The students use the slope-intercept formula for a line as $y = kx + m$, where k is the slope of the line and m is the y-intercept (standard Swedish notation).

Triangle area

In the transcription below, we present the students’ utterances in the first part of the discourse, which focuses on the construction of the first narrative, “Triangle area”.

1	Student 2:	I wrote that the area is equal to y times x but then I realised that y equals $f(0)$.
2	Student 1:	Yes!
3	Student 2:	Ok, but m is also equal to $f(0)$ [Pause] therefore you can take m times x divided by 2.
4	Student 1:	Well, I have written […] I have thought a little differently, but it is the same.

In the first row, Student 2 made two utterances. The first involved recalling the area of triangles, expressed as y times x. The second focused on identifying a pattern between the geometric representation of the triangle’s area and algebraically comparing the triangle’s height to the value of the function at $x = 0$. Student 2 modified the use of y in $f(0)$. Student 1 endorsed this narrative using a ritual routine. In the third row, Student 2 further modified the previously constructed sub-narrative by generalising that the tangent line’s intersection with the y-axis is the same as $f(0)$. This action can be viewed as a deed routine, involving a reorganisation of the mathematical object. Student 1, by saying “a little differently,” neither endorsed nor rejected the narrative in rows 1-3. This suggests that this narrative is inconsistently endorsed, indicating that the student is uncertain about the truth value of the utterance. This inconsistency is also tied to the lack of details in the student's expression.

There are two ambiguities related to m that become evident on three occasions during the communication, denoted as A1 and A2. A1 is about m not being equal to the value of the exponential function at $x = 0$. Here, Student 2 could be interpreted as choosing to call the straight line f, but this is not explicitly stated in what she expresses verbally. In addition, in the overall communication there are several indications that the student is referring to the exponential function when mentioning f. We refer to this as “ambiguity of sameness in different”. This concept represents the student’s focus on the same object using different notations that refers to two different objects. The object referred to depends on whether the value of the tangent line or the exponential function is calculated at $x = 0$. A2 is the ambiguity that arises when student 2 refers to $m = f(0)$ and uses this (endorsed) equality to rewrite the expression for the triangle's area. Since the student backtracks to the triangle's area, it is about the original object, y, as the length of the triangle's height. The "same" object is here y. We refer to this as "ambiguity of sameness". Right after the narrative about the area of the triangle, they shift to talking about the tangent line and different expressions of the equation of the line. They discuss the slope of the straight line and then return to m again.

The intersection of the tangent line and the y-axis

1	Student 1:	And as you said m is… no, m is…
2	Student 2:	Yes, m is the same as $f(0)$.
3	Student 1:	m is this [points at the intersection between the y-axis and the tangent line]. The function is there [points at the intersection between the y-axis and the graph of $y = e^{-x}$], can you see?
4	Student 2:	Yes!
5	Student 1:	The function is up there when zero, but m is a bit lower down…or do you mean a different m?
6	Student 2:	I was thinking m for the tangent line [shows $y = -e^{-x} \cdot x + m$ and $m = f(0)$].
7	Student 1:	If you think… $f(0)$ is this point…but m for the line [the tangent line] is further down.
8	Student 2:	Yes.
9	Student 1:	Therefore, it is not $f(0)$.

The constructed narrative starts by linking to the narrative about the triangle area and the ambiguity of sameness (A2) that $m = f(0)$. Student 1 starts to think about this and seems to realise that the utterance $m = f(0)$ cannot be endorsed yet. In the second row Student 2 sticks to her statement because it is only the algebraic expression that the student is looking at. The student has not yet verified her algebraic statements with any other representation. In the third row, which contains three utterances. The first, "m is this", and the second, "the function is there", recall a previous exploration involving a figure he had created. The third utterance, "can you see", is a ritual intended to ensure the other student's understanding of the arguments. On the fourth row, the student answers the question, which initially seemed like an endorsement of the narrative. However, it later became apparent that this was just a response to the question "can you see", rather than an endorsement of the narrative. This can be interpreted as a type of ritual, serving more as a phrase to indicate her participation in the conversation. We will refer to this kind of endorsement as a false endorsed narrative. Unlike the inconsistent endorsed narrative, the student here is aware that she did not fully understand the reasoning from Student 2. The awareness of the meaning of what the utterance can give as consequences differs from the inconsistent endorsement because the student is not aware of the falseness of the endorsement.

Student 1 refers to a previously drawn figure when trying to convince Student 2 that the difference between the two points on the y-axis can be interpreted as a substantiation through a justification. When asked if they are referring to the same m, Student 1 is interested in both getting the reasoning accepted and ensuring that they are discussing the same concept. In the answer in row 6, Student 2 confirmed the false endorsed narrative through a recalling routine and previous written statement, indicating that she still does not understand the meaning of m and $f(0)$ in the same way as Student 1. Student 1 tries to explain one more time but in other words. This is a ritual to overcome the false endorsed narrative by pointing out the difference between $f(0)$ and m using mathematical terminology. After several rituals, we can see that both students finally endorse the narrative.

The next narrative is about the slope of the tangent line, which takes place just before the narrative about the intersection of the tangent line and the y-axis.

The slope of the tangent line

This narrative precedes the narrative about the intersection of the tangent line and the y-axis.

1	Student 1:	[…] but the slope for this line should be the derivative of this function [point at the graph].
2	Student 2:	Yes!
3	Student 1:	but then I have written that $k = f'(a)$. I don't know where this point should be because it depends on where we move it.

The first utterance "but the slope for this line should be the derivative of this function" is a sub-narrative and serves as a substantiation through a kind of justification where the student use well-known knowledge about how the slope of the tangent line depends on how the tangent point is chosen graphically. The first utterance on the third row is a generalising exploration that refers to the function's derivative at $x = a$ and the slope of the tangent, with algebraic notations where the student tries to identify patterns. The last utterance on the third row is an exploration through a validation of how the tangents' slope depends on where the point $x = a$ is placed. This utterance is another sub-narrative and serves as a starting point to construct a new narrative. At this point in the discourse, a new ambiguity (A3) appears. A3 is an ambiguity of generalisation and is associated with the given point $(a, f(a))$, for which the students struggle with to express and make use of.

The tangent line

Student 1 notes $g(x) = kx + m$ in his paper and then the students became interrupted by something about the slope of the tangent in the picture, when pointing at the graph. The students lead into a discussion about the derivative, the slope of the tangent line which is another narrative, and a discussion about the value $f(a)$ of the function at $x = a$. After they have expressed the derivative of $f(x) = e^{-x}$ as $f'(x) = -e^{-x}$ they come back to talk about the expression of the tangent line as follows in the citation below.

1	Student 2:	If you look at this [points at the expression $e^{-a} = -e^{-a} \cdot a + m$] we can use, choose any x-value and insert into the original function and then you will get a y-value, then you have an x-value and y-value at that point [points at the tangent point].
2	Student 1:	Hmm
3	Student 2:	Then you can use the same y-value to figure out what this is [points at $g(x) = -e^{-x} \cdot x + m$].
4	Student 1:	Hmm? Yes, but then you get m for a specific point, and we want to be able to derive and find the minimum value...we need something more, we are missing something in order to move on […]. We need to find m! What is m?...

In row 1 the student has rewritten the expression further by replacing the notations $f(a)$ and $f'(a)$ by e^{-a} and $-e^{-a}$, respectively. This equation can be used to find an expression for m but they do not seem to realise that here. In row 2, Student 1 makes a double entendre which neither endorses nor contradicts the statement in row 1. When Student 2 then points at the erroneous expression $g(x) = -e^{-x} \cdot x + m$, Student 1 responds with a mixture of questioning and courtesy, using utterances like "Hmm" and "Yes, but then …". These responses can be seen as a ritual that do not lead to an endorsement of the narrative.

Discussion and conclusion

The purpose of this study was to investigate the characteristics of narratives in a problem-solving discourse involving derivatives, and the factors that contribute to the endorsement of these narratives. The results suggest that the narratives within the discourse exhibit two notable characteristics. First, each narrative begins with an explorative process and closes with a ritual. These rituals, often socially driven and reflecting students' acceptance of each other's utterances, do not consistently result in the endorsement of a narrative. Our findings indicate that the students' discourse differ from the discourse that typically occurs in lecture settings (e.g., Petropoulou et al., 2020; Viirman, 2021). The second characteristic is that the process of endorsement varies due to factors such as inconsistent endorsements – where details in the student's expression are lacking – and sometimes false endorsements, where one student appears not to fully grasp the reasoning of the other student. By identifying students' varied approaches to endorsing narratives, our results add to the existing body of research on the challenges students may face in understanding derivatives at the university level (Biza, 2021; Haghjoo et al., 2020; Lefrida et al., 2021; Zandieh, 2000).

The study identifies two main factors that influence the endorsement of narratives: rituals and ambiguities. Rituals, characterised by social interactions and the acceptance of utterances, play a significant role in endorsing sub-narratives that follow an exploration routine. Ambiguities, in contrast, can disrupt endorsement by introducing uncertainty or multiple potential interpretations into the reasoning process. This study expands the concept of ambiguity in mathematics as defined by Byers (2007) and categorised by Foster (2011). In the context of our research, ambiguity refers to situations where students experience uncertainty or encounter multiple potential interpretations in their reasoning process during problem-solving. This uncertainty can be triggered by word use, visual aids, unspecified details, or unspoken assumptions in the problem-solving process. Three distinct types of ambiguities are identified in this study: ambiguity of sameness in different, where students perceive a single mathematical object as two distinct entities by using different notations simultaneously; ambiguity of sameness, where a previously accepted statement creates ambiguity by treating several mathematical objects as one; and ambiguity related to generalisation, where uncertainty arises when students attempt to generalise concepts or objects. Although previous research has recognised the presence of ambiguities in mathematical discourse, this study advances our understanding by defining new types of ambiguities and exploring their implications for students' reasoning processes. Furthermore, studies by Peterson et al. (2020) and Tasara (2018) have emphasised the importance of resolving clarifiable ambiguities such as the correct use of vocabulary and symbolism in Calculus to support students' mathematical thinking within classroom settings. Our study focuses on understanding different ambiguities in students' discourse created by their actions. Based on the results, we can conclude that ambiguity provided opportunities for the two students to think critically and creatively, thus contributing to advancing their mathematical reasoning.

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Insights from Grade 10 students in a problem-solving classroom

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Problem solving is a well-established approach in primary and secondary mathematics education. This paper presents insights gleaned from a Norwegian problem-solving classroom, wherein we explore interviews conducted with three Grade 10 students who actively participated in two distinct mathematics classrooms each week—one dedicated to problem solving and the other mainly adhering to exercises with closed structures and the practicing of particular methods or techniques. These students' unique positions enabled them to articulate and dissect their 'take-ways' from the problem-solving classroom. Our analysis of three months' worth of observation notes and three interviews determined that the students perceived that the problem-solving classroom facilitated the integration of their knowledge while also leveraging the knowledge they already had. They found motivation in the challenging nature of the problems encountered in the problem-solving classroom and appreciated how the teacher of that class orchestrated the classroom.

Keywords: Problem solving, non-routine problems, routine problems, lower secondary

Introduction

Mathematical problem solving 'has been of interest to mathematics education researchers for as long as our field has existed' (Liljedahl et al., 2016, p. 1). The last three decades have proven to be a particularly prolific period in the history of research on problem solving in school mathematics, including research on the teaching of mathematics *through* problem solving (see, e.g., Liljedahl & Cai, 2021; Suseelan et al., 2022). While most studies speak in favour of teaching mathematics through problem solving (Lester Jr. & Cai, 2016), less is known about students' perspectives on the issue. A bibliometric review of 159 studies on problem solving from 1969 to 2021 by Suseelan et al. (2022) determined the foci of research on mathematics problem solving by conducting a keyword co-occurrence analysis. Notably, 34 keywords surpassed the co-occurrence threshold and the cluster size limit, none of which signposted student voices. In addressing the interplay between research and the practice of teaching mathematics through problem solving, we assert that student voices are important.

In this paper, we enrich the discourse surrounding problem solving in school mathematics by presenting findings from a Norwegian classroom in which mathematics is taught solely through problem-solving approaches. We offer insights by amplifying the perspectives of three Grade 10 students who participate weekly in both a problem-solving classroom and a classroom that focuses more on exercises with closed structures and the practicing of particular methods or techniques. The goal of this study was to broaden our understanding of students' perspectives on problem solving by capturing insights from those who could compare their experiences with ongoing instruction in an alternate classroom.

Literature review

Problem solving is an all-encompassing term for which it is difficult to provide a single prescriptive definition (Schoenfeld, 1985). Nevertheless, this study approached problem solving as the process of overcoming the challenges experienced by an individual (Polya, 2004) when engaging with a problem with which the individual lacks access to a straightforward solution (Schoenfeld, 1985). In this process, there is a need to make choices (i.e. no established methods seem immediately obvious) and draw connections between different mathematical ideas to effectively solve the problem. In this way, drawing on Mullis et al. (2003), we emphasise the non-routine character of these problems, which we see as 'a counterpart of solving routine problems aimed at getting practice in particular methods or techniques and in problem settings that are more familiar to students' (Kolovou et al., 2011).

Teaching mathematics using such non-routine, challenging problems often results in students enjoying mathematics (Cai & Merlino, 2011). These problems capture students' interests and curiosity (Cai, 2014), causing them to hold more positive beliefs about the importance of understanding mathematics (Cobb et al., 1991). Studies comparing results between students who were taught mathematics through problem solving with students who were taught mathematics using exercises with closed structures pre-established solution methods, show that students in the first category outperform the latter when it comes to conceptual understanding in mathematics both at primary (e.g., Cobb et al., 1991) and secondary level (e.g., Reys et al., 2003).

While acknowledging that mathematical problems can have both routine and non-routine aspects (Mamona-Downs & Downs, 2005), we assert that teaching mathematics through problem solving is a demanding task, especially considering that the extent to which a problem presents obstacles and challenges 'depends on the person who is dealing with the situation' (Zhu & Fan, 2006, p. 612). With this recognition as a backdrop, we present a case study that allowed us to draw on the perspectives of three Grade 10 students (aged 15–16 years) who were taught mathematics weekly in two different classrooms: one problem-solving classroom and one classroom focused on solving familiar routine problems by practicing particular methods or techniques (see Kolovou et al., 2011).

While there is a related body of research examining how the process of problem posing is experienced by students (see the 2024 scoping review by Cai and Rott on the issue), to the best of our knowledge, there is a lack of studies reporting on students' views on being taught mathematics through problem solving. Hence, we assert that this study offers a novel contribution by giving voices to students who are in the unique position of having weekly experiences in two different classrooms. In this paper, we pose the following research question: *What do Grade 10 students highlight in their accounts of a problem-solving classroom?*

Methodology

This paper draws on data collected for a larger project investigating students' mathematics learning in problem-solving classrooms in Norway. For the larger project, data were collected in two schools and three different classes, all taught by the same mathematics teacher, Peter, who teaches mathematics only through problem-solving tasks. Here, we utilised one of Peter's classes, which was taught by him and another teacher interchangeably. The 25 students in this Grade 10 class (aged 15–16 years) were in a novel position: each week, they were taught by Peter for one lesson (using a problem-solving approach) and another teacher for the three remaining mathematics lessons (focusing on familiar routine problems by practicing particular methods or techniques).

After three months of observation, three students were selected for the interviews. The first author recruited students who he had observed engaging in solving the problem in Figure 1 (which enabled him to revisit this problem during interviews) while also considering gender and ability level. Hence, *Ben*, a male student aiming for a Level 5 or 6 in mathematics (the Norwegian grading scale goes from 1–6, where 6 is best), and two girls, Eir and Dina, both on Level 4 in mathematics, were chosen for interviews. Their accounts of a problem-solving classroom were given against the backdrop of their ongoing experiences in the other classroom.

Most of Peter's problem-solving lessons followed a similar pattern. He began by presenting a problem on the board, providing little or no information on how to solve it. He would also rank the difficulty level of the problems, claiming, for example, 'this problem is a 5-er on the 10th grade exam' or 'this is a 7+' (going beyond the Norwegian grading scale). The 7+ marking was common, revealing how he drove them to engage in cognitively demanding tasks (gradings above 6 indicated that these tasks were more difficult than the tasks they would ever get on the 10th grade exam). Figure 1 is an example of a Level 7 problem that will be referenced throughout this paper. Next, the students were given time to solve the task either individually or in groups while Peter walked around the classroom, observing and giving feedback to the students on their progress. Later, one or more solutions were shown on the board (and carefully explained) before another problem was given. In their problem-solving classroom, the students chose with whom they would work.

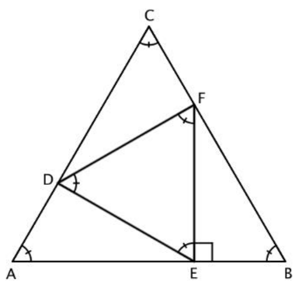

Figure 1. Problem: Find the area ratio between triangles ABC and DEF.

Unfortunately, we did not observe the other classroom that the students attended during the week. Hence, we could not describe that classroom in detail and instead relied on the characteristics featured in the interviewees' accounts of that particular classroom.

Data collection

We drew on two sets of data: three months of handwritten classroom observation notes and audio-recorded, semi-structured interviews with Ben, Eir and Dina. The classroom observations and observation notes were used to a) pick two problems that were revisited during the interviews (one of which we report on here, rendered in Figure 1), b) inform the interview guide and c) contextualise and validate the student interview responses. The first author's presence in the interviewees' classroom over a period of three months enabled him to connect questions to relevant problems that he knew the students had encountered and to ask follow-up questions about situations to which he knew the interviewees would relate.

The semi-structured interviews with Ben, Eir and Dina were conducted individually in the students' classroom and typically lasted for 30 minutes. The aim was to allow the students to share their thoughts and experiences from Peter's problem-solving classroom against the backdrop of their ongoing experiences from the other classroom they attended every week. Key topics for the interviews included the students' general thoughts on mathematics, their motivation and engagement, their experiences with problem solving and their reflections on the differences between learning through the use of challenging non-routine problems versus routine problems.

Analysis

The interviews were transcribed in full in the students' original language (Norwegian) and analysed in two steps. In step one, to gain an overview of the material, we conducted a round of meaning condensation to summarise statements and read them in the context of what had been observed. For instance, at this stage, the first author's notes from observing the students solving the problem in Figure 1 were connected to the transcripts of how they discussed the problem during the interviews. Next, we moved on to a holistic reading of each interview (including notes from observations) to uncover similarities and differences between the students' accounts of the problem-solving classroom that might be important for our results (considering that the interviewees' ability levels differed). We looked for utterances and reflections on what they learned, how they learned and the factors they believed contributed to their learning, all while tracking which features from the problem-solving classroom were highlighted in their accounts.

To ensure validity, the passages identified during the analysis were translated into English and discussed by the two authors to ensure that the original meanings were captured during the translation. In presenting our analysis, we are necessarily limited to illustrative quotes. Additionally, we are aware of the limitations of this study: the small number of interviews, observations from the problem-solving classroom only, the way in which the students' utterances can be biased (e.g. towards the teacher they like the most) and the fact that it can be difficult for students to distinguish between what they learn in one classroom compared to another. However, we believe that our decision to draw on interviews with students who have had weekly experiences from two different classrooms using two different approaches makes a unique contribution to the literature on problem solving in mathematics. This research project was approved by the Norwegian Centre for Research Data (now Sikt).

Results

Since the students were enrolled in two different mathematics classrooms every week, they were in a unique position to express what the problem-solving classes did for them. They appreciated that they were encouraged to work together to solve problems that made them engage with mathematics in a more connected way in the problem-solving class:

> … there's more like … cooperation in a way. You get to ask the students around you and work together … I learn the most when I work in groups that have the same level as I do and when I get to use several elements of the mathematics in the same problem. (Eir)

This observation of how the problem-solving classroom encouraged the students to exploit connectedness in mathematics became even more evident when Ben reflected on the differences between the two classrooms in which he participated every week:

> In traditional classes, it's more like … right now, we have a lot of repetition, but usually, we work on new things. Then, we do a lot of small, simple tasks, so we don't get to use it properly because it is the same pattern of solving them. You don't really learn how to use it in different situations … It is different … engaging in solving problems instead of doing a bunch of tasks, because you kind of work with it on a deeper level and learn different techniques. (Ben)

Notably, Ben's choice of words mapped the description of a classroom focusing on practicing particular methods or techniques, as set forth in the Methods section.

Eir, who said she 'feel[s] like I learn a lot more one Monday per week than I do in half a year [in my traditional classes]', provided further insight:

> He [Peter] expects you to know it. So he never stops for details, such as 'Now we have to remember how to solve parentheses'. It is kind of … more at a higher level. We do not have to deal with those simple details all the time. (Eir)

She went on to explain how they had to draw on already established knowledge while recognising that different mathematics ideas are connected:

> You get to use a lot more of the mathematics … if you are about to find the area, you need to remember how to do that, and maybe put it into an equation, or maybe you have to use that knowledge in similar triangles. You get to use all of it in one task. And it … requires a lot more knowledge. (Eir)

Dina supported Ben and Eir when reflecting on her own understanding and learning processes:

> I experience a big difference between Peter's classes and the traditional classes. … I think it is motivating—the way in which they [Peter's classes] are a bit more difficult. I feel that if you manage to do something, then it is more motivating than when doing traditional tasks … You have to use different methods to really get it … you use more parts of the mathematics and more of your brain. (Dina)

Dina went on to explain in more depth how the problem-solving approach enabled her to solve a given problem in several ways:

> … you don't limit yourself to one method all the time. I think it is easiest to solve [geometry] problems by using similar triangles, and I think it is hard to solve them using equations … You have to think about how to solve the problems, and you have to think, "Should I use similar triangles? Should I use ratios? Should I use the Pythagorean theorem?" or whatever. And there is no—it's not set for you. You kind of have to arrange it for yourself. (Dina)

In this way, Dina searched her memory, trying to connect different methods to the problem at hand. Collectively, the students' comments were in favour of the problem-solving classroom because a) it focused on connectedness in mathematics, b) it was challenging and c) there was collaboration.

When discussing the problem in Figure 1 during the interviews, the first author knew that the students had encountered these problems in Peter's class the week before. The observation notes stated that the problem was drawn on the board, while it was emphasised that the two triangles (ABC and DEF) were equilateral. Furthermore, Peter made it clear that there were no specifications for the lengths of the sides of the triangles. After that, the students worked in groups. This was how Ben saw his reasoning process for solving this particular problem:

> First, with the information given, [we see that] the two large triangles are equilateral. Because they are equilateral, we can say that the three smaller triangles are congruent … You can use calculations and find out that the angles are similar. Then, the longest side forming the right angle is equal in all three triangles because it is equal to the sides in the equilateral triangle. (Ben)

Ben had a precise and confident way of using mathematical terms. He seemed trained to share his reasoning and had no problem explaining his lines of thought. Ben's responses did not include the answer, calculations or procedural matters; instead, he focused on the concepts and problems within the task while including important information needed to put forward a logical argument, integrating his knowledge into interrelated (conceptual) systems. The first time Ben encountered the problem, he solved it in a different way: 'I did not realise that the smaller triangles had angles with values of 30, 60 and 90 degrees. So I ended up with something much more difficult' (Ben).

He improved his way of thinking between the class and the interview. Once he realised that the smaller triangles were congruent, finding the area of the two equilateral triangles was simple: by using the Pythagorean theorem ('You have to find the baseline for the outer triangles by applying Pythagoras' [Ben), he was able to calculate the heights of the triangles, which gave him what he needed to calculate the areas. The problem involved many difficult aspects that caused Ben to have a productive struggle, leading him to engage with the problem even after class.

Eir and Dina (both Level 4 students) were in a group with two other girls when encountering the problem in Figure 1. They struggled when determining how to begin. After approaching Peter, they were given help finding the properties of the three smaller triangles. Although they initially set the value of AB as X, they were still unable to calculate the area of triangle ABC. They continued by first trying to find the area of the smaller triangles. However, they encountered a problem when facing squared fractions using the Pythagorean theorem. During the interview, Eir commented:

> I thought it was special that it [the task] had no numbers. No measurements—nothing. The angles in the triangles are the same, and these three triangles [pointing at the figure] are congruent, and these two are equilateral … I tried to solve it like three times, and I made some careless mistakes every time. But [it was] so much fun when I finally nailed it! (Eir)

Encountering a problem without measurements or numbers forced Eir to look for patterns and connections. Trying and failing multiple times with the same problem showed a commitment to participation. The nature of the problem caused Eir to use her current mathematical knowledge in ways she had never done before.

Discussion and concluding remarks

We did not intend to compare the two different mathematics classrooms in which our interviewees participated on a weekly basis. Rather, we saw the opportunity to explore how students with ongoing experiences from both a problem-solving classroom and a classroom that focuses more on exercises with closed structures could articulate in detail what the problem-solving classroom did for them. In the subsequent discussion, we will provide a summary of the key points emphasised by the students in their accounts of what they learned, how they learned and the factors they believed contributed to their learning in the problem-solving classroom.

We assert that it is worth drawing attention to the way in which the two girls, both at Level 4 in mathematics, talked about their thinking and engagement in the problem-solving classroom. The challenging non-routine problems presented gave them, and Ben, opportunities to access materials that connected different mathematical concepts and ideas. This made them view problems from different perspectives, relate new ideas to previous experiences and integrate their knowledge, building on knowledge they had already internalised. The problem-solving classroom made them realise that there was more than one way to approach a given problem. These findings add to the list of positive outcomes from engaging with cognitively demanding problems reported in Cai (2014), Cai and Merlino (2011), Cobb et al. (1991), and others.

The observations and interviews clearly showed that the students appreciated the focus on cooperation in the problem-solving classroom. They were challenged to 'talk mathematics' and explain their thinking and reasoning in precise language. This was necessary due to the challenging nature of the problems presented. When engaging in complex problems, we found that the students showed perseverance in how they examined the logic of their arguments, presented their reasoning and searched for help from their peers when building on one another's lines of thought. As in Cai and Merlion's (2011) study, the students were motivated by the challenging tasks prepared by Peter to fit their levels (which is decisive; see Zhu and Fan, 2006, p. 612). This made their joy from solving them much more meaningful, with the challenging nature of the problems and their productive struggle causing them to engage with the problems even after class.

We acknowledge the limitations of the current research, which are connected to our focus solely on one problem-solving classroom. Nevertheless, we contend that perspectives from students experiencing weekly mathematics instruction in two different classrooms, where one focused on problem solving only, add valuable insights, especially considering how the students focused on what they gained from the problem-solving approach instead of what they lost.

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Expressiveness in an embodied learning activity in mathematics

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The idea of connecting mathematical learning and bodily performance has been previously investigated in embodied design studies and STEAM (Science, Technology, Engineering, Arts, Mathematics) research. Yet, little is known about the expressive qualities of connecting mathematics to bodily performance. To address this gap, we lean on the perspective of embodied learning to examine the expressive dimensions in a number-sense activity developed for seven- and-eight-year-olds. Guided by Laban/Bartenieff's framework for movement, our qualitative analysis shows how time (tempo), weight, and shape, along with rhythm and the phrasing of these expressive dimensions, can extend students' number sense in their full-body performance. This paper enhances our comprehension of the notion of expressiveness within the context of embodied mathematics.

Keywords: *STEAM, full-body performance, embodied learning, expressiveness, number sense.*

Introduction

A growing body of educational mathematical research embraces the view that learning is intimately connected to embodied experiences (Shapiro & Stolz, 2019). Embodied learning is recognised as a key domain for the innovation of student-active designs, integrating the moving body to foster emotions, engagement, and meaning-making in the learning of 21st-century mathematical skills (Paniagua & Istance, 2018). Embodied learning encourages students to innovatively plan their movements to illustrate mathematical ideas, fostering experimentation and creativity in the learning process. The idea of extending mathematical concepts into bodily performance has been investigated in research on the integration of gestures and upper-body movements (Novack et al., 2014; Radford et al., 2017). In embodied research within digital learning environments, some studies support the connection between full-body movement and mathematical learning (e.g., sideways movement to model the linearity of the mental number line; Dackermann et al., 2016; Ruiter et al., 2015). The integration of full-body movement is also observed in STEAM research concerning the intersection of mathematics and the arts, showing how creative dance and bodily performance support students' mathematical learning (e.g., Leandro et al., 2018). However, expressive aspects of full-body experiences, such as pace, rhythm, weight, and body shape, remain less explored (Gandell, 2022), especially in non-digital learning environments where number sense is a focal mathematical domain. Therefore, in the context of an embodied learning activity, we ask how students' number sense can be extended through the expressive dimensions of full-body performance. We aim to enhance our insight into the design of immersive learning environments that link students' creative bodily performance and dynamic movements with meaningful mathematical experiences.

Bodily performance within STEAM research

Previous STEAM studies have explored how performative arts and dance can be incorporated into mathematical education to engage learners aesthetically and emotionally, bridging abstract concepts with bodily performance (Fenyvesi et al., 2019; Kaufmann & Dehline, 2014). Wood (2008) used square dance as a pedagogical tool to enhance students' number sense, including concepts like fractions, counting skills, and doubling. In a related case study, Budinski et al. (2022) brought together mathematicians and dance choreographers to transform mathematical concepts into embodied and artistic expressions. In an inventive approach, the study integrated static body postures with dynamic movements to embody concepts like linear functions and exponential growth. Similarly, Parsley and Soriano (2009) explored the integration of mathematics and dance, focusing on students collaboratively translating geometric concepts into static, embodied, and artistic expressions. Further support for the mathematics-dance connection comes from the study by Leandro et al. (2018), which showed that 7- and 8-year-olds who integrated mathematical learning with creative dance exhibited significantly better learning outcomes than the control group. Other evidence from embodied design studies shows how dynamic movement qualities such as pace and rhythm can serve as embodied extensions of children's number sense (e.g., Bjørnebye, 2022b).

Embodied learning and Laban/Bartenieff's movement framework

The embodied perspective of thinking rejects the mind-body dualism and the separation between internal and external representations. Instead, it posits that mathematical thinking is a relational phenomenon, extending beyond the brain to include the sensorimotor system and the surrounding environment (Newen et al., 2018). From the embodied perspective, learning is situated within multimodal experiences, emphasising the importance of coordinating information from different sensory modalities and transitions between them (Barsalou et al., 2003; Nathan & Alibali, 2021). Accordingly, learning can be observed as moving in new ways (Abrahamson & Lindgren, 2014) and students' number sense can be extended through distinct countable actions. For example, combining verbal counting with rhythmic stepping in a square pattern, extends ordinal and cardinal concepts of numbers via the structured experience, with the final step symbolising the total.

This focus on movement qualities finds resonance in Laban/Bartenieff's movement framework (Hackney, 2002; Wahl, 2019), which comprises four key elements: Body, Effort, Shape, and Space (BESS). These BESS elements enable the classification of both quantitative and qualitative aspects of movement, where movement functionality is associated with Body and Space, whereas the expressive aspects of movement are operationalised through the Effort and Shape elements (Hackney, 2002). Effort is about how the movement takes place and captures its qualitative aspects, such as rhythm, force, pace, spatial orientation, and smoothness of movement. Effort comprises four factors, each positioned along a continuous scale between two dichotomies: Weight (Light-Strong), Time (Quick-Sustained), Space (Direct-Indirect), and Flow (Bound-Free). Shape describes the expressive qualities of the body, capturing static and dynamic forms and configurations created by the body. Individuals combine, sequence, and organise the different BESS elements to create a coherent and meaningful sequence of actions, which can be denoted as phrasing (Hackney, 2002).

Method

Educational design research concerns the theory-driven iterative development, testing, and refinement of learning environments (Gravemeijer & Cobb, 2006). Data were collected using hand-held video cameras, capturing students' embodied actions in detail. Informed by the embodied perspective, four design principles (DPs) guided five cycles of testing and refinement of the activities (Bjørnebye, 2022a). These were DP1) high degree of consistency between number sense and bodily performance; DP2) aesthetic and expressive qualities of bodily performance (e.g., elegance, rhythm, tempo, weight); DP3) play and collaboration, and DP4) speech, and dialogue. Based on our aim of investigating expressive movement qualities within the context of an embodied activity, this paper focuses on the first two DPs. The mathematical content involved key elements of 7-to-8-year-olds' number sense such as cardinality, ordinality, part-whole relations, and addition (Sarama & Clements, 2009). Strategies of addition involved direct retrieval (e.g., recalling the answer to *9+3* from memory), counting-on (e.g., counting "nine, ten, eleven, twelve" to solve *9+3*), and decomposition (e.g., solving *9+3* as *9+1+2=10+2=12*) approaches.

Participants

The embodied design project took place in a Norwegian primary school and involved first- to third-grade students and their teachers (Bjørnebye, 2022a). This paper focuses specifically on an activity developed in collaboration with fourteen second graders, aged 7 and 8. The research was conducted with approval from the Norwegian Centre for Research Data and adhered to the ethical guidelines established within the Norwegian context (NESH, 2016). Consent was given by the caretakers of the participating students and their teachers.

The BESS elements for qualitative analysis and data selection

Informed by Laban/Bartenieff's movement framework (Hackney, 2002; Wahl, 2019), our analysis employs weight, time (tempo), and shape, along with rhythm and composite descriptions for phrasing these expressive dimensions. To encompass the spectrum from quick to sustained time in relation to movement tempo, we employ descriptive terms like sprint, jog, quick/slow walk, rigid/fluent dance, and freezing/pausing. Apart from the term time, we will use the term tempo to be able to emphasise the movement effort. Similarly, to explore the continuum between light and strong weight, we utilise terms like heavy/light walk, and tiptoeing/tramping dance. Shape captures both static and dynamic forms and configurations created by the body and is analysed using descriptive notions like freezing body posture and forceful body rotations. The expressive movement elements described can be combined in various ways (Hackney, 2002), and through the individual's unique phrasing of these elements, we can gain a holistic understanding of the potential dynamics and relationships of expressiveness within the context of the embodied activity. In our analysis, we use two complementary approaches to phrasing. The first involves the concept of rhythm, defined as repeating patterns in movement such as walking, sprinting, and square dancing. Rhythm acts as a cohesive concept linking the different BESS elements, with effort changes creating dynamic shifts that align with verbal, auditory, and bodily rhythms. The second approach involves providing a detailed description of composite sequences of expressive elements in embodied action, which reflects students' number sense. Our selection of excerpts for analysis aims to capture common patterns and contrasts within each expressive element, conveying narratives of how individuals phrase these elements into coherent sequences of full-body performance that extend their number sense.

The embodied mathematical activity

The embodied mathematical activity was situated on parallel number lines sprayed on the asphalt, ranging from 0 to 30, with numerals written at one-metre intervals, and with icons for numbers 1 to 12. For example, in a square structure, 9 was illustrated by two complete 4-layers of distinct colours, and an additional colour layer on the dot in the upper right corner (*2·4+1*). The students worked in three teams, each provided with number line, flags, and dice. The learning objective was the concept of addition, but it also involved part-whole relations of numbers and properties such as cardinality, ordinality, odd, and even numbers (DP1). A cone placed at the number 9 determined one addend of the addition (i.e., $9+x$; where $1 \le x \le 6$). The students rolled the die, carried the flag while running past the addend 9, continuing along the line to add the value of the die, and square dancing the sum of the addition before returning to the start. The square dance involved using the right foot to touch the upper right dot of the iconic square structure, followed by touching the upper left dot with the left foot, and proceeding by moving the right foot backwards, and subsequently the left foot, to touch the dots behind, completing a 4-step cycle. In the final step, the student should shout out the answer (DP4) while performing a body rotation or a body posture if the sum was even or odd, respectively (DP2). The students had the autonomy to select their addition strategy (DP1), set their own pace, opt for individual or collaborative work, and incorporate graceful and creative movements (DP3 and DP4). The students had prior experiences with the square dance and the number line, although those activities did not focus on strategies in addition.

Results: Observations of the embodied activity

We first present observations of the addition strategies used in the embodied activity, followed by examples of students' extension of number sense through the expressive themes of time (tempo), rhythm, shape, weight, and phrasing. To illustrate these findings, we will use the figures below.

Figure 1: Rhythmic point-counting

Figure 2: Tramping

Figure 3: Stiff dance

Figure 4: Quick and light-weighted dance

Figure 5: Light-weighted body posture

Figure 6: Rotation after sprinting and slow dancing

Figure 7: Forceful rotation after quick dancing

Figure 8: Forceful rotation after slow dancing

Figure 9: Student A in quick tiptoeing dance

Figure 10: Student B in counting a new

Figure 11: Student C in slow tramping

Figure 12: Student C in light-weighted body posture

We observed three main approaches to solving the addition problems. 1) The counting-on strategy was observed as stopping at the reference 9, and then counting on ones equal to the number of the dice, for example, "nine, ten, eleven, twelve" in combination with stepping and/or finger-pointing (Figure 1). The cone at 9 was also used as a starting point for counting the die-number as a distinct unity, "one, two, three, four, five", then reading the answer "fourteen" after moving five units forward (cf. Figure 10). 2) The decomposition strategy was associated with breaking down the addition into two or three distinct movement trajectories. Examples include a fast walk to 9, pausing briefly, before proceeding two entities forward while saying "eleven". It was also associated with the shouting "six more than nine" while running to 10, then stopping for exploration, followed by a non-verbal sequence of 5 one-metre hard weighted steps (Figure 2), before concluding with verbalising "fifteen". 3) The direct retrieval strategy was related to straight movement to the solution, and it was frequently observed in answers in the interval 10 to 12 (e.g., Figure 4). But some used it consistently, for example, a student combining sprinting with shouting "nine and twelve", "nine and thirteen", "nine and fifteen", suggesting direct retrieval of *9+3=12* (cf. Figure 6), *9+4=13*, and *9+6=15*, respectively. The square dance encompassed a variety of expressive configurations, ranging from sustained and stiff light-weighted patterns (Figure 3), via slow, heavy-weighted, to more quick and lighter-weighted rhythmic dancing movements (Figure 4). In embodied extensions of odd-number answers, students balanced on one leg adopting body postures (Figure 5). Conversely, when dealing with even-number answers, they demonstrated forceful rotations (Figures 6-8). We also observed students collaborating, such as engaging in square dancing together, and peers providing verbal guidance like "walk one more" and "thirteen is freeze" to aid in performing addition and comprehending the rules. Next, we present three examples of students' phrasing of these expressive dimensions in composite sequences.

Student A: Running directly to 15, tiptoeing swiftly during a light-weighted square dance while whispering number words (Figure 9), executing a rapid and fluid rotation while vocalising "fifteen", and returning with a run. **Student B:** Quickly running to 9, then transitioning to slow forward walking while counting "one, two, three, four" (Figure 10), followed by a fast, articulated square dance, freezing in a body posture while saying "thirteen", and running back. **Student C:** Fast walk to 9, then slow walking and counting "nine, ten, … thirteen", then a slow and heavy-weighted tramping square dancing combined with a clear articulation of number words (Figure 11), followed by a lightly-weighted body posture for "thirteen" (Figure 12), before returning with a fast walk.

Analysis: Expressive movement qualities in the embodied activity

The results showed a tight interplay of expressive dimensions and students' number sense. The dimension time (tempo) was evident through variations in movement tempo, encompassing fast sprints and dance (Figure 4), quick rotations (Figure 7), as well as slower-paced walks and dance (Figure 3), deliberate steps (Figure 11), and even moments of freezing or pausing (Figure 5). This flexibility in tempo allowed the students to align their movements with their mathematical abilities, with faster movements potentially signifying quick numerical associations (e.g., direct retrieval) and sustained tempo movements reflecting a more thoughtful approach to numerical relations (e.g., counting strategies). Furthermore, the students seamlessly incorporated weight and shape in their movements along the number line. This ranged from dynamic bodily shapes seen during forceful sprints to more upright postures during heavy marching or light walking. Similarly, rhythm played a prominent role in the square dance, recurring as a movement pattern within students' extensions of part-whole relations as illustrated by the 12-dance involving three 4-step cycles (Figure 4). The dynamic interplay of the expressive dimensions time (tempo) and weight further contributed to the rhythmic qualities of their dance, spanning from fast and heavy-weighted stepping to rapid, fluent tiptoeing and ultimately to slower, deliberate tramping and stiff dance. Similarly, in the final step of the square dance, the weight and shape dimensions were evident in two extremes: light-weighted one-legged balancing in static body postures for odd numbers (Figure 5), and dynamic bodily shapes in forceful rotations for even numbers (Figures 6-8).

Students A, B, and C exhibited distinct actions and phrasing in response to the embodied activity. Student A opts for a swift and fluid sequence, moving directly to 15 with a dynamic blend of tempo and light-weighted tiptoeing during a square dance, adding a layer of verbal rhythm through whispering number words. In contrast, Student B showcases distinct shifts in tempo, starting with a quick run to 9 and then transitioning to slow counting and walking, ultimately integrating a fast, articulated square dance with freezing in a body posture for "thirteen". Student C's actions emphasise variations in tempo and weight, featuring a fast walk to 9 followed by slow counting and walking to 13 and a subsequent slow, heavy-weighted tramping square dance. In summary, these students' phrasing choices demonstrate unique interpretations of the expressive dimensions within the activity.

Discussion

Based on the embodied perspective, this paper explores how expressive dimensions of full-body performance can extend students' number sense in an embodied learning activity. Our paper builds on previous STEAM research, emphasising the significance of incorporating movement into mathematical education (Fenyvesi et al., 2019; Kaufmann & Dehline, 2014). As students navigated along the number line, they extended their ideas of ordinality and addition (e.g., direct addition, decomposition, and counting-on strategies) through bodily movements. Similarly, the rhythmic square dance extended the students' ideas of cardinality and part-whole relationships, while odd and even numbers were extended in body postures and rotations.

Just as Budinski et al. (2022) integrated static postures and dynamic movements to embody ideas of geometry and functions, our findings show how students can use a range of contrasting bodily shapes and patterns, from slow and deliberate to rapid and energetic forms in their extensions of number sense. In odd-number answers, the students adopted imaginative body postures, while forceful rotations were part of even-number answers. Similarly, in the students' extensions of ordinality, addition, and part-whole relations, we observed contrasting shifts in the expressive movement qualities of time (tempo), weight, and rhythm. This departure from the conventional classroom experience resonates with the study by Bjørnebye (2022b), which showed the integration of pace, rhythm, and rotation in 4- and 5-year-olds' full-body modelling of counting-based addition.

The analysis of Students A, B, and C's phrasing provides insight into how weight, time (tempo), and shape can be assembled into rhythmic, dynamic, and composite sequences of meaningful mathematical experiences. These various, unique, and contrasting extensions demonstrate a potential for creativity, emphasising the possibility to infuse personal expression and innovation into mathematical task design (Paniagua & Istance, 2018). This perspective aligns with Hackney's (2002) viewpoint that it is in the phrasing that individuals become personally expressive and form relationships with, in this case, mathematical ideas. The space revealed for individualised choreography of mathematical meaning aligns with previous STEAM studies, showing a connection between creativity, embodiment, and learning of mathematics (e.g., Leandro et al., 2018).

Movement and bodily performance are inherently intricate and dynamic phenomena; therefore, conducting research in this domain entails unique challenges. Given these complexities, it is important to note that our analysis provides only a glimpse into the multifaceted nature of movement-based expressive extensions of mathematical ideas. Our paper is further limited as it relies on descriptive narratives of students' movement qualities, and we recognise that incorporating quantitative measures could provide a more comprehensive analysis. However, our paper shows the practical utility of some BESS elements in fostering expressiveness within mathematical learning designs. Additionally, it contributes theoretically to the concept of expressiveness through the application of the Effort and Shape elements in Laban/Bartenieff's movement framework.

In summary, this paper adds to the growing body of STEAM research, delving into the integration of the body and movement in mathematical learning. Our findings not only underscore the potential for interdisciplinary approaches in mathematical education but also highlight the inherent limitations of traditional classroom settings in supporting richly expressive experiences. By incorporating expressiveness into mathematics education, we can create immersive learning environments that encourage students to engage with mathematics actively and creatively.

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Principles for and Approaches to Task Design in Mathematics Education: The Perspectives of Norwegian and Turkish Mathematics Teachers and Lecturers

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This study intends to shed light on the main principles mathematics teachers and lecturers use in designing, selecting, and/or adapting tasks for teaching mathematics and their approaches to address those principle(s). Through a phenomenological study, six mathematics teachers and lecturers from Norway and Türkiye were invited to semi-structured one-on-one interviews. The notion of praxeology in the Anthropological Theory of the Didactics (ATD) was used to analyze the data. Our findings revealed five main task design principles, including the consideration of inclusion, cognitive, affective, social, and curriculum and assessment aspects. Furthermore, different approaches were provided for different principles, and sometimes, the same approaches were offered to address different task design principles. Moreover, we observed that an approach for one person could be a principle for another, thereby reflecting the dynamic nature of the praxeology of task design on a personal level.

Keywords: Task design, principle, the Anthropological Theory of the Didactics, mathematics teachers, mathematics lecturers

Introduction

Tasks are the *"mediated artifacts"* through which students' mathematical learning is shaped and serve as an intermediary between the teacher, the student, and the mathematics (Rezat & Sträßer, 2012, p. 644). They are the cornerstones of classroom activities in cognitive, practical, and cultural terms as they allow students to make sense of the subject over their experiences (Watson & Ohtani, 2015). A task can be defined as a "written presentation of a planned mathematical experience for a learner, which could be one action or a sequence of actions that form an overall experience" (Watson & Thompson 2015, p. 143). Here, by tasks, we refer to any form of educational material, such as a question, problem, or activity that teachers or lecturers use for teaching mathematics. Selecting, adapting, or designing tasks for teaching or assessing mathematical knowledge is a regular part of teachers' and lecturers' professional activity (Sullivan et al., 2015). Due to the importance of this professional activity, the current study intends to investigate the principles that mathematics teachers and lecturers consider important in designing, selecting, and/or adapting tasks for teaching mathematics and the approaches for addressing those principles.

Theoretical Framework

The Anthropological Theory of the Didactics (ATD) is a theory introduced by Yves Chevallard. It provides a language to model the didactic world and has its own methodology for studying reality (Bosch et al., 2020). ATD perceives any human activity as a praxeology that can be broken down into praxis (i.e., task and technique) and logos blocks (i.e., technology and theory) (Chevallard, 2019).

The praxis block explains what people do (i.e., the task they engage with) and how they do it (i.e., the technique with which they undertake the task). The logos block explains why something is done in a certain way, which includes a technology justifying the technique and a theory justifying the technology (Chevallard, 2019).

As Bosch and Gascón (2014) argued, the theoretical part of any human activity is more elusive to capture because it is often overlooked. In the context of task design, Kieran et al. (2015) raised a similar concern, arguing that the principles of task design need to be clarified and expanded upon and that further theoretical investigation is called for in this field. Based on this assumption, we perceived task design, adaptation, and/or selection for teaching mathematics as a *praxeology.* We intended to carry out a *praxeological analysis* on this matter to obtain a fine-grained account of how and why tasks are designed, adapted, and/or selected in particular ways, thereby shedding more light on the praxis and logos of this endeavor. In this respect, designing, adapting, and/or selecting tasks for teaching mathematics could be considered the task component of the praxis block. The way(s) of designing tasks (i.e., approaches) for this purpose could correspond to its technique component, whereas the rationale (i.e., principles) for the chosen way(s) for task design refers to the logos block. We believe that using praxeology as an analytic tool could enable us to have a clearer picture of the principles and approaches that might have been overlooked and remained unarticulated so far. Moreover, important to note here is that praxeologies can change depending on the institutions, and even their elements can have different roles based on the praxeology in question (Bosch et al., 2020). For instance, Pythagoras' theorem could be the technological component of a mathematical praxeology called Trigonometry at the lower secondary level, whereas it could be used merely as a technique in the context of a topographic measurement at the tertiary level (Bosch et al., 2020). Uncovering these differences can allow us to reveal their dynamic nature, not only at the organizational level but also at the personal level. To this end, our purpose was to answer the following question:

> *Which principles and approaches do mathematics teachers and lecturers have in the praxeology of designing, adapting, and/or selecting tasks for teaching mathematics?*

Principles for and Approaches to Task Design in Mathematics Education

Principles could be regarded as "rules of design to effectively achieve the intended goals of a task" and "the goals of tasks refer to the intended goals for students' learning through completion of the tasks" (Yang & Ball, 2022, p. 13). Here, a principle for task design is defined as a consideration regarding the task that shapes and guides the planning and formulation of questions, problems, and activities used for teaching mathematics, whereas an approach for task design is defined as a way of addressing the target principle. Principles are important considerations for task design since they potentially close the gap between the teachers' intentions and their outcome on their students' learning (Coles & Brown, 2016).

Different viewpoints regarding the task design principles and the approaches for addressing those principles have been suggested in the literature. Informed by our analysis, here, we present the pertaining literature by organizing the principles and approaches under five major aspects: inclusion, cognitive, affective, social, and curriculum and assessment aspect.

Research to date has provided task design principles in relation to the inclusion aspect, such as considering students' prior knowledge and experience in and outside mathematics (e.g., Radmehr, 2023). One approach that the task designer could undertake is to identify the knowledge gap of the students that need to be addressed, thereby gaining insight into the students' prior conceptions (Van Dooren et al., 2013). Some principles align with cognitive aspects and can be based on knowledge or skills. For instance, some studies provided knowledge-based task design principles and approaches, such as in geometry (e.g., Krainer, 1993) or numeracy (e.g., Geiger et al., 2014), and several studies have also offered skill-based task design principles and approaches aiming to promote the development of target skills such as proportional reasoning (e.g., Chin et al., 2022). Affective and social aspects have also been considered in the design of tasks. In relation to the affective aspect, tasks could be designed in a way that they offer purpose and utility for the learner so that the task would become motivating for them (e.g., Ainley et al., 2006), while regarding social aspects, one example of task design principle could be to promote collaboration by working in teams (e.g., Doorman et al., 2019). Lastly, there are also task design principles that could be considered from the curriculum and assessment perspective. For instance, self-assessment could be considered an important principle in the design of mathematical modeling tasks (e.g., Patel & Pfannkuch, 2018), and the alignment of the tasks with the school curriculum is another principle that has been considered (e.g., Buchbinder & Zaslavsky, 2011).

Methodology

The current study utilized a phenomenological approach since it allows the investigation of a phenomenon that is experienced by a group of people through in-depth interviews (Creswell & Poth, 2018) and attempts to disclose the meaning of lived experiences in a textual format and provides implications for enhancing real-life practice (van Manen, 2016). Therefore, a phenomenological approach was chosen to investigate the phenomenon (i.e., designing, selecting, and/or adapting mathematical tasks for teaching mathematics) from those who have experience with it and who would give rich information pertaining to that issue.

Six participants from Norway and Türkiye participated in semi-structured one-on-one interviews. There were three participants from each country from different grade levels, ages, gender, and experience. The organization of the school system in Norway includes seven years of primary education (i.e., grades 1-7), three years of lower secondary education (i.e., grades 8-10), and three years of upper secondary education (i.e., 11-13) (European Commission, 2023), whereas in Türkiye, the educational organization includes four years of primary school (i.e., grades 1-4), four years of middle school (i.e., grades 5-8), and four years of high school (i.e., grades 9-12) (Gün & Baskan, 2014). The rationale for inviting participants from different countries and grade levels was to enrich our data with people having different experiences in task design based on their country context and grade levels at which they work. The interviews were recorded and transcribed verbatim. Pseudonyms were used for referring to the participants in the results section. Data analysis of interviews was done inductively (Creswell & Poth, 2018). Starting with an identification of patterns (i.e., how and why tasks are designed in a certain way), we have attached pertaining codes to data extracts and categorized them based on their similarities, leading to the formulation of five main principles. These principles are not treated as mutually exclusive; rather, we acknowledge that there are overlaps where one principle could interrelate the other(s).

Results

Regarding the logos block, our analysis identified five principles guiding the task design, adaptation, and/or selection by teachers and lecturers. These principles encompass the consideration of the following aspects: *(i) inclusion, (ii) cognitive, (iii) affective, (iv) social, and (v) curriculum and assessment.* In relation to praxis block, we identified several approaches that align with each of these identified principles. Here, we present only the main approaches in the space available in this conference paper.

Addressing Inclusion

Addressing inclusion was one of the main principles considered by the participants, which included tailoring the task design in alignment with the target group of students based on their cognitive level, grade level, and/or study program. Several approaches were provided to designing tasks based on this principle. One of the approaches was to design tasks that are accessible to all learners. For instance, Henrik, who has been teaching mathematics as a lecturer for 12 years at the tertiary level in Norway, explains as follows:

> I am trying to give exercises that are interesting so they are not just routine stuff but are accessible … if you know what has been going on in the lecture and think a bit about what the problem [is]… you can figure that out without … figuratively running against the wall.

It is seen above that Henrik tries to include his students in his lectures by using accessible problems so that students can manage to solve them without putting in extreme effort.

Addressing Cognitive Aspect

Addressing the cognitive aspect was another principle that was revealed from our analysis. This principle included subprinciples such as *(i) preparing students for the upcoming lesson, (ii) developing students' procedural skills, (iii) fostering students' critical thinking skills, (iv) helping students make connections between mathematical concepts,* and *(v) developing students' conceptual understanding*. Each subprinciple revealed its own approach. An example of an approach to the second subprinciple was given by Oliver as follows:

> We train them to be able to do these computations by hand, which is a very important skill. And so that means that some of the tasks will be designed with that in mind. Meaning that [it] will not feature very complicated numbers, very complicated algebraic expressions that need to be simplified, [or] very complicated integrals necessarily that only a computer can reasonably solve. So, you need to incorporate that into your design.

Oliver is a lecturer who has been teaching mathematics at the tertiary level in Norway for eight years. As evident from the above excerpt, he regards the development of procedural skills as an important consideration in task design. He believes that his learners should be capable of doing certain calculations by hand. Another example approach can be given from Ece's practice, for the subprinciple called helping students make connections between mathematical concepts. She is a mathematics teacher working at the primary level for two years in Türkiye, and her approach to addressing this subprinciple is to design tasks that include real-life examples. She explains as follows:

> I try to choose some daily life problems or [other] ways in which students can relate to previous or later topics.

Addressing Affective Aspect

Addressing affective aspect was another principle for task design proposed by participants. It included only one subprinciple in our sample, which was increasing students' motivation. For instance, Ece considers this subprinciple and designs tasks that involve active participation and real-life examples. She explains as follows:

> I also try to pay attention to the fact that the task or activity I prepare attracts the attention of the students because they [students] can get bored very easily or … they may have some difficulty paying attention. In that regard, I try to prepare real-life problems or tasks in which students could be active participants.

As the above excerpt demonstrates, Ece's main intention is to make her students more attentive to the lessons and be active participants. Her approach to doing so is to include real-life problems in her tasks that she thinks could attract students' attention and make them more active participants.

Addressing Social Aspect

Two subprinciples were identified in relation to addressing social aspect principle: *(i) encouraging active participation* and *(ii) encouraging collaboration*. For instance, Eivind, who has been a mathematics teacher at the primary level in Norway for 12 years, touched upon this principle as follows:

> I want the students to [be] actively involved with hands-on tasks and to talk about the tasks and the mathematical principles in the tasks at an early stage, and also let them cooperate and elaborate and give agreement together to see how they could solve the tasks. … And collaboration, they can learn from each other.

As seen above, Eivind wants his students to actively engage with the tasks, work collaboratively, and talk about the mathematical concepts in the tasks together, and his approach to doing so is to design hands-on tasks.

Addressing Curriculum and Assessment Aspect

Two subprinciples regarding curriculum and assessment also emerged in our analysis. These were *(i) considering the curriculum objectives* and *(ii) preparing students for the exam (i.e., university entrance exam or college exams)*. For the first principle, the approach was to design tasks in alignment with the curricular objectives. For the second principle, two approaches have emerged. As part of preparing students for the university entrance exam, one approach was to design tasks that included various types of questions. For instance, Bora, who is a high school mathematics teacher in Türkiye, explains as follows:

> I feel like I have to show more types of questions because 12th graders will take the university exam, […] because they need to see different types [of questions] at least. … we are working with sine and cosine theorems in trigonometry. Some questions include figures. In some questions, you need to draw the figure yourself. I also pay attention to that. I pay attention to

> the questions that are not all in the same type, but [for instance] some of them [are presented] without figures and the student has to draw it himself.

As seen above, Bora wants his students to be ready for the university entrance exam. He pays attention to providing his students with a wide range of question types, which he believes are important for them to see before the exam.

Discussion and Conclusion

The current study intended to unpack the praxis and logos block of the task design praxeology. When we looked at the logos block, we identified five main principles addressing inclusion, cognitive, affective, social, and curriculum and assessment aspects. Our results are in line with those observed in previous studies. Concerning the inclusion aspect, literature has also paid attention to the design of tasks that are built on student-related considerations such as their prior knowledge (e.g., Kieran et al., 2013). Our analysis revealed supporting findings in relation to the inclusion aspect since our participants touched upon this issue and mentioned how this influences their design. We also noted corroborating results concerning principles related to the cognitive aspect. Teachers and lecturers mentioned how important it is for students to develop mathematical knowledge or skills such as critical thinking. In the literature, some studies also reported task design principles in relation to the cognitive aspect, such as the importance of developing students' knowledge and skills (e.g., Yerushalmy, 2015).

In relation to the praxeological analysis, two important conclusions could be drawn. First, our analysis revealed that it was possible to see similar approaches for different task design principles. For example, designing real-life tasks was perceived to help students make connections between mathematical concepts as well as to increase their motivation (e.g., Ece). Similarly, designing hands-on tasks was perceived to help develop students' conceptual understanding along with encouraging them to participate and collaborate actively (e.g., Eivind). This finding indicates that within the task design praxeology, sometimes a technique can be supported by different logos. Another important conclusion is that a technique (i.e., an approach) for a participant could be a logos (i.e., a principle) for another. For instance, designing tasks that encourage active participation is a technique for Ece, and she uses it to increase her students' motivation, whereas it is a principle for Eivind, whose technique is to design hands-on tasks. This finding concurs with the postulation that depending on the praxeology at stake and the manner it occurs and develops in an institution, a particular object can be considered a task, technique, technology, or theory (Bosch et al., 2020). The institute in question here is the same for both teachers on a grade level basis, but the teachers work in a different country, which might have accounted for this variation. It is also possible to interpret this variation based on personal praxeologies, as personal differences play as much a role in shaping praxeology as institutional differences (Bosch & Gascón, 2014).

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Designing learning activities to promote algebraic thinking: Equation solving in Grade 5

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The aim of this study is to give insights into the design, implementation, and re-design of a teaching-learning activity on algebra at fifth grade. The mathematical topic was linear equations. Drawing on algebraic thinking as a theoretical construct, two progressions are analyzed, from arithmetic equations to algebraic equations and from actions on concrete objects to symbolic manipulations. The study reports design principles developed from a pilot study and theory, and the design-based research process in which an algebra learning-teaching activity (ALTA) was co-designed by mathematics teachers and researchers, through which the design principles were revised. The revised design principles are reported.

Keywords: Algebra, design research, fifth grade, learning and teaching algebraic thinking.

Introduction

The research reported here took place in the context of a research and development project called ALGEBRA – Algebra Learning: Generalization, Expressing, Balancing, Reasoning and Argumentation. The goal of the project was to generate insights about the nature of effective learning environments, tasks and tools for developing algebraic thinking in middle school mathematics teaching, and to produce such learning environments, tasks and tools. These were called ALTAs: Algebra Learning-Teaching Activities. The reported research is not an empirical study in a classical sense, but rather an account of the design of an ALTA for Grade 5 on equation solving. We will first give an account of its relevance, both with respect to the curriculum in Norway and within the research literature on algebraic thinking. Second, we will situate our study within a design-research methodology. Third, we will give an account of the evolving design principles of the ALTA.

The reasons for developing the project were found both in the research literature (see below) and in the current Norwegian curriculum (Ministry of Education and Research, 2020). The curriculum launches what is called "core elements," which are principles that should permeate the mathematics teaching at all levels. The three core elements most relevant for this research project are *Reasoning & Argumentation* (making sense of the reasons behind mathematical rules and relationships), *Representation & Communication* (representing mathematical ideas in various ways, concrete, visual, verbal and symbolic) and *Abstraction & Generalization* (gradually developing thinking and strategies from concrete descriptions to the use of symbols and formal reasoning).

The curriculum (Ministry of Education and Research, 2020) also put forward concrete competence goals for what fifth graders are supposed to be able to do after fifth grade, including:

- solve equations and inequalities through logical reasoning and explain what it means that a number is a solution of an equation

This competence goal is new in this version of the curriculum; in the past equations were only introduced at Grade 8, so teachers lack experience and resources to achieve this goal. Algebra is vital to academic and economic success (Moses & Cobb, 2001) and an essential part of professional skills for the 21st century (Blanton et al., 2011), but Norwegian students struggle with algebra (Kaarstein et al., 2020). Hence there is a need to explore new ways to teach algebra, and specifically to introduce equation solving in Grade 5. Based on these considerations, we have formulated the following research question for the current study:

> What design principles can be derived from empirical investigations of an ALTA for introducing the solving of equations in grade 5?

Theoretical principles

We situate our study within a sociocultural perspective on learning and development (Rogoff, 1990, Vygotsky, 1986) in which mediating tools such as language, artefacts, gestures, drawings, and symbols are used to reveal (algebraic) thinking. The design of the ALTA was based on research on algebraic thinking, and on a novel combination of two progressions described in the research literature: (1) from arithmetic equations to algebraic equations (e.g., Filloy & Rojano, 1989) and (2) from actions on concrete objects to symbolic manipulations (e.g., Bruner, 1966; Kotagiri, 2008; Tall, 1995). Here we outline the theoretical principles related to each of these.

Algebraic thinking

Many studies have investigated algebra interventions in schools, prior to the usual introduction of algebra in Grade 8. We are drawing particularly on two seminal studies. Blanton et al. (2019) coined what they call “four fundamental algebraic thinking practices” (p. 194) that are crucial for students’ learning of algebra. These are “generalizing, representing, justifying, and reasoning with mathematical structure and relationships” (p. 194). Generalizing is the process of extracting an underlying structure or relationship from specific instances. Generalizations can be represented in many ways, including traditional algebraic symbols, but also through concrete objects, drawings, gestures and natural language. Justifying involves building “an argument about the validity of a generalization within a given representational system” (p. 195). What we call algebraic reasoning corresponds to Blanton et al.’s “reasoning with mathematical structure and relationships” (p. 194) in which “one acts on generalizations as mathematical objects … themselves in novel situations” (p. 195). These four thinking practices were seen across three fundamental content domains, “generalized arithmetic; equivalence, expressions, equations, and inequalities; and functional thinking” (p. 195). The design principles of the ALTA reported in our study addresses reasoning with mathematical structure and relationships within the content domain of equations.

Radford (2018) analyzed students’ developing algebraic thinking in Grades 2 to 6. He identified and described algebraic thinking in the actions of students with limited ability to express themselves in writing. Thus, Radford paid particular attention to the various semiotic systems that students used to express mathematical variables. He found that the students revealed their algebraic thinking through different semiotic systems, both non-symbolic and symbolic “early algebraic generalizations” (Radford, 2018, p. 3). In our design principles we also emphasize students’ use of semiotic systems as mediating tools as we emphasize student actions with concrete materials, drawings, and symbols.

Arithmetic and algebraic equations

The distinction between arithmetic and algebraic equations was first made by Filloy & Rojano (1989). The key characteristic of an arithmetical equation, e.g. $2x + 1 = 3$, is that solving it involves only operations on numbers. This is not the case when considering equations of the type $\mathrm{A}x + \mathrm{B} = \mathrm{C}x + \mathrm{D}$, i.e. what is called an algebraic equation. Solving an algebraic equation, e.g. $2x + 1 = x + 3$, requires operations on the unknown. This distinction is related to another fundamental distinction in algebra research, between seeing the equal sign as connecting a calculation on the left with a result on the right, and seeing it as marking the equality of the two sides of the equation.

From actions on concrete objects to symbolic manipulations

It has long been recognized that learning can progress through stages from more concrete representations to more abstract representations. Bruner (1966) for example distinguished between three stages:

> Enactive (action-based). This stage is sometimes called the concrete stage and involves manipulating physical objects.
>
> Iconic (image-based). This stage is sometimes called the pictorial stage. It involves images or other visuals that are first used to represent concrete situations enacted in the first stage. These can be drawn images of the objects or imagined pictures. Later these images can themselves be acted on.
>
> Symbolic (language-based). This stage, sometimes called the abstract stage, takes the images from the second stage and represents them using words and symbols, that are later acted on directly.

A similar progression is found in the Suido Method (Tooyama & Ginbayashi 1971, cited in Kotagiri, 2008) used in Japan. Kotagiri (2008) refers to its stages as "Real World," "World of Models," "World of Schemas," and "Mathematical World". The "Real World" and the "World of Models" divide Bruner's Enactive stage according to whether the concrete objects being acted on are actual situations the students find themselves in, or representations of them. For example, actually distributing sweets among children is a "Real World" action, but distributing wooden blocks that represent the sweets is an action in the "World of Models". Kotagiri's "World of Schemas," and "Mathematical World" correspond to Bruner's Iconic stage and Symbolic stage, respectively.

Tall (1995) adds a stage beyond the symbolic, the Formal stage. The symbols in the Symbolic stage *represent* pre-existing concepts. At the Formal stage the symbols do not represent anything, and are manipulated according to syntactic rules.

Based on this literature, we used five stages of representations in our design of the ALTA, synthesizing the mentioned progressions. They are summarized in Table 1, along with correspondences to the literature. We adopted Kotagiri's (2008) distinction between concrete contexts and models, which Bruner (1966) and Tall do not make, as well as Tall's (1995) distinction between symbolic and formal actions which is missing from Bruner and Kotagiri.

In accordance with Radford (2018), we argue the importance of distinguishing between action and expressing an action verbally, either orally or in writing. Thus, at each stage mentioned in Table 1 we see a distinction between acting within that stage and expressing those actions. Furthermore, we see using representations to express actions at an earlier stage as being important in the transition from one stage to the next. For example, representing actions on concrete models using drawings can be a step towards acting on drawings independently of concrete models. Thus, we do not view these stages as static levels of any kind, but rather as dynamic. However, the stages are used pragmatically to address the students' various actions and expressions of those actions.

<table>
<tr><th>Stage</th><th>Bruner</th><th>Kotagiri</th><th>Tall</th></tr>
<tr><td>Actions in concrete contexts</td><td rowspan="2">Enactive</td><td>Real World</td><td rowspan="2">Enactive</td></tr>
<tr><td>Actions on concrete models</td><td>World of Models</td></tr>
<tr><td>Actions on drawings</td><td>Iconic</td><td>World of Schemas</td><td>Iconic</td></tr>
<tr><td>Actions on symbols</td><td rowspan="2">Symbolic</td><td rowspan="2">Mathematical World</td><td>Symbolic</td></tr>
<tr><td>Formal actions</td><td>Formal</td></tr>
</table>

Table 1: Stages of representations associated with the research literature

Design-based research

Our research is design-based, inspired by Bakker's comment that "[m]ost educational research describes or evaluates education as it currently *is*. Some educational research analyzes education as it *was*. Design research, however, is about education as it *could be* or even as it *should be*." (Bakker, 2018, p. 3, emphasis in original).

Design-based research is structured in cycles of four steps: specifying the content to be taught with respect to the teaching goals (the theory phase, i.e. solving of linear equations), designing what will be taught according to clear design principles (the design phase, i.e. mathematics teachers and researchers collaborate), and undertaking a design experiment (the implementation phase, i.e. the mathematics teachers conducting the teaching of the designed ALTA) and finally analyzing the data gained towards a theoretical perspective of the teaching-learning activity (the evaluation phase). This then informs revisions of the content, the design principles and the lesson design, completing the cycle. Several cycles are to be conducted to (1) improve and adapt the teaching-learning activities and (2) refine and test the theoretical knowledge of how teaching-learning activities may work best in the next design experiment (see Prediger, 2019; Bakker, 2018).

Analysis – The design, implementation and re-design of an ALTA

Three design principles were used to design the initial ALTA:

1) Move gradually from actions on balances to actions on tiles, to actions on drawings to actions on symbols,
2) With each mediating tool, move gradually from simple arithmetic equations to more complex algebraic equations, and
3) Increase the complexity of the equations along with the increasing abstractness of the mediating tools, but with overlap in the equations used with each tool.

Four levels of equations were specified: 1) addition with unknown answer ($1 + 2 = x$); 2) addition with unknown addend ($3 + x = 5$); 3) multiplication by an unknown factor ($2x = 4$); and 4) algebraic equation with unknown on both sides ($3x = x + 6$).

Design of the initial ALTA – Cycle 1

Based on these principles and levels, the ALTA as designed comprised:

Lesson 1: Physical experiences with balances

The students are presented with a balance, and bags of different colors and weights unknown to the students. Students must explore and discover ways to manipulate the bags so that balance is maintained. Questions for reflection are 1) Which actions maintain equality/balance?; and 2) What changes can be made, but the balance is maintained, while using the same bags?

Lesson 2: Using the balance to solve equations

The students must solve equations using a balance. The equations presented are at all four levels, i.e. 1) 1 marble + 2 marbles = bag with an unknown number of marbles; 2) 3 marbles + bag with an unknown number of marbles = 5 marbles, 3) 2 bags with an unknown number of marbles = 4 marbles; and 4) 3 bags with an unknown number of marbles = bag with an unknown number of marbles + 6 marbles. By manipulating the bags (adding/taking away/moving) and maintaining the balance, the students must find out how many marbles the bags contain.

Lesson 3: Manipulation of physical objects that represent the balance (algebra tiles)

Algebra tiles are introduced. The green (long rectangle) tile represents a bag with an unknown number or marbles (see Figure 1a). By manipulating the tiles, the students solve equations at all four levels.

Lesson 4: Drawings of physical objects that represent the balance

The students solve equations at levels 2–4, using drawings of tiles (see Figure 1b).

Lesson 5: Symbolic solution of equations

The students solve equations at levels 2–4, using symbolic representation of equations. Text tasks are also presented involving equations at levels 2–4.

Results from the implementation – Cycle 1

The ALTA was taught by an experienced teacher who had volunteered to participate in the ALGEBRA project. Formal consent was obtained for all students in his class to participate in the research. He planned his teaching based on the ALTA but made whatever modifications he felt were needed to fit the conditions in his classroom. For example, he taught the five lessons of the ALTA in four class sessions. Thus, his implementation deviated from the design, both when it comes to the introduction of symbolic equations earlier than the last session and when it comes to letting students to use tiles in equation solving in the last session. In the first session, the students were given balances and several objects of unknown weight. They used Numicon pieces as a standard measure to find the weight of each object. This class session combined Lesson 1 & 2. After working in groups, the students gathered as a whole group for a summary, as usual for this class. Actions that maintained balance were stated and the teacher introduced the notation x to stand for the unknown weight.

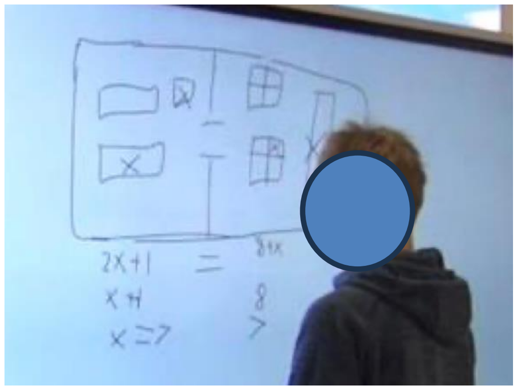

Figure 1a and 1b: Tiles and drawings representing the equation 2x + 1 = x + 8

In the second session, the students were given a worksheet with eight equations (levels 1–4) presented as pictures of tiles (see Figure 1a). Some were able to solve the equations by formal manipulation of the tiles while others needed to connect the tiles back to the balances. In the summary, students volunteered to solve the equations on the interactive whiteboard. The equations were represented as drawings of algebra tiles, and the operations they performed were shown by crossing out tiles. They also recorded their actions symbolically (see Figure 1b).

In the third session, since the students had used drawings and symbols to record their solutions in the second session, the teacher decided to move directly to solving equations symbolically and presented in texts. Algebra tiles were still available as an aid. The students were given a worksheet with 14 equations. The first eight equations were the same as the ones the students solved in the second session but represented symbolically. The last six equations involved numbers too large to be represented with tiles. The students first worked individually before joining together in groups to solve the tasks. Towards the end, the students were also given a worksheet with text tasks, where the task was to create letter expressions that represented the situation in the task. As before, after the group work, the students gathered for a summary. In the group work the students were frustrated by these tasks and became distracted. They were unable to solve most of the equations.

In the fourth session, the students were given a new worksheet with 14 equations. The first four were at levels 2 and 3, and the remainder at level 4, however tasks 5–8 did not require dividing. Only the last two tasks involved numbers too large to be represented with tiles. The students were instructed to use tiles but to write down their solutions symbolically and recording each step of the solution. They could, when they became confident, solve the equations without using tiles, but they should still record the steps of their solution. The students worked well in their groups, and most were able to solve all the equations.

Revised design principles

We set out to answer the question: "What design principles can be derived from empirical investigations of an ALTA for introducing the solving of equations in grade 5?" Based on our experiences and the results communicated above, the revised design principles are (revisions in italics):

1) Move gradually from actions on balances to actions on tiles, to actions on drawings to actions on symbols, *without omitting stages.*
2) With each mediating tool, move gradually from simple arithmetic equations to more complex algebraic equations*, from equations that do not involve division to those that do.*
3) Increase the complexity of the equations along with the increasing abstractness of the mediating tools, but with overlap in the equations used with each tool.
4) Distinguish between equations that can be represented, or not represented, with each mediating tool, for example, those that can be represented using tiles and those that cannot.
5) Emphasize the shift from the equal sign marking the answer of a calculation to the equal sign representing the equality of two expressions.

The most important change, inspired by Kotagiri (2008), was a strengthening of Design Principle 1. The importance of using drawings and symbols to record actions on tiles was shown by the problems encountered in the third session when the move to actions on symbols was made without much prior use of symbols to record solutions. While a few students had used symbols on the whiteboard at the end of the second session, most had only watched.

The levels of equations were also refined, with new distinctions being made between equations with unknowns on both sides that require division, and those that do not (Design Principle 2), and also between equations that can be represented using tiles and those that cannot (Design Principle 4). Eight levels of equations are distinguished, four arithmetic and four algebraic (See Table 2). Level R1 is used only in Lesson 1 with balances and levels L1–L4 are not used in that lesson. Levels R2 and R3 are not used after Lesson 2. Tasks in all lessons progress through the levels. These eight levels of equations thus testify to the second revised design principle and the fourth revised design principle, in moving gradually from the mathematically easier arithmetic equations to more complex arithmetic equations, then gradually from the easier algebraic equations to the more complex ones. Moreover, in Lesson 4 some tasks are given that cannot be represented with tiles (cf. Design principle 4).

Levels	**1**	**2**	**3**	**4**
Arithmetic (R)	X = B	X + A = B	AX = B	AX + B = C
Algebraic (L)	AX = BX + C (A–B =1)	AX = BX + C (A–B > 1)	AX + B= CX + D (A–C =1)	AX + B= CX + D (A–C > 1)

Table 2: Eight levels of equations - four arithmetic and four algebraic

Design Principle 5 is inspired by Blanton et al. (2019) and Radford (2018). This principle was present in the original design of the ALTA but was not made explicit.

We have revised the Grade 5 Equations ALTA, and developed a new one involving word problems, both of which were taught by another Grade 5 teacher in Spring 2024. Those ALTAs incorporate changes based on the revised design principles, as well as practical changes based on the experiences in Cycle 1.

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Playful learning in mathematics in the first years of schooling: Opportunities and challenges. A theoretical contribution

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The aim of this paper is to give a theoretical account of what may be called a playful approach to the teaching and learning of mathematics in the first years of schooling. According to the mathematics education literature, playfulness and mathematical inquiry in mathematical activities may establish coherence between kindergarten and primary school. Mathematics education research literature is, however, sparse with respect to in-depth analyses of the role of play in early mathematics learning. In this study we report from an ongoing project aiming to generate research-based knowledge about how teachers may teach playful and inquiry-based learning activities that will contribute to developing children's appropriation of central concepts and operations of early mathematics through mathematical reasoning.

Keywords: First years of schooling, Mathematical inquiry, Playful learning.

Introduction

The aim of this paper is to give a theoretical account of what may be called a playful approach to teaching and learning mathematics in the first years of schooling. Playful mathematical activities are discussed, as to testify in concrete terms what this approach may entail. This paper reports from a research and development project called SPLEMAS – Supporting Playful LEarning in MAthematics in School. The main aim of SPLEMAS is to develop research-based mathematical learning activities that are successful and effective during the first years of primary school. The basic idea in the project is to ground learning in activities that are characterised by playfulness and mathematical inquiry. There are two reasons for this emphasis.

Firstly, these approaches to mathematics teaching and learning for the first grades of school is advocated in the curriculum of Norway (Ministry of Education and Research, 2020). This emphasis is grounded both in theoretical and research-based potentials of play for mathematics learning and in the current Norwegian educational ambitions of facilitating children's transition from kindergarten to school. The examples below are consonant with the emphases in the curriculum, where play and games are explicitly mentioned as important resources for reaching learning goals during the first grades. Furthermore, the playful inquiry approach to mathematics teaching and learning is consonant with the emphasis of the curriculum on *exploration and problem solving*, *reasoning and argumentation*, and *representation and communication*, which are all explicitly mentioned as core elements of mathematics learning for students at this age.

Secondly, research supports the idea that systematically organized play may serve as an important source for learning academic knowledge for young children. This applies to mathematics, where the first years are critical for the future learning careers of students (Clements & Sarama, 2021; Fisher et al., 2010; Hirsh-Pasek et al., 2009). Research and political documents also point to the significance of continuity in curriculum and pedagogy between kindergarten and primary school as one of the most important determinants for the future school career of students (Lillejord et al., 2017; OECD, 2017). Furthermore, playfulness and mathematical inquiry in mathematical activities may improve the transition from kindergarten to primary school. Research has shown that this transition may have unfortunate consequences for children's school careers, resulting in anxiety and in social-emotional problems (Lillejord et al., 2017). The positive effects of early childhood education may be jeopardized if the transition from kindergarten to school is not well prepared (Lillejord et al., 2017).

In the Norwegian kindergarten pedagogy, play is regarded as essential for children's well-being as well as for their social-emotional and cognitive development. Play is associated with child-initiated activity, which is voluntary, intrinsically motivated and characterized by fantasy, and this makes it challenging to combine play and curricular learning. Nevertheless, in the literature play is argued to be an important source of learning of mathematics (Breive, 2019; Clements & Sarama, 2021). Despite this extensive literature, the mathematics education literature is sparse with respect to in-depth analyses of the role of play in learning in primary schools. Early mathematics learning plays a significant role in this context (Claesens & Engel, 2013; Weiland & Yoshikawa, 2013). Early mathematics learning is thus particularly emphasized in SPLEMAS accordingly.

However, there is a considerable gap with respect to research-based knowledge about how to support this transition. SPLEMAS aims to generate research-based knowledge about how teachers may teach playful and interactive learning activities. We take a longitudinal perspective to this transition, not just a limited view to the very early schooling. Activities that will contribute to developing children's mastery and appropriation of concepts and operations of early mathematics. For this particular paper, however, we have formulated the following research question:

> What design principles can be derived from developing mathematical activities emphasizing playfulness and mathematical inquiry?

Playful learning and mathematical inquiry

In this study and in SPLEMAS we adopt a sociocultural perspective on learning (Vygotsky, 1986). Thus, we view learning as a process of appropriation of knowledge by individuals, mediated through the use of tools and interaction with others (Rogoff, 1990). More specifically, we address the issues of how playful activities can be designed and conducted in manners that make it possible for children to appropriate mathematical concepts and procedures that are embedded in play and gaming.

At a general level, our interest concerns the learning of mathematics through mathematical reasoning (MR) as it emerges in learning activities within the context of teaching. We adopt Jeannotte and Kieran's (2017) view of mathematical reasoning in school mathematics, as these authors argue that MR is "a process of communication with others or with oneself that allows for inferring mathematical utterances from other mathematical utterances" (p. 7). Thus, in the project we focus on MR in the tasks and mathematical activities we design as our interest is both in the processes of engaging the children in MR and the products in terms of how children, through communication and bodily activities, master and appropriate mathematical concepts and skills. We argue that MR is fundamental to students' mathematics learning, including the use of mathematical concepts and procedures. The design of our project thus necessitates the documentation of 1) processes of engaging in play; and 2) products, i.e. what students learn about mathematics and about how to nurture mathematics learning.

Playful learning in mathematics in the first years of schooling

Playful learning may be defined in various ways (Zosh et al., 2018). We see playful learning as a process in which mathematics learning is combined with a playful way of engaging with the mathematics. The literature exemplifies how playful learning experiences may involve children in role-play situations, where the children adopt various roles in a sociocultural context for mathematics learning, e.g. a shoe shop (van Oers, 2010) and a supermarket (Hundeland et al., 2020).

According to Zosh et al. (2018), play may be seen as a spectrum in which free play, guided play, games, and co-opted play are different types of play where children are involved in playful learning. These types of play vary with respect to who is the initiator (child or adult), who is the director (child or adult), and to what extent there are explicit learning goals formulated for the activity. Free play is both initiated and directed by the children involved in the play. Adults are usually remotely present. Guided play (Games are similar) is initiated by the adult but directed by the children. Thus, it is the adult (in our case the teacher) who chooses what mathematical activity to engage with. There is thus formulated an explicit learning goal for the activity. However, the children may exercise their agency by directing the activity within the given context. In co-opted play playful learning situations are initiated by the children, however directed by the adult. In co-opted play, the adult (teacher) mathematizes the children's play and directs their attention to mathematical concepts and ideas involved in the play.

Nevertheless, in our project, which addresses mathematics teaching and learning in the first years of schooling, we advocate a guided play approach. It is the teacher who necessarily initiates the mathematical activities she wants her students to engage in and to address competence goals in the Norwegian school curriculum. The teacher needs to play a significant role in nurturing the students' mathematical learning processes. That is, to promote the students' appropriation of mathematical concepts through promotion of engagement, freedom, and participation (van Oers, 2010).

Mathematical inquiry

Within the project we use mathematical inquiry as a concept addressing the approach students may adopt when engaging with mathematical tasks and problems. Mathematical inquiry is thus seen as a stance and a way of being when confronted with mathematical problems, in line with how Wells (1999) defines dialogic inquiry: "a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them" (p. 121). These wonderings and questioning will constitute key features of the students' mathematical reasoning. Moreover, mathematical inquiry was part of the tasks and activities the students were to engage with.

We argue that the teachers can nurture the students' learning processes through mathematical inquiry. In this process teachers contribute to the playful, guided play activities by encouraging the students to work together, seek answers to meaningful questions, mathematically and individually, argue for their claims and make sense of the mathematics inherent to the playful activities.

Methodology

The project was inspired by a design-based research methodology (Kelly, 2003) as the ambition was to develop both the theory and the practice of teaching and learning mathematics during the first years of schooling. The project implied a close collaboration between researchers, teachers, and their students. Design research "is directed primarily at understanding learning and teaching processes when the researcher is active as an educator" (Kelly, 2003, p. 3). Furthermore, design research focuses on "simultaneous and parallel knowledge development and product development" (Hjalmarson & Lesh, 2008, p. 521). Knowledge development in parallel with product development, understood as mathematical activities for teachers to use, was at the core of the project (cf. Cobb et al., 2003).

Cai and his colleagues (Cai et al., 2018) strongly emphasise the complementary expertise of teachers and researchers as these different roles are to be blended in design research. Design research is thus a suitable approach for developing mathematics teachers' professional expertise (cf. Smit & van Eerde, 2011), in this case to teach mathematics through a playful inquiry approach in order to involve students in meaningful mathematical reasoning. Through the iterative design cycles of our project, we established, in partnership with the mathematics teachers, what were productive ways of teaching mathematics in the specific mathematical areas that we focused on. Then we reflected together with the mathematics teachers on the mathematics teaching carried out, students' responses to the teaching, patterns of collaboration and mathematical discussions going on. These reflections established a common ground for preparing further iterations of mathematics teaching activities, etc. (Cobb & Gravemeijer, 2008).

Setting and participants

In the reported project we collaborated with seven mathematics teachers, two first grade teachers, two second grade teachers, two third grade teachers, and one fourth grade teacher. The teachers were selected due to their interest in developing their teaching based on playfulness and mathematical inquiry. Preparatory work was done in the autumn of 2022 in which schools and teachers were invited into the project. We had a first workshop with the teachers in March 2023, in which we discussed how mathematics teaching emphasizing playfulness and inquiry may look like. Ideas for mathematical activities were presented by both teachers and researchers, and we started to develop a two-week lesson plan for the mathematics teaching. After the workshop, teachers and researchers continued their thinking with respect to developing mathematical activities incorporating the key ideas of playfulness and inquiry. These initiatives were developed further in a workshop in April. Here the different ideas and suggestions were formalized into a two-week lesson plan for the teachers to teach during May, adapted to the local schools and the grades in which the lessons were to be implemented. However, all lesson plans were designed within the same overall storyline (see Example 1 below), which emphasized fantasy and excitement nurturing the students' MR.

The teachers carried out the lesson plans and the researchers observed two of their lessons based on fieldnotes. In June, we all met for a new workshop in which we reflected on the mathematics teaching that had been conducted and observed, to what extent the teaching had been emphasizing playfulness and inquiry, opportunities within this type of mathematics teaching, and in what ways the students' mathematical reasoning was nurtured. A second cycle was initiated for the autumn of 2023. Example

2 and Example 3 exemplify developed mathematical activities that were included as part of the overall storyline represented in Example 1.

Example 1: Mattelandia – a Fantasy Country

In the first workshop it was agreed that fantasy and dramatization should feature the overall frame of the designed mathematical activity. One of the teachers introduced a story of a fantasy country - Mattelandia (translation – the Mathematics country), an idea that was adopted by all. Main characters were a king, a villain, and an agent. One of the teachers, dressed as the king, made a minute long video recording that was used as an introductory story in their teaching. This story was used by all teachers, but the further storyline was individually adapted by each of them:

Dear all. My name is Syver Åttesen and I am the king of Mattelandia. This night something terrible happened. Our mascot, the dog with the name Tusenlappen was stolen. We believe it has been stolen by the thief Billionus. I need your help! Unfortunately, as this situation stands, I have to remain here in Mattelandia, but I have sent you my special agent who will be the connection person between you and me. I hope you are ready to help me and my inhabitants in Mattelandia. Good luck!

The story included mathematical words: Syver Åttesen (syv = seven, åtte = eight); Mattelandia (matte = math); Tusenlappen (tusen = thousand); and Billionus (billion = billion). Mathematically, numbers and calculations were proposed as topic areas. Teachers and researchers collaboratively prepared several activities that were used in the different classes (adapted versions according to each teacher's class and context). All activities were introduced by the teacher, who reminded the students about their overall task, to earn points that in the end were to be used to free the mascot dog Tusenlappen.

Example 2: 'NIM' – an unfair strategy game

NIM is an old strategy game which can be organized in many ways. The activity can be differentiated to adapt to students' mathematical experience by variation of the rules. A simple version of NIM is the following: Two players organize 21 matches in a stack. Player 1 starts the game by removing one or two matches. Player 2 continues by removing one or two more matches. Player 1 then removes one or two matches, etc. The player who removes the last match(es) wins. The game is not fair, because as soon as one (or both) of the players realizes the winning strategy behind the game, the end result will depend on who starts and whether the players remove one or two matches in each round.

For example, a winning strategy is not to start the removing of matches. If Player 1 takes one match, there are 20 matches left. Then Player 2 takes two matches making 18 matches left. If Player 1 then takes two matches, Player 2 has to take only one match, making a reminder of 15 matches. The winning strategy is that you have to make sure that the number of matches after you have removed matches in your turn is 18, 15, 12 etc. That is, a number which is divisible by three. The students' MR was triggered when they realized the unfairness of the game nurturing a winning strategy.

Example 3: The Magic Box

The Magic Box was covered by glossy paper, making what happens inside the box visually hidden to the students. On the front, there was a sticker with magic words, a spell, written on it. Electric lights were turned on as an indication of the box working on the numbers put into it, se Figure 1.

Figure 1: The magic box used in third grade

The Magic Box served as a tool to playfully engage the students in functional thinking. The teacher put a number, i.e. a numeral written on a card, into one side of the box and another number was taken out at the other side of the box. After the teacher had put a number, for example 12, into the box, all the students and the teacher read aloud the spell in order to make the machine work. On the other side of the box, a new number appeared, for example 24. The task for the students then became to uncover the magic process inside the box. What had happened with the input number? The concept of doubling came up in the following plenary discussion. Some students drew attention to multiplication by two and some students suggested that the magic process was addition. The teacher then put the number 24 back into the magic box (from the same side as it originally came out). The number 12 then came out at the opposite side. These actions created opportunities to discuss the mathematically inverse processes of doubling, multiplication, and addition, i.e. halfling, division, and subtraction.

Discussion and concluding remarks

We set out to come up with answers to the research question: What design principles can be derived from developing mathematical activities emphasizing playfulness and mathematical inquiry? We will discuss answers to this question both theoretically and by referring to the examples of designed mathematical activities and stories. According to Zosh et al. (2018), there is a continuum between free play and direct instruction. We are making a plea for guided play (Clements & Sarama, 2021; Fisher et al., 2010; Hirsh-Pasek et al., 2009) reinforced by mathematical inquiry (Breive, 2019, Wells, 1999) as an approach to mathematics teaching and learning in the first years of schooling.

The three examples above illustrate various ways that playfulness and mathematical inquiry may be incorporated into certain activities. The overall story in Example 1 documents how the teachers and researchers designed a playful context characterized by fantasy and curiosity by way of dramatization. The teachers and their students co-created their fantasy world. Example 2 and Example 3 exemplify two of the designed mathematical activities that were interwoven into the playful context as a story of earning points to free the mascot dog. Furthermore, the activities nurtured mathematical inquiries through explorations and questioning. They involved mathematization due to the multiplicative structure of the number of matches left in each round in NIM and to the mathematical processes happening inside the Magic Box. The participating students' mathematical reasoning (Jeannotte & Kieran, 2017) was indeed nurtured. Many students find activities comprising competition motivating and fun and NIM challenged the students' mathematical reasoning. The Magic Box became a playful tool since it inspired the students' curiosity and interest regarding what was happening inside the box.

This first cycle resulted in six criteria with respect to the nature of playful inquiry-based mathematical activities nurturing students' mathematical reasoning in the first years of schooling. These criteria were informed both by the mathematics education literature (e.g. Jeannotte & Kieran, 2017; van Oers, 2010; Zosh et al., 2018)) and the practical implementation of mathematical activities by the teachers:

- Mathematical explorations characterized by student freedom balanced with structure (cf. van Oers (2010), NIM and The Magic Box).
- Potentials for mathematization through exploration and problem solving within, for the students, mathematical meaningful contexts (cf. van Oers (2010), NIM and The Magic Box).
- Potentials for mathematics argumentation and reasoning (cf. Jeannotte and Kieran (2017), NIM and The Magic Box)
- Teacher initiated, student centered, and student directed activities in which the teacher guides the students' learning processes to reach mathematical learning goals (cf. Zosh et al. (2018)).
- Engagement and motivation through the use of fantasy and imagination (cf. Zosh et al. (2018) and Mattelandia).
- Collaboration to nurture students' interest and curiosity (cf. van Oers (2010) and Mattelandia).

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Making changes in school as a mathematics expert teacher: framing problems of practice

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We address the interplay between research and practice in terms of expert teachers' roles in "cascading" their learning from professional development courses to their colleagues in school. Focusing on five teachers who have participated in two different professional development courses in Norway, we analyse how they frame problems of practice involving colleagues' practice, despite the fact that the role of "expert" sits uneasily for teachers working within Norway's flat social structure. We argue that expert teachers' ability to identify successful ways of working with colleagues through relational agency enables them to see as fully actionable problems of practice that stem from other teachers' practice even in the absence of a shared problem of practice.

Keywords: Expert teachers, change agents, problems of practice, mathematics.

Introduction and background

National programmes for credit-bearing professional development (PD) in Norway such as the Competency for Quality (Kompetanse for Kvalitet, KfK) and Teacher Specialist (Lærerspesialist, LSP) initiatives aim to support, in a research-based manner, not only teachers' own learning, but also the way in which they contribute to their school collectives (described by KfK as "sharing", and LSP as "leading"). However, this role is challenging for teachers working in Norway's flat social structure, in which individuals are hesitant to place themselves as in any way "better than" others. Lorentzen (2020) reports that the Norwegian teachers in her study felt uncomfortable with their designated 'teacher specialist' title, largely because it ran counter to the flat structure of the profession, in which teachers tend to hold on to their autonomy (Mausethagen & Mølstad, 2015). However, the teacher specialists in the study valued the opportunity for their own professional development and engagement with research and voiced a desire to build local cultures by sharing their learning with others. This was not always easy, partly because the specialist role was ill-defined, with no clear-cut leadership role or mandate to change practice. Hence the specialists aimed to support colleagues, but avoided challenging or critical comments. They developed resources and informed colleagues about research, but did not require major action from them. Reflecting on the situation, they remarked that change relied on individual teachers identifying a need for themselves (Lorentzen et al., 2020).

International research indicates uncertainty regarding the impact of teacher specialists on school practice. Mills et al.'s (2020) systematic review of research on teacher specialists in primary school science and mathematics identifies three foci in the relevant literature: how teacher specialists are defined and work; how they are prepared for the role; and impact on teaching and learning. The review concludes that, while policy reform assumes that training teacher specialists will positively impact teaching and learning, the evidence is inconclusive. A further systematic review of research on teachers as change agents (Brown et al., 2021) shows that the field focuses mainly on the top-down approach, where teachers have a formal mandate to bring about a specific change in their schools. Bottom-up examples are infrequent. Both reviews indicate the importance of capturing how teachers are positioned within their schools, and we contribute to this literature here in terms of expert teachers' perceptions of their ability to effect change within the Norwegian context.

Theoretical framework

In this paper, we follow Horn and Garner's (2022) view that teacher learning involves the development of situated judgement based on local contexts and concerns and involves a continual refinement of 'problems of practice' – practice-based narratives of what needs to be done or improved. Solutions for such problems of practice rest on how teachers frame the reasons for their existence (Bannister, 2015). Of particular interest to us in the context of teachers as (potential) agents of change is the issue of the resulting 'actionability' – is there something that can be done? – as a factor in defining teachers' agentic space (Babichenko et al., 2021; Solomon et al. 2023). Günal-Aggül et al. (2023, pp.1610-1611) extend these ideas with particular reference to teachers' contributions to social change in their schools. They conceptualise three levels of agency in teachers' workplace learning: [1] individual agency is the product of teacher identities, histories and goals (see also Biesta et al., 2015), while [2] collective agency is required to transform structural constraints through "collectively unraveled problems of practice" (p.1610) which enable joint recognition and development of new possibilities, strategies and resources (Horn & Kane, 2015). Finally [3] relational agency arises when different professionals work together on the same problems of practice, flexibly taking others' perspectives while making their own visible to others. In this paper, we keep an open mind as to whether the teachers in this study are in some sense 'different' professionals in comparison with their colleagues, given their expert teacher status and potential leadership roles in cascading research-based practice in their schools. Bearing these distinctions in mind and focusing on teachers' perceptions of their ability to effect change in their schools, we ask the following research questions:

RQ1 How do expert teachers frame problems of practice involving other teachers?
RQ2 To what degree are these problems of practice seen as actionable, and, if so, how?

Methodology

Teachers recruited to this study were attending an advanced mathematics pedagogy course for primary teachers, either through KfK or LSP, with LSP at master's level. Both PD courses have research-based reading lists (2200 pages per 30 credits) and aim to build on research emphasising the integral role of all students' informal understandings, classroom discussion and inquiry (see Solomon et al., 2023). We interviewed the teachers for a larger study of inclusive mathematics teaching in Norway (see acknowledgement), covering topics on becoming an expert in primary mathematics, on 'good' mathematics teaching, on supporting and challenging all students. We did not ask about working with other teachers, as our focus was on teachers' own practice; however, we were struck by the strength of narratives of concern about other teachers' practice which arose with 5 of the 10 teachers interviewed. These five interviews form the basis of this paper. All interviews were conducted on Zoom in Norwegian and were transcribed and translated into English. Appropriate translations were extensively discussed between the authors. At the time of interview, three of the teachers (Vetle, Håkon and Kristin) were at the end of their advanced course (KfK), while the other two (Beate and Vida) were halfway through their teacher specialist programme for primary mathematics (LSP), bringing them to a minimum of, respectively, 60 and 90 credits in mathematics.

During data analysis, we first identified references to problems of practice (PoPs) involving other teachers, and then analysed the content of references in accordance with our theoretical framework. We focused on how teachers framed issues in terms of [1] the perceived cause and possible ways of addressing a PoP; [2] its actionable/ inactionable qualities; and [3] the nature of action to be taken (by self or with/for others). We also noted their expressed levels of agency in terms of individual (personal histories, experience, expertise); collective (shared PoPs and joint strategising); and relational (focus on professional relationships and differences in perspective).

Findings

In this section, we present each of the five teachers in turn, illustrating the variation in their framing of PoPs that involve other teachers, and how this connects to their perception of the actionability and the extent of their agency in addressing it. We begin with Vetle, who sees the need for quality teaching as totally inactionable as long as other colleagues are involved. Kristin is at the opposite spectrum in her belief that the issues are not only actionable, but involve the teacher collective. We argue, however, that Vida has the strongest sense of actionability and agency, as she describes working on incremental and less ambitious changes in which she has a sense of her role as an expert professional.

Vetle (KfK). Vetle's PoP is that children are not offered the best opportunities to succeed. His first example is that of recent immigrants to Norway, who go to 'welcome classes', "until they know Norwegian enough as a learning language". Vetle disagrees with this policy, as it assumes that children only need to learn new vocabulary and overlooks their need to learn how to engage in mathematical discussion. This is hard in a classroom where "the language barriers are enormous and very complex". Vetle doubts the capacity of 'welcome class' teachers to support this and believes that these children would be better off in mainstream classes: "I think it's first and foremost about the competence of those teaching welcome classes. They may be good in some subjects, but not in all". He is committed to the idea that every child can become good at mathematics with the right teaching "I think that surely it must be possible to teach this! At least part of it", and frames this PoP in terms of his colleagues' shortcomings, as in explaining why he stopped an initiative for stronger students:

> I've stopped [sending them one grade up] now, because they get most out of the teaching here. It's about the competency teachers have or do not have. [...] Having talked to the students and their parents I concluded that they get more challenges by staying here than by going one grade up.

Vetle always describes acting on this PoP by taking over from other teachers. He is comfortable with explaining why publicly in his school, as he did in a discussion about how best to allocate resources:

> I talked to the [teacher] who had [the weaker students] for the past weeks and I said 'This is ridiculous. I am the best maths teacher in this school. I should have that group. Do you agree?' She agreed. So I sent a proposal to the principal and now it's decided. It was my initiative.

Vetle describes himself as acting when he believes students are disadvantaged, drawing on his individual agency, positioning himself as a mathematics expert. He never indicates any intention of acting to support his colleagues, apparently seeing this PoP as inactionable while they are involved.

Håkon (KfK). Håkon is often tasked with helping students with special educational needs and has little opportunity to make his mark on mathematics teaching in his school. He explains this in terms of tradition (the class teachers tend to teach mathematics) and training (he earned credits in special education), but notes that working with individual students limits his opportunities to experiment with ideas from the KFK-course. However, he says, observing others has given him opportunities to learn, leading to a pressing PoP concerning how students are positioned in mathematics:

> Following [individual] students into their [general] classroom allowed me to hear what their teachers said and did. I listened and observed while the students answered. And I sat there and … I could barely hold myself back at times! When you feel that a student says something very important, but the teacher doesn't notice it. And you just sit there and hope: 'Go on, hear it! Hear it!' Or, other times, when the teacher … sort of … humiliates students by dismissing their ideas.

He points to his reason for being in the class in the first place, to explain why he holds back:

> I am there to support specific individuals. I believe it's best for them to be in their ordinary class. My role is that of a supporting teacher [whereas] the one who leads the lesson, leads the lesson.

While he sees the situation as inactionable in the classroom, he suggests that it was slightly less problematic to broach outside of it, although he does not go into detail about how:

> If I was going to meddle and say … Well, it wouldn't have been easy. But I did say my piece in the teachers' lounge. I managed to say it […]. I could have taught the maths, but the [class] teachers wanted to do it, even though they are not qualified. And then it's really horrible to sit, listen and see [...] when it's obviously not good. [laughs] It's been hard to … keep quiet.

Håkon returned frequently to his PoP, making distinctions between teachers who value students' contributions, and those who do not, and remarking on how the difference seems to reflect teachers' qualifications in mathematics pedagogy. While his perception that this is largely inactionable leads to reliance on his own expertise and individual agency, he and others who "think alike, largely those who took [KfK]" can support colleagues by shifting their attention towards big ideas: "[many] often look at the page number [in the textbooks] but not at the tasks on these pages. [...] They count the tasks. [...] It's more about student thinking, not about how many tasks you do".

Beate (LSP). Beate's PoP, as a future mathematics specialist, is colleagues' neglect of problem solving. She frames the PoP in terms of the resources teachers use for problem solving:

> They work great [for the best students], we have excellent lessons – lots of discussions, hard thinking [...], justifying, arguing, everything we want to see them do. But then it's hard for those at the other end. They can't articulate their ideas, it gets too hard very fast, they give up. […] They can work with problem solving tasks for a much lower grade. But then you can't bring the [mixed attainment class] together in a discussion. That's why I thought of low threshold high ceiling tasks.

Here, Beate takes her colleagues' perspective: she does not teach mixed attainment class, but a group of high attainers occasionally, and 'her group' of five struggling students. She has followed the nurture group through several years, teaching them for 3 out of the 4 weekly mathematics classes, and has “in some sense always controlled the [learning experience] of this group”. Beate is unhappy with the one lesson when the group are taught in their ordinary class, the class teacher is in control:

> [T]hey try to follow what the others are doing […]. But, actually, that’s often too difficult for them and there isn’t enough help. And then they are put to work on tasks online or something like that.

Beate acts first to get the low threshold high ceiling initiative accepted: “I sent them tips – tasks, links to websites where you find this kind of tasks”. She reports that a co-teaching experiment in a mixed attainment class: “was very successful, I got the other teachers to have some faith that it might be nice to try it”. She finds that changing the approach to mathematics teaching is difficult; she recounts how she had tried to talk to a teacher about teaching for understanding: “I said to her ‘You are not to give them the recipe, how to do this. They are to understand what it means to divide, because it’s very important’”. Despite her elaborate explanation, the teacher ignored it:

> But then I had to leave for one hour. When I came back in, [the class teacher] had gone around to four of the students from [my] group and taught them the algorithm for division! [Laughs] Ah, I was so frustrated! … I had them again yesterday and then I didn’t let them use that method.

Beate thus envisages her PoP as actionable through collective agency, but while her experience of co-teaching low threshold high ceiling tasks is successful and generates enthusiasm, her more general attempt (individual agency) to influence another’s teaching by telling her what she should do fails.

Kristin (KfK). Kristin has a long-standing concern with students’ access to mathematics, as evidenced by her detailed account of an episode before she joined the course, where she tried to persuade her colleagues that attainment grouping in the early primary years was restricting some children’s exposure (“the students I had didn’t get the same teaching as the others”). Furthermore, Kristin felt that the practice hindered genuine collaboration in the teacher team: “We worked together, yet we didn’t work together. In maths, Norwegian and English, you followed your own trajectory, based on where your students were, apparently.”

Her initial concerns were shrugged off by the other teachers (“Well, we’ve always done it like that”) and leaving her unable to act. Later, the teachers raised a PoP of their own: the amount of time they spent orienting confused children in different groups at different times. Kristin took this as an opportunity to speak with the leadership on behalf of the teachers, and the grouping practice was disbanded. She notes that her defence of mixed attainment classes to one leader’s who questioned that they could not meet all children’s needs, was supported by her later learning on the KfK course:

> To that we said ‘No, I’ve worked in a normal [mixed attainment] school many years, it’s just a question of how to do it. And you can discuss this every which way. But, for first graders now - we’ve got this! [...] And based on what I heard and read since [on KfK] … [it was the right decision]. I’m sure we didn’t do everything right then, I want to make changes, I’ve started already.

In this case, Kristin’s PoP became actionable once she was able to speak on behalf of the teacher collective (overusing ‘we’ even as she tells the story: “we said [...] I worked [...]”). From her current teaching situation, she describes a PoP of mismatch between her colleagues’ practice and ideas from the course (e.g. “high cognitive demand [...], rich tasks that create an opportunity to share strategies”). She argues with her colleagues for a vision of teaching which goes beyond a focus on test scores:

> Early in the fall, when I started [the KfK course] I felt that this isn't what we are doing [in my school]. [...] We can't carry on like this! [...] We need a plan. You need to see – where are we heading to? We can't just work on this and that. There's a lot you can accomplish by following the textbook, we can get pretty good scores [...]. But that is not the goal.

She elaborates on her own experimentation in the classroom, as well as on her gaining her colleagues' trust as a mathematics expert: they have now asked her to write the lesson plans for all parallel classes. She sees this as an opportunity to share her ideas, but is also mindful of the possibility of her assertiveness alienating colleagues, and of her ideas being wasted without collaboration:

> I'm aware of feeling that 'It's great you all have confidence in me' but… They say 'You've got this!' But still, they do a bit what they want [...]. And I let them [hand out pages of routine tasks instead]. I just think 'Fine. Just give the task [I suggested] as an extra challenge then.'

Although she sees her PoP as actionable through sharing her ideas with others, her scope of action is limited by the lack of feedback from colleagues. While she endeavours to develop collaborative agency, she struggles to enact this on the general level of their mathematical vision.

Vida (LSP). Vida has a formal role as a mathematics specialist teacher, to be called on if particular input on mathematics teaching is required. In practice, this tends to result in her working with nurture groups, but she also works closely with some teachers and writes lesson plans for them. A major PoP for Vida is to pass on the emphasis on the LSP course on actively listen to students and understand their thinking. She is unhappy with comments that blame students ("But we've been working on two-digit multiplication on Tuesday and he could do it then – today he can't remember a thing"), and questions teachers' interpretation of what they see: "I think about this a lot in my job, it's not the case that if you can do something one day then you have learned it". Making a distinction between learning something and answering correctly, she sums up her message: "What I am really trying to convey to everyone in school is that [it's about] building a good foundation on one hand and working with problem-solving on the other hand. Balancing the two goes a long way!" However, unlike Beate and Kristin, she does not raise the issues directly in her collective; instead, she writes lesson plans that convey the message more indirectly, turning teachers' attention to the student thinking:

> I posted [for teachers and students] a string of tasks […] with questions to ask. I almost control the conversation in the other classrooms! [Laughs] Fascinating to see that the other two teachers come to me and say 'Oh, do you know what he said, do you know what she said?!' They are keen to talk about it and I hear what good discussions they had and how the students responded.

Vida takes this experience as a lesson on how to assess success in realising her role in the collective: "now I have to write down the questions I normally use, so that other teachers can take part in [my teaching]." Vida thus sees her PoP as actionable, drawing on her expertise and position to support change without confrontation. We see this action as based on a relational agency, in which she develops a learning situation for her colleagues, making her perspective visible to her colleagues.

Discussion and implications

In Norway, national initiatives for PD build on the assumption that one teacher's expertise can be beneficial for the entire school. Yet, there is resistance against establishing hierarchies among teachers, leaving the expert to find informal ways of contributing to the collective. This study explores cases of bottom-up teacher agency, which are poorly represented in the literature (Brown et al., 2021). Drawing on narratives of teachers that formulate PoPs involving colleagues, we aim to understand the relationship between the framing of the problem and subsequent attempts to address the problem in a way that creates opportunities for teacher learning in the sense of Horn and Garner (2022).

Our research questions focus on how expert teachers frame problems relating to other teachers' practice and their expressions of actionability and agency in terms of their experience of trying to act in a way that influences their colleagues. We find that the research input from their courses (e.g., on valuing and building on student contributions, promoting collective sense-making in inclusive classrooms) enables the teachers in our study to articulate PoPs involving colleagues' practice. However, the practical value of such observations was limited in two ways. First, in the case of Vetle and Håkon, we can see that they do not in fact attempt to change their colleagues' practice, preferring to 'do it themselves' to counterbalance 'bad practice' elsewhere. They draw on their expertise from the KfK course, as their source of individual agency in this situation – with Håkon held back by the school rejecting assumptions of expertise. Secondly, the three teachers who attempt to change colleagues' practice, struggled to get colleagues to share their (research-based) concerns. A similar difficulty was reported by Lorentzen: the teacher specialists in her study, acting in a top-down manner and with little direct contact with individuals, conclude that teachers need to identify their own needs first (Lorentzen, 2020). By contrast, our teachers, driven by their own PoPs, acting in a bottom-up manner and in close contact with some colleagues, seek ways to engage colleagues in a solution, in the climate of high autonomy of Norwegian teachers (Mausethagen & Mølstad, 2015).

By interpreting teachers' attempts in terms of their levels of agency (Günal-Aggül et al., 2023), we found that, while collective agency aligns better with Norway's flat social structure and the high autonomy of teachers (Mausethagen & Mølstad, 2015), it poses significant obstacles. While Beate succeeded in awakening colleagues' interest through a positive co-teaching experience with the shared goal of trying out some tasks (collective agency), she alienates them as she falls then into telling them what to do and why, promoting her goal (individual agency). Kristin presents a very different position: for her change is only possible through collective agency. This ultimately fails, for lack of shared concerns: her colleagues position her instrumentally as responsible for lesson plans (individual agency) but do not join in a discussion of an overall vision for mathematics teaching. Finally, Vida does not seek to generate discussion in the collective, instead working to reproduce her own learning on the LSP course, enabling her colleagues to experience similar insights.

Vida thus acts relationally, trying to make her perspective visible through annotated lesson plans, in contrast to Kristin's attempt to generate a collective vision of inclusive teaching when her colleagues have not had the opportunity to develop the point of view that will generate this. This contrast underlines a mismatch between Norway's flat social structure, with its emphasis on the collective, and what we can see as successful cascading of research as illustrated in Vida's success in enabling other teachers to join her in noticing student learning. In terms of implications, we see this as a delicate balance, illustrating the need to research and understand how expert teachers can work within a context such as Norway's, and to provide the policy support that they need.

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Expanding a task on base four to base ten in a collaboration between researchers and teachers

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In this paper, we focus on the expansion of a task developed in one educational context and moved into another educational context. The working process was conducted as a collaboration within a research team consisting of researchers and teachers. The aim of this paper is to describe how a task designed for students to work in base four, was collaboratively expanded into a task that deals with both base four and base ten. The overarching idea is the importance of early exposure to the structural aspects of the positional system and to advocate for exploring different bases to enhance learning. The theoretical framework of the project is based on learning activity, inspired by the El'konin and Davydov mathematics curriculum. The paper includes an analysis of the task staged in a Swedish classroom, highlighting collaborative design work, task features, and outcomes. The results indicate challenges and insights in students' understanding of base ten as one of many possible base numbers.

Keywords: Design study, number base system, positional system, learning activity.

Introduction

In this paper, we exemplify and discuss how a task developed in one educational context can be collaboratively tested and developed in another educational context.

The project reported here, funded by The Swedish Institute for Educational Research, can be described as a design-based project, and is driven in a close collaboration between researchers and teachers. The motive for this project stems from an ambition to address the challenges that teachers face in planning, implementing, and evaluating a mathematics education that lays a foundation for enhancing the youngest students' number sense. The overall aim is to develop professional support consisting of a series of tasks that teachers can use to introduce students to a theoretical understanding of the positional system. The idea for the tasks which we believe need to be included in the professional support that introduces the positional system, is based on a selection of tasks found in a textbook written by Davydov and his colleagues (2012). Both the selection of tasks and how these can be used in a Swedish school context are the focus of the design project. It is not possible to simply copy the tasks directly; a trial and transformation process is required.

The aim of this paper is to describe how a task, selected from the El'konin-Davydov Curriculum, was collaboratively expanded into a task that addresses both base four and base ten. The task concerned exploration of the positional system as a structure of the number base system. The research question we raise is: What aspects contribute to expanding and staging a task focusing on the structure of the positional system through a collaborative process involving teachers and researchers?

Background

The number base system: Researchers such as Vygotsky (1986) emphasise that understanding the positional system as a structure of the number base system is crucial for even the youngest learners to develop number sense. This can be promoted if students are given the opportunity to work in different bases (e.g., Chambris, 2018; Davydov, 2008; Slovin & Dougherty, 2004; Venenciano et al., 2015; Schmittau, 2004; cf. Björk, 2023). Traditionally, the positional system is introduced on a so-called arithmetic basis, often using laboratory materials (Chambris, 2018). Furthermore, base ten is difficult for students to discern as a general structure as it is often taken for granted. The project on which this paper is based follows the El'konin-Davydov Curriculum (EDC) in mathematics according to which students are initially given the opportunity to develop a pre-numerical understanding of numbers which includes working with models of structures in numbers, such as relationships in the positional system (Schmittau, 2004; Slovin & Dougherty, 2004; Venenciano et al., 2015). Structures important for the number base system are, for example, identifying the relationship between the different digits in a number and what to do, and when, when changing between different positions in a number (Chambris, 2018; Slovin & Dougherty, 2004; Venenciano et al., 2015).

Learning activity: EDC is dependent on the principles of learning activity (LA), which is based on giving students the opportunity to work theoretically with different content according to the model "ascending from the abstract to the concrete" (Davydov, 2008, p. 106). The abstract or the general in mathematics is explored starting with the measurement of quantities and the identification of units of measurement (Venenciano et al., 2015). The starting point of an LA is a problem situation – a rich and meaningful situation with inherent problems and contradictions and with the potential for new knowledge (Repkin, 2003). The problem situation is often designed in a playful format (Wettergren, 2022); for example, involving imaginary friends or the teacher pretending not to understand the students' answers. Initially, students need to analyse the problem situation to identify what the problem consists of and to assess what methods and tools they can use to solve the problem. Central to the students' work is the use of mediating tools – so-called learning models – which graphically, symbolically, or materially enable an exploration of the structures and connections that the students need to discern (Eriksson, 2021; Wettergren, 2022). A learning model (not to be understood as a mathematical model) is always dynamic and changeable.

Features for describing a task design: In this study, the research team collectively and iteratively strove to expand a pre-designed task using the experiences of the researchers and teachers in a way that should enable students to engage in an LA regarding the positional system. For this purpose, we used Ellis' (2003) five features to describe task design. The first feature when designing or redesigning a task concerns the capability the task aims to develop, which in our case was to discern the structure of the positional number system. The next feature concerns the input given in the task, for example, verbal or non-verbal information supplied by the task – pictures, maps, or written text. Ellis' third feature concerns the conditions of the task and the way the information is presented and used. The fourth feature concerns the procedure, for example, group versus pair work and whether the students should plan their work or follow stepwise instructions from the teacher. The fifth feature, the predicted outcome from the task, concerns the process and the product. The process is the linguistic and cognitive engagement that the task is hypothesised to generate. The product from completing the task, for example, tables, maps, or pictures, is the result (Ellis, 2003).

Methods

The overall research project, which aims to develop professional support, consists of data produced in planning seminars, research lessons, and interviews. The research lessons were conducted in two schools. The students who participated (n=40) in the study were initially in Grade 2 and were taught by their regular teachers. The guardians of the participating students had signed an informed consent form which is a requirement of the Swedish Research Council (2017). The research team conducted the seminars, which consisted of teachers who taught the research lessons, and researchers who collaborated on task design. The teachers had many years of teaching experience. The research group met roughly every two weeks and the lessons were usually staged and conducted the week after.

For this paper, we have chosen a single task from the textbook by Davydov et al. (2012, p. 72, see Figure 1). This task was chosen due to the potential for the students to elaborate on the number base system in relation to base four. In the textbook, students are asked this question: “What is the number in the given base?”.

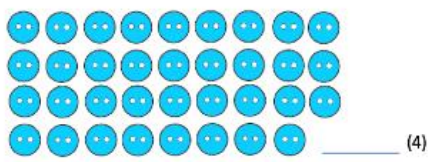


Figure 1: The original task for the lessons consisting of 35 buttons organised as 8 x 4 + 3 in base four (Davydov et al., 2012, p. 72)

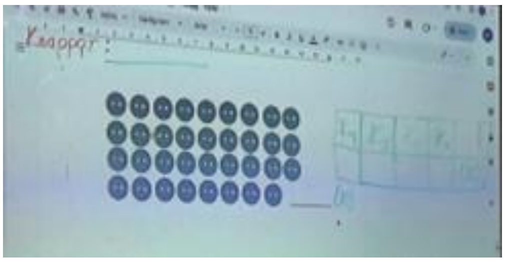
The work in the research lesson was staged jointly where the teacher led the whole class reflections. The task was projected on the board and the teacher asked the class to write a message to imaginary friends communicating the number of buttons in specific bases. The students reflected on each other’s arguments and on questions and challenges from the teacher. The result is presented as a narrative drawing on a research lesson where the task was tested. The analysis creating this narrative was conducted in several steps. First, we identified what content the students should work with and the choice of task that could highlight that content. The task chosen from the curriculum was aimed at the process of grouping according to base four (Davydov et al., 2012), but we also wanted the students to explore the base-ten system from the same situation. This analysis therefore yielded an expansion of the task to work in both base four and ten, which tentatively made it possible to group quantities according to the structure within the number base system, identify the number units used in this grouping, identify the quantity of the different number units, and represent the quantity in a base (Davydov et al., 2012; Slovin & Dougherty, 2004). The next step of the analysis was to identify central sequences in the lesson that seemed to enable an LA. In this analysis, it was also possible to identify sequences that did not work well. These sequences were then in focus for an exploration through an iterative intervention in the teaching that the research team designed. The third and final step of the analysis regarded the sequences that did not work well. This step of the analysis, which is addressed in the section Discussion, was conducted according to Ellis's (2003) features for designing tasks: goal, input, conditions, procedures, and predicted outcomes such as product and process. By focusing on these features, it was possible to understand what was necessary for the students to establish an LA.

Results

The result consists of two identified aspects regarding the process of developing a task focusing on the structure of the positional system. The staging of these aspects provided important information for developing the task in a Swedish context, creating opportunities for the students to identify the positional system as a number base system. In the following we present the two aspects.

Aspect 1: *The situation the students were engaged in and their identification of the problem.* The research lesson began with the task in Figure 1, above, projected on the board. As planned, the teacher asked whether the students remembered that they had previously sent a message regarding different quantities in different bases to some imaginary friends. At the same time, the teacher pointed to the number four in brackets beside the buttons on the board. The students, who recognised that the system referred to was marked (*n*) where *n* was a number showing the base, said "yes, four, it's there. But why are we doing this [working in base four]? Who are these friends?" Later in the lesson, the students asked if the imaginary friends were children. Another student confirmed this question: "Otherwise they would know", which can be interpreted as meaning that the imaginary friends were not children and would know how to count beyond four. When the teacher asked: "What are the consequences of using base four" one student answered: "How many are there?" The students' reflections on the need to use base four indicates that they identified the problem they should be working on during the lesson, that is, to use base four to note the number of buttons.

To explore base ten in the same situation as base four, the research team extended this task to determine the number of buttons in base ten (Figure 2).



K_4	K_3	K_2	K_1	
				(10)

Figure 2: The task on the board – base ten to the left and a reconstruction of the table suggested by the students to the right

On the board, there was already a table showing the number of units in different columns and the red text (see Figure 2) stated "Number of buttons". The information given on the board seemed to help the students to identify again that the number of buttons was the problem they had to work with. Next to the buttons was also written ten in brackets (10), which the students directly related to the fact that they had to work in and indicate the number of buttons in base ten. One student expressed this saying

"We will group ten and write in the table how many." The students worked in pairs or individually with grouping according to base ten in their books.

Aspect 2: *Tools and learning models developed by researchers, teachers, and students in collaboration:* The discussion that evolved together with the students focused on the structure of a number base system. In the research lesson, students proposed several tools for the joint exploratory work (see Figure 3). During the collaborative work on the board in relation to base four, the colour green was used to group K_1 (the representation for the number unit corresponding to the ones and the colour red to group K_2 (the representation for the number unit to the left of K_1 corresponding to tens in base ten). In this lesson the last K_1 was also given the colour red. One student suggested that the number symbols available in base four should be noted on the board (see the bottom right of the two board images in Figure 3). Another student started to construct a table next to the buttons to indicate the number of the different units, which the teacher then developed into the table in Figure 3. Referring to the table, one student reflected on how K_1 and K_2 could be grouped on a number line. The teacher responded to that idea and constructed a number line on the board below the table. When the number of buttons had to be represented as a number written outside the table, two different solutions were discussed. The students wanted both solutions to be written on the board, "to make it easier." The students gave two different proposals for numbers in base four as a solution to the task, $2\ 0\ 3_{(4)}$ and $2\ 3_{(4)}$ respectively. The recordings from the research lesson show how the students finalised the result by connecting the digits in the different positions in the number to the grouping by drawing lines (see the arrow in Figure 3).

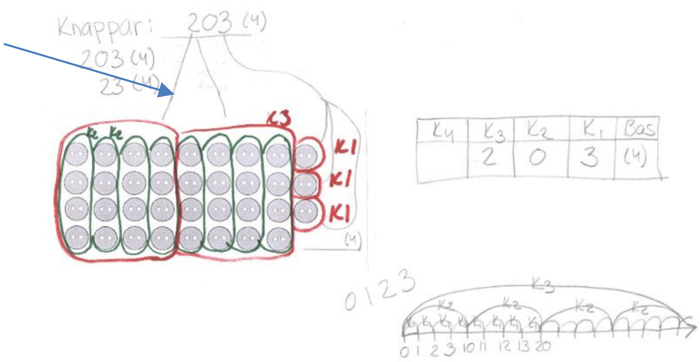


Figure 3: Reconstruction of the joint work on the board visualising the work in base four

Later, when the students and the teacher started to elaborate on how to use base ten, the students encountered great difficulties with grouping according to the structure of that base. Everyone seemed to agree that there should be ten buttons in each group, but the arrangement of the buttons in nine columns seemed to confuse them (Figure 4). When working in their workbooks, the students could not find a functional pattern for the grouping. Especially since they only used pencils for both groupings and all modelling work.

The students grouped the buttons in rows, that is, with nine buttons in each group, or in patterns that resulted in different numbers of buttons in each group (Figure 4). A great deal of effort therefore had to be devoted to constructing the groups of ten in the lesson. When students worked in their workbooks, in addition to the grouping of numbers, they also used both the table and the number line to explore and determine the number of the different number units.

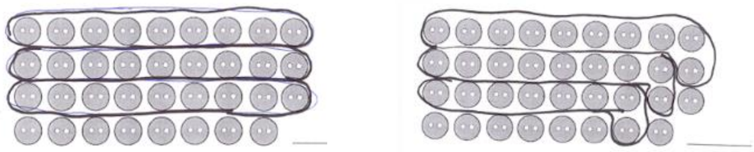


Figure 4: Two students' work illustrating difficulties when grouping in base ten

The joint work on specifying the number of buttons in base ten was completed on the board where students and the teacher helped to document the conversation (Figure 5).

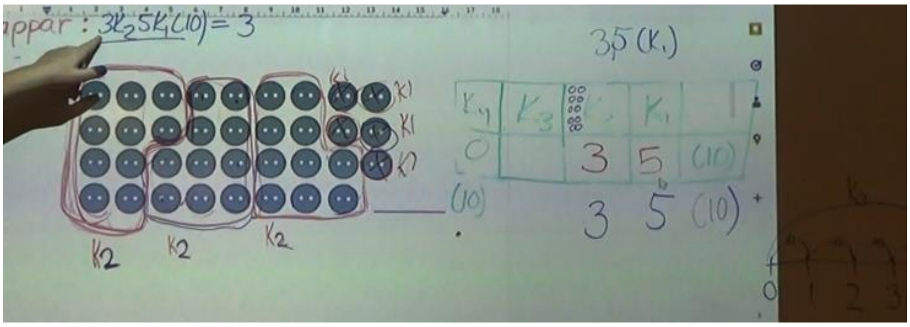


Figure 5: The joint work on the board with base ten

The students quickly realised that the number of buttons was 35. But in the work of grouping, entering the table, and showing their workbooks, where several students also constructed a number line, they expressed the number as "three five in base ten". None of the students explicitly related the three groups of ten to 30 in 35 or the last five ones to the five in the same number. They either talked about the number of buttons or reflected on the grouping work according to the base-ten system.

Discussion – features for task design

The narrative based on the chosen parts of the lesson (see above), is discussed below according to Ellis's (2003) suggestions about features for a task, but we also use these features to discuss the designing work of the task.

The goal (Ellis, 2003) of the task in our professional support was to establish a situation where the students collectively could explore base ten as one example of the number base system (cf. Chambris, 2018; Slovin & Dougherty, 2004). That is, to engage the students in a joint reflection on the structure of the number base system. Thus, the situation should enable the students to reflect on and explore that the structure is the same for all number bases (cf. Davydov, 2008). In the original task, the goal was to construct a number in a given base (Davydov et al., 2012). So already in the initial design work by the research team, the task was expanded to include the exploration of the positional system as the system of a base through a contrast with another base (base four and ten). Thus, the modified task enabled students to recognise that the same quantity can be represented as numbers in different bases, which was not required in the original task. To engage the students in this situation, the research team used *input* (Ellis, 2003) from the task presented in Davydov et al. (2012) and from the iterative work on two research lessons. Thus, details of the task the students struggled with in the lesson were important input to the research team's work (cf. Eriksson, 2021). Regarding *the conditions* in the task (Ellis, 2003), that is, discerning how the same quantity can be represented in different bases, the research team decided that the base should vary while the quantity, situation, and problem remained constant. Thus, the same quantity of buttons would be represented in both base four and ten (cf. Slovin & Dougherty, 2004). The conditions for the students to jointly work with the task included choosing some tools to explain and reflect on the structure of the number base system.

Because of the difficulties students had in grouping according to the base-ten system, the research team considered determining the number in base four only. However, the possibility of contrasting base ten with another base was considered central, so the research team instead decided to adjust the layout of the task by adding a column of four buttons. In this way, the task became 3x10+9 buttons, that is, three rows with ten buttons in each row that could be grouped into K_2 units and nine solitary buttons that constituted the K_1 unit. The *procedure* (Ellis, 2003) that consists of a playful format (Wettergren, 2022) seemed to both annoy and engage the students. They wanted to know why they should learn arithmetic like their imaginary friends and understand the structure of their way of representing numbers. They did not see the purpose of constructing the message to these friends in this base but became engaged in working on the task. The idea of sending a message may therefore continue to be part of the task, but the idea of the number base system as a theoretical mathematical idea needs more focus (cf. Björk & Berthén, 2024). *Predicted outcomes* (Ellis, 2003) regarding the process: the students had the opportunity to jointly reflect on the structure in the number base system and simultaneously reflect on the different number units in the two bases. A means for this was the joint work on the board. They had to interpret how the grouping corresponded to the representation of a number in a certain position in the number base system. We conclude that the students need to work with several tools combined to develop the tools into learning models to create opportunities for reflecting on structure in the base-ten system. Regarding the product, the new task was revised to enable grouping in tens. Also, the students' choices regarding what tools to use helped the research team to understand what needed to be highlighted in the professional support. However, given that the result is derived from work in only two lessons, more research on the tools and the students' construction of the learning models is required (cf. Coles, 2021).

Acknowledgement

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Identifying narratives of an individual mathematics teacher that might limit professional development

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This study explores characteristics of the identifying narratives of an experienced teacher (Roy), who participated in a professional development initiative that failed to reach its potential. Two identifying narratives – one about Roy's experience as a learner of mathematics and another about his work with problem solving as a mathematics teacher – illustrate tensions and issues to consider for future development. Roy points out the significant quality of the first narrative by repeatedly referring back to his experience as a learner of mathematics. The second narrative's significance is expressed when Roy highlights the importance of how the interpretation of a problem influences the kind of solutions you find. Identifying and carefully challenging such narratives could be vital for improving the continued professional development and support of practicing mathematics teachers.

Keywords: Professional development, in-service mathematics teachers, identifying narratives.

Introduction

When mathematics teachers participate in professional development (PD), they not only engage in developing new knowledge and skills. Teachers must also consider multiple reform ideas in relation to their own identity as mathematics teachers, which can be regarded as continually evolving narratives about mathematics and mathematics teaching and learning. Sometimes, the narratives of these reform ideas conflict with teachers' experiences and narratives of themselves as mathematics teachers or learners. This study is part of a larger project, *Partners in Practice,* where we seek to establish a collaborative space for continuous development of teaching practice across subjects. This particular study reports from an unsuccessful development initiative in this project, and we inspect the identifying narratives of an individual teacher. Through this analysis, we aim to identify issues that could inform ongoing professional development.

Theoretical background

Research on mathematics teacher identity has flourished in recent decades (Lutovac & Kaasila, 2018; Zhang & Wang, 2021), but this field of research has limitations. One limitation concerns definitions of identity, and Sfard (2019) claims that a lack of operational definitions makes it challenging to understand the different authors' messages. In an attempt to develop an operational definition of identity, Sfard (2019) highlights the notion of storytelling. Drawing on her previous work (e.g., Sfard & Prusak, 2005), Sfard (2019) defines identities as stories about individual persons, highlighting a collection of stories or narratives that are *reifying*, *significant* and *endorsable*. Sfard (2019, p. 557) points out that the reifying quality of a story speaks "about what a person is and has rather than what she does." The significant quality of stories "are considered by the storyteller as crucial, indeed, as defining features of the person", while endorsable narratives – pointing to the third quality – "are seen by the storyteller as reflecting the real state of affairs in the world, and thus as reliable guides for her future actions" (Sfard, 2019, p. 557). This conceptualization is also used by Graven and Heyd-Metzuyanim (2019), and they emphasize how "Sfard insists that identities *are* narratives. Identities are not *reflected* or *represented* by narratives" (p. 364).

In this paper, we build on this definition and focus on the significant and endorsable qualities of stories about "who teachers understand themselves to be in relation to mathematics and its teaching" (Lutovac & Kaasila, 2019, p. 506), since these qualities of the stories are prominent in our empirical material. More specifically and building on Westaway (2019), who defines identity as the way teachers express their roles as teachers or how they articulate their roles, we focus on the identifying narratives of one experienced teacher. We analyze what he tells about himself in relation to mathematics and mathematics teaching as he is collaborating with a group of mathematics teachers from the same school, discussing mathematical problem solving in the context of PD sessions within the overall project. The teacher's stories emerge through his interactions with the other participants in this particular community. Sfard (2019, p. 558) suggests that the same person's identity possesses various characteristics of stories, the person's self-identity and how "you consider identities persons have *in the eyes of other people.*" On the one hand, the teacher is the storyteller who puts words to his stories within this community. On the other hand, we as researchers attempt to tell consistent stories (identifying narratives) about the teacher based on our analyses of transcripts from the six PD sessions. With this foundation, we approach the following research question:

> What characterizes the identifying narratives of one experienced teacher when discussing mathematical problem solving?

In approaching this problem, we respond to the call for more focus on the individual teacher in research on mathematics teacher identity (Lutovac & Kaasila, 2018), and to the call for more emphasis on the content and tasks in mathematics identity research (Graven & Heyd-Metzuyanim, 2019). By approaching this, we also aim at better understanding how we can improve continued professional development of mathematics teachers in contexts like this.

Methods

In this paper, we explore data from a group of mathematics teachers working together in six PD sessions at a primary school, over a period of 18 months. Each session lasted approximately two hours. All sessions focused on a problem-solving approach to teaching mathematics. The discussions in the PD sessions ranged from what the Norwegian national curriculum from 2020 states about problem solving to how to lead whole class discussions as students worked on solving mathematical problems. The teachers were also invited to solve mathematical problems themselves, and to discuss their experiences as problem solvers (for details about the six PD sessions see Table 1).

The collaboration involved five teachers and three researchers (teacher educators, the three authors of this paper). Two from the school administration were also present in some PD sessions. We focus here on one experienced male teacher (pseudonym: *Roy*). Roy has 22 years' experience as a mathematics teacher and the year of the PD he taught students from Grade 5 to 7 (aged 10–12 years).

Table 1: The PD sessions

No.	Focus of the session
1	Providing first-hand experiences with mathematical problems, with a basis in how the curriculum emphasizes exploration and problem solving.
2	Further experiences with mathematical problem solving and planning the establishment of a space for exploration in the classroom.
3	Practicing how to present a mathematical problem to students and highlight observation of students' exploration of mathematical problems.
4	Listening to understand students' thinking and planning of a new enactment to stimulate different types of listening.
5	Modelling and discussions of students' and teacher's roles in whole-class discussions around mathematical problems.
6	Modelling and discussion of whole-class discussions that focus explicitly on problem solving and deep learning.

There are several reasons for exploring his voice in the sessions. First, he was one out of two teachers present in all six sessions. Moreover, when we compared the frequencies of terms used by the participants, we noticed that Roy used the term "I" more frequently throughout the six sessions than the other teachers. This indicates that his stories might include identifying statements to a larger extent than the stories of the other teachers.

When reviewing the literature for methods of studying mathematics teacher identity, Lutovac and Kaasila (2019) found that all studies applied qualitative approaches. The present study is also mostly qualitative, but we base our qualitative analysis on quantitative explorations of the same data material (see Mosvold, 2024). We analyze Roy's narratives based on transcriptions of audio and video recordings from the six sessions by using a qualitative approach to content analysis that is often referred to as conventional content analysis (Hsieh & Shannon, 2005). The first two authors read the transcripts and individually identified and coded themes in what Roy said. This careful examination of Roy's voice in the transcripts led to an identification of three main themes in his talk: His own experience as a *learner of mathematics*, his *own work with problem-solving tasks as a mathematics teacher* and his *own reflections on being a mathematics teacher*. Next, utterances related to these three themes were marked in different colors to visualize the identifying statements within these three themes. This part of our qualitative analysis resulted in several identifying statements (Sfard, 2019) of Roy within each theme. In the few instances of disagreement, we discussed to reach an agreement. Following Sfard (2019, p. 557), "for the purpose of telling a consistent story about the identified person", we consider these three themes as identifying narratives of Roy. Due to space restrictions, we focus on the two first themes in our findings below since the mathematical content is particularly foregrounded in these two themes.

Findings

Below, we present the two identifying narratives, as we interpret them to emerge through Roy's utterances across the six sessions. Our presentation of these narratives is not intended to be chronological, but we draw on utterances from multiple sessions to illustrate how they were consistent narratives rather than isolated utterances. The first narrative concerns Roy's experience as a learner of mathematics, and the second revolves around his own work of teaching problem solving.

Roy's experience as a learner of mathematics

During the six sessions, Roy described his own experience as a learner of mathematics. Both in session four and six, and when discussing the present national curriculum and referring to the previous curriculum, Roy said that he was surprised by what he experienced as "completely new methods [algorithms for adding, subtracting, multiplying and dividing multi-digit numbers]" (session 6, S6). In the discussion, an algorithm for dividing multi-digit numbers was used as an example when Roy said: "we are kind of stuck in what we learned as students" (S6). He continued to explain how he initially found these new algorithms (i.e., algorithms based on properties of operations and different from the ones Roy learned in school) "smart and clever" (S6), but that he wanted the students to learn "the proper way, eventually" (S6) – referring to the algorithm he himself learned as a learner of mathematics. Yet, Roy experienced that the new algorithms "worked much better than the ones I learned [in school]" (S6). Reflecting on how he learned the algorithms in school, Roy said that they "were told to do like this and like that and then we had to learn it [the algorithms] by heart" (S4) and that his understanding for what he was doing came in 2006. Having worked six years as a teacher, a new national curriculum was then introduced, and Roy's school got new textbooks. In these textbooks, "all of a sudden, there were totally new algorithms" (S4). At first, Roy thought that "ok, they [the new algorithms] are introduced" (S4) but he was surprised that "the textbooks never arrived at introducing the old ones [the algorithms Roy learned in school]" (S4). After having worked with the new algorithms for a while, he found that they "were actually much better" (S4), and that he did not need his old ones. He also realized that the new algorithms he had learned helped him "take [calculate] bigger numbers in my head, because I understood the algorithms" (S4). Understanding the algorithms helped him see "the old [algorithms] with totally new eyes" (S4). In addition, he had now come to understand the algorithms. He realized that he "had not previously understood them [the algorithms]" (S4), and that he "had worked as a teacher for six years without having a clue. I had just learned them [the algorithms] by heart" (S4).

When discussing explorative problem-solving approaches to mathematics teaching, the teachers referred to their students' eagerness to quickly find an answer to mathematics tasks. When one of Roy's colleagues expressed that the goal for her students was to find the answer, Roy replied: "That's why I loved math!" (S6). The colleagues agreed that the reason they liked the subject in school and became mathematics teachers, was the experience of using algorithms to quickly finish one task and move on to another one. The overall goal was to finish all the tasks first. Roy continued by referring to himself as a learner in upper secondary school, eager to get the highest grades in mathematics. According to Roy, an easy way to get high grades was to understand "how you should calculate it, and then you got the correct answer (…). That felt safe" (S6).

The cluster of interconnected utterances presented above constitutes an identifying narrative of Roy in which he seems to move back and forth between his own experience as a learner and as a teacher. As a learner of mathematics, he loved to use the algorithms he had learned by heart to quickly find answers to mathematics tasks. When challenged by national curriculum and textbooks that did not present the algorithms he learned in school, however, he realized that other algorithms might be even better. The algorithms even helped him understand that he had not truly understood how and why the algorithms he used in school, worked. Moreover, he learned that the new algorithms helped him improve his mental arithmetic. By repeatedly referring to his experience as a learner of mathematics, the significant quality of the story about Roy as a learner of mathematics is pointed out. His experiences as a learner are seen as "crucial, indeed, as defining features" (Sfard, 2019, p. 557) of Roy. What he loved as a learner seems crucial for his work as a teacher. The endorsable quality of the story becomes evident by how Roy keeps referring to himself as a learner in his reflections. Although Roy learned that his own algorithms could be successfully replaced by the new algorithms that were presented to his students in their textbooks, what he loved and learned (i.e., using rote learned algorithms to calculate quickly), indicates a direction for his future actions in his future work as a teacher.

In session four, Roy's voice confirmed the endorsable quality of this story. When working on a problem where the teachers are invited to find a general rule for the number of squares in a growing frame, the teacher educator who facilitated the discussion related to strategies for finding the number of squares using the letter *n* for the variable (i.e., the varying numbers of squares in frames of different sizes). Roy explained that he used the letter *x* and argued that the reason why he used *x* "actually was that I couldn't remember that we did anything else [in school]. I haven't had [learned about] this since upper secondary school" (S4). Referring to a digital resource (Campus Inkrement), which they use at Roy's school, his experience with using *x* for the unknown (when solving equations) as well as for variables had been challenged. He argued that "Campus varies a lot in what letters they use, and it's even not always letters. Sometimes it was squares and, suddenly, it was triangles and hearts. The point is to highlight that this is an unknown" (S4). When a colleague pointed out that the task they worked on was not an equation involving an unknown, but rather a task involving a variable, and that using *x* in this case might confuse the students, Roy replied by referring to his own experience as a learner: "Yes, but I've always believed that it is *x* for both [unknowns and variables]" (S4).

Roy's work with problem solving as a mathematics teacher

At the first PD session, before starting to collaboratively explore mathematics problems, the participants established three basic rules for the mathematical problem-solving activity. These rules were summarized by one of the researchers: "It is allowed to guess, it is allowed to make mistakes, and if you know the answer, you must wait to tell it, else you actually do not give others the opportunity to explore the problem" (S1). The following problem was then presented – verbally – twice: "If we look at sums of consecutive numbers, which numbers will we never get?" Below, we illustrate the cluster of thematic and interconnected utterances from the first session, giving insight into Roy's first attempt to work with problem solving as a mathematics teacher.

In a first utterance, Roy referred to one of the basic rules for the mathematical problem-solving activity: "But have I understood it correctly? If we think that we know the solution [to the problem], then we shouldn't participate?" (S1). Other participants reminded him about the importance of participating, but without revealing the solution. In a subsequent utterance, Roy indicated that he had already found a solution, but he was uncertain about what consecutive numbers were. The following utterance illustrates how he tried to make sense of the problem: "Is the difficulty with consecutive, what it [the word] means? I shouldn't say how I interpreted it. Because it's quite decisive for what kind of solution you get" (S1). Roy had understood and interpreted the problem "as two consecutive numbers." For him it could be one plus two, but it could also be: "seven hundred plus seven hundred and one" (S1).

When the researcher challenged the teachers to consider if the problem is limited to two consecutive numbers, other teachers in the group responded that it could also be three or more consecutive numbers. The researcher continued to challenge the teachers to consider if there were numbers they would never get when adding consecutive numbers, and Roy expressed his thinking as follows: "I thought well, but then the prerequisite is... if you have two numbers... which come one after the other, then one solution would be that you will never get even numbers" (S1). Roy was still considering the problem based on a prerequisite of two consecutive numbers, responding to the challenge by suggesting as a solution that you never get even numbers.

The teachers were first invited to work individually to come up with ideas on how to approach the problem and then discuss those ideas in pairs before sharing their thinking with the whole group. Roy expressed his thinking to the other group members as follows: "The more I think about it, the more … I basically lost the problem. I mean, it is likely because one is used to looking for the solution and not the problem, that one kind of thinks that the problem has been understood, and then: No, it wasn't like that! And then the problem kind of slips away!" (S1). Some group members considered the problem to provide many opportunities for explorative work. Roy then said: "Yes, it [the problem] is growing very much, so it almost disappears... where is it?" (S1).

The cluster of interconnected utterances constitutes an identifying narrative of Roy in which he seemed to be satisfied when having found a solution for sums of two consecutive numbers without spending too much time on finding further solutions. The more he tried to understand and make sense of the problem, the more it slipped away. Roy pointed out the significant quality of the story, expressing how he had understood and made sense of the problem. To him it seemed to be crucial how an interpretation of a problem leads to what kind of solutions you find. The endorsable quality of the story shows Roy's reflections, expressing himself in relation to the experience of working on the problem. He was eager to find a solution without working further on the problem, indicating a direction for his future actions in the upcoming sessions.

Roy participated actively in the group discussion in all PD sessions when working on mathematical problems. In the fourth session, after having worked on another problem, Roy seemed to confirm the endorsable quality of the story when he stated that, "One thing I discovered is how incredibly difficult it is to find more solutions when you have found one [solution], then you have found a solution that works, so why bother to look for one more." In one sense, he confirmed by this reflection that he was satisfied when having found a solution to a problem. On the other hand, when he expressed that it is extremely difficult to find other solutions, he acknowledged that this is a difficulty, and he seemed to be aware of this situation. This identifying statement suggests that Roy will be conscious of this difficulty for his future actions as a problem solver.

Discussion and conclusion

In this study, we have explored the identifying narratives of one experienced teacher when discussing a problem-solving approach to teaching mathematics during the school-based *Partners in Practice* professional development and research project. Our study is closely connected to practice (cf., Graven & Heyd-Metzuyanim, 2019), and it responds to the call for attending to individual teachers' identifying narratives (Lutovac & Kaasila, 2018). The two identifying narratives highlighted in our findings illustrate the significant quality of the stories about Roy as a learner as well as a problem solver. Roy points out the significant quality of the first narrative by repeatedly referring to his experience as a learner of mathematics. There is a strong narrative about mathematics as learning algorithms and applying them to quickly solve many tasks, and this narrative indicates a tension in Roy's learning and development. The significance of the second narrative is displayed when Roy highlights the importance of how the interpretation of a problem restricts the solutions you find.

The endorsable quality of the two identifying narratives reflects how Roy is expressing himself as a learner and problem solver of mathematics, indicating a direction for his future actions. For instance, the first narrative tells how Roy, even after having understood that the algorithms he learned in school are inferior to the new ones in his students' textbook, still prefers the algorithms he learned himself. This narrative of mathematics, and what it means to learn and be successful in mathematics, continually appears in Roy's discourse. Whereas Heyd-Metzuyanim and Shabtay (2019) identified tensions between two dominant discourses about teaching and learning of mathematics, Roy's narratives seem to indicate a tension in the underlying discourse about mathematics and what it means to do mathematics. The strength of these narratives could potentially limit development and learning for someone like Roy.

Whereas few previous studies of teacher identity attend to the mathematical content or tasks involved (Graven & Heyd-Metzuyanim, 2019), our study foregrounds the mathematics in Roy's identifying narratives. The first narrative, focusing on algorithms for the four arithmetical operations, shows that Roy moves back and forth between the algorithms he used as a learner, and the algorithms presented in the new textbooks. The second identifying narrative unveils some of Roy's work with problem solving as a mathematics teacher. It foregrounds mathematics by highlighting the importance of understanding the problem and looking back to possibly find a more elegant solution. After 22 years as a teacher, Roy still holds on to the narratives he developed as a learner of mathematics. This testifies to the persistence of these narratives of mathematics and what it means to do and to learn mathematics, and we believe that professional development efforts are doomed to fail if they cannot identify and productively challenge such narratives.

Our third-person stories (Sfard, 2019) about Roy are based on observations and analysis of the transcripts from the sessions. We recognize that our dual role as teacher educators and researchers might have constrained Roy's stories, and this is a possible limitation of our study. We are aware that the same author can be telling different stories about the same person on different occasions (Sfard, 2019). There could also be an identity gap, like in the study of Heyd-Metzuyanim and Shabtay (2019) where the first-person identity of a teacher is different from her third person identity constructed by these authors based on classroom observations. To mitigate these limitations, we have tried to be transparent in our careful elaboration of Roy's identifying narratives.

To conclude, the current study has illustrated one experienced mathematics teacher's identifying narratives, showing glimpses of his individual thinking when foregrounding his mathematics as a learner as well as a problem solver. The present study is a contribution to the call in the field to attend to mathematical content or tasks involved (Graven & Heyd-Metzuyanim, 2019). The tensions in the teacher's experience as a learner of mathematics and his work with problem solving as a mathematics teacher are examples of teachers' identifying narratives important to identify and carefully challenge for improving the continued professional development and support of practicing mathematics teachers. However, further studies are needed to explore how to productively call out such narratives in professional development.

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Broadening the concept of measuring through a video about Sámi traditional boat building

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Two Sámi pre-service mathematics teachers created a video about Sea Sámi boat builders' use of traditional ways of measuring. In a focus group interview, they suggest how to use this video in mathematics teaching. Cultural symmetry, a tool developed in Māori mathematics education, framed the analysis, which pointed at three important issues: i) The video provides insights into traditional boat building and its specialized language. ii) Students must experience individual body-based measuring's functionality before any standardized units of measuring are introduced. This is because students' own bodies are less abstract to them than the SI-system. iii) The two Sámi verbs for knowing something, "diehtit" (to have knowledge about something), and "máhttit" (to know a practical skill), must be considered.

Keywords: Sámi culture, Sámi traditional knowledge, measuring, body-based measuring units, ethnomathematics.

Teaching Sámi[1] traditional ways of measuring is part of the mathematics education at the Sámi University of Applied Sciences. The institution is improving Sámi higher education by developing studies that fulfil Sámi needs, and it is essential to combine Sámi traditional knowledge with academic and scientific knowledge (Sámi University of Applied Sciences, 2023). One approach is to focus on mathematising traditional boat building. In this study, pre-service mathematics teachers visited the Sea Sámi institution Mearrasiida (2019), which has worked systematically with traditional Sea Sámi boat building for years. The idea of studying Sea Sámi boat building was inspired by the Yup'ik[2] use of individual body measuring in kayak building (Lipka et al., 2010). Measuring is central in Yupiaq mathematical reasoning; a group of Yupiaq elders defined mathematics as "the process of measuring and estimating in time and space" (Kawagley, 2006, p. 51). The first author, Anne, showed the Yup'ik kayak building book (Lipka et al., 2010) to the Mearrasiida boat builders who confirmed that they used bodily ways of measuring when building a traditional *spissá*, a small rowboat. The Mearrasiida boat builders told how they make traditional boats, and one of them videorecorded parts of this for the pre-service teachers. One year later, two pre-service mathematics teachers Dina and Aile, (one of them participated in the first visit) returned for a two-day visit together with their teacher Anne, who also is a researcher. They were assigned the task of making a video or a poster that shows the boat

[1] The Sámi are an Indigenous people of the Arctic who inhabit northern and central parts of Scandinavia, and the Kola peninsula in Russia.

[2] The Yup'ik (also known as Yupiaq) are an Indigenous people of the Arctic who inhabit areas of Alaska.

builders' use of Sámi traditional measuring. The pre-service teachers chose to make a video, which they posted on YouTube (Karlsen et al., 2023) with Norwegian and English subtitles. Here the boat builder Hans Oliver uses the Sámi term *vuogas čalbmái,* which means to be satisfying to the eye/to look good. This is an oral way of expressing the beauty of functionality. Dora Andrew-Ihrke, who is Yup'ik, uses a similar phrase, "We made things that were pleasing to our eyes but that were also pleasing to the eye of the spirit watching us" (Lipka et al, 2011, p. 160).

After one year, the video had around 750 views. This number does not reveal anything about whether the video is useful in mathematics education. The teaching value of a video like this might come to the surface by, for instance, investigating how to use it in Sámi schools as well as in Norwegian classrooms throughout the country. Therefore, it is worthwhile to identify mathematics that is embedded in the boat builders' work. Anne decided to interview Dina and Aile about the possibilities for using the video in mathematics teaching. The research question is, how can a video of two boat builders' work contribute to Sámi mathematics teaching? For quality control, we used Meaney et al.'s (2022) framework of cultural symmetry. This framework is designed to prevent situations where culture is reduced to just a tool for teaching mathematics.

Literature review: Sámi culture and education

Reindeer herding and combinations of smallholdings and fisheries are two examples of traditional Sámi livelihoods. *Meahcceealáhus* (hunting, fishing, and gathering) and *duodji* (Sámi handicraft) have strong traditions. Norway has ratified the Convention for the Safeguarding of the Intangible Cultural Heritage (The United Nations Educational, Scientific and Cultural Organization [UNESCO], 2003/2020) in 2007, with a particular commitment towards the Sámi people. The Convention's phrase 'intangible cultural heritage' means practices, representations, expressions, knowledge, and skills; as well as the instruments, objects, artefacts, and cultural spaces associated with them. This means that Sea Sámi boat builders' language and ways of working are examples of intangible cultural heritage. In Sámi traditional knowledge, the term *knowledge* means knowledge as a process, not just as the outcome of a process. Guttorm (2011) distinguishes between two Sámi concepts of knowing, *máhttit* (knowing something as bodily knowledge) and *diehtit* (knowing about something). *Máhttu* and *diehtu* are mostly connected to practical and theoretical knowledge, respectively, but there is no sharp border between the two concepts. Fyhn and Jannok Nutti (2023) claim that *máhttu* and *diehtu* are two aspects of knowledge that are important to consider in Sámi mathematics education.

The first Sámi curriculum, published in 1995 (Ministry of Education, Research and Church Affairs, 1997), was based on Sámi culture and values. The mathematics curriculum, however, was just a simple translation of the national curriculum. Despite the Sámi curriculum, Keskitalo (2009) points out that much of the Sámi school system in its present state in Norway is still based on the ideas of the national Norwegian school system. Today, although there is no longer a separate Sámi curriculum, there are still Sámi issues in most curricula, but not in mathematics. Our paper aims at contributing to the improvement of Sámi mathematics education by providing examples of how Sámi boat builders' mathematical reasoning can function as a basis for Sámi mathematics education. The *máhttu* and *diehtu* aspects of mathematical knowledge constitute one frame for the analysis.

Traditional body-based measuring is still in use by *duojárat* (individuals who perform traditional craft) and reindeer herders. Sámi traditional measuring of length can mean individual as well as standardised measures, depending on purpose. Examples of measuring units are *salla* (fathom), *lávki* (step), *gardnjil mihttu* (cubit), and *goartil* (the finger span measure shown in Figure 1). Studies in the Swedish part of Sápmi have shown that reindeers can dig through a snow depth that is less than six *goartil*. If the snow is deeper, they do not reach the ground where the food grows (Ryd, 2007/2022; Jannok Nutti, 2007). The reindeer herder Johan Rassa explains how to measure snow depth by a ski pole (Ryd, 2007/2022). When you need a new *suohpan* (lasso), it is useful to relate it to your body size, and therefore your individual *salla* measure is used (Fyhn & Nystad, 2021; Juuso, 2022). The need for standardised units of measuring comes to the surface if you buy a *suohpan* in a shop, because the price is given in kroner per meter. To find the price, the shopkeeper must calculate the number of *salla* into meters (Fyhn & Nystad, 2021).

Figure 1. *Goartil.* Screenshot from the pre-service teachers' video

Theoretical framework: Cultural Symmetry

Cultural symmetry is a three-step framework for revitalisation of cultural practices in mathematics education, and it has been developed over time in Māori mathematics education research (Meaney et al., 2022). The point is that "[l]anguage, cultural practices, and mathematics, must be balanced if mathematics education is to contribute to decolonializing the education process" (p. 196). The authors point out that when educators include cultural practices in their teaching, there is a risk that they focus only on the mathematics. In such a case the cultural practice is reduced to merely being a vehicle for transmitting mathematics. In cultural symmetry, both culture and language are in focus and are valued before any mathematics is introduced, and thus intangible cultural heritage is valued. The first step of the framework concerns describing cultural knowledge, identifying actual cultural values, and recognising the importance of Indigenous language. The second step is to examine cultural practices and to discuss them from a range of perspectives, of which mathematics is but one. The third step is about considering how mathematics can add value to cultural artefacts and practices without detracting from the cultural understanding. In Sámi contexts, Fyhn and Steinfjell (2023) used cultural symmetry to show that the Sámi language and cultural practices include ways of reasoning that are highly relevant for learning combinatorics. They support the idea that including combinatorics in a prospective Sámi mathematics curriculum might increase the value of Sámi cultural practices.

Method

Our study is the first step in a design research study (Plomp, 2007) that intends to develop teaching practice and produce new theories. The idea is to design the starting point and premises for a teaching unit that later can be used in classrooms in collaboration with teachers. According to Bradley and Reinking (2011), design experiments invite teachers to enter a collaborative relationship with researchers. In contrast to the researcher Anne, the pre-service teachers Dina and Aile have a Sámi background and are native speakers of North Sámi. This means that including Dina and Aile in the analysis lowered the risk of misinterpretations and thus increased the quality of the analysis. As pointed out by Louis (2007), there is a need for research on Indigenous issues to be conducted in a way that is respectful and ethically sound from an Indigenous perspective. Aile and Dina were familiar with cultural symmetry through their studies, but that was not necessarily a guarantee for them to have internalised the framework when planning their teaching.

To design the basis and the premise for a teaching unit, Anne invited Aile and Dina to participate in a structured discussion. It was organised as a focus group interview where Anne was the moderator. The reason for choosing a focus group interview was, as described by Morgan (2001), because the data were collected through group interaction on a topic determined by the researcher. The interview guide consisted of five questions, with additional reminders for the moderator in parentheses.

1. What mathematics do you believe the video includes? (Measuring, geometry)
2. How do you suggest that teachers can use the video's measuring examples when teaching about measuring? (Individual and standardised units of measurement, transition between them)
3. How do you suggest that teachers can use the boat builders' angle tool in teaching about angles?
4. How do you make sure that culture and language are valued and not reduced to just a tool for teaching mathematics?
5. If you look at the two verbs *diehtit* and *máhttit*, what is the value of including both here?

The interview took place on Zoom and lasted 34 minutes. The interview guide's four first questions were on a shared screen for one minute before the interview started and then again during the last minutes. This was to clarify for the pre-service teachers what the interview was about. During the interview, Anne realised she had forgotten the fifth question when she made the interview guide. She added it during the interview. Within one day after the recording, Anne shared the audio files with Aile and Dina, asking them to send eventual corrections and missing issues. During the following week, Anne transcribed the interview and sent the transcriptions to Aile and Dina.

Analysis

The interview data revealed almost nothing about angles, so question three is not part of the analysis.

Question one – the mathematics content

Aile and Dina replied that the video is about measuring. It also shows calculations, for example, that a boat's length must be three times the width and the width must be three times the height. This estimation of ratios is a way of measuring as well. The pre-service teachers' next point considered measuring by eye whether a boat looks good. The boat builder Hans Oliver uses the Sámi term *vuogas čalbmái.* This means to be appropriate for the eye/to be good looking, which also refers to what works best in real life use. The connection between *vuogas čalbmái* and the ratio of 3:1 shows that even though the question was about mathematics, the pre-service teachers' answers concerned language terms and references to how cultural practice deals with creating a functional boat, which is step I in the cultural symmetry framework.

Question two – how to use the measuring examples in teaching

The answer was that the video documents how traditional ways of measuring are as good as using a measuring tape, because the result is a boat that works. Some weeks earlier, Dina and Aile presented Sámi mathematical thinking at a national conference for teacher educators. One participant commented that she had presented and tested out body-based measuring with her preservice teachers, who doubted the usefulness of these ways of measuring. Dina and Aile's answer included an example of discussing the cultural practice from different perspectives, which is step II in the cultural symmetry framework. They showed the importance of focussing on the traditional measuring's functionality. Aile and Dina emphasised the significance of a practical approach, stating that it is useless to just sit down and calculate what is one third of this and what is one third of that.

Figure 2. Birch bark boat made by John Kuhmunen. Photo: Dina N. Somby. Printed with permission from the *duojár*

Dina and Aile suggested how measuring by *goartil* (Figure 1) could be used in making small boats out of *beassi* (bark), like the one in Figure 2. They had seen fellow preservice teachers' similar boats made at a university course some weeks earlier. The idea was that all students use their personal *goartil* as a unit of measure when making their boats. They would also ensure that the ratio between the boats' length and width, as well as between width and height, would be 3:1. Then everyone in the class can see that the boats' sizes vary because the students' *goartil* vary, and still all boats are good boats. This is interpreted to include step I, valuing cultural practice, as well as step II, discussing the cultural practice from different perspectives. Each student would experience the functionalities of their body-based measures in a situation where they were expected to believe a measuring tape was the only tool that could work. Regarding individual versus standardised units of measurement, Dina immediately said that she would only use individual ways of measuring. Aile emphasised the importance of showing that individual measures work and are appropriate. She wanted the students to experience the functionality of individual body-based measuring before they meet centimetre-based measuring.

> Aile: Because, then [if we start with cm] we say, between the lines, well, it is ok that you know how to measure [by body-based measuring], but what counts is how many centimetres there are.

Thus, they both argued against including standardised units of measuring even though the question was about how to deal with transitions between individual and standardised units.

Question four – how to make sure that culture and language are valued

The pre-service teachers' immediate response was to follow the steps of the cultural symmetry framework. Aile and Dina appreciated valuing the boat builders' words and the investigation of culture before even looking at the mathematics. After that, it made sense to ask how mathematics could provide support for this cultural tradition. Dina also stated that for schools, the video could be an inexpensive alternative to inviting an expert in traditional knowledge.

Regarding language and terminology, Dina pointed out that the boat builder Hans Oliver Hansen used many words that are part of his traditional craft, "So when you use the video, the Sámi traditional knowledge will be there, … and the use of these words". She explained that the boat builder's professional terminology is intertwined with the boat's construction. She claimed that this reduces the risk that a teacher using the video will focus solely on the measuring part without respecting the language and cultural practice. She continued by pointing out the importance of including Mearrasiida's books about traditional boat building (Nilsen, 2021a; 2021b) since these provide information, and they provide opportunities for including an extensive part of the tradition. This is step I in the cultural symmetry framework.

> Aile: We were really dedicated when we created the video, we wanted those words, those boat words he used, to be written too in the video, so that those who watched the video did not just hear them but also could see them written. That we had a focus on the boat words, too, and not just the mathematics in the boat building.

These replies revealed that they really had considered cultural symmetry when editing the video.

Question five – the two Sámi words for knowing

According to Dina, very few students are expected to become boat builders; to *máhttit* boat building.

> Dina: Regarding tradition, it is very good that as many as possible know about it [boat building], *diehtit*. That one understands it and values it.

She also explained that you can start with *diehtit* and then move on to *máhttit* by trying out different ways of measuring. Aile agreed and added that the video only shows the *diehtit* part, so in order to *máhttit* a way of measuring, you have to try it out yourself in practical situations. Aile and Dina pointed out that students should share their experiences with body-based measuring with each other orally, through drawings, and through text. This may contribute to students' in-depth knowledge of how to perform and value body-based measuring.

Discussion and conclusion

The intention of this paper was to reply to the question of how a video of two boat builders' work can contribute to Sámi mathematical teaching. Two pre-service mathematics teachers describe how a video can contribute to a teaching unit where culture and language are in focus. They refer to the first two steps in cultural symmetry. They point out the need for a teaching unit that focuses on these two steps before step III is introduced; no standardised units of measuring should be introduced before students have experienced that their individual body measuring works. Regarding the two Sámi words for knowledge, *diehtit* and *máhttit*, watching the video can provide knowledge for students about boat building and body-based measuring, which represents *diehtit*.

Dina and Aile suggested that students could experience their individual measuring units by building small bark boats. Discussing this idea with *duodji* (Sámi handcraft) teachers, revealed that the material is inappropriate for compulsory school use. The preservice teachers are still searching for an appropriate material. However, the video can contribute to mathematics education as a part of a teaching project about body-based measuring because it provides examples of a) how body-based measuring units are part of Sámi cultural practice and b) how language is an integrated part of this practice. Students can investigate their own body-based units of measuring and exercise using these ways of measuring, *máhttu.* The video opens for students to talk about mathematics, to develop their *diehtu* by talking about how they experience the value of how body-based measuring is used. A group task for students could be to identify Sámi mathematical thinking in the video after having discussed the video's content. Next, the groups meet for a plenary discussion lead by the teacher.

The analysis shows that language is an issue in the replies to every interview question, not just question four where it was explicitly in focus. The discussion about standardised versus individual units of measuring is related to the introduction of standardised units of measuring. Sámi as well as non-Sámi students are expected to benefit from learning about individual ways of measuring before the standardised units are introduced.

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“On the learning train”: problematising “the good lesson”

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We problematise the interplay between research and practice through a close analysis of a Norwegian primary school lesson which in many ways appears to be a well-executed example of “the good lesson”, a model which is widespread in Norwegian schools. We argue that the research which underpins this model is filtered through mechanisms of New Public Management, reducing its usefulness to a tick-box exercise and thus disempowering the teacher and removing opportunities for responding to students’ contributions. We show how this process diminishes one teacher’s professional agency and results in actions which are not at all conducive to “the good lesson”.

Keywords: New Public Management, “the good lesson”, performativity, teacher agency.

Context and literature: a new era of education in Norway

“PISA-shock” hit Norway in December 2000, highlighting Norwegian students’ perceived underperformance in reading, mathematics and science in comparison to other countries (Bringeland, 2022). The ensuing educational debates triggered major policy-level attention to teaching quality, leading to several reforms aiming to improve students' learning. National tests in reading, English and numeracy were introduced in 2001, and a new curriculum was implemented in 2006 with a stronger focus on achievement and goals. Teacher education was upgraded to a five-year master’s qualification, and universities developed new master’s programmes on effective management for school leaders. Together with student and parent “satisfaction” surveys, these shifts were part of a general move towards New Public Management with an increasing focus on school standards and effectiveness, student learning outcomes and teacher quality (Solhaug, 2011). In Oslo, where our research took place, the education authority is well known for being ahead of the rest of the country in promoting competition among schools, especially at upper secondary school level, via recruitment (and associated per capita funding) based on student choice rather than geographical location. Student testing is also more widely practised as part of a strong New Public Management trend (Stavelie, 2021), and there has been a rise in competitive school ranking and branding (Dahle, 2021). Within this new education market, performance in mathematics has become a key indicator of perceived educational standards and teacher quality.

The impact of performativity on teacher practice

The impact of performativity on teachers’ practice and their sense of themselves as good teachers is internationally well-established (Ball, 2003). More recently, researchers have reported on the phenomenon of *post*-performativity, where teachers trained in the current climate develop practice which is inextricably bound up with prioritising test performance (Holloway & Brass, 2018). This development is also visible in Norway (Holmedal, 2023), and highlights the power of the discourse of New Public Management to transform teachers' perceptions of good teaching and its goals. A particular issue is the deprofessionalisation that results from demands for "best practice" or "evidence-based practice" which fail to recognise the situated nature of teaching and the need to exercise professional judgment rather than following set criteria for what a lesson should look like (Horn, 2020; Biesta, 2023; Holloway & Hedegaard, 2021). In this paper, we focus on "the good lesson" as a powerful example of prescribed "best practice" which has appeared in Norway.

The "good lesson" as a central driver of "quality" in Norwegian schools

The "good lesson" is frequently presented as a key component of what schools in the wider Oslo area offer their students, appearing on school websites as a major element of a school's "pedagogic platform". In Norway, the origins of this discourse of "the good lesson" lie in various sources. Most prominent, and easily accessible through a direct link on the Norwegian Directorate for Education and Training webpages (UDIR, 2013), is a report on observations in four Oslo schools published by the Nordic Institute for Studies in Innovation, Research and Education (Markussen & Seland, 2013). Drawing on Hattie's (2009) conclusion that teacher quality and the teacher-student relationship is most influential on student achievement, the report recommends that schools emphasise classroom management techniques, including clear statements of lesson objectives at the beginning and end of a lesson, highly focused class activity, and the importance of teacher authority in ensuring students stay on task. The report concludes that "good lessons" occurred when "schools are run according to a basic pedagogical idea, have a clear pedagogical management at the school and clear leaders in the classrooms" (p.7). Closely associated with these ideas, and also appearing on several school websites as part of their pedagogic platform, is reference to Thomas Nordahl, a professor of education and a prominent voice in Norwegian educational debates. His work is often connected to "the good lesson", listing the key features of lessons which optimise student learning outcomes. Like Markussen and Seland's report, Nordahl's work draws on Hattie's meta-analysis and defends the "what works" agenda, arguing for instance that "values-oriented pedagogy" such as that advocated by Gert Biesta "will, in most cases where it were to be used alone, not give a teacher any guidance on what he or she should and should not do … Values are of little use here …" (Nordahl, 2015, p.68, our translation).

The "good lesson" discourse thus plays a complex role in relationships between education policy, New Public Management and schools' self-presentation. It is noteworthy that the research it draws on does not explore the details of teachers' pedagogical practice, particularly subject-specific practice and how it promotes learning. The interplay between research and practice in this context is therefore both complex and unclear, leading us to ask the following research questions:

- How is the "the good lesson" model enacted in a teacher's mathematical pedagogy practice?
- How is teacher professional judgment impacted in this enactment?

Theory and methodology

In this paper we draw on data gathered as part of a larger study of mathematics teaching across Norway focusing on inclusion and classroom grouping (see acknowledgement). We focus on one 4th-grade lesson in Ask primary school [pseudonym, *ask* is Norwegian for the ash tree], located in a higher socio-economic status area of Oslo. The teacher - Arne [pseudonym] - had specialised in mathematics in his 4-year bachelor degree and had been teaching for 3 years. In addition to observing Arne teach, we interviewed him to gain his reflections on grouping practices and inclusive education, his educational choices and actions, and the nature of pupils' experience in mathematics classrooms. Ask School is no exception to the trend of presenting a statement of "the good lesson" as a major selling point on its website. In order to protect the school's anonymity, we present a generic form of this discourse in Table 1, translated into English and categorised in terms of organisational, pedagogical and educational principles.

Table 1: Generic form of "the good lesson"

Organisational	Pedagogical	Educational
● Pupils and teacher shake hands at the door. ● The class starts promptly. ● The teacher presents the lesson plan. ● Lesson objectives are clear and understood by students. ● The teacher establishes and maintains a calm work atmosphere.	● Use of learning partners is strongly encouraged. ● The teacher adapts methods, materials and tasks to student needs. ● Teaching includes rich tasks which are open to all pupils. ● The teacher incorporates pupils' contributions and facilitates "talk moves". ● The teacher summarises and concludes clearly.	● The teacher activates students' prior knowledge. ● The teacher creates active and thinking pupils. ● Pupils must develop tools for dealing with demanding learning situations. ● Trying and failing are essential parts of the learning process. ● The teacher presents subject matter in a motivating and relevant way. ● Teaching must develop different strategies and ways of thinking. ● The teacher encourages pupils to make connections to other subjects and topics.

Analytic framework

To analyse the data we draw on Gee's (2014) critical discourse analysis, which enables a focus on teachers' enactment of their classroom practice within the context of policy requirements and school organisation. Gee's theory emphasises the role of "big D" Discourses which capture actions as well as words, and hence identity performance, combining "ways of enacting socially situated identities and associated practices in society through language" with "ways of acting, interacting, valuing, knowing, believing and using things, tools and technologies at appropriate times and places" (Gee, 2014, p. 127). When people are engaged in Discourse, they use language to do things, but also to *be* things as they take on socially situated identities: saying-doing-being gains its meaning from the practice it is a part of and enacts. Enactment of a Discourse such as "the good teacher who delivers the good lesson" involves recognition as a certain kind of person engaged in a certain kind of practice.

As a theory of critical discourse analysis, Gee's theory provides insights into the role of social forces in the Norwegian mathematics classroom. Understanding teachers' practice requires analysis of the different Discourses that they draw on as they act out their roles within the school and the classroom. Gee's theory therefore offers an opportunity for exploring the teacher's enactment of the potentially structuring policies or systems of "the good lesson". Our operationalisation of Gee's concept of big D Discourse in the analysis of Arne's interview and the observation data involved identifying passages in which he enacted the "good teacher" Discourse as "saying, doing and being" in terms of his use of the language of the "good lesson" discourse ("activating prior learning" etc), and his sequencing and combining of actions.

Findings

Overview of the lesson

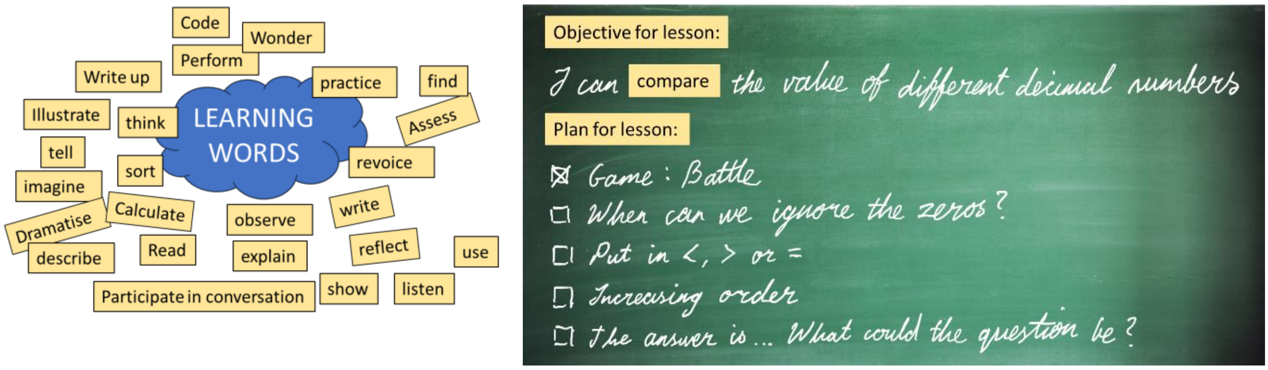

Figure 1: The learning goal and plan on the board, and the adjacent collection of ready-made "Learning Words"

The 70-minute lesson consists of 10 distinct sequences incorporating 4 classroom modes - Learning Pairs (LP), Whole Class Discussion (WCD), Whole Class Activities (WCA) and Individual work (I):

1. Start-up activity: Sorting decimal cards. [LP: Duration 8:30]
2. Teacher presents the learning goals, illustrated in Figure 1: "I can compare the value of different decimal numbers". The word "compare" is on a laminated card and has come from a collection of "learning words" placed on the wall beside the chalkboard. Students are instructed to sit in "the listening position" and read the learning objective in unison. Discussion: "What does 'compare the value of' [in the learning goal] mean?". [LP: Duration 2:40; WCD: Duration 2:20]
3. Set of three tasks: Identifying location of decimal numbers marked with 'a red ring' on the number line. Each image flashed up for 2 seconds followed by a short (5-20s) discussion in LP. [WCA: Duration 6:30]
4. Task: Discuss this statement: "'4.09 and 4.009 are the same as 4.9 because we can take away the zeros after the decimal point' – Agree or disagree?". Students are instructed to first think alone. Teacher holds up a red card depicting a mouth [talking not allowed]. [I: Duration: 1:10; LP: Duration: 00:40; WCD: Duration: 3:30]
5. Task: "When can we ignore the zeros?" Array of 10 numbers with zeros both before and after the decimal is presented on the digital board. Eg., 0.3; 3.0; 0.030. Students copy the task heading in their books. Teacher holds up the red card [no talking] and also indicates that students should stop others seeing their work. [I: Duration: 4:50; LP: Duration: 01:30; WCD: Duration: 5:10]
6. Task: "Which decimal is biggest?" Students copy the task heading in their books. Five pairs of numbers are revealed, students must insert <, > or =. Teacher circulates and looks at some students' work without comment; attends to one student who is off task. [I: Duration: 7:00]
7. Activity: "Increasing order" Students copy the task heading in their books. Students are instructed to think of a "which number is bigger" question. They then go round and pose their question to each other when they meet. Teacher circulates, ushering students around the classroom to ask each other questions, then moving on. [WCA: Duration: 5:20]

8. Task: Sort these decimals in increasing order (from the smallest to the largest): 0.1, 0.36, 0.4, 0.89, 0.90, 1, 1.01, 1.1, 1.11, 2.1, 2.89, 2.9, 3.0 (presented randomly in a cloud on the board). Teacher holds up a green sign and says "talk to your learning partner along the way". Walks around and engages with groups to ask if they were in agreement. Teacher picks one student to read their answer in the WCD. [LP: Duration: 05:00; WCD: Duration: 01:30]
9. Restatement of learning goal: "Today we have checked out, looked at several numbers, tried to find out who has the biggest value, the least value." Indicates learning goals on the board. [Duration: 00:10]
10. Task: "Find calculations that give 7.5 as an answer". Teacher explains the task [Duration 01:20]. Teacher circulates and engages with students mathematically as they do this task. In the WCD every student gives an answer. [I/LP: Duration: 05:30; WCD: Duration: 07:50]

Lesson structure – "the learning train"

The lesson is heavily planned and structured, proceeding at pace through the first 9 activities with no room for diversion. Arne organises all activities with rotating sequences of individual, learning pairs and whole-class activities/discussion. He uses multiple signals and instructions which the students clearly understand in order to control how they behave ('listening position', instruction cards [red/green for stop and go], etc). Transitions between tasks are smooth and often indicated verbally by Arne in a 'sung' jingle. Arne's actions are frequently focused on ensuring that students are on task and moving around appropriately. Indeed, the lesson is an example of 'the learning train' which Arne talks about in his pre-observation interview as he describes his use of the activity in Sequence 1:

> I try to have an activity first, which catches their attention. Which everyone can easily participate in. […] Yes. We just had decimal numbers. And then it could be: everyone must call out a decimal number. […] And not everyone will be able to do that. But then at least you have ... it's not scary, then. I think. Everyone can jump on the learning train. […] I'm looking for appetising (laughs) startups, […] simple tasks, just to switch on ... switch on prior knowledge and switch off free time.

Arne thus enacts the 'good teacher' Discourse in language and actions, in his sequencing and combining of the different tasks and activities with his accompanying commentary and directions. This is recognised by the students, who participate without hesitation in their designated roles. An exception to the overall pattern appears at the very end of the lesson, after Arne's recall of the learning goals, when he finishes the lesson in Sequence 10. We return to this observation below.

Missed opportunities for engaging with students' mathematical thinking

As the overview of Arne's lesson above suggests, there is little time for responding contingently to what students may offer as solutions or strategies. We zoom in here on the individual and pair work phase of Sequence 5 "When can we ignore the zeros". During the individual phase, a boy puts his hand up, calls Arne's attention and asks "If you kind of write 0.03 like…". Arne interrupts and says "Do what you think - trust yourself" and tries to move away. The boy persists but, although he returns to him, Arne merely makes an ambiguous rocking movement in response. Arne tells the students to check their answers with their learning partners, but does not respond mathematically to suggestions such as "for 3.00 you can either cross out one or two zeros and it will still be the same". Instead, he encourages learning partners to share their answers, asking "Are you in agreement? In agreement all the way?". When one boy responds "no", Arne replies "Then you need to find out who was right. So, then you can learn from the mistake". This kind of missed opportunity also occurs in Sequences 4 and 8. Emphasising that students can learn from mistakes is part of Arne's enactment of the "good teacher" Discourse, in which he draws on the language of learning from "trying and failing" in the "good lesson" list of key points. In the interview he says "I like tasks where I can plan on mistakes ... and celebrate that we make mistakes". However, when the interviewer responds "So you're saying it's fun – or it's good for you as a teacher, when you encounter student mistakes […]", Arne does not pick up on the invitation to talk about how he could use mistakes in order to build students' mathematical understanding. Instead, he focuses on how it is hard "to convince [many girls] that failure is a huge positive". Thus, it does not appear that Arne focuses on *how* the students can learn from their mistakes in his enactment of the "good teacher" Discourse, that he has had the opportunity – or time – to develop ways of working productively and mathematically with student mistakes.

There are three sequences where there is no whole class discussion: Sequences 1, 6 and 7. These activities account for nearly 19 minutes of the 70-minute lesson. In Sequence 1, Arne explains the rules of the game to some pairs, and checks others' understanding of the rules, but does not ask any questions about the mathematics. At one point he says to the class "Good, help each other and get going as quickly as you can". As in Sequence 5, we see Arne's enactment of the motivational goals ("learn from mistakes"/"trust yourself") which feature in "the good lesson" discourse. It is important to note, though, that these directions are not realised in terms of mathematical actions. In Sequence 6, he ensures that students are speaking to each other, but he does not listen to the conversations. Although there may be some information gathering here, the teacher's actions are mostly focused on keeping the students on task, as "the good lesson" requires.

Stepping outside of "the good lesson" discourse

It is noteworthy that the only time that Arne engages in mathematical discourse with the students is when they are working on the final activity (Sequence 10). Significantly, this activity falls outside of the learning objective (although it appears on the lesson plan) and takes place after Arne's summary in Sequence 9. This shift is further signalled by his announcement that "we are going to move a bit away from [the learning goal]". Unlike the other sequences that we have commented on, he slows down the pace and spends significantly more time on this activity (nearly 15 minutes). His interaction with the students is mathematically grounded, and draws on their work and ideas. He asks questions such as: "Can you make one with 5 + something"; "What about a different arithmetic operation than plus?"; "Have you used all the possible arithmetic operations?"; "Can you turn that division into a multiplication?". Our tentative interpretation of this change in his practice is that because this activity lies outside of the learning objective and the 'protocol' that the "good lesson" discourse generates, Arne is possibly freed from its constraints during this activity.

Discussion and conclusions

In response to our first research question "How is the "the good lesson" model enacted in a teacher's mathematical pedagogy practice?", we argue that Arne is driven by the individual elements of the "good lesson", almost as though he must 'tick off' delivery modes and organisational requirements. He ensures that the requirement to use learning pairs is satisfied, and much of his activity is focused on keeping the lesson going at pace, ensuring that the students are moving around the classroom and staying on task. He is preoccupied with controlling student activity and ensuring a calm working environment using terms such as the "listening position". As a result, his interaction tends to focus on movement and doing rather than mathematical discussion. We argue that this enactment of the "good lesson" discourse (and hence the "good teacher" Discourse) is a powerful indicator of how research enters into a teacher's practice but is translated into a set of directives via a "what works" approach as emphasized by Nordahl (2015) rather than offering opportunities for reflective teaching.

This brings us to our second research question, which asks: How is the teacher professional judgment impacted in this enactment? Our analysis suggests that, during the main body of the lesson, Arne is reduced to the role of deliverer and producer of the "good lesson show". This is emphasised by the contrast with his very different teaching style in Sequence 10, which, we argue, illustrates his ability to exercise professional agency as a teacher able to respond contingently to students' contributions (Horn, 2020). We suggest that this only happens once he has stepped outside of the multiple pre-planned activities addressing the learning objective. We understand this difference as a product of the pressure exerted by the "learning train" in the main body of the lesson, which prevents him from responding to unexpected mathematical initiatives coming from the students as in Sequence 5. We connect these observations to the background context of post-performativity (Holloway & Brass, 2018), in terms of its unquestioned emphasis on results and procedures. Arne is in this sense acting as a post-performative teacher as he pushes through the learning objectives agenda on the board.

In this paper, we have argued that the refraction of research through a performative lens and a focus on school performance and branding as highlighted in Norwegian research (Solhaug, 2011; Stavelie, 2021) means that teachers are disempowered through two mechanisms. One is that the research is so far removed from the final distillation into 'key points' that teachers are unable to engage with it and apply it reflectively in their particular classroom contexts. The second is the power of discourse to hide the mechanisms of this disempowerment and prevent discussion of what the research really means for professional practice. We argue that Arne is only able to exercise professional judgment when he is freed from the demands of "the good lesson" as a performative mechanism, and, like Holloway and Hedegaard (2021), we question the "best practice" assumptions which underly it, telling teachers how to teach according to "what works" rather than engaging with the complexities of applying research in their own context.

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Visualising challenging transformations – Two teachers in upper secondary school teaching exponential functions

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Abstract: The mathematics classroom in Swedish upper secondary schools has over the years undergone cumulative changes due to the digitalisation. In this study we use Duval's Registers of Semiotic Representations to analyse the mathematical transformations in a teaching practice. Episodes from two teachers using multiple digital resources in a whole class setting are in focus of the study. The study shows that the use of multiple digital resources in the teaching practice, makes cognitively more challenging transformations visible.

Keywords: Mathematics, teaching practice, transformations.

Introduction

The mathematics classroom in Swedish upper secondary schools has over the years undergone cumulative changes due to the digitalisation. The teacher has a computer, padlet or a cell phone, and so do the students. Most often there is some kind of interactive board, or a projector connected to the teachers digital device where the lecture can be visualised, activities can be introduced, and central aspects of a mathematical problem can be outlined and investigated. The software used, for example digital curriculum resources such as e-textbooks, and open digital educational resources such as GeoGebra, is embedded within one (or several) applications for organising, sharing, and communicating before, during, and after the lessons. The classroom becomes a 'connected classroom' (Clark-Wilson, 2010; Sahlström et al., 2019), where multiple digital resources are put into play in the teaching practice (Trouche et al., 2020).

When teachers explain mathematics to their students, they need to recontextualise and transform the subject knowledge into a form that is possible for the students to study and learn (Brousseau, 1997). This is advanced work:

> One can easily see two aspects of the teacher's rôle which are rather contradictory: to bring knowledge alive, allowing students to produce it as a reasonable response to a familiar situation, and, in addition, to transform this 'reasonable response' into an identified, unusual cognitive 'outcome' recognized from outside. (Brousseau, 1997, p. 227)

For instance, signs play a fundamental role in algebra, and thus in the learning process of upper secondary mathematics. However, the signs have no meaning on their own; the meaning of the mathematical sign has to be constructed, and discussed in the teaching situation (Steinbring, 2009). Steinbring highlights the relations between reference context, the sign/symbol, and the mathematical concept by describing it as the epistemological triangle of mathematics (Steinbring, 2009, p. 22). He argues that mathematical knowledge cannot be revealed by only reading mathematical signs and symbols; they have to be interpreted. Duval (2006) takes the argumentation a step further: without a semiotic system of representations, no mathematical processing can be performed; mathematical processing always involves the replacement of some kind of semiotic representation with another one. Consequently, an important aspect of teaching mathematics is not the representations per se, but their transformations. Teaching practice in connected classrooms given its digitalised possibilities for teaching and learning mathematics, is therefore an interesting milieu to study focussing such transformations.

In this paper we use data from video-recorded mathematics lessons in Swedish upper secondary schools to study how mathematics is made visible in connected classrooms. We focus on a specific episode of two teachers (teaching in a whole class situation) when they use multiple digital resources. In this setting, the teachers transform the mathematical content within and/or between different registers, that is, semiotic systems that permit the transformation of representations of the mathematical content (Duval, 2006). We ask: What kind of transformations are made visible?

Theoretical frame

According to Duval (2006) the learning process of mathematical thinking involves two types of transformations: treatments and conversions. Treatments are defined as transformations within one register, such as rephrasing or solving an equation using algebraic signs and symbols. Conversion is defined as a transformation where the register changes, such as going from an algebraic to a graphic representation of a function, while maintaining the same conceptual reference, that is, keeping the mathematical meaning of the object. Duval claims that performing a conversion (i.e., changing the representation register) is the most challenging transformation.

Registers are a semiotic system that permits the transformation of representations (Duval, 2006, p.111). Duval divided registers into two groups, monofunctional and multifunctional. Registers that enable mathematical processes such as algorithms (e.g. manipulations of algebraic expressions) are described as monofunctional registers. Semiotic systems that can be used for various cognitive functions, such as awareness and communication which mathematical processes that cannot be made into algorithms are called multifunctional registers.

Furthermore Duval (2006) distinguishes between discursive and nondiscursive registers. A discursive register involves statements of relations or properties, and a nondiscursive register consists of, for example, figures, graphs, and diagrams. All transformations, both treatments and conversions, can occur among these four register types (see Figure 1).

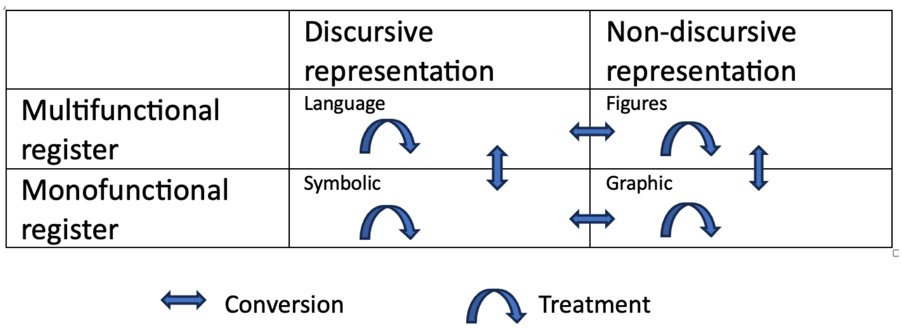

Figure 1: Duval's theory of transformations. Adapted from "A Cognitive Analysis of Problems of Comprehension in a Learning of Mathematics" by R. Duval, 2006, *Educational Studies in Mathematics, 61*, p. 110.

The curved arrows in Figure 1 symbolize treatments and the straight arrows symbolize conversions. The conversion is, according to Duval (2006), a more challenging process because of the cognitive complexity when shifting between registers. A conversion in which the transformation from one register to another can be performed via a sign-by-sign translation appears easier than one in which this is not the case. Duval referred to these as “congruent and noncongruent transformations”, respectively.

Method

The empirical material in this paper consists of recordings of classroom activities in two different mathematics classes in a Swedish upper secondary school. Two video cameras were used, one focusing on the teacher and the other on most part of the classroom. One part of the classroom was not recorded, hence students not partaking in the study could join the lesson. The teachers used the same learning management system for organising, sharing, and communicating before, during, and after the lessons making it in principle possible to compare and contrast the teachers’ teaching practices. Additionally, we have video material from focus students from these lessons and also interviews with both teachers and students.

The participation was voluntary. All participants in this study, the teachers as well as students, were informed orally and in writing about the study and what their participation would mean for them as basis for an informed consent.

The focus in this study is the teaching practice. In the analysis the video recordings were watched through several times, and sequenced into a series of activities. The activities were coded using Duval’s classification. Written summaries of each activity were made, together with a description of the outcome of the analysis.

Result and analysis

The learning goal in focus in the following examples is the exponential function. To each case we describe the situation. The analysis starts from the moment the teacher and the students start their work with a specific task (introduced by the teacher).

Teacher 1

The teacher uses *OneNote* and draws two coordinate systems. He picks up the *formula sheet* on the screen and shows what the formula for an exponential function looks like on the formula sheet. He then shows $y = 50 \cdot 1{,}22^x$ on *GeoGebra*. Uses the slider to show what happens if C changes to 100. He then changes $a < 1$ and explains what happens to the graph.

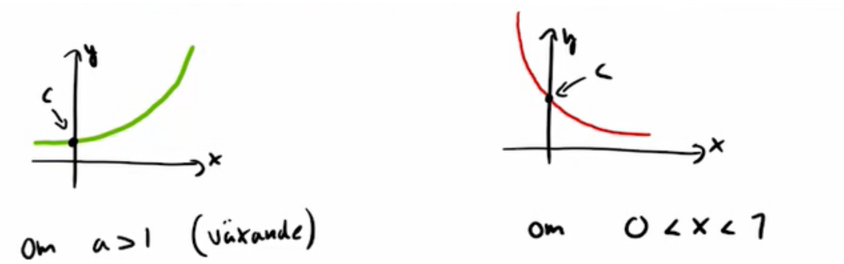

Figure 2. Teacher 1 sketches of an increasing and a decreasing exponential function

Then he *switches to OneNote* and sketches a growing function and writes $y = C \cdot a^x$ and explains that C is the starting value and "that's where it intersects the *y*-axis" and then makes a point there. Further he explains that if $a > 1$ "[t]hen we have an increasing function. Another alternative is a decreasing function", he sketches a decreasing function. The teacher explains that C is the same (marks it on the sketch) and *a* must then be less than 1 but *writes in OneNote* and says what he writes: $0 < x < 1$ and explains that it means between zero and one. No one of the students asked why he writes *x* instead of a.

The teacher writes, $y = C \cdot a^x$ draws arrows to C and writes 'start value', 'a change factor' and '*x* time' (says: "it is usually time factor"), draws arrow to *y* and writes 'new value'.

He then shows a task (our translation):

> The function $y = 12000 \cdot 1{,}13^x$ describes how the number of deer, *y*, changes within an area *x* years after the measurement started.
>
> a) How many deer were there initially in the area?
>
> b) By how many percent does the number of deer change per year?
>
> c) How many deer are there in the area, according to the model, after 9 years?
>
> d) After how many years will there be 100,000 deer in the area according to the model?

Table 1: Description and analysis of the teaching practice of Teacher 1

Activity	Analysis
The teacher explains the expression, the change factor and writes the expression in *GeoGebra.* Zooms out the image and shows how to read the answer of the question of how many deer there were. (task a). He changes to *OneNote* and writes the answer to the question on a and b.	The teacher switches registers from the natural language in the multifunctional register when he explains the algebraic expression. In Duval´s (2006) terms it is in the monofunctional register. The transformation is a discursive conversion. Because they have to change register from the multifunctional to the monofunctional register. Then the teacher writes the expression in *GeoGebra* and when he shows how to answer the question from the graph, he changes the register to the graphic representation. The transformation is a non-discursive conversion because it is a transformation between different registers. The conversion is non-congruent because they must change to the graphic representation which cannot be translated sign by sign.
Further he wants to find out about task c Writing in *OneNote* $\boldsymbol{y = 12\,000 \cdot 1,13^9}$ The teacher: "That's how you write". Change image to *GeoGebra* and enter	In solving problem c, the teacher uses *GeoGebra* as a calculator to solve the problem. We interpret this as a treatment as he does not make any transformation

f(9) and obtains 36 048.5 He also shows another way of entering it into *GeoGebra* to get the answer. Then he writes it on the calculator in *GeoGebra.* He rounds the answer to 36 000 when he answers the question in *OneNote*.	between registers, it only takes place within the monofunctional register as he enters *GeoGebra.*
To answer question d, he uses the graph and shows how to read the graph. (about 17 years) Also enter $\boldsymbol{y = 100\,000}$ in *GeoGebra* and use the intersection tool to read more accurately in *GeoGebra*. (17.35 years). Switch to *OneNote* and write the answer.	The teacher uses the graph to solve task d. The interpretation of the answer is a non-discursive conversion as the solution of the intersection is in the monofunctional register and the interpretation in the natural language is in the multifunctional register. The conversion is non-discursive since the intersection is a graphic representation.

Teacher 2

The teacher makes a short repetition about linear equations that the class have worked with before and further tells them that they will now work with another type of functions, the exponential function. He goes through what they will do this lesson and asks the students to take out their *notebooks* and write the heading *Exponential Functions*. The goal of today's lesson is that after today's lesson they will understand what an exponential function is. The teacher talks about what they should write. The teacher *writes in OneNote*, he writes a task (our translation):

The function $y = 320\ 000 \cdot 0{,}87^{x}$ describes the value of a car *y* SEK *x* years after it was bought.
a) How much did the car cost when new?
b) How much does the car depreciate in value per year?
c) What is the car worth in 8 years?
d) After how many years has the value been halved?

Table 2: Description and analysis of the teaching practice of Teacher 2

Activity	Analysis
The teacher invites the students to discuss task a, and b. Together with the teacher they come to the conclusion that the car cost 320,000 SEK as new and the car's value decreased by 13% per year.	To solve task a and b they interpret the expression, and they go back and forth between orally explanations of the algebraic expression and the algebraic expression in itself. The orally speaking is in the discursive multifunctional register and the algebraic expression is in the discursive monofunctional register which means in Duval's (2006) terms it is a discursive conversion.
Students are asked to start *GeoGebra*. The teacher goes around and makes sure they make it work. The teacher describes what they should write in *GeoGebr*a (period instead of comma). He shows how	

they should change the axes so that they get 320000 on the *y*-axis.	
To answer task c the teacher asks them to insert 8 instead of *x* gives $\mathbf{320000 \cdot 0,87^8 = 105000}$ SEK first he calculated it on the calculator and then he uses calculator *GeoGebra* to find the solution.	When solving task c they use a ordinary calculator and the calculator in *GeoGebra*. This is symbolic representation and it is in the monofunctional register both the equation and the solution which means that the transformation is a discursive treatment.
The teacher invites the students to discuss the question d first. The teacher stops the discussion and says; "Then we want the value to be SEK 160,000 We do it with the equation, $\mathbf{320000 \ \cdot 0.87^x}$ We are going to find out *x* and then we need to know *y* and then we replace *y* with a value so that we create an equation, $\mathbf{320\,000 \cdot 0,87^x = 160\,000.}$ The teacher says "We cannot solve this equation by hand, we have to do it with *GeoGebra*" Then he shows how to enter the equation in *GeoGebra* and says, "don't press enter but wait for my signal. Click on the three points and at approximately 4.98, i.e. approximately 5 years, the value of the car will be halved".	First, in the process of solving task d they interpret the task and put up an equation. This process is a discursive conversion since they start in the multifunctional register (orally speaking) and went to the monofunctional register (the equation). The solution of the equation is a discursive treatment since they use *GeoGerba* but just for calculate it. They did not use the graph to see the intersection point (graphic representation). The solution is in the symbolic representation.

Discussion

When the classroom becomes a 'connected classroom' multiple digital resources are put in play in the teaching practice. This means that teachers' use digital resources to recontextualise and transform the subject knowledge, the mathematical content, into a form possible for students to study and learn. In this paper we show how the mathematics becomes visible, by analysing which kind of transformations that occur when two teachers teach about exponential functions.

Even though the two teachers use different approaches when presenting the content, and stress slightly different aspects of the mathematical content, the analysis shows that treatments, and conversions across registers and representations are made visible in both teaching practices. This suggests that the use of multiple digital resources in the teaching practice, makes cognitively more challenging transformations visible despite differences in teachers' didactical and pedagogical choices (e.g., sequencing, level of student engagement, when and to what the digital resource is used, etc.).

When teachers use different mathematical representations and shift between registers in their teaching practice, they enable a deeper understanding of mathematical concepts by inviting the students into a mathematical processing that involves the replacement of one kind of semiotic representation with another (Duval, 2006). By using the graphical representation in *GeoGebra* to interpret the answers to the first questions in the task, Teacher 1 allows a deeper understanding as the students are connected through both the linguistic aspect and the graphic representation (conversion). Before he does that, he explains the mathematical expression in words, which causes another transformation (conversion) to be made from the linguistic aspect to the symbolic representation. In this case, the teacher makes two transformations between three different mathematical representation forms. These transformations visualise the concept of exponential functions in such a way that, in Steinbring's terms, the students have to interpret the mathematical content in a more profound way. Teacher 2 starts from a similar example and answers the first questions in the task by explaining the mathematical expressions into words. He then makes a transformation (conversion) between the linguistic and the symbolic representation. Unlike Teacher 1, he does not use the graphic representation in his explanation. However, an important aspect of teaching mathematics is not the representations per se, but their transformations (Brousseau, 1997; Duval, 2006). The study shows that teaching practice in a 'connected classroom' visualise cognitively challenging transformations that might have been difficult, to perform without multiple digital resources.

Further, by using Duval's theory of transformation we notice that small differences in teaching practice might enable different kinds of mathematical processing. To answer the last question in the task (when it comes to calculating after how many years each event has taken place), the teachers use different approaches. Teacher 1 uses the graphical representation in *GeoGebra* and answers the question in words. He makes a transformation (conversion) between the graphic and the linguistic representation. Teacher 2, on the other hand, does not use the graph even though it is drawn in GeoGebra, but enters the similarity, the equation, in *GeoGebra* and in this way shows the answer. Here, the transformation is done solely in the symbolic register, which becomes a treatment. Brousseau (1997) explains this in terms of choices the teachers do in a complex teaching situation when recontextualizing and transforming subject knowledge into teaching.

Limitations

In this study two teachers and a short teaching sequence are in focus of the analysis. We have not analysed the students' responses. There are probably patterns in the teaching practice (regarding conversions and treatments) over a series of lessons that is important to consider in relation to student learning.

Implications and relevance

Using Duval's theory of transformations, shows how the mathematical content is made visible in different ways depending on how digital resources are used. By reflecting on teaching practice from the lens of transformations over registers and mathematical representations, teachers, teacher educators and student teachers can be attentive to mathematical processing and the challenges involved. If digital resources are used more purposeful, it is possible for students to gain a deeper understanding of the mathematical content.

Acknowledgments

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Problem design based on Realistic Mathematics Education

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Learning based on Realistic Mathematics Education allows students to experiment and work creatively with open-ended and sometimes complex problems. School teachers' ability to design problems facilitating such learning is crucial for students to have the possibility of working with mathematics this way. We investigated how groups of preservice teachers designed Realistic Mathematics Education problems as part of their participation in a mathematics didactics college course at the master's level. Our findings showed that the preservice teachers generally found a lot of possibilities for students to reason independently and work with the problems. The learning pathways they proposed for the problems were based on formal and informal approaches. Some groups did not formulate the problems so that students could work flexibly on them. Also, many of the suggested learning pathways suffered from the lack of representations.

Keywords: Realistic Mathematics Education, preservice teachers, problem design, learning pathways

Introduction

Mechanist and structuralist perspectives are dominant in mathematics classrooms. Introducing open-ended problems can contribute to a process-oriented perspective on work with mathematics.

An instruction theory that takes the principle of mathematics as a constructional process into account is Realistic Mathematics Education (RME). The RME perspective builds on the ideas of Freudenthal (e.g., (Freudenthal,1991)). In RME mathematics is considered a human activity where students, building on well-adapted contexts and receiving adequate guidance, are supposed to formalize mathematics independently. Through this, they can reinvent essential mathematical ideas gradually. To start such learning processes, one recommended presenting students with rich contexts where various mathematical systematisation and reflection could be applied (Freudenthal, 1991; Gravemeijer & Doorman, 1999). Both problems embedded in contexts from the physical world and pure mathematical issues could be starting points for mathematisation if they were real in the student's mind (Van den Heuvel-Panhuizen, 2000, p. 4). Contexts had to be constructed so that they could both be approached by heuristics and generate formal solution strategies.

Students' possibilities to work with open-ended problems heavily depend on mathematics teachers' capabilities to facilitate them, a recurring theme in research (e.g., (Paredes et al., 2020)). To investigate how introducing a process-oriented view on working with mathematics could influence preservice teachers' task design, we introduced a class of preservice teachers to RME asking them to create problems that supported this perspective on how mathematics can be taught. Our research question was formulated as: *What traces of RME can we identify in preservice teachers' design of school mathematics problems and explanations of solution pathways?* The preservice teachers' target group was secondary school mathematics students. For practical reasons they did not have the opportunity to test their problems in practice teaching.

Theoretical framework

Two of the principles behind RME are the activity and reality principles (Van den Heuvel-Panhuizen, 2000). The activity principle (1) states that instead of students being recipients of ready-made mathematics, they should learn by actively participating in the mathematisation process. This means that instead of being offered descriptive plans of how to proceed, students should be perceived as active participants in the process of finding solution methods and developing insights that help and support them in their own mathematical work. The reality principle (2) states that students' mathematical learning processes should start from realistic situations or contexts that students can imagine. Further, the contexts should be "mathematically rich", so that students would have several opportunities to develop mathematical concepts.

A principle, central to the mathematisation process, is the level principle (3). It states that learning mathematics means passing through different levels of understanding. At the start of the process, the interpretation is linked to informal, practical-related contexts in the form of real or imagined situations. By creating shortcuts and schematisations, so-called "shifts" in the understanding are developed. Passing through a shift indicates that the learner has developed a mathematically higher and more abstract level of thinking about the problem. After some unspecified shifts, students can operate on a formal mathematical level with the use of symbols and generic mathematical concepts. The level principle is linked to horizontal and vertical mathematisation (Treffers, 1978; Van den Heuvel-Panhuizen, 2000). Horizontal mathematisation refers to performing transitions into mathematics from an outside extra-mathematical domain. Vertical mathematisation is manipulation within the abstract mathematical system itself, using established mathematical symbols, relations, results, and methods. Treffers (1978) pointed out that horizontal and vertical mathematisation were difficult to distinguish in practical situations because they would usually be intertwined (p. 71).

Realistic mathematics education was developed for primary school mathematics (Treffers, 1978). Gravemeijer and Doorman (1999) demonstrated how the approach could be used in calculus courses, as well. Their example involved horizontal mathematisation as the picture of a discrete graph in a time-velocity coordinate system. Understanding distance as the geometrical calculation of the area between the discrete graph and the time axis, could trigger students' interest for limit-calculations (p.121). This was presented as a "common-sense" situation that could be introduced to induce shifts in students' mathematical understanding.

Results of introducing RME to preservice teachers

Combining the results of two research studies, Wubbels et al. (1997) investigated how preparing preservice teachers for realistic mathematics education affected their view on mathematics and acting as teachers. The first study was longitudinal, following a group of 18 preservice teachers participating in an RME-based teacher education program for 4 ½ years. The second study considered comparing graduates from this programme with graduates from a traditional preparation programme. The results showed that the graduating teachers from the RME-based programme had altered their view on mathematics towards perceiving the subject as more inquiry-oriented. They also displayed a more effective teacher behaviour, for example by realising that students should be offered many possible explanations for problems. Still, many of these preservice teachers did not recognise, or use, the important RME principle of building on students' constructions.

A similar result was found by Yilmaz (2020). An investigation involving 32 preservice teachers showed that the teachers possessed a satisfactory theoretical knowledge of RME, but still could have problems posing suitable contextual problems. While some of the problems had an adequate context for further mathematisation, others could not be qualified as contextual problems. This was for example the case with the problem where students should pile four hoops over one another to examine their junction points (Yilmaz, 2020, p. 32). Despite many positive findings, Wubbels et al. (1997) concluded that changing preservice teachers' views of mathematics and mathematics teaching towards RME was a long-term issue.

Methodology

Data collection took place in the spring semester of 2023 in a master's level class consisting of 31 preservice teachers. The students were in their fourth year of teacher education, preparing to become teachers for lower secondary students. We had no previous knowledge of the class but noticed that they were familiar with the mathematical modelling concept. We introduced the class to RME in two 3-hour lectures where the article (Van den Heuvel-Panhuizen, 2000) was part of the literature. As an RME-task example, the preservice teachers worked in pairs on a problem downloaded from the website in (Manchester Metropolitan University, 2018-2020). This led to a class-discussion of formal and informal approaches to calculating the prices of two elements (caps and umbrellas). The total cost was given for two different situations. The first was a picture of two umbrellas and one cap. In the second situation one umbrella and two caps were pictured (https://rme.org.uk/what-is-rme/about-rme/). Here preservice teachers came up with different ways of swapping hats and umbrellas, and together with guess-and-try procedures, these were discussed in a whole class conversation. Gradually, symbolisation and more formal approaches to the problem were discussed. Afterwards, many examples from the article (Van den Heuvel-Panhuizen, 2000) were studied.

After these lectures, we presented the class with an obligatory assignment to be worked on in groups consisting of two to five class members. The assignment asked the preservice teachers to design an RME problem that fulfilled the activity and reality principles and argue why the principles were fulfilled. It also asked to give at least two learning pathways for the proposed problem. A learning pathway was explained as a solution proposal containing information about how students could work with the problem using both formal mathematics and heuristics. To work with the assignment the preservice teachers independently divided into nine groups. Each group delivered its' assignment in a document file. A first overview of the nine documents showed that they all contained a problem, two or three learning pathways and reflections describing how and why the problem fulfilled the activity and reality principles. After the course and exam had finished, we asked the groups for their consent to use their documents for research. The project was approved by the Norwegian Agency for Shared Services in Education and Research (SIKT).

To analyse the content of the documents, we induced some requirements that we used to characterise each assignment. The requirements were based on the principles characterising RME and the examples in (Van den-Heuvel Panhuizen, 2000). A prerequisite for stating a requirement was that it was possible to determine its presence from the given data. From this analysis we ended up with the list in Table 1 (first column). The answers to these requirements informed whether a given problem and pathway facilitated learners' development of formal mathematical concepts, formal and/ or different solution strategies, solutions strategies that included heuristics, different levels of understanding, use of representations, possibilities of suggesting new and interesting problems, and shifts in understanding. We overviewed problems, learning pathways and groups' judgements of these to determine which of the requirements that were fulfilled. Generally, the occurrence of shifts is difficult to detect and explain (Freudenthal, 1991; Treffers, 1987). This was a problem also in the analysis of our data material, where single solution pathways often were designed for a certain level of understanding. We decided to interpret that a shift in mathematical understanding was facilitated if one of the solution pathways was on a more formal mathematical level of understanding than others.

By a thematic analysis we also extracted group explanations of why their given problem designs and pathways fulfilled the activity and reality principles. Some general results from this analysis are presented in the discussion section.

Results and analysis

Here we present a closer analysis of two of the assignments. For each example, we present the designed problem and refer some of the solution pathways. In the first example we also include a transcription. We decided to select assignments where the two groups displayed different views on the use of representations to solve the problem. The following show how we applied the framework in Table 1 to identify traces of RME in the problem designs and solution pathways.

Example 1

The first example is from a group consisting of two preservice teachers, and the analysis is summarised in the second column in Table 1. The group had created the following problem:

> Emma's car has a petrol tank that can hold 60 litres of petrol. She usually drives to and from school, which is 10 kilometres each way. Her car uses 0.08 litres of petrol per kilometre. Emma wants to find out how many days she can drive to and from school before she must fil up with gas again. a) Create a mathematical formula that describes how much petrol Emma uses per day when she drives to and back from school. b) Create a mathematical formula that describes how many days Emma can drive to and from school before she must fill up with gas again. Use this formula to find an answer to the problem.

Two learning paths were suggested, both starting as follows: 1) Read and understand the problem 2) Identify relevant sizes and variables. Then the paths diverged in the sense that by one of the pathways, the student with guidance from a teacher was imagined arriving at the formula $N = V/(0.08 \cdot d)$, V being the volume of the tank, d the tour/ retour distance and N the number of tours to be taken. From this step we interpret that the group related the problem to generalisation, which is a *formal mathematical idea*. Solving the problem by making a general formula can be considered as the *introduction of a formal solution strategy*. By the other pathway the student was imagined tracing the above formula in two steps by first calculating the amount of petrol needed for a one-day school trip, then using the result to determine the number of back-and-forth school drives by trying and failing. Thus, this pathway includes *heuristics.* Together the pathways demonstrate *different solution strategies* and relate to two *different levels of understanding*. Because the two learning pathways target different levels of mathematical formality, we interpret that shifts were facilitated. There were no invitations for students to create *representations.* This could indicate a lack of attention towards facilitating students' constructions (Wubbels et al., 1997). The petrol price 0.08L/ km was not necessarily realistic. This could have been a typing error. The problem was formulated by the sub-questions a) and b). Taking these aspects into account, another interpretation is that this was a dressed-up mathematical problem (Yilmaz, 2020), meaning a lesser correlation with RME design.

The group also included this statement:

> The student uses the knowledge and skills they have learned through the task to solve related problems and situations, for example calculating how much petrol they themselves use per day or how they can reduce their own CO2 emissions by limiting car use.

Here the group demonstrates that they are thinking about how work with the problem could lead to the formulation of *new and interesting problems*.

Example 2

The second preservice teacher group, consisting of three preservice teachers, had designed the following problem:

> A class must sell paper towels to raise income for a class trip and they can sell 80 sacks of paper towels if the price per sack is NOK 200 or less. If the price is NOK (200 + 10x) per sack, then there are 2x sacks not being sold. The students must deliver the sacks to the customers door, which costs NOK 50 in fuel per sack. The sacks that are not sold cost NOK 70 to return to the producer. What price optimises paper towel sale?

x-verdi	…	6	8	9	10	12
Salgspris (200+10x)	…	(200+10*6) = 260 kr	(200+10*8) = 280 kr	(200+10*9) = 290 kr	(200+10*10) = 300 kr	(200+10*12) = 320 kr
…	…	…	…	…	…	…
Kostnad solgte sekker	…	68*50 kr = 3400 kr	64*50 kr = 3200 kr	62*50 kr = 3100 kr	60*50 kr = 3000 kr	56*50 kr = 2800 kr
Total inntekt	…	17 680 kr - 840 kr - 3400 kr =13 440 kr	17 920 kr - 1120 kr - 3200 kr =13 600 kr	17 980 kr - 1260 kr - 3100 kr =13 620 kr	18 000 kr - 1400 kr - 3000 kr =13 600 kr	17 920 kr - 1680 kr - 2800 kr =13 440 kr

Figure 1: Excerpt of a groups' table representing a trial-and-error process to find the optimal number.

The first learning pathway described a trial-and-error process, where the group suggested students arriving at a table (Figure 1) of eight rows systematized into the items: 1) x-value, 2) profit per sack, 3) number of unsold sacks, 4) number of sold sacks, 5) profit for the sold sacks, 6) costs to return the unsold sacks, 7) costs for the fuel bringing each sack to the customer, and 8) the total income for selling a given number of sacks. There were eight columns for the x-values (x = 0, 2, 4, 6, 8, 9, 10, 12). Only an excerpt of the table is given in Figure 1. The green column represented the optimal sale.

For the second learning pathway the group suggested more formal calculations. Here students were imagined arriving at the total income function $f(x) = -20x^2 + 360x + 12000$. (This function resulted from calculating the sum of the profits and costs as can be seen partly from the table in Figure 1 $[(200 + 10x) \cdot (80 - 2x)] - [50 \cdot (80 - 2x)] - [70 \cdot 2x]$). They were supposed to experiment with this function in the digital tool GeoGebra to find the x-value for the absolute maximum of the function. A properly conducted procedure would result in the same x-value as in the green column in Figure 1.

This example introduces the idea of optimisation and extreme values of functions, which is a *formal mathematical idea* (Table 1). The pathways present *different solution strategies* for students at *different levels of understanding*. Both pathways demonstrate how *representations* can be used (as the arithmetic table in Figure 1 or a function in GeoGebra). The first pathway, here represented by Figure 1, can be interpreted as an informal representation, while the second pathway referred to a formal representation in GeoGebra. Introducing strategies for the calculation of extreme values can be understood as making attention towards a *formal solution strategy*. Together the two pathways can be interpreted as facilitating a *shift* in the understanding of how to determine optimisation values. Using RME to understand aspects of calculus by referring to a common-sense situation was suggested in (Gravemeijer and Doorman, 1999). Though the example from this groups also facilitated calculus ideas, the introduction of the variable x in the problem formulation could speak for a dressed-up problem (Yilmaz, 2020). The problem formulation also included reference to the definite mathematical expression 200 + 10x, that would give students lesser flexibility to work on the problem.

This analysis is summarised in the third column in Table 1.

Requirements for the problems and learning pathways to align with RME	**Example 1**	**Example 2**
Formal mathematical concepts were expected to be developed	Yes (the concept of a mathematical formula)	Yes (the concept of optimisation)
Formal solution strategies were expected to be developed	Yes (solutions by general formulas; how to create an algebraic formula)	Yes (doing formal calculus; formal calculations of extreme values)
The learning pathways demonstrated different solution strategies	Yes	Yes
One or several learning pathways demonstrated solution strategies that included heuristics	Yes (trying-and-failing)	Yes (trying-and-failing or using tables)
Several levels of understanding were demonstrated	Yes (two levels)	Yes (two levels)
The use of representations was demonstrated	No	Yes (representing in a table or by GeoGebra)
The preservice teachers explained how new and interesting problems could be induced	Yes (problem concerning pollution from own car-driving)	No
The solution pathways facilitated different levels of understanding (so that shifts were supposed to occur)	Yes (moving from dealing with number calculations towards a general formula)	Yes (moving from experimenting with a table towards formal calculus)

Table 1: Traces of RME in problems and learning pathways designed by two preservice teacher groups.

Discussion

Some typical reflections concerning the activity principle were that it was fulfilled because of the openness of the task formulation. For example, one group wrote, *"…the wording of the task makes it possible for the students to be active in their own learning process when working. There are no guidelines for how the students can go about working on the task".* The group in our second example stated: *"The students are confronted with problem situations, where they can start simply by trial and error, and based on an informal way of working, they can gradually develop an algorithmic method in the form of equation expressions."* Here the argument is that the principle was fulfilled because the problem was formulated such that it could give students the possibility to arrive at a solution gradually. Also, when arguing for the reality principle many groups reflected in the same ways. Here is an example from the second group *"… this task, about finding the optimal price for the towels, is a real context that is closely linked to the students' world of experience. This allows students to mathematise an everyday context..."* Rooting the problems in everyday contexts was the preferred strategy to cope with the reality principle for most of the groups (examples were finding the cheapest driver's license bureau, comparing petrol prices with money for shopping, determining the amount of water to fill the school swimming pool, and calculating assurances).

Most groups embedded the problems into everyday contexts one could expect being experientially real to potential secondary school students. It was noticeable that no groups introduced stories or pure mathematical situations as starting points for mathematisation. Such contexts often characterise RME problems (Gravemeijer and Doorman, 1999), also for the primary grades (Treffers, 1987; Van den Heuvel-Panhuizen, 2000). This indicated a lack of attention towards creating contexts building on plausible mathematical experiences of students at this level. The preservice teachers seemed to equal everyday contexts and RME. These findings indicated lack of ability designing a complete set of suitable contexts for RME problems (Wubbels et al., 1997; Yilmaz, 2020). A positive finding was that the solution pathways were diverse and included various solution strategies. Particularly, the attention towards informal solution pathways seemed to have influenced the preservice teachers' development of useful techniques for designing RME problems. Still, some groups included sub-questions that could hinder students working flexibly with the problem.

We also noticed that the shifts in understanding were usually coped with by designing two pathways, each at a different level of understanding. Shifts were not detected within single pathways. This could mean that the preservice teachers had too few examples to lean on being able directly to exemplify shifts. Further, it was surprising that only four of the nine assignments contained representations. Amongst the four, the representations were often digital and "perfect". By this we mean that the representation did not have features that could presumably have been a student construction (Wubbels et al., 1997; Yilmaz, 2020). All the groups constructed both heuristics and formal solution pathways. It was probably our verbal request for both formal mathematics and heuristics that stimulated the groups to develop these different strategies.

We also found it interesting that some of the ideas about making generalisations and estimations were derived from the contextual descriptions, not from the mathematical problem. An example was the group in the first example that asked how students could use the knowledge and skill they had learned to solve the related problem of how to diminish CO2 emission in general. It was the wider context that stimulated to ask for this more general problem.

Conclusion

Our study showed that introducing preservice teachers to RME stimulated them to create open-ended problems and present formal and informal solution pathways. We only had a few lectures to introduce our preservice teachers to RME. The extended complexity and formalism in the groups' learning pathways indicate that introducing RME as a theme of development in mathematics education is beneficial (Wubbels et al., 1997). The learning pathways were important thinking tools for the preservice teachers to deepen various themes and reflect on how the learning could be carried out in practice teaching. Our information about the preservice teachers' former mathematical experiences was limited. The complexity of the mathematical themes in their created problems indicated that many of the preservice teachers had a relatively strong mathematical background. An assumption is that, since the assignment was an obligatory course workload, its requirements strongly influenced the designing of the tasks. Particularly, the instigation to create contexts that could be approached both heuristically and by formal mathematical methods seemed to have influenced the preservice teachers' mindset towards the task design. The opportunity to test RME problems in practice teaching could add valuable information about the potential of such tasks.

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Analysing how teacher students plan questions for choral counting and strategy sharing tasks: A comparative study

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90 mathematics teacher students engaged in either a choral counting task or a strategy sharing task. This paper explores the characteristics of their planned questions for middle school pupils related to the two different task types. The analysis of the students' individual planning forms revealed their connection to mathematical content, the degree of openness in the questions and the content of the teacher students' justifications of the questions. Students working on the choral counting task planned questions closely tied to mathematical content and justified them with concrete mathematical concepts. Teacher students working on the strategy sharing task advocated for open questions to validate all pupils thinking and to enhance oral skills. The results of this study contribute to the discourse on how types of task influence teacher students' planned questions for pupils.

Keywords: Mathematics teacher education, rehearsals, task type influence, lesson planning.

Introduction

In the Norwegian curriculum for mathematics for grades 1-10, we read:

> Exploration in mathematics means that the pupils search for patterns, find relationships and discuss their way to a shared understanding. The pupils shall place more emphasis on strategies and approaches than on solutions (Ministry of Education, 2020, p. 2).

This requires that teacher education programs develop the teacher students' competence to guide such discussions. This paper will explore teacher students' question planning when tasked to facilitate productive mathematical discussions, as defined by Stein et al. (2008) during two different problem solving tasks.

Kazemi and Hubbard (2008) recommend the use of collective lesson planning, rehearsals, and enactments in teacher training. This paper builds on data collected during a teaching session based on the "Cycle of enactment and investigation for professional development" as described by Wæge and Fauskanger (2021). This is a 6-step cyclical model: 1) *Preparation*, 2) *collective analysis,* 3) *co-planning, 4) rehearsal with Teacher Time Outs (TTOs), 5) classroom co-enactments with TTOs*, and 6) *collective analysis.* This paper focuses on teacher students' planned and enacted questioning. 90 students planned questions for a whole-class discussion, and the planning forms, rehearsal and enactment are analysed with the following research question: *What distinguishes the planned and enacted questions of teacher students in a choral counting task compared to a strategy sharing task?*

Theoretical background

Show and tell

When a teacher endeavours to guide discussions centred around the pupils' own problem solving strategies, a natural inclination is to let the pupils speak individually in a whole class setting about their solution strategies. Which pupils get the chance to speak is often based on how much the pupils showed desire to participate in the discussion. Stein et al. (2008) characterizes this whole-class discussion as "show and tell", characterised by subsequent pupil presentations, where the pupils are free to choose the solution strategy they prefer. Lack of guidance from the teacher to connect different solution strategies or relate them to larger ideas and a lack of motivation for peers to listen are weaknesses to this approach (Stein et al., 2008). The approach does not thoroughly assess pupils' understanding and does not emphasize the significance of different solution methods. Teachers need to become more active in guiding discussions towards important mathematical concepts, connecting various solution methods, and helping pupils better understand mathematics (Stein et al., 2008).

Five practices for productive mathematical discussions

Stein et al. (2008) introduces five practices that help teachers engage pupils in deeper mathematical thinking and reasoning, going beyond the traditional "show and tell" approach. These five practices are: 1) *Anticipating*: Teachers anticipate pupils' ideas, misconceptions, and possible solutions to mathematical problems before the lesson. By understanding pupils' thinking, teachers can better guide the discussion and respond to pupils' needs. 2) *Monitoring*: During the lesson, teachers carefully observe and listen to pupils' conversations and strategies. Monitoring allows teachers to assess pupils' understanding and progress, identify misconceptions, and determine when to intervene or expand the discussion. 3) *Selecting*: Based on the work in the two previous practices, teachers strategically choose pupils' strategies to highlight and discuss with the whole class. The selection process aims to promote various mathematical ways of thinking. 4) *Sequencing*: Teachers carefully arrange the order of pupils' contributions and questions to connect concepts and develop depth in the discussion. 5) *Connecting*: The final practice involves linking pupils' ideas, relating different strategies, and connecting the discussion to the mathematical concepts and goals of the lesson. This practice helps pupils see the relevance and connections between mathematical ideas.

Rehearsal in teacher education

In ambitious mathematics teaching, the aim is to foster pupils' understanding as well as the social, intellectual, and linguistic resources that pupils bring with them when studying subject-specific ideas (Kazemi et al., 2016). Kazemi et al. (2016) argue that teacher students should work hands on to develop the skills required to lead ambitious mathematics teaching. Teacher competencies needed to implement a practice in line with this form of instruction include eliciting information about pupils' thinking, responding to their ideas, and guiding pupils toward each other's thinking. This competency can be developed via rehearsals (Kazemi et al., 2016), where the teacher students engage in a teaching activity and practice its implementation, guided by the teacher educator (TE). This provides teacher students with an opportunity to collectively explore instructional choices. One or more teacher students take on the role of the teacher. The remaining students assume the role of pupils.

Task types to promote teacher students' mathematics teaching practice

Lampert et al. (2013) have developed several different instructional activities (IAs) to promote the development of novice mathematics teachers' skills regarding eliciting pupils understanding and performance of mathematics. The IAs are designed to include a wide range of mathematical topics relevant for the elementary grades and to enable the participation of all pupils. They are developed over time, parallel with their work on rehearsal as a teacher education practice.

The two IAs investigated in this study were chosen because they both present as rich tasks, but still differ greatly in how they are structured. 1) Choral Counting: In this activity, the teacher student guides pupils through a collective counting in a number sequence determined by the teacher (Lampert et al., 2010). The class is meant to work as a collective unit. The teacher student stops along the way to reflect with the pupils on number patterns and the next numbers in the sequence. Counting can proceed forwards, backwards, and by skipping over strategic numbers in the sequence. 2) Strategy Sharing: In a strategy sharing task, the teacher provides a task that can be solved in many ways (Lampert et al., 2010). The pupils' work can begin wherever the pupil chooses, and they are free to choose and/or develop any solution strategy and representation of their thought process. The focus of the teacher student is on how to elicit ideas from the pupils. The choice of strategies and models to present and targeted questions is central.

Planned questioning for rehearsals

In the co-planning phase of the "Cycle of enactment and investigation for professional development" the participants often discuss what types of questions to ask, the purpose of questions and timing of questioning (Mosvold & Wæge, 2022). The teacher must possess the skill to frame questions that unveil pupils' ideas and use their professional judgement to discern which ideas are worthy of further exploration (Grossman et al., 2009). Orr and Bieda (2023) revealed a pattern in a similar cycle where preservice teachers (PSTs) initiated a "question and answer"-sequence with questions with low academic rigor (AR) (Boston, 2012) to elicit pupil response. When the pupils were actively attentive, the PSTs could follow up with questions with a higher AR. However, Orr and Bieda (2023) also found that when students had planned fewer questions with high AR beforehand, the questions they asked in the lesson also had a lower degree of AR. This suggests that continually increasing the AR in the mathematical discussion might be a goal for PSTs, but if they have not prepared follow up high AR questions beforehand, their goal might not be met.

Method

During January 2023, 90 students planned, rehearsed, and co-enacted a mathematics lesson. This study is based on an analysis of the students' planning forms and TE's observations of the rehearsal and co-enactment related to this.

Tasks

The planned lessons revolved around two different tasks: A strategy sharing task and a choral counting task. The strategy sharing task was presented as follows: "You have 65 tennis balls to distribute into boxes of 3 and 7. How can you distribute the tennis balls?" In the choral counting task, the teacher and pupils count in unison from 0.3, with increments of 0.3, using a shared board. Both tasks were presented orally to the teacher students, and they were encouraged in the planning phase to reflect on how to phrase the task for the pupils.

Student preparation

All the students were studying to become mathematics teachers. The students were divided into three groups: 1) Pre-service, second year, campus-based teacher students. (PST/PSTC) 2) Pre-service, second year, session-based teacher students. (PST/PSTS) 3) In-service, session-based teacher students (ISTs). The PSTC had the lowest average age (19-24 years old) and had the least teaching experience. In contrast, the ISTs had the highest average age (25-45 years old) and had the most classroom experience. All ISTs were working in school related positions at the time of data collection, ranging from kindergarten to upper secondary school. The PSTSs constituted a more mixed group of students with regards to age and work experience.

The students were introduced to the concept of ambitious teaching (Lampert, 2010) in the preparatory stage of the cycle. The students were asked to implement four principles from ambitious mathematics teaching (Norwegian Centre for Mathematics Education, January 2023) when planning the lesson for the pupils: Engage pupils in reasoning and problem solving tasks, have pupils collaborate and participate in mathematical conversations, encourage the pupils to develop their own problem solving strategies and to foster a positive classroom environment by respecting pupils, listening to their ideas, and valuing their contributions. The PSTs worked on the strategy sharing task, while the IST students worked on the choral counting task. All students received a planning form containing three main sections for them to fill in: 1) Presentation and organization of the task. 2) An overview of expected solution strategies from the pupils. 3) Planned questions from the teacher to the pupils with justifications for each question.

The PSTC and PSTS were divided into six groups in total, the IST was split into two. The teacher students acted as pupils in each other's co-enactments of the lesson. The students filled out the planning forms during the planning phase. Some added changes during and after the rehearsal. The forms were collected at the end of the day. Students had the option to decline the use of their planning form for this data analysis. The students were instructed to sign their forms with group number only, ensuring individual anonymity.

Data analysis

The questions in the students' planning forms were analysed using a conventional content analysis approach (Fauskanger & Mosvold, 2014), with codes being defined in parallel with the analysis. Initially, the analysis aimed to examine the students' questions as a whole, but it became evident early in the process that there were significant differences in the questions linked to the different tasks. The analysis evolved to identify categories that clarified these differences. The three categories are:

1) Mathematical content: This category concerns the mathematical content of the questions the teacher students planned to ask the pupils. Examples of questions with a high degree of mathematical content include "What is the next number in the sequence?" or "Why isn't it 0.12?" In contrast, a low-content question could be "How have you thought about it?". The category is not defined by the amount of mathematical content that can be found in the *answer* to the question, but rather how closely the question itself is linked to specific mathematical content for the given task.

2) Grade of openness, talk moves and formats: The analysis assesses whether the questions are open or closed and if there are traces of talk moves and formats (Chapin, 2009) in the question phrasing.

3) Mathematical or dialogue-based justification: For a justification to be categorized as mathematical, it had to mention a specific mathematical topic. Even though "Sharing your problem solving strategy" could be argued to be a mathematical competency, such justifications were

categorized as dialogue-based, as they did not mention specific problem solving strategies. The justifications for the questions turned out to be closely related to the content of the questions.

Observation of implementation: The rehearsal and co-enactments occurred simultaneously across several groups. Five different TEs observed the different groups' rehearsal and co-enactments. After the observation, the TEs assessed whether the students were able to facilitate a productive mathematical discussion (Stein et al., 2008).

Results

Below are the results from the analysis of the teacher students' questions, sorted by task type (strategy sharing or choral counting) and category. There were no significant differences between the campus-based and the session-based PSTs results, therefore they are treated as one group in the following.

Category	Choral Counting, ISTs	Strategy Sharing Task, PSTs
Mathematical content	Strong connection to mathematical content. All questions were marked with where in the sequence they should pause and ask questions to the pupils. Questions were related to further numbers in the sequence, special numbers in the sequence, or pattern identification. Example: "After 0.9: What will happen now?" "What is the next number in the sequence?"	Low connection to specific mathematical content. None of the questions mentioned specific numbers, with one exception: "What methods would you use if you had 70 tennis balls?"
Open questions and talk moves	Talk moves as follow-up. Questions are mainly closed but with open follow-up questions. Talk moves and formats like partner talk, reasoning, and adding on (Chapin, 2009) were noted as follow-ups to mathematical questions. Example: "What is the next number now? How did you figure it out? Anyone thinking something else?". This was written as one question.	Mainly open questions standing alone. Examples: "What was your thought process here?" "Is it possible to do this in another way?"
Justification	Mathematical justifications. Sub-questions were justified with specific mathematical topics, e.g., the 3-times table, the relationship between fractions and decimals, the use of a number line as a representation, and place value.	Dialog-based justifications. Justification was mainly related to ensuring that all pupils were seen and heard and could articulate their own solution strategies.

Table 1: Table of question analysis by category

Notes from observations of the rehearsal and co-enactments

In the groups who worked with the choral counting task, the classroom conversation was characterized by shared curiosity where the ISTs in the teacher role ("teacher") asked questions, and students in the role of the pupils ("pupils") participated in the conversation with their contributions and interpretations of each other's contributions. The "teacher" mainly based the conversation around the planned questions with strong connection to the mathematical content and used talk moves as follow up questions. Although different explanations of the number sequence emerged during the discussion than what was planned for, the ISTs were prepared with a clear direction for the conversation.

Common to the sessions with the strategy sharing task was that "pupils" worked in small groups for a long time on their own solutions. Then followed a presentation round, where the "pupils" who presented received praise and validation, but the different strategies were connected to a limited extent. The conversation largely exhibited the characteristics of "show and tell" (Stein et al., 2008). The "teacher" mainly asked open questions and talk moves but asked very few questions directly related to the mathematical content.

Discussion

The task's influence on the level of productive mathematical discussions

Conducting a productive mathematical discussion (Stein et al. 2008) based on the pupils' own statements is a demanding task. The teacher should not only highlight individual statements so that the selected pupils are heard, but the pupils should also have an opportunity to learn mathematics through mathematical dialogue. Analysis of the teacher students' planning forms from this study shows that the ISTs planned a more mathematically focused conversation related to the choral counting task than the PSTs did with the strategy sharing task. ISTs working on the choral counting task had questions related to specific numbers and various mathematical topics as justifications for their choice of questions. Implementation differences were observed, with strategy sharing sessions characterised by individual recitation, while choral counting sessions fostered a shared sense of curiosity guided by the teacher.

The teacher students engaged in a session inspired by Stein et al.'s (2008) five factors for a productive discussions. The students came up with potential solutions strategies when they worked in factor 1, *Anticipating*. The questions created by the PSTs working on the strategy sharing task were largely about getting an overview of the pupils' different strategies and can thus be considered applicable to both factor 2, *Monitoring*, and factor 5, *Connecting*. Stein et al.'s (2008) third factor is to select different strategies. Here it seems that the ISTs working on the choral counting task was the more prepared of the two groups. All ISTs planned questions were marked with a number in the sequence, meaning they had a concrete plan for when to ask the question. The questions were also justified with specific mathematical topics that they envisioned bringing up in the conversation. Based on this selection, they had a planned goal for the conversation to follow. The PSTs on the strategy sharing task had no planned questions about concrete strategies related to the task. It may, therefore, seem that they had not prepared enough for Stein et al.'s (2008) third factor, *Selecting*. The following reflects on possible explanations for why that might have been.

The PSTSs did not make selections because they prioritized pupil engagement

While this study does not argue against strategy sharing as a starting point for a productive mathematical discussions, it notes that PSTSs did not progress sufficiently. The challenge might be that the PSTs very clearly prioritized the idea that all pupils should be heard and validated. This approach might lead to a loss of learning potential, as the conversation lacked guidance to connect the different solution strategies or link them to larger ideas. The well intended principle might have led to a teaching situation similar to Stein et al.s (2008) “show and tell”-structure. This resonates with the findings of Orr and Bieda (2023), that showed that PSTs asked fewer questions with high AR when the PSTs had not prepared such questions in advance.

The choral counting task has more structure than the strategy sharing task

The choral counting task’s structure is based on a number sequence that can be followed from one number to the next. This might be a more manageable starting point than an open strategy sharing task. The pupils working on the strategy sharing task may have widely varying solution strategies when the collective conversation begins. To lead a productive mathematical discussion (Stein et al., 2008) might be more challenging for the teacher student with an open strategy sharing task than with a choral counting task.

The teacher students’ prior teaching experience

The ISTs, working on the choral counting task, had more classroom experience. It is conceivable that, due to their experience, they had a more goal-oriented approach during the planning phase, were more accustomed to planning with specific learning objectives, and spent less time discussing general pedagogy during the planning phase compared to what might be expected from a second-year, PST with limited classroom teaching experience.

Further research

From my perspective as a TE, it appears that it may be a more manageable task for the teacher student to lead a discussion based on the choral counting task than with an open strategy sharing task. The factor related to the students’ work experience will be challenged by collecting a new dataset by myself and colleagues in January 2024. We will conduct the same teaching program with similar groups, but with the tasks swapped among the groups. If a new analysis of the questions in the planning forms shows a similar distinction in mathematical content in questions and justifications, it will strengthen the idea that the difference in the tasks have led to the distinction, and not the students' work experience as teachers. The results of these studies can affect further work with choosing tasks for rehearsals in mathematics teacher education.

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How mathematical strategies and didactic models for equation solving address coefficients

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Solving equations is a critical skill in mathematics education in upper secondary school. Studies have shown that some students' abilities to solve linear equations depend on the kind of numbers present in the equations. Didactic models for linear equations might not address this issue. In addition, there is no systematic analysis of how well didactic models represent coefficients in equations, and their potential to support students' recognition of structure in equations in different numeric settings. In this paper, we review existing mathematical strategies and didactic models of equation solving and explore how they address coefficients, with a special focus on decimal and negative numbers.

Keywords: equation solving, mathematical strategies, didactic models, coefficient.

Introduction

Mathematical models utilize a range of representational media, including symbols, language, and graphics, to describe systems of mathematical importance. They focus on two integral parts: the *conceptual system* and *accompanying procedures* (Lesh & Harel, 2003). The conceptual system encompasses the relevant mathematical objects, relations, patterns, and regularities. In equation solving, this system indicates the role of the equal sign, the variables, and coefficients (i.e., *a, b, c,* and *d* in the equation $ax + b = cx + d$). The accompanying procedures specify manipulations for achieving specific mathematical goals. In equation solving, this includes rearranging, simplifying, and transforming equations to effectively solve for the unknown.

Two broad approaches help learners understand the conceptual system and accompanying procedures of equation solving: *mathematical strategies* and *didactic models* (Filloy & Rojano, 1989). Mathematical strategies focus on the logic of equation solving and specific mathematical principles like preserving equality with *do the same operation on both sides.* Didactic models, on the other hand, translate algebraic concepts into more concrete forms, for example by using a balance scale to represent the abstract concept of equality. Filloy and Rojano argued that learning with didactic models involves a dual process, *translation* and *separation*. The translation phase involves using didactic models to help students understand the structural aspects of algebraic syntax based on the behaviors of these models. The subsequent separation phase is about students taking the algebraic syntax they learned from concrete models and applying them in more abstract or other novel mathematical contexts.

For some students, the difficulty of equation solving stems from the presence of non-natural coefficients, such as decimal and negative numbers (Holmlund, 2024; Vlassis, 2002). For example, some students who can solve x in $a = bx$ still cannot solve the equation $0.4x = 0.12$. As students usually learn equation solving with didactic models, we consider if didactic models that are intuitive with natural numbers might hinder students' recognition and understanding of equations that involve decimal or negative numbers. In a more general discussion on didactic models, Vig et al. (2014) stated that there is a point at which all models are not applicable or useful, *a breaking point*. Several attempts have been made to analyze the limitations of specific didactic models for equation solving, e.g., the balance scale (Otten et al., 2019) and the number line (Dickinson & Eade, 2004). However, there is limited research that systematically analyzes and compares how different mathematical strategies, or didactic models translate coefficients that are rational numbers, especially negative numbers and positive decimal numbers. Therefore, we conducted a comparative analysis of various studies in order to answer the central research question:

How do mathematical strategies and didactic models for linear equations address coefficients, especially decimal and negative numbers, and the relevant operations?

Mathematical strategies and didactic models of equation solving

We employed a snowballing sampling of literature to identify relevant studies discussing the mathematical strategies and didactic models of equations. This approach resulted in three formal mathematical strategies and three types of didactic models, as described below. Representing an equation as the intersection of two graphs is not included in the analysis as the current paper focuses on concrete didactic models that translate algebraic syntax.

Mathematical strategies

The strategy of *do the same operation on both sides (DSBS)* illustrates the principle of maintaining equality in an equation (Filloy & Rojano, 1989). This approach highlights the fundamental symmetry of an equation and underscores the concept of equality (Kieran, 1992). It involves strategically choosing procedures to solve *x* and operating them on the expressions on both sides to preserve equality.

The strategy of *transposing terms* moves terms from one side of an equation to the other to isolate the unknown variable. This approach focuses on inverse operations as a way of moving terms and numbers to the opposite side of the equality sign. This strategy, which is based on the principles of maintaining equality, can be considered an abbreviated form of DSBS (Kieran, 1992) or a potentially route learnt procedure without conceptual understanding of equality.

The strategy of *restore and/or confront* transforms coefficients to restore the standard form of an equation (e.g., in the linear case $bx = c$) and to transform the coefficient of the highest term into a "1" (e.g., $b = 1$) (Oaks & Alkhateeb, 2007). In this case, the simplification of the equations involves: moving quantities to the opposite side of the equation (e.g., $2x + 3 = 5$ to $2x = 2$) and increasing or reducing the coefficient of x (e.g., the 2 in $2x$). This method focuses on restoring the standard form through manipulating the coefficients in the equation.

Didactic models

The *balance scale* is a traditional metaphor used in teaching equation solving that emphasizes the mathematical strategy of DSBS. In the balance scale, the expressions on each side of the equal sign are linked to one plate of the balance scale, and the primary task is to determine the unknown weight. This metaphor is successful in representing the concept of equality (Elkjær

& Thomsen, 2022), since any change made to one side of the plate (i.e., one side of equation) must be mirrored on the other side to maintain the balance.

Geometric models utilize visual and graphical representations to depict the mathematical strategy of DSBS. We identified four main typical geometric models: the area model, algebra tiles, the number line and the block model. The area model utilizes the areas of rectangles (with length representing x terms and either length or area representing numeric terms *A, B,* and *C*) to represent algebraic expressions. Such visualization facilitates the comparison of two algebraic expressions (Filloy & Rojano, 1989). Algebra tiles contain three specific shapes designed instructionally and physically to represent the different terms in an algebraic expression (constants, x terms and x^2 terms). This model uses different colors to distinguish between positive and negative numbers, for example, red for positive terms and blue for negative terms (Saraswati et al., 2016). The number line is a model in which one side of the equation is described as 'jumps' on an ungraded line, where x is a series of equal jumps of unknown length (see Figure 1). The other side of the equation is represented by jumps below the number line and should cover the same distance (Dickinson & Eade, 2004). The block model, originally used for visualizing arithmetic word problems in Singapore primary mathematics curriculum, consists of a series of rectangles in various sizes, representing specific numerical values or x terms (Ng & Lee, 2009). This model can also be used to visualize the relation between algebraic expressions in a linear equation, where each algebraic expression is represented by a combination of rectangles (see *Figure 2*).

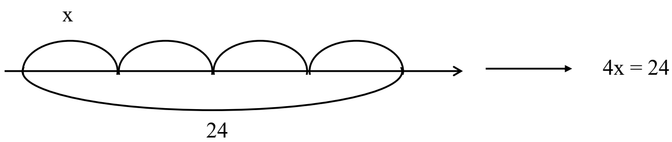

Figure 1 The number line model

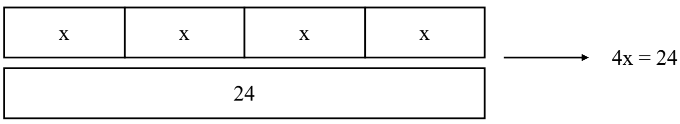

Figure 2 The block model

The *hidden quantity* model visualizes the distinction between the known and the unknown quantities in an equation (Radford, 2022). For example, imagine that two people have equal amounts of cards, but an unknown number of cards are hidden in envelopes and that the task is to figure out the number of cards in the envelopes. The process of solving equations involves removing objects from both sides, as well as combining similar objects and focusing on isolating one envelope on one side. This mathematical reasoning mirrors the ancient Arabic mathematical method; reducing the coefficient to 1 (radd) and combining like terms on opposite sides of the equation (muqābala) (Oaks & Alkhateeb, 2007; Radford, 2022).

Mathematical strategies and coefficients in equation solving

To analyze the roles of coefficients in mathematical strategies, we first pose questions regarding how the strategies address linear equations: (1) what are the central concepts in focus?

(2) what are the procedures for performing the strategies? From these two questions, we can analyze (3) what the role of coefficients is in these strategies (see Table 1).

The role of coefficients (see column 4 in Table 1) varies when using different strategies. We take $5 = 3x - 1$ as an example for illustration. DSBS focuses on maintaining equality. The coefficients are primarily used to describe the quantity of change when operating on both sides of the equation. Transforming it to $6 = 3x$ with DSBS would eliminate -1 on the right side, thus adding $+1$ to both sides. The coefficients 5, 3, and -1 are relevant after the choice of operation has been decided, indicating the extent to which this operation should be performed.

Table 1. The focus three strategies put on the coefficients in a linear equation

Strategy	Central concept	Procedure	The role of coefficients
DSBS	Equality	Operate on two expressions for equality	Quantify the operations on both sides
Transposing terms	Inverse operation	Move a term within the equality	Be moved along as part of the x term or constant term
Restore/confront	Structure of expression	Restore the standard form of equations bx=c	Be manipulated to transform the coefficient of the x term to 1

The strategy of transposing terms views the equation as a whole and requires moving the -1 to the left side to isolate the x term. The size of the coefficient is not in focus, but rather its position relative to x and its sign. The focus is on the inverse operation with procedural objectives, aiming to separate the knowns and the unknowns to different sides of the equal sign.

The restore and/or confront strategy perceives coefficients as central in deciding the next steps, as the goal is to reduce the highest term's coefficient to 1 and restore it to the standard form $bx = c$. For the equation $5 = 3x - 1$, the emphasis will be on how to move and combine similar terms (e.g., 5 and 1 in the equation) and how to change the 3 in $3x$ to 1 by reducing the coefficient (divided by 3).

Didactic models and coefficients in equation solving

To analyze how the didactic models translate the equations, we specifically analyze the translation of: (1) the equality concept, (2) the decimal coefficients, (3) the negative coefficients, (4) the addition of decimal or negative numbers, and (5) the multiplication of decimal or negative numbers. To display different possible positions of the negative or decimal number, we use some standard linear equations as baseline. For the translation of addition, we use $a + x = b$ $(a, b, x \in \mathbb{Z})$ for negative numbers and $a + x = b$ $(a, b, x \in \mathbb{Q}_{+})$ for decimal numbers. For the translation of multiplication with negative numbers, we use $ax = b$, $-ax = b$, $ax = -b$, $-ax = -b$, where a, b > 0. For the translation of multiplication with decimal numbers, we use $ax = b$ $(0 < b < a < 1)$ and $ax = b$ $(0 < a < b < 1)$. The focus of the analysis is whether the models can represent the equation, both the numbers and x terms.

The analysis reveals three levels of translation. *Direct translation* (D): The didactic model provides a one-to-one corresponding translation of the concept with either visualization or text description. For example, the balance between two plates in the balance model can directly represent the equality of the two algebraic expressions. *Adding feature* (A): The didactic model cannot represent the equation on its own without adding features. For instance, while a balance scale cannot represent negative weight, adding reverse gravity in a digital environment makes it possible. *Transformation* (T): The didactic model can represent the equation only when the equation is transformed. For example, for $-ax = b$, the balance scale cannot represent a negative number of weights (x). To represent it, the equation is transformed to $a\,(-x) = b$, or to $ax = -b$ by multiplying -1.

Table 2 presents an overview of the translations of coefficients and operations by different models. The result indicates that all the didactic models directly translate the concept of equality. This is done either through analogy of balance in a balanced scale, the visualization of equal areas or lengths in geometric models, or with direct descriptions, such as two envelopes have the same number of cards in hidden quantities.

Table 2. Overview of didactic models' translation of coefficients and operations

	Balance scale	Geometric models				Hidden quantities
		Area model	Algebra tiles	Number line	Block model	
Equality	D	D	D	D	D	D
Negative numbers	A	A	D	D	A	A
Decimal numbers	D	D	A	D	A	A
Addition with negative numbers						
$a + x = b$ $(a,\ b,\ x\ \in\ Z\ )$	A	A	D	D	-	A
Addition with decimal numbers						
$a + x = b$ $(a,\ b,\ x\ \in \mathbb{Q}_{+})$	D	D	A	D	A	A
Multiplication with negative numbers $(a, b\ >\ 0)$						
$ax = b$	D	D	D	D	D	D
$ax = -b$	A	-	D	D	A	A
$-ax = b$	A&T	-	T	T	A&T	A&T
$-ax = -b$	T	T	T	T	T	T
Multiplication with decimal numbers						
$ax = b$ $(0 < b < a < 1)$	T	D	A&T	T	A&T	A&T
$ax = b$ $(0 < a < b < 1)$	T	D	A&T	T	A&T	A&T

When translating *negative numbers*, some constructs like weight, area, or the number of cards do not naturally translate the concept of negativity. Some features can be added to these models to distinguish between positive and negative numbers, such as adding different colors in the area model, using reverse gravity for balance scale in a digital environment (i.e., negative value lifts the plate) (Elkjær & Thomsen, 2022), or using different colors to mark the objects in the block model or the hidden quantities model (A). Algebra tiles were specifically designed to represent negative numbers by using different colors (Saraswati et al., 2016). The number line model naturally includes all rational numbers by using directions.

In terms of the translation of *decimal numbers*, the qualitative difference among these models lies in their varying potential to split or convert the unit. For example, weights on a balance scale can naturally be replaced by smaller units, such as converting 0.9 kg to 9 hg (D). However, this is not feasible with algebra tiles, the block model or the hidden quantities model in a natural setting, where algebra tiles, rectangles and cards are considered intact objects. These models require adding features to translate the decimal number, such as splitting the algebra tiles or the cards in a digital environment (A).

The operations of *addition* are translated in all these didactic models through adding or taking away weights, areas, or cards and, in the case of the number line, making jumps in different directions. However, if the didactic models could not represent the decimal and negative numbers intuitively, problems might emerge in the translation of these operations on decimal and negative numbers. For example, in block model, with the premise of the same length as the equality, the block model cannot represent equations that involve both positive and negative numbers, such as $3 + x = -24$.

In multiplication, the coefficients of equations have intuitive meanings when translated into didactic models. In all didactic models except the area model, the coefficient next to x (i.e., the multiplier), represents the number of things. Such intuitive meaning makes translation with negative and decimal numbers difficult in some of these models, as we demonstrate below.

For multiplication by negative numbers, both the algebra tiles and the number line can represent $ax = -b$ as repeated addition of negative objects (D). For the balance scale and hidden quantities, adding features is required to represent negative values (A). The area model could not represent this model as x represents a side of an area, and there is no explanation how to combine a negative and positive length that would result in a negative area. As mentioned, in all models except the area model the multiplier represents a number of things that cannot be negative. Therefore, $-ax = b$ should be transformed into $ax = -b$ to be represented by the models (T or A&T). All models could represent $–ax = -b$ by transforming it to $ax = b$ (T). However, for the balance model, the algebra tiles and the block model, it is more natural to add features and transform it while retaining negative numbers.

For multiplication by decimal numbers, the area model has a natural representation with various areas and lengths. For example, in the equation $0.12 = 0.4x$, the model could be interpreted as one rectangle with an area of 0.12 cm² and two side length of 0.4 cm and x cm respectively (D). All the other models interpret the multiplier as the number of things or steps, wherefore they cannot represent equations with a decimal number directly. To resolve this, an equation with decimal coefficients, such as $0.12 = 0.4x$, is commonly transformed into $12 = 40x$, making the multiplier a natural number. With such transformation, the balance scale can represent the 40 smaller weights that amount to 12 kg, where the smaller weights easily can change unit from 0.3 kg to 3 hg. Similarly, the number line can represent 40 small jumping segments that together have a length of 12 and has a natural representation of 0.3 as the unknown length. (T). In contrast, algebra tiles, block model, and hidden quantities represent x terms with intact objects, which cannot represent decimal number unless extra features are added, such as simulation-based context that enables students to split objects (A&T).

Discussion

The results reveal how mathematical strategies and didactic models for linear equations address non-natural coefficients in diverse ways. The results show that coefficients are not the primary focus in mathematical strategies like DSBS and transposing terms, whereas they are central to the strategy of restore and/or confront. This is the only strategy that emphasizes the need for *one* unknown, restoring the 1. In addition, strategies are not restricted to the types of coefficients. However, the strategies alone might not address the problem that students can solve the equation $ax = b$ with natural numbers but not with decimal numbers despite calculator aid (Holmlund, 2024).

Didactic models could potentially support the understanding of the role of coefficients. However, our results show that all models have shortcomings in explaining certain operations of decimal and negative numbers. The results support the claim that all models have a breaking point and are, therefore, only useful for mathematical problems up to a certain point (Vig et.al., 2014). The results reveal that when direct translation is not feasible, other levels of translation, (i.e., adding feature and transformation) are involved to represent these coefficients and relevant operations. Even though these two levels of translation address the problem of representing, it is not clear if they always make the models more useful.

When features are added, the predefined rules for negativity can cause contradictions in some models. For example, adding different colors to the area model does not convey how a positive length and a negative length can become a negative area. For algebra tiles, the colors generate two meanings of negativity. The smallest tiles with negative color are always of negative value, whereas the x-tiles with negative color only indicate "the opposite of the positive x-tile". In contrast, the balance scale and number line illustrate negativity more consequently as the opposite value of the positive. However, no model can directly translate $-ax = b$ and they are thereby incapable of explaining the syntactic rules of two negative values becoming positive.

The need for transformation of equations with decimal numbers to be represented is also problematic since it might prevent students from engaging with the properties of non-natural numbers and their operations. For example, when a number is multiplied with a number between 0 and 1, the product can "become smaller", such as solving $0.4x = 0.12$. However, if the equation is transformed to natural numbers, it bypasses the explanation of the meaning of 0.4 and its relations with x. Moreover, it does not help students that can solve equations with natural numbers but not equations with decimal numbers. Furthermore, the need for transforming equations to fit within these models suggests a backward assumption: students understand the mathematics of these transformations before they use the models, rather than using the models to understand the equations. This is in line with the results of Vig et. al. (2014) that describe how "adaptions at the model breaking point can turn model use into nothing more than executing a graphical analogue of a not-well-understood procedure" (p.23). Consequently, it's illogical to rely on these models for instruction in cases where they do not effectively bridge students' comprehension gaps.

Conclusion

While didactic models aid students in developing a concrete understanding of concepts, such as equality and variables, the breakpoints in representing negative and decimal numbers might potentially cause insecurity whether the syntax applies for these numbers. To overcome such insecurity, teachers can guide students to actively reflect on the limitations of the models and bridge didactic models with mathematical strategies, so that students do not over rely on the models. In addition, employing different mathematical strategies might help students to notice different aspects of algebraic syntax in equation solving, for example, comparing the restoring of $1x$ with isolating x on one side, to notice the role of coefficients in the equation solving process.

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Danish preservice mathematics teachers' multidigit arithmetic strategy adaptivity and flexibility: A competence insufficiency

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The Danish national curriculum for mathematics privileges number-based strategies above standard algorithms for the solution of arithmetical tasks, the achievement of which is dependent on teachers acquiring appropriate dispositions and competence. In this paper we investigate Danish preservice mathematics teachers' (PTs) multidigit arithmetic strategy adaptivity and flexibility. PTs completed several tasks, each designed to elicit number-based shortcut strategies, at the beginning and, for those who completed their programme, at the end of their mathematics education programme. Statistical analyses indicated that the lowest levels of procedural competence are linked to lower levels of both strategy adaptivity and flexibility. The use of standard algorithms, despite falling between assessments, remained high. Importantly, PTs who completed both pre- and post-tests were more competent, adaptive and flexible than PTs who quit the programme.

Keywords: Multidigit arithmetic, strategy adaptivity, strategy flexibility, Denmark, initial teacher education.

Introduction

With respect to children's learning of arithmetic, the Danish national curriculum expects teachers (our translation) to "challenge and support individual students to develop their arithmetic strategies based on their number understanding rather than learning procedures for setting-up and calculating standard algorithms" (Børne-og Undervisningsministeriet (2019a, p.15). Despite such ambitions, national tests and independent research have found, across all grades, students privileging standard algorithms on multidigit arithmetic tasks (Børne-og Undervisningsministeriet, 2019b; Jóelsdóttir & Andrews, 2023). Recently, the Danish initial teacher education authorities increased the time for mathematics specialists from 40 ECTs to 50 ECTs (Børne-og Undervisningsministeriet, 2023) in order to ensure that preservice teachers (hereafter PTs) will develop the competence to teach mathematics in accordance with subject specifications and develop those insights that underpin the subject- and didactics-related knowledge of primary mathematics. This paper, therefore, reports on the impact of one well-regarded Danish teacher education programme on the development of PTs' understanding of and ability to undertake multidigit-related number-based strategies. In so doing, it employs the distinction between strategy flexibility (having a repertoire of strategies for a given task) and strategy adaptivity (selecting the optimal strategy from that repertoire) (Verschaffel et al., 2009).

With respect to children's multidigit arithmetic, research has confirmed that number-based strategies are frequently more efficient than standard algorithms (Hickendorff et al., 2018; Sievert et al., 2019; Torbeyns et al., 2017). However, these same studies find children abandoning number-based strategies for standard algorithms or basing their choices on perceptions of strategy competence rather than task characteristics (Torbeyns & Verschaffel, 2016). Further, warranting interest in Danish PTs, while Danish children's multidigit-related strategy flexibility increases across the grades, strategy adaptivity peaks at grade six before falling. Moreover, children's accuracy, which improves with maturation, is influenced positively by both adaptivity and flexibility, with flexibility having the greatest influence in grade three and adaptivity in grade six (Jóelsdóttir & Andrews, 2024).

Sadly, little is known about preservice teachers' multidigit arithmetic strategy flexibility and/or adaptivity, although there are indications that primary preservice teachers are more likely to invoke number-based strategies than secondary (Van Dooren et al., 2002). Away from arithmetic, while preservice teachers' strategy flexibility has been found to influence positively their achievement (Segura & Ferrando, 2023), their deployment of high level mathematical process skills (Daher & Anabousy, 2020) and their ability to produce rich learning tasks for children (Sevinc & Lesh, 2021), little is known about preservice teachers' strategy flexibility, strategy adaptivity and achievement.

Acknowledging the above, this paper reports on an investigation driven by two questions.

> In what ways has a well-regarded mathematics teacher education programme impacted on Danish PTs' multidigit arithmetic-related strategy flexibility and strategy adaptivity?
>
> Are there differences in strategy flexibility and strategy adaptivity between PTs' who completed the whole programme and those who did not?

Methods

In this study, concerned more with the extent of PTs' strategy repertoire than their speed of execution, no constraints were placed on participants' strategy choices, an approach that has proved successful in identifying children's strategy repertoires in various European countries, including France (Lemaire & Brun, 2017), Germany (Heinze et al., 2009), and the Netherlands (Hickendorff, 2018). Participants undertook assessments of their multidigit arithmetical competence at the beginning (pre-test) and end (post-test) of their programme. The assessment, detailed in Jóelsdóttir and Andrews (2024), comprised two sets of nine multidigit tasks, three each of addition, subtraction and multiplication. One set was used for the pre- and the other the post-test. The tasks, designed to elicit number-based shortcut strategies, were used because their procedural simplicity should have optimised the opportunities for PTs to identify those task characteristics indicative of shortcut approaches, where a shortcut strategy is defined as a modification of a task's features in ways that both simplify the arithmetic and reduce the number of steps from those of a standard algorithm (Hickendorff, 2018; Torbeyns et al., 2009). For example, the task 482 + 218 would be solved more simply and in fewer steps than the standard algorithm if the solver recognised the complement to one hundred implicit in the task. Each assessment, which comprised two phases, was completed during a single teaching session. During phase one, PTs solved each task once by whatever means they thought was appropriate. PTs who offered shortcut strategies were construed as adaptive on that task. During phase two, PTs solved each task again by whatever alternative strategies they were able to generate. PTs able to offer two or more strategies were construed as flexible on that task.

Participants

Participants initially comprised a cohort of 55 teacher education students with a mathematics major at one campus of a large, multi-campus university college in Denmark. However, the flexible nature of Danish undergraduate study meant that a high proportion of PTs transferred to different programmes after completing the pre-test. Thus, the results below draw on the data from two sets of PTs: 30 who completed both pre-test and post-test, and 25 who completed only the pre-test. Importantly, all participants began their programme in 2021, two years before the introduction of the new law for Danish teacher education, meaning that their full programme equates to 40 ECTs.

Coding and analysis

As indicated above, for each item, a student was defined as adaptive whenever a shortcut strategy was used during phase one, and flexible whenever two or more distinct strategies, including standard algorithms, were offered during phases one and two. With nine items implicated in each test, a student's score for both adaptivity and flexibility, whether pre-test or post-test, lay between zero and nine. However, for ease of communication, PTs were assigned to four adaptivity and flexibility groups – none, low, medium and high – according to the protocols shown in Tables 1 and 2. Further, as students' answer sheets were constrained to five solutions per task, the mean number of solution strategies per item ranged from zero to five. In addition to descriptive statistics, and acknowledging the small sample sizes, non-parametric Wilcoxon rank-sum tests were used to compare data from pre- and post-tests. The original intention was to use the pre-test as a baseline measure of students' multidigit arithmetic adaptivity, flexibility and competence and the post-test to determine whether/to what degree these outcomes had changed. However, distinguishing the cohort according to which tests PTs completed, and quantifying their use of standard algorithms, led to unexpected insights.

Results

Adaptivity, accuracy and standard algorithms

The figures of Table 1, which show for each group descriptive statistics for PTs' strategy adaptivity, solution accuracy and use of standard algorithms, yield both expected and unexpected insights. First, the combined pre-test score shows that across the cohort PTs were arithmetically competent, scoring on average 90% on across the nine items. However, while not statistically significant, the mean score for the PTs of group 1 was five points higher than that of the PTs of group 2. Second, the lowest achieving PTs in both groups were those exhibiting low levels of adaptivity; the mean accuracy scores for these PTs in groups 1 and 2 (83% and 63% respectively) were considerably lower than those of all other levels of group membership. Third, the most accurate PTs were either the eight (7 from group 1 and 1 from group 2) who exhibited high levels of adaptivity or, unexpectedly, the 29 (12 from group 1 and 17 from group 2), who exhibited no adaptivity. However, while the former's success was due to their efficient use of shortcut methods, the latter relied almost entirely on the successful application of standard algorithms. Fourth, when the two groups are compared, an adaptive distinction emerges, with those who completed their teacher education programme showing higher levels of adaptivity on their pre-test than those who dropped out. Indeed, 46% of the former group showed medium or high levels of adaptivity, compared with only 20% of the latter. Fifth, the PTs of group 2, who left the programme, were marginally more reliant on standard algorithms than those of group 1, particularly at the low and medium levels of adaptivity.

When comparing group 1's pre- and post-test results, no change can be seen in either accuracy, which remained high, nor the number of solution strategies offered (means of 2.66 and 2.72 respectively).

There was a tendency towards higher levels of adaptivity, although the means for the pre- and post-tests (3.10 and 3.97 respectively) were not significantly different ($z = 1.09$, $p = 0.274$). Also, the number of PTs who exhibited no or little adaptivity remained high, as did their use of standard algorithms. Overall, there was a decline in PTs' use of standard algorithms from a mean of 4.27 tasks to 2.93 tasks, albeit not statistically different ($z = 1.62$, $p = 0.106$).

Table 1: PTs' adaptivity, pre- and post-test accuracy, and use of standard algorithms (SA)

			Pre-test			Post-test		
		Items solved adaptively (out of 9)	Mean items solved by means of SA (out of 9)	Number of students (percentage)	Mean percentage of items solved accurately	Mean items solved by means of SA (out of 9)	Number of students (percentage)	Mean percentage of items solved accurately
Adaptivity of students who completed pre- and post-test (Group 1, N=30)	None	0	8.08	12 (40)	94	7.71	7 (23)	92
	Low	1-2	3.5	4 (13)	83	4.6	5 (17)	84
	Medium	3-6	2.0	7 (23)	90	1.1	10 (33)	91
	High	7-9	0.43	7 (23)	98	0	8 (27)	99
		Means	4.27		92.5	2.93		92.2
Adaptivity of students who completed pre-test only (Group 2, N=25)	None	0	7.59	17 (68)	90			
	Low	1-2	4.33	3 (12)	63			
	Medium	3-6	4	4 (16)	92			
	High	7-9	0	1 (4)	100			
		Means	6.32		87.5			

Flexibility, accuracy and standard algorithms

Table 2 shows the results for PTs' flexibility, accuracy and use of standard algorithms. With respect to the pre-test, every PT, irrespective of group membership, was able to offer at least two strategies for each task. That is there were no PTs demonstrating no flexibility. However, three PTs, all members of group 2, exhibited low levels of flexibility. These, along with the six members of group 2 with medium levels of flexibility, demonstrated disappointingly low mean levels of accuracy (78% and 76% respectively). Moreover, in comparison with their peers in group 1, this relatively poor performance, when viewed alongside their frequent use of standard algorithms, indicates that some algorithms were poorly executed.

With respect to the PTs of group 1, those who completed both pre- and post-tests, no changes were observed; levels of both flexibility and accuracy remained constant. The only difference, confirming the earlier results in relation to adaptivity, was a reduction, albeit statistically insignificant, in the use of standard algorithms.

Table 2: **Levels of flexibility and accuracy for pre- and post-tests**

			Pre-test			Post-test		
		Items solved flexibly (out of 9)	Mean items solved by means of SA (out of 9)	Number of students (percentage)	Mean percentage of items solved accurately	Mean items solved by means of SA (out of 9)	Number of students (percentage)	Mean percentage of items solved accurately
Flexibility of students who completed pre- and post-test (Group 1, N=30)	None	0		0 (0)			0 (0)	
	Low	1-2		0 (0)			0 (0)	
	Medium	3-6	4.25	4 (13)	92	3.75	4 (13)	91
	High	7-9	4.27	26 (87)	93	2.81	26 (87)	94
		Means	4.27		92.5	2.84		92.3
Flexibility of students who completed pre-test only (Group 2, N=25)	None	0		0 (0)	0			
	Low	1-2	6.33	3 (12)	78			
	Medium	3-6	4.83	6 (24)	76			
	High	7-9	6.88	16 (64)	94			
		Means	6.32		87.8			

Comparing strategy use across groups

Figure 1 shows graphically, for each assessment, the variation in the strategies employed by each group of PTs. The blue columns highlight well the extent to which the strategies of the PTs of group 2 differed from those of group 1. In particular, they show an almost complete lack of adaptivity alongside a strong reliance on standard algorithms. Also, acknowledging a maximum score of nine, the columns confirm the relatively high levels of flexibility across all groups. However, when the condition that the multiple solutions that define flexibility must include a shortcut strategy, the means fall across all groups, particularly for the PTs of group 2. Overall, the data indicate that the PTs of group 1, who undertook both tests, are more likely to recognise and act upon different task characteristics by employing a shortcut strategy than a standard algorithm. Finally, in what appears a bizarre coincidence, the column totals for 'adaptivity' and 'standard algorithm as preferred strategy' sum to, subject to minimal variance, a constant of around 7.25. In other words, given one score for a particular group on a particular test, the other can be inferred relatively accurately.

Discussion

In this paper we have evaluated preservice teachers' (PTs) multidigit arithmetic-related strategy flexibility, strategy adaptivity and procedural accuracy. Each task was intended to be procedurally simple so that PTs' attention would be paid less to task difficulty than the task characteristics that invite the intended shortcut strategy. The original aim of investigating any strategy-related changes prompted by the teacher education programme, was modified in the light of high levels of attrition. In the following, we discuss, albeit briefly, first the outcomes in relation to the teacher education programme and second implications for the tool as a screening device for teacher education.

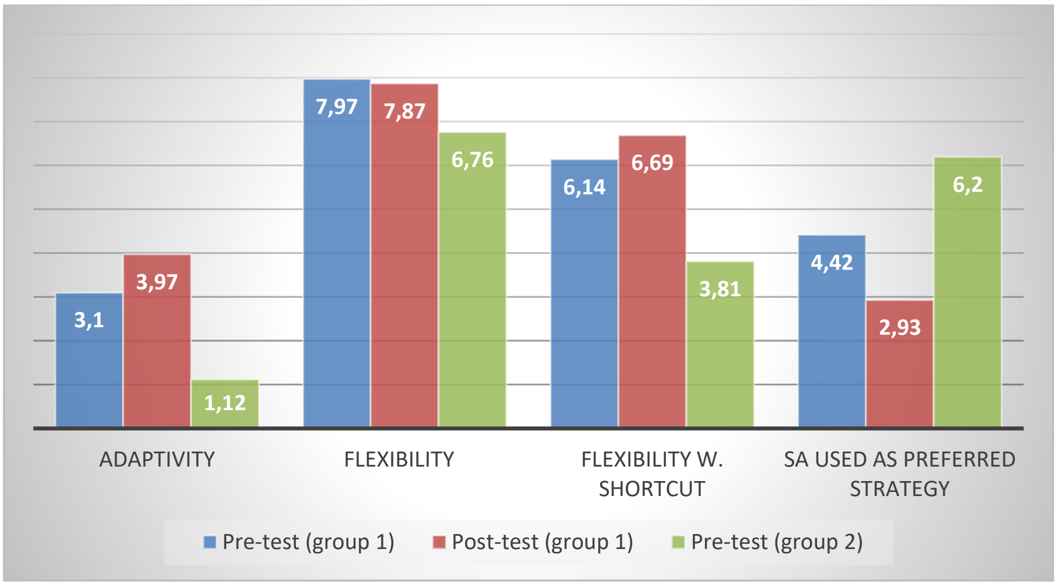

Figure 1: Mean adaptivity, flexibility, flexibility with shortcut and standard algorithm (SA) as preferred strategy for the two student groups

First, the 30 PTs of group 1, who completed both pre- and post-tests, typically exhibited high levels of flexibility, as measured by the extent of their strategy repertoires, and accuracy across both assessments. However, their generally lower pre-test levels of adaptivity, as measured by their use of shortcut strategies, showed little growth by the end of their programme. In other words, in respect of an educational system with statutory expectations that children should learn number-based strategies rather than standard algorithms (Børne-og Undervisningsministeriet, 2019a), the PTs' programme seems to have had limited success, particularly with those procedurally very competent PTs who began their programme with a preference for the use of standard algorithms. Such findings indicate a need for the programme to address those culturally-situated and unquestioned beliefs and practices that teachers hold and enact simply by dint of being members of a particular cultural group (Andrews, 2011). In this respect, the increased time for mathematics in the new teacher education programme should help the development of PTs' strategy flexibility and adaptivity.

Second, of the 55 PTs who undertook the pre-test, 25 quit the mathematics teacher programme and, of these, 19, or 35% of the original cohort, left teacher education. Consequently, any insights studies like this can offer are important, as there is a "dearth of research on retention and attrition in the initial teacher education (ITE) context" (Keane et al., 2023, p.7). In this respect and in comparison with their peers in group 1, PTs in group 2 were more likely to resort to standard algorithms, were the only group with members exhibiting low levels of flexibility and, importantly, problematically low levels of competence when exhibiting low levels of adaptivity and low to medium levels of flexibility. Such findings could be construed as flags for concern, not least because low levels of preservice teachers' mathematical competence are known to impact negatively on children's achievement (Campbell et al., 2013). In conclusion, in addition to highlighting the need for an increased focus on number- and arithmetic-related skills in Danish mathematics teacher education, this study has yielded a simple to operationalise tool for identifying likely dropouts or potentially problematic PTs.

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Listening to student voices of participation in mathematics - a teacher's reflections on own inclusive mathematic teaching

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Inclusion is a widely interpreted concept in schools and has many roles and definitions in mathematics education. By listening to and reflecting on students' experiences of participation in mathematics, new understandings and inclusive ways of teaching mathematics can be fostered. This study explores a Norwegian teacher's reflections on her own practice within the framework of formal teacher-student conversations. Through a collective qualitative analysis of the teacher's reflections on conversations with three different students, two main themes emerged: teacher's commitment to the students' experiences in her math lessons, and disharmony between ideal practice and what she actually does. The findings confirm that the students' voices can be a valuable source for the teacher to develop inclusive practices, and how change processes can sometimes result in discomfort for the teacher, where sufficient time can appear to be a significant factor.

Keywords: Inclusion in mathematics education, students' voices, formal teacher-student conversations, teacher reflection, teacher change.

Introduction

The Salamanca Statement (UNESCO, 1994) defines inclusion as offering equal opportunities and full participation for all students in regular schools. However, the interpretation of this concept varies widely and a literature review on inclusion in mathematics education revealed that the term has many uses and roles (Roos, 2019). Thus, Roos (2023) suggests that listening to and reflecting on students' voices can provide new insights into the complexity of teaching mathematics inclusively. Although some studies provide new understandings of inclusive practices in mathematics through student voices (e.g., Lange, 2009), few studies seem to explore how teachers experience listening to and reflecting upon student voices in this regard. A study on a professional development initiative found that students' perspectives on learning mathematics provide insights central to teachers' cognitive dissonance and reveals how this plays a key role in changing practice (Treacy & Leavy, 2023).

This paper presents results from a qualitative study that aimed to explore a teacher's reflections about her inclusive teaching practice in mathematics, because of engaging in conversations with three of her students about their experiences of participation in the mathematics classroom. The study took place in a research-practice partnership between a Norwegian public primary school and a research group at a Norwegian university with an overarching aim to explore inclusion and inclusive practices in a school with a student body diverse in terms of ethnicity, religion, culture, language, and learning needs. Two other studies within the larger partnership inspired the development of the study this paper reports from. Justnes & Gätzschmann (2023) studied a partnership between a mathematics teacher and a researcher and explored how a teacher could involve more of the multilingual students in mathematical conversations. They found that the multilingual students came up with suggestions on contexts and explanations that were valuable for all students' participation. Furthermore, Uthus & Aas (2024), studied how students can contribute to understanding their experiences of inclusion within the framework of Formal Teacher-Student Conversations (FTSC). With this, we decided to extend this framework to encompass students' experiences of participation in mathematics, aiming to study student voices as resources for the teacher to reflect on inclusive practices in mathematics teaching. Following Roos' definition of inclusion in mathematics as a process of student participation (2023), the research question was formulated as follows: *What reflections does a teacher make about her teaching practice in mathematics by engaging in conversations with students about their experiences of participation in the mathematics classroom?*

Theory

Inclusion in mathematics education

Recent research overviews on special educational needs in mathematics (SEM) (Lewis & Fisher, 2016,), and inclusion in mathematics (Bishop et al., 2015) indicate an increased research interest in the manifestation of inclusion connected to mathematics education. According to Roos (2019), inclusion can refer to both an ideological perspective and a way of working in mathematics that aims to facilitate mathematical learning for every student. By connecting and interrelating various meanings of inclusion, students' participation and access to mathematical learning can be placed at the core of inclusion in mathematics education. Thus, key aspects of inclusion in mathematics education include access for everyone, equity through fair distribution of learning opportunities, awareness of power dynamics in the classroom, student-centred approaches, selecting appropriate tasks and ways of interacting with them, positive interactions between students and teachers, and a sense of belonging (Roos, 2023). Different types of instruction have different impacts on students' opportunities to engage in rigorous mathematical activity. In a study that identified forms of practice that appeared important for supporting all students to participate in and learn through solving complex tasks, Jackson et al. (2013) found that how a task was introduced affected which students could participate in solving it. When students did not understand a task, teachers often spent valuable time reintroducing it, which could reduce its cognitive demands, and limit opportunities for whole-class discussions and collaboration which are both key aspects of inclusion in mathematics (Roos, 2023). However, if the introduction of a task supports a common language to understand the contexts and mathematical relationships, it may maintain the cognitive demand, and whole-class discussions can lead to higher-quality learning opportunities for everyone (Jackson et al., 2013).

The potential of student voice for teachers' reflection and change

United Nations Convention on the Rights of the Child states that teachers should systematically and consistently practice confirming every child's right to express themselves and have their views given due weight in all matters affecting them. According to Lundy (2007), this requires that teachers facilitate students' opportunities to express their voice, that teachers have the skills to actively listen to the students, and that teachers act appropriately according to the students' views (Lundy, 2007). With a definition of dialogue as: "reciprocal interactions between participants that lead to authentic engagement with each other views, creating new meanings and further questions", Messiou (2019) argues that greater dialogue between teachers and students is in itself a manifestation of being inclusive (Messiou, 2019, p. 769).

Listening and responding to students' voices about their participation in mathematics, could according to Flutter (2007) change the way teachers think about students and their learning, and thus result in the development of more effective teaching and learning in the classroom (p. 345). As demonstrated by Treacy and Leavy (2023) students' insightful perspectives about learning mathematics might increase teachers' engagement with professional development. However, in this study, the teacher change process was found to be complex, and cognitive dissonance played a key role in changing practice.

Teacher change can be defined as a gradual process that requires a concerted effort from teachers and comprises nonlinear messiness (Fullan, 2015). According to the theory of 'cognitive dissonance' (Festinger, 1957), this messiness is manifested as an experienced discomfort due to a discrepancy between what is perceived as an ideal practice and what teachers actually are practicing in their work (Fullan, 2015). However, such discomfort is viewed as an inevitable step in the process of teacher change (Fullan, 2015). Once this discomfort arises, the teacher will feel the need to lessen the dissonance in some way. This can be achieved, for example, by the teacher ceasing to do what causes the discomfort or by convincing oneself that the perceived ideal practice is not ideal after all. With this in mind, student voices appear to be an obvious, but challenging, starting point for creating more inclusive practices in mathematics lessons.

Methods

To explore the teacher's reflections on students' experiences of participation in mathematics lessons, we designed a qualitative study with one teacher and five of her 3rd-grade students. The teacher was voluntarily recruited from a partner school and the teacher selected three students she considered as different according to personal traits and learning prerequisites. We received consent from the teacher, parents, and children and informed all participants that they were free to withdraw from the study at any time.

In Norway, teachers are required to ensure formal teacher–student conversations (FTSCs) twice a year, normally focusing on students' well-being and learning. A FTSC, was developed for this study with open questions about students' experiences of participation in mathematics that built on four sub-categories from Uthus & Aas' study (2024). They were 1) a sense of relatedness to peers and teachers, 2) a sense of mastering learning activities, 3) a sense of mattering, and 4) a sense of agency (for elaboration on this process, see Uthus and Aas, 2024). Data includes three recordings of FTSCs, each approximately 30 minutes, and field notes of two unstructured interviews with the teacher and both researchers. The first was carried out three days after the FTSCs, and the second after 6 months.

In the first interview, we were inspired by stimulated recall interview as a data collection method, a method suggested suitable when the focus is normally on the reflections on selected sequences, rather than on the recordings themselves (Lyle, 2003). In this study, all of the recorded FTSCs were used to support the teacher's recall of thoughts in, and reflection on, the situation. During a four-hour long listening-through, to treat the teacher as an expert on own thoughts and reflections, we encouraged the teacher to freely comment during and after each FTSC. The researchers posed open questions or asked for clarifications when needed, but also offered their own reflections or theoretical input when the teacher asked for them. In the second interview, 6 months later, we did not listen through the recorded FTSCs but asked the teacher to reflect on the experience of conducting such FTSCs about the students' experiences of inclusion in mathematics.

We chose a collective, open, and inductive approach to analyse our data. Collective qualitative analysis gives room to develop our understanding of data in creative ways when researchers collaborate across disciplines (Eggebø, 2020), in our case researchers with backgrounds in mathematics didactics and special needs education. The researchers went individually through the data. Then we met on several occasions during the analysis process to share, summarise, and discuss emerging themes. This process was repeated after the second interview. We arranged segments of the data into tables to track the relationship between the student's responses and the teacher's associated reflections from both interviews. As we analysed the data, we identified two primary themes that relate to our research question. We also identified relevant theory (Festinger, 1957; Fullan 2015) and research (Lampert et al., 2010; Jackson et al., 2013; Kazemi et al., 2021) that could guide us to make sense of the data. In the following sections, we will present the two themes and discuss their interpretations in light of theory.

Findings and interpretations

Teacher's commitment to the students' experiences in her math lessons

The first theme we identified is the teacher's reflections on what the students tell her about how they feel when they are in her math class, and how those experiences are intertwined – and influenced by her role and practice. When the students were asked about how they were doing in math class, the students had different responses like "good", "tip-top" and "boring". From the teacher's point of view, this is valuable because it allows her to live up to her expectations of taking the students and their feelings in math class seriously. The teacher emphasizes this by referring to the student who answered that her lessons are boring, once again in the second interview:

> Looking back, I think that I give him too easy math. He has some unused potential. I need to be aware so that I can connect more with him to avoid that he loses interest in the subject.

The extract illustrates that the teacher becomes aware that the student's need for more challenging tasks can keep him motivated for the subject. She also takes responsibility to make sure it happens. The phrase "connect more to him", suggests that differentiation is not only about providing challenging tasks but also involves creating a connection between the two. Based on her reflections, we find that the teacher wants to prioritise fair distribution of learning opportunities and positive interactions between student and teacher, which are key aspects of inclusion in mathematics, as described by Roos (2023).

In the recordings from the FTSCs, we found that the teacher makes promises to the students, according to the experiences they tell her about. Examples of promises stated by the teacher were *to take into consideration who talks most*, *to let them cooperate more*, *or to let them choose more themselves*. In the first interview, the teacher reflects on the teacher role, where taking students seriously is an aspect of this role:

> It is important for me to show the child that I take their input seriously. I want to show them that what they say to me really means something. That's why I intend to change practice and promise the students to do so.

Here, it becomes clear that the teacher's promise to the students is not about being self-critical, but about confirming to them that she cares about them. The phrases "taking seriously" and "what they say really means something" suggest she not only intends to listen to their voices but to give their voice influence by acting upon their views and changing her practice accordingly, like Lundy (2007) advocates. In the second interview, the teacher reflects further on the potential dilemmas of making such promises. The teacher says:

> It will be impossible to hold such promises to all of my students. Promises make the conversations good the first time they are conducted, but not the second time if I haven't held my promise.

She is aware that by not holding the promises made in the first conversation, she can potentially reduce the positive outcomes of the second one. Given student diversity the teacher admits to herself that she is unable to carry out what she has promised everyone in the FTSCs. Her reflections are in line with research on teachers engaging with evidence as students' voices, which might be the first, but not easy step, towards change in classrooms (Flutter, 2007).

Disharmony between ideal and actual practice

Secondly, we have identified the teacher's reflections on disharmony between ideal practice of inclusion in mathematics and what she actually does as a main theme. The teacher's reflection suggests an understanding of inclusion as access to mathematics learning through a process of participation, that is every student having opportunities to engage in mathematics activities in the classroom, in line with Roos (2023). These reflections are initiated by engaging with evidence, listening to students telling her how they experience her as a teacher in math class, and leading to the teacher realising that the students' experiences are not in line with what she perceives as an ideal practice. An example that illustrates this disharmony is that all three students reported that she is the one who talks the most in math lessons, and in response to this feedback, the teacher made promises to the students to try to change this. In the first interview, the teacher says that she already knows she speaks the most, and that she needs to, because she wants to explain until she is sure that all students understand both the mathematics in question and the instruction on what to do. This can be understood as learning goals that encompass both conceptual understanding and procedural fluency, however *explaining until she is sure* indicates reintroducing tasks which could affect students' participation (Jackson et al., 2013).

However, she claims that she is feeling guilty about it. The teacher highlights one situation and reflects on it:

> I carried it with me for a long time afterward, the student said that her body gets silly and wiggly when the teacher talks a lot. I relate it to their age and the school's seriousness. What can we really demand from students when it comes to being physically quiet?

Her description of carrying it with her, like she can't stop thinking about it, illustrates a discomfort in this situation, in line with Festinger's theory (1957) of cognitive dissonance. At the same time, she rationalises the student's restlessness as being due to their age and school's demands to conform, instead of reducing discomfort by what she, in fact, promised the students; to change her practice. While she understands the students' perspectives in the sense that it is unrealistic for them to live up to the expectation of sitting still while she talks for so long, she does not reflect on the possibility of alternative instructional strategies, for instance facilitating more student dialogue. According to Fullan (2015), we can understand this as the teacher reducing the dissonance by understanding the discomfort as aspects of something outside herself and thus cannot be changed (age and school system), instead of changing the cause of the dissonance (her practice).

In the second interview, the teacher says that she believes facilitating student talk in mathematics lessons is a valued practice, and that meaningful dialogue about mathematical ideas plays an important role in developing understanding. With this we find traces of theory in her reflections, e.g. that whole class discussions are connected to higher-quality learning opportunities for everyone (Jackson et. Al., 2013). Despite this, she admits that she still hasn't changed her practice. However, she seems to be more aware that this is more about her and not about the students:

> I still feel that I talk too much, and I can observe them drowning a bit in it. But I am so afraid that they won't get everything. I need control.

The teacher emphasizes in both interviews that it is important to her that every student understands the mathematical content and instruction. When she simultaneously reflects that her chosen approach may not be the best option in this respect, a new self-awareness arises, of being afraid of failing to support her students' understanding. Therefore, although she knows that other practices can lead to a better understanding of mathematics for students, she is not ready to talk less, let go of control, and step back and rather center the work of teaching on student engagement and participation (Lampert et al., 2010). When she repeats her practice of explaining, it indicates an intensity in the teacher's experience of discomfort or cognitive dissonance, as described by Fullan (2015). This arises from the gap between the teacher's ideal and actual teaching practices. However, if this discomfort is lessened by reexplaining, it can negatively impact student participation (Jackson et al., 2016), which is a key aspect of inclusion in mathematics, as defined by Roos (2023).

Discussion

This study is confined to the voice of only one teacher and three of her students in which much is missing in order to truly understand the phenomenon studied. The teacher who participated in this study found the students' feedback in the conversations thought-provoking, confirming previous research demonstrating how students' voices can be valuable for teachers' reflections and changes in their teaching practices in mathematics (e.g., Treacy & Leavy, 2023). The teacher seemed committed to the students' experiences and emotionally affected when she was reflecting on the students' voices on how her practices hampered their learning in mathematics. Through engaging with student voices, the teacher becomes aware that her much-used practice of 'talking' met her own need to ensure (control) the students' understanding, but not necessarily provided fair distribution or higher quality learning opportunities for everyone.

In transitioning from traditional to alternative teaching practices in mathematics, “Stepping back” or “letting go of control” are known challenges that teachers can face. This can arise due to habits and traditions of mathematics teaching and might lead teachers to view alternative approaches as additional tasks to undertake, making it difficult for them to step back from traditional instruction. When viewing mathematics in a broader sense, where the learning goals encompass both conceptual understanding and procedural fluency, goals which are also stated by the teacher in this study, the teachers’ work should center on students’ engagement and participation (Lampert et al., 2010). This leads us to the point that incorporating communication in and for mathematics learning (as the opposite of monological teacher talk/explaining) is challenging and complex work (Lampert et al., 2010). While the teacher is aware of the importance of dialogical interactions, she still needs support to implement these ideas in her teaching. Kazemi et al., (2021) found that teachers’ familiarity with routines such as choral counting and "turn and talk" enabled them to experiment with broadening student participation and dialogue. This suggest that having knowledge of such routines can be productive for teachers in their work of implementing student participation and dialogue in mathematics teaching.

However, the teacher’s resistance to change her practices seemed to be reduced in the second interview. This confirms other research indicating that teachers can develop their inclusive mathematics practices related to student voices (Roos, 2019). Teachers are often deeply committed to the best interest of the students and to fully support their learning and when it is the students themselves who make it clear that the teacher is not contributing to this, it might be impossible for teachers to ignore. Exactly therefore, their discomfort of continuing as before will also naturally be stronger. In such a situation, according to Fullan (2015), the teacher will try to reduce the discomfort following from the discrepancy between current practice and students’ voices (of experiences of current practice). Through our analysis, it became clear that the teacher used two different strategies to reduce dissonance. From rationalizing the students’ feelings of boredom with their age and the school's demands, the teacher changed how she thought about it, and went on to state she was willing to change her practice. Based on this, it might be reasonable to ask whether it is necessary for time to pass, for a teacher to become ready to reduce dissonance by changing practice instead of finding reasons to continue current practice. We find support for such an interpretation in research on teachers’ professional development, where teachers’ change is seen as a process that occurs over time (e.g., Treacy & Leavy, 2023).

Teaching is associated with various dilemmas, as well as in mathematics. One of the dilemmas is balancing recommendations of alternative teaching approaches and the teacher’s expectations of inclusive mathematics teaching, against the impossibility of fully accommodate all students’ prerequisites and needs and effectively influencing their experiences in mathematics. It is important to note that if teachers allow students to express themselves, affirm their experiences, and promise to act, but fail to act on their promises, it can have a negative impact on students. As Lundy (2007) emphasizes, giving students a voice is important, but it is equally crucial for the teacher to be willing to be influenced by or prepared to give what they say have an impact. The findings in this study illuminates how the teacher’s commitment to the students as they share their experiences in her math lessons through FTSCs can lead to process of change in teaching approaches, and ultimately positively impact the students' sense of belonging in mathematics as suggested by Roos (2023). In this way, our findings illuminate how FTSCs regarding students' experiences of participation in mathematics are an aspect of inclusive practices in themselves, as Messiou (2019) argues. Finally, it remains to be said that the teacher's potential learning and change in this study must be seen in light of her participation and close collaboration with researchers over time. We can assume that the opportunity for dialogue and reflection on one's own role and practice in mathematics teaching forms a valuable starting point for learning and change for all teachers. However, the question is whether there are framework conditions for this in schools in general.

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Preservice teachers' negotiation of acceptance criteria for mathematical arguments in the context of number theory

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This paper investigates the discussion of preservice secondary mathematics teachers (PSMTs), as they work on a task in the context of an elementary algebra course in a Norwegian university. At this point in the course, they are in the process of shifting from working on generalizing to justifying conjectures. The PSMTs negotiate acceptance criteria for arguments, but rarely consider validity criteria for mathematical arguments in their discussions. They hold a local perspective in their negotiations, while the criteria that they come up with include both structure-oriented and meaning-oriented perspectives. The PSMTs recognize three aspects of arguments as important: correctness of the argument, explanation of the mathematical content and if generality is addressed.

Keywords *mathematical argument, criteria, acceptance, preservice teachers, number theory*

Introduction

Most preservice secondary mathematics teachers (PSMTs) are hesitant when they have to judge if a mathematical argument is valid (Weber, 2008), while they struggle with understanding the relationship of empirical arguments and deductive proof (Buchbinder & McCrone, 2023). This is a challenge for students as: i) the transition from school mathematics to university mathematics comes with different mathematical norms (Corriveau & Bednarz, 2017), ii) criteria for valid mathematical arguments and proofs are less evident and clear for students than one might expect (Dawkins & Weber, 2017), iii) judging the validity of a mathematical argument deeply depends on the understanding of the mathematical content which the students are about to learn (Abels, in print). Research has revealed that all these dimensions play a role when it comes to PSMTs' judgements of mathematical arguments and formulation criteria for mathematically valid arguments (Sommerhoff & Ufer, 2019). Little research has focused so far on how preservice teachers' discussions about validation criteria for mathematical arguments can foster students' awareness for validation criteria. Addressing this challenge explicitly and providing opportunities for preservice teachers to negotiate what constitutes valid mathematical arguments might allow PSMTs to come to a common understanding of validation criteria for mathematical arguments. In the context of standards and curricula that emphasise processes like mathematical argumentation, as recently in Norway, future mathematics teachers need to better understand these processes in the context of mathematics, also as a community of teachers. This paper focuses on PSMTs' negotiation of validation criteria for mathematical arguments in the context of a first-year university course on algebra and number theory in Norway. Particularly we are interested in investigating the criteria that PSMTs negotiate when asked to discuss and review students' arguments in number theory.

Research on acceptance criteria and socio mathematical norms for proofs

What constitutes a mathematical argument or proof varies in practice and research (Reid & Knipping, 2010); (Sommerhoff & Ufer, 2019). Acceptance criteria for mathematical argument and proof also vary, among mathematicians (Weber & Mejía Ramos, 2011), university students (Inglis & Alcock, 2012); (Selden & Selden, 2003) and school students (Healy & Hoyles, 2000). Thus, agreeing on acceptance criteria is a social practice, as is the practice of proving and mathematical argumentation within a community (Manin, 2010). Becoming a member of such a community, as for example a community of PSMTs, is an enculturation process (Schoenfeld, 1992). Individuals get familiar with social mathematical norms in a local context, interacting in a community and negotiating in an ongoing process what constitutes a mathematical argument and what are criteria for a valid argument. Research on acceptance criteria has differentiated two perspectives with specific strands of criteria (Sommerhoff & Ufer, 2019): i) global vs local perspective; ii) structure-oriented vs meaning-oriented.

i) Global vs local perspective - Global criteria refer to a normative ideal, where a deductive, logical-structural perspective is prominent. Acceptance criteria along this global perspective focus on the structure of a proof, for example the correctness of formal derivations and the valid application of inference rules (e.g. (Heinze & Reiss, 2003). In contrast, for a local perspective looking for shared knowledge and commonly accepted criteria is characteristic. Acceptance criteria are here analysed from a social-descriptive stance characterising the practice of argumentation, the acceptance criteria and norms for arguments in a given context. Specific warrants used to justify a claim, are analysed thoroughly. Imre Lakatos (1976) famous work "Proofs and refutations" takes such a local perspective in characterising proving practices of Euler's Formula and their acceptance.

ii) Structure-oriented vs meaning-oriented – In a structure oriented perspective the correctness of derivations and the valid application of inference rules is characteristic as in a global perspective. Different to the global perspective, here the mathematical content and its structure are focal points and do not need to be represented formally. The aim of structure oriented criteria is generally to establish absolute conviction, by referring to logical and mathematical structures. This may emphasize formal criteria over mathematical ones. In contrast, meaning-oriented criteria aim for understanding rather than absolute certainty (Weber & Mejía Ramos, 2011). For example, an iconic argument may provide understanding, but not indicate an implicit deductive inference. Meaning-oriented criteria relate rather to functions such as explanation and understanding than to formal correctness (Hanna & Jahnke, 1996). Structure-oriented and meaning-oriented criteria are not necessarily exclusive, they may in a local perspective be seen as complementary. From a global perspective a structure-oriented view generally emphasizes a formal, axiomatic interpretation.

In their research, Sommerhoff and Ufer (2019) found that first-year university students struggle with applying acceptance criteria properly. Instead, the students focus rather on surface features and hold a local perspective when reviewing a proof. In the context of our research on first year PSMTs we were interested to find out what perspectives novice preservice teachers would emphasise when negotiating criteria for reviewing students' arguments in the context of number theory. The following research question guided our investigation: What criteria do PSMTs come up with in a group discussion when reviewing fictional written arguments in the context of number theory?

Methods and study context

This study is a part of a three-year doctoral study. The data presented here were collected as part of a pilot study, from an obligatory elementary algebra course for PSMTs in their first year at a University in Norway. In the course, which also includes number theory, there is a focus on mathematical processes. In the first part of the course the PSMTs are continually encouraged to engage in problem solving. Building on teaching principles developed by Mason et al. (2005), mathematical norms for solutions of tasks and for validation criteria are established in a local perspective, in a context of teaching and learning algebra. In the second part of the course the focus is on mathematical reasoning and proving in a global perspective, which the PSMTs encounter for the first time. Based on a script for the course the PSMTs get familiarized with logic and formal forms of inferences. The part of the study presented here investigates peer discussions of acceptance criteria for justifications of mathematical statements and to what extent these criteria reflect a global or a local perspective.

For the pilot study, an entire course of first year PSMTs got divided into groups of four to five PSMTs and each group was asked to discuss whether they agreed with the reviewing comments on justifications for tasks they had seen and worked on before (see Figure 1). In these discussions acceptance criteria for justifications can be expressed, as well as criteria emerge and are negotiated. This allows us to reconstruct such criteria and to analyze whether the acceptance criteria resonate with the PSMTs' criteria for valid arguments expressed in their own previous solutions of the tasks. Our research approach reflects a social-constructivist paradigm (Cobb & Yackel, 1995; Paul Cobb, 1995) , where the emergence of meaning is emphasized, in the context of a community of PSMTs, who are learning to shift from generalizing their conjectures to justifying them.

In this paper the focus is on the groupwork session where the negotiation of acceptance criteria takes place. More specifically, the data presented in this paper were collected from the course's groupwork session, which was a mandatory part of the course. The tasks were initially given to the PSMTs in the university classroom. They had to work individually and produce arguments to support their conjectures. We gathered the written responses and developed fictional answers based on the inferences, common and unique mistakes and representations in their own arguments. For the groupwork session, we gave the PSMTs these peer-reviewing tasks and they worked on each task for a maximum of 10 minutes. They had to decide as a group which of the fictional answers in the task was the strongest, allowing them to explore what they see as strength of an argument.

For the analyses of the groupwork sessions we entirely transcribed the group discussions and then translated them into English. We followed a deductive-inductive content analysis approach (Mayring, 2015). For the deductive coding of the transcripts, we used our theoretical codes for acceptance criteria (see above). For including students' further commentaries and their idiosyncratic ideas related to acceptance criteria, we created new inductive codes, while going through the transcripts. In this paper we focus on the results related to our deductive coding.

Task design

The PSMTs were presented with ten tasks and they worked on these during two forty-five minute sessions, meaning approximately 10 minutes for each task. In this paper we will focus on the responses on task 9, which illustrates students' intense discussions of acceptance criteria related to a challenging mathematical topic for the students (see Figure 1).

Figure 1. Translation of task 9

OPPGAVE 9. Modulo.
A teacher gave the following task: *Find x in the following expression: 17 ≡ x(mod 4).*

Harry:
17 and x have the same remainder when divided by 4.
Then 17 ≡ 13 ≡ 9 ≡ 5 ≡ 1 (mod 4).

James:
If we divide x by 4, we will have the same remainder as 17.
So 17 ≡ 1(mod 4) which means that x = 4k + 1, k ∈ Z.

a) Which answer is most similar to what you would write?
b) For each answer, do you agree or disagree? Explain why.

Student A
I agree with Harry, because the task asks to find x.

Student B
I agree with James because a≡b (mod n) means that a-b=kn, k∈ Z.
In other words, a and b have the same remainder when we divide by n.
So x can be any number that has the same remainder as 17 when we divide it by 4.

In the task there were four fictional answers, as prompts for the PSMTs to negotiate acceptance and validity criteria for arguments: (I) two fictional answers to a mathematical problem (Harry and James) and (II) two purported student commentaries on these answers (student A and B).

Harry and James' arguments were mathematical while student A and B's arguments were narrative reviews of Harry and James' answers. In Harry's answer we wanted to showcase an approach that was commonly used in PSMTs' individually produced written responses, where they provided specific numbers as values that satisfy the expression 17≡x (mod 4). Harry's argument is meaning-oriented and local, as it is tied to the specific context of the task and does not provide an indication of generality. Student A agreed with this answer providing a narrative structure-oriented argument.

James' argument included more formal mathematical symbols in comparison to Harry's argument and led to a general expression as an answer, while it was structure-oriented. The same approach was also prominent in the PSMTs' written answers. Student B agreed with James but provided an elaborate explanation that was more formal than Student A's and was structure oriented as it was based on the definition of modulo. These answers were constructed based on the most common written answers that the PSMTs produced individually and aimed to provide the opportunity for PSMTs to reflect on their own as well as their peers' arguments and review them.

Results

When the PSMTs were given this and other similar tasks as a group, they took about ten minutes to discuss whether they agreed or disagreed with "Harry/ James" and with the commentaries of "Student A" and "Student B". In most cases they summarized their group discussion in a brief written answer.

i) Global vs local perspective

Overall, the PSMTs do not take a global perspective when discussing Harry and James' answers or Student A and B's commentaries on these answers. Although the contrast of these answers and commentaries constituted an invitation to move to a global perspective and stimulate a discussion on validation criteria, in fact the PSMTs conjointly tend to a local perspective. The PSMTs do not analyze and comment on the structure of the arguments, possibly (in-)valid inferences and (in-)correct applications of definitions, but they focus instead on their common understanding of the answers by Harry and James and the commentaries on these by Student A/B.

The PSMTs invent categories like “weak/strong answer”, “right/ partly right”, “weak/strong rationale” to characterize the answers and commentaries.

526 Peter: Student B says he agrees with James because he has found a good answer. Also, he gives a rationale for why that is right, what James has said. Student A hasn't. So the best thing about B is really the rationale. That he has made a rationale.

The PSMTs acceptance criteria incorporate three main categories: i) correctness of the answer, ii) quality of the explanation for the answer, iii) generalization provided in the answer.

All of these categories are conceptualized in a local perspective by the PSMTs and their meaning is emerging in the process of discussing the fictional answers provided, tied to the mathematical context of the task. While they seem to agree early on that the fictional answers need to be mathematically correct, it was first less clear for the PSMTs whether Harry’s argument is sufficient as it is based on specific instances instead of a general claim. This debate seems to be connected to the way they perceive modulo. Even though some look for a general expression that yields every possible value for x, others expect an answer with a specific value for x, as that was what the task was asking for.

446 Roger: But he hasn’t written what x might be.
447 Peter: Which means x equals 4k+1.
448 Roger: And, okay then.
449 Peter: So x can be several things.

They give credit to Harry, who is “finding x in the given expression”, and interpret Student A’s comment as an acknowledgment that not more was requested in the task.

430 Peter: He has a good expression of x. Student B says he agrees with James because he has found a good expression to find all the answers.
437 Eric: I also think in B, where he mentions at the end that x can be all numbers that have the same remainder, is part of that task to understand. Because A hasn't, or, Harry then, says 1, basically, or 17, 13, 9, 5, and 1. Yes, he's found an x, but he hasn't mentioned that there could be more than those too.
440 Peter: It's short, yes. But is it Harry who has found x or is it James who has found x. Harry may have found the numbers on all of them. But James has found a better answer, I'd say.
457 Peter: Harry has found all the x's separately. But had it been a bigger number, he would have struggled to write down the long line. Agree?
482 Peter: He has not written concrete value, but when there are many values, then the easiest thing is to do it in a general way.

The PSMTs then come to the conclusion that if a generalization is provided in an argument or not, is rather context dependent and not per se a rejection criterion. This relative stance the PSMTs take is also visible in their discussion of the quality of an explanation. They agree that an explanation is provided (and not left out), but they are not quite clear about what this might mean in the given context. The importance of the inclusion of an explanation in an argument is also highlighted by the fact that it is the only criterion the PSMTs report on in their written answer as a group.

Looking closer at the PSMTs’ acceptance criteria for the provided arguments both structure-orientations and meaning-orientations are evident, but not explicitly formulated as criteria.

ii) Structure-oriented vs meaning-oriented criteria

When the PSMTs consider the correctness of answers and the quality of the explanations for these, orientations related to both structure and meaning occur, but are not explicitly formulated as criteria. When the PSMTs compare and analyze the arguments of Harry/James and the commentaries of students A and B, the following criteria are explicitly addressed:

- The length of an explanation is discussed as a characteristic of an argument which might possibly burden or highlight the structure of an argument.

435 Peter: After all, B has a longer (...) a bigger explanation.
436 Chris: It's true.
437 Eric: I also think in B, where he mentions at the end that x can be all numbers that have the same remainder, is part of that task to understand. Because A hasn't, or, Harry then, says 1, basically, or 17, 13, 9, 5, and 1. Yes, he's found an x, but he hasn't mentioned that there could be more than those too.
438 Peter: Yes, he hasn't used x when he's solved it. In a way.
439 Eric: And Student A's argument (…) In short, he should have.

- The mathematical content of the responses is often considered as important when the PSMTs discuss, whether the answers are acceptable or not. The PSMTs refer to the meaning of the mathematical content, as well as to the mathematical structures at stake, but it becomes evident that they still struggle with the concept of modulo. Their mathematical uncertainty lets them hesitate, whether the fictional answers respond to the task, so formulating criteria is not evident for them.

478 Roger: I always go to the task, I do. "Find x in the following expression". And then Harry is right. But James hasn't written any x, or he hasn't written what it is.
479 Layla: He has written that x is 4k+1.
480 Peter: He's written it.
481 Roger: He has not written specifically.

- It is also important to the PSMTs, whether the answer includes a good rationale, without elaborating on what precisely they consider as a "good" rationale.

485 Peter: If we are to consider the strongest argument in response to the whole of problem 9, it will be B who has answered best. Strongest. Because he has a very good rationale.

The PSMTs do recognize the need for clarifying what qualifies for an argument and reject the idea to "blindly trust" a claim that a conjecture is correct. But they don't agree yet on clear criteria for acceptable mathematical arguments in this context. They seem to use the terms rationale, justification, and explanation, still without explicitly separating between them.

488 Eric: In and of itself, A blindly trusts Harry, then. "I agree with Harry because the task is to find x." Harry has found an x, but is that x correct? Technically speaking.

The PSMTs also hesitate if students A and B are actually aware of the correctness of the arguments by Harry and James. They do not necessarily see if and how such an understanding is represented in the students' commentaries. Even though the PSMTs discuss the students' answers both from a structure-oriented and meaning-oriented perspective, the PSMTs cannot solve their dilemma of evaluating the student commentaries and coming up with agreed upon acceptance criteria.

Discussion

The PSMTs went into a review process without clearly established criteria for accepting or rejecting arguments. While they had some experiences in the course with proof, mathematical validation criteria from a global, logical point of view and producing arguments for mathematical statements and conjectures themselves, the PSMTs were hesitant about how to use these to evaluate the quality of arguments. In the group discussion, they comment on surface features of the provided arguments, not on the validity of the inferences in the arguments, a finding that agrees with Sommerhoff and Ufer (2019). While, in their discussion validation criteria do not come up, acceptance criteria are discussed from a local perspective. Here, the PSMTs show both a structure- and a meaning-orientation, but they struggle with formulating these orientations in terms of criteria. The criteria, which the PSMTs attempt to explicitly express, are oriented towards whether the arguments are convincing or explanatory and whether they provide a sufficient response to the given questions. In the reviewing process, the PSMTs keep a local perspective, commenting on the arguments in a way that is tied to the mathematical context and to the task. They avoid a global perspective, applying rules of logic, proof structures and theorems they encountered in the course. Instead, the PSMTs focus their discussions on surface features of the given arguments such as the length of the explanation, or on characteristics indicating whether the task and the meaning of modulo are understood.

It seems that the first year pre-service teachers need further opportunities and more supportive prompts to come up with explicit criteria for what constitutes acceptable mathematical arguments in the envisioned local context. Our ongoing research project, presented here only in its first stage, aims for further insights into what might support preservice teachers in this complex endeavour. This might reveal a need for more structured opportunities and more explicit criteria within teacher education programs to help PSMTs develop a deeper and more global understanding of what constitutes a valid mathematical argument. It also indicates the critical need for research-based approaches in teacher education.

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"But like all models, it has its limitations"— Pre-service teachers' evaluation of validity of mathematical models

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Mathematical modelling involves creating models to describe real life situations as well as critically evaluating whether the models are appropriate for those situations. This study explores pre-service teachers' evaluation of mathematical models in their written solutions of three modelling problems. Validation is conceptualised as evaluating a result by comparing it with another result or an element of any stage in the modelling process, the standard of validation. We found that most teacher students used multiple models in their written work. We argue that this allowed for a greater variation in validating activities. Alternate models were sourced from the Internet or developed as a result of comparisons between various stages of their own modelling process. Comparisons between real results obtained from different models were performed to validate the models.

Keywords: Validation, mathematical modelling, revised models, pre-service teachers.

Introduction

Validation plays an important role in shaping students' mathematical modelling processes (Blum & Niss, 1991). Research shows that students often perceive validating as challenging and neglect to validate results even when the results seem unrealistic (Blum & Borromeo Ferri, 2009; Schukajlow et al., 2023). The ease or difficulty of validation may vary depending on modellers' interpretation and connection with the given problem. In some cases, even multiple attempts to validate may not lead to satisfactory results because the problem situation cannot be mathematised in a sensible manner (Blum & Niss, 1991). Czocher highlighted a need to understand "how and why individuals revise their models" (p. 153). Our work addresses these issues by investigating pre-service teachers' (PSTs) solutions to three modelling problems as written in a notebook.

Our aim is to identify the nature of validating activities in PSTs' written solutions to modelling problems and in what ways these activities support (or not) the development of sound models. The following research question is addressed: How do pre-service teachers (grade 8–13) validate mathematical models in their written solutions to three modelling problems and what role do the validation play in their solution processes?

Mathematical modelling

According to Blum and Niss (1991), a mathematical model consists of an extra mathematical domain D, a mathematical domain M, and a mapping or relation by which objects, relations, and phenomena of D are related to objects, phenomena, and relations of M. Mathematical modelling refers to all processes, including structuring of the real problem in D, finding a suitable mapping from D to M, working mathematically within M, interpretating and evaluating solutions in relation to D, and repeating the cycle several times if needed (Niss et al., 2007, p. 4).

Taking a cognitive perspective (Schukajlow et al., 2023), we conceptualise the process of modelling as a cycle involving multiple stages (Blum & Leiβ, 2007; Borromeo Ferri, 2006). In the process of solving a real problem, a modeller envisions a *situation model* based on an understanding of the problem situation. The modeller then structures, simplifies, and makes necessary assumptions according to their experiences and interests. This leads to a *real model* of the original situation, which contains essential features of the situation but is structured to allow for mathematisation. Through mathematisation, the modeller then translates the relevant objects and relations of the *real model* into mathematical objects and relations, applies mathematical methods and analysis, and calculates the *mathematical results*. The mathematical results are interpreted in relation to the problem situation to obtain *real results*. At this point, the modeller validates the model to see if it was appropriate to use in the given situation and revises or modifies the original model. Critically analysing alternative models is an essential part of mathematical modelling (Blum & Niss, 1991). In this study, we consider validating as critical analysis of each model and alternative models.

Validation in mathematical modelling processes

In modelling cycles, validating is often depicted as a final stage in the process as verifying the real results (Schukajlow et al., 2023). Czocher (2018) demonstrated that validating is also an ongoing activity wherein modellers monitor their evolving models at different stages of the solving process. Investigating four undergraduate engineering students' solutions of modelling tasks, Czocher found that students' validating activity always included an element of comparison and they mapped these comparisons onto different stages of the modelling cycle. Czocher defined validation "as a comparison between an object of validation (an intermediate result) and a standard of validation (an (implicitly held) ideal)" (p. 144). The object of validation could be the result at any stage in the modelling cycle including the situation model, the real model, the mathematical expression, the mathematical results, or the real results. The standard of validation is also rooted in the stages being compared. For example, if the modeller compares the mathematical results to the mathematical expression, the (implicit) ideal may be mathematical rigour. Czocher (2018) derived a typology of five validating activities that include: comparing mathematical results and mathematical expression; mathematical expression and situation model; mathematical expression and real model; real results and situation model; and real results and real model. Inspired from Czocher (2018), we conceptualise validating activities as comparisons between different stages of the modelling cycle.

Methods

Twenty PSTs' notebooks, which were submitted for evaluation in a mathematical modelling course, are the empirical basis in this study. The PSTs participated in a six-week introductory modelling course as part of a year-long study in mathematics when enrolled in a five-year program to qualify as teachers in secondary school mathematics. The course was organised as classroom seminars and group work sessions. The PSTs encountered twenty modelling problems including estimation problems such as Fermi problems (Ärlebäck, 2009), modelling problems involving proportionality, model fitting through regression analysis, and modelling through experimentation (Giordano et al., 1997). The PSTs chose four problems for their notebook. The modelling processes, as well as meta-reflections on these, were included in the notebooks. The PSTs had six weeks to work on the problems. In this study, we analysed the PSTs' written work regarding the three modelling problems most frequently attempted in the notebooks. In all, this included nineteen solutions in fourteen notebooks (see Table 1). The individual PSTs, numbered from S1 to S20, will be referred by their respective labels in the following text.

Table 1: Overview of the tasks and the students that solved them

Problems	PSTs who solved these problems
Euro-coin problem (Fermi problem)	S2, S3, S8, S9, S13, S16, S17
Tea-cup problem (Modelling with experimentation)	S4, S10, S15, S17, S20
Bondi beach problem (Modelling with proportionality)	S1, S2, S3, S4, S6, S16, S19

Taking an interpretive stance, we present an account of the PSTs' solution processes based on similarities and differences in the models they used, the evaluation of validity of the models, and the role of validation in their solution processes. For each problem, we studied how PSTs validated each model, and what role validation played in their solution processes. The validating activities are conceptualized as comparisons between elements of different stages in the modelling cycle. We report all the comparisons that the PSTs made in their work rather than confining to the comparisons listed in the typology by Czocher in a deductive manner. This decision was made due to the emergence of different comparisons in PSTs' work which are not covered in Czocher's typology such as comparisons between results of two or more models (e.g., S8 and S16 in Euro-coin problem), comparisons between mathematical results and real model (S3's comparison in Euro-coin problem), among others. Thus, we document instances of all the comparisons that validate the models in PSTs' work, while also noting missing comparisons as indicators of the absence of validation. For the role of the validation, we see how the presence, nature and absence of comparisons played a role in rejecting, accepting or revising the models in PSTs' solution processes, as evident in their written work.

Analysis and results

Euro-coin problem: How many one-euro coins fit in a 1-m^3 safe-box?

All seven PSTs tried more than one mathematical model in their solution process. Five PSTs' work (S2, S3, S8, S9, and S13) included two mathematical models based on two ways of organising the coins in a cube-shaped box: (i) placing the coins in horizontal and vertical rows on the base and stacking up, (ii) placing the coins in a hexagonal pattern (see figure 1, left) on the base and stacking up. S16 tried the first arrangement pattern (i) and then tried several shapes of the box in their multiple models. S17 arranged coins in a hexagonal pattern in the first solution without explicitly stating the reasons for this arrangement.

Constructing the real model of their first attempt at making a mathematical model, S2, S3, S8, S9, S13, and S16 made assumptions about measurements and uniformity of the coins and the cube-shaped box. However, they did not consider the arrangement of the coins as a factor. The PSTs used the arrangement (i), calculated mathematical results, and found the number of coins as the real result.

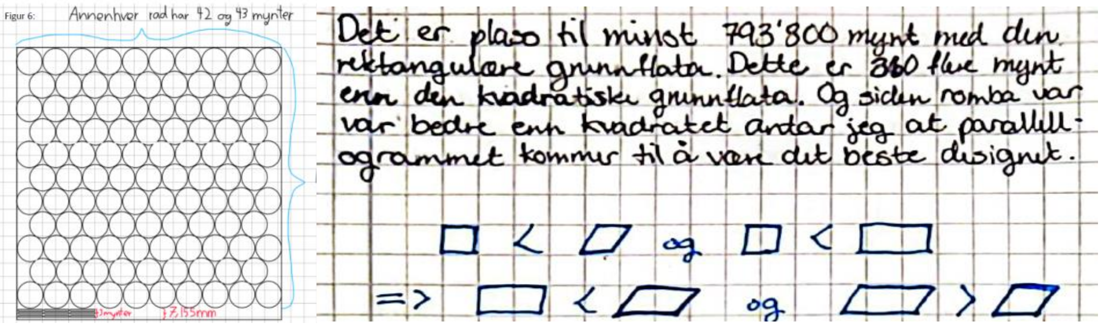

Figure 1: S8's arrangement of coins (left), S16's validation as comparison of several models (right)

Regarding validation of the first model, S2 found on Internet that a hexagonal pattern of arranging the coins could optimise the use of the base area, without performing any comparison. S3 compared partial mathematical results of the first model (number of coins fitting in the first layer on the base) to the ideal number of coins that could be fitted on the base in their real model without considering the air-pockets. S9 asked themselves if the space between the coins could be used up applying another way of arranging the coins. S8 and S13 did not state any reason for using another arrangement of coins, thus their solutions lack validation. S16 first calculated the number of coins that fit side by side in one row on the base, and then they calculated how many coins fit in one stack vertically. Based on these calculations, S16 conjectured that "there is room for more coins stacked on top of each other than placed side by side. This means that a smaller base allows for more coins". Here, S16 made a comparison between the partial mathematical results obtained and the real model (the assumption that the safe was a cube). S17 used a hexagonal pattern (ii) in their first model and wanted to consider alternative dimensions of the coins in the next model, without stating any specific reasons.

Developing the second mathematical model, using a hexagonal pattern (ii) (see Figure 1, left), S2, S3, S8, S9, and S13 calculated mathematical results and found that the total number of coins in the safe increased. Concerning validation of the second model, S2 did not reflect on the result any further, while S3, S9, S13 simply stated that quite many more coins could fit in the box or commented that the second model in general was more space effective, without questioning and comparing the results of the two models developed. However, S8 compared the difference in results of the two models. S8 asked themselves: "It's about 100,000 more coins than before. Could it be true that as few as five rows added to the base make room for so many [extra] coins?". S8 compared the increased number of rows (5) and the partial mathematical results with the mathematical formula they had at start to validate their second model.

S16 tried out different shapes of safe-box (see Figure 1, right), one by one, which are usually found in a rectangular or a cubic shape, thus making unrealistic assumptions. As a second model, S16 tried a rather extreme case of a cylindrical box with base area equal to that of one coin but discarded this after considering the real results. S16 explained that such a cylinder is not realistic due to its height. S16 continued using different shapes of safe-box in new models and validated each model individually. At the end, S16 compared the real results of the different models and concluded that the parallelogram was the most space effective base.

Tea-cup problem: A cup of tea with a temperature of a 100° Celsius. One minute later, it's 93°. When is it 65°, so that you can drink it?

Five PSTs attempted solving the Tea-cup problem in their notebook. S4, S10, and S20 used more than one model in their solutions. In the first model, all five PSTs made similar assumptions about room temperature and used the temperature function, i.e. Newton's cooling law ($T'(t) = -k(T(t) - \theta), k > 0$), in their mathematisation of the problem. Four PSTs obtained the real results as a duration of approx. 6 min, while S4 obtained 1 min and 20 s. S15 and S17 presented the real results as their solution without validating their model. However, S10 validated both the mathematical results and the real results. For the mathematical results, S10 checked that the result of the calculations matched the value depicted in the graph of the temperature function. For the real results, S10 reflected on possible sources of error and limitations in their model. S10 first reevaluated their assumption of a room temperature of 22°C in their real model and wondered if the room temperature could be calculated from the two temperatures given in the situation model. Thus, S10 compared the situation model and the real model. S10 realised, "then the problem emerges that I either have to look up what the constant k for a cup of tea is (which can vary greatly) or guess blindly". S10 also considered the impact of additional factors, including cup properties and the assumption of a constant cup temperature, on real results, thus again comparing the real model with the situation model. S10 finally concluded: "This is a fairly realistic model, but like all models, it has its limitations when compared to the real situation it simulates."

S4 compared the real results of 1 min and 20 s with the situation model and concluded that the real results were inaccurate. S4 then examined their temperature function and discovered that an omitted term in the expression caused the error. S4 found as their real result 6 min and 10 s and concluded that it was a realistic result. S4 also compared the calculations done with the graph of the temperature function, like S10 did. S4 extended the problem by asking "what if I put tea in the refrigerator?", changing the real model. S4 now found as their real result 3 min and concluded that the result was realistic, arguing that it took half the time compared to what was calculated using room temperature. S4 then attempted to validate the result experimentally, however, the duration was observed to be about 26 min. The PST was unable to track the error in the experiment and concluded that Newton's cooling law did not apply in this situation. S20, in their respective models, tried the conditions of refrigerator and freezer and observed that it took 1 min to cool down the tea to 93°C in both models. S20 reflected on this error while comparing the real results with the situation model and considered the impact of other factors like blowing on the surface of the cup to cool it down.

The Bondi beach problem: **How far can the lifeguard see? Does it make any difference if he is in the lifeguard tower or on the ground?**

Five of the seven PSTs that modelled the far-sight of a beach lifeguard included validation in their modelling process. The validation activities often led to developing or finding (online) an alternate model, which in turn could be used for validation by comparing results of the two models or comparing the models themselves. The most common approach to mathematise the real model was to sketch a triangle using an estimated radius of the earth (see Figure 2) and then apply Pythagoras theorem.

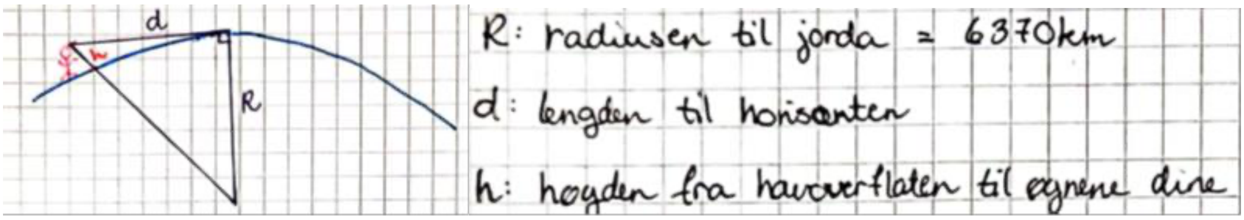

Figure 2: S16's mathematical model of the problem

S1 described a real model in terms of the beach having sea level altitude and that the positions in the tower (*2 m* high floor) and on the beach (height of lifeguard eyes *1.75 m*) have equal distance from the shoreline. Using Pythagoras theorem, S1 found that the lifeguard could see about *4.7 km* when standing in the sand, while the lifeguard could see about *6.9 km* from the tower. S1 found these estimates to be "surprisingly large results" and revised the real model to include one-meter-high waves. In mathematising the revised real model, S1 added one meter to the distance from the centre of the earth to the tangent-point and found that the lifeguard could see approximately *1.15 km* and *3.34 km* respectively, results the PST argued to be more realistic.

S4 started by listing assumptions in the real model, such as perfect vision, good weather and light conditions, etc. Estimating that the floor of the tower is *5 m* high, S4 found that the lifeguard can see *4.79 km* and *8 km* respectively. The PST argued that the large difference made sense because of the curvature of the earth. Thus, validating the difference by comparing it to the real model. S4 then presented another model which was found online $a \approx 2.07\sqrt{h}$, in which *a* is distance to the horizon in nautical miles and *h* is the height of a person. Using the formula, the distances found were *5.15 km* and *8.56 km* respectively. Comparing the results from the two models, S4 found the results close enough to argue that the results are reasonable. In the conclusion S4 compared mathematical results to the situation model and argued that the situation cannot be as ideal as assumed in the real model and adjusted their mathematical results to about *4 km* and *7-8 km* respectively.

S6 started by using an online "horizon formula" ($d = 357 \cdot \sqrt{h}$, in meters). Comparing the mathematical results to the situation model, S6 argued that these results are not very relevant as it is the lifeguard's vision of the shoreline that needs to be described. For this reason, S6 argued to include the curvature of the earth in the real model to see if that would give different results. Using Pythagoras S6 reported to get identical results to the ones using the formula. S6 validated by comparing the results of the two models. S16 made a drawing of the mathematical problem (see Figure 2) without explicitly stating assumptions and validated their mathematisation, i.e. the use of a right-angle triangle, arguing that a tangent and the radius of a circle are perpendicular. From this model, S16 derived a general formula for calculating the distance *d* in meters to the horizon based on the height above sea level of the eyes of the person looking: $d = 3569 \cdot \sqrt{h}$. S16 then stated assumptions and calculated the results for a female lifeguard to be *9.87 km* and *11.91 km* respectively, while for a male lifeguard *9.95 km* and *11.98 km* respectively.

S19 also formulated the problem mathematically and validated it in the same way as S16, however starting out by describing the real model in detail. After finding the results *4654.2 m* and *7738.7 m* respectively, S19 reflected that the distance calculated was aerial distance and not the 'walking distance'. The PST then formulated a more complex model by using radians to derive an expression for the length of the arc and the length underground. He called the three different lengths x (aerial distance), y (through the ground) and z (the arc length on ground) (see Figure 3). In validating their second model, S19 found that the line y did not form a right angle with the radius and thus the model is incorrect. However, comparing the results of the two models, and finding minimal differences are, the PST argued that if the height above ground (h) and the distance x are small then the angle will be close to 90°.

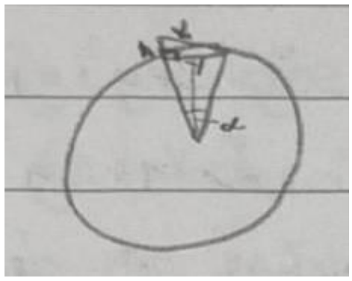

Figure 3: S19's mathematical representation of the problem

Discussion and conclusion

In our study, we conceptualised validating activities as performing comparisons between different stages of modelling cycles in the solution processes (Czocher, 2018). We found that PSTs did not only compare real results with the situation model, as suggested by modelling cycles (Blum & Leiβ, 2007), but also made other comparisons in line with Czocher (2018). For example, in the Euro-coin problem, we found comparisons between partial mathematical results and real model (S3 and S16). In the Tea-cup problem, S10 performed comparisons between the situation model and the mathematical formulation. In addition to the comparisons between different stages in modelling (cf. Czocher (2018), the PSTs also performed comparisons between solutions of different models, which have not been documented in earlier research. PSTs in our study included more than one model in their work and compared results of those mathematical models at the end to find the most suitable model (S16 in the Euro-coin problem), and to validate the first model (Tea-cup problem) or results of the second model (S8 in the Euro-coin problem). These additional comparisons in our study offer a new insight into students' validating activities. The comparisons may have arisen due to the differences in the nature of tasks, the conditions for task solving, as well as the student group, compared to those in Czocher's study.

Regarding the role of validation in solution processes, we found that in the instances where PSTs did not make realistic and critical assumptions in the real model at start, validation did not contribute qualitatively to the evaluation or revision of the model. For example, in the Euro-coin problem, all PSTs missed a crucial assumption regarding arrangement of the coins at start, so the space effectiveness was not compared systematically in most solutions for creating the new revised model. The PSTs considered 'anything better' as an alternate model. Moreover, some PSTs focused on making complex mathematical models overlooking the demands of the real situation. S16 used alternate models involving rather complex mathematical formulations without making realistic assumptions regarding the situation and discarded the models later due to reasons that were attainable from the start (unrealistic height of the cylinder). Similarly, working with the Bondi beach problem, S19 developed a new and more complex model upon comparing the results of the first model with the situation model, arguing that real results should quantify the length of the arc rather than the ariel distance to the horizon. S19 decided to also involve a line cutting through the earth and thus calculate three different lengths. S19's reflections on the validity of the second model and the mathematical results did not consider the situation of the lifeguard and were rooted in purely mathematical considerations instead. Thus, the solutions of S16 and S19 can be seen to emphasise mathematical complexity and accuracy at the expense of formulating relevant mathematical models and results that address the real situation. We also found instances of missing validation, evidencing PSTs' lack of critical thinking (Blum & Borromeo Ferri, 2009). Some PSTs used alternative models found on Internet (Euro-coin and Bondi beach problem) or without including any justifications (Tea-cup and Euro-coin problem).

In conclusion, the PSTs performed several comparisons, between different stages of a modelling cycle and between results of different models, to validate their work. However, the validating activities did not lead to improvement of the solution when the models lacked grounding in a careful and critical consideration about the real problem situation. The findings imply that teacher educators can support PSTs' development of sound mathematical models by emphasising the importance of the real problem situations addressed. In other words, they should help PSTs recognize that the real problem situation is not merely a backdrop but an essential component of the modelling process.

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Insights from Preservice Teachers' Perspectives on Mathematical Argumentation

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In our study, we examined the written responses of Norwegian Preservice Teachers (PSTs) when asked to identify and expand students' key ideas about the sum of two odd numbers becoming an even number into a mathematical argument. The results highlight different approaches taken by PSTs. Our findings suggest that the PST tried to base their argumentations on students' ideas and used various definitions and representations. However, the lack of mathematical generalization observed in the PSTs' responses indicates a need for further development in their understanding of mathematical argumentation, definitions, and representations. Our findings reflect the current state of practice, where PSTs may rely on informal understandings and explanations and need help articulating coherent arguments.

Keywords: argumentation, representations, generalization, preservice teachers.

Introduction and theoretical background

The Norwegian curriculum LK20 (Kunnskapsdepartementet, 2019) highlights the importance of formal and informal mathematical argumentation in learning mathematics. Argumentation in mathematics is about students providing the basis for procedures, reasoning, and solutions and proving that they are valid. The students must be able to use mathematical representations in various contexts and explain and justify their choice of form of representation. There is a focus on developing reasoning skills to construct informal valid arguments in the curriculum. This requires teachers to understand mathematical argumentation and see the logical structure of students' arguments to offer practical guidance to help students reason, argue, and refine their arguments. This study focuses on preservice teachers (PSTs), their understanding of mathematical argumentation, and their feedback on students' arguments.

According to the definition in Stylianides (2016, p.13), a mathematical argumentation in school has the following characteristics: it is based on the ideas accepted and understood by the classroom community; it uses forms of reasoning accessible to students; it is communicated with grade-appropriate language. Based on this definition, students in middle grades are not expected to craft formal deductive arguments. They should have opportunities to provide argumentation compatible with their understanding of mathematical ideas and use language that makes sense to them. Schifter (2009) argues that reasoning from visual representations to justify general claims appears accessible to students. There are three criteria a representation-based argumentation needs to cover: the meaning of the operations is represented in diagrams, manipulatives, or story context; the representation can accommodate a class of instances; and the conclusion of the claim follows from the structure of the representation.

The complexity involved in mathematical argumentation in school brings us to the challenging task of teaching and the necessary preparations teachers must undertake, especially when intertwining content with pedagogy in teaching and learning mathematical argumentation. As

Ball and Bass (2000) argue, teachers must be able to deconstruct their mathematical knowledge into a less compressed form where elemental components and connections are visible and assessable. This requires flexibility and the ability to see mathematics from the perspective of someone who is still learning and developing their understanding. The students' arguments may be difficult to express, lacking underlying premises or a conclusion. Therefore, it requires some effort in addition to mathematical knowledge to grasp and assess the logical structure of students' arguments (Anderson et al., 1997, p. 166, as cited in Morris, 2007, p. 484). PSTs must be ready to identify key mathematical ideas in students' solutions, choose representations that can demonstrate these ideas through visible patterns, and be able to guide students' reasoning further.

Observations of regularities in the number system and discovering properties of symbols and operations can engage students in the process of proving generalizations about numbers. In our study, the complexity of defining even and odd numbers described in Ball and Bass (2000, p. 83-85) and Levenson et al. (2007, p. 85) underscores the importance of PSTs' understanding of various definitions of the same mathematical concept. This includes knowing what critical attributes in the definition of even and odd numbers are and how to use definitions in argumentation to obtain implications and make reasoning explicit, understanding the underlying structure of each definition, recognizing the conventional domain of these definitions, and having a sense of when each definition might be useful. Additionally, PSTs should compare and explain the correspondence between the definitions and the students' ideas they want to expand.

Based on research on mathematical argumentation in school mathematics and previous studies that have shown that PSTs are relatively weak and not competent in constructing arguments by themselves (Zao et al., 2021; Morris, 2007) and evaluating students` arguments (Morris, 2007), we state the following research question: What characterizes the Norwegian preservice teachers' written answers when asked to identify and expand students' key ideas about the sum of two odd numbers becoming even numbers into a mathematical argument?

Methodology

A set of problems regarding algebraic thinking was devised for 764 PSTs as part of the national examination in mathematics. The national examination in mathematics tests PSTs at all of Norway's teacher training institutions to determine whether they possess knowledge of algebraic thinking. Algebraic thinking spans various mathematical topics taught in grades 1 to 7. One of the learning outcomes in the national guidelines (NOKUT, n.d.) emphasizes the PSTs' ability to analyze and evaluate students' argumentation from different perspectives on knowledge and learning.

In the present paper, we analyze the Norwegian PSTs' written answers to one of the tasks in the national examination in algebraic thinking (Figure 1). The task was chosen for this study for two reasons. First, it is one of the common argumentation tasks PSTs can meet in university textbooks (see, for example, Solem et al., 2023, p. 96) during their initial training programs. Second, it provides unique opportunities for the researcher to explore PSTs' understanding of teaching mathematical argumentation in depth. This task pushes PSTs to be in dual positions, making both didactical and mathematical decisions simultaneously. For example, they need both to evaluate students' argumentations and construct their mathematical arguments based on students' ideas.

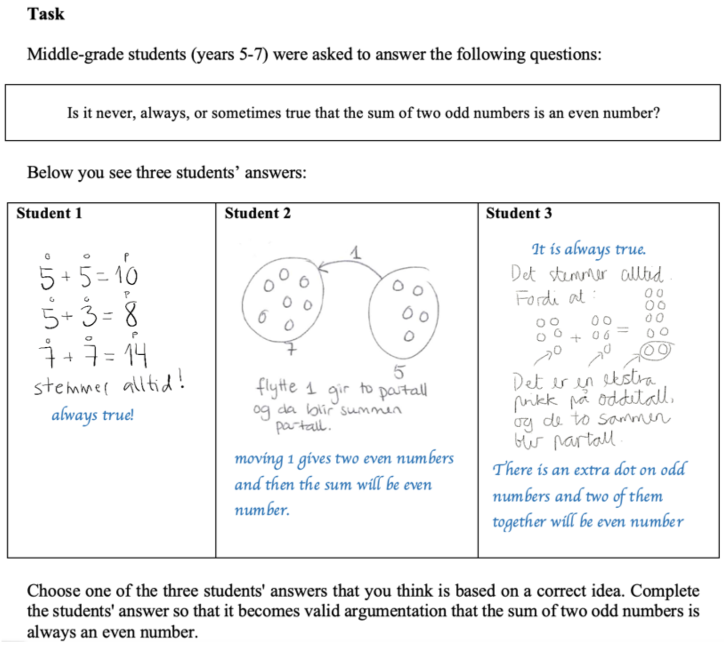


Figure 1: Task in the national examination in mathematics for PSTs (NOKUT, n.d.)

According to the task in Figure 1, PSTs might choose any of all three students' answers that were built up concerning different expressions of the same claim. Student 1 used empirical generalization, demonstrating three examples of adding odd numbers, focusing on numerical patterns. Student 2 visualized the addition of odd numbers with unarranged dots and demonstrated how it transforms into the sum of even numbers. Student 3 relies strongly on structural representation, demonstrating how extra dots from two identical odd numbers can form a new pair. This approach met two of Schifter's (2009) three criteria for representations-based argumentation, lacking only a representation for all cases.

A subset (N = 162) of the PST answers was selected randomly and analysed in the current study. Our analysis was grounded in the established definition of mathematical argumentation in school (Stylianides, 2016, p.13), as well as the criteria for representations-based argumentation (Schifter, 2009). We wanted to identify how the PSTs described the mathematical ideas on which the three chosen examples could be built and expanded them to the mathematical argument and what form of reasoning and language they prefer. When PST chose one of three students' answers, we expected them to use their mathematical knowledge to grasp and describe the idea behind the student's solution and connect, for example, a definition of odd and even numbers to it and then derive a statement about the sum of two odd numbers based on that idea. The appropriate forms of expression might be verbal, visual, or algebraic, and the conclusion of the claim should follow from the structure of the definition or representation.

To address our research question concerning the mathematical argumentation present in the written answers of the PSTs, we coded and categorized their written descriptions of the students' solutions. The two authors carried out the coding independently. The disagreements in coding were discussed, and the coding guidelines were revised for further clarity. The first stage involved organizing and familiarizing ourselves with the data. We categorized the response of each PST into three unequal groups according to their choice of students' solution. Then, we started to look for patterns within each group. At this stage, we identified and coded mathematical ideas PSTs recognized and described in students' solutions, for example, which definitions and representations of even and odd numbers were used. We looked for PSTs` ability to identify students' solutions' "key ideas", i.e., heuristic ideas that show why a particular claim is true and can be used as the basis for constructing a valid mathematical argument (Raman, 2003, p. 323). For example, student 2 and student 3 answers could be described as empirical generalizations. However, both rest on the key mathematical idea that when two odd numbers are added, the two remainders of one form a group of two, resulting in an even sum. Afterward, we looked at PSTs' reasoning about how the students' answers should be completed so that it becomes a valid argumentation that the sum of two odd numbers is always an even number. Morris (2007) states that: "…the ability to expand a key idea into a formal proof involves understandings about generalization, the ability to use language that refers to a class of objects, and the ability to present the argument in a logically coherent way that makes the reasoning explicit". We tried to identify within each of the three groups from the first stage if PSTs used the language or representations that covered a whole class of whole numbers and how they used definitions or representations of even and odd numbers to present arguments logically and make their reasoning explicit - e.g., by explicitly stating or showing that all even numbers can be represented as some number of pairs because even numbers are divisible by two. We present and describe our results in the next section.

We are aware, however, that our results are based on written responses, which might not fully capture PSTs' understanding or reasoning. Some PSTs may struggle to articulate their thoughts in writing, which could affect the analysis and interpretation of their responses. Additionally, the sample size we selected randomly may not represent the broader population of PSTs. This could limit the generalizability of the findings to all Norwegian PSTs.

Results and discussion

In the following sections, we present the results of our study divided into three distinct categories according to PSTs' choice of students' solution. Each category captures different approaches PSTs took when asked to identify and expand students' key ideas about the sum of two odd numbers into a mathematical argument. Our analysis revealed that PSTs based their argumentations on various definitions and representations of even and odd numbers within each category — moreover, those who selected students' solutions 1 and 2 encountered difficulties expanding them into valid arguments. We state that the approaches PSTs take when expanding students' ideas may reflect their understanding of mathematical argumentation and can provide insights into their perspectives on argumentation in school. We will review the results for each category and provide three empirical examples to illustrate PSTs' mathematical argumentation, one each based on students' solutions 1, 2, and 3.

Approach 1: Let us count all sums of two odd numbers

PSTs in this category described the key idea in the student's solution as "The student 1 has included several examples and has also informed that the numbers that have been added are odd and the sum is even". According to Raman (2003), this idea is based on informal understanding, e.g., grounded in empirical data that may give a sense of understanding but does not necessarily lead directly to formal argumentation. Another group of statements was connected to the use of variables: "I think student 1 is based on a correct idea because it has variables. Above each of the numbers, it says o + o = e". We can interpret this as PSTs in this category do not have a grade-appropriate language or representation to talk about numbers in a general way and may think that valid argumentation must be based on a formal algebraic notation. The most popular suggestions about completing student's solution in this category were producing more different empirical examples and using variables O and E, respectively, for odd and even numbers to extend arguments "formally". We could not identify the use of definitions or representations of even and odd numbers, which makes reasoning explicit in these suggestions.

At the same time, we want to draw attention to another type of answer in this category where PSTs presented and expended the key idea that shows why the sum of two odd numbers is always even. The answer in Figure 2 encapsulates how some PSTs expanded the student 1 solution:

Student 1 is based on the correct idea, but for it to be a valid proof, the student must show the calculation of all odd numbers. Below, I have created a table that adds up all odd pairs from 1-9, proving that odd + odd numbers are always even.

odd + odd =	even
1 + 1 =	2
1 + 3 =	4
1 + 5 =	6
1 + 7 =	8
1 + 9 =	10
3 + 3 =	6
3 + 5 =	8
3 + 7 =	10
3 + 9 =	12
5 + 5 =	10
5 + 7 =	12
5 + 9 =	14
7 + 7 =	14
7 + 9 =	16
9 + 9 =	18

Figure 2: PST's argumentation based on the solution of student 1

In our example, the candidate changed the task from one with infinitely many odd numbers to one with just one-digit odd numbers and extended the student's reasoning to consider sums of all the one-digit odd numbers systematically. The candidate could consider (without mentioning it) the so-called units digit definitions: "an odd number is a natural number whose units digit is 1, 3, 5, 7, or 9" and "an even number is a natural number whose units digit is 0, 2, 4, 6, or 8" (Ball & Bass, 2000, p. 84). The units digit definition depends on decimal representations of numbers and could be useful for large numbers. It gives the opportunity to prove the statement by considering systematically the sums of all the different combinations of the last digits of two odd numbers, confirming that every such sum yields a natural number with a last digit of 0, 2, 4, 6, or 8, i.e., an even number. This reasoning corroborates the classification of proving tasks according to the number of cases involved in a task (a single case, multiple but finitely many cases, or infinitely many cases) developed by Stylianides (2016, p. 34). The number of cases refers to the scenarios mentioned in the task that must be considered. The systematic consideration of all the cases involved in a situation is a form of reasoning accessible to students. It can help them effectively tackle tasks that require analyzing multiple but limited cases. That is why we consider this argumentation attempted to be built on the student's solution and went beyond specific examples, but the PST needed to complete it coherently.

Approach 2: Useful alternate numbers

When PSTs chose and analyzed the solution of student 2, their explanations focused on restating what the student did rather than addressing the underlying concepts. They stated: “Student 2 builds his reasoning around moving one dot over to the other side, and this may, by some modifications, show why two odd numbers always become even numbers” or “I believe student response 2 is based on a correct idea that the sum of two odd numbers is always even”. However, this confusion between description and explanation may stem from misunderstanding the task requirements. The PSTs were not explicitly asked to explain why the idea was correct; they were asked to choose only one solution and complete the argument. In Figure 3, there is an excerpt that presents a different argument.

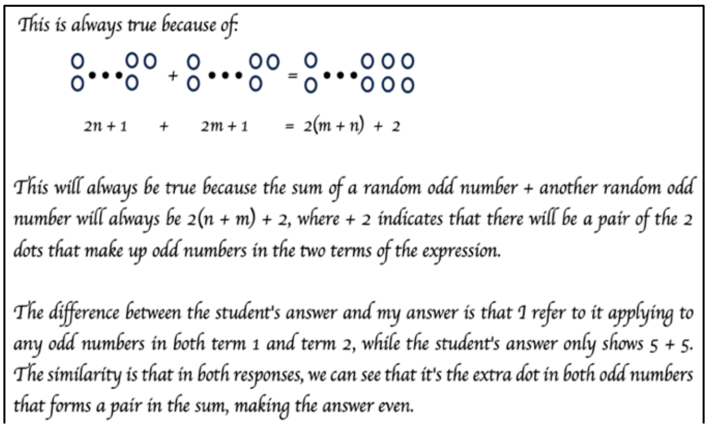
> *I think student response 2 is based on a correct idea. The student believes that moving 1 gives two even numbers, and then the sum becomes even.*
> *This strategy in the student answer will work well, as the example is generalized and can work on any addition with two odd numbers. If you remove 1 from one factor and add it with the other, you will now go from two odd to two even, and even plus even numbers will also always be even.*
> *Even and odd numbers are alternate numbers on the number line, so when you "go one notch forward" on one side, and "one notch back" on the other side of the plus sign - we now have two even numbers!*

Figure 3: PST's argumentation based on the solution of student 2

The PST recognized that student 2 already understood that the sum of two even numbers is even and considered the definitions of even and odd numbers, which state that the numbers alternate on the number line. This approach can be classified as valid argumentation, as it is based on the student’s assertions about the sum of even numbers and the correct use of alternating definitions to derive a statement.

Approach 3: All you need is a pair

The PST who chose to complete the student 3 answer either distributed the drawing to the student, wrote an algebraic expression, or created a written explanation based on the student´s answer. Alternatively, they choose a combination of drawing, algebraic expressions, and text to complete the student's answer. Often, the PST used a drawing as a starting point and added dots so that the drawing went from being a drawing of five circles plus five circles, which together make ten circles, to show what needs to be done with the drawing to bring out the general connection. This is usually done by placing dots between the even numbers and the one extra that makes the number odd. Below is a typical example of the PST proposals to make the student's reasoning more general. PST drew dots in the extension of the drawing (Figure 4) to show that the even numbers can continue ad infinitum and that now the illustration shows that the even numbers can continue ad infinitum. The limitation of this proposal is that it will apply the idea that two equal odd numbers will always become an even number. In this answer (Figure 4), it comes out even more clearly that it is the one extra dot in each of the even numbers that together will form one pair and, therefore, two odd numbers become one even number.


This is always true because of:

2n + 1 + 2m + 1 = 2(m + n) + 2

This will always be true because the sum of a random odd number + another random odd number will always be 2(n + m) + 2, where + 2 indicates that there will be a pair of the 2 dots that make up odd numbers in the two terms of the expression.

The difference between the student's answer and my answer is that I refer to it applying to any odd numbers in both term 1 and term 2, while the student's answer only shows 5 + 5. The similarity is that in both responses, we can see that it's the extra dot in both odd numbers that forms a pair in the sum, making the answer even.

Figure 4: PST's argumentation based on the solutions of student 3

Student 3 does not show that the example applies when the odd numbers are equal. However, in PST's answer, the drawing is combined with an algebraic representation that shows that the odd numbers do not have to be equal for the sum to be an even number. The drawing is described, and the algebraic expression below shows that this will apply generally. This approach is based on a pair definition of even numbers and meets three criteria for representations-based argumentation (Schifter, 2009, p. 76).

Concluding remarks

The study examined the written responses from Norwegian PSTs, focusing on their ability to identify and elaborate on students' key ideas, precisely the idea that the sum of two odd numbers results in an even number, and how they transformed these ideas into a coherent mathematical argument. Our findings suggest that PSTs tried to base their argumentations on students' ideas and used various definitions and representations accepted and understood by the middle-grade students; for example, we identified units digit, alternating, and pair definitions of even and odd numbers. At the same time, the levels of mathematical reasoning vary. Some PSTs relied on informal understandings and empirical data to explain the concept, indicating a need for formal argumentation skills. Another focused on using variables and algebraic notation but did not explicitly use grade-appropriate definitions or representations of even and odd numbers to support their reasoning. However, they may need help choosing appropriate forms of reasoning and communicating how the conclusion of the claim follows from the structure of the definitions and representation.

Our study also highlighted the relationship between practice development and research in the context of PSTs' understanding and use of mathematical argumentation. The discrepancies observed in the PSTs' written answers indicate a need for further development in their understanding of mathematical argumentation and the use of grade-appropriate definitions and representations. Examining different approaches to constricting mathematical argumentation preferred by PSTs can provide valuable information about their perspectives on argumentation and potential teaching styles. Prior studies (Zhao, G. et al. 2021; Morris, 2007) have also noted limitations in the abilities of PSTs to evaluate and construct arguments and the importance of teaching them how to expand a key idea into a valid argumentation. The findings reflect the current state of practice, where PSTs may rely on informal understandings and explanations and need help articulating coherent arguments. Reflecting on the previous research (Zhao, G.

et al. 2021; Morris, 2007; Raman, 2003) can inform the development of practice by guiding the design of teacher education programs and instructional approaches to better support PSTs in developing their mathematical argumentation skills. PSTs may need to interact more with students' solutions and encourage discussions around mathematical concepts that form the basis of students' arguments. PSTs must delve deeper into the thought processes behind students' argumentations, such as understanding how students perceive the regularities in the number system and which forms of reasoning are accessible to students.

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Intended and interpreted meanings in the discussion of an algebra task

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We analyse the discussion of two students about a pattern task. We show that the ambiguity of the word pattern, which was never addressed in the conversation, and unclear use of reference words such as "it" and causal relations are not just formal issues the teacher should ignore. They are indications of the way the students think about mathematical concepts as procedures and of possible sources of confusion if the students were to give a precise algebraic description of a number pattern.

Keywords: Algebra, language as resource, pragmatics, discourse analysis.

Introduction

With a high focus on oral activities such as group discussions and argumentation, the study of language in mathematics classrooms has come to receive a lot of attention in mathematics education research. Learning to communicate about mathematics is complex for all students, not only those who are learners of the language of instruction. Rubenstein and Thompson (2002) identified twelve categories of language issues that arise in the mathematical register in English. Most identified issues concern vocabulary: words used both in everyday language and mathematical register, but with different meanings (e.g. *factor*); words with similar meanings in everyday language and in mathematical register, but with a more precise mathematical meaning (e.g. *difference*); words from mathematical register with more than one meaning (e.g. *square*). Not all of the issues concern single words: modifiers can change meanings of words (*polygon* and *regular polygon*). Language is a medium used for knowledge transfer, but also a tool for thinking when new understanding is being developed. In addition to the typical register of a field of knowledge such as mathematics, so-called academic language is used for schooling. It is characterized by the use of words not frequently encountered in everyday language, complex syntactical constructions, and higher order discourse practices that follow certain norms (Prediger et al., 2019). Therefore, Prediger et al. (2019) distinguished students' challenges with the mathematical language on four different levels: word level, sentence level, text level and discourse level. While the word level is concerned with vocabulary, words must be arranged to construct valid sentences about mathematical objects and connections. This requires knowledge about grammatical structures. This includes to understand and construct verbal references to already mentioned objects, for example by using pronouns. Competence on the text level and discourse level enables students to be aware of the requirements of specific mathematical genres, so that they can follow others' written and oral genre-specific contributions and compose their own. The most important discourse practices in mathematics classrooms comprise reporting procedures, explaining meaning, arguing, and describing patterns (Prediger et al., 2019).

In this article, we have a special focus on ambiguities of meaning in a task text and in students' talk about this task and on how these ambiguities of meaning impact on the quality of the students' talk about a pattern task. Whenever students are confronted with a task, with their teacher's or peers' utterances, or when they are asked to communicate mathematical ideas, the analysis of how meaning is communicated by a speaker/writer and how it is understood by a listener/reader is important. Such questions are studied by the linguistic branch of pragmatics (Yule, 1996), and we will therefore use categories from linguistics in this article to analyse two students' discussion about an algebraic thinking task. Our analysis was guided by the following research question:

How do linguistic ambiguities impact on two students' discussion about an algebraic thinking task?

Results from earlier research

The communication about mathematics makes use of multiple semiotic systems: natural language in both written and oral form, mathematical symbols and formulas following rules of transformation, visual representations such as graphs and drawings. In addition to the challenges posed by the interplay of different semiotic systems, linguists and mathematics educators have found that learners of mathematics have some challenges with the features of natural language as they are used in the communication about mathematics, such as (Schleppegrell, 2007):

- special meanings of words in the mathematical register that are different from the use of the words outside mathematics (e.g., product, place, borrow)
- dense language, such as long noun phrases ("the volume of a right rectangular prism")
- relational processes involving the words be and have can be either attributive (i.e., describing properties, "Three and four are factors in twelve.") or identifying (i.e., providing a definition, "A prime number is a number that can only be divided by 1 and itself.")
- objectified discourse (pronouncing processes as things), caused by the double nature of mathematical objects as processes and objects (the square of a number is again number, but is practically found by the process of multiplying a number with itself)
- conjunctions such as *if*, *when*, and *therefore*, have specialized, precise meanings

Spontaneous mathematical talk is often unprecise and contains words like "thing" or "it". Expressions like "you plus two and then you times it" are common and can be perfectly understandable for all participants in a conversation although the object that is the starting point for the calculation is not specified. But to make an expression disembodied, i.e., understandable independently from a context, the designated objects must be specified (Pimm, 1987, p. 31). This can put the teacher into situations where one must decide when unprecise or incorrect expressions in a student discussion are acceptable and when the teacher should focus on the development of precise mathematical language. Language can be treated as an inconspicuous medium of communication as long as it functions successfully in the communication, even if unprecise or incorrect. Mathematical language can also be made visible as an explicit object of learning, with the risk of interrupting processes of collective meaning making. This is known as the dilemma of transparency and was studied by Adler (1999).

Barwell (2005) contrasted a formal perspective on mathematics and mathematical language with a discursive perspective. According to the formal perspective, mathematical terminology consists of pre-existing precise definitions, and it is the teacher's role to teach these to the students at the appropriate point in time, detecting ambiguities through questioning and correcting them. From a discursive perspective, meaning arises through verbal interaction, and Barwell (2005) showed by analysing a discussion about 2- and 3-dimensional objects that ambiguity can be a fruitful source of concept understanding. He stated further that the formal perspective results in a linear model of the teaching of mathematical terminology, while the discursive model results in a cyclical model, in which terminology is revisited and enriched. Marmur and Zazkis (2022) described how ambiguity in definitions, representations, and language can enrich classroom discourse.

Theoretical underpinnings

Pragmatics is a branch of linguistics that analyses language in use, in contrast to syntax that studies grammatical aspects of composing sentences, and semantics that studies the relationship between words and the designated objects or concepts. Pragmatics is concerned with the users of language, the intention of the speaker and the way that utterances are conceived by the listeners, in the context of a situation. In this article, we focus on the aspects of *deixis* and *reference*. Deixis means "pointing" via language and is concerned with expressions that refer to persons, things, places or points in time that are tied to the speaker's and listeners' context, in the moment when an utterance is made (Yule, 1996). Examples are: I, you, he (persons); this, that (things); here, there (places); now, then, yesterday (time). The term reference denotes a speech act in which a speaker/writer uses some linguistic expressions that enables a listener/reader to identify something (Yule, 1996). Referring expressions can for example be indefinite or definite nouns or noun phrases (a fraction, the denominator, the rectangular triangle), pronouns (he, it, they, them). When speakers make a reference, the listener is required to make an *inference* in order to understand what the speaker intended. In complex texts, with several references to the same entity, references can be anaphoric (pointing back to an entity that was already mentioned) or cataphoric (pointing forward to an entity that will be mentioned in a following sentence), as shown here:

(1) Choose a number and square it. (Anaphoric: 'it' points back to 'number'.)
(2) I found it. The number that solves the equation is five. (Cataphoric: 'it' points forward.)

A special type of anaphoric reference is the zero anaphora or ellipsis. In this case, no linguistic expression is present, but still there is an entity that is referred to and must be identified by the listener/reader (Yule, 1996). Example: "Think about a number. Multiply by two and subtract three." Although not mentioned explicitly, *it* is the number that must be multiplied by two, and it is *the result of the multiplication* that is to be diminished by three.

The words *it* and *there* do not only function as deictic expressions or references, but also as so-called dummy (or formal) subjects (Bloor & Bloor, 2013). They appear in expressions like:

(1) Is there a solution to the problem?
(2) It is possible to make a change of variables.

In (1) and (2) *there* and *it* take the place where the subject of the sentence is grammatically expected, but carry no semantic meaning: no corresponding entity outside the sentence can be identified.

Corresponding sentence constructions can be found in Norwegian (Faarlund et al., 1997), with *det* (3rd person singular pronoun, neutral grammatical gender) that can serve as a deictic expression or as a formal subject (*Det ringer i telefonen*, lit. "It rings in the phone", meaning "The telephone rings"). Constructions with a non-stressed *det* used as a dummy (or formal) object are also possible (Faarlund et al., 1997, pp. 708-709), as in *Ta det med ro* ("Take it easy").

Method

Algebraic thinking tasks often involve the observation and generalization of patterns. These can be regularities in number sequences or figural/geometric patterns (Dörfler, 2008). The word pattern has no clear definition in mathematics, and the work with patterns is usually based on examples (of number sequences or geometric formations) having some regularity. In everyday English, the word pattern has the following two meanings that both are relevant for the work with patterns in the mathematics classrooms (Collins dictionary, n.d.):

A. A pattern is the repeated or regular way in which something happens or is done.
B. A pattern is an arrangement of lines or shapes, especially a design in which the same shape is repeated at regular intervals over a surface.

In mathematics, when students are asked to describe number sequences, the word pattern refers to the rule that has to be applied to get the next element (recursive rule) or to get any element of the sequence (explicit rule), in alignment with meaning A. When studying patterns in geometry, the focus lies on the entire picture and its structure, in alignment with meaning B. The word pattern in the latter sense is also a word frequently used in everyday register, for example used to describe clothes, quilts, etc.

The task

According to Radford (2010) the activity of generalizing patterns algebraically "rests on the capability of grasping a commonality noticed on some elements of a sequence S, being aware that this commonality applies to all the terms of S and being able to use it to provide a direct expression of whatever term of S". The task shown in Figure 1, adapted from Cramer (2001), provides a description of a non-linear pattern by a fictitious student who observed a repeating trait in a pattern.

A student described a pattern like this:

First one adds 6 and then 8 then 10 and so on. This is a pattern, and one adds two more than last time each time.

a) Draw a pattern that can be described like this.
b) Explain why you think your pattern fits this rule.

Figure 1: Task given to the students (translated from Norwegian)

The task required to reverse the process described by Radford (2010) and construct a geometric pattern with a given property. A difficulty lied in the fact that the description referred to the difference between consecutive elements, not the elements of a sequence themselves.

Method for data collection and data analysis

The algebraic-thinking task shown in Figure 1 was given to three students (aged 11-12 years) at the end of their 6th grade in a school in Norway. Their discussion was videotaped, transcribed and then translated to English. One of the students was mostly silent and had no contributions in the excerpt presented below. The translation of the transcript is faithful to the linguistic phenomena that we focus on, such as deixis, use of references, and formal subjects and objects.

The task text and the transcription were analysed by both authors first separately and then combined. We looked for occurrences of the words "pattern" (mønster), "rule", and their synonyms. We also identified all deictic expressions: all those expressions that need a context to be interpreted. We looked for personal (e.g. pronouns referring to an object), spatial (e.g. "here", "there") or temporal (e.g. "all the time", "now") deixis. We used grammatical rules and context to interpret the meaning with which these words and references were used.

Results

The task text in Figure 1 contains both meanings of the word pattern, and the intended meaning in each case must be inferred by the reader: In "a student described a pattern like this", the word pattern could take both meanings A and B. While the fictitious student's "This is a pattern" refers to a regular way of building a structure in the sense of meaning A, part a) of the task asks the students to draw a (geometric) pattern, thus making use of meaning B. In part b) of the task, the word 'pattern' keeps this meaning as it points back to the drawn geometric pattern asked for in part a), while the word 'rule' refers to a regular law for building a number sequence. The word 'rule' functions therefore as a synonym for the pattern in the sense of meaning A. The fictitious student's description also contains a zero anaphora, i.e., a word that is missing and must be inserted mentally by the reader: in "first one adds 6 and then 8", it is not specified what one adds those to. A reader who is experienced with this type of task, can infer that the fictitious student was given a number sequence or a figural pattern, e.g., made up by squares or dots, and that 6 must be added to the first item to get the second one, while 8 is added to the resulting next element.

Our analysis of the students' discussion about this task is centred around the following excerpt:

12	Student 1:	Ok, it is six, eight, ten, twelve.
13	Student 2:	No, but it is exactly the same pattern. We must use another pattern.
14	Student 1:	Then we can take four, eight, twelve, sixteen.
15	Student 2:	Or a bit more difficult. Three, five, seven, nine, eleven.
[*Student 1 repeats these numbers while he writes them down in his notebook.*]		
17	Student 2:	[…] a: "Draw a pattern that can be described like this".
18	Student 1:	That's what we've done. b: "Explain why you think your pattern fits this rule". I think it fits because you double it. Well, you don't double it, but…
19	Student 2:	But you always write two more.
20	Student 1:	Yes, well…You always add the same number several times. So, here [*points to their handwritten notes*] it was that you added two all the time, and here you added four all the time. So, no matter what number it is, you always plus it with the same all the time, so the pattern actually fits quite well.
21	Student 2:	Yes, so the pattern actually fits quite well.
		[…]
25	Student 2:	"Explain why you think that the pattern fits…"
26	Student 1:	Why do you think that 3, 5, 7, 9, 11 fitted to this [*points to the task text*] here?
27	Student 2:	Because of the pattern, you always add two after the first number.
28	Student 1:	You add the same all the time, so, no matter what, so you get the same pattern.
[*Some comments by student 3 and student 1 about their writing into the notebook.*]		
33	Student 2:	We think it fits when one adds them together.
34	Student 1:	Yes, adds them together.
35	Student 2:	You don't have to write 'because'.
[*Some more comments by student 1 and student 2 about their writing into the notebook.*]		
38	Student 1:	We think it fits because one adds them together.
39	Student 2:	We add the same number.

In turns 12-15, the students discussed the validity of certain number sequences as solutions to the task. We infer from this part that the students' study of the task text was superficial: they ignored the fact that the task text informed about first order differences of a sequence S, not the elements of S. Their engagement with the task was therefore restricted to a more elementary level than was the intention of the task. The request to make a drawing was overlooked by the students (turns 17 and 18), who claim to have done what they were asked to do. Consequently, in turns 12-15 the students referred to number sequences and not a geometric formation. Their understanding of the word pattern was nevertheless, in accordance with meaning B, focused on the perception of the sequences as a whole and on surface similarities between sequences and not on building rules. However, the three sequences followed a common building rule that the students seemed to agree upon, although it was not yet formulated at this stage. Later in the discussion, the word pattern was used parallelly in both meanings. In turn 27, Student 2 used the word "pattern" synonymously to "rule that describes a number sequence", i.e., in the sense of meaning A. In the following turn 28, Student 1 switched back to the holistic understanding of patterns (meaning B) that was dominant earlier in the excerpt.

Answering task b), the students found a rule to build the suggested number sequences. The rule was reformulated several times in turns 18-20: "you double it", "you always write two more", "you always add the same number several times", "no matter what number it is, you always plus it with the same all the time", reaching a higher degree of correctness and generality. The students answered part b) by expressing a regularity in what they did (or in what someone might actively do) to get these number sequences, not as an abstract, timeless description of the structure of the pattern. The repetition taking place in the process was globally described with temporal deixis (e.g., "always", "all the time"), not by precisely pointing out where the additions take place, i.e., at each number in the sequence.

The students' lack of specification of where the addition takes place is related to their use of the word "it" with sometimes unclear reference. In turns 18-20, the word "it" was used six times. In turn 18 ("[…] it fits because you double it"), the first "it" can be interpreted as an anaphora for the previously mentioned "pattern", or the student could have switched to a construction with a dummy subject ("it fits", could be used as "it works" in Norwegian). The second "it" could again refer anaphorically to the pattern, raising the question of how the speaker and the listeners each interpret "double a pattern". This "it" could also deictically refer to each number in the sequence, meaning that one adds two to each term of the sequence. This would imply an incorrect use of the word "double", as an operator that augments a number by two, instead of multiplying it by two. The fact that Student 1 corrects himself ("Well, you don't double it, but …") supports this interpretation. Finally, the second "it" could be a grammatically incorrect variation on the structure of formal object: the speaker has not a clear idea about what the "it" that is being "doubled" is, but recognizes a repeated addition by two, and, without much thought, verbally expresses this by formally completing the transitive verb "to double" with a dummy object. The reformulation provided by Student 2 in turn 19, only clarified the correct type of arithmetic calculation ("two more" for addition); we get no information about how Student 1's use of "it" was interpreted by Student 2. A clarification was provided by Student 1 himself in turn 20 ("no matter what number it is, you always plus it with the same all the time"). Here, the second "it" seems to refer in general to all (already constructed or future) entries of the number sequence. In turns 20 and 21, the students state that the pattern fits quite well, without specifying what the pattern should fit to. The word pattern could here both refer to the number sequences that they constructed or the rule that they formulated. This raises the question of what exactly should fit what, which is an ambiguity that was never addressed by the students. It is also possible that the students did not give meaning to every word in the sentence and that it could just be interpreted as a generic "this works", that the student thoughtlessly formulated using elements already present in the discourse.

Another source of unclarity is the logic behind the use of "because". In turn 18 ("[…] it fits because you double it") the "because" indicates that the student is answering the why-question posed in part b). The answer should depend on what one interprets "this rule" to be, but up to this point of the discussion, no rule was explicated. The answer is an action, and the connection between the action and the rule is not made explicit for the listener; "you double it" can be interpreted both as the rule, and the action performed on a specific sequence, resonating with double nature of mathematical objects as processes and objects (Schleppegrell, 2007). Likewise, in turns 33 and 38, Students 1 and 2 say almost the same sentence, but differing by "when" and "because", where the use of "when" indicates performing the rule as an action, while "because" expresses a causal relationship.

Discussion

In this article we analysed a small part of a student discussion which unfolded without any teacher support. The students' language contained several ambiguities and inaccuracies on the word level, sentence level and discourse level that restricted the possibilities to engage in a deeper discussion. They used the word pattern both referring to a rule and in a more holistic way, referring to the resulting object. While we agree with Barwell (2007) that ambiguity can be a source for concept understanding, teacher intervention would be needed to initiate the conversation. Clarity about this point in the students' language could go hand in hand with their understanding of processes as entities and not as sequences of actions performed by someone. The fact that the two meanings were thoughtlessly applied in parallel might have been one of the reasons why the students did not notice the higher complexity of the task.

Incoherence on sentence level (the unclear use of "it) together with general temporal deictics such as always affected the nature and quality of the students' discourse. Instead of describing the pattern as a structure, the students engaged in the discourse practice of describing actions which reminded of reporting procedures (Prediger et al., 2019). This became also visible in their traces of argumentation, where an interchangeable use of the temporal conjunction "when", referring to an action, and the causal conjunction "because" occurred. Inviting students to clarify such uses of pronouns and conjunctions, and making sure subjects are explicitly stated, would not be important just at formal level, but also at conceptual level, supporting abstraction and the transition from a superficial perception of a pattern to a deep mathematical awareness about the structure behind the sequence.

A challenge for teachers might be that of noticing ambiguities on the spot, and exploit or clarify them, without affecting the flow of the students' discussion. To support the teachers, future research might be needed to identify sources of ambiguities, and suggest teacher moves.

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Challenges and opportunities for algebraic reasoning in a 6th grade programming lesson

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In this paper I describe challenges and opportunities that arise as two 6th grade teachers attempt to use programming to develop algebraic reasoning in their classrooms. The lesson is part of a collaborative design project. I use the Teaching Triade to characterize the challenges and opportunities identified by the teachers both in the design workshop and the evaluation workshop. I also analyze the implementation of the lesson in two classrooms. My main findings are that developing algebraic reasoning through programming is difficult and relies on both programming skills and a culture for generalizing and justifying.

Algebraic reasoning, programming, teaching practice development

Introduction

Algebraic competence is important for proficiency in all areas of mathematics and is seen as a gatekeeper for college access and career prospects (Grønmo, 2013). At the same time, algebra has enduringly been a topic students struggle with in the Nordic countries (Hemmi et al., 2021). Research has shown that it is possible to foster algebraic reasoning from the start of formal schooling (e.g. Blanton et al., 2015; Kaput et al., 2017). Changing the curriculum, however, does not mean that the teaching in schools change. To facilitate a real change in classrooms we need to learn more about what kinds of practices support or hinder the development of algebraic reasoning.

This paper focuses on using programming to develop algebraic reasoning in the classroom. Bråting and Kilhamn (2021) state that "Within the research field of algebraic thinking, the connection between the learning of algebra and computer programming is rarely discussed" (p. 171) and that research on how programming can be used to learn mathematical ideas in general is also rare. This study contributes to this topic by investigating the following research questions: 1) What characterizes the challenges and opportunities for developing algebraic reasoning through programming that the teachers voice while planning and reflecting on their teaching? 2) What challenges and opportunities for developing algebraic reasoning through programming do the teachers face in the classrooms?

Literature and previous research

The recent decades there has been a substantial interest in algebraic thinking, including algebraic reasoning and problem-sloving in the early grades (Wettergren, 2022). Kieran et al. (2016) propose that supporting students in noticing regularities, articulating generalizations, and explaining and proving conjectures is vital to developing algebraic reasoning. Mason (2005) claims that algebraic thinking is largely the recognition and articulation of generality, and that the recognition of generality is often done by specializing. This means working with specific examples to test a pattern or a relationship and get a sense of what is going on. Generalization is then the articulation of the underlying general pattern that one has tested by specializing. Mathematical thinking is to a large extent this constant interplay between specialization and generalization (Mason et al., 2010). When you have described a general pattern, you must then produce an argument that proves your articulation of the pattern is valid. Mason (1980) calls this reasoning. Algebraic reasoning is also one of four essential algebraic thinking practices identified by Blanton et al. (2019).

In this study programming is used as a tool to develop algebraic reasoning. Previous research on using programming to learn mathematics shows that there are some common challenges articulated by teachers. The skill level of students and teachers in programming and in mathematics is mentioned, as well as time pressures and technical issues (Bråting et al., 2020; Humble et al., 2020). Elicer and Tamborg (2023) show that even when the teacher is an expert in programming, the connection between mathematics and programming prove difficult. Boylan et al. (2018) reported that their effort to use programming to learn mathematics improved programming skills, but had no clear impact on mathematical learning.

I used the Teaching Triad (Jaworski, 2017) to do a thematic analysis of my data. I chose it because it is a tested framework used in similar research and it provided me with categories for my coding. The Teaching Triad has three categories; Management of Learning (ML) describes the teacher's role in creating a learning environment, including classroom groupings, task and activity planning, use of resources, and setting of norms. Sensitivity to Students (SS) describes the teacher's knowledge and attention to students' affective, social and cognitive needs. Mathematical Challenge (MC) describes the challenges offered to students promoting mathematical thinking and activity, including tasks set, questions posed, and emphasis on metacognitive processing. These three domains are interlinked and interdependent (Potari et al., 2023, p. 3). Jaworski et al. (2017) claim to have shown that to teach efficiently there needs to be a good balance between these three.

Method

My study is part of a collaborative project (the ALGEBRA project) where in-service teachers work with researchers on designing and implementing process-oriented Algebra Learning-Teaching Activities (ALTAs). The teachers and researchers engage in cycles of design, testing, and redesign of the ALTAs, participating in workshops before and after the teachers use the ALTAs in their classrooms.

I am working with two teachers that I will call Elisabeth and Hans. The teachers are from two different schools. At the point of data collection, I have worked with the teachers for three months and have observed both their classrooms and the classrooms of colleagues at their schools. I have found the general norms of the classrooms to be quite different. In school A, where Elisabeth works, I see that there is a strong focus on exploration, and that the teachers tend to ask questions instead of explaining. The class consists of 15 students and all students are native speakers. School B has a heterogenous student population with a greater difference in mathematical skill level and not all are native speakers. The norm in the classrooms is that the teacher explains, attempting to include all students.

The data is video and audio recordings of the design workshop, the evaluation workshop, and the classrooms, complemented by research notes. The classroom and workshop dialogue were transcribed and coded using the Teaching Triad as either SS, ML, MC or a combination of these. The coding was done by identifying thematical sequences in the dialogue where opportunities or challenges for learning in the ALTA were addressed. These were assigned to one or more categories from the Teaching Triad. I used this categorization to establish the characteristics of the opportunities and challenges. I also looked at the balance between the occurrence of ML, SS and MC, and evaluated this according to the frequency of sequences coded as ML, SS, MC or a combination of these. Finally, I analyzed the sequences coded as MC in both workshops and classrooms to look for instances of specializing or generalizing as defined by Mason (2005).

Results

The ALTA starts with the students writing instructions and 'programming' their peers. One student reads the instructions, and another walks the outline of a square and an equilateral triangle. They are then supposed to create this program in the block programming environment of 'Scratch' (https://scratch.mit.edu/). They then are asked to make the program draw any kind of regular 2D shapes.

The ALTA document lists the following generalizations: 1) the angles turned add up to 360° as the human computer goes around the shape once turning a total of 360° 2) The total exterior turn in any polygon is 360°, and 3) each exterior angle in a regular polygon is 360° divided by the number of sides. This paper focuses on the second lesson of the ALTA, where they are discussing the angles turned and making a computer program. The ALTA document suggests that the students should receive a finished program that draws a square. They should modify this program to draw an equilateral triangle. They should then make a conjecture about the sum of turns in a polygon and test this out by drawing an equal-sided pentagon. Finally, they should explain why the conjecture is true for any polygon.

Design workshop

Hans is concerned about his own and the students' poor skills in programming (SS). He spends most of the workshop asking questions to help him understand the technical aspects of the task. He also expresses that the students tend to want to complete the tasks and move on (SS), and that this can pose a challenge: "They have no interest in talking about why this happened". He says their discussions in mathematics are "on the lowest level", that they can "discuss anything else, but not mathematics" (SS). Hans suggests that it is important to get the balance between mathematics and programming right in the task design (ML). "The programming part of it is oversized compared to the math". "If they can't understand the programming the math will be secondary. This is connected" (MC). This shows that he finds discussion vital for mathematics learning (MC). Elisabeth listens more than she speaks. She contributes with help for Hans, as he asks questions about how to program. She says that the class has programmed before, and that they have just worked with geometry and should know the facts (SS). She mentions that "It cannot be just the part where they play or explore, they have to in a way understand what is behind it" (MC).

The discussion in the group focuses on how to help the students with programming (ML). The teachers also focus on what the students need to know about equilateral polygons (SS). There is very little discussion on how to support mathematical discussion, and no focus on how to challenge the students mathematically and how the programming connects to algebraic reasoning (MC).

Classrooms

Hans starts by walking the shapes of a triangle and square (ML). He asks "Could I make a rule for that? To find out how many times I must, how many degrees I must turn to get back to the start position? I will give you a hint, I will tell you. For a triangle: 360 divided by three" (SS). He then tells them that to find the angle to turn in a square they will have to divide by four. He uses a whiteboard to express the pattern in the calculations of the angles by calculating it for triangles, squares, pentagons and hexagons (MC). The students then work in pairs (ML). He lets the students copy a program (SS) from the digital board that draws a square and modify it to draw an equilateral triangle. He tells them to use the rule of dividing 360 by the number of sides to draw different equilateral polygons. He suggests using a calculator to calculate the angle to turn and insert this number in the program (SS). After a short while he leads them through the modification of the program on the digital board. He asks them to make a program that can draw any equilateral polygon. He guides them through writing the program and then lets them specialize by testing out the program making different polygons. He tells them what blocks to use, but not in what order to put them (MC). He then helps the students by moving from pair to pair (SS). As they try different numbers, they see that as the number of sides increases the shape starts to look like a circle. They also see that as the number of sides increases, they must decrease the number of steps to move to keep the drawing in the visible area. (MC) Hans picks up on discoveries the pairs make and communicates them to the class (MC) to let all students see the patterns (SS). After a while he shows them how to make a new variable and gives them more blocks to copy and use to make a program that also takes the number of steps walked as a variable (MC).

Elisabeth marks a triangle on the floor with tape, and the students sit on the floor in a circle while she acts as a human computer (ML). She extends one of the sides in the equilateral triangle to show that the angle turned is the supplementary angle of the angle inside the triangle (SS). They spend a long time discussing how to find the angle to turn if you know the internal angle (MC). She then moves on to the conjecture of how to find the external angle from number of sides. Elisabeth walks on the triangle and says: "I walk, I turn, I walk, I turn, I walk I turn". One of the students suggests that they can multiply 120 by 3. Elisabeth then asks about how it would work in a pentagon (MC). Another student suggests that they can divide 360 by five. She does not make the students explain their thinking, and she does not help them formulate a conjecture. She lists orally the structure as 360 divided by 3 in a triangle, 360 divided by 4 in a square and 360 divided by 5 in a pentagon. The rule is not explicitly formulated as 360 divided by the number of sides.

After a short introduction repeating some blocks to be used (SS), the students program in pairs trying to replicate what they did in the unplugged programming, making a program that draws a square and a triangle. A finished program is provided as a possible support, but none of the students refer to this. The students discuss while they program. Most pairs manage to make a functional program. As Elisabeth walks between the groups her focus is initially on technical programming issues, using the 'erase all' block and where to start drawing. She then challenges them to make the program more efficient (MC). She does this by asking how many blocks she would have to change or add to make the triangle program draw a square. Most students have made it by adding one block for each move. They see that they can use the 'repeat' block, drawing on earlier experiences with programming.

Elisabeth then teaches the whole group and shows them how to use operational blocks to calculate the angle by using 360 divided by number of sides. She then introduces the ‘ask’ and ‘answer’ blocks. The students are challenged to make a general program drawing all equilateral polygons (MC).

Evaluation Workshop

Hans expresses that there were too many students in the class and says that smaller groups would have allowed him to support them better (ML, SS). He proposes that giving the students a finished code would limit the time spent on programming issues (ML). He is impressed by the perseverance and energy the students brought, and that they were really “wringing their brains” (SS). The students need to practice using their own language to generalize, and that this takes time: “You have to practice multiple times” (SS). Hans claims that: “they have to do it consistently. It is hard to talk about math if they are not used to it” (SS). Hans finds that generalizing in this topic is hard, because it is mathematically challenging with many elements to grasp (MC). “It is challenging enough to understand internal angle, external angle, turn”. He mentions that he would have liked to put algebraic expressions as another representation of the relationships used in the programming. “Not that they have to find the formulas themselves, but just mention it”. “If it is on the blackboard, they might get curious. Then you can explain later” (MC). He says that the new teachers are conscious of the importance of emphasizing the relationships, “why things are happening”. “I am also very aware of this myself: can they explain why they are doing it? Not just: this is the way it is” (MC).

Elisabeth explains how she had walked the triangle taped on the floor and that this led to nice discussions (ML). “My experience is that they understood more”. Elisabeth poses that expanding the lines in the triangle to visualize the angle turned was a good idea to move the discussion further. “It let us talk about internal and external angle” (MC). She says that taping a triangle on the floor and walking on it: “helped a lot to understand the mathematics in it” (SS, MC). Elisabeth says the students were able to use what they saw in the first lesson when programming in the second lesson (SS). She says the students need to practice expressing themselves mathematically. “They have to get used to it” (SS, MC). She proposes that the students could make a collection of explanations for mathematical concepts in the students’ own words (ML).

Analysis

The design workshop focuses on the preparation of the lesson. Using the Teaching Triad, I find that the discussion focused mostly on ML and SS. There are nearly no mentions of MC. The participants discuss programming and the level of programming skill, but only briefly mention the connection to mathematics. They express a concern that programming can get in the way of the mathematics if the level of programming skill is not high enough. In the evaluation workshop analysis shows that MC is more prominent in the discussion. They focus on challenges and opportunities for engaging students in generalization and justification of conjectures.

Both teachers utilize the affordances of the programming environment to help develop algebraic reasoning. Elisabeth makes use of the instant feedback of the program and discussion in pairs to let the students generalize on their own (MC). The programming provides the appropriate support for the students (SS) and the appropriate challenge (MC) to do this. However, the conjectures are never explicitly formulated. Hans uses the program to support the students (SS) in testing out conjectures through instant feedback that let them test out many examples (MC). This give them an opportunity to validate conjectures by specializing, while providing the appropriate level of MC. Hans finds the relationship between mathematics and programming to be important and that the balance between them should be attended to. This is consistent with the fact that he utilizes the blackboard throughout the lesson, representing the patterns found in the programming by drawings and symbols.

Discussion

The focus in the design workshop is on programming, with emphasis on the Management of Learning (ML) and Sensitivity to Students (SS). Both teachers and researchers seem to think that this must be tackled first if any learning of mathematics is to be achieved (MC). There is little to no mention of algebraic reasoning, even if that is the expressed goal for the project. This is consistent with previous research stating that teachers find that the level of programming skill is an obstacle to mathematical learning through programming (Boylan et al., 2018; Bråting et al., 2020; Humble et al., 2020). Hans spends a significant amount of time instructing his students, showing them patterns, relationships, and formulation of conjectures. He does not seem to think that students can use programming as a tool to generalize. Hans emphasizes the importance of explicitly explaining the mathematics involved. Due to the varying skill levels of Hans and his students, he struggles with using programming to generalize and formulate conjectures. Instead, he focuses on connecting mathematical notation with programming. Hans suggests presenting the finished program to the students so they can explore mathematical patterns. He seems to find programming a good tool for specialization, allowing students to search for patterns and relationships.

Elisabeth focuses on the internal angles instead of the angles to turn and their connection to the number of angles. Here the discussion focuses on the taped triangle. This could have been done outside of a programming lesson and did not utilize either the unplugged programming or the Scratch programming. The part of the lesson where they focus on the angle turned is very short and include no discussion, just questions and answers. The conjecture is not formulated, and the students then have to figure the generalization out on their own. Elisabeth's focus on the mathematical challenge and the culture of posing questions, not answers, might have led to her not prioritizing explicitly formulating the conjectures made. In the discussion in the pairs the students might have generalized from the unplugged programming, but their possible generalizations is left implicit and never revealed to the other students or justified. This might be an inherent problem with programming as posing and testing conjectures are done quickly and then you move on. Leaving the conjectures implicit also hinders the discussion of them. As mentioned above, the discussion and validation of conjectures are at the core of algebraic reasoning. When validation is done by testing the program, it can pose a challenge to using programming to develop algebraic reasoning.

Conclusion

I find that using programming to develop algebraic reasoning is challenging in many ways. The potential to use programming to support the exploration of functional relationships and variables might seem intuitive but can be hard to do in practice. The students and teachers need some level of programming skill to be able to use programming to achieve other goals. The focus on generalization also needs to be explicit, and not just implicit in the examples. However, the opportunities to use programming to specialize seems to be easier to achieve. Specialization is part of understanding the general and is not to be discounted. To be able to utilize programming for generalization, however, the teachers propose that the students need to work on their skills for justifying, explaining and discussing mathematics in their own words. I also think that the teachers need to focus on generalizing and asking questions that support students' algebraic reasoning, and not do the conjecturing for them. Supporting the development of algebraic reasoning through the use of programming demands a certain degree of programming skill and a culture for generalizing and justifying. The findings suggest that teaching development in this area should afford enough time for skill development in programming, but that programming skills are not enough to ensure a focus on generalizing and justifying that can help develop algebraic reasoning. Making conjectures explicit and spending time justifying these conjectures might lead to the development of algebraic reasoning.

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Mathematical modelling in an interdisciplinary project on solar panels on the school roof

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In this paper we discuss an interdisciplinary project where Norwegian lower secondary school students explored whether the school should install solar panels by using concepts from science and mathematics to find answers. Given previous research's warning on the mathematics becoming invisible in such projects, the focus in the paper is on using modelling perspectives to make it visible. The extended modelling cycle made it possible to identify the mathematical processes students went through. One way the students engaged with modelling was by using an existing mathematical model, a solar panel calculator that both helped and hindered them in exploring the problem and making sense of mathematical concepts. The findings are discussed in relation to prior research on the mathematics' role in interdisciplinary projects in the classroom.

Keywords: Mathematical modelling, interdisciplinarity, solar panels, authentic problems

Introduction

Modelling and applications is of significant importance in the Norwegian mathematics curriculum as one out of six core elements (exploration and problem solving, modelling and applications, reasoning and argumentation, representation and communication, abstraction and generalization, and mathematical knowledge areas) which are interdependent and should be seen in context with each other. The curriculum is furthermore built on values such as ethical awareness, environmental awareness, and critical thinking. Furthermore, mathematical modelling and the values are in line with the three interdisciplinary topics in the curriculum: democracy and citizenship, health and life skills, and sustainable development (Ministry of Education and Research, 2020). Preceding the current curriculum, there has been a clear focus on recruiting more students to the natural science and mathematics subjects as expressed in the *Realfag* strategy (strategy for mathematics and natural sciences) and for contrasting mathematics teaching where the focus has been on procedural knowledge rather than problem solving approaches (Ministry of Education and Research, 2015). Giving students opportunities to work with mathematical modelling and interdisciplinary problems might motivate them to engage in and use mathematics and sciences in meaningful real-life problems, to develop their knowledge and attitude towards mathematics, as well as to help the students develop critical citizenship and successful participation in their current and future worlds (English, 2016).

However, mathematical modelling and interdisciplinarity, both aiming to engage students with challenges and dilemmas in the topics representing prevailing societal challenges (Ministry of Education and Research, 2020), are not clearly defined in the curriculum. From an international perspective, there are various interpretations of interdisciplinarity in STEM (Science, Technology, Engineering and Mathematics) projects. Mathematics tends to either be hidden or understated, and it is not always easy to make connections across different STEM disciplines (English, 2016; Kelley & Knowles, 2016; Maaß et al., 2019a; Shaughnessy, 2013). More research is required on various uses of mathematics in STEM education. In the project that the current paper comes from (https://argument.uib.no/), one particular focal point has been on providing examples of interdisciplinary resources and on supporting teachers in implementing them in the classroom in grades 8-10 in Norway, and by that providing opportunities for interplay between research and teaching practice in STEM. In the current paper, we focus on one implemented example of students working on solar panels in grade 10. We explore the research question: Can mathematical modelling act as a lens to reveal the mathematics that students work with in an interdisciplinary project about solar panels?

Theoretical considerations

Shaughnessy (2013) defined STEM education as "solving problems that draw on concepts and procedures from mathematics and science while incorporating the teamwork and design methodology of engineering and using appropriate technology" (p. 324). A common element in STEM education was identified by Maaß et al. (2019a): the problems must be situated in real life contexts to ensure the concepts learned are meaningful, and to establish connections between the academic content and the contexts in which it is applied. Tytler (2020) identified a shift of STEM ideas towards "advocacy of interdisciplinary curriculum practices built around authentic problems, involving some or all of science, technology, engineering, and mathematics" (p. 31) that seems to have some of the elements from Shaughnessy (2013) and Maaß et al. (2019a). Vasquez (2014/2015) discussed different degrees of STEM integration as increasing from disciplinary to multidisciplinary, interdisciplinary and transdisciplinary integration. Interdisciplinary integration in STEM is described as the interrelation and interdependency of concepts and skills from different subjects to work with a real problem (Vasquez, 2014/2015). This is how we also have chosen to look at the concept in our study.

Research has pointed out challenges with interdisciplinary integration in STEM: the mathematics needs to be made visible for the students as they don't automatically see it or integrate the different disciplines (English, 2016; Honey et al., 2014; Maaß et al., 2019a; Shaughnessy, 2013), and there is the risk that mathematics is just a mean that serves the other disciplines (English, 2016; Maaß et al., 2019a). For productive mathematics learning to happen, Tytler et al. (2019) identified the following principles: the teacher should be aware of the mathematics in the tasks; the tasks should be something the students need and want to do and they should require the use of mathematics in unfamiliar ways or such that the use of known mathematics brings new insight to the problem; and the problems should be open-ended and thus require real problem solving. English (2016) and Maaß et al. (2019a) have argued for the close connections between mathematical modelling and STEM education, and Maaß et al. (2019b) proposed mathematical modelling of socio-scientific issues with inquiry-based teaching methods to promote active citizenship in mathematics teaching. Mathematical modelling in the classroom can be considered as the process of engaging with real-life problems and transitioning from reality to the mathematics to find solutions. To understand and explain the students' work within the solar panel project, we specifically used the extended modelling process (Maaß et al., 2019b). The model builds on the modelling process by Blum and Leiß (2007) with its phases of constructing a mental model of a real problem and simplifying it by making assumptions, mathematizing it by using relevant mathematical concepts and working mathematically to obtain mathematical results, then interpreting the results towards the real problem, validating them with the situation at hand and exposing the final solution. For socio-scientific issues, such as the feasibility of installing solar panels on the school roof is, Maaß et al. (2019b) have added new elements to the modelling cycle. One such element is collecting information and analysing sources while simplifying the problem, and another one is considering social, economic, cultural, and ethical aspects, in addition to the scientific aspects, to decide on the final solution. From previous research (Blum & Borromeo Ferri, 2009) we know that students have difficulties especially with the mathematization phase. Figure 1 represents the Maaß et al. (2019b) cycle with our data superimposed. The extended cycle is used in the findings section to give a view of students' interdisciplinary work with the solar panel from a modelling perspective and to locate the different mathematical processes.

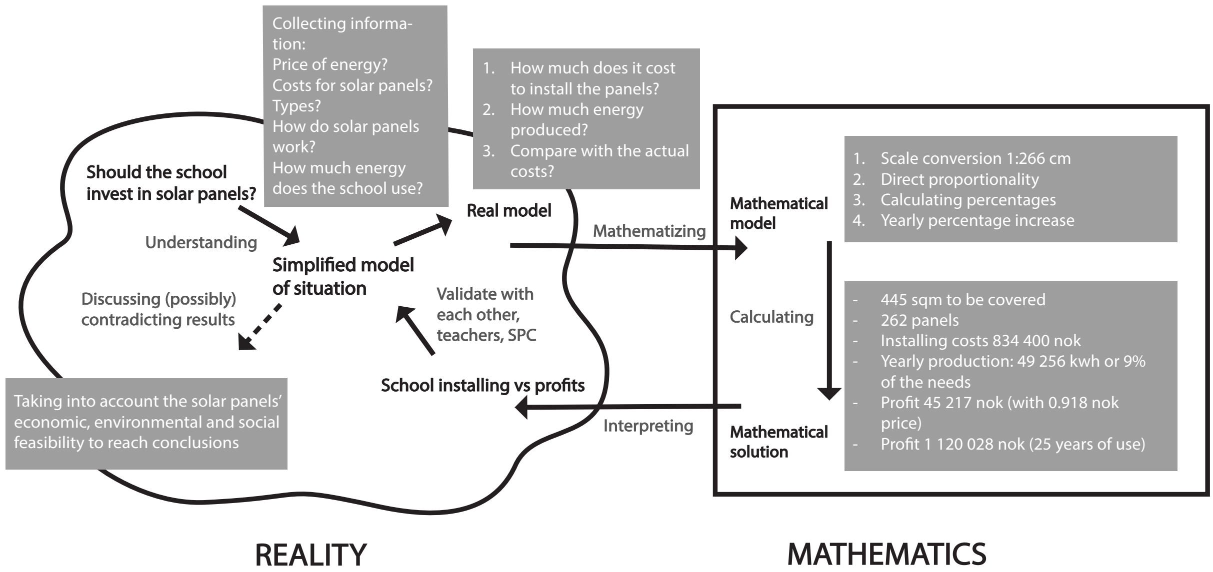

Figure 1 The extended modelling cycle by Maaß et al. (2019b) with our data super-imposed

While a modelling cycle can explain students' work with mathematics to find answers to a problem, there are also other ways they can use mathematical modelling, such as using existing mathematical models as part of their work with the problem. Such an example is presented by Steffensen and Kacerja (2021) where students explored a carbon footprint calculator (CFC), an existing mathematical model with unfamiliar mathematics, while working on climate change issues. Even though the students did not see and thus did not explore all the mathematics behind it, the authors argue for the relevance of using such mathematical models in the classroom to help students be aware of the socio-scientific issue the model aims to quantify and of the hidden mathematics behind such models. In our paper we will thus use the extended modelling cycle and existing mathematical models to make visible the students' work with mathematics in the interdisciplinary project.

The research setting

Researchers involved in the "Argument project" from science and mathematics education formulated the solar panel activity (among other activities, https://argument.uib.no/) and adjusted it together with lower secondary teachers that were involved in the project. Then the project was implemented in classrooms, with the same teachers and researchers (including one of the authors) participating. Having a plan for the activities within the project gave us the chance to foresee and make some mathematical concepts that the students would meet in the project visible, to avoid its invisibleness as reported in earlier interdisciplinary projects (Tytler et al., 2019). The students were given the task to help the school decide whether they should consider installing solar panels on the school roof. The part of the project reported in this paper was run for around 12 hours with 10th grade students during the autumn of 2019. The question used for the project was not a real problem since the school was not really considering the shift to solar power, but it could however be a characterized a *realistic problem* in the sense that students could connect to it and recognize it from the societal discussions. Students were given the possibility to learn more about solar panels by visiting a company that used solar panels and by exploring electricity concepts and measuring electricity productions of miniature panels. They also collected data about prices and production of energy from different real solar panel companies by calling or emailing them. The students worked in groups of three to four students. One main teacher followed the class all the hours, and another one to two teachers followed the project some of the time, together with one mathematics education researcher and one science education researcher. First, the students visited a business that had installed solar panels, then they discussed how panels work and used miniature versions to produce electricity, covering relevant scientific concepts. They made their own measurements of the energy produced by miniature panels using an app and collected further information. The students came across a solar panel calculator (SPC) (Pedersen, 2022) that some of them used for calculating the economic profitability of solar panels. This model was not included in the original plan for the activities and was thus not scrutinized beforehand by the research team or the teacher. Students worked in groups exploring the task, planning how to find a solution, and preparing a final written assignment. The teacher's role was to support the students' exploration. The main teacher introduced the task in the beginning and followed and supervised the group work, making sure everyone was engaged, organized some whole class moments to present parts of the task, explained some relevant concepts in mathematics and science, and asked groups to share their work with each other. The resulting data for this paper consists of transcriptions of video-taped class discussions and audio recordings of discussions in two student groups working on the project, in addition to their written group-assignments in the form of a project-report. We identified the modelling phases in the transcripts using the extended modelling cycle by Maaß et al. (2019b) and focused on the mathematical work in the written texts also. The data was analysed thematically based on the different phases of the extended modelling cycle.

Findings and discussion

In the first subsection we use the extended modelling cycle to explore how the students work with mathematics in the solar panel activity. Then we zoom in on students' mathematical calculations and how they solve the problems they encounter on their way. We end this section by discussing how students answer the main question using a solar panel calculator, i.e. an existing mathematical model as a different way of engaging in modelling and what kind of challenges that brought to the students.

The extended modelling cycle in the solar panel activity

The students' process with the task was not as linear as it might seem by the description here presented (see Figure 1), they went back and forth between the different phases of the extended modelling cycle. They worked with concepts from science, such as electricity, and mathematics, such as scale conversion and percentage increase, to establish whether they should recommend the school to install solar panels on the roof. This stage corresponds to the extended cycle, *collecting information* to understand the problem (Maaß et al., 2019b). In *simplifying* the problem, the students searched for relevant questions to find answers to, such as the total costs of installing solar panels on the school roof, the amount of energy produced and comparison with the school's actual energy costs. They engaged in finding the area of the school roof to be covered by panels and the number of panels needed by also using information from the solar panel companies such as how to consider the space between the solar panels. They had to find the installation costs for the panels, the amount of energy the panels would produce, the electricity price's yearly increase and explore the feasibility of the installation compared to the school's actual costs for buying energy. This was then *mathematized* by using different formulas and mathematical knowledge to then calculate the solutions. One such example was measuring the area of the school roof in a map provided by the school and engaging with the scale 1:266 cm to convert that to square meters area. Here the teacher intervened and explained the concept of scale conversion for the whole class as he saw that the students were struggling with the concept. Once they had calculated the area of the school roof, they would find the area to be covered by solar panels by halving the total area. Another example would be the use of proportionality to find out the amount of electricity produced by the solar panels: if one panel produces a units of electric energy, then b panels would produce $a \cdot b$ units. They further worked on *calculations* and came up with mathematical solutions that are presented in the next subsection. The students *interpreted* their solutions in terms of the school installing the solar panels or not, and *validated* their answers by comparing with other groups, with the teacher or with the online SPC, as well as relying on their knowledge about solar panels. The students themselves decided which criteria to follow for finding qualified answers. One of the groups e.g., suggested that given the elevated costs for installing solar panels and the little amount of electricity produced, the school should wait for some years for the technology behind the solar panels to become better and the panels would last longer. In this case, the economic feasibility became a prioritized factor for this group of students. In another group, the students advised to install panels because of the school's advantageous flat roof, environment considerations and low maintenance costs for the panels. But this group did not present detailed calculations and thus prioritized the environment rather than the economy. Considering the extended model, these examples account to "*Taking into account ethical, societal, cultural and economic aspects*" (Maaß et al., 2019b).

The task given to the students is open-ended, and the students need the mathematics to find answers, in line with Tytler et al. (2019). The project can be characterized as interdisciplinary (Tytler, 2020): it is authentic (realistic), and it requires concepts, procedures, tools, skills and values from the different subjects to explore a real-life problem. The project also has all the three requirements pointed out by Shaughnessy (2013): The students analysed real data, provided their own measurements and arguments for the feasibility, and considered different variables. In the following, we focus on the students' calculations.

The calculations and the mathematical solutions

Here we summarize the students' work based on their written assignments. We have chosen the assignment that had most details while also showing the calculations to support their arguments about installing the solar panels. The assignment was from a girl group. First, since the school roof had different levels, they decided to install solar panels on the two highest roof parts to capture as much sunlight as possible. This area was measured to be 890 m^2. Using information from the company they visited, the students argued that only half of the area of the roof, thus 442 m^2, could be used for solar panels, due to the need of space for maintenance and snow shovelling during winter. Given that each panel measured 1.7 m x 1.0 m, the students concluded by *proportional reasoning* that the roof could hold 262 panels. Based on information from a solar panel company, a package of 250 panels costs 800,000 Norwegian kroner (NOK). The students found out that 262 panels would cost NOK 838,400 using proportionality. It is not clear whether this total price includes mounting and maintenance. Proportionality ideas were used to deduce that 262 panels would produce 49,256 kWh knowing that 250 panels produce 47,000 kWh energy annually. The school spends NOK 491,000 each year on energy, and given the price of 0.918 NOK/kWh, the students deduced that the school each year uses 535,000 kWh of energy. Therefore, the solar panels would produce 9% of the schools' required energy (49,256/535,000).

Exploring an existing mathematical model, SPC, to answer the problem

After some calculations, one group concluded that it is not economically feasible to install solar panels and decided to look back and search for mistakes. This is a *validating* moment in the extended cycle (Maaß et al., 2019b). They used the online SPC (Pedersen, 2022) by entering prices and amount of energy to determine the panels' feasibility. The students did not seem to question the calculator, or the variables considered in it. The variables are the costs of investing in solar panels, the amount of energy produced and used, costs for producing and selling energy, yearly percentage increase of the energy prices and life expectancy of solar panels. Based on these values, the SPC gives the profitability in NOK. The calculator was unfamiliar to the students, they needed to get a sense of it and of the mathematics involved, therefore the task corresponded to the requirements by Tytler et al. (2019). This is also similar to the study by Steffensen and Kacerja (2021) where students explored the CFC online. Following is a conversation between a student S and the teacher T.

Student:	If I insert the price of power, or I don´t know what it is, of 50 [øre] instead of 90 [øre] here, we will go into minus here [the profitability in the calculator].
Teacher:	So, you earn less than what you had there [higher price]. Yes. Then it is computed in such a way that if the price of power is 50 øre, you will save less on having solar cells than if the price was 90 øre. But what do you mean by "we go into minus", are the expenses too high?
Student:	That it is not profitable [the result in the calculator].

The student has found that if the energy price is lower, 50 øre instead of 90 øre, then it is not profitable for the school to install solar panels since they "will go into minus". Here the student is pointing at the economic aspects. Using the SPC, the students can vary the price of the electricity (and other parameters), opening the power of mathematical modelling and helping the validation process.

Afterwards, the students talk to the teacher about another variable they used in the SPC, the official information from authorities that there will be a 4% yearly increase in the energy price because of economic inflation. In the following they discuss with the teacher what this means. Perhaps the energy price will be higher, which can make the installation profitable.

Student:	According to Statnett and the Ministry of Petroleum and Energy, the price of power will increase by 4%.
Teacher:	Then you can use a higher price than 50 øre since it increases by 4% every year. 50 øre is the average price this year. So next year the increase is 4% of 50 øre. 4% of 100 øre is 4 øre.
Student:	So, then it will be 2. So, it increases by 2 øre per year? So, 2 øre in 25 years. 52 øre the next year.

The teacher proposes higher prices given the new piece of information and starts calculating himself the new price from 50 øre to a 4% increase. So, 4% of 100 is 4, which the student continues by calculating half of the increase, so 2 øre per year. But afterwards, the student does not seem to capture the exponential nature of the increase and instead reasons linearly, an increase of 2 øre each year for 25 years. The mathematical model underlying the SPC is not transparent for the students, thus hindering them from understanding the concept of exponential growth. As in Steffensen and Kacerja (2021), the mathematics remained hidden. From the available data, the teacher does not seem to follow on the students' reasoning, and as in previous research (Honey et al., 2014; Tytler et al., 2019), the students lacked support to elicit the mathematical ideas of the percentage increase. In the two examples the SPC both supported and hindered the students' exploration of the mathematics.

Conclusion

Previous research about mathematics in interdisciplinary projects has pointed out the risk of mathematics either being hidden or just being reduced to serving the other subjects (see e.g., English, 2016). In this paper with data from our interdisciplinary project, we used mathematical modelling perspectives to make visible the work the 10th grade students did with mathematics. We found that the students were able to see where and how the mathematics was needed at different stages to explore the feasibility of installing solar panels on the school roof. A careful planning of the project and the mathematics and science concepts needed in it was helpful in making mathematics visible and supporting the students as previous research advises (Honey et al., 2014; Tytler et al., 2019). Some of the mathematics involved in the project was not very challenging for the 10th grade students, but the new concept of exponential growth was challenging.

The use of the extended modelling cycle made it clearer how students worked with mathematics and how they engaged in the different phases. It also allowed us to zoom in into special moments of their work with mathematics. Such a moment was when the students were stuck in their use of mathematics as in the scale conversion moment. Previous research has pointed out the mathematization stage as one of the challenging ones (Blum & Borromeo Ferri, 2009). In our study the students were stuck in how to mathematize the problem to find out what the area of the school roof was and at that point the teacher intervened and talked about the concept of scale conversion with the whole class. Given this, we conclude that both the planning of the project and teacher intervention to teach the mathematical concepts are part of limiting the challenges of mathematizing.

Another way students engaged with modelling was their use of the SPC, an existing mathematical model with its variables, to calculate the economic feasibility of the solar panels and to validate their work. The model both helped and hindered the students in their work with mathematics. However, as also Steffensen & Kacerja (2021) argued for, working with such models is worthy to help students engage with the many mathematical models in our society and be aware of and explore the hidden mathematics within the models. Tytler et al. (2019) argued for STEM projects to be prolonged in time so that the teachers involved get opportunities to develop appropriate skills and pedagogies. The students' common experience with the SPC can later serve as a basis for the teacher to bring the concept of exponential growth into students' attention.

The results from our study point to the opportunities such a project has for students' engaging with current societal issues and use mathematical and natural sciences concepts to find answers. Further research is required for finding ways to support students' use of mathematical concepts and their process of making meaning to fully exploit the potentials of the projects. This also requires more ways to support teachers in developing their skills in both noticing mathematics and supporting students in noticing and engaging with it.

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Misaligned communication among partners in practice – identifying potential impediments for development efforts in mathematics

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Reform efforts in mathematics education often emphasize investigations of mathematical problems, with the aim of enhancing student participation in productive discussions, but it has proven difficult to make real and sustainable changes of practice. This paper reports from a project that aims at establishing a system for educational development in four subject areas, through a partnership between researchers and schools. Data from the pilot phase of this project are analyzed with computational text analysis, to identify potential misalignment in communication, which can prohibit the establishment of such a system for educational development in mathematics.

Keywords: Mathematics teachers, professional development, topic modelling.

Introduction

For several decades, mathematics educators have tried to change practice from a traditional toward a more reform-oriented practice (e.g., Wright, 2021). Recent research on professional development in mathematics education emphasizes the challenges of establishing partnership and community among diverse participants, and an entire volume of the *International Handbook of Mathematics Teacher Education* is devoted to this. In one chapter of this handbook, Santagata et al. (2020) started out by reflecting on a professional development initiative that failed to reach its potential. That study appeared to echo one of Clarke's (2007) ten principles of professional development in mathematics education: recognizing that development can be both challenging and painful. Decades of research on professional development of mathematics teachers underscore the value of working with groups of teachers (not only individuals) in partnerships, focusing on issues that are of relevance to the teachers, and letting the teachers participate actively in activities that are close to their practice (Clarke, 2007). Some studies have facilitated lesson studies for this purpose; others have used action research (e.g., Wright, 2021). Many studies focus on teachers and their learning or development (e.g., Hwang et al., 2024), and they typically apply qualitative methods and analytic approaches (e.g., Hwang et al., 2024; Wright, 2021). Few studies, if any, focus on the teacher educators and their involvement in professional development. The present study does, and, like the study of Santagata et al. (2020), aims at scrutinizing and learning from a professional development initiative that failed to reach its potential.

This professional development effort is part of a larger project, *Partners in Practice*, which aims to develop collaboration among researchers and teachers, and to facilitate collaborative and continuous development of teaching practice. A core idea is to develop student participation in teaching with problems (Lampert, 2001), and to facilitate collective explorations of what reform ideas about problem solving, classroom discussions, and student participation might look like in regular

Norwegian primary classrooms. From the outset, we[1] clarified that our intention was not to provide solutions or directions that the teachers could straightforwardly implement, but we wanted to create a space where we could explore these issues together with the teachers. After three semesters of collaboration, the teachers reported in a post-interview that little had changed, and they were disappointed that they had not received any recipes for how to carry out discussions of mathematical problems in a way that could facilitate student participation. However discouraging such feedback might be, it made us realize that we must have talked past each other in the sessions. The present study thus aims at identifying possible areas of misalignment in the communication between facilitators and teachers in the professional development sessions. The following research questions are approached:

1. How does the communication of facilitators and teachers differ across sessions?
2. What are key topics in the communication of facilitators and teachers across sessions?
3. How do facilitators and teachers communicate about these key topics across sessions?

Since it can be challenging to analyze communication in a context where the researcher is also participant, the present study explores application of techniques from computational text analysis.

Methods

The collaboration involved a group of elementary mathematics teachers in six two-hour professional development sessions over 18 months. The sessions' overall progress was organized in three stages, with the objective of developing a teaching practice that involves student participation in productive discussions of mathematical problems. The initial stage provided the teachers with opportunities to get first-hand experiences with mathematical problem solving. The second stage consisted of guided practice, where participants engaged in cycles of planning, teaching, and reflection—as described in common frameworks for professional development (e.g., Kavanagh et al., 2020). Following the principle of gradual release of responsibility, teachers then were expected to continue exploring and developing their own practice more independently in the final stage of the collaboration. Five teachers and three researchers participated in the group. In some sessions, the school principal and vice-principal joined in on the discussions. The focus here is on the participants who were present in all six sessions: two experienced male teachers (pseudonyms: "Roy" and "Ted"), and one facilitator (Reidar, the author of this paper).

Efforts to identify potential communication misalignment can be influenced by bias, particularly when the author is one of the communication partners. To avoid some of the bias, the transcribed recordings from the six sessions were analyzed by using techniques from computational text analysis with Quanteda (Benoit et al., 2018)[2], which is a package for doing quantitative text analysis in R (R Core Team, 2023). The analysis in this study primarily involved three techniques from computational text analysis. The first technique—keyness analysis—was used to answer the question about differences in communication between participants. Keyness analysis is a technique that allows for

[1] "We" here refers to the group of researchers that facilitated the professional development.

[2] Source files, including R scripts with output from the analysis, are available at https://github.com/rmosvold/norma24/

computing relative frequencies of all words used by one participant and compare it with a reference group. In this study, keyness analysis enabled comparing the relative frequencies of words across the three target participants (Roy, Ted, and Reidar).

The second technique was topic modeling (Blei et al., 2003), which is a type of unsupervised machine learning that applies a probability model to identify patterns in a set of documents or texts. Applying such probability models generates lists of words that are likely to occur together, and these lists of words constitute "topics" or categories. In this study, topic modeling was used to identify key topics in the communication of the three target participants.

The third set of techniques, used to analyze participants' communication around the key topics identified through topic modeling, was a combination of dictionary analysis, keyword-in-context analysis, and frequency analysis. Based on the identified topics, the list of all words that were used by the three participants in the data material was carefully considered, and the author generated "dictionaries" of words that related to each of the three topics. The dictionary analysis function in Quanteda was then used to identify how often each participants' communication (their word use) related to each of the topics. Keyword-in-context analysis was then applied to investigate how participants used keywords from the topics in a smaller context—five words before and after the keyword. This provided a better understanding of how the words were used in context and not only focus on individual words (often referred to as a "bag-of-words" model).

Results

Difference in communication between participants

A 'keyness' analysis compared the relative frequency of the words used among each participant as compared with the others (see Table 1). The keyness analysis made it possible to identify differences in the three participants' talk. Reidar[3] often used words like "perhaps" and "try", and these words were used in contexts that communicated possibilities and things to try out. He often used the phrase "we can try" to suggest things the group could explore as part of the partnership. Reidar also used the word "students" relatively more often than Roy and Ted. This involved directing attention to the students, prompting the group to consider how the students might encounter the problems they discussed, and how they might be thinking about it.

[3] In the results section, I refer to myself as "Reidar" and "he" to signal that I now consider the analysis of my communication as one of the participants in the study.

Table 1: Relative frequency of words across participants

Participant	Key words	Frequency	Relative frequency
Reidar	perhaps	166	.86
	try	123	.84
	students	126	.78
	thing	115	.76
	talked	61	.85
	different	45	.87
	further	45	.85
	important	48	.83
Roy	class	15	.41
	multiply	8	.62
	fourth	5	.83
	drove	5	.83
	parenthesis	5	.83
	struggle	4	1.00
	hear	6	.67
	particular	6	.67
Ted	held	30	.75
	happen	17	.89
	math	33	.66
	sit	22	.69
	motivation	14	.82
	right	58	.50
	come	36	.55
	work	31	.57

Whereas Reidar frequently attended to students and their thinking more in general, Roy often talked about what he experienced in his own "class." When investigating this further, his talk often focused on challenges he experienced, and how his students struggled; the word "struggle" came up high on the list of words he used relatively more often than the two other participants. When considering the words Ted used more often than the others, it can be noticed that the word "held" was often part of a phrase that, translated from Norwegian, means something like "I was about to say". The word "happen" is more interesting, however, and Ted often used this word when he talked about how students were experiencing mathematics, and how they would be confused and wonder about "what is happening" in the context of a particular mathematical task. Ted also talked more about "motivation" than the two other participants—mostly with reference to students' motivation, or lack thereof. Like Roy, Ted often talked about challenges that he experienced with students who were not getting it and lacked motivation for doing mathematics.

Key topics in the communication of participants

Topic modeling was then used to identify key topics in the communication of participants. Tuning of different models indicated that three topics were prevalent in each participant's communication. The list of words that were generated by topic modeling was validated by checking the transcripts for how these words were used together.

For Reidar, the first topic features "problem" as the top keyword, followed by "number," "get," and "try." This topic was most prevalent in the first two sessions, where the participants talked a lot about a problem on consecutive sums. A second topic had "try" as the top keyword, followed by "students," "believe," and "think," and this topic was prevalent in sessions 3, 4, and 6. Finally, a third topic featured “students”, “believe”, and “think” as the highest featuring keywords. This topic was most prevalent in session 5. At the outset, the second and third topic appear similar, but they are different. The second topic focuses on trying out a problem and attending to students’ thinking while they work on it, whereas the third topic highlights anticipation of students’ thinking.

When considering the topics that emerge from the analysis of Roy’s communication, the first topic has “problem” as the most prevalent keyword. This often came up in the first two sessions, where Roy talked a lot about the problem and shared his thoughts about trying to understand the problem. A second topic involved the keyword “student,” along with keywords like “say” and “answer,” which indicates a focus on what the students might say and what answers they might get (anticipating). A third topic for Roy involved several action words, like “get,” “do,” and “find,” and these topics were prevalent in sessions 4–6.

For Ted, one topic involved top keywords like “right,” “think,” “find,” “do,” and “students.” This topic was most prevalent in sessions 1 and 4. Another topic had “find” as a top keyword, and it also involved keywords like “other”, “very”, “much”, and “get”. This was most prevalent in sessions 2 and 3. A third topic features “times” as one of the top keywords, and it was mostly used in talk about multiplication. Another featured keyword was “think,” and this topic was most prevalent in sessions 5 and 6. In session 5, these keywords were used when discussing a particular problem, and he reflected a lot about how students might think about this problem.

Altogether, there appeared to be three main themes—two that were prominent across all participants. All three participants talked a lot about the *problem* and about the *students*. Roy and Ted also talked a lot about *doing/finding* as a third topic. Differences in how the participants talked about these three topics can be seen already from the above interpretations of the topic models. These differences in communication are further investigated below.

Participants’ communication about key topics

Based on the analysis of key topics, additional analyses were conducted to learn more about how the participants communicated about these key topics. All the words used by participants were considered to create dictionaries, which included every word that could be related to each topic. For instance, the dictionary for the *student* category included words like student, students, class, fifth-grader, group, boys, and girls. These were all words that participants used when talking about students. A dictionary analysis then showed how much the participants talked about each category (Table 2).

Table 2: Frequencies (and relative frequencies) of words used across categories

Participant	Problem	Students	Doings
Reidar	304 (.010)	295 (.010)	602 (.020)
Roy	99 (.011)	68 (.008)	167 (.019)
Ted	140 (.007)	114 (.006)	383 (.020)

The dictionary analysis indicated that all participants talked more about *doings* than any of the other two topics. Roy used relatively more words that belonged to the *problem* category than Ted, whereas Ted used relatively fewer words about *students*. Both Ted and Roy talked more about the problem than about the students, whereas Reidar’s talk was more evenly distributed among the categories of problem and students. To learn more about differences in the way participants talked about these topics, keyword-in-context analysis was applied, with particular emphasis on identifying possible variations in the participants' talk about *students* and *problem*.

The kwic function in Quanteda was applied to analyze word use in a context of five words before and after the keywords "problem" and "student" (and their derivatives). These text strings were then analyzed with word-frequency analysis. One clear difference was the use of pronouns. For Reidar, the second most frequently occurring word in the context around "problem" was the pronoun "we." Much of his talk prompted teachers to reflect on the problem, and his frequent use of the word "we" prompted participants to reflect on the problem as a group. For Roy, the most frequently occurring pronouns were "they" (pointing to the students) and "I." Both pronouns were among his ten most frequently occurring words in the context around the keyword "problem." The word "they" often pointed to the students. For instance, he talked about finding a "method to solve a problem that they don't already know," and "if they don't understand the problem." His use of the pronoun "I" often involved reflections about how he had struggled or failed to understand the problem himself. Ted, on the other hand, rarely used the pronoun "I" when talking about the problem, but he frequently used the pronouns "they" and "we." He often reflected on what the students (they) would do and what they would or would not understand. When using the pronoun "we," it mostly referred to them as teachers and not the entire group as a community. In other words, Reidar's "we" included both teachers and researchers participating in the group, whereas Ted's "we" did not.

When talking about the problem, Roy relatively often used words like "method," "solve," and "algorithm." He often focused on finding a method or an algorithm to solve the problem, and he explained that the ability to use algorithms to solve problems was an aspect of mathematics that he had always liked. Ted, on the other hand, used words like "solve," "find," and "work" more frequently in this context, which points to the category of *doing/finding*. Reidar frequently used words like "students," "present," and "try." This involved talk about considering how the students might understand the problem, what to think about when presenting the problem (to the students), and things the group might consider trying when solving the problem, or when presenting the problem to students. Such perspectives did not come up when Roy and Ted talked about the problem.

Analysis of participants' word use in a context of five words before and after the keyword "student" also revealed some clear tendencies concerning use of pronouns. The most frequently used pronoun by Reidar was again "we," whereas Roy and Ted more frequently used the pronouns "they" and "I." Roy and Ted rarely used the pronoun "we" in this context, which indicates that they primarily refer to their own personal experiences and thinking about students, rather than reflections about how "we" as a group can consider and relate to the students and their thinking in these contexts.

After removal of stop words from the analysis, Reidar's most frequently used words were "get" and "give," and he also frequently used words like "various" and "experience." When looking more into these contexts, Reidar often talked about how to "get" students engaged, and how to "get" students to participate. The word "give" was used several times in the context of giving students ownership, and he often talked about giving students particular "experiences," which was another frequently used word in this context. Words like "various" or "different" were often used to highlight how students are different, but Reidar also used the word to highlight differences in student strategies, and different ways of facilitating students' experiences. When Roy and Ted talked about the students as "they," their talk often involved characterizations of students. For instance, Ted described students as "strong" or "weak," and he said that there are "lots of concepts that the students don't understand." Instead of focusing on what they as teachers could do to stimulate students' thinking and exploration, Roy and Ted often talked about what students would not be able to do or understand.

Concluding discussion

Whereas previous research has focused on how the views of participants can hinder successful professional development initiatives (Santagata et al., 2020), the present study focuses on the communication and word use of participants and investigates misaligned communication as a potential impediment for successful development efforts in mathematics education. The discussion below highlights two areas of misalignment in communication and one methodical perspective.

A first area of misaligned communication relates to how participants talked about the problem. *Partners in Practice* aims at developing partnerships between teachers and researchers to collaboratively explore what it might look like to develop teaching with problems (Lampert, 2001). All three of the target participants in this study talked a lot about "problem," but the analysis identified some differences in communication worth noticing. Whereas Reidar, as the facilitator, talked a lot about the problems in terms of opportunities for the group ("we") to collaboratively explore and try out things, Roy and Ted often turned their attention to the students' lack of motivation and ability as constraints to teaching with problems. In a partnership like this, it can be difficult to develop productive collaborative exploration of problems and the teaching with problems if the teachers stay in a talk about how students' lack of motivation and ability would prevent such efforts to succeed. This kind of misaligned communication must be identified and worked on for a professional development initiative to succeed.

A second area of misaligned communication is how participants talked about students. Attending to students' mathematical thinking and being responsive to it is emphasized in the research literature (e.g., Kavanagh et al., 2020), and Reidar's talk about students highlighted attending to students' thinking and the ideas and strategies that students might bring and considering ways of facilitating their learning and exploration. Roy and Ted, on the other hand, continued to talk about students as struggling, and they often used labels like "weak" or "strong" to describe students. If teachers maintain deficit views of students, it is difficult to facilitate productive investigations and problem solving, and this is also a potential misalignment in communication that needs to be identified and addressed in professional development.

The implications of this study are twofold. First, although perfect alignment of word use is not to be expected, this study implies that it is important to carefully attend to teacher educators' own communication and the correspondence with the communication of participants in professional development. Second, analyzing data from professional development initiatives where the researchers themselves are participating can be challenging, and there is always a danger of bias. In such a context, it can be useful to apply analysis techniques from computational text analysis, which provide a more objective perspective on the data. A benefit of applying techniques from computational text analysis is that they allow for hypothesis testing and use of statistical tests that for instance provide opportunities for exploring the significance of differences in word use between participants or groups. Another advantage is that computational text analysis works with large collections of texts, and Quanteda is developed to deal with large textual data (Benoit et al., 2018). Topic modeling, for instance, works best with large text corpora. This is contrary to qualitative analysis, which is more time consuming and more demanding when the datasets are larger. A possible disadvantage of computational text analysis lies in the “bag-of-words” approach, where data are typically considered as collections of words rather than sentences and paragraphs. It is therefore necessary to validate the quantitative findings by revisiting the textual data to check if the patterns identified through the quantitative analyses are meaningful. Future studies should consider exploring techniques from computational text analysis, but possibly in combination with more conventional and qualitative analytic approaches.

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How were parents in Norway informed about New Math?

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Parents' role in implementation of educational reforms is important. In this article, we study how parents in Norway were informed about New Math in 1967–1974. We have identified many initiatives to inform parents, including a TV series and a stand-alone TV program, courses for parents, books, and articles. In sum, the initiatives must have reached many parents. However, the initiatives did not have a consistent message about what New Math consisted of, why it was introduced and whether New Math included new content or just a new way of presenting mathematics. Such inconsistencies may have contributed to scepticism towards New Math among parents.

Keywords: Curriculum implementation, curriculum reform, educational history, mathematics education, parents.

Introduction

What make some educational reforms successful, while others have little impact? Comparative studies between countries on educational reforms of the past, might give some answers to this. Such comparative studies require detailed knowledge of educational reforms in each country, and this article is a contribution, by looking at one aspect of one educational reform in Norway.

When educational reforms are implemented, parents may play a significant role in supporting – or not – the reform efforts (Peressini, 1998). Parents are supposed to help the students with homework and to underscore the importance of schooling, but this task may be impeded by changes in content that make parents feel insecure. Informing parents may help. The introduction of New Math, which was based on research in mathematics and mathematics education, is an interesting case to study. New Math involved major changes in mathematics education, not least in the basic concepts and notations. After the second world war, there was a desire among mathematicians in many countries to include new mathematical developments into the school subject, particularly in upper secondary school. Considering the need to meet the rapid change in technological development, the academic content was considered to be outmoded (van der Blij et al., 1981). A change in emphasis was proposed: from developing skills in technology and computation to insight into mathematical structure, "often but not exclusively pursued by the study of abstract concepts such as sets, relations, algebraic structures, number bases other than 10, etc" (De Bock, 2023, p. 2). Two leading countries when it came to New Math were US and France (De Bock, 2023, p. 2). Other countries were afraid of being left behind in what they thought could be "a general 'revolution'" (van der Blij et al., 1981, p. 107).

In Norway, experiments with nine-year compulsory school (Grunnskolen), an expansion from the previous seven-year compulsory school (Folkeskolen), began in 1955. In 1960, an Experimental Plan was published. Also in 1960, the Nordic Committee for the Modernization of Mathematics Education (Nordiska kommittén för modernisering av matematikundervisningen) (NKMM) was appointed by the Nordic Council (Bjarnadóttir, 2023). Throughout the 1960s, experiments with New Math were carried out in classrooms in the Nordic countries. The textbooks used were partly produced by people associated with NKMM and partly translated US textbooks. In 1967, the NKMM's report was published, containing a proposal for a common curriculum for the Nordic countries from first grade to upper secondary school.

A temporary curriculum with two alternatives for mathematics – a traditional plan, and a plan with New Math – was valid from 1971 until the final curriculum (Mønsterplanen) came in 1974. In the final curriculum, set theory, which is what many associate with New Math, was mostly gone. This development contrasts with countries (for example Denmark) where New Math kept its position longer (Smestad & Opsal, 2023). To do comparative research on such reforms, detailed studies of various aspects of the reforms need to be performed in each country.

Therefore, we want to examine this research question: "How were parents in Norway informed about New Math 1967–1974?". We will look at both the means of information and the content of the information. We have chosen to study the period from 1967, when the NKMM's report was published, to 1974, when the new curriculum was introduced.

Previous research on information to parents on New Math in the Nordic countries

Parents' reactions and attitudes have been discussed by several researchers, for instance Bjarnadóttir (2011, 2012). Efforts to inform parents have received less attention. In Iceland, there were meetings and newspaper articles, including an infamous newspaper interview with two experts on New Math, Agnete Bundgaard and Karen Plum, documenting the advice that children should not bring their textbooks home, as their parents would tend to confuse them (Bjarnadóttir, 2011, p. 1673). On the other hand, Unenge (1978) mentions that study circles for parents were established in Sweden, independent of the regular school system (p. 224). Gjone (1985) gives detailed accounts of the planning, content, and reception of the Norwegian TV series "Alternative 2", which was a series for secondary school mathematics teachers.

Information to parents about New Math

Schools had a key role in informing parents about New Math. These efforts, which would mostly occur in regular meetings with parents, cannot be expected to have left written documentation. However, there were also significant national efforts at informing parents through different media. By means of historical methods, including searches in the Norwegian National Library and various archives, we have found the following initiatives. We will discuss all of these in this article:

- The TV programme "Mathematics in a new way" ("Matematikk på en ny måte"), broadcast by the state TV network NRK March 3, 1969.[1]
- The correspondence course "New Mathematics for parents" ("Ny matematikk for foreldre") from 1970 and the accompanying book (1972).

[1] A radio series with the same name were broadcast 1966–1967 but had teachers and teacher-students as intended audience.

- The TV series "Alternative 2. Modern mathematics in primary school" ("Alternativ 2. Moderne matematikk i barneskolen"), broadcast by NRK January 18–March 3, 1972.
- The book "What is modern mathematics?" ("Hva er moderne matematikk?") by Erling Myrmo (1971).
- An article on mathematics in an issue of Consumer Report ("Forbruker-rapporten") (1972).
- Two articles in *Norwegian school encyclopaedia for parents* ("Norsk skoleleksikon for foreldre") (1967).
- Two articles in the journal *The Children and We* ("Barna og vi") (1969–70).

In addition, there were some debates in newspapers. We will not discuss the debates here.

To answer the research question "How were parents in Norway informed about New Math 1967–1974?", we have analysed using the following categories:

- Who took the initiative, what were the initiatives, and what were their duration?
- What were the target groups (parents of which children)?
- How was New Math presented? Did authors stress continuity with or a break from previous curricula? And what reasons were presented for New Math?
- What mathematical topics were stressed in the initiatives?

The TV programme, 1969

The TV programme "Mathematics in a new way" (NRK, 1969) was broadcast only once and lasted 16 minutes. The state broadcaster, NRK, was the only television channel in Norway at the time. We do not know who took the initiative and who planned this programme. The programme was targeted at parents and explained that the new mathematics "appears to be radically different from the traditional mathematics." The programme presented an upper secondary school classroom, suggesting that the target audience would be parents of upper secondary school students. The main mathematical content of the programme was set theory and some logic. As an example, first one student used Venn diagrams to prove $C - (A - B) = (C - A) \cup (C \cap B)$, then another student proved this using set algebra (Figure 1).

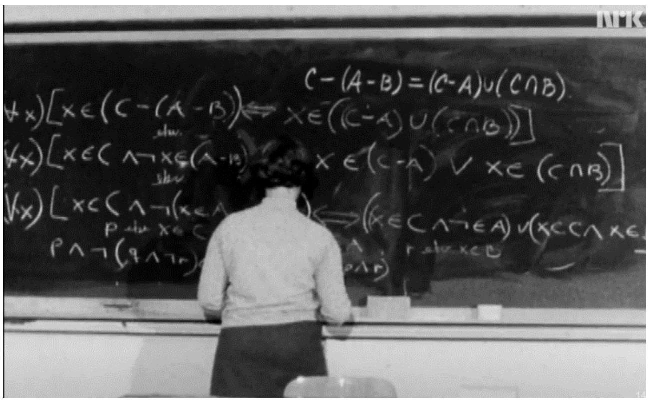

Figure 1. Screenshot from the TV programme "Mathematics in a new way".

Figure 2. Screenshot from "Alternative 2. Modern mathematics in primary school".

The correspondence courses, 1970–1982

From a search in Norwegian newspaper archives, we have found a course called “New mathematics for parents” mentioned in 65 different newspapers. The earliest mention was in Akershus Amtstidende August 31st, 1970, and the last one was in Aftenposten, January 19th, 1982. The course had three components:

- A book, also called “New mathematics for parents,” published in 1972, based on the Swedish book with the same name (Ny matematik för föräldrar), published in 1970.
- A series of tests that participants could send to the organizer for feedback.
- Local courses, which may or may not have used the book as material.

The local courses were often advertised in local newspapers, all over Norway. Sometimes, small newspaper notices give ideas of their popularity. The interest in the courses for parents in New Math, varied greatly. Rudberg (1992) wrote that in 1973, there were planned for 15 study groups in New Math in Hedemark, but only one course was started. On the other hand, a newspaper notice from 1973 noted that in the region of Nedre Romerike, “the course in New Math has got 225 participants, divided into 10 classes” (Romerikes blad, 1973, p. 4).

As we have no sources describing the content of these courses, we have looked at the textbook that some of these courses probably have used. The book, written by Gunnar Mjaaland and Karl Erik Sandvold (Mjaaland & Sandvold, 1972), had 183 pages. It targeted “parents who have children in school” (p. 4), mentioning both primary, lower secondary and upper secondary school (p. 5). The authors reassured parents that most of the mathematics was the same as before. The changes consisted of some things being expressed in a new way and by means of new symbols. There were also some new subjects in primary school, but according to the authors there was no reason to worry too much about this. The content of the book went far beyond set theory, including for instance functions, statistics and probability, and the use of slide rule.

Courses with the same name were given until 1982, long after the end of the period we discuss in this article. The content of the courses changed over time, however (Bergens Tidende, 1981, p. 3).

The TV series, 1972

The TV series “Alternative 2. Modern mathematics in primary school” (Grunnskolerådet og NRK, 1972) targeted parents of children in primary school (years 1–6). The development of the series was a collaboration between Primary School Council (Grunnskolerådet) and NRK (Gjone, 1985, Del V:28). It was broadcast from January to March 1972. We have found no sign that it was ever broadcast again. In the opening scenes, small children (aged 6–7) were sorting geometrical forms on the floor (see Figure 2), in a way that prepared for more mathematical work later. The series had six episodes: 1) Set theory; 2) How many are there?; 3) Understanding numbers; 4) How big is it?; 5) Statistics; 6) Summary. Each episode lasted from 23 to 30 minutes. (Episode 3 is not currently available on NRK’s website, and therefore has not been studied for this article.)

In the series, a new way of approaching mathematics was presented, emphasizing the use of concrete materials as for example set circles and Cuisenaire rods. In each of episodes 1, 2, 4 and 5, students from different classes worked with a mathematical theme in an inductive way. They were given a task/problem to solve in groups and used various kinds of concrete material to solve them. The teachers in the classes explained what the aim of the teaching was and how this differed from previous teaching. In the last episode, key figures in the introduction of New Math, for example general manager of the Primary School Council Erling Slaatto, and teachers who had tried this in their own teaching, were interviewed. Slaatto argued that the mathematics was not "new," but there was a new form of presentation and a new use of language. Also, there was an emphasis on inductive working methods instead of deductive ones which had been more common in traditional mathematics teaching. Those who were interviewed talked about parents being positive because the students were excited. "Even if the parents do not understand everything, they see that the students manage this." However, according to Slaatto, it was difficult for students at home when their parents did not understand what students were talking about. The teachers also found it challenging since they had no education in teaching New Math. Nevertheless, a teacher in the series responded "no" when asked if she would like to return to the "old" mathematics.

In the program magazine, under the heading "Mathematics can be fun" (Matematikk kan være morsomt, 1972), the program series was described. Here it was also highlighted that the mathematics was not new, but that the teaching methods and use of aids had changed. The teaching followed the inductive method, also known as "the discovery method," where students were to do experiments by using concrete materials.

Book: What is modern mathematics?, 1971

Myrmo's book "What is modern mathematics?" had the subtitle "An introduction for teachers and parents," and was thus explicitly targeting teachers and parents, although parents were not mentioned anywhere in the book itself. The first sentence in the preface stated that results from experiments and research from a lot of countries had shown a need for a major restructuring of the traditional mathematics subject, and that the new curriculum demanded that "teaching in primary school [must] include a number of new topics, concepts and symbols" (Myrmo, 1971, p. 3).

The chapters in the book were: Some set concepts; Statements; Union of sets; The number concept; Some geometry; More about statements; Ordered pairs; Numbers; More geometry; and Some statistics.

Article in special issue of *Consumer Report*, 1972

A major initiative to inform parents of the general ideas of the new curricula, was a special issue of the "Consumer Report" (Forbruker-rapporten) in 1972 (Hva blir den nye grunnskolen?, 1972). The content of this issue was planned and prepared in collaboration with the Ministry of Church and Education (Gulbrandsen, 1972). The circulation was 730,000, intended to reach every Norwegian household.

The special issue included a five-page article on "And this is how the subjects will be" ("Og slik blir de enkelte fag"). Less than one page was devoted to mathematics, some of which was dedicated to explaining the ideas behind individualistic teaching with self-instructing and self-correcting materials. Some reasons for New Math were given: the development within technical subjects, new pedagogical insights and that more students would probably more easily follow the new plan.

However, it was stressed that the two alternatives had many things in common. The only mathematical topics mentioned were set theory and logic, which "has become fundamental in the subject" (p. 14).

Articles in *Norwegian school encyclopaedia for parents*, 1967

The 1967 book "Norwegian school encyclopaedia for parents" edited by Einar Ness, included an article on Mathematics in the nine-year school by Asbjørn Ryen and Th. Faarlund (1967) as well as an article on "Modern Mathematics" ("Moderne matematikk") by Ragnar Solvang (1967). The first article described the mathematics content of the 1964 plans and concluded that several topics had been moved to earlier ages. However, New Math was not mentioned, except that in the future, mathematical logic, set theory and vectors would be more prominent, while number processing and practical calculation would be less prominent. They claimed that mathematics would face a major transformation (p. 187). Solvang's article, however, described the research efforts in the US and Europe as well as the Nordic committee. Solvang stressed that what was novel was the way of presenting mathematics, with concepts from set theory and mathematical logic. He did not explain the concepts but stressed strong reactions to New Math's extensive use of symbols and concepts. He claimed that the experimental teaching did not show any negative impact on the teaching by the extensive use of symbols and concepts. On the contrary, students liked precise representations and it was easy for student to learn (p. 195).

Articles in *The Children and We*, 1969–1970

"The Children and We" (Barna og vi) was a journal with the subtitle "for collaboration between school and parents" (for samarbeid mellom skole og foreldre). In an article about homework, Sivertsen (1969) mentioned the problems parents could have when the content of a school subject was changed. As an example, she highlighted the terminology in New Math which would be unfamiliar to most parents. She noted that parents had created study circles and learned the new subject content so that they could keep up with their children's schoolwork. In the same journal, Dahr (1970) referred from a course for parents in New Math in Oslo. The participants were mostly mothers who were learning the "new" mathematics; they solved equations using the number line and drew set diagrams.

Discussion and conclusion

The efforts to inform parents about New Math in Norway in the period 1967–1974 included TV programmes (one standalone programme and a series of six episodes), a correspondence course with an accompanying book, many local courses, four articles in a magazine, and a book targeting parents. While it is impossible to ascertain the impact and reach of the individual initiatives, in total, most parents must have seen the special issue of *Consumer Report*, followed the TV series or attended courses. Still more would have become aware of the changes by seeing these initiatives mentioned in newspapers.

The target audiences varied. The 1969 TV programme clearly targeted upper secondary school, while the 1972 TV series targeted primary school. The other initiatives studied did not seem to target a particular group of parents.

The Consumer Report special issue gave three reasons for the introduction of New Math: The need for new competences to answer the demands from technical subjects, new pedagogical insights, and that research and experience showed that students found New Math easier. We can find these three reasons in some of the other initiatives as well, but often, just one was mentioned.

Most of the initiatives seemed to stress that New Math was much the same as the traditional mathematics, except that the form of expression was different. This message was not consistent, though, as words like "radically different" and "new topics" were also mentioned. Set theory and logic was stressed as new topics in some of the initiatives (The 1969 TV programme, the articles in the Consumer Report and in the Norwegian school encyclopaedia), while broader views of New Math were given in the books and the 1972 TV series. Thus, parents could get quite different understandings of New Math depending on which source they were informed by. This could, in turn, result in different opinions on the reform.

The idea that parents should not see the children's textbooks, to avoid confusing them, is nowhere to be found in the Norwegian sources, in contrast to Iceland. The initiatives we have listed, show a sustained commitment to inform parents about the changes that were occurring. The initiatives were not consistent and failed to secure support from parents.

New Math was an effort to base curricula on new research. In Norway, it reached the temporary curriculum and was supported by many initiatives to inform teachers and parents. These initiatives were often partly organized by the government. Still, New Math did not last. To be able to understand why New Math had different trajectories in different countries, detailed research on many aspects is needed. Understanding how parents were informed forms one piece in such a puzzle.

Detailed comparative research will be needed to see whether the quality and consistency of information to parents could help explain the different trajectories of New Math between countries. Such a finding would be of interest also today, for instance when programming is reintroduced into mathematics teaching, but with unclear goals.

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Pre-service teachers’ experiences of working with learner-generated example tasks through a computer-aided assessment system

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This study reports from 21 pre-service teachers’ (PST) experiences of working with learner-generated example tasks for the first time, through a computer-aided assessment system. Three PSTs volunteered to a follow-up interview after engaging with the digital tasks. Results indicate that most PSTs found these tasks difficult and struggled to grasp the new approach. Additionally, the one PST who solved all tasks correctly also was the only one that enjoyed the new working strategy. This suggests that new working methods can be overwhelming for PSTs, and that it takes time to engage and implement mathematical tasks designed in a new way.

Keywords: Learner-generated examples, pre-service teachers, computer-aided assessment

Introduction

In the recent decades, there has been an ongoing discussion about how to make mathematics education at all levels more student centred (Freeman et al., 2014). One aspect of this is to provide exercises where students need to be creative, rather than just practicing standard methods. A type of task that offers this approach is learner-generated examples (LGE), where learners are asked to construct their own examples (Watson & Mason, 2005). We adopted this approach and presented a series of LGE-tasks to a group of pre-service teachers (PSTs). Each task required an answer in the form of an example that satisfied some given properties (see Table 1). Watson and Mason (2005) argued that engaging PSTs in LGE-tasks help them deepen, enhance, and expand their grasp of mathematical structures, concepts, and interrelations between topics. Thus, it is notable that students in secondary school, and even undergraduate students at tertiary level, rarely work with tasks where they construct own examples (Wagner et al., 2017; Watson & Mason, 2005). An analysis of the exercises in a first-year undergraduate mathematics course, showed that only 2.4 % of the exercises dealt with constructing examples (Pointon & Sangwin, 2003). This supports the claim that constructing examples is not central in mathematics courses. Consequently, students get few opportunities to practice and hence to develop their skills in example constructing. These assertions caught our interest in how students, especially PSTs, work with LGE-tasks.

LGE-tasks encourage the learner to explore multiple solutions. This exploration promotes learning as students engage in decision-making and adopt the thought process commonly employed by mathematicians (Breen et al., 2016). Further, practicing providing examples is important for PSTs, since giving examples will be necessary in their future work, when their students ask for clarifications. This competence is often overlooked in teacher education programmes (Zodic & Zaslavsky, 2008).

Recognising the potential of LGE-tasks tasks, Sangwin (2019) proposed CAA systems as a powerful tool for assessing such tasks. CAA is a common notion for technology that have opportunities to offer automated assessment of student responses (Fahlgren & Brunström, 2023), by applying algebraic equivalence. This is beneficial for assessing LGE-tasks, since a correct response might be any expression that fulfils the requirements. STACK (a System for Teaching and Assessment using a Computer algebra Kernel) is regarded a CAA system and will be utilised in this study.

This paper is an extension of a poster presentation at CERME 13 (Ovedal-Hakestad & Larson, 2023), addressing the following research question: What are PSTs' initial experiences of working with learner-generated example tasks in the computer-aided assessment system STACK?

Literature review

Exercises with learner-generated examples can take various forms, for example, generating and using examples in a proving process (Sandefur et al., 2013), or a series of tasks as 'write down two numbers that differ by 3', 'give another pair', 'give a pair that no one else in the student group will suggest', 'describe the class of these numbers' (Watson & Mason, 2005). The example with a series of task will both enable students' creativity and aim for a general conclusion, where algebra might be a valuable tool. This paper builds on such series of tasks (Table 1 shows the version used in phase two of this study).

Exercises employing LGE can be implemented both with 'pen and paper' (e.g. Breen et al., 2016; Wagner et al., 2017) and with digital tools including an online assessment system (Fahlgren & Brunström, 2023; Kinnear et al., 2022). A strength with using computer-aided assessment (CAA) is its ability to check any answer given in the right form by applying algebraic equivalence. Tests with pen and paper must be checked manually, and if 'solutions' are given to the students 'on paper', only examples of correct answers or a description of properties of the correct answers can be provided. When students work in a system utilising CAA, immediate feedback, or hints of how to solve a task, can be provided (Ovedal-Hakestad, 2023). The most common feedback types studied are knowledge of results, knowledge of correct response, and elaborated feedback (Attali & van der Kleij, 2017). Feedback of the type 'knowledge of results' only reveals if an answer is correct or not. The type 'knowledge of correct response' presents the correct answer, while 'elaborated feedback' offers extra information like hints or examples. However, for a series of similar tasks, detailed feedback risks to decimate the learning potential of the subsequent tasks.

There are several arguments that LGE might enhance students' learning (Watson & Mason, 2005), such as "greater appreciation of mathematical concepts" and "development of a sense of agency in self-determining correctness" (Wagner et al., 2017, p. 214). Generating examples may also serve as a lens on students' understanding of a concept (Zazkis & Leikin, 2008). However, Iannone et al. (2011) found no evidence that LGE should be beneficial for students' learning. In addition, Breen et al. (2016) reported on students describing LGE-tasks as "hard, tricky and even overwhelming" (p. 34), and concluded students' lack of experience with this type of tasks was likely to mainly explain their negative impression of what they referred to as 'the backward ones' or 'working backwards'. Further, Wagner et al. (2017) reported on students initially being negative and struggling with LGE-tasks. Nonetheless, a vast majority of the students in the study by Wagner et al. (2017) expressed a positive change of their views of LGE by the end of the teaching experiment. Moreover, Breen et al. (2016) declared "students' reflections on their own learning through the unfamiliar tasks were generally positive" (p. 37). This strengthens the claim that a lack of experience is likely to hinder students learning from LGE, and hence that teachers need to introduce students to this type of tasks, so that the students gradually can develop their skill in solving LGE-tasks.

Method

Aiming at exploring how PSTs view working with LGE-tasks in STACK for the first time, we created a series of LGE-tasks in STACK. A series of tasks will hereafter be referred to as a quiz. The study took place at a university in Norway, involving PSTs (for grades 1–7) during their second semester of mathematics in teacher education. These PSTs had completed quizzes in STACK prior to this study (Ovedal-Hakestad, 2023). The new component in the current quiz, was working with LGE-tasks. The STACK-quiz was carried out in two phases. In the first phase, a group of seven PSTs volunteered to test this quiz outside of regular classes.

Following the first quiz, the first author had an informal talk with the participants. Based on feedback from phase one, we made two changes in the quiz, resulting in the quiz presented in Table 1. First, we added four tasks in the beginning of the quiz, where the PSTs were asked to give an example of two numbers whose sum was equal to 4 and 2, respectively. The reason was to get the PSTs 'into' an LGE-based format by some possibly simpler tasks, before giving tasks about a linear function. Second, the point the straight lines should go through (e.g. task 5 below) was changed from (3, 2) to (2, 1), which could make some calculations easier. In addition, the PSTs in phase two were given oral information about the task design before the quiz started, to prepare them for this new type of tasks. Table 1 provides an overview of the tasks presented to the PSTs in phase two.

1	Give an example of two numbers whose sum is 4, that is, $x + y = 4$.
2	Give another example of two numbers whose sum is 4, that is, $x + y = 4$.
3	Give an example of two numbers whose sum is 2, that is, $x + y = 2$.
4	Give another example of two numbers whose sum is 2, and that you think no one else in the class will give. That is, $x + y = 2$.
5	Give an example of a straight line that goes through the point (2, 1).
6	Give another example of a straight line that goes through the point (2, 1).
7	Give another example of a straight line that goes through the point (2, 1) and is horizontal.
8	Give another example of a straight line that goes through the origin (0, 0) and the point (2, 1).
9	Give another example of a straight line that goes through the point (2, 1) and has negative rate of increase.
10	Give another example of a straight line that goes through (2, 1) and you think nobody else in your class will give.
11	What is different/similar for all straight lines you have given. Can you generalise? Explain.

Table 1: The LGE-inspired tasks given in phase two

The whole class was invited to complete the quiz in phase two, and 14 PSTs volunteered. Both quizzes were close to their final exam, which may explain that only 7 + 14 out of 83 PSTs chose to participate in either phase one or two. All 21 participants were in the age of 19–25 years. During both phases, the first author was present when the PSTs completed the quiz.

Following phase two of the quiz, the first author conducted interviews with all three PSTs that volunteered to be interviewed. They had participated in phase two of the quiz, but not in phase one, and are hereafter called Alan, Beth, and Cara. The semi-structured interviews aimed at capturing the interviewees' experiences of their first interaction with LGE-tasks in STACK. The questions guiding the interviews focused on identifying initial thoughts, the overall sentiment (positive or negative) regarding engagement with the quiz, and their perception of the feedback provided in STACK.

The first author transcribed the interviews, followed by both authors making an individual examination of the transcripts, to foster an in-depth understanding of the content. Subsequently, they conducted a collaborative analysis, focusing on expressions and themes related to "immediate reaction", "positive or negative experiences", "procedures/strategies in use", and "feedback", in line with the interview guide. This analysis aimed to identify relevant episodes relating to the PSTs' experiences, aligning with the research question of this paper.

Results

After the informal talk that followed the completion of the STACK-quiz in phase one, it became evident that the PSTs were not only discouraged and feeling hopeless about the tasks, but also rarely managed to give a correct answer. Some PSTs explained they had no strategies when they met tasks that were unfamiliar to them. This affected the whole quiz, because if you could not respond to the first task, also the remaining tasks were hard to answer. Further, most PSTs in this group wanted STACK to provide the correct solution immediately after submitting their answer, as they had experienced in the previous semester's STACK-quizzes. However, the tasks in the current quiz have no unique correct answer and, furthermore, providing one possible answer would undermine the subsequent tasks. Hence, the instant feedback in this quiz only revealed if the answer was right or wrong, without further explanations. Some of the PSTs mentioned this made them insecure about the approach to the tasks, and they felt discouraged and unmotivated for doing more mathematics.

We continue with results from the follow-up interviews with three of the PSTs participating in phase two, focusing on their views of tasks 5–11 that concerned functions (see Table 1). The tasks asked the PSTs to give examples of straight lines that go through the point (2, 1). Beth and Cara struggled with these tasks, getting few answers correct. Beth confirmed in the interview that she found the tasks challenging. Her immediate reaction was "I had no idea what to do, how to start working with the tasks. I had no clue …". She also explained she tried different strategies to try to solve the tasks, although she neither knew what she was doing nor had any objective of choosing these particular strategies. Cara had a similar experience and said: "I don't know (…) I don't know what I thought here. I actually don't know what I thought on the other tasks either." Both Beth and Cara explained they lack strategies when they face non-standard tasks. Beth stated that, in general, giving own examples was difficult: "And when I was going to create it myself, it was a completely different way of working. I have never done it this way before." She said that this reminded her of working backwards. She explained: "I thought the subject was hard initially, and now we had to solve the tasks in a completely different way."

Further, Cara said that she experienced the straight line-tasks to be worse than the STACK-quizzes she completed the previous semester. She wanted the tasks to contain more information and commented that in standard tasks you can figure out what to do, and how to solve the task, by reading the information given, combined with some mathematical knowledge. Cara also expressed "I struggled to figure out what the tasks were asking (…) because when I am used to work with functions, I create a table (…) and I put x and y into the table", which indicates she felt that she did not understand what was asked of her. One strategy she applied was to try to draw the graphs. However, this did not in any way help her to solve the tasks, because she did not know what to draw more than a coordinate system and the point (2, 1).

In contrast to Beth and Cara, as well as to all the other four PSTs participating in phase two, Alan enjoyed working with LGE-tasks and got every answer correct. He said that he particularly liked those tasks where you should give an example that no one else in the class thought about, because this forced him to think on his own. Alan explained: "instead of those learnt processes where you sort of go through what you have memorised, what you remember, here you actually have to think in order to get the correct answer". He also said that this type of tasks was exciting and depending on your prior knowledge you could choose level of difficulty for your answer. One strategy Alan applied was to draw graphs on his note sheet to structure the information and to double check his solution.

For reasons mentioned, no changes were made regarding feedback to the students after submitting their answer, despite the frustration among the seven PSTs participating in phase one of the quiz. In contrast to these reactions, Beth explained that she appreciated the immediate feedback of right or wrong. When it showed her answer was wrong, she realised she had to change her strategy of solving the tasks. Because she gave the wrong answer several times, this resulted in her trying several approaches to the tasks, even though she explained that she had no strategy behind choosing a new strategy. However, repeatedly getting wrong answers eventually influenced Beth's motivation negatively. She said she did not like to spend time on things you cannot manage and thus easily gave up when facing challenges in mathematics, as in this quiz.

Discussion

The aim of this study was to explore how PSTs initially experienced working with LGE-tasks in the CAA-system STACK. The results revealed that the PSTs had limited experience and lacked strategies for working with LGE-tasks, which led to frustration and decreased motivation for doing mathematics. Our findings align with previous research that suggests a lack of experience with LGE-tasks can be demotivating and hinder students' learning, while working with such tasks (Breen et al., 2016; Wagner et al., 2017).

It is well known that learning new concepts and solution strategies in mathematics takes time. It is likely that this also applies to unfamiliar working methods and new types of tasks, even in cases where the student actually is familiar with the topic. This might explain why so many PSTs in this study perceived LGE-tasks as challenging. It was their first exercise with LGE, and they did not get enough time to develop understanding of this new type of task, unlike the students who after an eight-week teaching experiment with LGE expressed more positive experiences (Wagner et al., 2017). In addition, it is possible that the participants in this study found the topic of linear functions to be challenging, although they had met it in secondary school. Nevertheless, independent of their pre-knowledge of the topic, it is reasonable to assume that the new type of task was perceived as demanding. The PSTs in this study expressed a preference for tasks that provided a clear, step-by-step process or a single correct answer. They struggled with the open-ended nature of the LGE-tasks, which required them to think critically and be creative. This finding is consistent with previous research that reports students may find LGE-tasks challenging and overwhelming (Breen et al., 2016), which, though, can be overcome by working with LGE-tasks continuously (Wagner et al., 2017).

Breen et al. (2016) reported that students said tasks dealing with generating examples reminded them of working backwards. The notion of 'working backwards' was also pointed out by Beth in this study. This association highlights the contrast between standard mathematical tasks that students are used to and the LGE-tasks in this study. Building on this contrast, Alan's experience with LGE-tasks has a different perspective. Unlike the rest of the group, he found these tasks more engaging, valued the opportunity to create his own examples and express creativity within mathematics, and appreciated the chance to think independently rather than merely applying previously learnt procedures. However, as our current study also indicates, for most students, it is not enough to deal with such tasks only occasionally and without a specific purpose (cf. Wagner et al., 2017). Thus, it should be stressed that generating examples is an essential part of PSTs' future profession, and consequently a critical skill that every PST must develop and practice. Furthermore, getting used to LGE-tasks might also entail the students becoming more positive to such tasks (Wagner et al., 2017). Hence, we argue that working with LGE-tasks is important for PSTs, although the study by Iannone et al. (2011) claimed students' construction of own examples did not lead to enhanced learning of mathematics.

The PSTs from phase one, as well as Beth and Cara, made several mistakes in the quiz, which, as mentioned, led to frustration and hopelessness. They expressed the experience of getting multiple wrong answers in the same quiz as demotivating. Although CAA enables giving immediate feedback to the students, there are several ways of how this feedback can be provided (Attali & van der Kleij, 2017), and how to instruct or 'programme' the CAA. In this quiz, we decided to give feedback as 'knowledge of results', i.e. only to show if the PST's response was right or wrong. We concluded that for these LGE-tasks, where the subsequent subtask asked for a new example with similar properties, providing feedback as 'knowledge of correct response' or 'elaborated feedback' would make the following task essentially meaningless. In addition, feedback as knowledge of results has the advantage of encouraging the learner to reconsider their chosen strategy (Attali & van der Kleij, 2017), as Beth reported she did when she got the feedback her answer was wrong. On the other hand, if we could provide elaborated feedback that does not undermine the following subtasks, maybe more students would experience this LGE-exercise as more manageable than the participants of this study did. This highlights both the issue of what kind of feedback that enhances students' learning, and the difficulty of 'programming' CAA to give proper feedback on non-standard tasks.

Integrating CAA with LGE-tasks is a possible way to enhance the advantages digital technology offers, particularly in evaluating students' work. The tasks are easy to design, and CAA is able to assess all examples students submit as answers. In line with the objective of our study, we claim that combining CAA with LGE-tasks offers students autonomy in selecting their level of challenge, as exemplified by Alan. In addition, since LGE-tasks can serve as a lens on mathematical understanding of concepts (Zazkis & Leikin, 2008), they raise awareness of students' knowledge level, while CAA contributes by offering an opportunity to reduce teachers' workload. Another approach is to highlight some students' answers in a whole-class discussion. This will give the teacher, and other students, an immediate possibility to evaluate the suggested answer, and also entail that students can learn by listening to their peers' answers. Additionally, working with LGE-tasks in a whole-class setting might be a way for students to become familiar with such tasks, and hence reduce an issue mentioned by the participants in this study as well as in previous research (Breen et al., 2016; Wagner et al., 2017). Thus, supplementing LGE-tasks in STACK with whole-class discussions could be beneficial.

Conclusions

Our research question was: What are PSTs' initial experiences of working with learner-generated example tasks in the computer-aided assessment system STACK? It is clear that the PSTs were not familiar with working on LGE-tasks. Introducing a new method to the PSTs resulted in frustration and decreasing motivation for doing mathematics. Many of the PSTs, from both phases of the intervention, commented that they did not have strategies to apply when they faced new challenges in mathematics. They also expressed they were used to getting the one right answer after completing a task, which here led to insecurity, since this was not provided. However, Alan, who understood the tasks, experienced joy of creating own examples and saw this as an opportunity to challenge himself.

Our results show that PSTs' initial experiences with LGE-tasks vary. This study supports earlier findings (Breen et al., 2016; Wagner et al., 2017) that integrating LGE-tasks to students' mathematical repertoire is challenging. To be successful, it is essential to prepare the PSTs for the strategy, that they practice LGE regularly and to stress that generating examples is an important competence for teachers. Further, despite the report from Iannone et al. (2011), it is not unlikely that working with LGE-tasks at an appropriate level, so they are perceived as challenging yet manageable, can enhance students' learning. Hence, for future studies, we find it relevant to explore how LGE-tasks and the feedback offered can be designed to optimise students' learning, synchronous with their role of practicing the skill of providing examples.

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Semiotic registers as access to students' concept images being part of their mathematical thinking competency: The case of differentiability

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The mathematical thinking competency of the Danish competency framework (KOM) has shown challenging to integrate in the practice of the teaching and learning of mathematics. This paper argues that using the perspective of semiotic registers as a theoretical lens assisting to identify students' concept images of differentiability can make aspects of the mathematical thinking competency a more visible part of students' work. We provide and analyse an empirical example to illustrate in what ways the perspective of semiotic registers can specify students' concept images. The notion of conversions between registers can work as a tool to identify the relation between a student's concept definitions and concept images, offering a bigger picture of the student's concept image as a total cognitive structure being part of the students' mathematical thinking competency.

Keywords: Semiotic registers, concept definition and concept image, differentiability, mathematical thinking competency

Introduction

The Danish competency framework, so-called KOM, has had a profound influence on the curriculum of Danish upper secondary school. KOM describes mathematical mastery in eight distinct, yet mutually interwoven competencies. One of them is the mathematical thinking competency which concerns *engaging in mathematical enquiry*. KOM is of a cognitive nature and focuses on physical and mental actions carried out by the individual. The cognitive actions of mathematical thinking competency involve relating and posing questions and answers characteristic to mathematics; distinguishing between different types of mathematical statements and claims, such as definitions and if-then, universal and existence claims; navigating with logical connectives and quantifiers in these statements; and relating to a mathematical concept's varying scope within different contexts (Niss & Højgaard, 2019). Hence, the mathematical thinking competency involves students' articulated, but not least unarticulated hypotheses about the involved concepts and their properties and relations. The mathematical thinking competency seems more subtle to express and assess in the work of teachers and students (Lindenskov & Jankvist, 2015; Pedersen, 2023). With KOM's wide-ranging purpose and influence in Denmark, it is interesting to examine on a local level, how mathematical thinking competency is expressed in practice as part of the students' mathematical work. In practice, mathematical competencies cannot be put into action without mathematical subject matter (Niss & Højgaard, 2019). The curricular area of differentiability is particularly interesting for upper secondary school mathematics as an entrance to more advanced calculus. It builds heavily on both algebraic and visual registers and ways of thinking as well as the capabilities and types of support that computer algebra systems (CAS) can provide. To achieve deeper insight into the individual's actions and thoughts behind the students' exercise of mathematical competencies, it can be useful to apply other theoretical approaches of mathematics education research (Niss & Jankvist, 2022).

Students' conceptualisation of differential calculus and its related concepts such as continuity, limit and derivative are topics well researched in mathematics education related to the distinction of concept definition and concept image (e.g., Tall & Vinner, 1981). This distinction has a crucial role for the processes of advanced mathematical thinking, since concept images inspire one's thinking, and concept definitions and logical deduction formalise it (Tall, 1995). Yet, the mental process of establishing concept images is difficult to describe and analyse. An important element of mathematical thinking is the access to the mathematical objects that mathematicians have through representations (Duval, 2006, 2017; Tall, 1995). Therefore, analysing students' work with mathematical representations can provide access to their concept images. The purpose of this paper is to illustrate that the two theoretical perspectives of concept definition and image and semiotic registers can aid in elaborating the underlying actions and thoughts of students' mathematical thinking competency. We address the interplay of the theoretical approaches by the questions: *In which ways can the perspective of semiotic registers as a theoretical lens assist to identify students' concept images of differentiability? And what may this add to the perception of students' mathematical thinking competency in practice?* We approach this question with the analysis of an empirical example of two Danish students working with tasks on differentiability and non-differentiability using the mathematics software TI-nspire. In Denmark, CAS are well integrated in upper secondary school and exams. Therefore, CAS are also part of the students' concept formation in the empirical example, but not of importance as such for our answer to the questions above.

The theoretical perspectives of concept image and semiotic representations

We aim to combine the two theoretical perspectives to elaborate on students' mathematical thinking competency. Semiotic registers and concept definition-concept image address cognition, which is in line with KOM. Moreover, the two frameworks stress the importance of mental imagery for thinking but through transformations of mathematical representations respectively the relations between mental images and formal definitions. The focus on transformations of representations within and between semiotic registers provides a tool to analyse the cognitive processes of mathematical thinking (Bach et al., 2021; Duval 2017). Semiotic registers may add to the terminology of concept definition and concept image, contributing with a perspective on students' mathematical thinking competency. Thus, there are no indications that the three frameworks conflict and combining these seems possible.

The cognitive processes of getting insight and making mistakes when learning mathematics can be approached by the distinction between concept definition and concept image (Tall & Vinner, 1981; Vinner, 1983). A *concept definition* is a phrasing in words specifying a concept and can be formal or personal. The formal is the definition accepted by the mathematical community, and the personal is an articulation of the individual's own explanation of the concept. The *concept image*, however, is "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). The mathematical thinking competency concerns students' work with concepts and the associated properties, relations, conditions, and contexts. Handling mathematical concepts requires concept images and in thinking these concept images are often evoked (Vinner, 1983). An entire concept image is not necessarily coherent, and various parts of a concept image can be evoked in different situations, referred to as the *evoked concept image*. Various parts of the concept image, the personal or formal concept definition can conflict with one another. If two conflicting parts of the concept image or concept definitions are evoked at the same time, a cognitive conflict can emerge, and the learner need to adjust those parts of the concept image to restore balance. An important part of an individual's concept images is the mental picture, which "is the set of all pictures that have ever been associated with [the given concept] in the [person]'s mind" (Vinner, 1983, p. 293). External mathematical representations are essential to create the mental picture. Through external representations the individual gets experience with the mathematical object they represent, which then becomes part of our concept image (Tall, 1995). Duval (2006) does not distinguish as strictly between internal and external representations, as "mental representations that are useful or pertinent in mathematics are always interiorized semiotic representations" (p. 126). We consider Duval's mental representations constituting Vinner's mental picture, which is part of the concept image.

Duval (2017) focuses on understanding the difficulties of learning mathematics from the perspective that in mathematics we can only access the objects through semiotic representations. Semiotic representations of mathematics can be distinguished in two ways, dividing them into four registers in a two times two matrix (Duval, 2006, 2017), also termed the linguistic, symbolic, figurative and graphic register (Pedersen et al., 2021). The first distinction is between *discursive* and *non-discursive* representations. The former are of written or spoken form, such as language and symbols, whereas the latter are visual, like figures and graphs. The other distinction is between mono- and multifunctional representations. *Monofunctional representations*, namely symbols and graphs, are algorithmic and follow specific rules and formalism, whereas *multifunctional representations*, such as language and figures, cannot be thought of in terms of algorithms. Treatments within one register (e.g., symbol manipulation) and conversions between registers (e.g., from a symbolic expression to its corresponding graph) are key to mathematical activity and the mathematical way of thinking. The notions involved in Duval's theory of registers of semiotic representations constitute a theoretical framework for analyzing the cognitive processes of developing mathematical thinking. Hence, analyzing students' treatments and conversions between semiotic registers can provide access to the students' mental pictures and the associated properties and processes, thus their concept images, which offers a terminology elaborating on the mathematical thinking competency.

Method and case selection

To illustrate how the two perspectives of concept definition-concept image and semiotic registers can aid in elaborating on the mathematical thinking competency we analyse two students' work with different representations of differentiable and non-differentiable functions. We analyse what representations the students are working with and the treatments and conversions between the different semiotic registers they are carrying out to identify the students' evoked concept images, the cognitive processes related to differentiability and how these are connected as part of their concept image as a whole. Thereby, we gain insights into the students' mathematical thinking competency and their view on the involved concepts and what they take as necessary and sufficient conditions.

The example stems from an empirical study carried out in Danish upper secondary school, grade 11 (age 16-17). The presented students work on the final tasks of a two-lesson sequence on differentiability, each of 90 minutes duration. All students were divided into groups of two or three sharing one computer. Their actions on and with the computer were recorded on screen, including audio and webcam to capture who is talking as well as potential hand gesticulations. Out of the total number of 14 groups, six groups managed to work on the final set of tasks. The case provides an illustrative example of the connection between working with semiotic representations and the students' concept images and was selected out of the six, due to these students' thorough work with representations. The students explicitly referred and made use of representations from the linguistic, the symbolic and the graphic registers. However, the analysis of their work illustrates a persistent graphical view on the concept of differentiability. Hence, the case indicates a potential to dig into the details of the students' work with representations and their related concept images of differentiability.

Identifying students' concept images through semiotic registers

Two students, Amy and Beca, are working on three tasks. First, they are to explain in their own words when a function is differentiable and when it is not. Afterwards, they should construct an example of a function that is not differentiable, and finally a function that is continuous but not differentiable.

For a function to be differentiable, the students write the following explanation: *When all points on the graph are differentiable – which is, when* Δx *goes towards 0 from both plus and minus, the derivative / tangent slope should be the same*. For a function not to be differentiable, they give the explanation: *If it does not satisfy* $a_s \rightarrow A$ *for* $\Delta x \rightarrow 0$ [a shortened version of the definition of differentiability, where a_s is the different quotient and A is the derivative]. *That is, if one point does not have the same derivative for minus and plus going towards 0*. These two written explanations are given in the linguistic register representing the students' personal concept definitions of differentiability and non-differentiability. The students interpret the concept of differentiability and the symbolic representation $a_s \rightarrow A$ *for* $\Delta x \rightarrow 0$ in terms of graphic representations, such as 'point(s)', 'graph' and 'tangent slope'. This indicates that their personal concept definition is inspired from concept images building on representations from the graphic register. Also, their evoked concept images related to constructing examples of non-differentiability build on graphic representations.

To construct a non-differentiable function, the students make a conversion from their interpretation of the "one point" of non-differentiability in the linguistic register to the symbolic register by using the split function template in TI-*n*spire to obtain a graphic representation of the function, converted by TI-*n*spire. To define the split domain of the function, the students make use of inequality signs.

1	Beca:	$4x + 5$ is from, when x is…
2	Amy:	No, we can write like –5 for instance, and then such equal to stuff.
3	Beca:	No, because if we do not write equal to, then there will be, where they do not meet.
4	Amy:	But, here it is okay, if they meet. Then we shall use those signs [≤ and ≥], x shall be larger than –5, right? But, it is also allowed to be equal to –5, so I think that one [≤]. Can it not be 5, or… it can be –5 or larger than –5. And, then is should be less than…
5	Beca:	Less than 0 [such that the first part of the function is defined as $4x+5$ for $-5 \leq x < 0$].
6	Amy:	That is okay. Then 0 does not go into that interval.

The two students end up with the function defined by

$$f_1(x) = \begin{cases} 4x + 5, -5 \leq\ x < 0 \\ 7x^2 - 3, 0 < x \leq 9 \end{cases}$$

The function, f_1, is not defined for $x = 0$ and has the associated graph depicted in Figure 1 (left).

7	Beca:	Then it is not differentiable, because of that there [points to the screen].
8	Amy:	But it is a bit annoying.
9	Beca:	Mmm… it is, but I suppose it is not differentiable.
10	Amy:	But can you also see it, because there is that jump.
11	Beca:	Yes, that one there.

12	Amy:	Hmm, couldn't we… Yes, we could just… So, for 0 we could just show that it is not defined if it isn't. It shouldn't be, right? [Amy types in the command to find the derivative for $x = 0$ in the CAS, TI-*n*spire, see Figure 1 (right)]

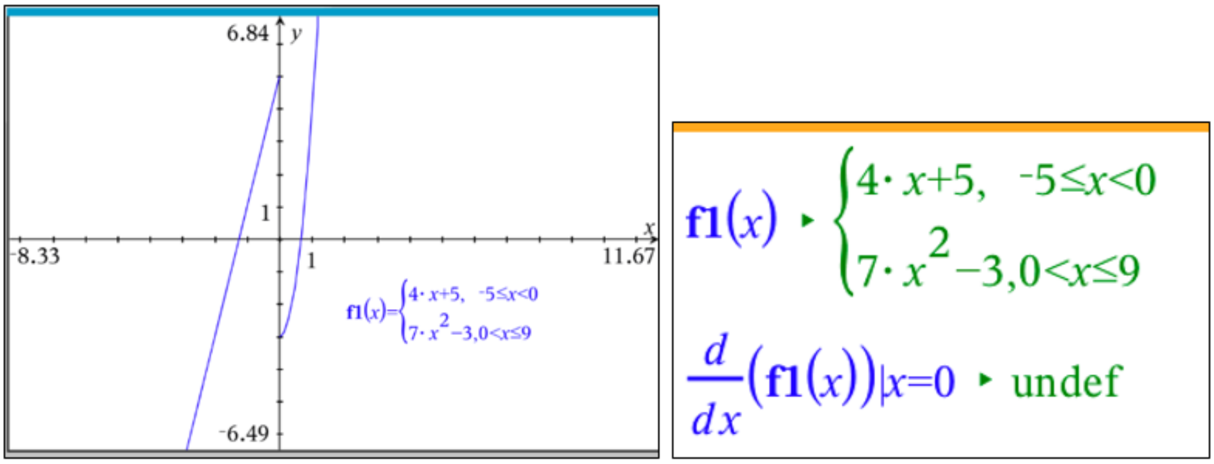

Figure 1: TI-*n*spire screenshots of the graph (left) and of the calculation of the derivative (right)

Lines 7-10 indicate that simply by looking at the graph and the fact that the graph "jumps" make the students presume non-differentiability. It seems the students rely on the graphic register, which indicates an evoked concept image of non-differentiability being when the graph of a function "jumps". CAS confirms this concept image, since it states 'undefined' [Figure 1 (right)], as the students predict [line 7-12]. However, it does not show that this is because f_1 is not defined for $x = 0$.

Their discussion of the inequality signs [lines 1-6] indicates that Beca connects the less-than-or-equal-to inequality sign with whether the two parts of the graph of the function meet [lines 3]. When the students want to connect the two pieces of graph to make a continuous function, Beca exchanges the different inequality signs [<] in the domain of f_1 with the ≤ while observing that the graphs of the functions all look the same as in Figure 1 (left). The treatments of the symbolic inequality signs indicate an evoked concept image of continuity involving the implication that defining a split function using ≤ is a sufficient condition for continuity. This evoked concept image becomes a cognitive conflict factor and Beca experiences confusion [line 13], as she cannot interpret why the graphs of f_2, which is defined for $x = 0$ [highlighted in grey below], and f_1 look the same.

$$f_2(x) = \begin{cases} 4x + 5, -5 \leq x \leq 0 \\ 7x^2 - 3, 0 < x \leq 9 \end{cases}$$

The function, f_2, does not solve the problem of constructing a continuous function. Later, when Amy suggests changing the expression of the function [line 14], which solves the problem, Beca does not return to the meaning of the inequality signs.

13	Beca:	But it cannot be, x cannot be… Argh…
14	Amy:	They just mean that it can be –5 and it can be 0. There, everything has to be larger than 0 … do you think it is just because we should write… try to write –3?

Beca changes the value of the constant from +5 to –3 [highlighted in grey below], such that:

$$f_3(x) = \begin{cases} 4x - 3, -5 \leq x \leq 0 \\ 7x^2 - 3, 0 < x \leq 9 \end{cases}$$

With f_3 the students succeed to connect the two pieces of graph. As before, they test differentiability at $x = 0$ with the use of CAS and get the expected output 'undefined'. For calculating the derivative and drawing the graph, TI-*n*spire performs the treatments within the symbolic register and the conversion to the graphic register. Issues with the computer doing the actual transformations of representations are that students may not understand the transformations themselves (Duval, 2017). When TI-*n*spire carries out the transformations, the students do not see the issue with the domain and its relation to the derivative, neither symbolically nor graphically. TI-*n*spire hides a potential cognitive conflict, which could have helped the students re-evaluate their concept image.

Changing the symbolic expression creates a continuous, non-differentiable function, f_3. For both discontinuity and continuity, the students look at the graph and assume non-differentiability, which CAS confirms. The situation of writing (using natural language and symbols) seems to evoke a concept image of differentiability including the process of limit, whereas in the situation of constructing a function evokes a concept image connected to the look of the graph. When CAS gives the expected output 'undefined', the students do not experience a need to check why. The conversion from the linguistic to the graphic register through the symbolic is in a sense like "the whispering challenge", where children in a circle pass on a whispered message. Here, the situation of writing their own explanations evoke one part of their concept image which lose its relevance later in the situation of constructing their own functions. As the students do not convert the graphic representations back to their linguistic representation, they miss the opportunity to recollect the lost content and re-evaluate the relation between their evoked concept images. Vinner (1983) argues that concept images build on previous examples, and the concept definition becomes inactive and forgotten. This seems to be the case for the two students in the presented situation. Studying the graph and whether it has a "jump" or "bends" is considered easier than working with the limit process of the secant slopes. The graphically explained process of limit included in their personal concept definition of differentiability is superficial and not involved in their further work.

Discussion

The purpose of the paper is to address the two perspectives of semiotic registers and concept definition-concept image for elaborating on the mathematical thinking competency. The perspective of semiotic registers offers a tool to specify students' actions with the mathematical concepts, through their work with representations. We find that this can help identify students' concept images which can provide a terminology for the mathematical thinking competency. The notion of conversions allows for deeper insights into how the student connects their evoked concept images related to the same mathematical concept and thereby how they experience the scope of the given concept in the different situations. The empirical data indicate three ways in which the perspective of semiotic registers makes it possible to identify evoked parts of students' concept images. The first way is to analyse a student's treatments within the linguistic register to identify evoked concept images related to a student's personal concept definition. A personal concept definition is "the form of words that the student uses for his own explanation of his (evoked) concept image" (Tall & Vinner, 1981, p. 152). Thus, compared to the formal concept definition the personal seems more related to the linguistic register than the symbolic, as it is the student's own words and articulation. Thereby this analysis can provide access to students' interpretations of mathematical statements and the involved claims. Amy and Beca's personal concept definitions are analysed as a treatment within the linguistic register from explaining differentiability to explaining non-differentiability. This helps identify their evoked concept images of differentiability as a limit of secant slopes towards the tangent slope, and as being aware of a potential critical point of non-differentiability, indicating an awareness to the difference between “for all” and “one point”, being part of the mathematical thinking competency.

The second way is to analyse treatments within one of the four registers to identify evoked concept images not directly related to concept definitions. For instance, analysing the symbolic treatments Beca carries out, when changing the inequality signs to obtain continuity, or analysing the students’ interpretations of the graphic representations of the different functions. This indicates the students’ considerations of conditions for the concept and the scope of the concept as perceived by the given semiotic representation. The third way, which we find the most important, is the analysis of a student’s conversions between the four semiotic registers to help identify how students relate and connect the independent and situated evoked concept images. By this, we can obtain a fuller picture of a student’s total cognitive structure of the concept image. The empirical example illustrates this by the students’ activities and interpretations in their sequence from the written answer, the students’ personal concept definition, which they convert into the symbolic representation of a function with a split domain, converted into a graph that “jumps” by TI-nspire. This way contributes to elaborate on the students’ ability to relate to the varying scope of a concept within different situations.

Based on the three identified ways, we suggest that a fourth way could be to analyse students’ treatments within the two discursive registers, the linguistic and the symbolic, when working with formal concept definitions to gain insights into their interpretations of such statements. The empirical example does not include work with the formal definition, however, a concept definition being a form of words used to specify a certain mathematical concept (Tall & Vinner, 1981), a formal concept definition is represented explicitly by language and symbols. The students are working with TI-nspire and in some situations, treatments and conversions are performed by the computer. Concept definitions, both formal and personal, are formulated in the linguistic register outside CAS, but the personal may be inspired by the work with CAS. The treatments and conversions significant for the second and third way can be performed by CAS, initiated by the students. Thus, the students’ work with and interpretations of the representations and transformations (even if produced by CAS) can help to specify and identify their concept images.

In the first way of identifying students’ concept images through their transformations and interpretations of semiotic representations, students work with mathematical statements where they indirectly need to navigate regarding logical connectives and quantifiers as well as relate this to the expected answer of the task, which are all aspects of the mathematical thinking competency. In the case of differentiability, working with personal concept definitions requires students to distinguish between the definition of differentiability as a for-all-statement and the opposite, non-differentiability, as an existence-statement. This would also apply for the fourth way of working with the formal concept definition. For the second and third way, students also get to work with mathematical statements through their (often unarticulated) conjectures of the concepts, accessible through their transformations of representations and the associated evoked concept images. The students’ written answers and how they work with representations of the other registers indicate their considerations of the conditions of differentiability and how they relate to the varying scope of the concept of differentiability in the context of “differentiability at a given point” and “differentiability of a function" which is an aspect of the mathematical thinking competency. The lens of semiotic registers can elaborate the students' mathematical thoughts and actions indicating which concepts images they work from. In line with the suggestions of Niss and Jankvist (2022), we find that having this view on the students' work can bring forward aspects of the mathematical thinking competency making it a more visible part of students work.

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Examining the effectiveness of teacher professional development in the BM and MIST initiatives

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Teacher professional development (PD) plays a pivotal role in bridging educational research and practice. This paper compares two large-scale PD projects, Boost for Mathematics (BM) in Sweden and Middle-school Mathematics and the US's Institutional Setting of Teaching (MIST). Despite sharing the mission of improving instructional practices, these projects unfold within unique cultural and institutional contexts. The analysis focuses on the strategies employed by each project to facilitate teacher PD. Drawing on Darling-Hammond and Richardson's work (2009), the study evaluates six aspects of effective PD. While both projects address these aspects, MIST demonstrates a broader scope and stronger support structure for teacher learning. However, we find that deficiencies in coherence may potentially undermine the effectiveness of MIST.

Keywords: Professional development, effective PD, implementation, mathematics innovations.

Introduction

Implementation is a crucial step for educational innovations, aiming to improve students' learning outcomes. One effective model for achieving this is educating teachers to implement these innovations in their classrooms. Consequently, teacher professional development (PD) becomes a cornerstone in this process, bridging the gap between educational research and tangible improvements in practice (e.g., Kennedy, 2016; Krainer, 2015). The challenge of building a robust collaboration between researchers and teachers lacks clear answers. However, valuable insights can be gleaned from studying completed large-scale PD projects. For instance, Popova et al. (2021) evaluated 139 PD programs from 14 countries, revealing that programs linked to career incentives, with a specific subject focus, incorporating lesson enactment in training, and including initial face-to-face training tend to show higher student learning gains. Additionally, the program implementers highlighted "follow-up visits as among the most effective characteristics of their professional development programs" (Popova et al., 2021. p. 107). Another study by Lindvall (2017) examined the impact of two large-scale Swedish projects comprising one-year-long PD programs on students' achievement.

The analysis indicated that changes in student achievement vary by grade level and project, with more substantial effects observed at the secondary level (Grades 8 and 9) than at the primary level (Grade 2). Such comprehensive studies are essential to guide future initiatives and policy decisions in teacher PD. However, there is a noticeable gap in studies comparing PD initiatives, especially in mathematics education, between different contexts. To address this gap, our paper explores teacher PD models within two distinct educational landscapes – Sweden and the United States. Our comparison centers on two PD projects, the Swedish Boost for Mathematics (BM) and the US Middle-school Mathematics and the Institutional Setting of Teaching (MIST). Both projects share a joint mission: to harness research findings to empower teachers to elevate their instructional practices. However, they unfold within unique cultural and institutional contexts, offering a rich background for analyzing the diverse strategies employed in teacher PD on a global stage. Within this paper, our focal point narrows to the strategies used by each project to facilitate teacher PD. Our research question is: *How effective were the different PD program strategies in BM and MIST in supporting teachers' professional development, as analyzed through the framework provided by Darling-Hammonds and Richardson (2009)?* We adopted Darling-Hammonds and Richardson's (2009) framework since it encompasses a broad spectrum of aspects within PD programs that align well with the aims of this paper. The comparison of MIST and BM reveals that while comprehensive PD (as in MIST) offers extensive support, coherence among PD activities is crucial. PD designers should balance diverse PD initiatives with unified objectives to prevent overwhelming teachers and ensure program effectiveness.

Conceptual framework

We used six crucial aspects of a successful PD according to the framework provided by Darling-Hammonds and Richardson (2009), addressing *content knowledge, understanding student learning, active learning opportunities, professional learning communities, curriculum alignment, and sustained support*. The *first* aspect involves what to teach and how to teach. A PD program should empower teachers with subject content knowledge and effective methods for conveying this knowledge to students. It is equally important to help teachers understand how students comprehend mathematical concepts, which is the *second* aspect of a successful PD. The *third* aspect considers how teachers learn, emphasizing the provision of active learning opportunities through participation, as opposed to one-way lectures. The *fourth* aspect focuses on improving professional learning opportunities by establishing communities of practice that foster collaborative discussions, sharing experiences, and reflection on each other's practices. The *fifth* aspect underscores the importance of aligning the curriculum, assessment, and standards with the teachers' professional learning. Teachers are then more likely to embrace innovations coherently with national or regional curriculum, assessment, and standards. Lastly, the *sixth* aspect emphasizes the need for a PD program to provide coherent and intensive professional learning opportunities over an extended period to ensure sustainability.

Method

Context: The Boost for Mathematics (BM) Project

BM was a PD program for K–12 mathematics teachers in Sweden, from 2013 to 2016, with a high participation rate of 76% of all Swedish mathematics teachers. Although participation was voluntary for school principals, teachers were required to participate once their principal opted in. BM was launched in response to international assessments like TIMSS and PISA, identifying limited teacher control over instruction and students' individual work as contributing factors. BM aimed to transform teaching and PD cultures by broadening teachers' methods and decision-making skills and was managed by the National School Agency. BM followed a cascade model for teacher PD, where expert teachers, known as "coaches," were trained by university researchers. These coaches led school-level teaching cycles structured around web-based modules, each comprising four sessions. Teachers studied PD materials, met with coaches to reflect and plan teaching activities, carried out the activities with their classes, and then reconvened to discuss their experiences. The program covered various aspects of the mathematics curriculum, such as arithmetic, geometry, algebra, functions, problem-solving, and digitalization. Modules were tailored to different grade spans and specific educational needs. Swedish mathematics education researchers played a central role in developing these modules, and expert teachers facilitated peer learning sessions in each participating school.

Context: The Middle-school Mathematics and the Institutional Setting of Teaching (MIST) Project

The MIST project, conducted in partnership with four urban school districts in the US from 2007 to 2015, had two phases (Cobb et al., 2018). Initially, all four districts collaborated, but in the subsequent phase, only two continued. The project aimed to develop inquiry-based instructional practices among middle-grade mathematics teachers in ethnically diverse urban districts and create a Theory-of-Action applicable to Research-Practice Partnerships (RPPs) in various disciplines. A provisional Theory-of-Action was formulated during the first phase for district-wide instructional enhancement. The concept of a theory of action, influenced by Argyris and Schön's (1974, 1978) seminal work, is widely used in the US, particularly in efforts to enhance classroom teaching and learning (Cobb et al., 2021). Such a theory outlines explicit goals for student mathematical learning and teacher improvement, offers strategies to support learning, and includes testable hypotheses about how implementing these strategies will enhance instruction and student learning outcomes (Cobb et al., 2021). In the second phase, the project tested, revised, and expanded upon the conjectures established (Cobb et al., 2018). The project involved 50 participants per district in the first phase and 100 per district in the second. Based on mathematics education literature, preliminary conjectures were iteratively refined using a design-based implementation research approach with four cycles. District leaders selected high-quality instructional materials aligned with rigorous learning objectives defined by the MIST project team (Cobb et al., 2018). The MIST project also followed a cascade model for teacher PD, where multipliers were trained to conduct the PDs for teachers (Maaß & Artigue, 2013).

Notably, the decision-making process was different in MIST than in BM. In the MIST project, researchers provided recommendations to district leaders to enhance the quality of teacher PD strategies, and district leaders had the autonomy to choose suitable methods for supporting teachers' instructional practices (Cobb et al., 2018). In the BM project, researchers trained coaches to organize teacher professional learning opportunities within schools according to the adopted lesson study strategy (Lindvall et al., 2023; Prytz, 2021).

Data collection and method of analysis

Data was collected using the qualitative research method of document analysis to systematically examine documents or texts and extract meaningful insights (Bowen, 2009). It involves selecting relevant documents, such as reports, articles, policies, or historical records, and establishing clear research objectives to guide the analysis process. For BM, we referred to three recently published papers (Grönqvist et al., 2021; Lindvall et al., 2023; Prytz, 2021), and for MIST, a journal paper (Cobb et al., 2021) and a book (Cobb et al., 2018). The documents were chosen because they provide a comprehensive description of the projects together and, therefore, provide sufficient data to answer our research question. The first author searched for the six aspects of designing a PD suggested by

Darling-Hammonds and Richardson (2009) in each document to identify how the six aspects were dealt with in each program, respectively. The six aspects functioned as codes for instances highlighting successful PD in the programs. We then compared the two projects under each aspect.

Analysis and results

The *first* aspect of Darling-Hammond and Richardson (2009) emphasizes that PDs should concentrate on *teachers' content knowledge,* particularly in mathematics within our context, and how to convey this knowledge to students effectively. In the BM PD, the design was based on modules created within the program, structured around core mathematical content in the Swedish K-12 school curriculum (Prytz, 2021). The module for numbers was mandatory, but teachers were free to choose one additional module. In the MIST project, district leaders had the benefit of selecting high-quality instructional materials from various sources, such as textbook series, online resources, and district curriculum frameworks (Cobb et al., 2018). In the MIST project, high-quality instructional materials are designed to assist students in achieving rigorous learning goals by providing challenging tasks sequenced to support the development of significant mathematical concepts (Cobb et al., 2018). Three out of four districts opted to adopt the Connected Mathematics Project 2 (CMP2) as their primary resource, while the fourth district chose a traditional textbook as their primary resource and CMP2 as the secondary resource (Cobb et al., 2018). Hence, both projects addressed *what to teach* mathematical content knowledge.

Regarding *how to teach*, the BM modules encompassed four didactical perspectives: assessment for teaching and learning, socio-mathematical norms, teacher routines for interaction in the classroom, and teaching in accordance with mathematical competencies (Lindvall et al., 2023). In the MIST project, in addition to adopting preferred resources, two districts developed comprehensive curriculum frameworks to offer guidance on what, when, and how to teach, while the other two districts created pacing guides to provide instructions on what and when to teach (Cobb et al., 2018). In comparison, the BM project designed modules and made them available for school leaders to choose from (Lindvall et al., 2023), and in the MIST project, district leaders determined the resources their teachers should use (Cobb et al., 2018). Regardless of the approach, the instructional materials (or modules) addressed both *what* and *how* to teach the content.

Consistent with the *second* aspect, *helping teachers understand how students comprehend mathematical concepts*, the BM modules guide "how the content in the modules should preferably be taught" (Lindvall et al., 2023. p. 5). Moreover, teachers in the BM project had opportunities to discuss their teaching experiences for each module during weekly collegial meetings (Lindvall et al., 2022; Prytz, 2021). These meetings served as a platform to exchange best practices, address challenges, and identify areas for improvement (Lindvall et al., 2023). In the MIST project, in addition to the guidance provided in the instructional materials, teacher learning subsystems, including pull-out PDs, teacher collaborative time, and one-on-one coaching, supported teachers in understanding how to teach the content effectively (Cobb et al., 2018). Like the BM project, teachers in all four districts of the MIST project had opportunities to share their experiences, and they found collaborative time valuable (Cobb et al., 2018). It appears that, in each project, there was an emphasis on guiding teachers in effective instructional methods.

The *third* aspect focuses on how *teachers learn*. For example, it involves creating opportunities for active learning through participation and reflection. Both projects included this component in their implementation programs but used different approaches. In the BM project, an adapted version of the lesson study was used to foster active learning (Grönqvist et al., 2021). The teachers studied a module selected by school leaders and then collaboratively planned lessons with other teachers. They taught a lesson in their classrooms and shared their reflections with peers. These collegial discussions were guided by coaches, in which teachers exchanged teaching practices, highlighted difficulties, critically examined their instruction, and received feedback from colleagues. Teachers in the MIST project also had opportunities for collaborative time (Cobb et al., 2021; Cobb et al., 2018). The design and implementation of teacher collaborative time varied across districts and schools. However, all four districts participating in the MIST project valued this approach as an effective method for teacher PD and an integral part of their teacher subsystems (Cobb et al., 2018). Teachers met regularly as departments, grade levels, and so on to work collaboratively on shared teaching-related tasks. These tasks included, among others, weekly lesson plans, task design, student data analysis, and specific instructional practices (Cobb et al., 2018).

The *fourth* aspect centers on enabling teachers to expand their knowledge, apply it in practice, and engage in collaborative reflection (Darling-Hammond et al., 2009). Both projects addressed this aspect but through different methods. In the BM project, teachers had opportunities to implement modules in their classrooms and reflect on their teaching (Grönqvist et al., 2021). The teachers primarily reflected on their teaching rather than being observed by other teachers, school leaders, or coaches. However, they had opportunities to receive feedback for self-reflection from their peers and coaches during teaching cycles. In contrast, the MIST project utilized instructional coaches, who were hired full-time or part-time, to support teachers in improving their instructional practices. These coaches provided one-on-one coaching, which started with co-planning and co-teaching lessons with teachers and then gradually transitioned to observation and feedback (Cobb et al., 2018). In both projects, reflections on teaching occurred regularly, whether biweekly, weekly, or fortnightly. However, while the BM teachers primarily reflected on their teaching and received feedback from peers and expert teachers within teaching circles, the MIST teachers received one-on-one coaching.

The *fifth* aspect links curriculum, assessment, and standards to teacher professional learning. In the BM project, this aspect was displayed by the fourth didactical perspective, *teaching in accordance with mathematical competencies*, focused on nurturing five mathematical competencies outlined in the national curriculum (Grönqvist et al., 2021; Lindvall et al., 2022). The project's modules aligned teacher professional learning with the national curriculum, emphasizing problem-solving, reasoning, and whole-class discussions (Lindvall et al., 2022). The modules were trialed with a group of in-service teachers. In the MIST project, Theory-of-Action was implemented in each district, adopting high-quality instructional materials, such as textbook series and curriculum frameworks with high cognitive demand to support students' conceptual understanding and procedural fluency (Cobb et al., 2018). These materials were aligned with the school curriculum. However, "more than half of the teachers reduced the cognitive demand of the tasks during the instructions, typically by demonstrating specific steps … to solve the tasks" (Cobb et al., 2018, p 151).

The *sixth* aspect was evident in both projects, albeit with slight differences. In the BM project, teachers spent 24 to 32 hours on learning activities per module in addition to teaching time (Lindvall et al., 2022). They were encouraged to meet weekly, leading to an estimated 60 hours per year during four years of professional learning, excluding teaching time (Lindvall et al., 2022). In the MIST project, the time allocated for teacher professional learning varied across districts and schools (Cobb et al., 2021; Cobb et al., 2018). Nevertheless, all districts conducted pull-out PDs, school leaders organized teacher collaborative time weekly or biweekly, and principals or recruited coaches observed lessons and provided weekly feedback. Researchers interviewed all stakeholders, reviewed recorded lessons, and offered recommendations to each district individually to enhance the quality of their Theory-of-Action for supporting teachers' instructional practices.

Discussion and conclusion

We have utilized the six aspects of successful teacher PD identified by Darling-Hammond and Richardson (2009) in order to investigate: *How effective were the different PD program strategies in BM and MIST in supporting teachers' professional development?* The results show that both projects have incorporated all six aspects in their PD designs but have done so with varying approaches.

Regarding teachers' content knowledge, the support was stronger in the MIST project than in that of BM. In MIST, support for the entire teaching was provided through the curricular guides on what, when, and how to teach and the pacing guides on what and when to teach. In contrast, the four didactical perspectives encompassed in the modules in BM were used in adopted lesson studies and did not function as a checklist for what, when, and how to teach but rather as a source of inspiration for a changed teaching practice in line with the competencies. Both projects provided platforms for teacher collaborative work to support *teachers' learning in understanding how students comprehend mathematical concepts* (aspect two) and make possibilities for *teacher learning* (aspect 3). The collaborative meetings had different agendas. In BM, the focus was on reflecting on one lesson at a time, whereas in the MIST meeting, weekly lesson plans, task design, and student data analysis were closely examined. In both projects, the collaborative work provided opportunities for reflection on teachers' practice (aspect four). Again, the support was more comprehensive in MIST, where the teachers received coaching, help with co-planning and co-teaching, and individual feedback.

Both projects linked their respective Standards-based curricula (e.g., focus on problem-solving, mathematical reasoning, and whole class discussions) to professional learning. However, it is not possible to determine which project had the most significant impact on *teaching in accordance with mathematical competencies* (aspect five). Since we have no information on the extent to which the teachers omitted parts of the curriculum resources (modules or textbook series) in their implementation in practice, we cannot know how the support worked. From MIST, there is information that the teachers gave too much instruction to students, thus undermining the inquiry-based processes advocated by the curriculum materials (Cobb & Jackson, 2021). From studies of BM, we learned that teachers who participated in the BM program, in self-reports, on average, spend more time discussing problem-solving strategies with the students as well as organizing other types of teaching activities (Grönkvist et al., 2021). However, we also know from previous studies that what teachers say they do and what they actually do, do not necessarily match (Boesen et al., 2014).

The *sixth* aspect concerns how the projects ensure *coherent and intensive professional learning opportunities over an extended period to ensure sustainability.* Here, we can state that while BM included two modules expected to influence teachers' overall practices, MIST embraced the entire teaching practice. The intensity of MIST is clearly more comprehensive. But how about coherence? Each district of the MIST project implemented teacher learning subsystems comprising four distinct types of support for teachers' instructional practices (Cobb et al., 2018). These support mechanisms ranged from the district to the school, including individual one-on-one coaching. Their findings demonstrate that teachers value each form of support (Cobb et al., 2018; Cobb et al., 2021). However, the findings also suggest that the lack of coherence between district pull-out PDs, school-based teacher collaboration meetings, and one-on-one coaching may inadvertently hinder teacher improvement in instructional practices (Cobb et al., 2018). Thus, while valuing and incorporating different methods for teacher PD in the MIST project, it is also crucial to maintain coherence among the methods. Introducing several different methods of PD to achieve the goals of implementation research projects may not be helpful when improving teacher learning. Both BM and MIST address all six aspects of the Darling-Hammond and Richardson (2009) framework for effective PD, yet MIST has a broader scope and a more comprehensive support structure for teacher learning. At the same time, there are indications that MIST had deficiencies in coherence, possibly working against the extensive efforts. According to the resources we examined, the BM project incorporated only an adapted version of the lesson study for teacher professional learning, and coherence among different PD activities was not an issue for this project. After all, the goal of PD is to increase students' achievement. Cobb et al. (2021) say that MIST was effective, but the only way to truly know this is to evaluate students' performance. Student evaluation has not been carried out in MIST, and the results from Swedish national tests on the school cohorts whose teachers have recently undergone BM show no significant improvements (unfortunately). Our claims are based on available literature, but we carefully selected resources that thoroughly address each project's PD components.

The comparative analysis of MIST and BM shows that while a comprehensive approach (like that of MIST) offers extensive support, coherence among PD activities is essential. PD designers should balance a variety of PD activities with consistent, unified objectives to avoid overwhelming teachers and ensure program effectiveness. Additionally, a significant implication for future PD evaluations is the need to include measures of student achievement to validate PD effectiveness comprehensively. Although both MIST and BM offered varied support for teacher development, the absence of direct student performance data in the literature about MIST complicates the assessment of their impact on student outcomes—a primary measure of PD success. Without evaluations of students' achievement, the question of which type of PD is best remains unanswered.

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Developing algebra learning-teaching activities in collaboration with middle school teachers: Preliminary design principles

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In this paper we outline the theoretical and methodological foundations for a research project that aims to develop algebraic learning-teaching activities in cooperation with teachers. We describe a vision of how strands of algebraic reasoning may be integrated across topics in grades 5-7. Drawing on the Vygotskian concept of sense making we formulate design principles for algebra learning-teaching activities (ALTAs). We discuss the importance of collaborating with teachers in a co-learning agreement to make an impact in schools. Through working together on concrete lesson plans we came to learn more about each other's different worlds and over time our goals became more aligned. Through our collaboration we developed design principles, such as to include unit plans that are detailed but flexible to suit different classrooms and nurture teacher autonomy, and the ideas, representations, and solving methods ought to be relevant beyond the grade where introduced.

Keywords: Algebra, Co-learning agreement, Design research, Sense-making

Introduction

In this paper we demonstrate how teachers and researchers co-developed algebra learning-teaching activities (ALTAs) for the middle grades. As a result of the collaboration, we present preliminary design principles. Algebra is a topic in which many students fail to make sense as they are asked to engage in equation solving or express and manipulate generalities. The results of international (TIMMS, 2019) and national tests (algebra exam for first year pre-service teachers) show that Norwegian students struggle with algebra whether they participate in primary, secondary or university mathematics (Kaarstein et al., 2020; Haakens & Bråten, 2023).

Kaput (1998) described the problem of school algebra in terms of an institutionalized and lifeless algebra (constituted by meaningless manipulations) and argued that it had to be replaced with algebraic reasoning as a way of developing conceptual understanding within topics, as well as connections between them. Although research has shown that young students (years 5-12) can develop algebraic reasoning, Kieran (2018) argued that new teaching practices as well as new ways for learners to participate are required. The current challenge is to find effective ways to develop these practices among teachers and students in everyday mathematics classrooms.

It is widely recognized that it is difficult to achieve change in educational systems. Reviews of research on professional development consistently point out that long-term effects of interventions are difficult to maintain (see Sztajn et al., 2017). Sowder (2007) noticed in her review on professional development an emergence of a new paradigm that moved "from providing workshops and focusing on techniques" (p. 215) to connecting teacher learning and the practice of teaching. Sztajn et al (2017) summarize that "the close collaboration between researchers and practitioners in design and implementation can help increase the likelihood of participants' buy in and successful scale-up" (p. 818). This close collaboration is an important aspect of developing algebra teaching in schools. As

Kieran (2014) notes, there is a “need for a certain kind of support by the teacher in order to promote students’ algebraic reasoning – support involving both task design and whole-class teacher questioning” (p. 31). Furthermore, Kieran advocates that “research involving the development of such support in teachers has been hindered […] by a lack of appropriate methodological and theoretical tools” (p. 31). Therefore, there is a need to integrate teachers into research processes, and to develop the theoretical and methodological tools needed to allow the sustainable implementation of new algebra learning-teaching activities in Norwegian schools.

Acknowledging the demands of the everyday algebra classroom and teachers’ expertise, we have formulated the research question: What design principles for ALTAs in middle school classrooms are agreed upon in a collaboration between teachers and researchers drawing on both practical and theoretical considerations? We report on how these preliminary principles were derived during our collaboration with the teachers. These may later be elaborated and refined upon empirical analyses.

Algebra in middle school (what is it)

Studies of school algebra in the 1970s and 1980s showed that students failed to make appropriate and useful connections between arithmetic and algebra (Lee & Weeler, 1989). In a reform effort, *algebraic reasoning* became a central concept, which over time transformed school algebra from a specific topic to a way of reasoning that scholars argued should permeate school mathematics (Kaput, 1998; Kieran, 2018; Mason, 1996). Studies in the new millennium have shown that students in the early grades can engage meaningfully in algebra as a way of reasoning (Kieran, 2018).

Kaput (1998) advocated for a radical change in how algebra was taught as he believed algebra as a school subject had been reduced to meaningless manipulations (beyond repair). He suggested that there are five interwoven strands of algebraic reasoning that students should develop from the early grades. In short, these included: (1) generalising and formalising patterns and constraints; (2) syntactically guided manipulation of formalisms; (3) the study of structures and systems; (4) the study of functions, relations and co-variation; and (5) algebra as a modelling language.

Kaput’s (1998) first strand is about generalising and formalising patterns in real situations and within numbers and operations on numbers. It is central that students learn to notice and express generality. Formalisation involve expressing and making sense of the perceived generality in terms of a formal mathematical structure. For example, by drawing a diagram and describing a half and then a third, a student may express that $\frac{1}{2}=\frac{2}{4}=\frac{4}{8}$, and $\frac{1}{3}=\frac{2}{6}=\frac{4}{12}$ and argue that one may double the numerator and the denominator and still be able to describe the same part of the diagram. The teacher can talk about equivalent fractions and give the student an opportunity to discern and express a connection between the formalization introduced and his experience with the diagram. Becoming familiar with formalisms, the second strand, allows for effective manipulations and a reduction of cognitive demand, i.e. being able to reduce or scale up fractions will often be helpful in solving problems.

The third strand is looking at mathematics structurally. Kieran (2018) coined the term *structure sense*, which is to see through mathematical objects and discern structural decompositions. In terms of learning fractions, this can involve relating fractions to natural numbers by discerning connections between them, i.e. natural numbers can be expressed as fractions and they can all be placed on the number line, and by discerning what is different, i.e. $3 > 2$ while $\frac{1}{3} < \frac{1}{2}$ and $a \cdot 2 > a$, while $a \cdot \frac{1}{2} <$

a. Thus, manipulating formalisms can lead to another level of generalizations and abstractions. Acts of classifying and labelling are part of formalising experiences and important in developing structure sense. Kaput (1998) argued that developing awareness of different mathematical structures and ways of relating these to each other is both a content strand and a reasoning strand in school mathematics.

The fourth strand involves co-variational reasoning and a family of functions (linear, quadratic, exponential, etc.). In grades 5-7 students can develop co-variational reasoning when working with relationships between quantities within less or more complex contexts, such as figural sequences or real situations. In the modelling of real situations such as growth of a plant over time, the spreading of a disease or the cost of buying a phone with different payment plans, students can explore different forms of co-variation between quantities. Using proportionality and rate of change as formalizations and means for structuring, students can make initial inquiries into this complex topic. It is within this strand that students can make sense of the variable as taking on a set of values.

The fifth strand is the use of algebra as a modelling language. Although, it is useful for learners to express generalities in everyday language, it is neither efficient for manipulations nor sufficient for reaching new levels of generality (Radford, 2018). Thus, algebra as a modelling language permeates all the other strands. Kaput (1998) suggested that students first use available language and semiotic means in algebraic reasoning, secondly the students can make sense of a gradual introduction to the algebraic language. In grades 5-7 this involves encountering indeterminacy as an *unknown*, a *generalized number*, and as a *set of varying values*.

These five strands coincide with several core elements of mathematics teaching in the current curriculum of Norway (Ministry of Education and Research, 2020). Particularly, core elements such as modelling, generalising and argumentation are new in the Norwegian context and teachers are looking for ways to integrate these in their teaching practices. In our project we have focused on these practices by generating different types of ALTAs. So far these types include: (1) generalizing involving cycles of exploration, conjecturing and justifying (see Figure 1); (2) modelling real situations with the aim to generalise relationships between co-varying quantities; (3) algebra as a modelling language in the context of equation solving.

Algebra as sense-making in process-oriented teaching

A main goal for the project is for the students to make mathematical sense through their involvement in ALTAs where these reasoning strands are main parts, an approach to teaching and learning that we describe as *process-oriented*. (Mathematical) sense-making is at the core of the sociocultural process of appropriating cultural tools such as algebraic notations, algebraic ideas symbolized through letters, etc. For the students to make sense of the algebraic ideas, they need to be actively and collaboratively involved in algebraic reasoning in which they share their own ideas, listen to and critically engage with others' ideas. Furthermore, the mathematics teacher plays a major role in organizing activities, asking questions and facilitates meta-discussions regarding what it means to reason algebraically, modelling how to act in algebraic reasoning, and so on.

In the project we advocate a so called process-oriented approach to the teaching and learning of algebra, following Lampert's (1990) seminal argument for bringing the practice of what it means to know mathematics in school closer to what it means to know mathematics in the discipline. This approach is in accordance with the Norwegian curriculum and Kaput's (1998) five strands.

A co-learning agreement as a foundation for the project

Research within mathematics education is a complex field. However, researchers have made huge efforts to improve mathematics teaching. Nevertheless, these developments do not transfer easily to mathematics classrooms (Artigue, 2008; Breiteig & Goodchild, 2010). Moreover, an apparent disconnection between research and practice has permeated education for a long time (Silver & Lunsford, 2017). A teacher has practical obstacles to take into consideration, which are not always evident for a researcher (Ruthven & Goodchild, 2008).

While past research has focused on teachers' knowledge and beliefs (Ball et al., 2008), there has been an increasing acknowledgement of the complexity in which teachers work. In her plenary talk at ICME 13, Ball advocated a perspective where we need to look beyond these traits and examine what is actually going on in the classrooms (2016). In our opinion, this means that when we want to develop learning materials in algebra, we need to do this together with the teachers. We therefore build upon the notion of co-learning agreement by Wagner (1997). This means researchers and teachers are equally important in developing the ALTAs and are engaged in both action and reflection. Most importantly, we see teachers and researchers as learners of each other's worlds. This means that not only will the teachers learn from participating in this project, but we as researchers are also expecting to learn from the collaboration with the teachers. This is what makes it a co-learning agreement.

Researchers and teachers bring different expertise to the project. The teachers serve as experts in identifying and formulating educational challenges in classroom mathematics teaching and learning and its specific contexts. Researchers bring theoretical background and knowledge of empirical research, as well as providing "useful alternative perspectives that shape the way problems of practice are perceived" (Cai et al., 2018, p. 515).

About the ALTAs

The Algebra Learning Teaching Activities (ALTAs) are teaching sequences that might take 6 – 8 lessons to complete. This is to allow students to go through phases of exploration, hypothesizing and justifying on various occasions on the topic. A general example of a type 1 ALTA (generalizing as the central process), derived from literature, looks like Figure 1. This general example of an ALTA gave the research group a starting point for discussions and task design together with teachers. It provides a frame for noticing various important aspects of task design in algebra and allowed us to focus on how to facilitate exploration, hypothesizing and justifying in the classrooms. Algebra is a topic that can be understood in various ways, as previously pointed out, but this general ALTA highlights what we emphasize and why for instance fractions can be a topic in an ALTA.

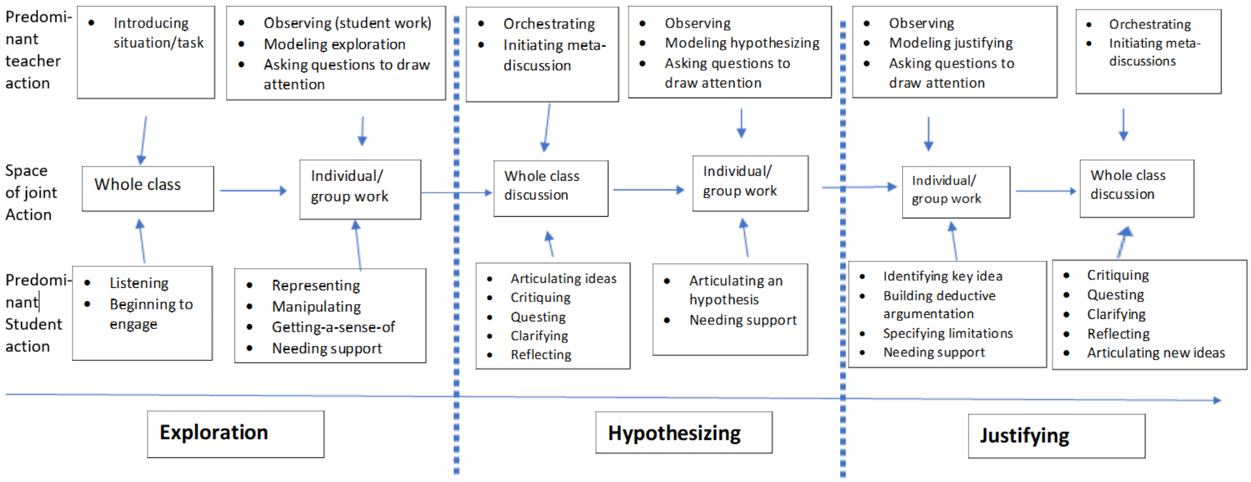

Figure 1: A general example of a type 1 ALTA

The general ALTA in Figure 1 is divided into three phases: exploration, hypothesizing and justifying. Within each of these phases, actions of teachers, students and joint actions are described. For example, in the exploration phase the predominant teacher action is to introduce a situation/task for the whole class, which is the joint action. The predominant student action at this point, is to listen and begin to engage. When the exploration phase shifts to where the joint space of action is individual or group work, the predominant actions of teachers and students changes accordingly. However, teachers' and students' predominant actions change when moving into the phases of hypothesizing and justifying. The teacher moves between orchestrating and observing, but also initiating discussions and modelling justifying, among other actions. The students are, on the other hand, expecting to develop actions characterizing mathematical hypothesizing and justifying. Therefore, this general ALTA emphasizes the design principles (see Table 2) and the many challenging actions teachers and students must master in a process-oriented classroom (Lampert, 1990).

Collaboration with teachers in the design of ALTAs

The overall objective of the two-year project was to generate knowledge about the nature of effective learning environments, tasks and tools for developing algebraic thinking in middle school mathematics teaching, and to produce such learning environments, tasks and tools. Building upon our acknowledgement of the complexity of teaching and the notion of co-learning agreement, we collaborated with five teachers from two local schools in this project. Together we developed ALTAs designed for grades 5, 6 and 7. The outline of the ALTAs were designed based on teachers' and researchers' discussions of learning challenges, initial ideas for classroom activities and theoretical emphases. These ALTAs were refined through cycles of enactment, observation, analysis, and redesign, with systematic feedback from teachers and researchers. The design cycles consisted of a design workshop, teaching in schools and a reflection workshop. Each cycle was done within a time frame of 4-6 weeks, and in 2023 we have completed three full design cycles.

We initiated the project by asking the teachers what they felt was difficult when it came to teaching algebraic thinking in middle school. The researchers and teachers aimed to develop a joint understanding of algebraic thinking in middle school, and the researchers were mindful of not imposing their views on the teachers. Thus, initially we designed starting points for ALTA's but encouraged teachers to make changes and further develop the ALTAs in collaboration with us. The ALTAs included clear learning goals. However, the teachers were often more concerned about what to do in the classroom, thus putting the process-oriented learning goals on the 'back burner' – consequently the research team focused the evaluation discussions back to the learning goals and how the algebraic thinking aimed for might be recognized in the classroom. The teachers asked for detailed lesson plans and expressed uncertainty with respect to how to execute initially open plans in the classroom. When working with more detailed lesson plans the teachers made more changes to the activities, responding to their and their students' needs. Furthermore, the teachers acknowledged the contingency involved when implementing an ALTA and emphasized achievements rather than failures. The design cycles we have completed so far, are on the following topics:

First cycle (January/February 2023): 5th grade – Equation solving with balances and algebra tiles; 6th grade – Relationships between quantities in modelling; 7th grade – Solving word problems with block drawings and spreadsheets. *Second cycle* (March/April 2023): 5th grade – Fractions with concrete materials; 6th grade –Programming, angles and shapes; 7th grade – Fractions with concrete materials. *Third cycle* (September – November 2023): The teachers wanted to further develop the ALTAs from the second cycle. So, these ALTAs were taught in new classrooms by other teachers in the project, learning from the experiences in the second cycle.

Through this collaborative effort we developed design principles for ALTAs considering (1) the promotion of students' development of algebraic thinking, based on the research literature; (2) the relevance with respect to curriculum goals and progressions, as well as flexibility regarding the inclusion of contingent mathematical ideas; and (3) teachers' needs and the allowance for use of the activities in various contexts. Together with the participating teachers we formulated the current principles for designing ALTAs, see Table 2. These principles have been organized into three strands: Promoting algebraic reasoning, Relevance and Pragmatic considerations.

Promoting algebraic reasoning	**Relevance**	**Pragmatic considerations**
Provide students opportunities for coming up with and expressing their own generalizations and reasoning. Structure the activity in phases of exploration – conjecturing – justification, including whole class and group/individual work. Provide students opportunity to use representations that are meaningful to them and support a gradual use of more formal representations. Provide support for concretization processes and abstraction processes. Provide appropriate challenges for all students. Formulating clear learning goals and ways of assessing student progress to provide appropriate learning opportunities.	Closely connected to the curriculum. The ideas, representations and solving methods are relevant beyond the grade were introduced. Openness regarding the inclusion of different but relevant ideas/topics.	Can be completed in two weeks. Suitable for classrooms with few or many students. Include unit plans that are detailed but flexible to suit different classrooms and nurture teacher autonomy.

Table 2: Preliminary design principles

Conclusions

Design principles promoting algebraic reasoning were derived from previous literature. However, the last principle listed in this category, addressing learning goals and formative assessment, was formulated as we (teachers and researchers) reflected on our experiences of implementing ALTAs in the classrooms. We found that there was a need for explicating student actions that are signs of them reaching the learning goals of the ALTA, with an openness in terms of teachers formulating these signs for their own classrooms. This can aid teachers in monitoring the students' progress and help them re-focus their teaching as they implement the lessons in the ALTA. The principles addressing relevance and pragmatic considerations grew from addressing the needs of the classrooms. Particularly important to the teachers were unit plans that are detailed but flexible to suit different classrooms and nurture teacher autonomy. Furthermore, the ideas, representations, and solving methods in an ALTA ought to be relevant beyond the grade where introduced, to make the effort invested in those useful again in new situations with new learning goals.

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Categorising types of programming in mathematics education

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In this study, Puentedura's (2006) SAMR model for technology integration is used to categorise two programming activities and provide insights into the distinction between what Puentedura termed the Enhancement area and Transformation area by using the sublevels of Substitution, Augmentation, Modification and Redefinition. Identifying different uses of programming can increase the knowledge about when and how the use of programming can be of value in mathematics education. The analysis shows that the model does help identify characteristics of different types of programming. Still, there are also challenges connected to the hierarchical grouping into four strict sublevels in the model.

Keywords: Programming, mathematics, integration, categorisation, SAMR model.

Introduction

Programming has during the last decade become a part of school in general and mathematics education in particular. In 2012, Estonia was the first country to start teaching students programming from grade 1 and onwards (the Republic of Estonia, Ministry of Education and Research, 2021). England did the same in 2014 and Finland in 2016. By 2017, more than 20 European countries had followed suit and included programming in their curricula (Balanskat et al., 2017). In Norway, despite excellent access to digital technologies in school and society and almost all students having their digital devices (Norwegian Directorate for Education and Training, 2022), programming was not included until a new curriculum was implemented in 2020. Norway then chose to let programming play a substantial role, including programming in the competency aims, implicitly for grades 2–4 and explicitly for grades 5–13. Additionally, computational thinking was included in the core elements, i.e., central didactic elements, and programming became a key part of digital skills, one of the five basic skills for mathematics subjects across all years and types of schools. Programming could have been an independent subject, but Norway has, like many other countries (Bocconi et al., 2022), chosen to include programming in other subjects and predominantly in the mathematics curricula.

During the last few years, an increasing number of studies have provided good examples of the productive use of programming. Gadanidis et al. (2021) highlighted that programming environments allow for interactive and visual ways of solving mathematical problems that would be challenging to solve manually. Herheim and Johnsen-Høines (2020) described how a Scratch programming environment can foster powerful investigations of polygons. Identifying positive outcomes of using programming is one thing, being able to distinguish between different ways of using programming in mathematics education is another. The latter constitutes the main rationale behind this study.

Trgalová (2022) argued that asking whether to use digital technology or not in mathematics education "is not an issue anymore nowadays, the question rather shifted to how to use it more efficiently and how to benefit the best from its affordances" (p. 417). Trgalová referred to technology in general and used the example of dynamic geometry. But what about programming? How can programming be used efficiently and how to benefit from its affordances?

Moreno-León et al. (2016) found that "there is a lack of empirical studies that investigate how learning to program at an early age affects other school subjects" (p. 283). Among mathematics educators, there is a clear interest in how programming can improve the teaching and learning of mathematics. Forsström and Kaufmann (2018) and Popat and Starkey (2019) found in their reviews of research on programming in mathematics education a lack of convincing evidence for the educational potential of programming. Forsström and Kaufmann (2018) therefore argued that "we need to know more about how we can integrate such technology in a school environment" (p. 28). One way to get to know more about this need is to better be able to distinguish between different uses of programming in mathematics education. We therefore address the following research question: *How can the use of programming in mathematics education be categorised?*

We decided in this study to try out Puentedura's (2006) model for technology integration, the SAMR model (Substitution, Augmentation, Modification, Redefinition). Blundell et al. (2022) did a scoping review of how the SAMR model has been used between 2009 and 2021 in research on the use of digital technologies in education. They found several studies within mathematics education, but none on the use of programming. Consequently, using the SAMR model to categorise programming activities in mathematics education introduces an element of novelty to the existing literature. To gain a better understanding of the different uses of the programming, we argue that Puentedura's categorisation serves as a model for understanding the different ways programming is used when designing activities in mathematics classrooms.

Conceptual framework: The SAMR model

Puentedura (2006) developed the SAMR model (see Figure 1) to inform teaching and learning with digital technology in education. The framework can be used to categorise how teachers integrate technology into their teaching (e.g., Ingram et al., 2018) or to classify and evaluate activities developed using technology (e.g., Pfaffe., 2017). The framework distinguishes between four different levels of technology integration grouped into two areas: *Enhancement* and *Transformation*. Substitution and Augmentation are sublevels of Enhancement, Modification and Redefinition are sublevels of Transformation.

In the area of Enhancement, technology is used as a substitute for traditional tasks and can replace or improve analogue teaching and learning approaches. Technology at this level is being used to perform a task that can either be seen as a substitute (Substitution) or having some functional improvement (Augmentation) (Puentedura, 2014). Kirkland (2013) wrote, based on the description of the SAMR model, that within the area of Enhancement, a "task may serve a particular purpose, but chances are it does not leverage technology for richer learning experiences" (p. 16). This implies that technology plays a small part in improving the tasks in the classroom and does not lead to major changes as a result of the integration of technology. In programming activities, this means that programming is used to solve mathematical problems that also could have been solved without programming (Substitution) or that technology is used as a direct tool (Augmentation) that generates some functional improvement.

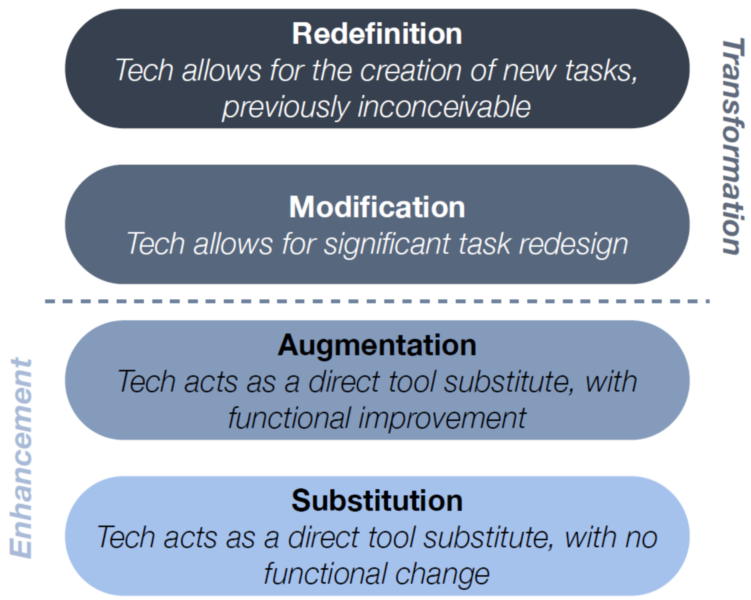

Figure 1: The SAMR model (Puentedura, 2006)

Figure 2: Sphero Ball

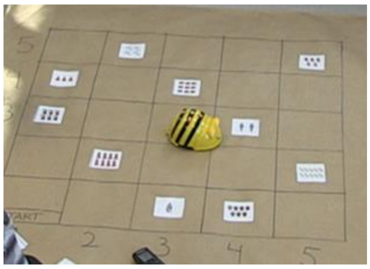

Figure 3: Bee-Bot in a grid

In the area of Transformation, teaching and learning are transformed. Technology creates new learning opportunities and allows for activities that could not be done without technology (Puentedura, 2014). Technology allows for significant task redesign (Modification) or the creation of new tasks previously inconceivable (Redefinition). Programming activities at this level lead to a substantial modification of an activity.

Hamilton et al. (2016) pointed out three challenges when using the SAMR model: lack of context, rigid structure, and an emphasis on product over process. Lack of context and theoretical foundation can lead to overgeneralisations and significant differences regarding how people understand and use the model, they argued. The rigidity is connected to how the model appears as a taxonomy with four hierarchical levels and how a certain technology integration belongs to one level only. Although teaching and learning are often complex and dynamic and thus difficult to categorise, models like the SAMR model can still prove valuable in order "to organize and label teaching and learning using specific categories" (p. 438). The third challenge concerns how an emphasis on the product can result in too little attention paid to students' learning processes. In this paper, we in particular follow up the critique about the rigidity of the four strict levels and the challenges connected to using them to categorise teaching activities.

Methodology

The two activities presented in this paper come from two out of many studies we have conducted with primary school students working with programming and mathematics in small groups or pairs. The activities are not chosen based on a classification process, but because they are representative of the studies as a whole and they serve as concrete examples that can be categorised by using the SAMR model. The aim was to better be able to distinguish between different uses of programming.

The Sphero Bolt activity, conducted by Røgelstad (2021) as a part of her case study, involved grade 6–7 students (10–12 years) working in pairs or groups of three. The task assigned to the students was to program the Sphero Ball by using the SpheroEdu app to create a quadrilateral and subsequently calculate its area based on the sensor data (perimeter) from the app. The students were then challenged to make new quadrilaterals with areas either half or twice as large as the original quadrilateral. The Sphero Ball (see Figure 2), a spherical robot designed for educational purposes, is roughly the size of a tennis ball and with the app, one can program the ball using a tablet, computer, or mobile device. In the app, the students could choose between coding the Sphero Ball with block coding, text coding, and even drawing.

The Bee-Bot activity was conducted by one of the authors of this paper as a part of her multi-case study (Nordby et al., 2022). This activity involved grade 1 students (5–6 years) working in pairs or groups of three. The students were asked to solve arithmetic tasks and program the Bee-Bot to move within a coordinate system. They calculated the given tasks and then programmed the Bee-Bot to move to the square with the correct number of icons representing the answer to the given addition or subtraction task. The Bee-Bot is a robot designed for educational purposes and looks like a bee (see Figure 3). Programming the robot involves manually pressing directional buttons on its back. The Bee-Bot has a limited repertoire of movements, meaning it can only turn 90 degrees at a time, and one step forward or backward is approximately 15 cm. This can be regarded as a somewhat rigid frame for the Bee-Bot's actions and movements, which in turn can make it difficult to make activities and tasks that are challenging and complex enough. Nevertheless, these limitations can also be seen as a strength, as the simplicity of the directional buttons on the robot's back enables students to be introduced to the grid system and programming without overwhelming complexity.

When using the SAMR model to analyse the designed activities, the emphasis is directed towards evaluating the extent to which programming is integrated into the mathematical activity. Puentedura (2013) made several questions to distinguish between the four SAMR levels. We have used these questions to identify and categorise the two programming activities. At the *Substitution level*, the question is: What will I gain by replacing the older technology with the new technology? From the *Substitution to Augmentation level*: Have I added an improvement to the task process that could not be accomplished with the older technology at a fundamental level? How does this feature contribute to my design? At the *Augmentation to Modification level*: How is the original task being modified? Does this modification fundamentally depend upon the new technology? How does this modification contribute to my design? At the *Modification to Redefinition level*: What is the new task? Will any portion of the original task be retained? How is the new task uniquely made possible by the new technology? How does it contribute to my design? Analysing the activities using the SAMR model provides insight into how programming activities can be categorised and contributes to a better understanding of how to use programming more efficiently in mathematics education.

Results and discussion

In the following, the two programming activities are presented and categorised. We identify some of the key characteristics in order to be able to say something about how certain uses of programming can be categorised according to the SAMR model.

Enhancement: Bee-Bot in a grid system in Grade 1

A baseline practice (no use of technology) of the Bee-Bot activity could be to guide a fellow student on the grid instead of using the Bee-Bot. Then simply replacing a student with the Bee-Bot will not make you gain much; it contributes with little to no functional change to the activity. Both guiding a student and programming the Bee-Bot entail some understanding of simple number sense and navigation within a grid system. It can, therefore, be argued that the activity is not transformed in any sense. Adding that the programming of the Bee-Bot is not directly connected to the arithmetic problems as they are already solved before the students start programming the robot, gives ground for categorising this activity at the Substitution level.

Compared to manually moving an object in the grid system, it can be argued that the programming offers some functional improvement to the activity. The main reason for this is that the students must first provide instructions and then convert them into a code string before making the robot move. Engaging in several steps to navigate the robot within the grid system can facilitate better learning opportunities and a process that encourages the students to articulate and justify their actions. To find the most efficient route for the Bee-Bot, the students must discuss and propose a code sequence. Since the Bee-Bot has buttons on its back to program it and no screen to visualise the block codes, the code sequence is not visible to the students. The lack of visual representation of the codes and the use of the buttons to do the programming requires a tactile perspective and challenges the students to strategically plan for the code sequence without being tempted just to press buttons until the Bee-Bot reaches the correct spot on the grid. Based on this argumentation, this integration of programming can be categorised to be at the Augmentation level.

This shows some of the difficulties with using the SAMR model to categorise because a particular use of programming can be categorised into several categories depending on what is emphasised. This aligns with one of the three challenges Hamilton et al. (2016) pointed out. This challenge was also identified by Blundell et al. (2022), who found several examples of inconsistent use of the SAMR levels. The Bee-Bot activity belongs to the Enhancement levels, but which one is challenging to say.

The activity introduces programming through a tactile user interface with the Bee-Bot's buttons. Including a screen to also get a visual representation of the code sequence would provide additional benefits, such as making it possible to observe the Bee-Bot's movements and the code sequence on the screen simultaneously.

Transformation: Programming a Sphero Ball to make a quadrilateral in grades 6–7

The Sphero Ball allowed the students to simultaneously observe the ball's movement on the floor and the app on the screen that showed the associated drawing, which immediately also provided the length of the perimeter of the shape. The use of the Sphero Ball contributed to several representations of mathematical concepts such as area and the quadrilateral: the students' oral explanations, the code, the drawing on the screen, and the ball's movements on the floor. The multiple representations proved valuable for facilitating students' mathematical discussions (Røgelstad, 2021).

In comparison, drawing a quadrilateral by hand to calculate the area will give fewer representations of a mathematical concept than with the Sphero Ball activity. The representation provided by the movement of the ball is not available, the immediate info about perimeter length is lost, as well as the opportunity to simultaneously see the movement of the ball and the drawing on the screen unfold. Using the Sphero Ball when working with students' understanding of polygons like this not only enhances the activity but also transforms it. The use can be regarded as a Modification, but it can also be argued to be a Redefinition because the students work with a new task. The requirement to make a polygon is retained from the original tasks, but the process of getting there is new and made possible by the new technology. They created quadrilaterals in ways they had not done before. Before they could start running the code, they had to find out the correct number of sides and vertices and which side lengths and angle sizes were needed in the code to make a quadrilateral that had four sides and was a closed shape where the first and last sides connected. They had to use the aiming function, similar to orienteering with a map and compass when working with angle sizes.

Elaborating further on the potential benefits of the multiple representations, the flexible functionalities of the Sphero Ball's graphical user interface with the opportunity to choose between various programming languages makes it possible to develop different ways of transforming the students' activities. These features allow students to engage in ways that would not be possible to achieve with e.g. analogue teaching techniques.

Conclusion

Our research question concerns how to categorise programming in mathematics education. Based on the discussion above, we argue that the Sphero Ball activity can lead to Transformation due to its flexible features and several representations of a mathematical concept simultaneously. The Bee-Bot activity is limited to Enhancement, although it offers a tactile user interface and clear frames for movement. In the context of these two activities, Enhancement refers to some improvements to the activity, while Transformation signifies a more fundamental change in the use of programming. It is therefore fair to say that the Sphero ball activity is the one that aligns the most with what Gadanidis et al. (2021) and Herheim and Johnsen-Høines (2020) stated: that programming environments can offer rich representations and investigations of mathematical concepts. Understanding the distinction between the two areas of Enhancement and Transformation in the SAMR model, together with the sublevels and the associated questions, can provide valuable insights into differences and similarities between different programming activities and how tasks done with programming can differ from similar tasks done without technology. Practical recommendations for the different uses of programming, such as implementing real-world problem-solving scenarios and societal concerns, can further enhance the applicability of programming activities in mathematics education.

Hamilton et al. (2016) questioned the hierarchical order of the SAMR model, the classification into four strict levels, and how the model "dismisses the complexity of teaching with technology" (p. 346). The challenges that we in this paper and Blundell et al. (2022) faced regarding how particular teaching activities can be categorised into several categories align with this critique. However, this critique about too strict levels and the model not being sufficiently dynamic and flexible is a typical challenge for many models, including well-known models such as the van Hiele (1984) levels for students' geometrical thinking and Piaget's (1936) stages of cognitive development. Still, these models have been widely used because they offer language to identify and distinguish between certain understandings students might have. Similarly, we argue that the SAMR model adds value to the process of identifying characteristics of different uses and, by that, one's ability to distinguish between different ways of using programming in mathematics education. While this study sheds light on a categorisation of programming activities and on how the use of the SAMR model requires a critical awareness of its strengths and weaknesses, there remains a need for further research on the applicability and relevance of programming in mathematics education.

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Investigating curriculum materials and their implementations: An analytic framework

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This methodological paper proposes an analytic framework for investigating written curriculum materials and teachers' implementations. As distinct from previous studies of interactions between teachers and curricula, we aim to improve curriculum materials in light of the analysis. Accordingly, we suggest analytic steps with the potential of uncovering strengths and weaknesses in the written material, where both critical examination of the material itself and teachers' implementations are essential. We illustrate the proposed framework by applying it to data from an intervention-based research project on reasoning and proving in primary education. Thus, we exemplify how the suggested analytic steps can display passages in written curriculum materials needing revision. The paper also sheds light on how teachers can be resources for developing curriculum materials.

Keywords: Fidelity, curriculum materials, implementation, argumentation, proving.

Introduction

Teaching mathematics is a complex endeavour, and teachers engage with a wide range of resources in this work. Curriculum materials are considered promising for structuring content and affecting teachers' teaching (Remillard et al., 2019). Thus, their designers can influence classroom activity and thereby impact student learning (Taylor, 2016). Previous research has studied the relationships between how guidance for teachers is communicated in curriculum material and teachers' uptake of the guidance. For instance, Brown et al. (2009) studied enacted whole-number lessons in 1st- and 2nd-grade classrooms, focusing on how the written curriculum material supported or hindered teachers' enactment. Grant et al. (2009) examined how curriculum material supported teachers in eliciting and extending student thinking. Their findings indicated that curriculum materials should explicitly discuss teachers' role in facilitating whole class discussions. Remillard et al. (2019) investigated how mathematical goals were articulated in four U.S. teachers' guides and how this influenced teacher's enactment. She found that the teachers were more likely to steer instruction toward the mathematical goals of a lesson if the material provided purposing support for these goals.

These studies make essential contributions regarding general principles and recommendations for curriculum designers in their work. Nevertheless, there is always a discrepancy between written and enacted curricula, as "a curriculum unfolds over several temporal phases, from the written, to teacher intended, to enacted" (Remillard et al., 2019, p. 102). Consequently, teachers' implementations could serve as a basis for further development of the curricula. Yet, none of the above-mentioned studies aimed to develop the curriculum materials under investigation. In this paper, we propose an analytical framework for investigating written curriculum materials and teachers' implementation, with the purpose of further developing the material. The research question is: *How can teachers'* *implementation of a curriculum material be analysed with a view to further developing the curriculum material?*

The analytic framework takes Brown et al.'s (2009) reconceptualisation of fidelity as its starting point, a notion referring to teachers' faithfulness towards curriculum materials. We account for modifications and further development to Brown et al.'s (2009) analytic procedure, and we illustrate how the proposed analytic framework can be applied to data from a Norwegian research project, ProPrimEd[1] – Reasoning and Proving in Primary Education. The intervention-based project aims to promote reasoning and proving in primary school. Written curriculum materials (tasks, detailed lesson plans, and student handouts) were developed and tried out in collaboration between teachers and researchers. Three teachers who did not participate in the design of resources used these in their teaching to increase the amenability for scaling up the intervention. The development of the current analytic framework originates from a wish to use data from these teachers as a resource to further develop the material.

Proving tasks relevant to primary school classrooms can be classified according to the number of cases involved: single case, multiple but finitely many cases, and infinitely many cases (Stylianides, 2016). The illustrative case in this paper concerns the implementation of the task "Brick towers," asking how many different towers of height four bricks one can build using bricks of two different colours. The task is a combinatorial problem with 16 solutions. Stylianides (2016) argues that a proof in school mathematics needs to take into consideration both mathematical aspects and the knowledge of the community in which the proof is constructed. More specifically, for an argument to qualify as a proof, it must build on mathematically valid definitions and results that are accepted by the classroom community, the mode of argumentation must be valid and within the conceptual reach of the classroom community, and the representations used must be appropriate and known to the classroom community. In the case of "Brick towers", a valid mode of argumentation is proof by exhaustion, and suitable representations for middle school students (with little experience working with combinatorial problems) includes systematic listing or drawings.

Teachers' fidelity to curriculum materials

To develop an analytic approach that enables us to further develop the written material, we build on Brown et al.'s (2009) study on how written curriculum materials can support or hinder teachers' enactment. Accordingly, we follow their distinction between, on the one hand, curriculum authors' ideas, and, on the other hand, written lessons in curriculum materials, meaning that written lessons are not a set of ideas themselves but rather a description of ideas (Brown et al., 2009; Remillard & Kim, 2020). Moreover, we assume there is a participatory relationship between the teacher and curriculum materials (Remillard, 2005), meaning that curricula are not static objects in the hands of teachers. Enacted lessons arise from classroom interactions between students and teachers when engaging with the curriculum material (Remillard & Bryans, 2004).

[1] The project is a collaboration between the Norwegian University of Science and Technology (NTNU) and Trondheim municipality, partly funded by the Norwegian Research Council, see: www.ntnu.edu/ilu/proprimed

The term *fidelity of implementation* refers to “a measure of faithfulness between something that is implemented and actions taken by an implementer” (Brown et al., 2009, p. 365), providing a conceptual basis for studying teachers’ implementation of curriculum materials. However, the term has been criticised due to narrow conceptions of it in previous research; see Brown et al. (2009) for a comprehensive review. To study interactions between teachers and curricula, thus, not limiting their research to, for example, teachers’ curricular coverage, Brown et al. (2009) distinguish between *fidelity to the literal lesson* and *fidelity to the authors’ intended lesson*. Fidelity to the literal lesson refers to the degree of alignment between an observed lesson and the underlying written instructional material. Fidelity to the authors’ intended lesson refers to the degree of alignment between the curriculum authors’ intended opportunities for students to learn specific content and the opportunities to learn observed in the enacted lesson.

Using Brown et al,’s (2009) reconseptualisation of fidelity, one can analyse teachers’ faithfulness to a written curriculum material and compare it to their faithfulness towards the authors’ intended opportunities to learn. Especially, Brown et al. (2009) concluded that the level of fidelity to the literal lesson does not determine the level of fidelity to the authors’ intended lesson, and vice versa. Moreover, their study indicated that, if the level of fidelity to the authors’ intended lesson is generally higher for parts in the written curriculum material where teachers display high level of fidelity to the literal lesson, the material might offer sufficient support for these parts. On the contrary, there might be activities included in a written curriculum material that score worse, regardless of teachers, on fidelity to the authors’ intended lesson. This suggests that the material might not provide sufficient support for implementing the current activities.

The analytic framework

As mentioned earlier, Brown et al. (2009) did not aim to develop the material under investigation. For a more thorough description of their analytic procedure, see Brown et al. (2009). Since our purpose is to further develop a given written curriculum material on reasoning and proving, we include analytic steps suggested by Brown et al. (2009), yet, with modifications and further development.

Phase 1 – pre implementation

Phase 1 aims for uncovering possibilities for students’ learning of a given mathematical content as communicated in the written curriculum material. Accordingly, it contains three analytic steps: 1) Identifying the literal lesson through lesson steps; 2) Developing and identifying Opportunity to Learn (OTL) codes; 3) Mapping lesson steps with OTL coding.

1) Identifying the literal lesson through lesson steps. The *literal lesson* is defined here as “the particular lesson steps included in the written material” (Brown et al., 2009, p. 375). *Lesson steps* refer to explicit instructions, material recommendations, and suggestions given to the teacher. Thus, the analytic procedure consists of reviewing the written curriculum for instructions, recommendations, and suggestions to the teacher, such as: “Ask the students how they intend to work to find all the solutions, and observe whether the students systematize their work, and if so, how.” The list of identified lesson steps constitutes a Literal Lesson Protocol (LLP).

2) Developing Opportunities to Learn (OTL) codes. Based on the distinction between written and intended lessons, a separate analytic step is needed to identify the authors' intentions communicated through the written material. The analytic step involves examining the written curriculum material for specific statements about how students are expected to engage in the current activities or the mathematical content, from which OTL codes are inductively developed. The development of OTL codes can be further supported by aspects highlighted in the research literature regarding the teaching and learning of the current mathematical topic.

A statement about how students are expected to engage in a combinatorial problem (to promote reasoning and proving) is shown in the following passage from the written curriculum developed in ProPrimEd: "After the student pairs have got a good start on their work, it can be nice to make them chat with another student pair, to compare how many different brick towers they found." The associated OTL is here identified as "Opportunities to discuss and challenge peers' arguments." The OTL is supported by literature arguing that challenging peers' ideas, clarifying conflicting ideas, and proposing alternatives are useful and go beyond what can be denoted as typical requests of explaining one's answers (see, e.g., Makar et al., 2015).

3) Mapping lesson steps with OTL coding. To reveal potential strengths and weaknesses in the curriculum material per se, we present a procedure for identifying what we define as *critical lesson steps*. The procedure was not undertaken by Brown et al. (2009) and is thus an extension of their work. A lesson step is considered critical if mapped with one or several OTL codes. Thus, some lesson steps will not be paired with OTL codes and will be considered less critical. For example, the lesson step "Ask the students how they intend to work to find all the solutions, and observe whether the students systematize their work, and if so, how" can be paired with the OTL "Opportunities to represent systematicity in proof by exhaustion." Potential strengths in the curriculum material can be when OTL codes are repeatedly paired with lesson steps, while potential weaknesses can be OTL codes being sporadically paired with lesson steps. In this latter case, the teacher is given a considerable job of interpreting the material to implement the lesson as intended. The critical lesson steps also inform Phase 2 of analysing teachers' implementation of the curriculum material, to assess whether the material offers support to implement the lessons as intended.

Phase 2 – post implementation

Inspired by Brown et al. (2009), this phase aims for, on the one hand, describing how closely teachers act to the instructions and steps stated in written materials, and, on the other hand, studying their implementation of the intended lesson. Phase 2 contains the following steps: 1) Coding teachers' implementation of lesson steps; 2) Applying OTL codes to classroom transcripts; 3) Comparing the results from 1) and 2) to uncover possible strengths and weaknesses in the curriculum material.

1) Coding the teachers' implementation of lesson steps: Classroom transcripts are coded on basis of the LLP. Three codes are used for each lesson step: *implemented*, *implemented partially*, and *not implemented*. A lesson step is rated as *implemented* if the teacher has carried out the specific lesson step. *Implemented partially* means that a lesson step has been implemented in part or with modifications. *Not implemented* refers to no observation of a lesson step, even in a modified form.

2) Applying the OTL codes to the classroom transcripts: The step aims at uncovering teachers' fidelity to the intended lesson. This is done by first segmenting transcripts due to shifts in activity, and second, by coding each segment with OTL codes and rating it as either *arose*, *limited,* or *missed.* E.g., the OTL "Opportunities to deductively structure an argument" can be coded as *arose*, if the teaching emphasises the development of a chain of inference from premise to conclusion; *limited*, if there are traces of inferences, but less explicit than in the previous category; *missed*, if there are no traces of inference.

3) For each critical lesson step, comparing the results from 1) and 2). The analytic step builds on Phase 1, step 3, and is an extension to Brown et al.'s (2009) work. The purpose is to further elaborate on strengths and weaknesses in the curriculum material. For systematicity, a coding scheme, exemplified in Table 1, can provide an overview of lesson steps involved, the mapped OTLs, and the coding of teachers' fidelity to the literal lesson and intended lesson respectively.

Lesson steps	OTLs	L.F T1	I.F T1	L.F T2	I.F T2	L.F T3	I.F T3
<Insert the identified lesson step>	<Insert OTL mapped to the current lesson step (could be more than one OTL)>						
...	...						

Table 1: A coding scheme, providing an overview of teachers' (T1, T2, T3) fidelity to the literal lesson (L.F) and the intended lesson (I.F).

Reviewing an edited version of Table 1, i.e., comparing teachers' implementation of lesson steps with their fidelity to the intended lesson, can reveal various scenarios. For example, there might be passages of the written material for which lesson steps and authors' intentions were implemented across teachers. Thus, the written material may be supportive regarding the teaching and learning of the current mathematical content, or the material may be redundant. For other passages, the analysis can reveal that neither lesson steps nor authors' intentions were implemented across teachers. Thus, the written material might not offer enough support, e.g., in motivating the current lesson steps. For yet other passages, lesson steps were implemented by teachers; however, the authors' intentions were not. Again, additional support in the written material will be needed, for example, in the shape of communicating more clearly the intended outcome of the current lesson steps, providing additional examples of moves for the teacher to implement, possible questions to ask, or presenting anticipated student solutions and ways to respond on them. In the upcoming section, we give illustrative examples from our analyses of scenarios pointing to the need to revise the written curriculum material.

Illustrating the use of the analytic framework

We apply the analytic framework to data consisting of written curriculum material for a 60-minute lesson and transcripts of video recordings of three teachers implementing the lesson in grades 5, 6, and 7. The implemented task was "Brick towers", see the introduction of this paper. The curriculum material consisted of a lesson plan (approximately three pages), student handouts, and a notebook file for interactive whiteboards. Four different ways to systematize the towers were included in the material. According to the teachers, the students had not worked explicitly on mathematical reasoning and proof before the data collection, nor had they worked on combinatorial problems. Thus, the students were expected to approach the task by listing random solutions. They were expected to recognize the need to systematize their work to keep track of the solutions and convince others they got them all.

We elaborate on two scenarios where the analytic approach shows the potential to direct researchers' and curriculum material developers' attention toward strengths and weaknesses in the material. Due to the limitations of this paper, a complete analysis of the written curriculum material and the teachers' implementation is not presented. The first scenario concerns OTLs that were missed for all three teachers. During the activity, the OTL "Opportunities for students to discuss and challenge peers' arguments" did not arise in any of the observed classrooms. From Phase 1 of the analysis, one of the lesson steps to which this OTL was mapped, was "organize for students to share their thinking with another group". Phase 2 of the analysis revealed that all three teachers omitted this step. The reason for this might be time constraints or practical and organisational challenges. However, it can also be that the material did not sufficiently communicate the rationale behind the suggested lesson step. Organising students to share their thinking with another group was motivated in the material by the following statement: "It could put students on the track of systematizing their towers, which they can further work on." One can thus claim that the rationale provided for this step was relatively scarce.

The second scenario concerns OTLs not arising as desired, even though the related lesson step was fully implemented. Here, we refer to the analyses of one of the teachers' implementations. The lesson step requested the teacher to monitor students' group work by "Ask[ing] the students how they intend to work to find all the solutions, and observe whether the students systematize their work, and if so, how". In Phase 1, this lesson step was mapped with the OTL "Opportunities to represent systematicity in proof by exhaustion." The students' initial approach to the task was drawing different brick towers randomly. The teacher pointed to the insufficiency of this approach by giving examples of towers they had missed out on and challenging the students to determine whether they had included a given tower in their list of solutions, as in the following excerpt:

Dan:	I have found all [towers], teacher!
Teacher:	[Takes a look at the solution and adds a tower]
Dan:	I have done that one.
Teacher:	Where?
Dan:	It is on the other sheet. No, there it is.
Teacher:	Ok. Then I will give you another one.

In summary, the teacher was rated to fully implement the offered lesson step. Students' drawings of solutions held the potential to be reorganized into groups, for instance, by grouping all towers having an equal number of bricks of the same colour, or by grouping all towers starting with the same colour. Each group could then be further investigated and systemized. However, the teacher's feedback did not push in this direction. Instead, the students continued to search for random solutions, and thus, the opportunity to represent systematicity in proof by exhaustion was rated to arise in a limited way.

The discrepancy between the teacher's fidelity to the literal and the intended lesson calls for a further examination of the written material. In the case described here, the material can be said to communicate desired student solutions. However, it provided few suggestions for teacher actions (questions and other types of feedback) that could extend students' initial attempts. One way to support students in representing systematicity in proof by exhaustion could be to include in the material one or more ready-made, structured, but incomplete solutions for the students to discuss and complete during group work.

Final reflections

The analytic framework presented in this paper is developed due to the need to improve curriculum materials in light of teachers' implementation. We build on the work of Brown et al. (2009); however, we present additional analytic steps. For example, in what we denote as Phase 1, pre implementation, we suggest mapping lesson steps and OTLs, which made it possible to identify parts of the material potentially needing revision. The comparison of teachers' fidelity to the literal and the intended lesson conducted in Phase 2 of the analysis, post implementation, enabled pointing at additional passages or features of the material needing improvement. Two scenarios were presented based on our data: the example of a lesson step that few teachers implemented, even though the step held the potential to address students' opportunities to learn, and a lesson step where the intended OTL did not arise as desired even though the lesson step was fully implemented. We claim that the analytic approach has the potential to reveal other scenarios not illustrated here, requiring, for instance, a critical examination of how well the material communicates the lesson step's objective or a review of whether the amount and the form of teacher support offered is in line with teachers' needs. Becoming aware of such passages might bring the material closer to what is denoted as *educative curriculum materials*, i.e., curriculum materials designed explicitly to support teacher learning as well as student learning (Davis & Krajcik, 2005). For the purpose of re-evaluation, revised versions of the curriculum material should ideally be re-tested, meaning that the post implementation analysis should be undertaken on new data.

We argue that the analytic framework enables a perspective on teachers as resources for further developing written curriculum materials. Teachers' enactment might influence curriculum material designers to include a more varied repertoire of guidance. For passages where the analysis reveals the support to be insufficient, a detailed examination of classroom data where the intended OTL did arise might bring up teacher actions that are required or seem promising and which can be included in a revised version of the material. However, there can be various reasons behind teachers' choices during implementation. For instance, some aspects of the mathematical content can make teachers reluctant to implement lesson steps: They might find the content mathematically challenging or consider it too demanding for their students (or both). Revised versions of the curriculum material can include additional examples of typical student work and related teacher moves to support the students. The text can be pared down for other passages where the material appeared redundant. Teachers are selective in what they read and likely treat educative features as optional (Remillard et al. 2019), which can be an argument for keeping the text as short as possible. Moreover, teachers might depart from the written text while implementing the intended lesson. In this case, their adaptations can be included in new versions of the written curriculum material.

The analytic framework is dependent on a curriculum material that enables researchers to identify the literal lesson in the text, i.e., it must provide instructional suggestions for how to implement tasks, orchestrate discussions, and follow up on students' solutions. More research is needed to consider the framework's potential for uncovering scenarios of teachers' fidelity to curriculum materials and, thus, possibilities for developing the materials. Still, we argue that the framework can inform curriculum authors, contributing to what Remillard et al. (2019) have described as a currently scarce amount of research on designing influential materials for mathematics classrooms and students' learning.

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Conceptualizing critical thinking in a lesson study of an inquiry-based learning of mathematics

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This paper presents an analytical framework for conceptualizing the various aspects of critical thinking demonstrated in the process of facilitators and teachers' collaborative learning and planning an Inquiry-based learning lesson study. Data reported in this paper was part of a larger project involving two primary schools in Norway, with a key goal to enhance the use of inquiry-based learning pedagogies in mathematics and science through engaging teachers in lesson study. We develop subcodes of the analytical framework to highlight the various aspects of critical thinking.

Keywords: Lesson study, inquiry-based learning, critical thinking, mathematics education.

Research background

This paper concerns the issue of conceptualizing critical thinking in an inquiry-based learning (IBL) lesson study (LS) in Norway. The concept of critical thinking has been introduced into the national curriculum in Norway as one of the overarching goals and is emphasized in varied situations of school education. It has thus become one of the buzzwords in the school field over the last decade. For instance, in Section 1-1 of the Education Act of the Norwegian national curriculum it is stated that "The pupils and apprentices must learn to think critically and act ethically and with environmental awareness. They must have joint responsibility and the right to participate" (The Education Act, 1998, Section 1.1). A subchapter of the Core curriculum is called Critical thinking and ethical awareness. Here we read that "The teaching and training shall give the pupils understanding of critical and scientific thinking. Critical and scientific thinking means applying reason in an inquisitive and systematic way" (Kunnskapsdepartementet, 2017, Paragraph 1.3). Looking specifically at the mathematics curriculum, we find that "Deliberation or critical thinking in mathematics includes critical assessment of reasoning and argumentation and can equip the pupils to make their own decisions and take a stand on important questions in their own life and in society" (Utdanningsdirektoratet, 2020, 2nd paragraph). However, it is challenging for schoolteachers to understand what critical thinking really means, or how to foster or implement it in schools across the different disciplines and different grades. For example, is critical thinking something that transcends the different school subjects, and school itself, so that it is thinking that is connected to questions about society? Or can critical thinking be limited to developing academic thinking within subject matter, for example thinking related to doing mathematics? IBL is an approach where the intention is to make the student a more active learner, by having students engage in and explore situations, discuss procedures, evaluate and thus making students work more like scientists/mathematicians (e.g., Artigue & Blomhøj, 2013). This approach is intended to be in contrast to "transmissive" approaches where students are considered passive recipients of knowledge transfer by listening to the teacher, taking notes and then practice by solving (simple) exercises. In our lesson study project, we tried to implement a main idea with the IBL approach, viz. to get students to think, look for alternatives, and discuss with their peers. In this way IBL is closely connected to critical thinking: by engaging with IBL students develop their thinking and thus may be able to apply their thinking in critical ways. Nevertheless, schoolteachers are not scientists or mathematicians. There is thus a need to negotiate the meanings of developing pupils' thinking in critical ways by the IBL approach between schoolteachers and the curriculum goals via facilitators, for example in lesson study. Our study is largely oriented by the following questions: Can one distinguish critical thinking in a lesson designed with an IBL approach? What does it mean to schoolteachers and facilitators in the context of lesson study in primary classrooms? How to develop an analytical framework to increase the power of analyzing and interpreting the first two questions? In this paper, we report an initial data analysis of a lesson plan meeting in which six schoolteachers and three facilitators learned and worked together to build up their shared knowledge and teaching goals through an IBL-based lesson study at grade 1. Our question in this paper is: Which aspects of critical thinking are discerned in the analysis of the schoolteachers and facilitators' collaborative learning and working on IBL LS? This question is necessary and important to lead us to build up an analytical framework to study the following questions: What is the role and function of critical thinking in IBL? How can we conceptualize critical thinking as part of IBL to develop a framework for guiding the design of the development of critical thinking among students?

Schoolteachers' learning and working in lesson study

During the past two decades lesson study has spread throughout the educational systems, as exemplified by the book Theory and Practice of Lesson Study in Mathematics–An International Perspective (Huang et al., 2019), which has contributions by scholars from 21 countries or regions (p. ix). Even if lesson study can take many forms for supporting schoolteachers' professional learning, there seems to be agreement that it entails a cyclic process of four stages, namely studying, planning, teaching, and reflecting (Lewis et al., 2019). There have been problems inherent when trying to implement a practice in one part of the world that was originally developed in another part. That is, the ways of schoolteachers' learning and practice situated in the culture and traditions of East Asian countries cannot be mechanically transferred as a cooking recipe to other countries. As lesson study has been a part of Japanese school for more than a century (Makinae, 2019), and is a mandatory part of the school professional activities of teaching research groups in China (Yang & Ricks, 2013), taking part in lesson studies is an inherent part of professional learning for schoolteachers in a lifelong career. This is rather different from the West, where it until the last couple of decades was most common to regard a teacher with his/her classroom as a closed entity, and teachers planning and working together, not to say observing each other, was rare. Fujii (2014), for instance, revealed several misconceptions about Japanese lesson study in countries outside Japan (more specifically in Malawi and Uganda). Given the research question in this paper, we aim to contribute to developing a new insight into the process of schoolteachers' learning and working in collaboration through the IBL lesson study from the theoretical perspective of critical thinking.

The problematic definition of critical thinking

Critical thinking is not a well-defined concept in mathematics education. In an investigation of what critical thinking is about and amounts to, Mulnix (2012) concluded that "Critical thinking is the same as thinking rationally or reasoning well" (p. 477) and "That repetition is central should be no surprise given that critical thinking is a skill" (p. 477). Maričića and Špijunović (2015) likewise defined "critical thinking as a complex intellectual activity which emphasizes the following skills: problem formulation, problem reformulation, evaluation, problem sensitivity" (p. 654). It is, however, problematic to define critical thinking in terms of skills. The skills listed by the above cited authors do not bring more than what is inherent in IBL or may be inherent in problem solving skills (e.g., Artigue & Blomhøj, 2013). Bailin et al. (1999a) precisely regard defining critical thinking as a skill as one of the misconceptions of critical thinking. A main argument against identifying critical thinking with a set of skills is that it "separates critical thinking from the development of knowledge, understanding and attitudes" (Bailin et al., 1999a, p. 271). On the contrary, it is necessary to have knowledge in a subject to be able to think within that subject, and therefore knowledge is a precondition for critical thinking. Having a skill is related to mastering certain discrete processes, procedures, or operations, which according to Bailin et al. (1999a) is a misleading way of treating critical thinking. Developing critical thinking is related to developing knowledge, developing strategies for tackling problems, developing commitment, and developing understanding of quality criteria and standards of quality. This also involves making judgments, and making judgments cannot be made a routine task. So, it is the quality of the thinking that distinguishes critical thinking from uncritical thinking (Bailin et al., 1999b, p. 288). It is then a fundamental question what the relevant standards are that the quality of the thinking must be judged against. From a review of different philosophical positions Mason (2007) extracts three aspects with all together five elements of critical thinking (p. 343–344). We use these elements to set up the following five codes:

Table 1: Five elements of critical thinking (Mason, 2009)

(1) Ability to reason critically	Code 1: Ability to assess reasons properly
(2) Dispositions	Code 2: Critical attitude, like scepticism, tendency to ask probing questions, commitment to express such attitude. Code 3: Moral orientation which motivates critical thinking.
(3) Knowledge of content	Code 4: Knowledge of a particular discipline. Code 5: Knowledge of concepts of critical thinking

We follow Mason (2007) in that an integrated view of critical thinking must take all these five elements into consideration. This wholesome and integrated view of critical thinking fits in well with thinking related to IBL and lesson study. It is thus a good starting point for our analysis. However, it should be emphasised that critical thinking in mathematics is a complex and not well-defined notion and that there are other ways of characterising critical thinking in mathematics education research, e.g. in the Danish KOM-project as part of mathematical reasoning competency (Niss & Højgaard, 2019) and in the critical mathematics education research tradition Skovsmose (2013).

The IBL LS in Norway

The IBL LS reported in this paper was part of a larger project involving two primary schools in Norway. The project has as its goal to enhance the use of IBL pedagogies in mathematics and science through engaging teachers in lesson study. Explicitly, the goals of IBL were being defined in the project as focusing on inquiring and explorative activities, student engagement, student wondering, reflection, critical thinking, asking questions; open tasks with multiple solutions (or solution strategies), use of professional language, cooperation, and communication. Critical thinking was not extracted as a separate goal but seen as an inherent part of IBL. In the IBL lesson study reported in this paper, there were six participating schoolteachers: TP (the school principal), TI and TR (grade 1 classroom teachers), TS and TT (grade 2 classroom teachers), TM (special education teacher who supports the classroom teachers). There were three participating facilitators (university teacher educators): DS (mathematics teacher educator), DR and DJ (science teacher educators). The study follows the four stages of a cycle of lesson study (Lewis et al., 2019). Prior to the first meeting of planning the lesson, the teachers had agreed on a proposal for theme for the lesson study and communicated this to the facilitators via email. The facilitators also met to study and discuss the preliminary plan received from the teachers before the planning meeting with teachers. At the second stage, the teachers and facilitators met together in the school. Discussions focused on the openness of the tasks, the possibilities for inquiry and exploring for students to perform during the activities, and the relation to the mathematical goals. The planning meetings are audio-recorded. In this paper, we focus on the analysis of the planning meeting at the second stage. As explained in the research background section, we need to firstly build up categories and codes for analysing and interpreting schoolteachers and facilitators' collaborative learning and work through each stage of the IBL LS. In so doing, we build up an analytical and interpretive tool to further study the broader questions of the study.

Data and data analysis procedure

Given the initial stage of our data analysis from the critical thinking framework perspective (Mason, 2007), we select the first lesson plan discussion meeting to demonstrate how we build up the categories and codes for the study. The plan meeting is around two hours long audio-recorded data and is converted to written transcript for the analysis. On the one hand, we use the five codes developed from Mason (2007) to analyse the interactions between the teachers and facilitators in the plan meeting. On the other hand, we are open to enrich Mason's framework to help to understand and describe the various aspects of critical thinking demonstrated by the teachers and facilitators in the lesson study. Thus, we seek to build up subcodes under each category and code, and if necessary, new categories and codes, developed from the data analysis. For instance, what kinds of critical thinking can be revealed in the lesson study planning meeting? How is the critical thinking related to teachers' classroom practice? Do teachers bring critical comments to the discussion with the facilitators? What kind of criticality do the teachers display? What kind of criticality do the facilitators display? Building on these questions leads us to our broader research question, namely how to conceptualize critical thinking as part of IBL in order to develop a framework for guiding the design of the development of critical thinking among students. Two researchers (the authors of this paper) independently coded and analysed the data according to the two basic thinking processes, both from theory to data and then data to new analytical framework categories and codes. Afterwards, the data analysis was shared and discussed between the researchers and is presented in detail in the following section to increase the validity and reliability of the study.

Findings

In this section, we briefly report how we develop the subcodes of the categories and codes for conceptualizing the various aspects of critical thinking in the analysis of the interactions between schoolteachers and facilitators in planning the lesson, with emphasis on student inquiry, exploration, and discussion. The interactions between the facilitators and schoolteachers can be characterized to predominately fall into aspects (2) Dispositions and (3) Knowledge of content, with Codes 2, 3, 4 and 5, but also with examples related to (1) Ability to reason critically and Code 1. DS showed his attitude in responding to teachers' dialogue about how to set up a shared mathematical learning goal by the IBL approach in LS, emphasizing that discussions should continue even if one idea has been launched:

DS: In this type of discussion that we are having now, very often when planning, someone gets an idea and everybody thinks it is a good idea, and it is accepted without further discussion, thus closing for other possibilities. So, I think it is important that you launch more ideas.

We consider this an example of Code 2, displaying critical disposition, with an attitude to encourage discussion, by encouraging participants to think freely instead of submitting to ideas launched by others. We identify a possible subcode related to encouragement of others critical thinking, responding about how to set up a shared mathematical learning goal by the IBL approach in LS. Another example is teacher TR showing commitment to engage students in discussion:

TR: Yes, in the beginning, when we start the cooperation, we wanted to look at discussion in pairs and how the children discuss among themselves, and this is something we can include here. (…)

This example is also related to commitment to let others, in this case students, discuss among themselves, which is also an expression of orientation towards an IBL classroom. Here we identify a possible subcode about commitment to development of student critical thinking. TM expressed concern about the hypothetical trajectory of pupils' learning situated in the designed tasks, also emphasizing that it is important to engage students in talking to each other:

TM: I think it be important to spend quite a lot more time on this reflection, no, the hypothesis bit ahead of this one, when they make hypotheses that they sit together two and two or four and four, that one should in a way spend a good amount of time on it, or as much time on it as when they start drawing or things like that, so that we get to organize it well I think, because it is often the part that is challenging, that everyone gets to talk.

Making sure that all students get a chance to take active part in the planning of the work ahead is fundamental in an IBL classroom, and is related to Code 3, namely having a moral orientation to motivate critical thinking, among others. This points us to a possible subcode about moral orientation to motivate student critical thinking. That everybody gets a chance to talk is a challenging task for teachers. Acknowledgment of this challenge is related to knowledge of the particular discipline of teaching mathematics, and thus leads to a possible subcode related to Code 4.

In the transcript below TI commented that from her understanding classroom discussion and open tasks are important in planning and teaching an IBL lesson, therefore steering students towards inquiry instead of pure guesswork is important. TI thus shows commitment to develop students' critical attitude, which we identify as a possible subcode to Code 3. This was followed by DR displaying knowledge of content in the form of scientific method by responding to the teacher's question whether making hypotheses is used exclusively in situations with single correct answers or solutions (Code 4):

TI: I think discussion is most important, when they have an open task, since if they start guessing, but they are supposed to make drawings and explore, more inquiry instead of what they believe or guess, since there is no concrete answer here, and hypothesis is more related to when there is only one correct answer, isn't it?

DR: It does not have to be. (…)

TR shared her thinking about letting students freely express and share their thinking, emphasizing that when discussing beliefs, there are no right or wrong answers:

TR: What we have done sometimes, is when we have presentations, is turn to the one sitting next to you and discuss what you think, then we have a plenary round afterwards. You believe this, and you believe that, but nobody is wrong since it is about thinking. (…)

We consider this an example of commitment to express critical attitude (Code 2), and commitment to develop students' thinking, building onto a possible subcode to Code 3. It is also related to Code 1 about assessment of reasoning, in the sense that an important part of proper assessment of reasoning is to accept other's views and not focus on whether an utterance is right or wrong. To assess reasons properly is a goal within an IBL lesson, related both to the teacher assessing student reasoning and encouraging students to assess each other's reasoning properly. As such, it would be an important part of the classroom practice that emerges when we analyze classroom data. From this we see the need for a subcode about students'/teachers' assessment that encourages differing views. DR displayed knowledge of content in the form of knowledge of IBL (Code 4) by emphasizing that inquiry can be used no matter what the subject or theme is:

DR: But inquiring discussions can be implemented no matter which theme we choose

TM emphasized her opinion that the process is as important as the product in learning when working within an IBL frame:

TM: The process here must be as important as the product, the process itself is especially important.

This can be considered an example of Code 5, as it displays knowledge of the concept of critical thinking as part of IBL. As related to classroom practice it is a possible subcode related to the implementation of IBL and critical thinking among students.

Discussion and conclusion

In this paper we take as our starting point the aspects of critical thinking identified by (Mason, 2007). We develop those elements for conceptualizing the various aspects of critical thinking demonstrated in the process of facilitators and teachers' collaborative learning and planning an IBL LS. We develop subcodes from Mason's (2007) article to further highlight the distinct aspects of critical thinking of the facilitators and schoolteachers in an IBL LS setting.

Table 2: Subcodes of the distinct aspects of critical thinking of the participants of LS

	Subcodes of Mason's (2007) main codes of critical thinking
Code 1	1.1 Ability to let students discuss and launch ideas without assessment of being right or wrong. (Schoolteachers)
Code 2	2.1 Responding about how to set up a shared mathematical learning goal by the IBL approach in LS, encouraging teachers' critical thinking. (Facilitators) 2.2 Commitment to express critical attitude in planning lessons. (Facilitators and Schoolteachers) 2.3 Commitment to engage students in discussion (talking to each other; letting students freely express and share their thinking). (Schoolteachers)
Code 3	3.1 Moral orientation to develop students' critical thinking. (Schoolteachers) 3.2 Steering students towards inquiry, thus commitment to develop students' critical attitude. (Schoolteachers)
Code 4	4.1 Knowledge of content in the form of scientific method, the nature of mathematics and science. (Facilitators) 4.2 Knowledge of content in the form of what IBL entails and offers. (Facilitators) 4.3 Knowledge of the art of teaching school mathematics so that all students are included and get a chance to take part in discussions. (Schoolteachers)
Code 5	5.1 Knowledge of the concept of critical thinking in the classroom, the process is as important as the product when it comes to learning. (Schoolteachers)

Collaboration between schoolteachers and facilitator researchers in lesson study enhances the strengths of both parts, and combines to increase the chance that IBL is implemented in the classroom in ways where development of students' critical thinking is an important part. We have seen how several aspects of critical thinking can be identified during a lesson study planning meeting for an IBL lesson. In so doing, we have identified how the aspects extracted from the literature should be refined to focus on the particular setting of teachers collaborating, thereby contributing to a framework for guiding the design for the development of critical thinking among students. Our data suggests that critical thinking cannot simply be regarded as a skill, placing us in line with Bailin et al. (1999a, 1999b) and Mason (2007). We aim to continue investigating teacher and researcher collaboration in lesson study to further develop the subcodes, with possible further additions. This will enable us to use the analytical framework to gain new insight into the complex processes of schoolteachers and facilitators' learning and working through the IBL LS from the theoretical perspective of critical thinking.

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Introducing multiplication using a rich task

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In this paper, I show that lower primary school students might take significant developmental steps in their multiplicative thinking through their engagement in a rich task, even before they are formally introduced to multiplication. As part of a study of rich tasks in lower primary school, the results presented here stem from a videorecording of two 7-8-year-olds as they collaborate to create as many rectangles as possible using exactly 24 centicubes. The data were collected before they were formally introduced to multiplication. Drawing on Mulligan and Mitchelmore's description of multiplicative thinking, I present results showing that over the course of one lesson, the students go from using a direct counting approach to using more advanced multiplicative thinking. These results point to the potential benefits of using rich tasks in introducing new mathematical topics, providing teachers with insight to students' relevant competencies and initial conceptual understanding.

Keywords: Multiplicative thinking, rich tasks, inquiry, arithmetic.

Introduction

Many pedagogical strategies operationalize the principle of going from the concrete to the abstract, as discussed and problematized by Coles and Sinclair (2019). Intuitively, it makes sense to introduce a new concept in direct and tangible ways, which explains the relatively high use of manipulatives in primary education (e.g., Puchner et al., 2008). Moreover, teachers often present new mathematical concepts using simplification, defined by Kirch (2000) as the process of making accessible. To reduce complexity, concepts are defined through imprecise statements, as when multiplication is introduced as "repeated addition". This might be appropriate, but the limited applicability of this model is apparent when students later engage in multiplication of negative numbers (e.g., Sfard, 2007).

Teachers tends progress from the concrete to the abstract and from the simplified towards the sophisticated. The students learning outcome will depend on the starting point and the pace of the progress. By spending instructional time at a basic mathematical level and progressing slowly, one might limit the students' final achievement level. This calls for attention to finding effective ways to unveil students' relevant competencies and initial conceptual understanding when introducing a new mathematical concept. In this study, we use the case of multiplication and explore the potential of an inquiry-based approach (Artigue & Blomhøj, 2013), namely, the potential of using a rich task (Hagland et al., 2005).

Rich tasks

Rich tasks are defined in different, yet related ways (e.g., Aubusson et al., 2014; Hsu et al., 2007). In this study, we employ the approach of Hagland and colleagues (2005). They describe seven defining characteristics of rich tasks. To illustrate each of these, I will use the "24-cubes task" assigned to the students in this study. They were provided with 24 centicubes and asked to create as many rectangles as possible using all 24 centicubes.

1. Rich tasks address important ideas. The 24-cubes task address multiplicative thinking and fundamental multiplicative properties, including the commutativity property.

2. Rich tasks bridge mathematical topics. The 24-cubes task bridge arithmetic topics, including counting, addition, multiplication, and division. Moreover, it bridges arithmetic and geometry.

3. Rich tasks promote related problem posing. The 24-cubes task extends to related problems, for instance through varying the shapes and the number of cubes.

4. Rich tasks initiate further discussions. The 24-cubes task might be applied to initiate discussions about the commutativity, about ways to prove that all solutions are found, or about prime numbers.

5. It is easy to get going. The 24-cubes task might be solved through trial-and-error, and it is easy to get going by experimenting with centicubes.

6. It is challenging to reach the top. The 24-cubes task might stimulate further investigations of systematic approaches to identify all solutions of two-digit numbers.

7. Rich tasks can be solved in different ways. For instance, the 24-cubes task might be solved through manipulating centicubes, through repeated addition, through utilization of multiplicative number triplets, through division, and through a systematic approach.

The seven characteristics provided by Hagland and colleagues (2005) establish rich tasks within an inquiry tradition. Specifically, Characteristic 1 to 4 address the mathematical richness inherent in rich tasks, where important ideas are addressed and bridged, and where subsequent problems and discussions arise. Moreover, Characteristic 5 to 7 promotes inclusive pedagogy, as rich tasks provide challenges for both low and high achievers and for students with different learning approaches.

As part of a study of rich tasks in lower primary school, this paper addresses the possible benefits of using a rich task as a way of introducing a new mathematical concept. This potential is suggested by the foregoing paragraph: The richness of rich tasks provide an open ground for exploration of the concept being introduced and its connection to concepts and skills already known. Moreover, the inclusive approach provides insight into the mathematical thinking of all students. This will support teachers in identifying appropriate starting points and common misconceptions. Moreover, it enables teachers to draw on students' contributions as they progress.

Research question

Development of multiplicative thinking is a main target in primary school mathematics (e.g., Anghileri, 1989) and continues to occupy researchers (e.g., Callingham & Siemon, 2021; Polotskaia & Savard, 2021). Thus, drawing on the discussion above, the following research question is addressed in this paper: What characterizes the multiplicative thinking of two 7-8-year-olds with no prior instruction about multiplication, as they collaborate to solve the multiplicatively structured 24-cubes task?

Theoretical framework

Multiplicative thinking is undisputed as a decisive competence in everyday life and for the further learning of mathematics, partly due to the frequent occurrence of situations and problems with multiplicative structures. Greer (1992) identifies four main categories of such situations. Most prominent in primary school teachers' introduction to multiplication is the case of *equal groups*, which facilitates the well-known "repeated addition" approach to multiplication. *Multiplicative comparison* addresses situations like "Vincent has four times as many figures as Sven, who has three", and the *Cartesian product* is typically addressed by tasks like "With two pair of pants and three t-shirts, how many different outfits can you create?". Finally, exemplified by the 24-cubes task, are the situations related to *rectangular arrays*.

These different situations, all holding a multiplicative structure, give rise to different mathematical problems. Such problems have been utilized to scrutinize students' multiplicative thinking. In this paper, the results of Mulligan and Mitchelmore's (1997) research will serve as a theoretical framework to characterize the multiplicative thinking among students who have not yet had a formal introduction to multiplication. They studied a group of second and third grade students over the course of two years. Mulligan and Mitchelmore assigned the students with tasks with semantically different multiplicative structures (Greer, 1992) and analysed the students' interactions with these tasks. Three main models of intuitive multiplicative thinking emerged, namely, direct counting, repeated addition, and multiplicative operations.

Direct counting refers to instances when tasks are solved through direct modelling, for instance through physical modelling using centicubes or through drawings, and where the result is derived by counting the elements of the model one by one. This model is considered as holding a minimal degree of abstractness (Kouba, 1989). Many teachers' instructional approach to multiplication is *repeated addition*. According to Mulligan and Mitchelmore, repeated addition entails more than sequential additions like "3+3+3+3". They distinguish between four different strategies: Rhythmic counting following the multiplicative structure of the task (e.g., "1, 2, *3*, 4, 5, *6*, 7, 8, *9*, 10, 11, *12*"), skip counting (e.g., "3, 6, 9, 12"), repeated adding (e.g., "3+3+3+3=12"), and additive doubling (e.g., "3+3=6 and 6+6=12"). Finally, *multiplicative operation* refers to the application of known multiplicative facts or the deriving of new multiplication facts.

Mulligan and Mitchelmore's (1997) characterizations bring forth the gradual growth in multiplicative thinking. Only nuances separate direct counting from the rhythmic counting of repeated addition, there is a natural progression within the four different strategies of repeated addition, and the additive doubling of repeated addition represents a transition towards multiplicative operations.

Among different approaches to multiplicative thinking, Mulligan and Mitchelmore's results are used as a theoretical framework as it focuses on the intuitive thinking models, that is, on models that children derive from experience rather than instruction. Moreover, the characterizations offered by the framework holds an appropriate degree of specification with respect to the available data in this study. The framework is easy to operationalize and observable in the context of students collaborating to solve a multiplicatively structured task.

Method

This study holds a socio-cultural perspective, focusing on interaction and language. According to this view, students' dialogues during mathematical activity may be analysed to provide insight into their mathematical thinking (Mercer et al., 1999), including gestures, writing, drawing, and the use of physical representations, as these are important elements in the mathematical reasoning of primary school students (Jeannotte & Kieran, 2017).

Data collection

The data presented in this paper was collected in November 2022. The participants were 7-8-year-old students in a third grade and their teacher. The school is in a medium-sized Norwegian city with limited ethnic diversity and moderate to high socio-economic status. All students, their parents and caretakers, and the teacher were informed about the study. The parents provided written consents, while the teacher and the students provided oral consents to participate in the study. Approval of the study, including data collections, analyses, and disseminations, was provided by Sikt (The Norwegian Agency for Shared Services in Education and Research).

The rich task was developed in a collaboration between the author and the teacher. The data collection was scheduled to be the very first activity of the students' introduction to multiplication. The teacher introduced the task, assisted the students during, and led subsequent plenary discussions, while the author and a research colleague observed the students. The students worked in groups of two and three, each group equipped with exactly 24 centicubes. They were instructed not only to make as many 24-cubed rectangles as possible, but also to write down the solutions they found. To facilitate an analysis of the students' multiplicative thinking, a videorecording was made by one of the groups. This group, consisting of "Holly" and "Michael" (pseudonyms), was identified by the teacher as average achievers known to collaborate well.

Analysis

The videorecording was transcribed by the author. The transcription included the students' gestures, what they wrote and drew, and all actions related to the centicubes. Drawing on the process of analysis as described by Powell and colleagues (2003), the author and the research colleague viewed the recording several times and coded independently the students' dialogues and gestures according to Mulligan and Mitchelmore's (1997) framework. Firstly, the events were classified according to the three main categories (1) direct counting, (2) repeated addition, and (3) multiplicative operations. Secondly, the events coded as repeated addition were coded according to the subcategories (2a) rhythmic counting, (2b) skip counting, (2c) repeated adding, and (2d) additive doubling. The results of the two independent coding processes were compared. Only a few discrepancies were identified. These were discussed, and a final coding was agreed upon. Subsequently, excerpts from the transcript and the result of our coding, including all excerpts in this paper, were read and validated by an experienced colleague.

Results and analysis

Here, I will characterize the multiplicative thinking of "Holly" and "Michael" as they collaborate to solve the 24-cubes task. The analysis suggests two distinct phases in their work. Initially, they apply a "building rectangles" principle, focusing on the manipulation of the centicubes, making sure that they build valid rectangles. The second phase, indicating an advancement in their multiplicative thinking, is characterized by a "grouping cubes" principle. Rather than spending time on tedious manipulation of the centicubes, they merely cluster the centicubes physically, or work symbolically.

The "building rectangles" phase

Having no knowledge of formal multiplication, the task of creating a rectangle using all 24 cubes is not abstracted into a question of finding divisors of 24 or drawing on known multiplicative number triplets. Holly and Michael approach the task with the direct interpretation of the question, namely, to "use all cubes to make a rectangle". Anchoring their work in a direct model, Michael initiates a trail-and-error attempt of building a rectangle. He places four cubes in a row, upon which Holly places cube number five. She regrets, says "four!" and places the fifth cube beneath the first cube:

Michael:	Yes, 4 and 4 upwards.
Holly:	It doesn't… [Removes the fifth cube]
Michael:	Doesn't… Doesn't it? It might actually work. (…) If we just take 4 and 4 and 4 and…
Holly:	You do downwards, and I'll do upwards.

Having built a 4*4 square, Holly hesitates as she is about to place a centicube number 17:

Holly:	Hold on. But this isn't right. This became a square. And that's not what we need. Can't we have 5 in a row?
Michael:	Okay!
Holly:	Oh, then it's 4 again [now refering to the height of the square]. And then it won't… Yes! If we take, do like this [expands all rows to 5 centicubes], how many 1's do we have here [recounts the height], 4? Then we take 4 more here [places the remaining 4 cubes to extend all rows from 5 to 6, creating a 4*6 rectangle], then we have used all!
Michael:	Aaaah.
Holly:	Haven't we?
Michael:	Now we have used all cubes. But we need to coun… We need to count it all. Okay, let's see, 1, 2, 3, 4, 5, 6… [Having counted all cubes in the upper row, he hesitates, and Holly leans over the cubes.]
Holly:	So, let's say 1, 2, 3, 4, 5, 6 [recounts the upper row], 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24.
Michael:	That's good.

This excerpt clearly shows the students' direct and physical approach to solve the problem. They work "upwards" and "downwards" to reach a point where "we have used all" to create a valid rectangle. This direct model is accompanied by direct counting. Even though the physical placing of the cubes facilitates a rhythmic counting, Holly does not place any emphasis on the numbers 6, 12, 18, or 24.

As Michael goes on with new trail-and-error attempts using the cubes, Holly draws their solution. Rather than drawing the edges of a rectangle followed by drawing lines creating 6 columns and 4 rows, she draws one single column, divides it into six pieces, and repeats the procedure three times. Having finished the drawing, she writes "24" above. Taken together, these results indicate that the multiplicative structure of the solution, 4*6=24, is not in Holly and Michael's attention. This is confirmed by the fact that later in their collaboration, they repeat the attempt with rows of 4 and to "build 6-ers".

The first indication of advancement in multiplicative thinking comes in the late stages of their fourth attempt of reaching a new solution, namely, the 2*12-rectangle. They are building "2-ers", not knowing how many rows that will appear. Holly places the last two cubes:

Holly:	Yes! It worked! Okay, how many do we have? [Starts counting the cubes in the first column] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.
Michael:	12 plus 12 equals …24.

Michael's strategy of validating the solution through adding two equal groups is accepted by Holly and not validated any further. Increasingly, they start suggesting solutions not only referring to the physical "4 and 4 upwards" but to the mathematical "8 plus 8 plus 8", which is their fifth attempt.

The "grouping cubes" phase

Michael's "8 plus 8 plus 8" represents a turning point in their work. From this point on, they approach the task symbolically, adding equal groups of numbers. The centicubes are used to validate solutions. Notably, Holly finds that 16 plus 8 equals 26, causing them to leave the 3*8 rectangle solution. Michael goes on to add "six, six, six, six" using his fingers, but fails to reach 24 as a solution. Notably, the shift from physical to symbolical approach caused them to reject two valid solutions due to calculation errors. Two failed attempts later, having experimented with groups of 7 and groups of 10, Holly has yet again returned to the possibility of using "6-ers", still not realizing that they already found the 4*6 solution. She builds two rows of 6, keeping the remaining 12 centicubes in a single pile. She looks at the pile:

Holly:	Well, how many do we have here? [Counts as she drags cubes from the large pile into a new pile] 1, 2, 3, 4, 5, 6. [Counts the remaining cubes similarly] 1, 2, 3, 4, 5, 6! It works!

Holly and Michael start arranging the cubes into a 4*6-rectangle, and when she is about to draw the solution, she asks herself: "How many 6'ers are there? Four 6'ers." The shift from having to build rectangles to just piling centicubes into equal groups and talking about "four 6'ers" are indications of a more abstract way of solving the task. Without doing actual repeated adding, Holly infers from direct, physical grouping of centicubes that they will form a rectangle if arranged by 4 rows of 6. This represents an interesting case with respect to the theoretical framework of Mulligan and Mitchelmore (1997), as she combines the direct modelling and counting in groups.

Michael starts looking for new solution while Holly is drawing, returning to the "groups of 8" case:

Michael:	Have we tried 8s? [Holly finishes her drawing] What is 16 plus 8?
Holly:	16 plus 8? [Writes "24" above her drawing. Michael starts counting using his fingers.] Count! 16…
Michael:	Yes, it works! Since three 8s are 24!

This solution, the 3*8-rectangle, was derived by Michael without reference to the centicubes. Drawing on repeated addition, he solves the problem purely symbolically. Holly draws the solution, and the teachers breaks off the working time to by initiating a plenary discussion.

Discussion and conclusion

The purpose of this paper was to explore the potential of using a rich task (Hagland et al., 2005) in the introductory phase as primary school students are about to learn a new mathematical concept. Aiming to unveil the students' relevant competencies and initial conceptual understanding, a group of 7-8-year-olds worked on the 24-cubes task. The students' interaction with this multiplicatively structured task was analysed drawing on using Mulligan and Mitchelmore's (1997) results about multiplicative thinking as a framework. “Holly” and “Michael”, not having received any formal instruction about multiplication, collaborated to solve the task. Over the course of a single lesson, they transit from a phase focused on building actual rectangles and relying on direct counting into a phase focused on grouping the centicubes and, eventually, deriving solutions symbolically without referring to cubes.

Indeed, these data confirm research indicating that students intuitively employ direct models and direct counting to solve multiplication problems (e.g., Kouba, 1989). Through interacting with this task, however, Holly and Michael display other competencies relevant for multiplicative work. Holly’s grouping of centicubes in piles of 6 and Michael’s symbolic approach provides valuable insight into their emergent conceptual understanding. Michael’s “three 8s are 24” makes a great starting point for introducing the formal multiplicative notation. Notably, Greer (1992) presents four different multiplicative structures, where the 24-cubes task represents the rectangular arrays case. Rich tasks holding other structures might have prompted other competencies and opportunities.

Conclusively, the 24-cubes task provided the teacher with insight into the students’ relevant competencies and initial conceptual understanding. Drawing on such knowledge, the teacher might identify an appropriate starting point and is provided with several opportunities to build on student thinking in subsequent instruction.

Limitations and further research

One limitation of the results presented in this paper, regards the students’ knowledge at the time of the data collection. They had not yet received any formal introduction to multiplication, but most likely, they have experienced multiplicatively structured situations informally through interaction with parents or peers, and through play or everyday tasks. However, I argue that the data provides an insight into the multiplicative thinking prompted by the rich task.

Further research might explore how rich tasks may be employed in the introduction of other mathematical concepts. Moreover, case studies might unveil how teachers utilize insights generated from such activities and whether they find opportunities to draw on students’ thinking. Theoretically, Holly’s thinking as she piled the remaining 12 cubes into two piles of 6 represented an interesting case, as she combined a direct, physical strategy while displaying insight into the idea of grouping. This might be explored further as a theoretical contribution.

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Conceptualising practical literacy in mathematics teaching

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Recently, a new concept, practical literacy, was introduced at the political level in Denmark. The concept was not clearly defined, and—despite several attempts—it remains unclear what practical literacy entails, especially in the more traditional school subjects such as mathematics. To develop a clearer understanding of the concept, we conducted a literature review that resulted in mostly general perspectives on the concept. Based on the review, we suggest a conceptualisation of practical literacy in mathematics teaching as spanned by three dimensions: involving students' senses and embodied experiences, using authentic problems from the outside world or work life, and producing a product or an event. To further support and qualify the conceptualisation, we relate it to three theoretical approaches within mathematics education research. We argue that this conceptualisation is in line with the political framework and has the potential to support students' learning of mathematics—at least from a theoretical perspective.

Practical literacy, lower secondary education, embodied cognition, mathematical representations, mathematical modelling.

Introduction

In 2018, the concept *practical literacy* (in Danish *praksisfaglighed*[1]) was introduced in a broad political agreement (Ministry of Children and Education [BUVM], 2018). The concept was not precisely defined in the agreement; rather it was stated as a two-fold ambition. One ambition was to attract more students to vocational educations by making students attentive to the potentials of these educations. The other ambition was to strengthen a more application-oriented approach in teaching to increase the students' learning and motivation for especially the natural sciences and technology and support their personal development.

The agreement contains five initiatives of which one relates to the more traditional school subjects such as Danish L1 and mathematics. This initiative was the project, *Practical literacy in schools.* Its main purpose was to give teachers inspiration to strengthen aspects of practical literacy and vocational elements in all school subjects, which was to be achieved through collaborations between schools and vocational schools (BUVM, 2018). Later, several other projects (e.g. Højlund, 2020) have been conducted along with investigations of the concept itself. However, there exists not yet a clear description of the concept, and especially not one that applies to the teaching of mathematics.

In 2023, a new 3-year project was initiated, *Practical literacy in Danish L1 and Mathematics teaching*. Its purpose was to produce 12 teaching units on practical literacy in Danish L1 and

[1] The Danish compound word 'praksisfaglighed' is difficult to translate into English. Its first word ('praksis') means practical and is often used in contrast to theory. Its second word ('faglighed') means work-related and refers to a specific subject area that must meet certain standards characteristic for a profession (BUVM, 2020). For lack of better options, we chose to use the translation *practical literacy*.

mathematics teaching in grades 7-9 through close collaboration between coaches, teachers, teacher educators and researchers (the authors). In order to produce these units, a conceptualisation of practical literacy in mathematics (and Danish L1) is needed. In this paper, we aim to develop such a conceptualisation from a theoretical perspective—a conceptualization that to some extent meet the political ambitions and at the same time has the potential to support students' learning of mathematics. Against this background, we pose the research question: *How can practical literacy in mathematics teaching be conceptualised?*

Methodological approach

Apparently, it is not possible to identify a term in English that resembles 'praksisfaglighed', nor does it exist as a pedagogical or educational term in the research literature in the English-speaking countries or the Nordics countries (BUVM, 2020). The latter was confirmed by our search of related terms in the databases SwePub and Cristin that gather all educational publications in Sweden and Norway. Therefore, we decided to focus only on Danish publications. In our literature review, we used a hermeneutic approach. Such an approach focuses on developing understanding of a problem, concept etc., and views this process as iterative in nature, alternating between processes of searching and acquiring texts and of analysing and interpreting these texts (Boell & Cecez-Kecmanovic, 2014). We conducted three iterations and included 22 texts (see Table 1).

Iteration	Search strategies	Results	Inclusion	Exclusion
First	Google-searching for texts we already knew	11 texts	Books, reports, policy documents	
Second	Database-searching: Bibliotek.dk and NORA. String: "praksisfaglig*"	87 texts reduced to 9 texts	Grades 1-10 Also, peer-reviewed articles	Doubles Other subjects than Danish L1 and mathematics
Third	Snowballing	2 texts		

Table 1: Our three iterations with search strategies, inclusion and exclusion criteria

Between each iteration, we divided the included texts between us and carried out a full text reading, focusing on what practical literacy is, why it is important, and how to teach it. Early in this reading, we realised that a model of practical literacy suggested by one of the included texts (BUVM, 2020) summarised most of the previous attempts of defining practical literacy. We therefore (re)read the 22 texts against this general model and analysed to what extent the model could accommodate their approaches to practical literacy without subscribing to exactly the same line of thinking. Based on this, we made a synthesis, which inspired our conceptualisation of practical literacy in mathematics.

As expected, the 22 texts included only a few research publications, and these publications dealt with practical literacy in general, not in relation to school subjects. We therefore decided to select three overarching theoretical approaches in mathematics education research (embodied cognition, modelling, and representations) to further support and qualify our conceptualisation. In what follows, we present the synthesis and our conceptualisation, argue for our choices of theoretical approaches and describe these approaches, and discuss our conceptualisation in relation to other approaches.

Synthesis of the literature review

One outcome of the first *Practical literacy in school* project initiated by the Danish Ministry of Children and Education, was a general model of practical literacy (Figure 1, BUVM, 2020). This model was inspired by one of the other texts also included in our review (EVA, 2019), and therefore we start by presenting its main points. EVA (2019) investigated which areas of research and practice were part of the political ambition of strengthening practical literacy in teaching at grades 7-9. Based on interviews with 12 national researchers, they concluded that there were various and often conflicting conceptions of practical literacy. Nevertheless, EVA highlighted two aspects as possible features of a common concept. The first aspect dealt with how students should be active during such teaching. The researchers' answers ranged from students communicating their solutions in class to students producing products relevant beyond the school context. The second aspect dealt with the problems students should work with. The researchers' answers ranged from fictive problems that require the use of theory in practical situations, to authentic problems, which solutions are important to other than the students. Based on this, EVA (2019) concluded that practical literacy is a dimension of teaching, in which students participate in at least one of four forms of activities: 1) embodied and physical activities, 2) problem-based and application-oriented activities, 3) activities focused on producing a product, and 4) activities oriented towards a craft profession or another profession.

Our reading of the included texts confirms the various conceptions of practical literacy highlighted by EVA (2019). To exemplify the different conceptions, Tanggaard (2020) draws on ideas of situated learning and conceived practical literacy as learning by participating in the practices of a craft or profession. Others, such as Højlund (2020), draw on Dewey's concept of experience and connected learning with practical experiences, stressing the importance of students' understanding of the meaning of their actions.

Despite these different conceptions, our analysis suggests that the general model can accommodate them, although they do not subscribe to exactly the same line of thinking. The model describes practical literacy in and across subjects as students' competence to connect *theory and subjects* consciously and reflectively with *body and experience* when doing *actions and products* (see Figure 1). Moreover, BUVM (2020) emphasise that teaching based on these three elements relate to six pedagogical keywords: learning by doing, problem-oriented learning, creativity, aesthetic learning processes, the relation between practice and theory, and materiality. All the included texts refer to some of these elements and keywords, although they place different emphasis on them than the model. Therefore, we think it is appropriate to say that the model can accommodate their conceptions at least to some extent.

Figure 1: Model of practical literacy (BUVM, 2020, p.6)

Practical literacy in mathematics teaching

The model of practical literacy in Figure 1 is of a general nature. It has practical literacy in its centre and applies to all school subjects. As we want a conceptualisation specific for mathematics education, we suggest (inspired by the general model) to conceptualise practical literacy in mathematics teaching as a sphere suspended by three dimensions: *1) involving students' senses and embodied experiences, 2) using authentic problems from the outside world or work life, 3) producing a product or an event* (see Figure 2). Its origo consists of pure mathematical content that focuses on concepts and their internal relationships and uses conventional notation, but with no references to the outside world. The further the teaching is from the origo and the more it involves aspects from all three dimensions, the more it will potentially accommodate the students' practical literacy. The first dimension concerning students' senses and embodied experiences ranges from activities where students illustrate mathematical concepts with icons or concrete materials, to activities where students use their bodies (e.g. investigating the functional relationship between the throwing angle and the throwing distance through experiments). The second dimension that regards authentic problems ranges from word problems or stories that have semi-real references, to the outside world to authentic problems from the outside world or work-life, which solutions are important for others than the students (e.g. designing a parking area with areas for specific vehicles, such as SUV-cars, electronics cars, petrol cars, bicycles and motor-bikes). The third dimension related to producing products or events ranges from activities where students analyse existing products or build models of products (e.g. building a model of tiling in cardboard), to activities where students produce the products (e.g. crafting a table based on a model) or carry out an event (e.g. organising a tournament at the school's sports day). When students produce a product or an event, other school subjects will be involved, such as craft and design or physical education, making the teaching interdisciplinary. This is illustrated by the boundary of the sphere.

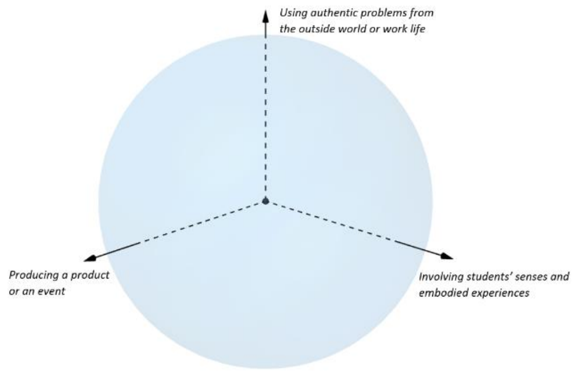

Figure 2: Our conceptualisation of practical literacy in mathematic teaching

Although our conceptualisation builds on the general model, it presents a specific view on practical literacy in mathematic teaching. To further support and qualify the conceptualisation, we relate it to three selected theoretical approaches in research on mathematics education.

Relating the conceptualisation to mathematics education

We start by arguing for our choice of embodied cognition, modelling, and representations as theoretical approaches to qualify our conceptualisation in relation to mathematics education. We claim that the first dimension—involving students' senses and embodied experiences—can be supported by theories of embodied cognition in mathematics (e.g. Radford et al., 2017). The reason is that this growing body of research recognizes that "students' gestures, body posture, kinaesthetic actions, artefacts, and signs in general are […] a fruitful array of resources to be taken into account when investigating how students learn and how teachers teach" (Radford et al., 2017, p. 700). This is particularly important in our conceptualisation as it attempts to highlight the importance of artefacts and students' gestures and senses and not just their cognitive abilities. The second dimension—using authentic problems from the outside world or work life—shares ideas and views with approaches to mathematics education that builds on mathematical modelling (Kaiser, 2017). Hence, the reason for our choice of modelling is that students' work in mathematising real-world problems plays a decisive role in the development of their understanding of mathematical concepts and methods. The third dimension—producing a product or an event—requires creation of relationships between various representations of a mathematical phenomenon. Although the product/event is non-mathematical, it originates from different mathematisations and representations of a phenomenon. Therefore, students' use of various representations plays a crucial role in the production, as well as in their development of mathematical understandings (Mainali, 2021). Next, we outline each of the theoretical approaches.

Embodied cognition

Embodiment in mathematics education is an emerging and developing research area, but the interest in the area is growing (Radford et al., 2017). The variety of theories into the cognitive role of the body share the premise that "meaning and cognition are deeply rooted in a physical, material, embodied existence" (Radford et al., 2017, p.717). All the theories attempt to explain how meaning and thought are related to these existences, and their differences can be tracked back to the long-standing philosophical problem concerning the relationship of the body, the senses, and the mind. In their overview, Radford et al. (2017) describe several emerging theories such as multimodality, enactivism, phenomenological approaches, and approaches based on materialism. Across these theories embodiment ranges from being embodied actions that are to be superseded by flexible actions with symbols, over a faculty of the body that support the creation and learning of mathematical constructs (multimodality), to more encompassing approaches with a nondualist view on the separation between the body and the mind. As we in our conceptualisation aim for a strong relation between students' senses and embodied experiences and their mathematical understanding, we seek support in a nondualist and materialistic view, namely Radford's *sensuous cognition*.

Sensuous cognition goes further than most nondualist theories by claiming that cognition and bodily experiences are almost two sides of the same coin, "We think (practically and theoretically) with and through our senses … perception, tactility, gestures, sounds, movement, and material objects do not mediate thinking. They are part of it" (Radford et al., 2017, p. 713). The theory understands material objects and cultural artefacts (concepts) from a cultural-historical perspective, that is, as bearers of historical intelligence developed and refined in the course of cultural development. These objects and artefacts are, thus, not conceptually neutral; they affect in a definite way the manner we come to know about the world. Cognition is, therefore, to "be understood as a culturally and historically constituted sentient form of" (Radford, 2014, p. 350) individuals' interactions with the world such as sense-making. In addition to providing an argument for the importance of involving students' senses and embodied experiences, this perspective also strengthens our conceptualisation by highlighting the close relationship between material objects, cultural artefacts, students' senses, and their cognition.

Modelling

Since the 1960's, modelling and application has played an important role in mathematics education at least in Europe and North America. The promotion of modelling as a creative process of making sense of and solving real-world problems using mathematics is now accepted as a central goal of mathematics education worldwide and is included in many national curricular (Kaiser, 2017), also the Danish. Kaiser (2017) identified four widely used goals related to the teaching of mathematical modelling: 1) Pedagogical goals: imparting abilities that enable students to better understand central aspects of their world. 2) Psychological goals: fostering and enhancing students' motivation and attitude toward mathematics learning. 3) Subject-related goals: structuring learning processes and introducing new mathematical concepts and methods. 4) Science-related goals: imparting a realistic image of mathematics as science. These goals distinguish the various approaches to mathematical modelling that existed at the beginning of the 21st century and that span two extreme positions (Kaiser, 2017). At one extreme, *realistic or applied modelling* stresses the pedagogical goal, whereas at the other extreme, *epistemological or theoretical modelling,* emphasises the subject-related goals and assigns low importance to the reality aspects of the problems. Of the four approaches that are placed between these two extremes, *conceptual modelling* supports our conceptualisation the most. The reasons are that it focuses on the three goals: pedagogical, psychological, and subjects-related, and that it uses "real-world examples to introduce new mathematical concepts and to foster [understanding]" (Kaiser, 2017, p. 273). Conceptual modelling thus emphasises the importance of using real-world problems to strengthen students' understandings of the world, enhance their motivation to learn mathematics, and support their understanding of mathematics.

Representations

In his overview, Mainali (2021) describes representations as "multiple concretisations of a concept and its common practices" (p. 2). A representation is something that stands for something else. It may be internal, in the form of a mental image or process, or external, in the form of an embodied internal representation such as signs, characters, icons, diagrams, graphs, or objects (Mainali, 2021). A representation illustrates certain aspects or properties of a mathematical concept, object or process, and the use of a number of representations of a phenomenon can provide a more holistic understanding of this phenomenon and contribute to a deeper and more nuanced understanding. Duval (1995) even states that "no knowledge […] can be mobilised by an individual without a representation activity" (p. 15). Considering that mathematics plays a central role in conceptualising the world, one might say that representations act as connections between the outside world and mathematical concepts. For our purpose, especially approaches that take account of the relationship between representations and the world are important. Such approaches include, for example, manipulatives, static figures, spoken language, written symbols, and experience-based real scripts models (Mainali, 2021). These models provide outside-world examples whose contexts may enable students to generalise and transfer their ideas to other situations.

Frameworks that take account of the relationship between representations and the real world support the importance of the third dimension: producing a product or an event. When producing a product or an event, students will inevitably need to translate between the physical object or event and one or more representations (e.g. verbal, numerical, graphical or algebraic). This may provide a concrete response in relation to the applicability of the chosen representations. Thereby, students gain insight into different representations and their strengths and weaknesses, as well as experiences in translating between them, all of which supports deeper mathematical understandings (Mainali, 2021).

Discussion

Our conceptualisation of practical literacy in mathematics teaching is, among other things, supported by conceptual modelling. It therefore shares some of the potentials that studies in this research area show. One such potential is the psychological goal of motivating students by using real-world problems (Kaiser, 2017). For our purposes, this goal is particularly important, especially for the large number of students in lower-secondary education who are tired of going to school or see no meaning in learning mathematics. By using real-world, or authentic problems as in our case (problems that are important for others than the students), our approach aims to make it possible for these students to make sense of mathematics in relation to problems from their own world and work life.

In studies on modelling, there is a consensus that modelling is an activity that students must experience themselves; it is not possible to learn by being told how to do it (Kaiser, 2017). Therefore, the first dimension—involving students' senses and embodied experiences—will also typically be included in classroom teaching of modelling. However, our conceptualisation distinguishes itself from modelling approaches in terms of the envisioned product. The product of a modelling process is a mathematical model, typically a function that describes the relationship between selected unknowns in a real-world problem (Kaiser, 2017), whereas in our conceptualisation, the product is not mathematical, but rather a material object, such as a specially designed table or a prototype of a table, or it is an event, such as a flea market at the school. Hence, our conceptualisation distinguishes itself from conceptual modelling and other modelling approaches by explicitly combining authentic problems and students' embodied experiences with a non-mathematical product.

Our hypothesis is that especially this dimension—producing a product or an event—is fruitful, as students must experience that mathematics can be helpful and often necessary when creating a product or an event that solves problems in our world or work life. Furthermore, it is important that students create the product or event themselves as it requires other competencies of a more practical nature, and often there will be a need to co-ordinate these competencies with mathematical thinking to create the best product or event.

There are other attempts to emphasise more practical aspects in approaches to mathematics teaching such as the 21st centuries skills, mathematical literacy as used in PISA, and modelling. We will argue that our conceptualisation distinguish itself from such approaches by combining three key features: 1) focusing on mathematics in relation to the outside world, work life, and other subjects, 2) promoting students' mathematical understanding by encouraging them to engage sensuously and bodily with various representations, 3) focusing on the role of mathematics in processes that result in a non-mathematical product or event.

Conclusion

In this paper, we propose a way to conceptualise practical literacy in mathematics teaching based on a literature review and a general model, and we support and qualify our conceptualisation by relating it to three theoretical approaches in mathematics education: embodied cognition, modelling and representations. The proposed conceptualisation is spanned by the three dimensions: involving students' senses and embodied experiences, using authentic problems from the outside world or work life, and producing a product or an event. We suggest that the more aspects from each dimension mathematics teaching involves, the more it accommodates students' practical literacy. It is thus not a question of whether mathematics teaching is practical literacy or not, but to what extent it is. Based on the general model published by BUVM (the Ministry of Children and Education), our conceptualisation seems to comprise the political ambitions, and the three theoretical approaches supporting it suggest that it has the potential also to support students' learning of mathematics. Further studies will show if this is also the case in mathematics classrooms targeting practical literacy.

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Lower secondary mathematics teachers' understanding of quality in inquiry-based mathematics

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Despite the fact that all teachers probably have an intuitive sense of the concept of quality, and the fact that inquiry-based mathematics can be described as containing a substantial and mastery-oriented concept of quality, it calls for a nuanced and systematic approach when working with quality. By using a mixed-method exploratory case study design we have applied a multidimensional model that implies a prescribed, experienced and documented perspective on quality to investigate lower secondary school teachers' understandings of quality in inquiry-based mathematics. Results show two teachers with very different understandings of quality in inquiry-based mathematics and also that the applied method can be an important tool for professional development.

Keywords: Quality in mathematics, Inquiry-based mathematics, Professional development.

Introduction

What can be considered as quality in mathematics teaching? The definition of quality in mathematics teaching is not unique and there is a risk that quality in mathematics teaching is considered equivalent with efficient instruction and students' improvement i.e., higher grades and more correct answers (Hansen, 2023). Now there is an increased attention to observe teaching quality with specific observation systems (Agergaard & Graf, 2022), for example the observation system PLATO (Protocol for Language Arts Teaching Observations) from Stanford University. PLATO was originally developed for observations of middle and high school English as Language Arts (ELA) in the USA but is also used for observing mathematics and social science in the Nordic countries (UiO, University of Oslo, n.d.). Plato derives its codes from value-add research in effective instruction and provides scores that characterize teaching quality across a number of conceptually independent dimensions (Grossman, 2015). Yet, quality in mathematics teaching calls for more than efficiency and the use of observation systems.

In this paper, we will investigate the quality of inquiry-based mathematics (IBM) in lower secondary school. We will investigate this theoretically informed and empirically grounded with the perspective of lower secondary school teachers' understandings of IBM and what opportunities and challenges the lower secondary teachers have in relation to understanding the quality of IBM.

Theories

IBM concerns problems, experiments and investigations, and is based on a question, hypothesis or a problem that is investigated, documented and interpreted and finally conclusions are made. These conclusions or findings can then lead back to the original question or hypothesis and lead to a new investigation. In this way, inquiry-based teaching consists of a cyclical process (Artigue & Blomhøj, 2013; Blomhøj, 2021; Hansen & Hansen, 2013). Inquiry-based teaching is broadly defined and calls for a categorisation of various investigative activities. Larsen & Lindhardt (2019) defines and describes five investigative and/or explorative activities: the puzzle, the discovery, the measurement, the product and modelling.

Despite an intuitive sense of the concept of quality, Harvey and Green describe quality as a "slippery concept" that is difficult to concretize (Hansen, 2023). At the same time, Nikolaj Elf, calls for "quality in the quality discussion" by seeing quality from several perspectives (Elf, 2022).

A multidimensional model, which makes it possible to view quality from several perspectives, was presented in the KiDM project (Quality in Danish and Mathematics) (Hansen et al., 2020). The model is based on three perspectives on quality: Prescribed, Experienced and Documented quality. The prescribed quality relates to the political and cultural values (for example the curriculum for the lower secondary school, contributions from policy makers and researchers) and focuses on quality in the future. The perspective of experienced quality is based on the experiences here and now, for example students', teachers' or parents' experiences and focuses on simultaneous quality. Finally, the perspective of the documented quality is based on results and observations in connection with investigations (e.g., competency tests) and hence reflects a past (Elf, 2022; Hansen, 2023).

This leads us to the following research questions: How do lower mathematics teachers understand quality in inquiry-based mathematics from the perspective of prescribed, experienced and documented quality?

Methods and data

This paper is case based and concerns two teachers named M and L. They are both participants in the 2-year development project Faglige Udviklingsskoler [Professional development], where we together with four experienced lower secondary mathematics teachers, design, perform and reflect on interventions concerning IBM (Artigue & Blomhøj, 2013). To qualify designing the interventions we discussed certain aspects of the theories concerning IBM; the cyclical process of IBM, categorisation of investigative activities and degrees of freedom in IBM (Artigue & Blomhøj, 2013; Larsen & Lindhardt, 2019; Thomsen & Skånstrøm, 2017). The aim of the project was to construct a long-term collaboration between schools and the teacher education through a common experimental development of the teaching of mathematics. The data for this project was collected within the first year of the described development project above. The two teachers were selected to participate in this research project by us based on their different perspectives on quality in IBM. Teacher M has an education as mathematics teacher and has taught mathematics in lower secondary school for 19 years and does also teach Sports and Biology. At the time of the project, she taught 8th grade. She does not have further education or other roles at the school in relation to mathematics. Teacher L has an education as an English and Social studies teacher but has taught mathematics for 25 years. She teaches mathematics in 4th grade and 8th grade at the time of the project. Teacher L also has a diploma in mathematics supervision (30 ECTS) and serves as a supervisor of mathematics. She is appointed external examiner, attends the yearly conference for examiners, is leader of the mathematics team and facilitates the mathematics developments laboratories at the school.

We audiotaped the workshop-meetings in the second part of the first year of the project (n=3), which were the core of the development project. Additionally, we observed one lesson with each of the teachers and their 8th grades, respectively, which was video-recorded with one static camera focused on the teacher and afterwards coded with respect to the PLATO-manual (Grossman, 2015) (n=2) and two lessons where the teachers from the development project taught their classes together, which was video-recorded with two handheld cameras following the two teachers (n=4). The observations were conducted with a non-participating approach (Fangen, 2010). We also obtained data through semi-structured interviews with each of the teachers, although the interview with teacher M was conducted together with one of the other teachers from the development project, (n=2) and we also interviewed a group of students from each of the teachers' classes (n=2). The students were selected by their teachers. The interviews were audiotaped and transcribed in full. Based on the multidimensional model of perspectives on quality (Hansen et al., 2020), we developed thematical codes (Tanggaard & Brinkmann, 2020), with respect to the prescribed, the experienced and the documented quality perspective by the different time periods in which the perspectives explore quality. Since the teachers did not know the time perspectives, we are aware that this analysis required an interpretation of the teachers' statements, e.g. passive voice that indicates the prescribed quality. The transcripts from the interviews, the workshops and the video-recorded lessons were thematically coded. We include both explicit and implicit statements and actions in the interpretations of the teachers' understanding of quality in IBM regarding the cyclical process of IBM, the categorisation of investigative activities and aspects on degrees of freedom in IBM.

The design of the research project was based on a mixed-method exploratory case study design, because of its possibilities to empirically explore a selected phenomenon in the context in which the phenomenon takes place (Johnson & Christensen, 2014). The aim of this case study was to obtain a detailed and specific description of how the teachers talked about and acted in relation to IBM. The case study method made it possible for us to go into more detail and depth. The mixed-methods design was chosen based on the methods being complementary to each other and being an analytical supplement to each other (Frederiksen, 2020).

Prescribed, experienced and documented quality in inquiry-based mathematics

When we consider the coded transcripts, certain themes emerge. We will first highlight the themes for teacher M, then for teacher L, all quotations from the transcripts and from articles are translated into English by the authors - in terms of meaning, not word for word.

Teacher M's personal interpretations

Teacher M expresses personal interpretations with respect to the investigative activities, in particular the measurement and modelling (Larsen & Lindhardt, 2019). The definition, which the teachers have seen and worked with, for the measurement is "to apply mathematics in a scientific study, as well as considering reliability and validity of the obtained data and calculations" (Larsen & Lindhardt, 2019, p.12). The teachers start planning an investigative activity the 22nd of February and revisit their plans the 12th of April, both times while planning and reflecting teacher M instead addresses the aspect of measuring in relation to the students working with non-standard units – a rubberboot - as an argument for their activity being a measurement: "I think our aim has to do with measuring" (220223, 01:30), "the students work with understanding the concept of measuring" (120423, 04:54), "that is our argumentation for measuring" (120423, 07:05). Hence Teacher M's seems to equalise the investigative activity measurement and the subject measuring.

The same applies to the investigative activity called modelling, which the teachers also have seen and worked with, which is defined as: “describe and analyse real-life situations by developing and applying mathematical models” (Larsen & Lindhardt, 2019, p.13). Here teacher M understands modelling as presentations with artifacts in mathematics: “We have talked about modelling, but I have done that a lot in biology, for example when [my students] had oral presentations, they had to do some modelling to show the other students some things” (220223, 17:36). When asked what she considers important, when she plans IBM, she answers: “it must result in a presentation with a model, or a written work” (Int., 260423, 25:00).

The third personal interpretation considers the requirements for when an activity is investigative. While planning, teacher M relates to whether there is one and only one answer to the given problem. If that is the case, then it is not IBM and she rejects the activity: “Here there is only one answer, one possibility, then it is not investigative” (120423, 27:48). Here she only considers the degrees of freedom regarding answers and not methods (Larsen & Lindhardt, 2019; Thomsen & Skånstrøm, 2017). Yet, earlier that day, she indirectly used the scaffolding tool from Hansen & Hansen, they mention confirmatory investigations, investigations with scaffolding, guided investigations and open investigations, by saying: “There are both scaffolded investigations, where you sort of guide the students, and then it ends up, when they hopefully have gained some basic knowledge, in a much more open and investigative way” (120423, 08:56). Somehow, she doesn´t consider scaffolded investigations as part of IBM, only as an important steppingstone on the way to the more open-ended problems. These personal interpretations affect her understanding of the prescribed quality of IBM and also inhibits her opportunities for experiencing certain aspects of quality of IBM (Elf, 2022; Hansen, 2023).

Teacher M’s actions in class and statements about actions

Both authors are PLATO-certified, and we have scored an inquiry-based teaching situation consisting of 4 consecutive segments of 15 minutes each for teacher M with respect to the codes: Purpose, Intellectual Challenge and Modelling (Grossman, 2015). The students are working in groups and are supposed to construct a plot of land and a house with specific requirements, and they must use Geogebra. All along she gives precise instructions to the students and even shows them how to do it: “You can't, you'll find out when you try (…) Watch me construct”, Student 2:” I figured it out, or M did” (280922, 28:15), ”Try to watch, we can drag the figure. I will show you how to drag to the right area” (280922, 50:02). The score for Purpose and Modelling is constantly high, the score for Intellectual Challenge is significantly reduced throughout by teacher M`s scaffolding to manageable sub-elements, with an entirely technical focus. We also see the same tendency in the implementation of the developed course. Here teacher M only asks the students about their plans for the inquiry, she does that several times, “Do you have a plan? Have you written down your hypotheses? Have you shared the document with each other?” (310523, 21:00-26:00), or she makes specific choices for the students: “Last time you found out how long the string was in rubber boots. If the string measured 40 rubber boots, then you could lay out the string, measure, and reason that the length was 40 and subtract that amount of rubber boot lengths” (310523, 35:37). Her students describe her teaching like this: “We have a book, our teacher puts the page on the board and then we usually just do the tasks individually” (Int., 310523, 00:35).

We see little signs of her understanding of the experienced and the documented quality (Elf, 2022; Hansen, 2023). For her IBM is a “new alternative type” (Int., 260423, 19:11) and she “works differently” (120423, 23:26), yet here she and a fellow teacher talks about how they observed the students playing a game concerning probability as an inquiry-based activity: “we were walking around and writing notes of little things we'd heard them say and we could see that they were referring to something they had done in class” (Int., 260423, 17:26). She also reflects on the difficulty of documenting the quality of inquiry-based learning (Elf, 2022; Hansen, 2023): ”I guess it's hard to prove, I can't explain why, but I think it sticks better somehow” (Int., 260423, 26:39).

Teacher L: Let us have fun in mathematics!

Teacher L seems to value having fun in math-classes, here they are preparing for Barbie to make a Bungee jump out of the window, an example of a reflective inquiry process (Artigue & Blomhøj, 2013): “We could do it with filled water bottles, but somehow it is much more fun with Barbiedolls” (091122, 02:03). At the interview she elaborates on the importance of having fun, having fun is also relevant to herself: “They really like it, especially when it gets a bit silly (…). They like that. <u>I like that</u>. (…) we all like it when you do something that is fun and a little different” (Int., 260423, 10:53). The students are used to these aspects of mathematics teaching, so much so that they sometimes hesitate: “I think I made a point of telling them that math can be fun too. Yesterday, I gave them sweets, it was my birthday. (…) The students said: Are we going to use it for math? Or is it just because you are nice to us?” (Int., 260423, 11:30). While working with the bungee jump, the students were loud, all over the place experimenting and teacher L said: “That's what always goes wrong when you're doing something like this [the last group enters the class]. It is so much fun and then I forget to stop you in time [looking at her watch]. Now we have 3 minutes to recap” (091122, 50:10).

Teacher L: Inquiry-based mathematics is fun

L is very aware that there is a difference between just having fun and having fun with math, and she thinks that her students can make that distinction too: “I think that they start to understand that when we make fun, it's a part of math-teaching. Previously when we had fun, it usually only had a social purpose. Now it also has a mathematical purpose” (Int., 260423, 12:21). She also refers to specific characteristics of IBM, i.e., experiences with the growth of further mathematical experience (Artigue & Blomhøj, 2013): “But then the students become curious and get some unexpected mathematical experiences. It's amazing when they're sitting there and then suddenly, they say: ““YES, that's the way it is.” It is wonderful.” (Int., 260423, 09:53). When planning and reflecting with her fellow teachers, she indirectly distinguishes between typical tasks which build up skills and inquiries, here she reacts to a suggestion where the students have to figure out how many rubber boots is equivalent to a ladder: “I find it difficult to proceed, it is boring and I'm not really into that right now” (120423, 05:37), yet she doesn´t mention specific theories, just that she finds it boring. She herself suggests inquiry-based tasks: “It could be fun to let them measure the volume of the rubber boot without telling them how” (120423, 12:43) and reacts positively to inquiry-based tasks: “Now, that could be fun!” (220223, 30:13), when one of the other teachers suggest throwing different rubber boots from different steps from the ladder and measure the different throwing distances. She keeps turning more traditional tasks into open-ended problems: “I think the task will be more interesting if you don’t say anything about how to find areas of the football pitch. If you don’t say anything about how to do it. After all, that's also the thing about how open ended the task should be. I think that could be a fun task” (120423, 30:13), but she doesn´t mention specific theories as evidence for her suggestions and changes, just that it could be fun. Her understanding of the experienced quality of IBM has depth, even if she just distinguishes between fun and boring, more importantly it is without a discernible connection or attachment to the prescribed quality (Elf, 2022; Hansen, 2023). When we analyse her actions while implementing the developed course, teacher L exclusively asks the students open-ended questions, that leads to further investigations, e.g. “Why does it not make sense to you?”, “What is the problem?”, “Does it make sense to you?”, What is the confusing part?” (240523, 24:00-27:00).

The Plato-scores for Purpose, Intellectual Challenge and Modelling (Grossman, 2015) of the teaching situation with Barbie-dolls, confirms her understanding of the experienced quality of IBM. The score for purpose is high, yet she addresses the focus on inquiries instead of concepts or results: “You need to make several experiments to find out how many rubber bands are needed for Barbie to bungee jump out of the window without hitting her head” (091122, 00:38) and “Remember to take notes, don’t focus on answers but on explanations and the choices you make” (091122, 01:25). We see no Modelling at all, and the Intellectual Challenge is high throughout the lesson: “make some assumptions before trying” (091122, 03:25), she also gives some of the students a new challenge: “Here is a new challenge. Try to calculate how many rubber bands are needed when bungee jumping at the stairway instead” (091122, 38:10).

Statements from her students support an inquiry-based teaching: “then we experiment a bit, it’s very good, because you find out yourself how to figure things out” (Int., 310523, 02:00). She has many experiences with IBM, they are being generalised and is now a part of her cultural and value-based foundation for mathematics in general, hence her own prescribed quality for IBM (Elf, 2022; Hansen, 2023). She has also noticed that the final tests in mathematics have open-ended tasks “For example, some of the tasks where you have to investigate” (Int., 260423, 16:45). We see little signs of her understanding of the documented quality, she emphasises the experienced quality (Elf, 2022; Hansen 2023): “On a scale from 1 to10, how interesting has it been working with Barbie dolls?” (091122, 54,30).

Conclusions and perspectives

Teacher M’s personal interpretations of the theories regarding IBM makes it difficult for teacher M to have a nuanced understanding of what IBM is. As a consequence, there are aspects of the prescribed quality that are not possible for teacher M to link to IBM. Together with her personal interpretations, teacher M’s actions in class limit her possibilities for experiencing quality in IBM significantly. Even though teacher M describes one experience with an inquiry-based activity and how she explored and documented the students work, her understanding of the experienced and documented quality is barely established. Teacher L on the other hand has a more nuanced understanding of the experienced quality and her experiences has over time become a cultural value associated with IBM, hence her own prescribed quality of IBM. ‘Fun’ is what makes up the foundation for the experienced quality, and therefore also the prescribed quality, but also teacher L’s investigation of the students’ opinion about working with IBM, hence the documented quality. Their understandings of the prescribed, experienced and documented quality of IBM are illustrated in Figure 1 below.

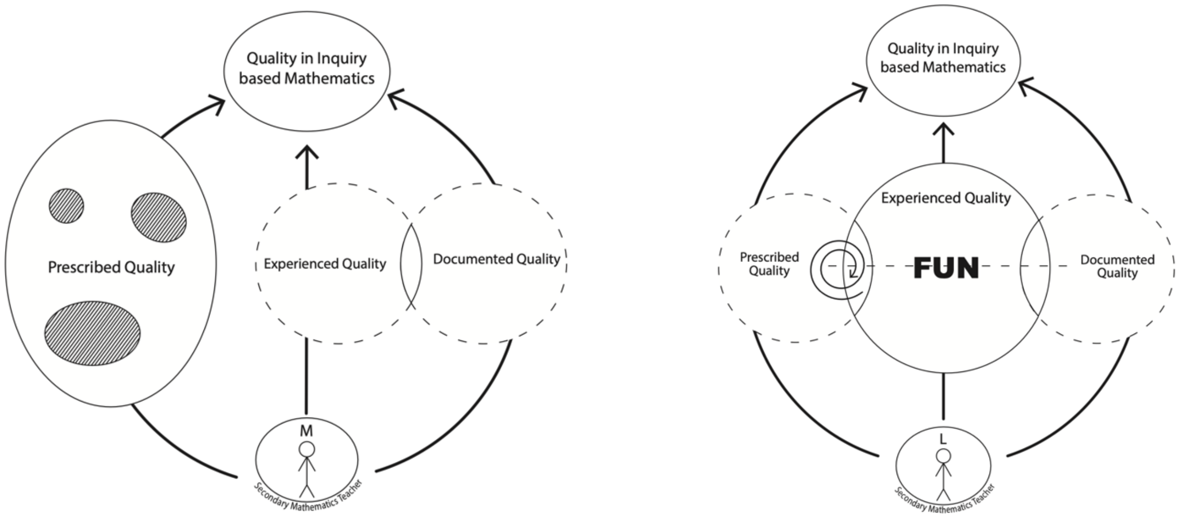

Figure 1: The understanding of the prescribed, experienced and documented quality of inquiry-based mathematics for teacher M and teacher L

As mentioned, teacher M and teacher L are part of the development project Faglige Udviklingsskoler and the aim of this project is to construct a long-term collaboration between schools and the teacher education through a common experimental development of the teaching of mathematics, where we in collaboration with the teachers have chosen IBM as our object of development. In a literature review by Dreyøe et al. (2018), the authors present key-elements for accommodating an inquiry-based approach in mathematics. They point out that issues arise when teachers don't have any personal experience with IBM and highlights the need for a broader repertoire of ways to evaluate students' mathematical competencies. Our findings above illustrate examples of these issues and the mapping of the teachers' understanding of quality in IBM gives us insight to support and qualify the further work with the professional development of the teachers in the development project. For example, we see that it is crucial for teacher M to get more personal experiences with IBM and teacher L lacks nuanced methods for evaluating students' work.

Dreyøe et al. (2018) also mentions that good results are achieved concerning the teacher's ability to reflect on certain aspects of IBM by utilizing already planned and targeted inquiry-based teaching activities. Despite the usage of already planned and targeted inquiry-based activities in our development project, we have identified a need for a more systematic approach and a common language linked to IBM and the concept of quality to obtain deep reflective discussions. It is relevant to consider whether the specific multidimensional model of the perspectives of quality and the corresponding method could be a concrete tool for researchers as well as teachers to obtain depth in the reflections on the quality of IBM.

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Unpacking teacher agency in communities of inquiry in mathematics teacher education

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Teacher agency is essential for educational change, empowering educators to customize their teaching methods to meet diverse student needs, cultivating inclusive and engaging learning environments. In the realm of mathematics teacher education, a Community of Inquiry (CoI) describes a collaborative professional development approach rooted in inquiry-based practices. CoI encourages questioning, problem-solving, exploration, and inquiry as an epistemological stance on teaching practices. Although teacher agency has been linked to taking a critical stance in CoI, the concept of teacher agency in CoI remains underexplored. This paper delves into the nexus between CoI and teacher agency, focusing on the relationship between them. The conceptual work in this paper is exemplified by a CoI exploring digital fabrication for making mathematical manipulatives.

Keywords: Teacher agency, Professional development, Mathematics teachers, Community of Inquiry.

Introduction

Change in education is inevitable in teachers' professional lives. For instance, the access to and use of digital tools has significantly influenced and fundamentally transformed our educational practices (Lund et al., 2014). To stay up-to-date with societal and technological shifts and address economic pressures, teachers must innovate and evolve their practices. This involves embracing new professional roles, nurturing professional identities, and integrating new insights into teaching and learning (Day & Kington, 2008). This ability to change teaching practices is commonly referred to as teacher professional agency (Cong-Lem, 2021). However, cultivating teachers' professional agency requires deliberate efforts. Professional development approaches should not only offer an opportunity to gain new competencies but also a means to nurture professional agency.

In the realm of mathematics teacher education, a Community of Inquiry (CoI) describes a collaborative professional development approach rooted in inquiry-based practices. CoI encourages questioning, problem-solving, exploration, and inquiry as an epistemological stance on teaching practices (Jaworski, 2006). Although teacher agency can be linked to taking a critical stance in CoI, the construct of teacher agency in CoI remains underexplored. In this article, my goal is to compare teacher agency to CoI and initiate a dialogue on how a perspective on agency can contribute to developing CoI theory. I analyzed both theories using networking strategies (Prediger et al., 2008). To illustrate my analysis, I present a CoI exploring digital fabrication (DF) for making mathematical manipulatives. The remainder of the paper is organized as follows: The second section provides the theoretical foundations of CoI and teacher agency. Subsequently, I outline the methodology employed for my analysis, followed by a brief overview of a CoI in DF for making manipulatives. Lastly, I argue how agency is rationalized in CoI and briefly discuss my contribution.

Community of Inquiry

CoI originates in mathematics teachers' professional development, focusing on how teachers and teacher educators can collaborate to improve mathematics education towards an inquiry approach. Jaworski (2004, 2006) emphasizes that teaching, learning, and research are fundamentally interconnected and that teachers and teacher educators learn through a shared engagement in a CoI. The application of CoI extends across a spectrum of professional development projects, addressing mathematics education from primary school to university levels (Jaworski & Potari, 2021). These projects underscore the central constructs of inquiry and collaboration as essential elements for the development of and transformation toward effective teaching and learning practices.

The construct of inquiry in CoI is fundamental and draws on Wells' (1999) dialogic inquiry and "indicates a stance toward experiences and ideas – a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them" (p. 121). Jaworski (2006) addresses three levels of inquiry practice and collaboration between students, teachers, and teacher educators, all participating in a CoI. Collaboration and learning in CoI can be described by Community of Practice (CoP) constructs (Wenger, 1998). This involves articulating three modes of belonging: engagement, imagination, and alignment. Engagement describes the active involvement in mutual processes of negotiation of meaning by "mutual engagement in shared activities" (p. 184) and "the production of a local regime of competence" (p. 184). Imagination implies creating images of the world and seeing connections through time and space by extrapolating from personal experience. This requires the ability to look at one's engagement through the eyes of another, disconnect oneself, and reflect on what one is doing in relation to a broader system. Alignment describes coordinating activities "in order to fit within broader structures and contribute to broader enterprises" (Wenger, 1998, p. 173), becoming part of something big, and doing what it takes to play one's part.

In contrast to CoP, Jaworski (2006) suggests alignment to be critical alignment in a CoI. Modes of being in a CoI are based on a critical stance where individuals question their role and norms as part of their participation. This is characterized by participants having an awareness and ability to reflect, step outside themselves, and question their own practices to explore and try new ways of teaching to avoid "perpetuation of undesirable practices" (Jaworski, 2006, p. 187). It is when participants are not aligned but are critically aligned that defines a "community of practice to become an inquiry community, [and] critical alignment is both a goal and an outcome of activity" (Jaworski, 2006, p. 206). "Through the exercise of imagination during engagement, alignment can be a critical process in which the individual questions the purposes and implications of aligning with norms of practice" (Jaworski 2006, p. 190). In summary, a CoI can be characterized using four key constructs: inquiry, engagement, imagination, and critical alignment.

Jaworski (2006) comments that personal agency in a CoI is "feelings of confidence and power in our own professional domains, the ability to inquire into aspects of our practice and to control our own activity and development, at least in part" (p. 197). In this aspect, CoI differs from CoP, according to Biza et al. (2014), stating that "agency is not addressed in the CoP theory, namely how self-directed individuals respond and affect the learning environment within which they practice" (p. 174).

Goodchild (2014) contends that agency poses a challenge within CoP theory and notes that "CoI is a development using adapted constructs from CoP/CPT, but the adaptation introduces new constructs such as agency, goal directed actions, contradictions and tensions" (p. 178). Furthermore, Goodchild et al. (2021) connect CoI to agency through the notion of critical stance, which "can be distilled into three components: awareness, self-evaluation, and agency" (p. 231). Nevertheless, neither Jaworski nor Goodchild delve deeper into the meaning of agency and how a CoI can nurture agency.

Teachers' professional agency for change

Agency is an important aspect of teachers' transformative practice, where teachers develop or change their way of teaching (Hökkä & Eteläpelto, 2014). Generally, it is seen as people exercising their agency to make a difference by actively shaping their lives, challenging the way things are, or taking a stand against undesirable conditions (Goller & Paloniemi, 2022). Vähäsantanen (2015) proposes three perspectives of teachers' professional agency for change. Firstly, agency "can be understood as the teacher's opportunities to influence his or her own work" (p. 2), secondly agency "encompasses the choices and decisions made by a teacher concerning her/his involvement with an educational reform during its implementation … [which] involves a process of sense-making through which teachers make meaning from their work environments" (p. 2), and thirdly, "professional agency is connected to the nature of professional identity amid changes" (p. 3). Current research on teacher agency can be categorized into interventions for developing teacher agency; agency change trajectory; outcomes of teacher agency; influential factors on teacher agency; teacher agency enactment; and teacher cognition (Cong-Lem, 2021). Teacher professional development approaches that emphasize teachers' self-reflection practices (Lambirth et al., 2019; Wallen & Tormey, 2019) are important in nurturing teacher agency, facilitating teachers' inquiry into their own conceptions and past experiences. The transformation in teacher agency typically starts with teachers inquiring into their own pedagogical beliefs, conceptions, and sense-making of their prior experience (Cong-Lem, 2021). However, studies on agency often do not clearly define how teachers' agency can manifest to achieve educational change (Pantić, 2015).

Inspired by Giddens' (1984) and Archer's (2000) work on agency, Pantić (2015) proposes a framework defining teacher agency as a process whereby teachers act strategically to their teaching, positioning teachers' agency within interrelations of teachers' individual and collective *sense of purpose*, *competence, autonomy,* and *reflexivity* as units of analysis. Teachers' sense of purpose is reflected in the active involvement of participants in the project, as well as their perception of their identity as professional educators concerning their capacity to enhance their teaching methods, ultimately leading to a better understanding among their students. Competence is displayed through teachers' engagement to change or plan to change their practices (Pantić, 2015). Applying an inquiry-based teaching mindset can be viewed as such competence (Woodruff, 2021). Autonomy "depend[s] considerably on the contingencies of the contexts (of school, policy or broader societal and cultural environments) that can be seen as structures and cultures" (Pantić, 2015, p. 768). The scope of autonomy influences teachers' ability to implement new practices, make decisions, and enact their plans. Autonomy can be identified by teachers' possibility to adapt their teaching in relation to their experienced competence. Reflexivity "including the capacity to articulate tacit knowledge, meaning-making of structures and cultures, critical reflection on own practices and transformative learning" (Pantić, 2015, p. 773), is an essential part of teacher agency for change. Teachers' reflections make tacit knowledge useful for individuals and the community. The ability to step outside oneself and reflect on the practice is essential for leaving well-established practice and making sense of one's actions (Thompson & Pascal, 2011). Indicators for reflexivity can be teachers discussing how, when, or why they have utilized a specific teaching approach or not.

Method

In this section, I present a CoI in DF for making manipulatives. DF is referred to as "the process of translating a digital design developed on a computer into a physical object" (Berry et al., 2010, p. 168). I applied networking strategies as a method to realize how CoI can nurture teacher agency.

CoI in DF for making manipulatives

Our CoI included three teacher educators, one DF expert, and eight in-service elementary teachers (referred to as teachers) with previous experience using manipulatives. We conducted eight monthly full-day workshops to collaboratively inquire into DF and its potential to develop practices for making and using manipulatives. In the workshops, the participants inquired into DF tools and techniques including: DF sharing platforms, 3D modeling and printing, 2D modeling and laser cutting, and design thinking processes to support the design of new manipulatives. Adopting an open and exploratory approach and drawing on previous work (Goodchild et al., 2013; Jaworski, 2004, 2006), our aim was threefold: 1) inquiring how to introduce DF technologies in the context of making manipulatives for mathematics, 2) inquiring into teachers critical reflections on DF for making manipulatives. 3) building competencies in DF technologies, manipulatives, and the use of manipulatives in the classroom. Each workshop comprised four parts: reflections, lecture, group work, and presentations inspired by Frank et al. (2011). The teachers used their manipulatives in their classrooms between the workshops. A more detailed description of the overall workshop design can be found in H. Stigberg et al. (2024) and S. K. Stigberg et al. (2022).

Theoretical analysis

I utilized Prediger et al.'s (2008) networking strategy, comparing, to unpack teacher agency in CoI. This involves comparing constructs from both theories to gain a comprehensive understanding of how CoI nurtures teacher agency. Ravitch (2017) defines constructs as carefully selected elements used to encapsulate a phenomenon comprising multiple concepts, operating at a high level of abstraction. For instance, the construct of *inquiry*, as articulated by Jaworski (2006), is delineated by concepts such as asking questions, seeking to understand, and attempting to find answers. Constructs within a theory are logically arranged to form a cohesive framework. For instance, in the CoI theory, constructs like inquiry, engagement, imagination, and critical alignment are all organized in the overall theoretical structure. Comparing theories aims "to provide a base for inter-theoretical communication" and "contributes to a better understanding of the foreign and the own theories" (Prediger et al., 2008, p. 171). Theories can be compared using specific criteria. I compared the constructs of inquiry, engagement, imagination, and critical alignment from CoI, related to constructs in Pantić's (2015) framework for teacher agency: sense of purpose, competence, autonomy, and reflexivity. Through deductive use of Pantić's (2015) framework, I systematically examined how agency constructs manifested in CoI activities. The synthesis of these reflections, with my understanding of how inquiry, engagement, imagination, and critical alignment were realized in the CoI, led to my insights into how constructs from both theories can be compared, as presented in the next section. The analysis is exemplified using my own reflections on both planning and participating in our CoI.

Unpacking teacher agency in CoI

To comprehend how CoI compares to constructs of teacher agency for change, as outlined by Pantić (2015), I provide an individual exploration of each construct, followed by descriptive examples from previously conducted CoI workshops, before synthesizing my main insights.

Sense of purpose

A CoI approach nurtures teachers' sense of purpose mainly through teachers' engagement and imagination. Teachers engage in CoI, asking questions that are relevant to them, making decisions based on their prior experiences, and imagining how new insights can transform their teaching practices (Jaworski, 2006). In our CoI, teachers engaged in group work to find and model appropriate manipulatives, print or cut, and plan how to implement them. During these making activities, teachers were engaged in a mutual process of meaning-making and imagining both manipulatives and their use in the classroom. They shared their insights during presentations, taking a broader perspective, e.g., imagining how it fits their ongoing teaching practice and the curriculum.

Competence

A CoI inherently aims to enhance teachers' competencies. Inquiry, engagement, imagination, and critical alignment are all essential for cultivating the teacher's identity and learning (Jaworski, 2006; Wenger, 1998). While engaging in collaboration with teacher educators and other teachers in the endeavor of their inquiry, teachers develop competence in what they are doing. They gain a comprehensive understanding of the phenomenon they inquire into, enabling them to position themselves within it (imagination), prompting critical reflections on their roles and practices (critical alignment). Teachers become more competent to question their role and practice. According to Jaworski (2006), inquiry and critical alignment are closely related. In our CoI, teachers inquired into DF, got experience with DF technologies, and also benefitted from using resources made by other teachers uploaded to a DF-sharing platform. All workshop activities (reflection, lectures, group work, presentation) and use in the classroom contributed to competence.

Autonomy

A CoI nurtures teachers' autonomy through engagement and inquiry. Teachers are not passive listeners but actively engaged in deciding what questions to ask in the inquiry relevant to their teaching based on their situation at school and "get involved in research into self-chosen aspects of their own teaching" (Jaworski, 2006, p. 195). Research has reported that the main issues for not using manipulatives are lack of time and materials (Marshall & Swan, 2008). In our CoI, teachers had the time and resources needed for them to inquire into DF nurturing their autonomy. Teachers decided which manipulatives to make and how many to fit into their teaching practices. This collaboration between teacher educators and teachers is essential to enable teachers' inquiry and strengthen their autonomy in schools.

Reflexivity

In a CoI, both critical alignment and imagination nurture teachers' reflexivity. Teachers reflect and question what they do, being critically aligned with their practice. This involves participants being conscious, distancing themselves from their own practice, and critically evaluating their methods and approaches. It encourages them to try new teaching methods to prevent the continuation of undesirable practices (Jaworski, 2006). Critical alignment permuted all parts of the workshops, from articulating their reflections, group work, presenting their manipulatives, and evaluating them in the classroom. We encouraged teachers to take a critical stance on workshop design, DF technologies, and manipulatives. Imagination promotes reflection by envisioning future use (Wenger, 1998), relating group work insight to a broader perspective, and reflecting on how new manipulatives can be applied in teachers' own practices.

Discussion and Conclusion

In this article, I attempted to unpack teacher agency in CoI using networking strategies (Prediger et al., 2008). Comparing theories proves challenging due to their independence and utilization of distinct constructs in describing a phenomenon. My findings should be understood as an endeavor to initiate a dialogue on the intersection of teacher agency and CoI rather than as an outcome of combining these theories.

In summary, inquiry, engagement, imagination, and critical alignment as constructs in CoI nurture teacher agency, which is important for changing practices (Cong-Lem, 2021). Constructs such as competence and reflexivity are inherently in CoI, whereas others need special attention. Autonomy requires particular careful consideration, being conscious of providing teachers with the freedom to adapt to their practices and support them to enable change. Likewise, it is essential that participants are able to relate the inquiry to their own practice to enable a sense of purpose.

CoI originates in teachers and teacher educators collaborating and inquiring into how to enhance mathematics teaching. Teacher agency is essential for teachers to transform their practices (Hökkä & Eteläpelto, 2014). Previous studies have mentioned a connection between CoI and teacher agency, mainly through critical alignment and teachers ability to take control over their own activity and development (Biza et al., 2014; Goodchild, 2014; Goodchild et al., 2021; Jaworski, 2006). However, they have not clearly defined teacher agency and how it relates to CoI. In this article, I provide a clear definition of teacher agency, using Pantić's (2015) framework. I delve deeper into the intersection by comparing both theories, offering a foundational understanding of their relationship. My work suggests that both theories can be combined as a unified framework for analysis to provide an empirical understanding of how CoI projects can foster teachers' agency in transforming teaching practices.

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Functional explanations for the understanding of mathematical results

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Although understanding is often sought after, it remains an elusive ideal that is difficult to define, measure, and cultivate. Here, an attempt is made to use a philosophical conceptualization of understanding in terms of knowledge of mathematical items and their relations to each other. We focus on explanatory understanding of mathematical results and review theoretical and empirical research from educational psychology and mathematics education to construct a conjecture map useful for initiating a design research project.

Keywords: Understanding, mathematical results, explanations, design, conjecture map

Introduction

Researchers and reform advocates have repeatedly called for teaching and learning which emphasizes deeper understanding of mathematical ideas, as opposed to focusing on routine application of procedures (e.g., Richland et al., 2012). Researching teaching for understanding is however quite challenging, because it is hard to know which teaching produces understanding and because it is hard to evaluate students' understanding. In this paper, we will present a preliminary design for a teaching model aimed at teaching for understanding mathematics. As this is a far too general aim, we will first need to limit the scope by delimiting our notion of understanding and of mathematics.

Previous research on conceptual understanding, as reviewed by Crooks & Alibali (2014), rarely agrees on how understanding should be defined, operationalized, or measured. Mellin-Olsen's distinction between relational and instrumental understanding (as described in Skemp, 1976) is closest to our interests. This distinction allows us to focus on understanding mathematical ideas in relation to each other as part of a coherent body of knowledge.

We use the framework by Michener (1978) to conceptualize the contents of mathematical understanding, or *what* we understand when we understand mathematics. This framework describes mathematical understanding in terms of knowledge of connections within and between three distinct yet interconnected categories of mathematical items: concepts (characterized by having definitions), results (statements about concepts, arrived at using deductive logic), and examples (specific illustrations of more general concepts or results). To our knowledge, this distinction between concepts and results is unusual in the research literature. However, we find parallels in Vergnaud's (1998) *concepts-in-action* and *theorems-in-action*. Here, concepts are "held to be relevant" and results (which we view as equivalent to Vergnaud's *theorems*) are "held to be true" (p. 168). These are then epistemologically different and, we claim, may benefit from different teaching models.

While concepts, results, and examples are intricately interconnected, it is possible to emphasize one or another in teaching and research. For example, by asking students to justify which fraction is greater, Jones (2013) examined understanding of the concept of fractions, but *not* of the rules with which fraction operations are carried out nor of the examples students use to think about fractions. In this article, we are specifically interested in the understanding of mathematical *results*. There are a few reasons for focusing on understanding results rather than concepts: understanding results requires an understanding of concepts but the reverse is not usually held to be true, much of mathematical teaching, learning and assessment are based on the use of results such as applications of established procedures, and finally there seems to be little previous research on students' understanding of results.

As a framework for describing our design ideas, we use a conjecture map (Sandoval, 2014). Here, the set of designed components of the learning environment that are supposed to realize some *overarching teaching intention* is called an *embodiment.* How the learning environment embodiment is supposed to lead to the mediating processes taking place is termed the *design conjectures* of the system. The prescribed mediating processes are then supposed to lead to some *outcomes* in the form of learning, and the understanding of how and why this will happen is termed the *theoretical conjectures* of the system.

There is a wealth of research on understanding of concepts, and on applications of procedures in problem-solving. This paper introduces a somewhat innovative approach by addressing the following question: *what constitutes a teaching model that effectively cultivates relational understanding of mathematical results among upper-secondary students?*

Theoretical perspectives on understanding in mathematics

Writing in philosophy, Baumberger (2014) distinguished between propositional, interrogative, and objectual understanding. In school mathematics, understanding that the area of a triangle is given by the formula $A = \frac{1}{2}ab\sin C$ is propositional understanding, indistinguishable from propositional knowledge. Understanding why the formula is true, as well as when and how to use it, are examples of interrogative understanding. Together, these elements of interrogative understanding add up to "knowing both what to do and why" (Skemp, 1976, p. 2), thus meeting Skemp's definition of relational understanding. Objectual understanding is about nouns, such as one might speak of understanding complex numbers, which is broader and encompasses the propositional and interrogative types of understanding. Here, we will focus on interrogative understanding, specifically *explanatory understanding* (Baumberger, 2014, p. 71) characterized by "why"-questions.

In relation to mathematics, Michener (1978) described how mathematical results are connected to previously established results and definitions of related concepts using *deductive reasoning*. Mathematical results are often illustrated by and inspired from specific examples. Explanatory understanding of mathematical results should therefore answer the "why" question using deductive logic based on characteristics of mathematical concepts, applications of previously established results and with the use of illustrative examples.

An example may serve to illustrate how we bring together the ideas of Baumberger and Michener: $A = \frac{1}{2}ab\sin C$ is a formula for the area of any triangle ABC. An explanation demonstrating relational understanding could be:

1. The height h of a triangle can be imagined from one vertex (B) perpendicular to the opposite side (b).
2. h can therefore be found using $h = a\sin C$.

3. The area of a triangle is previously known as $A = \frac{1}{2}bh$, which leads to $A = \frac{1}{2}ba \sin C$ and on to $A = \frac{1}{2}ab \sin C$.

In this explanation, the properties of the concept of triangle and interior angles are used in Step 1, and connects to the concept of perpendicularity. In Step 2 we see connections to another concept—the sine of an angle. In Step 3, a previously established result and the procedure of substitution are used together with the commutative property of multiplication. Thus, we can see that the explanation uses deductive arguments based on characteristics of multiple concepts and a previously established result. By working with results, the learner simultaneously reviews and deepens their understanding of concepts and previously learned results.

The potential of explanations to give rise to understanding is also considered in the work of Inglis and Mejía-Ramos (2021). Using generally accepted and somewhat simplified models and theories of human memory—the multi-store model, the working memory model, and schema theory—they suggested that *functional explanations*, i.e., explanations which support understanding, should fit with human cognitive architecture. To do this, explanations should a) focus the reader on some central and already known characteristic of concepts, b) direct attention to relevant prior knowledge so as to bring it to awareness, and c) be structured in ways that respect the limited capacity of the working memory.

A summary of the main terms mentioned in the above sections is included in Figure 1.

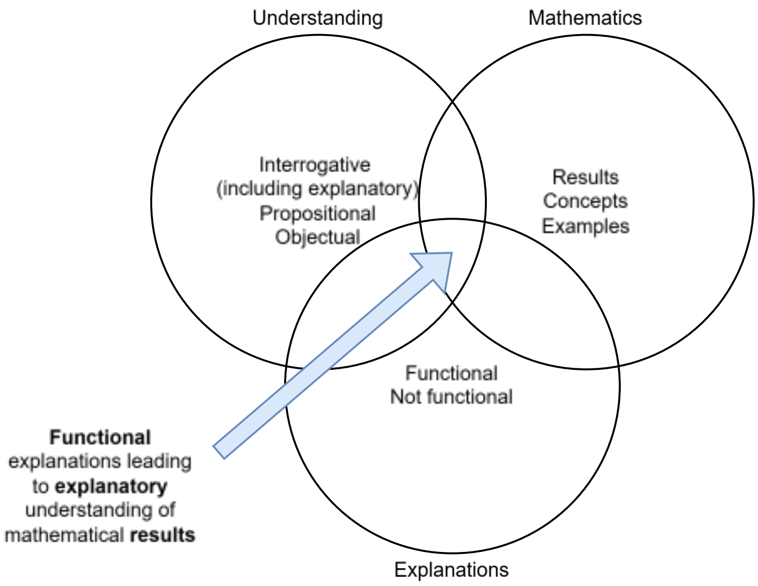

Figure 1: Venn diagram illustrating the position of the topic of interest in this paper.

Empirical research on explanations in mathematics education

The most common mode of explanation in mathematics education is that of instructional explanations given by teachers to students. However, Wittwer and Renkl (2008) reviewed existing research and demonstrated that such explanations very rarely succeed in eliciting understanding. Consequently, researchers have turned to other forms of explanation such as self-explanation and student-produced explanations directed at a real or imaginary audience.

Research on self-explanations in mathematics education has for the most part been focused on self-explanations of procedures and problem solving, with the goals of increasing understanding of concepts and procedures and achieving near and far transfer. In a meta-analysis, Rittle-Johnson et al. (2017) found highly variable results between different studies investigating the effects of self-explanation on procedural fluency, transfer, and conceptual understanding. In an attempt to explain the inconsistent effects seen in these studies, Rittle-Johnson et al. suggested that students are often unable to construct explanations, to understand what an explanation is as opposed to a descriptive account, and that students therefore must be supported by scaffolding in the form of prior training or structured questions.

Another line of research has explored learning-by-teaching, in which students explain mathematical results, solutions, or concepts to a real or fictive audience that may or may not interact with the explainer. Learning-by-teaching can be considered a form of peer interaction in small groups, and there are many reasons to believe that such interactions, when characterized by the giving and receiving of highly elaborated explanations, is beneficial primarily to those giving the explanations to responsive peers (Webb, 1989). While many experimental studies show impressive effect sizes from learning-by-teaching, the findings are inconsistent, especially when comparing learning-by-teaching to self-explanations, when learning-by-teaching is done without an audience, and when learning-by-teaching is conducted in writing rather than orally (Lachner et al., 2022).

Students can produce explanations either in writing or orally, and there is little research to suggest which of these could be superior for developing relational understanding. In the few existing studies on this topic, there are somewhat conflicting results. Pugalee (2004) found, in an experimental study, that students' written descriptions of their problem solving showed more metacognitive elements compared to the same students’ oral accounts. Writing was also associated with more frequent correct solutions to problems. On the other hand, Stylianides (2019) found that students demonstrated more proof-like reasoning in oral presentations to classmates, compared to the written accounts they had initially produced. It therefore remains unclear whether oral or written explanations are best for developing understanding, whether they should be used in conjunction and if so in what order.

Implications for design

Based on the theoretical perspectives presented above, we view understanding as explanatory understanding (Baumberger, 2014) that requires relational understanding of connections between mathematical concepts, results and examples (Michener, 1978). From the empirical research described in the subsequent section, our central hypothesis is that such understanding can be built by engaging students in generating explanations of mathematical results. We then have, from Michener’s (1978) conceptual framework of mathematical knowledge, that such explanations should include a) logical deduction, b) reference to previously known results, and c) connections to the concepts and examples that explain and illustrate the result. For such explanations to be cognitively manageable, psychological models of memory suggest that explanations should be accompanied using visual representations wherever possible, explicitly connect to previous learning, and be organized around one or a few clearly identified central ideas (Inglis & Mejía-Ramos, 2021).

The findings from research on self-explanation (e.g., Rittle-Johnson et al., 2017) and learning-by-teaching (e.g., Lachner et al., 2022) are useful for task design purposes in a rather vague way. Because both lines of inquiry show overall promising results, it is worthwhile to integrate similar approaches in our teaching design.

We will now describe our initial design ideas using a conjecture map (Sandoval, 2014), which is a method of visually communicating important or functional elements in a task or learning environment, thereby illustrating how they interact and lead to understanding of results. A conjecture map starts with a higher-level conjecture about the general if/then of the design and contains the following parts: embodiment (including some or all of: materials, task, participant actions and discourse practices), mediating processes (the observable behaviors and/or artefacts that the embodiment is meant to elicit) and the desired outcomes. These parts are connected to each other with conjectures to be tested: design conjectures about how the embodiment will lead to the mediating processes, and theoretical conjectures about how the mediating processes will lead to the desired outcomes. Such a conjecture map is presented in Figure 2 to illustrate the causal relationships suggested by the research considered in this paper in a way that can be repeated as often as students encounter new results for which it is possible to formulate explanatory justifications.

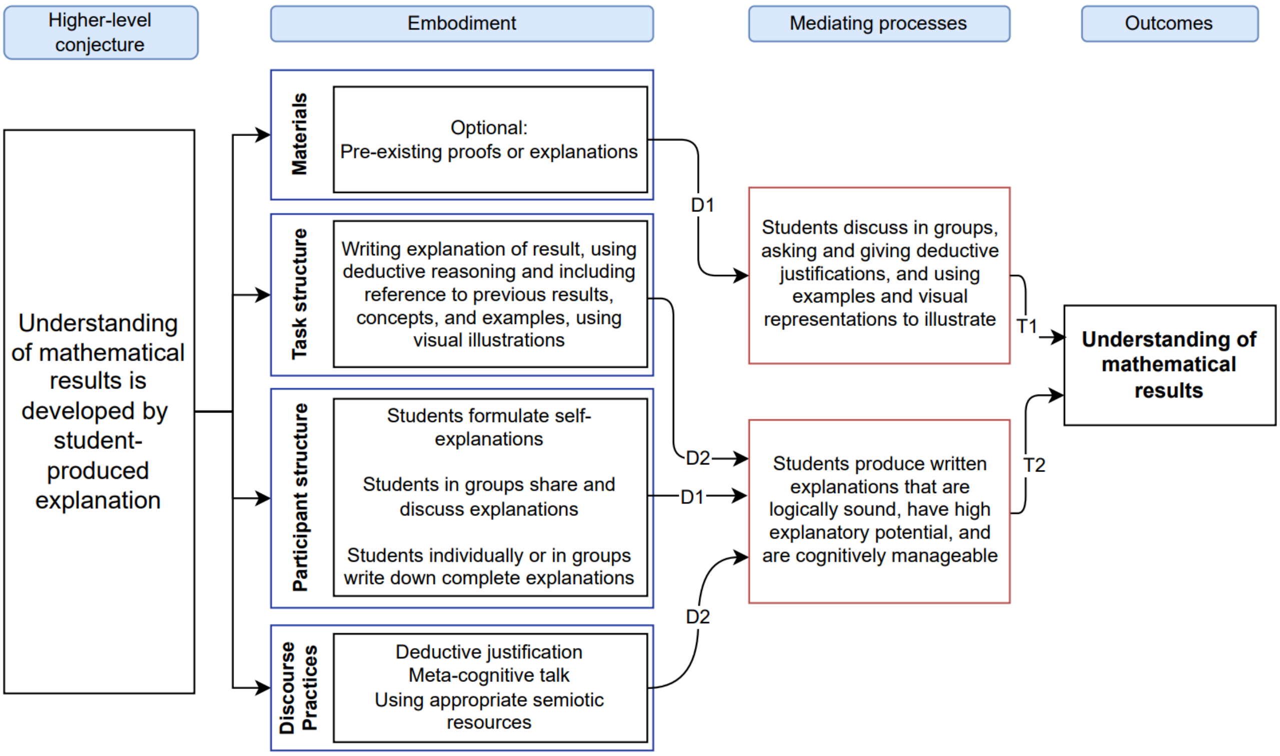

Figure 2: Conjecture map

Some of the embodiment elements in this conjecture map did not arise from considerations of the reviewed research but came rather from the initial attempts to try out the higher-level conjecture in situations with students. The participant structure sequencing (starting with an attempt at self-explanation, followed by group work and ending with the production of an instructional explanation) is an embodiment feature arising from practicality concerns after initial attempts demonstrated that many students were unable or unwilling to construct a full self-explanation on their own, and needed to collaborate in groups. In discussions with classmates, students identified and filled in their own gaps in understanding, as well as elaborated on and exemplified their explanations. This initial finding aligns easily with previous research (Webb, 1989) on the benefits of peer interaction in collaborative problem-solving.

Our design conjectures are of two kinds. The conjectures labeled D1 address scaffolding strategies, suggesting that a) when students struggle to generate explanations on their own, providing them with existing materials such as proofs or explanatory texts can aid in formulating their own explanations, and b) sequencing participant structures by having students first attempt self-explanation, followed by group collaboration, helps them identify and address gaps in their reasoning. The D2 conjectures focus on the quality of explanations, proposing that when students adhere to task instructions and engage with discourse practices, the resulting explanations are logically sound and clear.

The theoretical conjectures are closely linked to the background research, T1 suggesting that through group discussions, students are able to construct complete and explanatory self-explanations, incorporating self-selected examples and visual illustrations. The T2 conjecture suggests that by writing down their explanations in a coherent and organized form, using those same examples and illustrations, students can structure the information in a way that facilitates integration with existing cognitive frameworks in long-term memory, ensuring the explanation remains accessible over time.

Insights from informal implementations of early prototypes

The first author, being the mathematics teacher of students in a challenging mathematics course in an upper-secondary program in Sweden, was able to test initial prototypes of the design with two classes of students (18 in one class and 27 in the other). We designed a few different tasks based on the embodiment elements in the conjecture map in Figure 1. The tasks were focused on logarithm laws, probability laws, and Bayes Theorem. We report reflections from the task on probability laws.

The task aimed to help students understand probability laws. The design was embodied in a page with instructions, prompting students to “Explain why this result holds. In your explanation: Illustrate with a specific example. Refer to the concepts of probability, union, intersection, and event. Use at least one visual representation.” The students were instructed to work according to the sequence in the participant structure of the conjecture map.

In terms of mediating processes, students willingly engaged in ten minutes of self-explanation at the end of which they came to the group discussion with full or partial explanations of one or more of the probability laws. They shared their explanations in groups, using general claims and justifications and seeking examples to help clarify ideas. Working in groups, they produced coherent written explanations.

An interesting finding is that students spontaneously invented creative visual illustrations but sometimes resisted using words and specific examples in their mathematical explanations, even though they had used words in discussions. Figure 2 illustrates a student-produced explanation, which demonstrates a group of students making sense of the result in terms of concepts, using simple deductive reasoning in sequential steps, but without exemplifying and with minimal use of words.

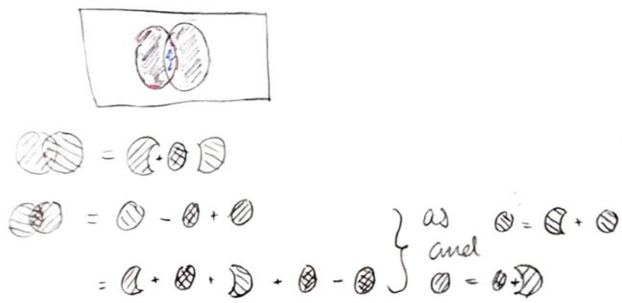

Figure 2: Students' group response to prompt to explain why

$$\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

Discussion

We asked the question: *what constitutes a teaching model that effectively cultivates relational understanding of mathematical results among upper-secondary students?* We have seen how previous research gives rise to recommendations for using student-generated explanations. Our teaching model includes a task sequence through which students first attempt to self-explain, then construct mathematically sound explanations through group work, and finally organize the explanations coherently in writing, thereby facilitating the development of understanding. We submit that the teaching model we have presented is well situated in earlier research and shows promise of working in practice. Due to how the model is presented in a conjecture map structure, the hypotheses the model builds on are also made researchable. Some important questions remain.

More sophisticated or complex results require more time and effort to explain them, and the teacher will need to prioritize between results or choose time-effective modes of explanation. If no explanatory proof exists, or it is not possible for students to come up with an explanatory rationale for the result, spending time on writing explanations will probably not lead to improved understanding. It should therefore be a priority to develop the conjecture map to account for such differences between various available explanations of mathematical results.

Assessment of explanations (as mediating processes) and understanding (as desired outcomes) will be central to subsequent research on our teaching model. While some promising attempts are being made in assessing conceptual understanding of concepts, such as comparative judgement (Jones, 2013), there seems as of now to be little research on how to assess understanding of results.

Conclusion

This paper presents the start of a design research project that aims to improve students' relational understanding of mathematical results. We used Michener's (1978) framework for conceptualizing mathematical understanding to delimit understanding in general to the understanding of results in particular. We found that self-generated, coherently structured explanations containing visual as well as verbal representations are a promising route to developing understanding, and we arrived at a preliminary conjecture map for a teaching model. Finally, we determined, through informal implementation, that this teaching model shows practicality for inclusion in secondary mathematics education.

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Developing analytic talk through video-based reflection conversations in teacher education

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In this paper, we use Horn and Little's conversational routines as an analytical framework on two reflection conversations between preservice teachers (PSTs), practicum teachers and campus teachers discussing PST-selected videoclips from their attempts to lead productive mathematical classroom discussions in primary school. The categories normalizing, specifying, revising, and generalizing are used to study how the participants talk about classroom situations the PSTs perceive as golden moments used or lost. We look for traces of development in the analytic talk from the first to the second conversation. We find that the development is most visible in the PSTs' willingness and ability to problematize, specify and discuss the golden moments regarding students' mathematical thinking. The practicum teachers and the campus teachers play an important part in modeling specification and generalization in both conversations.

Keywords: Practice-based teacher education, preservice teachers, conversational routines, video-based reflection.

Introduction

It is well documented that preservice teachers (PSTs) find it challenging to learn to facilitate classroom discussions that elicit and enhance students' mathematical thinking (e.g., Lampert et al., 2013). Such ambitious teaching requires a nuanced understanding of how to identify and build students' thinking through dialogue, for example, by spotting "the golden opportunities" (Walshaw & Anthony, 2008, p. 539) that a teacher can use to support students' engagement in worthwhile mathematics. However, PSTs can only be responsive and act on what they notice. Jacobs and Spangler (2017) conceptualize *teacher noticing* as focusing attention on and making sense of situational features prior to responding in-the-moment in classrooms. This is a skill that is underdeveloped among PSTs, but learnable if appropriate opportunities are provided (Jacobs & Spangler, 2017).

The practice-based teacher education project ReTPro[1] aims to answer to Jacobs and Spangler's (2017) call for more research on how to provide appropriate learning opportunities for PSTs and proposes that these skills can better be learned if such in-the-moment decision making can be brought forth and investigated afterwards. Providing PSTs with opportunities to engage in analytic talk about their own teaching using videos as representations of practice has shown to be a powerful tool (van Es et al., 2017). In ReTPro both practicum teachers and campus teachers engage in video-based reflection

[1] ReTPro (Researching teaching professionally) is a design-based research project funded by the Norwegian Research Council that follows a strand of research that see enacting and analyzing teaching as core elements for learning to teach (e.g., McDonald et al., 2013; van Es et al., 2017).

conversations with the PSTs to contribute to the PSTs abilities to notice, discuss, analyze, and generalize from their teaching experiences.

Horn and Little (2010) analyzed teachers' analytic talk about teaching and learning mathematics in the context of professional development at their workplace. They found that even though teachers are committed to improvement, the productiveness of the analytic talk varied. Particularly the extent to which the talk provided access to problematic instances in the classroom and how the specific situational features in these instances were linked to more general aspects of ambitious teaching. Horn and Little (2010) define the pattern and recurrent ways that conversations unfold within a social group as *conversational routines*. The routines are constituted by linguistic *moves*, turns of talk that shape the interaction's progress by setting up and constraining the response of the subsequent speakers. Horn and Little (2010) were particularly interested in conversations where teachers' talk revolved around classroom interactions experienced as dilemmas or problems because of their potential for generative learning. Their analysis led to the following categorization of types of talk: *Normalizing*, *specifying, revising,* and *generalizing*. *Normalizing* happens when participants in the conversation give responses that define a problem as an expected part of classroom work and teachers' experiences. This type of talk can be used to turn the conversation *away from* the problem by reassuring that the problem is known (e.g., "that's just the way 9^{th}-graders are"). In productive analytic talk, however, normalizing moves are used to turn *towards* the problem. They function as a starting point for a detailed discussion of the problem (e.g., "can you identify the source of…?"). By asking questions and eliciting additional information about the situation, the problem can be *specified*, and the account of the problem, and its nature and causes, can be *revised* through discussion. *Generalizing* is explained by Horn and Little (2010) as a conversational routine through which the group develops a general category of classroom experiences that the problem fits into and links them to generic principles. They claim that it is the movement between specific, linked accounts of classroom practice and general lessons from experience that characterize productive analytic talk about teaching. We found that these categories responded well with the reflection conversations in our material and wanted to try them out as an analytical tool. This paper presents an analysis of two video-based reflection conversations between PSTs, practicum teachers and campus mathematics teachers. We study the reflection conversations from the beginning and the end of the PSTs' mathematics course, hereafter called conversations 1 and 2. We ask the following research question: *How do the participants use conversational routines to develop productive analytic talk in the two conversations?*

Methods

The teacher learning cycle presented in McDonald et al. (2013) is used as a methodological tool in ReTPro. This cycle has four PSTs engagement phases: (1) studying and modelling in campus; (2) rehearsals in campus; (3) enactments in the practicum classroom and (4) analysis and reflections on the enactments using videos. These four phases are organized around predesigned instructional activities (Lampert et al, 2013), and all phases are videorecorded. During the one-year mathematics course the PSTs have engaged in four learning cycles. In the following, we present the instructional activities in the first and fourth cycle, then the data used in this study and the analysis of the material.

The first cycle was completed in early spring 2022, the PSTs' second semester, and was based on a quick image-activity (Lampert et al., 2013). The teacher shows an image displaying groups of dots for a short time to encourage students to find effective strategies for determining the number of dots. The teacher, then, leaves the image visible and engages students in a classroom discussion about different strategies. The activity was modeled through a movie, then all 23 PSTs planned, rehearsed, and enacted a quick image activity in their practicum class. The fourth cycle was completed in late autumn 2022, the PSTs' third semester, and based on a 3-act-task activity developed by Dan Meyer. In act 1 the teacher shows an intriguing short video that introduces a problem (e.g., somebody opens several small packages of Skittles and pours them in a big jar), then, the teacher invites the students to wonder, pose questions and estimate probable and improbable solutions, and finally, the class selects one question to work on mathematically. In act 2, the students gather information that will allow them to develop a mathematical model of the situation, and in act 3, the teacher leads a classroom discussion about students' models and possible solutions. The campus teacher modeled a 3-act-task activity, then 19 PSTs planned and rehearsed a 3-act-task activity in campus, in phase 3, each practicum group planned and enacted a 3-act-task activity during their practicum.

In phase 4 the PSTs were given the task to pick one-minute videoclips of what they perceived as golden moments from their videorecorded try-outs. We used the term golden moments to describe classroom situations, that if pursued by the teacher, may develop students' mathematical thinking, or what Walshaw and Anthony (2008) termed "golden opportunities". We included golden moments used and lost to embrace both successes and failures in PSTs attempts to respond ambitiously. The chosen golden moments were ground for the video-based reflection conversation between the PSTs and their practicum teachers and campus mathematics teachers. The goal for these conversations is to enhance the PSTs analytic skills; to learn to notice and exploit golden opportunities during teaching, and to generalize from one experience to similar experiences and to general principles. However, it is difficult to determine what the PSTs take away from engaging in learning cycles. One way of learning more about this is to analyse the reflection conversations.

Data collection

The case group of PSTs was chosen out of a class of 23 PSTs the first year of the ReTPro project. The class worked together in practice groups of two to four PSTs. Sometimes the group combination was changed, some PSTs did not complete all the tasks, and some had several different practice teachers. Our case group stayed together through the whole year and completed all tasks. We named them Steve, Sandra, and Sonia. One of the campus teachers, Celine, participated in both conversations. In addition to the stable members of both conversations, there is a second campus teacher, Carla, present in conversation 2, and the practice teachers are Pam in conversation 1 and Peter in conversation 2. Both conversations were conducted and recorded on Zoom, conversation 1 lasted 29 minutes, and conversation 2 lasted 61 minutes.

Data analysis

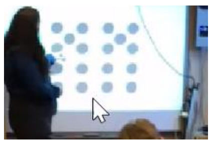
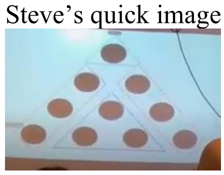
We transcribed the conversations, divided them into episodes based on the topics being discussed and coded them individually and collectively through several rounds of coding. We were particularly interested in the conversations around the PSTs' chosen golden moments because they represented what features of instruction the PSTs noticed when they studied the video. We looked for potential problems or dilemmas that surfaced during the discussion, and we investigated how the group's response oriented the collective attention towards or away from the problems using the four types of moves suggested by Horn and Little (2010): *Normalizing* a problem, further *specifying* moves to elicit more details and invite further analysis, which can result in *revising* the account of the problem and support the process of *generalizing* to principles of teaching. We then interpreted in what ways the conversational routines contributed to the analytic talk, and what characteristics about the talk constituted quality of analysis and productiveness in the conversations, as presented in the following section.

Analysis

Conversation 1: Quick Image Activity in 2nd grade

Sonia, Sandra and Steve had different quick images in their try-outs. See Table 1. The PSTs started by introducing five videoclips, each clip was followed by a short explanation for why they perceive them as golden moments. The first clip showed how Steve introduced his quick image activity. He did not identify it as a golden moment, he said he "simply wanted to give a glimpse of what was expected of the students." Then, they presented four videoclips which they identified as golden moments. They all showed how the PST in charge responded to a particular student contribution during the classroom discussion. Three of them were not problematic situations and the PST managed to respond rather well. For example, in the second clip, when a student started to explain his strategy to find the number of dots in Sonia's image: "First, I counted 4, like that (he points), then, I counted those two, so it is 4+2 or 2+4," Sonia noticed that the student grouped the dots in a different way with his fingers. The clip shows how she exploited the situation by prompting for the other strategy.

Table 1: Quick Image Activity in 2nd grade

Sonia's quick image	Sandra's quick image	Steve's quick image
		

However, when Sonia was asked why that golden moment was considered used, she did not attend to that golden opportunity at all. Instead, she highlighted a more general pedagogical feature: "I showed a good calculation from a student who usually does not say much." In conversation 1, the PSTs were mostly interested in presenting moments where they "played the students good," not problematic situations that could potentially become productive learning situations to reflect on.

The most potential golden moment was in the fourth videoclip, when Steve prompts for new calculating strategies by showing different groupings of his quick image on the board. A student takes on the invitation and suggests using multiplication. Steve does not pursue the student's statement. Instead, he moves on and presents a new grouping on the board. The same student then suggests an adding strategy, searching for groups of tenths: “It is 6 plus 4, it is a ten-friend[2]”. Steve repeats the answer with a questioning intonation and the videoclip stops. In the conversation, Steve points out that it is the group’s opinion that “we think this is a nice moment” because the class had been working on ten-friends recently. However, Steve signals that he is unwilling to say more about the situation: “Eh, I don’t have much to add.” It looks like Steve experienced the situation in the classroom as troublesome. Sonia starts to normalize the problem and draws attention to the task they were given, that they were only allowed one minute-clips to present the golden moments. She claims that there were many golden moments in the videos, but they were too long. The campus teacher, Celine, on the other hand, turns towards Steve’s problematic situation by challenging him to specify why he thinks it turned out the way it did. Steve does so by specifying the problem – explaining how the student sees the quick image as a multiplication rather than a pattern or an addition. He was taken by surprise because he was not expecting this from 2nd grade students. Steve is aware that the student’s response is a consequence of his action: “I did propose multiplication…but I did not manage to help him share what he meant.” Steve thus revises the account of the problem and calls the situation a “misused moment,” because he was the one who failed to exploit the moment. Steve criticizes himself and his own teaching, and Sonia takes him into defense, normalizing his choices by saying that Steve was able to turn it into a relevant example, assuring him that he made the best choice. By this, she turns the conversation away from Steve’s original problem, at the same time, she provides another perspective on the situation. She concludes that it was not a misused moment, because Steve did manage to “play him good”. Celine does not prompt for more in-depth considerations of Steve’s lost golden moments.

The generalization found in conversation 1 is brought forward by Pam, the practice teacher. She starts by acknowledging the students’ teaching and gives positive feedback on what she has noticed. She uses general features of ambitious instruction when she reasons about their teaching: “I think it is nice that there is room for different strategies, where you act as supervisors […] that you have a classroom environment where everybody can share their thinking.” Pam does not refer to specific events from the PST’s golden moments as evidence for her generalization. Instead, she links ambitious principles of teaching to general accounts of her own practice: “I have started to challenge them a bit, saying that we are allowed to disagree, but then we always have to give a reason for why we disagree… They are only in 2nd grade, but if you give them time and space, you see that they are quite receptive. That is nice to see.” By giving examples from her own teaching, she indirectly makes a comparison between her own and the PSTs’ experiences, and she links it to more general principles.

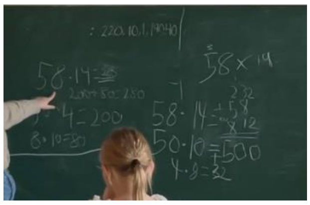
Conversation 2: Three-Act-Task activity in 5th grade

Conversation 2 has a different structure than conversation 1. It is twice as long but contains discussions about only three videoclips – each clip is presented and then discussed at length. Two of the three golden moments picked out by the PSTs are characterized as lost, and one as used. In the first videoclip, Steve decides to deviate from the 3-act-task activity structure in act 1 where the class is supposed to select one of the three questions that were on the board. Instead, without any

[2] A ten-friend, or «tier-venn» in Norwegian, is a concept used in schools. It is natural numbers whose sum is ten.

explanation Steve picks the question for them, which was the one the PSTs had planned to use: “How many Skittles fit in the jar?”. The other questions on the board were: “How many packages are needed?” and “How many Skittles in one package?” Table 2 shows the activity that the PSTs planned and enacted in 5th grade. The PSTs characterize this as a golden moment lost, because they failed to follow the plan, due to lack of time. In the following discussion, the PSTs defend Steve’s choice and argue that his decision was insignificant because any of the questions on the board can be used.

Table 2: Three-Act-Task activity in 5th grade

Act 1: Posing and selecting questions to solve. “How many Skittles fit in the jar?” 	Act 2: Gather information and find solutions. Information given: 58 packets and 14 Skittles in 1 packet. The problem is 58·14. 	Act 3: Sharing and discussing student solutions. 

Peter and Carla normalize too, by pointing out that “lack of time” is a classic dilemma in school. Carla then turns towards the dilemma and prompts for further specification asking what they gained and what they lost by their decision. This is followed by a sequence of specifications, in which the PSTs revise the problem and conclude that it was the mathematical relationship between the questions that was significant for the students to notice, and therefore the moment was, in fact, lost. However, the reason for the moment being characterized as lost was revised during the conversation.

The third videoclip shows a group of students having problems multiplying 58 times 14 in act 2. They wrote 4·8 = 32, then 50·10=500, and then they added 500+32 =532 on their worksheet but signalized that they were uncertain about their solution. The PSTs identified this as a golden moment lost, because Sonia, who filmed the clip, did not know if and how she should intervene in the students’ solving process. She chose not to intervene, but Sonia remarks that the situation had a lot of potential if it had been used. Sandra elaborates on the situation and concludes that they “could have pushed them more. They were right to multiply, but they forgot to multiply a couple of numbers.” Sandra is here referring to the missing products, 50·4 and 8·10, in the students’ solution. Peter, the practicum teacher, suggests that his students instead should have used other decadic units (58·10 and 58·4) in their decomposing, because that is what they had worked on recently. He points out that students in 5th grade in general have varying knowledge about multiplication with multiple digits. By saying this, he provides relevant background to better understand why these students are struggling. Peter then expands the problem, remarking that in act 3 there were four different student answers that were presented on the board. Sandra adds to Peter’s comment: “They had all worked on the same problem, but they all had different answers.” Then follows a sequence of specifying turns elaborating on the nature of students’ mathematical problems. As a result, the PSTs noticed that the students’ mistakes were similar, they all forgot to multiply some of the components when they decomposed the factors. Based on this, Carla revises the problem: “Then I think this is a fantastic golden moment!” She explains by generalizing: “This is just it; these different groups of students do make these mistakes. When they decompose, they forget which components they have. These are golden moments precisely because they bring out the importance of the fact that when we multiply, we must include all the components”. In this way Carla generalises how one can understand the students’ problem by relating it to situations that often happen, and she clarifies the generalisation by naming the components that are problematic for students: “It is a typical mistake they make when multiplying two-digit numbers by two-digit numbers”. She then challenges the PSTs to think of how they could have acted to make use of the golden opportunity that occurred.

Discussion

In our research question we asked *How do the participants use conversational routines to develop productive analytic talk in the two conversations?* According to Horn and Little (2010) normalizing can be used both to turn towards or away from a problem. In our material we find both kinds of normalization. In the first conversation, the PSTs are the ones doing the normalizing moves, both times turning away from Steve’s “misused moment” where he, according to himself, failed. Several times they express an eagerness to move on to the next situation, rather than dwell on the more problematic aspects as an “object of collective attention” (Horn & Little, 2010, p. 192). In the second conversation, however, we find normalizing used in a slightly different way. Here, the practicum teacher Peter and the campus teacher Carla use normalizing to support and encourage the PSTs when they introduce more problematic situations from their try-outs. In conversation 2 we see that the PSTs are more willing to engage in problematic situations than in conversation 1. This is visible in the moments they choose to present – moments lost rather than moments used – and by the way they specify the problems and engage in noticing activities such as “taking on different perspectives to gain deeper insight into what is observed” (van Es et al., 2017, p. 167). As a consequence, the level of noticing seemed to be higher in conversation 2, because of the level of details and the extent to which the observations were integrated in the noticing practice (van Es et al., 2017).

Horn and Little (2010) highlight the interplay between accounts of specific experiences and generalizations in productive analytic talk about teaching. We find that the generalization processes in conversation 1 are quite different from those of conversation 2. The generalization made by Pam in the first conversation is about relating PSTs’ teaching experiences, as well as her own, to the overarching theoretical framework for the intervention the PSTs are a part of – ambitious teaching. She does this without digging deeper into or specifying the experiences that the PSTs report. In that way, the generalization is not induced from the PSTs’ specific experiences. Pam’s account starts on a general level and is linked to the specific experiences as examples of practice by deducing. If we look at Carla’s generalizations in conversation 2, there is developed a more thorough understanding of the characteristics of the specific experiences before the generalization is given. First, Peter expands one student group’s mathematical problem to be relevant for other students as well. Then, Carla asks the PSTs to specify the mathematical problems. In that specifying process, the PSTs become aware that the student groups had similar problems with the decomposed components in the calculation. In this way, Carla’s generalization is underpinned by the PSTs observations, and the generalization expresses the problem in a general manner. It is important to notice that in both cases the movement between the particular and the general is prompted by the practicum and campus teachers and the generalization is not in either case developed by the PSTs. In both reflection conversations, the practicum and campus teachers are modeling ways to conversationally construct “general frameworks for thinking about teaching problems” (Horn & Little, 2010, p. 202), and hopefully these frameworks can become durable tools for the PSTs to think through future problems of practice (Horn & Little, 2010).

In this article we have shown some glimpses of how conversational routines can be used to analyse reflection discussions and we have outlined some traces of development in the productiveness of the analytic talk from the first to the second conversation. This study has given us valuable experience in using the Horn and Little concepts as an analytical framework, in the process discovering a need to further clarify what the conversational routine of generalization entails. An obvious limitation to the implications that can be drawn from this, is the fact that the analysis is limited to a small case group of three PSTs. However, such a detailed analysis gives insight into specific challenges and situations which in turn can be used to develop the practice–reflection-approach used in ReTPro. Such challenges can be connected to PSTs’ teacher noticing of pedagogical problems (van Es et.al., 2017) and their reasoning about them in productive analytic ways (Horn & Little, 2010). The analysis also shows that video-based reflection conversations between PSTs, campus and practicum teachers can be fruitful learning opportunities for all participants because of their potential to create shared frames of reference on how to identify and interpret problems of practice and what constitutes “situationally relevant, appropriate action” (Horn & Little, 2010, p. 209).

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How being assigned the role of initiator influences a silent student's participation in a problem-solving group discussion

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This study explores how given roles affect discourse in group work. An intervention was carried out where groups of eighth-grade students worked on problem-solving tasks, with one lesson without and one with the assigned roles: democratic leader, initiator, curious, and sceptic. We use a thematic analysis with positioning theory to compare the students' participation without and with roles. This article focuses on one student who, in the first lesson, barely contributed to the group discussion. We describe her position as a silent observer, an extension of Barnes' (2004) framework. In the second lesson, the student is assigned the role of "initiator". In this lesson, she initiates group work and suggests solutions, in contrast to the previous lesson. What enables and hinders her in fulfilling her rights and duties as initiator are discussed.

Keywords: Drama, roles, positions, problem-solving, group work

Introduction

This article derives from empirical data gathered as part of the Theatre in Mathematics (TIM) project financed by Erasmus+ and partners from Italy, Norway, Greece, and Portugal. The aim is to develop a mathematical teaching methodology that engages the students actively in their mathematics lessons by incorporating drama techniques. A part of this is using roles in ordinary mathematics teaching to change the discourse pattern toward more active student participation. Assigning and playing roles is fundamental to drama, but our use of roles also relates to naturally occurring roles, or positions, in the classroom. We are interested in studying how given roles affect discourse in group work and using positioning theory (Davies & Harré, 1990) to seek explanations for the observed changes. Consequently, our research question is: How does the assignment of roles influence students' participation during group work on problem-solving tasks in mathematics?

Theory

Positioning theory (Davies & Harré, 1990) offers tools to explain the dynamic nature of social relationships through positions people take up. It emphasises that these positions are not fixed but emerge and adjust through interaction. Positioning theory suggests a framework of three parts (Davies & Harré, 1999): communication acts, positions, and storylines. Communication acts, which include speech acts and other forms of communication, are the observable part that provides us with information to describe positions, and positions describe how one acts at a particular time in a social setting. Further, positions are defined by the rights and duties to perform certain actions in this setting (Harré & Moghaddam, 2003). For example, suppose students in a group work in mathematics agree that one student should act as a secretary who writes down solutions and reasons. In that case, this student arguably has the right to request repetition and elaboration and duties to write down what the group agrees on. Storylines are loose clusters of narrative conventions that emerge during interaction (Harré & Moghaddam, 2003). However, storylines do not emerge strictly inside one social setting but are affected by past experiences and cultural aspects, which means that each participant is simultaneously in or affected by different storylines without necessarily realising this (Wagner & Herbel-Eisenmann, 2009). Herbel-Eisenmann et al. (2015) examines the application of positioning theory in mathematics education, suggesting refinements for its use. The authors emphasize the importance of recognizing multiple, co-existing storylines and positions in interactions. Verbal and non-verbal communication are highlighted as essential in shaping and challenging positions in real-time. Additionally, Herbel-Eisenmann et al. introduce the concept of scale, illustrating how various narrative layers interconnect and how broader educational discourses—such as views on the nature and purpose of mathematics—influence local classroom interactions.

Another layer to positioning theory is how willingness, capability, and power might explain differences in taking up positions during interaction (Davies & Harré, 1999; Huang & Wang, 2021). Willingness describes how prepared a participant is to take up a position or try to position others, capability describes a person's ability to take up a position, and power to which extent individuals are permitted or encouraged to take certain positions. These three aspects of why positions are or are not taken up might explain why some group work goes well while others do not. Further, the idea of power illustrates that positioning during interactions may exhibit a high level of competitiveness.

Barnes (2004) offers a valuable framework for analyzing classroom positions, classifying student interactions in small groups into 14 positions and organising them into six categories. The first category, **getting down to work**, includes positions that promote focus and progress: *the manager*, who initiates work and assigns tasks, and *the helper*, who performs routine work. The second category, **maintaining group cohesion**, includes *the facilitator*, who resolves conflicts and provides support, and *the humourist*, who makes amusing comments to maintain positive group dynamics. The third category, **speaking for the group**, consists solely of *the spokesperson*, who communicates with the teacher on the group's behalf. The fourth category, **thinking about mathematics**, encompasses five roles related to intellectual engagement: *the expert*, who provides or verifies answers; *the outside expert*, who connects the math to broader contexts; *the critic*, who seeks explanations and challenges ideas; *the collaborator*, who engages actively with others; and *the in need of help*, who seeks or accepts assistance. The fifth category, **being distracted**, involves positions that divert focus: *the entertainer*, who initiates off-task activities, and *the audience*, who is receptive to these diversions. The sixth category, **not fully belonging**, describes partial participation, with *the networker* monitoring other groups and sometimes engaging off-task, and *the outsider*, who is either ignored or refrains from participating.

Barnes' classification also connects to broader mathematics education research using positioning theory, such as Wood (2013), who explored mathematical identity in group work, and Campbell and Hodges (2020), who examined group dynamics. Drageset and Ell (2024) identified five teacher positions, based on research on mathematical discourse patterns, and described their corresponding rights and duties, emphasizing how positioning theory uncovers the underlying powers shaping the classroom. These frameworks illustrate how positioning theory can be used to conceptualise the classroom and offer tools to study in-the-moment classroom interactions. It supports our goal of understanding how roles like "initiator" can transform participation in problem-solving.

Method

This article reports on an intervention study that investigated how assigned roles influenced students' participation in groups working to solve mathematical problems. To explore this, we collected data from four eighth-grade classes in a Norwegian school, first one lesson without roles and then one lesson with assigned roles. In both lessons, students were given a problem-solving task and worked collaboratively in groups of three to four students for 40-45 minutes. The group discussions were filmed and transcribed.

The four role categories used were first developed by Allern and Drageset (2017): the democratic leader, the curious, the sceptic, and the initiator. Each student was assigned one of these roles by their teacher without directions from the researchers. Between the lessons was the intervention, where we worked with the students on using these specific roles in group discussions and developing role cards describing each role. The role card of the initiator describes the role in this way: The initiator shall take the initiative to start the work on the task and help others get started. The initiator takes initiative and explores ideas and possibilities, is engaged, helps the group move on by suggesting other alternatives if the progress stops, and asks questions like "What if we rather do it like this?", "Shall we try to do it this way?", "What if we start by…". An ethical dilemma is that being assigned a role that is not natural for the student might cause discomfort. However, the students decided how and to which extent they would take on the role. The researchers and teacher also monitored the groups and followed up on signs of discomfort.

The tasks given in the lessons with and without roles involved the same mathematical topic and similar types of problem-solving. The group in focus in this study first worked on a task about an unknown number of cabins being equally furnished with stools with three legs and chairs with four legs and asked about the number of each type of furniture and the number of cabins. The second lesson had a task about an unknown number of friends buying the same amount of candy at various prices, asking what they could buy for 36 and 54 kroner. Each task had three sub-tasks, marked a, b and c.

In the initial phase of the analysis, we found one group particularly interesting due to their significant change in participation, with twice as many utterances related to the task in the second lesson. Further, we found one student particularly interesting based on a change from almost no involvement to more active participation. To explore this change, we applied positioning theory to analyse how the group interacted using a thematic analysis (Braun & Clarke, 2012). First, this was done mostly deductively based on the positions described by Barnes (2004). The first and second authors did the analysis together, coding student positions based on communication acts, both utterances, facial expressions, and gestures. Secondly, we inductively analysed the students' enacted rights and duties, in connection to their positioning.

Results

The group consisted of four members. Table 1 shows how large part of the time each participant spent working on the task. The time measured was the time of dialogues in which they participated, including the time they observably worked on the task by themselves, observed through gestures such as counting on their fingers or writing. After being assigned roles, all members participated in the second lesson. As explained above, we will focus on the initiator in this article and the change from the first to the second lesson. To make it easier to follow the changes, we will name the students by their roles in both lessons, even though they were only assigned roles for the second lesson.

Table 1: Proportion of the two lessons each participant spent working on the task

	Demo. leader	Sceptic	Curious	Initiator
Without roles	53%	12%	12%	1%
With roles	80%	36%	43%	19%

Lesson without roles

The first lesson, without roles, was characterised by many distractions from the task. Two of the students, who were assigned the roles of sceptic and curious in the second lesson, spent a significant amount of time talking about other things. The democratic leader took on a leading role and came up with suggestions and explanations or answered questions from the other group members. His communication with the group implies that he had a high status in the group, acting as an expert and speaking "with authority about mathematics" (Barnes, 2004, p. 8). When the others withdrew from the collaboration on the task, the democratic leader kept working on the task on his own.

In the first lesson, without being assigned her role, the initiator only had three utterances that could be linked to the task. Once, she was asked by the curious if she had anything to say and answered "No". The other time, she asked the following questions to the democratic leader:

1 Initiator: Are we on task a now?
2 Demo.leader: No, now we are on b
3 Initiator: Yes, but have we found out how many stools there must be in each cabin?

The initiator's position in this excerpt can be interpreted as *in need of help* since she asks questions that might make her able to follow and participate in the group discussion. The democratic leader, however, is placing her in an *outsider* position by ignoring her attempt to join the discussion and by not answering her question.

The rest of the time, when the group worked on the task collaboratively in the first lesson (without roles), the initiator said nothing. She often looked at the person who was talking, at the paper someone was writing on or at the task sheet, so albeit staying silent, she might have followed the group discussion and tried to understand the others' suggestions. This differs from the *outsider* position, as Barnes' describes it, where a person "withdraws mentally from the group, saying nothing for a long time, and giving no sign of seeking to participate" (2004, p. 11) or is ignored when trying. Because a student who pays attention to the group discussion without saying anything is not included in the *outsider* position and cannot be described as mentally withdrawn from the group, we see this as a new position that we call the *silent observer.* We suggest the *silent observer* as an addition to Barnes' framework. Since this position does not involve actively engaging in the group discussion, it could be placed in the "Not fully belonging"-category. The initiator often held a position as *silent observer*.

In addition to this, the initiator took on some other positions. When the others took up the position *entertainer,* she looked at them and laughed together with them, thereby acting as *audience.* She also talked to or listened to members of other groups, taking on the position of *networker*. None of these were positions that are related to "thinking about math".

Lesson with roles

In the second lesson, after being assigned the role of "initiator", the initiator takes a more active part in the group discussion. As shown in Table 1, from almost not participating at all, she now participates 19% of the time. Her number of utterances linked to the task rises from three to 15.

On two occasions, she initiates work. To begin with, after they have all read their role cards, the initiator says "okay, what if we start by reading the task". This is defined by Barnes (2004) as a *manager* position. On the role card, it is stated that "The initiator shall take initiative to start the work on the task". It is therefore a clear duty of hers to initiate the work, a duty that she acts on.

Her initiative is disputed by the sceptic, which causes a short period of distraction before the democratic leader picks up on the initiator's initiative:

4	Demo.leader:	Shall we take the initiative to read?
5	Initiator:	Some friends are in the kiosk to shop; the kiosk sells candy for 2 kroner.
6	Sceptic:	Are you sure it is candy?
7	Initiator:	It says so here
8	Sceptic:	Are you sure it says so there?
9	Initiator:	Can I change?

Here, the democratic leader invites her to take the initiative. One can interpret this as an act of support, which is an important part of the democratic leader's role. The sceptic asks questions but does so in a way that causes disturbance. This is not part of his role, as he is supposed to be a constructive critic, but it might be a way for him to explore his role. Anyway, this causes the initiator to ask if she can change roles as we interpret her utterance in line 9. The sceptic makes it hard for her to perform her duty to take initiative and violates her right to be heard and taken seriously when doing so.

Later, when the group have worked on task a and b, and are taking a short break after finishing a solution on task b, the following dialogue happens. Because of the significance of non-verbal communication acts, we include gestures and body language in this excerpt.

10	Demo. leader:	[Looks at the task sheet]
11	Initiator:	[Speaks with a low voice, directly at the democratic leader] *Now it is c -* [Looks at the task sheet and points to task c on the task-sheet]
12	Demo. leader:	Speaks with high volume, interrupts initiator] Okay. Now we do C. [Writes c on their solution paper].
13	Initiator:	[inaudible] [Looks at the task]
14	Demo. leader:	*If we are going to get 54 together, does anyone know a multiplication that makes 54 that we can use?* [Looks at the solution paper and then looks at the rest of the group]
15	Curious:	[Counts on her fingers silently]
16	Initiator:	*9 times 6* [Looks at the democratic leader]
17	Demo. leader:	*9 times 6, that is 54? Then we take 54, with 9 times 6.* [Looks at initiator]
18	Curious:	*Why 54?*

In line 11, the initiator tells the democratic leader that they should start working on task c. By doing this, she helps the group get back on track, an important duty as initiator. An interpretation of this is that she feels obliged to do this as part of her role. She says, "Now it is c", with low volume, and the democratic leader amplifies her initiative by saying it louder. This can be interpreted both as a positive recognition of her initiative, supporting her in her role, or as the democratic leader taking over her position as manager and interrupting her to manage the group's work himself.

In line 16, we can also see that the initiator actively engages in the discussion by suggesting “9 times 6”. One can say that she is a collaborator here because she listens to the question from the democratic leader and responds to this by making a suggestion, thereby working closely with him. Another way to look at it is that she is a helper, answering the democratic leader’s call for a pair of factors with 54 as the product. This concrete question from the democratic leader might make it easier for her to contribute to the discussion. The initiator might have interpreted that her role involves initiating new solutions. Thereby, she fulfils one of her duties when she says “9 times 6”. The democratic leader and the curious both respond to this suggestion and, by doing so, take her suggestion seriously, as is their duty as group members.

About two-thirds into the second lesson, the democratic leader turns to the initiator and says, “You initiator? You do not take so much initiative”. An interpretation of the democratic leader’s actions may be that part of his duty as a democratic leader is to include everyone and that he wants her to join the discussion. The statement can also be interpreted as a criticism from the democratic leader for not fulfilling her duty as initiator. The initiator seems to react negatively to this; she leans against the wall and turns her body away from the group. Her position in the group is probably quite fragile. Such criticism from a high-status person can cause her to lose some of the security she has established with support in the initiator role. The initiator does not return to the dialogue about mathematics after this episode. One interpretation is that she takes on an *outsider* position and withdraws.

Discussion and conclusion

Our findings show that assigning roles, especially the initiator, significantly impacted student participation. The silent observer, who barely contributed in the first lesson, became an active participant when given the initiator role in the second. The role cards, inspired by drama pedagogy, helped students understand their roles, facilitating this change. These roles also include rights and duties, in the same way as defined in positioning theory (Harré & Moghaddam, 2003). The initiator role can be seen as closely related to the manager position, which “suggests that they begin work, or recalls them to work after a diversion” (Barnes, 2004, p. 5). Being assigned the role of initiator gave the initiator certain duties and rights. The name of the role gave her a duty to take initiative. This duty was further elaborated in the role card, which emphasised the duties of starting the work, helping the group move on, and exploring new ideas and methods. To effectively initiate work, the other group members must listen to and follow up on the initiative. This gives the initiator a right to be heard, respected, and taken seriously.

The initiator answered her duties in several ways. Firstly, she was required to initiate the work on the task, and she did so by suggesting that they would start by reading it. Secondly, she helped the group back on track when they got distracted after finishing task b, another duty mentioned in the role card. Thirdly, she came up with suggestions for solutions. This was not stated on the role card but may have been a part of how the initiator interpreted her role.

The initiator’s position in the group collaboration went from mostly being a *silent observer* in the first lesson to actively participating in the group discussion in the second. Her duties and rights as the initiator might have made it easier for her to take part in the discussion. All this illustrates how rights and duties, as defined by (Harré & Moghaddam, 2003), seem to have been an important reason for why and how she participated much more in the second lesson.

The initiator's exercise of the role happened in connection with the other group members and their roles with rights and duties. A useful tool to understand this is the dimension of willingness, capability, and power to take on positions (Davies & Harré, 1999; Huang & Wang, 2021). The group members sometimes responded to her initiatives and suggestions by repeating them, building further on them, or questioning them. By doing so, they respected her right to be heard and taken seriously and gave her room and power to participate. Especially the democratic leader did this. He invited her to initiate reading of the task and suggest solutions and repeated her initiative to start on a new task. In this way, he supported the initiator's enactment of her role, and as the democratic leader, his duty is to invite and allow others, thus giving them the room and power they need. This also illustrates how the duties of one role (the democratic leader) were to secure the rights of another (the initiator). While our data cannot determine which of willingness, capability, and power (as defined by Davies & Harré, 1999) prevented her from participating in the first lesson, we see traces of power as the main aspect when she is invited and allowed into the discussion with her role as initiator.

However, there were also examples of interactions that hindered the initiator in exercising her role. Being assigned the role of initiator gave her a right to be heard and taken seriously when initiating a solution or managing group work. The sceptic asked unserious questions at the start of the task, which might be seen as an exploration of his duties as a sceptic. However, he thereby did not respect and take the initiator's initiative seriously. He, therefore, did not fulfil his duties as a group member in relation to the initiator role. This caused the initiator to ask if she could switch roles, which can be seen as an expression of her feeling that her rights as initiator were being violated. The democratic leader criticised the initiator for not taking enough initiative. This might have hindered the initiator, as she did not take part in the discussion after this point. Both these are examples of how the two roles, and especially the democratic leader, used his power in a way that resulted in the initiator withdrawing from the discussion and role. The explanation of this change in the democratic leader, from supporting to criticising, might be in storylines on another scale (Herbel-Eisenmann et al., 2015), perhaps about who is the mathematical authority. As mentioned earlier, assigning roles carries ethical implications, as it involves granting power that can be used to include or exclude others—both of which are evident in this case. This highlights the importance of teachers being attuned to group dynamics and power structures to ensure a supportive and respectful learning environment.

The conclusion is that the initiator's change in participation between the two lessons suggests that giving a student a defined role with concrete tasks might enable the student to take part in collaborative work on an unfamiliar task. The interaction with the other group members at the same time influenced the possibilities she had to use her rights and perform her duties, and this both supported and undermined her power to take on the role. Also, this analysis illustrates how positioning theory can be used as a tool to understand changes in participation, and further research might address how the use of roles can affect the power dynamics in group work, when applied over longer periods.

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Bridging the empirical-deductive gap – the mathematics content matters

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Research has shown that young students can engage in proving, and it has identified general guidelines for designing tasks and teacher practices that can support their work on proving. However, the specific mathematical content matters in a proving situation. In this paper, we give an example showing students struggle with proving, and the struggle is related to the complexity of the content. The task has features that can support students' deductive reasoning, and the features do that to some level, but only part of the way. We discuss what is challenging for the students and why giving general suggestions for supporting students toward deductive reasoning can fall short.

Keywords: Reasoning and proving, empirical-deductive gap, teacher support, primary school.

Introduction

Proving is a crucial mathematical activity recognized by mathematics researchers and authorities as an activity that should permeate all mathematics teaching, also in primary school (e.g., Kunnskapsdepartementet, 2019; Stylianides et al., 2017). However, proving is challenging for students, as documented in several studies (see Stylianides et al., 2017). One persistent challenge in work on proving is students' inclination to provide empirical arguments and regard them as valid mathematical proofs (see Reid & Knipping, 2010). Several classroom-based studies have examined how to support students in the transition from empirical to deductive argument. Especially the role of the teacher has been studied – how particular teaching practices can help students towards deductive arguments (e.g., Ellis et al., 2022; Valenta & Enge, 2022) and how to design proving tasks (e.g., Buchbinder & Zaslavsky, 2018; Knox & Kontorovich, 2023). By teaching practice, we mean a teacher's action to solve what she perceives as a particular pedagogical task (see, e.g., Valenta and Enge, 2022).

Thus, the research has shed some light on what makes proving difficult for learners and how teachers can support them. However, Dawkins and Karunakaran (2016) point out that in much of the research on learning and teaching proof, the researchers discuss proving as a generic activity without paying enough attention to the specific mathematics involved. Dawkins and Karunakaran (2016) give examples of how students' understanding of a particular mathematical notion (and not only their understanding of proving per se) can influence their process of proving, and they argue that framing mathematical proving as a single, content-independent practice may downplay the significance of the mathematical content. We agree with Dawkins and Karunakaran's (2016) comment on the importance of the mathematical content and problems with framing proving as content-independent practice. We notice that studies (also our own) on supporting students in developing deductive arguments describe their findings in general terms.

Our research question in this study is: *How can the mathematical content influence the development from empirical to deductive argumentation?* We will discuss the question using an episode from a 7th-grade classroom, and we believe that the episode can be helpful when discussing the complexity of teaching proving in mathematics teacher education.

Theoretical background

Given a general mathematical claim, Stylianides (2008) states that the argument for its validity can be mathematically valid and non-valid. *An argument by generic example* is a valid argument in which a particular example is used to represent a whole class to explain *why* the claim holds for a given example and that it will hold for all other examples. Another type of valid argument is a *demonstration* where no specific example is used, and the argument is expressed in general terms (often using algebraic symbols). Both argument by generic example and demonstration have a deductive structure. A type of argument that students often use is *empirical argument*, which is not valid. In such an argument, based on checking on some cases that the mathematical claim is valid, one concludes that the claim is generally valid without explaining why it should be the case. Several teaching experiments have aimed to develop the learners' skepticism toward empirical arguments (e.g., Brown, 2014). However, learners sometimes use empirical arguments even when they know they are not valid mathematically because they struggle to find another way to verify general statements (Stylianides & Stylianides, 2009).

To support students toward deductive reasoning, Mason (2019) suggests that students' attention needs to be directed to see structural relationships in the examples. He points out that once the student can "see the general in the particular," in other words, see the example as generic and recognize a general structural relation in a particular example, one barrier toward proving has been overcome. In their study, Valenta and Enge (2022; see also Ellis et al., 2022) identify several teaching practices that can support students towards deductive argumentation, such as communicating the limitations of empirical arguments, highlighting for the students a general structure across the examples, suggesting for students how to proceed to recognize a structure in the examples and directing students' attention to pattern searches in their work on examples. Knox and Kontorovich (2023) discuss features of a learning environment that can support students' bridging of the empirical-deductive gap, and they present their design for young students' work on parity of numbers. The features are that the wording of the task emphasizes the difference between the validity of the claim always and sometimes and that the manipulatives and illustrations are incorporated in the task to promote the structure of even and odd numbers and to help students in expressing generality.

The short overview above shows that teaching practices and task design principles for supporting students in transitioning from empirical to deductive reasoning are usually framed in general terms, even though the underlying empirical studies are related to some concrete mathematical content. This is reasonable since there are features associated with proving that are not dependent on the content, e.g., understanding the limitations of an empirical argument. Nonetheless, challenges in students' development of deductive argument can be related to the content, and the teacher's moves and task features can fall short of supporting the students, as we will show in the following example concerning a particular relation within multiplication.

Different contexts in everyday life can be described by multiplication. One often used to introduce multiplication in primary school is a context of repeated addition where there are some groups with an equal number of objects in each group, and the product is defined as a total number of objects. Another context can be seeing the product as an area of a rectangle or a number of cells in an array. As Rønning (2012) points out, different contexts give different opportunities to reason about properties of multiplication, and, for instance, while commutativity is rather obvious using the array model, it is more intricate when using the equal group model.

Method

The data we analyze in the current study is collected as a part of a larger research project aiming to develop resources that can support teaching and learning to reason and prove in primary school. As part of the project, the researchers in collaboration with three primary school teachers designed lessons and tasks to be tried out in their classes. The episode we analyze here is from the fifth lesson designed and tried in the seventh grade. The class consisted of 20 students, 12–13-year-olds. In the designed lessons, the teacher led the whole-class discussions, and the researchers participated as additional teachers in interactions with the students during their group work. For each lesson, whole-group discussions and group work of two to three groups were videorecorded. The data we analyze here is the transcription and written work in one group. The work of the same student group from the previous lesson, about the sum of three consecutive numbers being divisible by 3, is analysed in Valenta and Enge (2022). The task designed for the lesson is shown in Figure 1.

The relation between the products $a \cdot a$ and $(a-1)\cdot(a+1)$ promoted in the task has been suggested by several researchers as appropriate to promote mathematical reasoning, for example Lannin et al. (2011). They suggest two ways to investigate and prove the given relation within multiplication: one using the equal-groups model and the other using the array model of multiplication (p. 31-34). The 7th-grade students in this study were familiar primarily with thinking about multiplication as equal groups (according to the teacher).

In Dina and Jan's class, they made caramels that they would sell to raise money for a school trip. Dina and Jan were assigned the task of packing the caramels in bags. Eventually, they found out that Dina had packed seven bags with seven caramels in each bag, and Jan had packed six bags with eight caramels in each bag.

Jan: Ugh, we cannot have different numbers in each bag. We must do it all over again. But look here: We have packed almost equal number of caramels!

Dina: Yes! Seven times seven is 49, and 6 times eight is 48. Only one is the difference.

Jan: One 7 went up, and one seven went down, and there was a difference of just 1.

Dina: Weird. Let us take a number other than 7. We take nine times 9, that is 81. One up and one down, eight times 10, and that is 80. Same again!

9 × 9 = 81

8 × 10 = 80

Jan: Maybe it is always the case that when one multiplies a number with itself and compares it to the multiplication with one up and one down, it becomes 1 in difference? Or is it just something that happens sometimes?

Task: Investigate Jan's question, write a conjecture, and argue for it. Remember the criteria for a good argument.

Figure 1: Task given to the students

The teachers and the researchers envisioned that one possible solution within the reach of students could be to use an example, let us say, 7 · 7, and see it as seven groups of seven objects. Then, unpacking one of the 7-groups and distributing one object into each of the remaining six groups would result in six groups of 8 objects and one object left. Figure 2 shows this and compares 7 · 7 and 6 · 8. The argument could then be generalized and thus become an argument using a generic example.

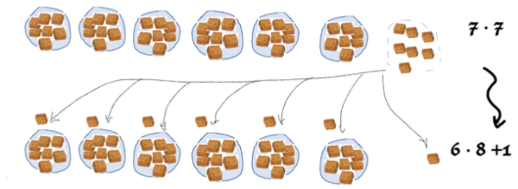

Figure 2: The solution envisioned by the designers

To support students' transition from empirical to deductive reasoning, an uncertainty was introduced in the task between *always* or *sometimes true* (as in Knox & Kontorovich, 2023; Valenta & Enge, 2022). Furthermore, the equal group model of multiplication, packing, and unpacking of bags, was incorporated in the task to invite students to use these features (similarly to visual mediators for work on parity in Knox & Kantorovich's (2023) study). Also, in the joint planning of the activity, possible teacher support during students' group work was discussed.

Episode

Three students were in the group: Marie, Aida, and Loran. In the previous lesson, about the sum of three consecutive numbers being divisible by 3, the same group, supported by the teachers, realized the limitations of empirical reasoning and developed a deductive argument (see Valenta & Enge, 2022).

The group started by checking several examples, as the excerpt below shows:

13	Loran:	OK. 10 times 10, 11 times nine.
14	Marie:	11 times nine is 99, and 10 times 10 is 100.

The students checked two more examples (8 · 8, 5 · 5). On the solution sheet, they wrote the examples the have checked and "It is always true" and "It goes one up and one down." Since they wrote this on the solution sheet, it seems that their argument would be like this (thus empirical) if there were no teacher intervention. However, the teacher came by:

62	Teacher:	So, guys think this is true? [The group confirms.] Why is that?
63	Loran:	Because nine times nine is 81, and … [The teacher interrupts]
64	Teacher:	Yes, but that is just an example.
65	Loran:	Oh yes. Well, if you take ten times eight and nine times nine. Then in a way that you can just add one, but instead of adding one … and the other number you multiply by, and then you add it to the answer.
66	Teacher:	Can you use these bags here? [Pointing to the illustration on the task sheet] How can you see that there will be one more here than here? That seven bags of seven is one more than six bags of eight.
67	Loran:	Ugh … Because six bags are one bag less. And then it will be minus eight. Also, you have seven more.
68	Teacher:	Yes, try to think in that direction. What is happening, seven bags of seven and six bags of eight?

In turn 64, the teacher pointed out that only checking examples would not be enough as a valid argument, and Loran took the hint (it was discussed with the teacher the week before) and started to explain what was happening in another way by pointing to some relations between numbers and operations. It was not easy to understand what he meant, and the teacher pointed to the illustration given in the task and suggested trying to use it. After that, the teacher left the group. The continuation of the group work can be described as Loran's repeated attempts to explain his argument and Maria's and Aida's repeated attempts to understand it. The first attempt is as follows:

75	Loran:	If we take six times eight, equal to 48, then we take seven times seven, equals 49, so we do... six bags, it becomes six bags. It is not the same as seven, because we lose a bag. So, we lose a bag, for example, and then it becomes eight less. So, it will be seven times eight, which is 56, so it will be seven less. Therefore. [Marie: What?] I do not know how to explain it.
76	Marie:	You need to explain properly.
77	Loran:	How many bags are there here? [points to the illustration in the task]
78	Aida:	12.
79	Loran:	I am not talking about altogether.
80	Marie:	It is seven and six, OK!
81	Loran:	Yes! How many bags will be less?
82	Marie:	One bag less.
83	Loran:	And how much less will it be? Eight.
84	Marie:	Where do you get eight from?
85	Aida:	What?
86	Loran:	It is minus eight. Seven times eight is 56, minus seven, no eight, is 48.

Loran pointed out that there was one bag less in $6 \cdot 8$ than in $7 \cdot 7$. Then he compared the two products to the third one, namely $7 \cdot 8$. He pointed out that $7 \cdot 7$ is seven less than $7 \cdot 8$, and that $6 \cdot 8$ is eight less than $7 \cdot 8$ (thus, $6 \cdot 8$ is one less than $7 \cdot 7$). While his approach of comparing the given products to the third one holds the potential for forming a valid argument supporting the conjecture, it is marred by several critical gaps. And Maria and Aida said that they did not understand. Loran repeated the same reasoning once more, again without success. In the third attempt, he expanded:

123	Marie:	I do not understand what you mean. OK, we are starting over here. You mean it is seven bags, also you take away one, then we have six. Also, it will be 48 somehow. Since it is six times eight. Um. Also, you have seven muffins, or seven bags, which becomes seven times seven. [Aida: And you take away one] Then it becomes 49.
124	Loran:	Now look, if you take as many bags as on this, as on seven, it will be 49.
125	Aida:	And six times eight is...
126	Loran:	Then eight bags, with seven pieces in them [Aida: aha!], you take away that one bag, and you have 49. But if you do it the other way around and have seven bags of eight pieces in each. And take away one, then it becomes 49.

From what Loran was saying, particularly in turn 126, it seems that his thinking was as presented in Figure 3: He started from $8 \cdot 7$ (eight bags, seven objects in each) and compared it with $7 \cdot 7$ and $6 \cdot 8$, using commutativity ($8 \cdot 7 = 7 \cdot 8$) on the way. The definition of multiplication and the proposed model is used throughout the whole argument (contrary to the argument in turn 75-86). Multiplication was presented as equal groups, the first factor being the number of groups and the second number of objects in each group. If we take the commutativity of multiplication as known before, the argument can be generalized and lead to a valid proof for the conjecture $a \cdot a = (a-1) \cdot (a+1) + 1$.

We are to compare 7 · 7 and 6 · 8.

Consider 8 · 7, thus eight bags, seven objects in each.
We know that 8 · 7 = 7 · 8, the latter being seven bags, eight in each.

Now, 7 · 7 is one 7-bag less than 8 · 7. That is, 7 · 7 = 8 · 7 – 7
And 6 · 8 is one 8-bag less than 7 · 8. That is, 6 · 8 = 7 · 8 – 8
} (8 · 7 = 7 · 8) ⟹ 7 · 7 = 6 · 8+1

Figure 3: Loran's argument

However, Maria and Aida still did not understand, and Loran repeated the same argument as above. Maria and Aida continued to ask for a better explanation, and the group work ended as below:

173	Loran:	OK. You have seven bags with eight pieces in each bag. Also, you have seven bags with seven pieces in them. There are just as many bags.
174	Marie:	Seven bags of eight and seven bags of seven.
175	Aida:	No, you have six.
176	Marie:	Yes, you have six bags with seven in them. That is how it is, Loran.
177	Loran:	No, or in a way, we can do it. Then, for example, we take one bag away and give it …
178	Marie:	From which?
179	Loran:	You have seven bags.
180	Marie:	But from which ones do you take away, six or seven? Seven times seven or six times eight?
181	Loran:	Not like that, you have seven bags of eight, so it is seven times eight. And six times seven.
182	Marie:	And?
183	Loran:	Also, take away one bag.
184	Marie:	From which?
185	Loran:	Seven times eight. And give it to [Aida: Six?] six times seven.
186	Aida:	Yeah!
187	Marie:	Yes (nods).
188	Loran:	Then it will be the same as seven times seven and six times eight. Do you understand now?
189	Marie:	Yes, you just move it! You could have said it from the start! Move over the bag! You can; you have been sitting and saying that you take away the bag, that it will be gone forever! [Aida and Loran laugh] and suddenly add one from who knows! Has it fallen from the sky?

All three students seemed pleased with the developed argument, and Loran wrote on the solution sheet, "7 bags of 8 muffins, and 6 bags of 7 muffins. Then it is the same as moving over that one bag", and they finished the work here. However, the argument provided in this last excerpt differs from Loran's earlier arguments and does not explain why 7 · 7 is one more than 6 · 8. In this argument, seven 8-bags and six 7-bags are compared; one 8-bag is moved from the first group to the second, resulting in six 8-bags and seven 7-bags (plus one object, which the students do not mention). This does not show the relation between 7· 7 and 6· 8, as can also be seen by noticing that they start with a set of 56 objects and a set of 42 objects, move eight from the first to the second, and end with 48 and 50 objects in the different sets (while it is to be shown that 7 · 7 = 6 · 8 +1).

Discussion

This study aimed to shed light on the complexities of bridging the gap between empirical and deductive arguments and highlight the importance of mathematical content. The students in this study is the same as one group from Valenta and Enge's study (2022). In the episode we present here, the students work on a task designed to support their development of a deductive argument. Similar features of task design have been used in other studies and shown to be effective in the sense that students, with support from the teacher, managed to develop a deductive argument (e.g., Knox & Kontorovich, 2023; Valenta & Enge, 2022). As indicated by the analysis, the students first developed an empirical argument, and the work on the task would have ended here if the teacher had not intervened to clarify that these were just examples. A week before, the teacher and students discussed the limitations of an empirical argument in a similar task, including a distinction between always and sometimes true, as described in Valenta and Enge (2022). Thus, this feature of the task and the teacher's interventions, both in the episode the week before and this one, lead students to abandon empirical argument and start to look for some structure, a first step towards deductive argument.

Further, Loran identified a key idea that could lead to a proof. Nevertheless, he struggled to express it and communicate it to the other students in the group. After several attempts, he abandoned the idea. The group's final argument, while not empirical, still lacks deductive reasoning. Although the students went beyond an empirical approach, their attempted deductive argument proved too challenging, and the empirical-deductive gap was not bridged. We argue that the particular aspects of the mathematical content involved made the bridging difficult, as we elaborate below.

Both factors change in the two products that are to be compared, $7 \cdot 7$ and $6 \cdot 8$. Using the context, both the number of bags and the number of objects in each bag are changing. This led perhaps Loran to introduce a third product ($7 \cdot 8$) to compare with the original two products. Even though the designers thought it should be possible to compare directly, by opening one of the bags and distributing, the students did not do that. Loran's introduction of the third product made the argument more complex, adding several steps. In particular, Loran's approach involves the use of commutativity, which is not so easy to see when multiplication is modelled as equal groups (Rønning, 2012). The argument becomes too complex to communicate (for Loran) and to comprehend (for the others), as shown in the expression of relief at the end when a bag is "just moved." Another reason students struggle with comprehending and expressing the argument is that the same numbers (6, 7, 8, 1) appeared in several different roles. While the first challenge (more complex argument) could be avoided by choosing some other approach, this second one is difficult to avoid when trying to examine the relation between the products $a \cdot a$ and $(a-1) \cdot (a+1)$.

More support from the teacher could perhaps help Loran and the student group further in their work. Particularly, the teacher could support Loran in structuring and expressing his argument, clarifying various steps, and filling the gaps (actions described in, e.g., Yackel, 2002). The episode analyzed in this paper happened during group work; the teacher could not stay with Loran's group. Even if the teacher came by and Loran had shared his thinking with her, there is no guarantee that she would have understood it immediately and managed to support him further since his argument was unexpected and somewhat complex. The episode illustrates why it can be challenging to teach proving. We believe developing general guidelines for designing tasks, providing visual mediators, and putting forward teacher moves in connection with teaching proving is worthwhile. However, proving is strongly connected to the specific mathematical content. Therefore, researchers should consider the mathematical content when researching the teaching and learning of proving.

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Design-based research: The challenge of executing what others have designed

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In this paper, we describe various challenges experienced by the authors when conducting a design-based research (DBR) project. The challenges are primarily elucidated from the perspective of a teacher who assumed the roles of codesigner and implementer in a developmental DBR project. Many of these challenges can be attributed to a singular factor: the teacher's lack of ownership over a lesson plan for which the detailed planning was performed by the researcher. While the difficulties in such cases may vary depending on the teacher's prior experiences with the subject matter, some of the challenges experienced may be embedded in the DBR approach itself.

Keywords: Developmental design-based research, collaboration, fraction instruction.

Introduction

Design-based research (DBR) is an increasingly prevalent method in educational research (Gundersen, 2021). Barab and Squire (2004) define DBR as a series of approaches designed to develop theories and practices with the potential to explain and improve learning and teaching in classrooms. Although DBR has the potential to bridge the gap between teaching practice and theory (Bakker & van Eerde, 2015), conducting DBR projects is not easy – particularly when it comes to collaborating on the development and implementation of lesson plans.

As part of the second author's PhD project, she conducted a small-scale DBR project in collaboration with a teacher (the first author). The DBR project focused on the development of principles to guide the use of different mathematical representations in initial fraction instruction in primary school settings (Grades 2–3, ages 7–9). The project was a collaboration between the two authors in which the teacher was a codesigner and implementer (Gundersen, 2021). Towards the end of the lesson implementation, we discussed some challenges the teacher faced throughout the DBR project. This paper aims to summarise these challenges, provide examples that elucidate them, expound on their origins, and discuss how the research community can, to some extent, mitigate these difficulties. The examples presented in this paper are not intended to pinpoint the specific moments in which the challenges occurred. Instead, they are provided as illustrations of where such complications might have emerged in hindsight.

We first provide a brief overview of the Danish school context within which the DBR project was conducted. Subsequently, we outline the specific DBR approach to which we adhered. Following this, we describe how the iterations of the project were executed. We illustrate some of the emergent challenges, focusing on two distinct yet interconnected challenges experienced by the authors. These challenges are likely rooted in the design choice and will primarily be described from the perspective of the teacher.

Background

In Denmark, teachers have considerable autonomy in their classrooms. This is partly because there is no strictly defined curriculum but rather some general goals to be achieved over a 3-year period. For example, it is expected that, by Grade 6, the student should be able to 'use rational numbers and variables in descriptions and calculations' (Danish Ministry of Education, 2023, p. 7). It is then up to the teacher to choose when and how this goal is translated into practice. On the positive side, this means that Danish teachers can be creative, independent, and reflective.

Fractions, which fall under the subject of rational numbers in primary school settings, are one of the most challenging topics at this level of mathematics (Behr et al., 1983). Research often suggests that the restricted introduction of rational numbers as part–whole comparisons is one of the major reasons that students have difficulties with fractions. This remains true even though numerous studies have highlighted the importance of conveying the many different interpretations of rational numbers (Behr et al., 1983; Kieren, 1993; Lamon, 2007). Additionally, Danish textbooks often focus on presenting fractions as part–whole comparisons (Færch & Pedersen, 2023). This might be why Danish students are very good at colouring parts of a whole, or naming the fractions when parts are shaded, but do not see fractions as numbers, cannot reason about fractions, and have trouble performing simple calculations involving them (Færch & Hodgen, 2023).

Design-based research

According to Gundersen (2021), DBR approaches all recommend four key activities:

1. Planning and implementing interventions
2. Developing these interventions in iterative cycles
3. Cooperating with the practitioners who implement the interventions
4. Testing and refining design principles based on data from the interventions

DBR is intervening in practice meaning that the development, evaluation, and improvement of the relevant design takes place in concrete practice. Furthermore, DBR consists of iterative processes: design, testing, evaluation, analysis, and (re)design. During these iterative processes, the design is tested in practice and then adjusted with the purpose of improving the design's robustness so that it can be applied in different contexts (diSessa & Cobb, 2004). DBR is collaborative at several levels. First, all participants can be involved in identifying the problem(s) to address. Subsequently, the researcher formulates a plan and then collaborates with the practitioner on designing this plan. The testing of the design is again a collaboration – the practitioner is not a subject but a participant. That is, the practitioner plays an active role in the reflections and analyses that lead to the redesign stage. Since the goal of DBR is to improve both theory and practice, the theories developed should address concrete problems that practitioners experience in their work. Whether the goal is to develop domain-specific theories, design frameworks, or design methodologies, DBR aims to illuminate practice. The goal is not to show that a specific design works but rather to develop and improve it. In a literature review, Gundersen (2021) identified three general categories that explain the different roles a practitioner can assume in a DBR study: codesigner, coresearcher, and cooperative partner and implementer. While practitioners who assume the *implementer* role may implement a design developed by the researcher or provide expert feedback on the design, a practitioner functioning as a *codesigner* can help identify the problem, develop a solution to the identified problem, or both. In the final role, *coresearcher*, the practitioner participates in all stages of the DBR process.

Method

In 2023, a developmental DBR study was conducted in which two design iterations were implemented, each of which involved three sequences (Table 1). All three design sequences focused on developing the students' initial conceptions of fractions. The overall goal was for students to understand that fractions are rational numbers. During the three sequences, the students encountered fundamental ideas related to ratios and proportions, which built on their existing everyday knowledge. The first two sequences took place in January and March, when the students were in the second grade, which in Denmark means that most students are 7–8 years old. The third sequence took place in October 2023, when the students were in third grade. All sequences consisted of 10–12 lessons with a duration of 45 minutes each.

Table 1. Overview of the design iterations.

	Sequence 1	Sequence 2	Sequence 3
Iteration 1 (Class 1)	January	March	October
Iteration 2 (Class 2)	February	April	November

In this DBR project, the practitioner (the first author) was both a codesigner and implementer in that she helped plan the lessons, provided contextual knowledge about the school, and implemented the lessons. When planning each of the three sequences, an overall goal was collaboratively determined by both authors. These decisions were informed by the academic literature on fraction education. The main ideas for each lesson were articulated during a meeting between the authors. After the meeting, detailed lesson plans were developed by the second author, who incorporated previous research findings into the ideas discussed. The teacher reviewed and commented on each lesson plan to make sure that it was realistic when the students' prerequisites were taken into account.

The second author then finished the lesson plan and developed the necessary materials before the teacher executed the lesson plan in a classroom (Iteration 1). The second author was present during these lessons and collected data by filming, taking pictures, and interacting with the students during group work. After each lesson, the two authors held a short meeting to discuss whether the lesson worked and what changes should be made to help the students learn. The first author then implemented the lesson again in another classroom (Iteration 2). Iteration 2 was typically carried out three weeks after Iteration 1 (see Table 1), so each sequence was completed in the first classroom before being repeated in the second classroom. Only teacher notes and student productions were collected during the second iteration; the second author could not be present due to time constraints and other practical obstacles.

The first lesson described in this paper was inspired by the ‘orange juice problem’, which was first described in Noelting (1980) and has been used in several Danish elementary school textbooks. In this version of the task, the mixture is presented as continuous data. As with the rest of Sequence 1, the focus is on students’ initial judgements of proportionality (Spinillo & Bryant, 1991) and ranges from part–part comparisons (ratios) to part–whole comparisons. The second lesson was inspired by Japanese textbooks. This type of task is typically found in third-grade textbooks in Japan when introducing the concept of the fraction as a measure. Since the aim of this paper is not to thoroughly analyse the tasks, but rather to use them to exemplify the challenges experienced by the teacher, the design decisions will not be further elaborated on here.

The challenge of executing what others have designed

Teachers make a wide variety of decisions when teaching students. Many of these decisions are sanctioned in advance if the teacher develops a detailed plan before entering the classroom. However, when involved in a DBR project, a practitioner loses some of the autonomy they normally enjoy, since a researcher or research team is involved, at different levels, in planning the lessons. Halfway through the third sequence of the DBR project described in this paper, the authors talked about the different lessons and discussed how and why some of them had been challenging for the practitioner to readminister in the second iteration. A predominant theme of this talk was the challenges of teaching a lesson that someone else had primarily responsibility for planning. The way the challenges came to the fore in the different lessons depended on their content.

During the three sequences, different challenges were encountered. For instance, the teacher would give some examples or use representations that the researcher had not expected or did not want to be included in the teaching; alternatively, the teacher may have focused on a different learning goal than the one intended in the lesson. Even when the lesson plan made sense to the teacher during the planning stage, to cite another example, some things did not make sense once the instructor was implementing the lesson. When a teacher is not involved in the actual planning of a lesson, they do not have the same opportunity to predict what the students might say and do in different scenarios, or what approaches different students might use. They might not, to the same degree, get to think about which children in the classroom could have trouble following the lesson points; thus, they might not be able to provide the same assistance to these students that they would in a normal, self-planned lesson. However, for the practitioner, the most crucial determinant of difficulties was whether she had taught a similar lesson earlier in her career – that is, the familiarity of the lesson content. In the following section, we illustrate some of the challenges with reference to two different lessons.

Executing a challenging lesson the second time around

Lesson 7-8 in Sequence 1 revolved around a widely used ratio task based on a mixture of juice and water. In accordance with the Danish context, we used a red juice called *saft* in Denmark. In advance, markings were made on six bottles to indicate the proportion of each mixture that consisted of water. In a whole-class discussion, the students were asked with respect to each sample of the mixture, What fraction of the whole bottle is filled with water? After they responded, the teacher added juice to the mixtures. The students had to determine which of the six different mixtures of juice would make the strongest drink. The students were paired two and two, and each group was assigned two bottles. The task for each group was to (1) figure out which mixture was strongest and (2) give reasons for why that mixture was strongest. They could use drawings of the bottles to support their arguments or use 2-cm linking cubes to build a representation of the mixture.

Figure 1: The six bottles of mixture used in Sequence 1, Lesson 7-8.

In the lesson prep phase, we had expected the students to be able to argue using terms such as 'less juice', 'more water' (distinguishing between different parts), 'half as much water as juice' (part–part comparisons) and 'the juice is one-quarter of the whole mixture' (part–whole comparison). However, during the lesson, only a few students tried to provide arguments using these different types of comparisons. Instead, most students' reasoning referred to the colour of the mixture. Their everyday experiences with juice mixtures appeared to get in the way of the mathematical arguments, leading them to focus solely on the colour. After the lesson, we talked about changing the type of juice to elderflower so that the students could not argue using colour alone. Upon further consideration by the teacher, she decided not to attempt the lesson – even in an altered format – in the second classroom. Several difficulties led her to this conclusion:

1. The students needed too much help to grasp the mathematical points.
2. Since she would be alone in the second classroom, it would be too difficult to assist all the students who needed help.
3. The teacher's lack of ownership over the task, and perhaps also over the overall aim of the lesson, made it difficult to apply the necessary changes on her own.

All in all, this left the teacher feeling that the changes needed before the task could work in the second classroom were too much to take on.

Executing unfamiliar content

In the third sequence, the aim was to get the students to see fractions as an extension of the number system they were already familiar with – that is, to shift from natural numbers to rational numbers. According to academic research, this is a crucial but relatively uncommon topic of fraction instruction in schools. It is not only unfamiliar to students, but also to teachers, since many of them do not encounter the concepts of fractions or rational numbers from this point of view in their own school days. Furthermore, it is not explicitly addressed in Danish textbooks, and Danish teachers thus do not have many examples to readily guide their teaching.

Figure 1. A student using a 1-metre piece of paper to find the length of the remaining part.

In Lesson 1-2 of Sequence 3, the students were asked to measure the length of their table using a 1-metre-long piece of paper. They were not allowed to convert units by dividing their paper into centimetres, but instead had to answer with the number of whole metres (1) and the remainder, given as a fraction of that metre (1/5). Although the students were quick to begin the investigation, and several students found an answer to the task, none of the students could see the relevance of answering using fractions. Instead, they would answer using the more familiar (to them) metric system, leading to the answer 1 metre and 20 centimetres. In their argumentation, they would state that ‘5 times 20 centimetres is 1 metre’, indicating that 1/5 of one metre is 20 centimetres. However, many students could not see the point of knowing how long the remaining part was. For them, it was enough to know that it was more than one metre and less than one-and-a-half metres, and since neither the task nor the content was familiar to the teacher, she found it difficult to motivate the children. Instead, we changed the context of the task and tried using caramel sticks in the next lesson. The reasoning here was that, if they could figure out how much each group member would get (one whole and the remaining part), then they would be allowed to eat the caramel sticks. This change in context motivated all students to find an answer that included a fraction of a whole stick. In the second iteration, only caramel stick tasks were included.

Discussion

In this DBR project, several challenges arose during lesson implementation, particularly in relation to repeated iterations of lessons in which mathematical concepts were difficult for students to grasp. According to Bakker and van Eerde (2015), DBR can bridge the gap between theory and practice, but it also presents unique challenges, especially when the practitioner assumes multiple roles, such as *codesigner* and *implementer* (Gundersen, 2021). In this context, the teacher’s role as *implementer* proved difficult when she was tasked with delivering lessons involving complex mathematical ideas – particularly those that students found challenging, such as fractions.

Fractions are notoriously difficult for students to master (Behr et al., 1983). In Denmark, there is significant emphasis on the part–whole interpretation of fractions, which is reflected in both teaching practices and curricular materials (Færch & Pedersen, 2023). Since multiple interpretations of fractions are essential in fostering a robust understanding of the concept (Behr et al., 1983; Kieren, 1993), the focus on part–whole interpretations might be one of the reasons why Danish students continue to struggle with reasoning about fractions, understanding them as numbers on a number line, and performing basic calculations involving fractions (Færch & Hodgen, 2023). This underscores the need for a shift beyond part–whole interpretations in fraction instruction.

When students struggled to understand key concepts in the initial iteration of a lesson, the teacher hesitated to repeat the lesson in the second classroom. This reluctance stemmed from concerns about reproducing a negative experience and the significant amount of assistance required during the first iteration. Even after discussions about how to modify each of the problematic tasks, the teacher ultimately decided against reimplementing reattempting them, fearing that the students would again struggle with the mathematical objectives. The complexity of these tasks, particularly those involving unfamiliar resources, highlights a core challenge of the DBR design. In this project, the researcher only actively participated in the first iteration, leaving the teacher to navigate the subsequent implementations on her own.

Another key challenge emerged when the teacher was asked to teach mathematical content with which she had limited personal or professional experience. In this case, the lesson required students to understand that $\frac{1}{5}$ is a number, and while the teacher had read and understood the lesson plan, she did not feel confident in delivering the key points as intended. This reflects the inherent difficulty teachers face when teaching unfamiliar content in DBR projects: they lack the mastery over the lesson plan that enables them to anticipate challenges and provide the necessary scaffolding for students. As a result, in this case, the teacher's focus during the lesson shifted towards aspects of the mathematics that were more immediately observable, rather than those that were pedagogically critical.

Gundersen (2021) pointed out that when practitioners serve as *codesigners*, they have more control over the content and lesson structure, which can mitigate some of these challenges. In contrast, teachers who primarily serve as *implementers* might struggle with lessons planned by others, especially when the content is unfamiliar. This was evident in our project. While the teacher was more comfortable delivering lessons on familiar topics, assuming responsibility for implementing the lesson design proved difficult. Danish teachers typically enjoy significant autonomy in their lesson planning and decide how and when to teach specific topics. This may have made it harder for the teacher to adapt to the externally developed lessons in this project. Even when the content was familiar and the lesson plan appeared clear, the teacher occasionally found it difficult to translate the plan into practice, suggesting a disconnect between the researcher's design and the concrete reality of the classroom.

These challenges highlight the need for careful consideration of the teacher's role in DBR projects. The introduction of unfamiliar content by a researcher without sufficient support for – and involvement of – the teacher in the design process can create problems for both teachers and students. As DBR aims to bridge the gap between theory and practice (Bakker & van Eerde, 2015), it must also address the complex dynamics that connect teacher autonomy and lesson ownership with the practicalities of classroom implementation.

Conclusion

In this paper, we have described some of the challenges we faced when conducting a DBR project. These challenges varied in frequency and severity in accordance with both parties' knowledge and prior experiences with the topic at hand. Due to time constraints, we initially divided the workload between us, which left much of the burden of lesson planning and material construction on the second author. This result can also be attributed to the (misleading) idea that when a teacher sacrifices their time to help with a DBR project, the least the researcher can do is make it as easy for them as possible. On the contrary, the workload delegation left the teacher with a lack of autonomy in delivering the lessons. Her limited involvement in the details of the lesson planning stage also left her less well equipped to predict students' thought processes, existing knowledge and experiences, and possible difficulties with the lesson. Normally, these contextual factors are anticipated during lesson planning.

While the types of challenges accounted for in this paper might have been more conspicuous in our specific DBR setup, where the researcher only participated in one of the two iterations, this complication could be experienced by other practitioners using DBR materials or lesson plans after a project has finished. However, the challenges that emerged in the second example could be distinct flaws embedded in DBR projects in general. To ensure better opportunities for cooperation and a more natural integration of the relevant theories in the execution of lessons, we see a need for teachers to be given the time needed to read key background literature relevant to the development and implementation of the lessons. This would lay the foundation for the teachers to be a more active players in planning lessons and choosing the materials needed, which would allow them greater autonomy over the lessons and the project in general.

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Flexible instructional trajectories in school mathematics

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This paper reflects on interim results of EduFlex ("Flexible learning paths to support learner-centred learning in schools") project where a set of flexible instructional trajectories were designed for 9th grade algebra and implemented by 5 teachers and 88 students in authentic classroom situation. These trajectories included a set of interactive resources that were used in the trials, and the learning analytics collected from the digital resources will help to explore how to support the change towards more personalised and flexible, learner-centred learning. A first, shorter-term pilot-study with digital interactive resources for multiplication of algebraic fractions showed that most of the students who participated in the study appreciated their self-directed learning and considered their interactive learning experience as effective.

Keywords: Flexible learning, flexible learning path, individualized learning, instructional trajectory, interactive learning resources.

Introduction

Although digital and open educational resources (OER) have been available in Estonia for a while, the existing educational resources are at times not relevant and up-to-date (Education Strategy, 2020). Also, digital learning resources are often not used by teachers to personalise learning or to implement self-directed learning. However, learners benefit when technology is applied to teaching in a research-based way: an effective learning process requires intrinsic student motivation and good independent learning skills in order to achieve satisfactory results, while independent learning skills are developed and maintained when teaching and learning activities are delivered in a variety of ways, using both different models and interactive learning tools (Sumbawati et al., 2020). Research also shows that merely providing a learning environment, learning resources and tasks in which students are given freedom of choice with little or no guidance, is not sufficient for students to feel effectively supported and able to learn successfully (Kirschner et al., 2006).

In order to support the development of digital learning resources and to eliminate learning gaps caused by the Coronavirus pandemic, the Ministry of Education and Research launched the MathDigilessons project in 2021 as part of the ‘Development and deployment of modern and innovative learning resources’ programme. In cooperation with the Tallinn University, the Tartu University and Videoõps Oü[1], comprehensive open learning resources supporting teachers and students were developed in the form of digital mathematics lessons for learning trigonometric functions at the upper secondary level and simplifying rational expressions at the lower-secondary school level. The learning resources were created specifically for mentioned sub-topics, as the

[1] https://videoops.ee/

Ministry of Education and Research identified that achieving learning outcomes in these sub-topics suffered the most during the Coronavirus pandemic. In the Tallinn University team, we focused on creating digital mathematics lessons for simplifying rational expressions at the lower-secondary school level. The content creation tool H5P was used to author the teaching and learning resources, which were published in 2022 on the national open repository e-Schoolbag and promoted to the community of mathematics teachers through various channels.

In an attempt to promote the development of established digital learning resources, explore the flexibility of learning, and provide learning analytics to support students and teachers, the research team at Tallinn University continued to develop digital lessons in lower-secondary school mathematics. We focused on students' perceptions of the flexible delivery-mode of the teaching resources created, and the role of technology, teacher, the fellow peers and learner in this mode of learning. Therefore, we set the following research questions:

1) How do students perceive the effectiveness of using flexible digital learning resources in learning to multiply algebraic expressions?
2) How do students perceive the roles and importance of technology, peers, teachers, and themselves in enhancing their learning experience within flexible, technology-enhanced learning environments?

In this paper, we introduce an initial conceptual approach for designing flexible instructional trajectories within an H5P-based learning environment, guided by a domain model in mathematics. We then evaluate these trajectories through an initial pilot study, aiming to capture students' perceived engagement and evaluate their effectiveness from the students' perspective, providing insights into how such instructional designs can enhance learning experiences.

Theoretical background and didactic elements of flexible delivery modes

There is no explicit definition of flexible learning, but there is consensus on three aspects that describe flexible learning: place, pace and mode (Hammersley et al., 2013). A number of researchers also agree that flexibility in learning can also consist (in addition to the above) in the choice of learning materials, the entry and exit points (in terms of learning outcomes, learning materials, etc.), and the choice of learning activities (Naidu, 2017).

In order to explore flexibility, we use two concepts in the research group: instructional trajectory and learning path. An instructional trajectory (see Figure 1) is a forward-looking part of a learning, designed by a teacher or instructional designer based on a domain model, which includes proposed learning objectives, learning outcomes, learning activities, learning resources, assessment methods, modes of instruction, etc. The instructional trajectory is designed to provide options for both teachers (compulsory vs. non-compulsory tasks, choice of learning activities, inclusion of supporting material, making decisions based on learning analytics, etc.) and students (use of supporting material, hints, recall, choice of pace and repetition, possibility of doing exercises, making decisions based on learning analytics, etc.). The learning path, on the other hand, is the digital footprint of the learner's learning activity, documenting in a machine-readable way the learner's actions, choices and outcomes. The learning analytics collected from the learning path allows the construction of a learner model and the formulation of recommendations for learner-centred choices for both teacher and student. Thus, at some point, the elements of an instructional trajectory (learning activities, exercises, etc.) become the elements of a learning path, from which the data collected are the input to the learning analytics provided in the instructional trajectory.

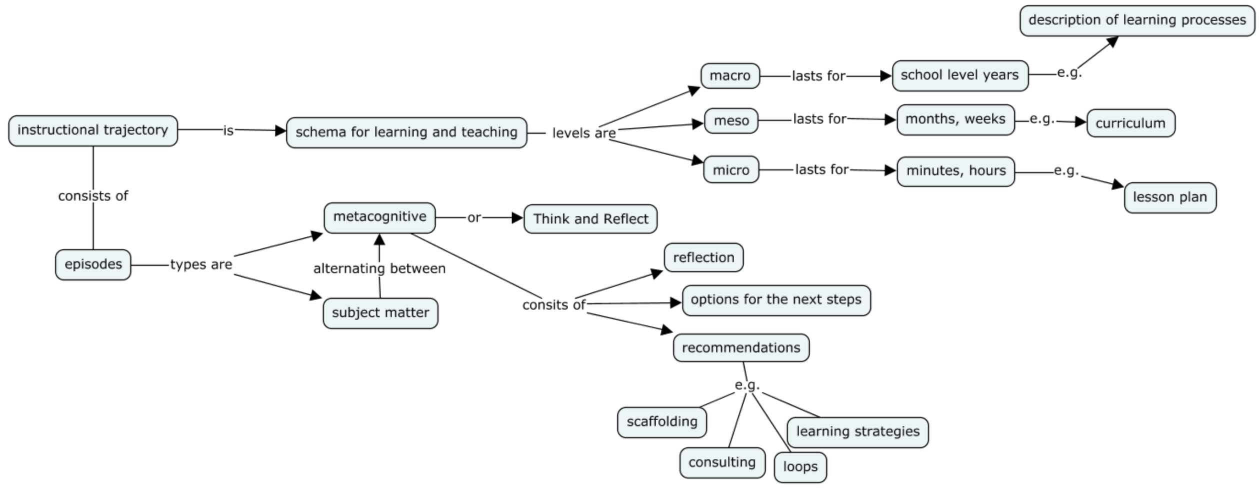

Figure 1: A concept map of the instructional trajectory

The prerequisite for the creation of an instructional trajectory is a domain model based on didactic principles and developmental psychology, which describes in a machine-readable way the expected cognitive development trajectory of the learner, presenting the units of knowledge to be learned (concepts and sub-skills) with their interrelationships.

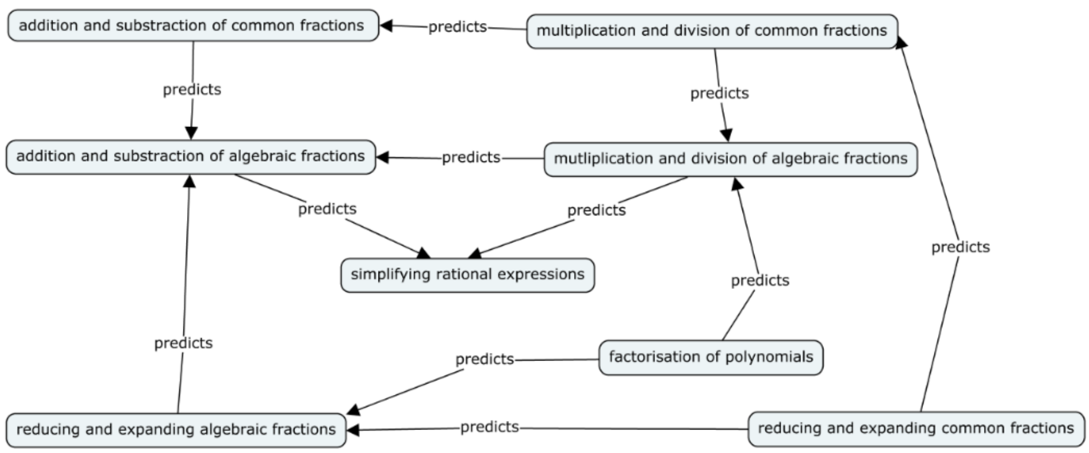

Figure 2: An example of domain model for 9th grade algebra

Figure 2 shows a domain model created for the 9th grade mathematics learning outcome in national curriculum for simplification of two-item rational expressions. In order to create a domain model, the learning outcomes need to be broken down into subcomponents (concepts and subskills), and define the relationships between them in a machine-readable form (e.g. as a knowledge graph). The domain model is the basis for the development of an instructional trajectory and for the provision of learning analytics.

As the lack of support and feedback in the learning environment can lead to cognitive overload and thus confusion for the student, which also can lead to misconceptions (Kirschner et al., 2006), we invested in the design of learning tools. In designing the instructional trajectory, we applied the 4C/ID (Four-Component Instructional Design Model) framework for instructional design (van Merriënboer & Kirschner, 2007) and Merrill's (2002) core learning principles. According to Merrill (2002), learning is more effective when new knowledge and skills are linked to prior learning and demonstrated through real-life examples, and when students have the opportunity to consolidate their learning through experimentation. Based on Merrill's basic principles of learning, the 4C/ID is used to create an instructional trajectory, according to which the acquisition of more complex skills should be supported by four main components: learning tasks, supporting information, procedural information and subskills practice.

Based on this, we designed an instructional trajectory consisting of learning episodes. The episodes include:

- supporting information and cues, to which the student can return at any moment, to give an overview of what is being learned and to provide cognitive strategies;
- procedural support materials and cues, which are designed to help the student to carry out specific and routine activities and which fade into the background as the student's skills increase;
- learning tasks that are organised in increasing order of complexity to avoid cognitive overload for the student, with sufficient number and variation to consolidate the student's skills, knowledge and attitudes;
- exercises that help the student to achieve a high level of automaticity in a particular skill;
- Think and Reflect (see on Figure 1) activities that include cognitive feedback, analysis based on the student's learning path, demonstration of learning strategies, support, suggestions for further action and student decision making.

Based on the learning analytics, recommendations and learning strategies, the student decides, alone or in collaboration with the teacher, how to move forward based on the information displayed at the end of the episode: e. g. to go back to the completed episode and do some sub-skill exercises, or to move to the next episode at the same level, or to move to the next episode at a more complex level. In this way, the teacher can adapt the episode by making some learning activities within the episode compulsory for the student and some optional.

Methods

Our study is part of the larger project funded by Estonian Ministry of Education and Research: Student-centred learning and flexible learning paths in primary schools. In this paper we report the results of one phase in our design-based research (DBR). As the DBR is implemented in a real context of social interaction, rather than in a socially isolated laboratory setting, it is therefore considered suitable for both research and the design of technology-enhanced learning environments (Wang, Hannafin, 2005) and its deployment for the study of mathematics education has promising potential (Fowler et al, 2022). In this phase, we piloted the initial pedagogical and technological framework to design flexible instructional trajectories in mathematics. Pilot-study was carried out in the winter of 2023, focused on the national curriculum learning outcome sub-skill multiplication of algebraic fractions.

Participants and classroom interventions

A convenience sampling was used to find participants for conducting study and offers for participating were made to teachers who already were more active users of digital learning resources, of which five teachers agreed to test created resources and flexible delivery-mode with their students. Five schools participated in the study, with a total of 88 students in 9th grade. For some students it was a repetition of what they had learnt in the autumn and for others it was a completely new topic. The students participated in the study in a pseudonymised way: the teachers assigned the students' usernames to the study group at their discretion, and the researchers were able to associate the username with the student's gender, the name of the teacher and the data collected on the student's track record. The survey was conducted in Tallinn University's closed environment vara.h5p.ee, where students logged in with their username to record choices made in learning process and the inserted solutions to the given tasks, but also to collate them with the data from the questionnaire responses. The study lasted 3-5 lessons, depending on the subject teacher, and consisted of digital measurement and learning resources (Figure 3 shows an example of a fragment of the task used in the study to teach multiplication of algebraic fractions), a final test and questionnaires. In addition to the flexibility provided by allowing movement within the learning material, the digital lesson plans created by the researcher were made available to students and teachers in the LePlanner[2] online environment to support flexibility in learning process so that moving between learning activities could be comfortable.

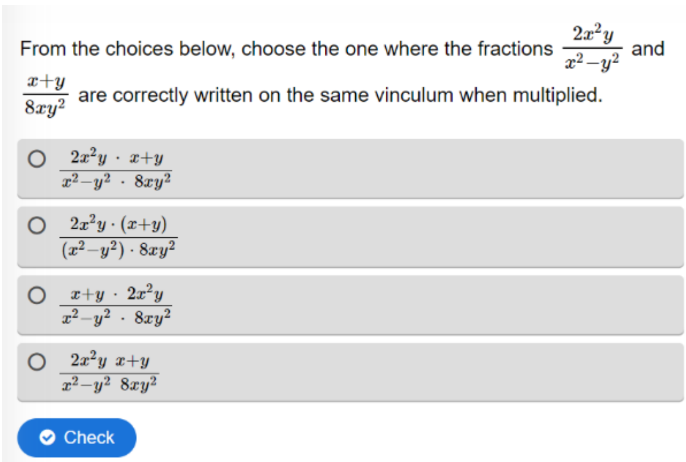

Figure 3: An example of a fragment of the task for 9th grade algebra

Although the design of the learning trajectory suggested one possible route for the students, the students were free to decide, alone or in collaboration with the teacher, how they would like to complete the instructional trajectory. Therefore, students were free to choose their own pace of learning, the learning resources they used and the order in which they completed the tasks. Students also had the freedom to use hints and additional support material where necessary.

[2] https://leplanner.ee/en/

Data collection and analysis

A questionnaire was used to investigate the perceived learning effectiveness of students and the perceived importance of technology, peers, the teacher and students' own role in learning. The survey was structured into various sections, each addressing aspects of the students' learning experience: the role and support of peers, instructors, and technological tools, the perceived efficacy of flexible teaching methodologies and the perceived challenges of learning, as informed by the research of Tammets et al. (2021). This approach allowed us to explore how the interplay between various components of the instructional design (such as the integration of technology or the involvement of teachers) and the characteristics of the students (their perceived challenges in learning) influence their views on the effectiveness of innovative teaching strategies. Students evaluated the survey items using a 1-5 scale, where 1 represented the lowest level of agreement or satisfaction and 5 indicated the highest. Descriptive analysis was conducted in this phase to understand students' experiences with flexible instructional trajectories in math lessons. In this study, we do not report the data collected with H5P and focus only on self-reported data about perceived effectiveness of trajectories and role of different stakeholders.

Results

In this pilot-study we aimed to understand the students' experiences with novel digital flexible instructional trajectories. First, we aimed to understand students' perceptions on the effectiveness of such methods (including compared to traditional mathematics classes). Figure 4 illustrates students' responses regarding their perceptions with effectiveness items.

The data reveals that students generally found the novel flexible instructional strategies quite effective for learning new topics (M = 3.41; SD=0.8), anchoring mathematical concepts (M=3.37; SD=0.92), enhancing their understanding of mathematics (M=3.30; SD=0.91) and supporting their overall learning process (M=3.21; SD=0.95).

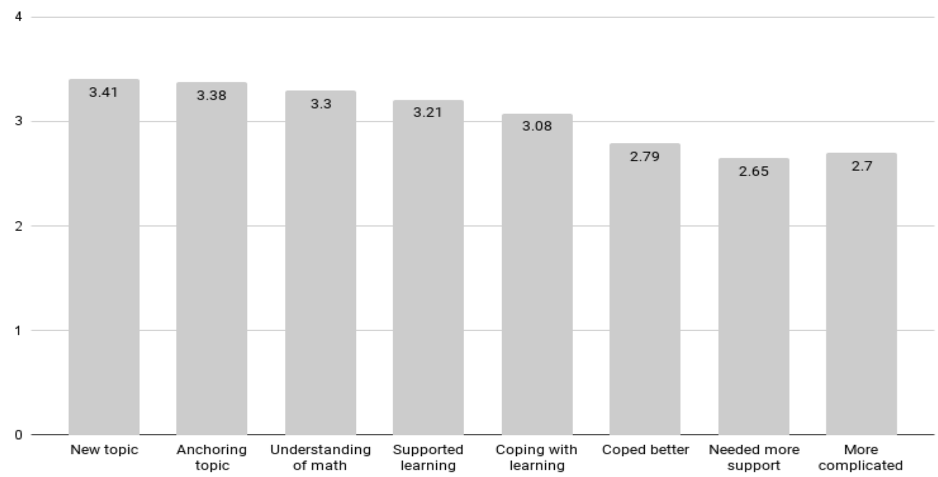

Figure 4: Students' responses regarding their perceptions with effectiveness items in math lesson

On average, students felt they coped well with learning in such settings (M=3.08; SD=0.86). Interestingly, when compared to regular mathematics lessons, the majority of students disagreed with the need for additional support (M=2.65; SD=0.98) or found the lessons more challenging (M=2.70; SD=1.18).

Subsequently, we aimed to understand students' perceptions of the roles different stakeholders played in these flexible instructional strategies. Given that the digital trajectories were carefully designed based on learning pedagogy and didactical principles, offering scaffolding and promoting students' autonomy in learning, we were particularly interested in students' views on this setting. Students rated their own involvement as the most critical factor in the learning process (M=4.27; SD=0.93), highlighting the significance they placed on their personal engagement. The next most valued element was the well-structured digital environment with mathematical tasks (M=3.86; SD=0.92). Students viewed their ability to manage their learning flexibly as the most important (M=3.93; SD=0.90), with technology being considered the second most crucial factor in facilitating their education (M=3.78; SD=0.83). Peer support was rated as the least significant (M=3.30; SD=0.92), while the role of teachers was seen as slightly more important (M=3.57; SD=0.91).

Discussion

Our results indicate that students generally perceived these new teaching methods as effective for various aspects of learning mathematics, from acquiring new topics to reinforcing existing knowledge. Although the intervention was short-term, the overall positive response suggests that these methods have potential to support and enhance the learning process. Students, on average, disagreed with the need for additional support and did not find the lessons more complicated than traditional mathematics classes, indicating a level of adaptability with the flexible instructional strategies. This suggests that students perceived themselves as being sufficiently supported and instructed in the learning environment. Kirschner et al. (2006) also argued that students learn more deeply from guided instruction. At the same time, we consider it is crucial that students have the opportunity in the form of self-regulation to opt out of support and guidance when they no longer need it and to use them again when they are in difficulties.

Students rated their own involvement and the digital learning resources as the most important elements for their learning. The relatively lower importance assigned to peers and slightly higher to teachers indicates a refined understanding of the learning in flexible instructional strategies. While peer interaction was not viewed as a primary factor, the teacher's role suggests that the direct instructional role of teachers may be evolving in these digitally enhanced learning environments. The findings highlight the importance of student autonomy and engagement in the learning process. It also implies the need to explore the evolving role of the teacher in such contexts in order to effectively support teachers and such teaching practices (Sumbawati et al., 2020).

Overall, our results indicate a positive adoption of flexible instructional strategies among students, emphasising the importance of students' own regulatory processes, the effectiveness of digital learning resources and the potential need for adaptive approaches to consider individual student needs and challenges. The study also points towards future directions for research and practice, particularly in understanding the nuances of how different factors influence the effectiveness of innovative teaching methods.

Acknowledgment

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Designing relevant tasks, tools, and communicative strategies – a teacher–researcher driven learning study project

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The aim of this paper is to exemplify how relevant mathematical knowledge can be developed through teacher–researcher collaboration in a learning study. Data come from a Swedish primary school, and focus on algebraic thinking in Grade 1. The project designed and tested tasks, tools, and communicative strategies to improve students' algebraic reasoning. The learning study emphasises an iterative process of planning, implementation, analysis, and dissemination of knowledge. Learning activity was used as a design theory that focuses students' theoretical thinking. The result includes two types of knowledge products that are relevant when promoting younger students' theoretical thinking in relation to algebraic equalities. The importance of teacher–researcher collaboration is discussed.

Keywords: Algebraic thinking, collective reflections, learning activity, learning study.

Introduction

In this paper, we start from the premise that practice-based research produces results that are rarely directly relevant for practising teachers. This, we believe, should be a given outcome. The results are often presented in such a form that they need to be translated to be used in concrete practice (Hultman, 2015). One of the reasons for this is the common and deeply rooted tradition of focusing educational research on teachers' shortcomings. Today, there are relatively high societal expectations that both teaching and student performance needs to improve. Thus, there is increasing interest, not only from the research community but also from governments[1] in linking practice-based research to local school development initiatives regarding improvement of teaching, where teacher participation is an essential feature. However, local school development projects and research projects are often seen as two different things, where teachers are assigned responsibility for the local development project and researchers study the outcome, for example, in terms of teacher learning or changes in practice. This means that even when researchers and teachers collaborate on, say, a teaching developmental project, it is not obvious that they share the same object (Eriksson, 2018). In line with Carlgren (2019), we argue that a shared research object in practice-based research should be about the problems and challenges that teachers face in realising their knowledge assignment. Bulterman-Bos (2008) also favours teachers' involvement in the research process. Their starting point is medical clinical research where the experience, the connoisseurship, and knowledge of the physician is crucial for the realisation and outcome of the research. In Bulterman-Bos's argument, the involvement of teachers

[1] Swedish Institute for Educational Research, https://www.skolfi.se/other-languages/english/research-funding/; ULF-Avtal, https://www.ulfavtal.se/

in practice-based research appears to be not only desirable but necessary. Thus, in a project with a common research object, the knowledge and experience of both the teachers and researchers are of complementary value when searching for knowledge that is useful in teachers' daily practice.

However, there is still a lack of in-depth knowledge about what can be seen as fruitful results, that is, what can be seen as professional scientific knowledge products that could be used by other teachers (cf. Lindberg et al., 2023). The aim of this paper is to exemplify how relevant mathematical knowledge can be developed through a teacher–researcher collaboration in a learning study. As an example, we use the teaching developmental research project *Developing algebraic reasoning capability* (2017–2019) funded by the Swedish Institute for Educational Research. The aim can be specified with the following research question:

What can be seen as relevant mathematical knowledge products emanating from a learning study project?

Methods

The overall aim of the research project was to explore how teaching, in terms of tasks, tools, and communicative strategies, can be designed and developed to promote and enhance students' algebraic reasoning in different grades. The research project was conducted in three different municipal primary schools (in grades 1, 5, and 7) and in one upper secondary school (Eriksson et al., 2019, 2021, 2024; Wettergren, 2022; Wettergren et al., 2021). The project focused on the problems teachers encounter when developing students' capability to argue and reason about various mathematical phenomena, and the fact that this is particularly difficult in terms of algebra and algebraic reasoning.

The project reported in this paper used learning study – a collaborative, interventionist, and iterative research model that uses the research lesson as a laboratory (Carlgren, 2019; Kullberg et al., 2024). Four learning studies, each corresponding to a grade in different schools, were conducted. During the first project year, two parallel learning studies were conducted, one in Grade 1 and another in Grade 5. The following year, two additional learning studies were conducted, one in Grade 7 and the other in the first year of upper secondary school.

To get an overview of how students in preschool class and Grade 1 experienced algebraic equalities, several group interviews were initially conducted. The interviews were organised around illustrations of algebraic equalities, for example, $5x=y$, $5c=z$, and $k=3e$. The students were shown both correct and incorrect illustrations (Figure 1; Wettergren et al., 2021, p. 11). The interviewer asked the students to explain how three imaginary "students in another class" could have argued. The interviews were analysed phenomenographically (Kullberg et al., 2024) to identify qualitatively different conceptions of (or ways of experiencing) a phenomenon such as an algebraic equality.

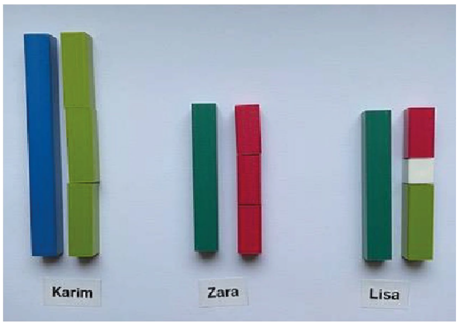

Figure 1: Illustration of the algebraic equality *k*=3*e*

The intervention consisted of designing tasks and research lessons that were iteratively developed, refined, and retested. Three to four iteratively adjusted research lessons were conducted in each learning study. Each research lesson was videotaped and transcribed, including statements, tone of voice, and gestures (Radford, 2010). In the iterative learning study process, the analysis of the research lessons was directed towards how the discussions, and the mediating tools used, were collectively initiated and driven by the students together with the teacher. The analysis had two foci: 1) how the actions took the form of arguments, and 2) the function of the different learning models in the research lessons. Data for this paper are from interviews and research lessons in Grade 1. Student names have been pseudonymised.

Learning activity as a design theory

In a learning study, a learning theory should be used both as a design principle and as a tool for analysis (Kullberg et al., 2024). Most commonly, variation theory is used, but recently there has been a growing interest in using learning activity (LA). As described by Davydov (2008), LA emphasises the importance of designing tasks considering that key aspects vary. In this way, LA can be said to have aspects of variation theory built in. In this project, we chose to use LA as developed by El'konin and Davydov (Davydov, 2008; Eriksson, 2017). The overarching idea of LA is the development of students' theoretical thinking (i.e., algebraic thinking; see Kaput et al., 2008, Kieran, 2004). Further, the development of students' agency in relation to their own learning is important. Thus, students must experience a need for new knowledge. Another key principle for students to be able to develop theoretical thinking is that they must elaborate the structural (essential) aspects of the knowledge content. According to Davydov, such essential aspects can only be available for the students through specially designed tools – learning models – and collaborative modelling work (Broman et al., 2023). LA builds on Vygotsky's argument that our thinking is first social before being transformed into individual thinking. This provides an argument for collective reflections that Zuckerman (2004) argues allow students to examine their own arguments in the light of others.

Using LA as a learning theory when designing tasks places demands on both the formulation of the task and its content. To promote students' agency and willingness to engage in a task, that is, to create a learning activity, the task needs to be presented in the form of a problem situation the students find challenging and meaningful (Repkin, 2003). A problem situation is characterised by being content rich and sufficiently challenging. Starting from a well-designed problem situation, students must first identify what the problem in the given situation consists of. Once they have identified the problem, they need to agree together on what methods and tools they know and what is new in the problem, and for which they do not yet have tools.

Result

Below, we present the result identified in the form of two mathematical knowledge products. First, we present the result from a mapping study concerning students' conceptions of algebraic equalities and critical aspects. Second, we present findings in the form of tasks, tools, and collective reflections from a research lesson in Grade 1 (Eriksson et al., 2019, 2024; Wettergren et al., 2021).

Conceptions and critical aspects

To design the tasks and research lessons, we initially needed to explore students' conceptions of algebraic equalities. Some of the students expressed conceptions that were not mathematically relevant, which has also been shown in other similar studies (e.g., Küchemann, 1981; Stacey & MacGregor, 1997). The conceptions that were categorised as non-mathematically relevant were, for example, that the students experienced that the letters in the expressions should be understood in relation to the alphabet or that "an imaginary student" had illustrated the Italian flag. To focus the mathematical content more clearly in the teaching design, we chose to disregard the conceptions that were not mathematically relevant.

The phenomenographical analysis thus revealed three mathematically relevant conceptions (see Figure 2, Wettergren et al., 2021, p. 14, our translation). By comparing the different conceptions with each other, we also were able to identify three critical aspects (cf. Kullberg et al., 2024). The critical aspects identified were 1) to discern that an expression consists of different components which have different roles/functions, 2) to discern that the same variable in an expression has the same value, and 3) to discern that the value of a variable in an expression is determined relationally.

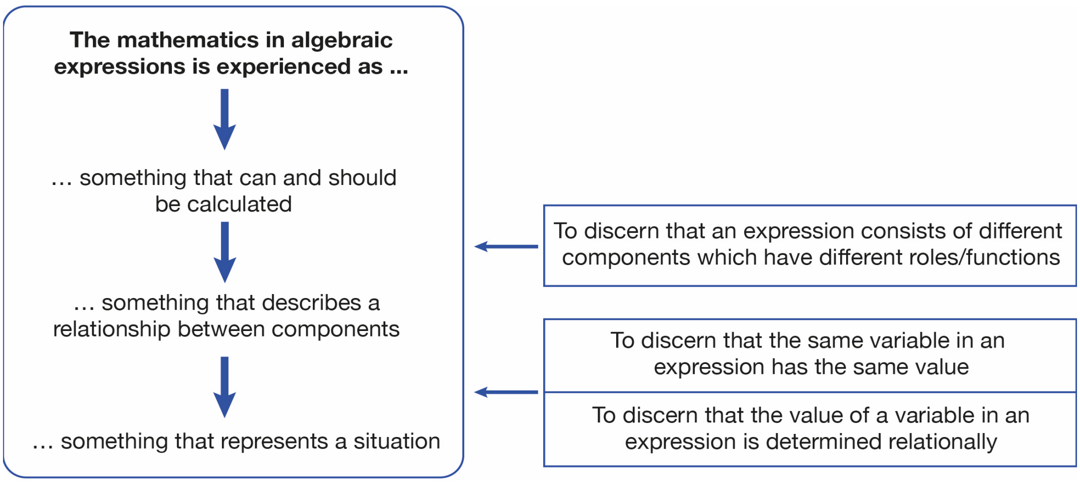

Figure 2: Schematic of identified conceptions (left) and critical aspects (right)

As mentioned, we used the identified critical aspects and key principles for an LA when designing tasks, tools, and communicative strategies to be explored in the learning study. In the next section, we provide examples of results from the research lessons.

Tasks, tools, and collective reflections

In all the video-recorded research lessons in Grade 1, the teacher, based on the research group's joint planning, staged a problem situation including a non-numerical equality to be explored in a whole-class discussion. So that the students would not feel personally responsible but would rather reflect generally (but substantively), the research lessons drew on what van Oers (2009) calls a playful format. Thus, the teacher presented the problem situation on the board with the equality *C*+*A*=*B* together with the yellow pieces of paper representing how two fictive students "in another class" found the equality to be appropriate (see Figure 3, Eriksson et al., 2019, p. 88). The teacher said: "What could these students have been thinking when they put their pieces of paper like this?"

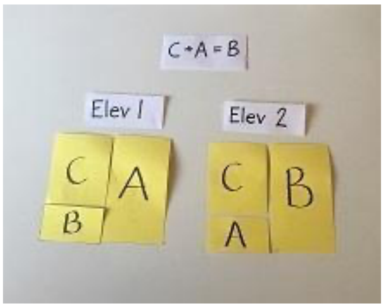

Figure 3: The problem situation as presented to the students

After a short whole-class discussion, the students were given a complementary task to create representations of another similarity, *C*+*B*=*A*, now using tools in the form of Cuisenaire rods. The rods, like the pieces of paper, represented unknown quantities and the aim was to visualise possible relationships and structures in the given equality. The students' rod constructions were used in the collective discussions (Wettergren et al., 2021).

When we analysed the first research lesson, we noticed that the students were not engaged in each other's proposals, which meant that they missed opportunities to reflect on each other's proposals and ideas (Zuckerman, 2004). This led us to adjust this element for the revised research lesson. Instead of the students presenting their solutions, the teacher had copied their various rod constructions onto a large table – a collective workspace (Figure 4, Eriksson et al., 2019, p. 88).

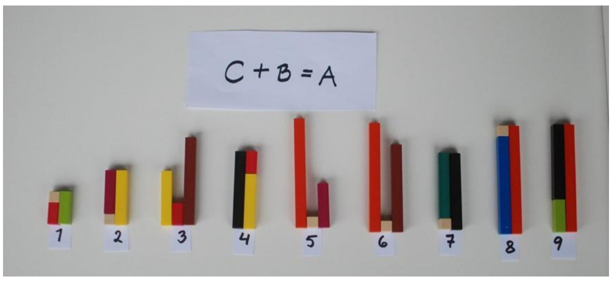

Figure 3: Copies of students' rod constructions of the equality *C*+*B*=*A* (both correct and incorrect)

To enhance conditions for collective reflections, the teacher provoked the students to explore the various rod constructions by saying “All of these are correct, right?” Several students immediately replied “No, they are not all correct”. If all student suggestions had been correct, that is, corresponded to the equality presented, we planned for the teacher to say: “But these aren’t all correct, are they?” or “They’re not all correct, are they?” With this approach, with all the rod constructions copied onto a collective workspace, students now actively engaged in trying to understand and explain how various constructions worked or did not work.

Discussion

In the following, we first discuss the two identified knowledge products and then, with reference to the project as a whole, we discuss some thoughts about what could also promote students’ theoretical thinking.

The first knowledge product was the students’ qualitatively different conceptions of algebraic equalities which helped us to identify critical aspects that they needed to discern. The critical aspects then informed the iterative work of designing and testing different tasks. Both the conceptions and the critical aspects may be useful as guidelines for other teachers who are interested in teaching students about algebraic equalities (Wettergren et al., 2021). In addition, the staging of the tasks themselves were designed in a way that would promote the students’ reasoning. The examples given in the result are from Grade 1. However, in relation to the overall project (Eriksson et al., 2019, 2021, 2024), we found that the critical aspects, with some modifications, were also relevant for students in other grades.

The second knowledge product was related to tasks, tools, and collective reflections. The iterative work in the research lessons showed the importance of using well-designed tasks that needed to be elaborated on with appropriate tools. Such tools could function as learning models (Davydov, 2008), when promoting students’ collective reflections on equalities. Consequently, tasks, tools and collective reflections highlight the interdependence and interrelationship between them. In Eriksson et al. (2019) we showed, for example, how the classroom discussions were facilitated by the students having learning models in the form of Cuisenaire rods that they could use to elaborate on the different tasks. By rearranging the rods in different ways, the students could explore algebraic equalities (see Figure 3). They could then more easily assess whether an argument was appropriate. Having the various constructions and proposals on a common workspace also created conditions that enabled students to remember and understand what other students explained or proposed. We call this materialising a collective memory (Eriksson et al., 2024).

In addition to tasks, tools, and collective reflections, the overall project indicates that the teachers’ communicative strategies are important (Eriksson et al., 2019, 2021, 2024; Wettergren, 2022; Wettergren et al., 2021). That is, we found that it matters how teachers orchestrate and promote students’ collective reflections. For example, we saw that it is important that the teacher does not interpret the problem for the students or explain to them which tools to use. Thus, it is important that the teacher, with their communicative strategies and available tools, promotes and deepens the students' discussions (Zuckerman, 2004) by, for example, claiming that a correct solution is incorrect. Another communicative strategy that we found effective was to present the task in a playful format – for example, presenting solutions as if they were from "some students in another class" (van Oers, 2009; see also Wettergren, 2022).

To conclude, a research project in the form of a learning study that from the beginning and in all stages of the research process is based on teachers' experiences and connoisseurship is crucial for quality, relevance, and the development of knowledge products that can be used by other teachers. The researchers contributed with theoretical and methodological knowledge. Thus, in a collaborative, iterative, and interventionist research process with a common research object, it was possible to keep the focus on the challenges identified by the teachers and jointly produce knowledge products.

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Real people in mathematics textbooks of Cyprus and Norway

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The mathematics education literature suggests different ways of linking school mathematics to the real world. Here, we focus on how real people/figures are presented in mathematics textbooks of Cyprus and Norway, as a way of establishing links of this kind. We explore (a) the demographic characteristics of these people, and (b) ways in which people are associated with mathematics. Findings indicate that women are absent in Cypriot textbooks, while in Norwegian textbooks, gender representations are balanced. In both countries, the representation of real people is highly Euro-centric, and in favour of athletes over other professions. Finally, in Cyprus, mathematicians are presented in relation to their mathematical ideas. In Norway, all figures are presented as generic points of reference, with no explicit relation to the mathematical concepts involved. We conclude with recommendations for textbook authors, teachers, and teacher education.

Keywords: real people, representation, mathematics textbooks, Cyprus, Norway.

Introduction

The importance of connecting school mathematics to the real world has been stressed internationally in several academic articles and policy documents. It can also be seen in the foci of large-scale comparative studies. For instance, the Programme for the International Student Assessment – PISA assesses "how well students can apply what they learn in school to real-life situations" (OECD, 2020, p. 6). Yet, no single answer exists on how connections between mathematics and the real world can be established. Our readings of the literature suggest that these can take various forms. Among other things, such connections can be facilitated by realistic contexts in word problems (Verschaffel et al., 2020), the incorporation of modelling problems based on actual data from the real world (Niss & Blum, 2020), the use of elements from the history of mathematics in the teaching of the subject (Xenofontos & Papadopoulos, 2015), and linking mathematics to sociopolitical issues, by using learners' own experiences and funds of knowledge (Gutstein, 2016). Regarding the rationales for contextualising mathematics, Rubel and McCloskey (2021) identify four types: (1) support the learning of mathematics by building on pupils' contextual knowledge, (2) motivate pupils to learn mathematics by making connections to their perceived interests, (3) support the teaching of everyday problem solving, and (4) supporting teaching mathematics for social justice.

When connections between mathematics and the real world are attempted, several references to human characters are often made. These references may concern realistic, yet made-up, characters, such as Jim, in the marbles task by Van Den Heuvel-Panhuizen (2005, p. 5): "Jim has 16 marbles and wins 10 more. How many does he have now?". They may also regard actual people from the real world, such as Australian swimming competitors and their results in 100 m freestyle events during 2012-2014, as presented in a modelling task by English and Watson (2018). Research informs us that references to human characters do not always take issues of diversity (e.g., gender, ethnicity, religion, functionality, sexuality, and class) into account (Smestad, 2021). One such example can be found in Parise (2021) and her analyses of US high school statistics textbooks, from a gender perspective. Her findings indicate that, when references to real humans are used, men are described as "founder of modern psychology", "one of the greatest statisticians of the twentieth century", and "great philosopher and hall of fame catcher", while women are mentioned merely by their job title: "researcher", "psychologist", "actress". References of this type, we reckon, can be problematic, as several children (including girls, queer children, children with disabilities, and children with diverse ethnic backgrounds) cannot see themselves represented in the tasks used in the mathematics classroom, and consequently may feel excluded from the subject (see Parise, 2021; Sleeter & Grant, 2016; van Veen et al., 2023). With these in mind, the exploratory work presented here seeks to answer the following research questions: 1. What are the demographic characteristics of real people used in upper primary mathematics textbooks in Cyprus and Norway? 2. What is the connection between these people and mathematics?

Our study focuses on real people in textbooks instead of realistic, made-up characters. This is a conscious decision, inspired by Vos (2018) and her concept of *certification of authenticity*. For Vos, some certification is needed to establish the authenticity of a mathematical task. Certification can take place in different ways, for example, the use of actual data from the real world, the use of actual artefacts in the classroom, or through references to places (e.g., a neighbourhood) pupils are familiar with (Vos, 2018). We acknowledge and agree with Vos that no task can be entirely authentic once it enters the mathematics classroom, as it is modified to serve educational purposes. However, we believe that references to real people (historical figures or living individuals) hold a greater degree of authenticity than references to made-up characters.

The study

There are two main reasons why we compare textbooks from Cyprus and Norway. Firstly, each of us has familiarity with at least one of these contexts, which allows us to bring insider and outsider perspectives to the discussion (Dhillon & Thomas, 2019). Secondly, the two educational systems present some interesting differences. The Cypriot system is highly centralised, with all public primary and secondary schools using the same national textbooks administered by the Ministry of Education (Xenofontos & Papadopoulos, 2015). In contrast, the Norwegian system gives schools and teachers a great degree of autonomy to decide which textbook series to use among those developed by independent publishing companies (Bakken & Andersson-Bakken, 2021).

In this study, we used textbooks for the last two grades of primary school, from the two countries: grades 5 and 6 for Cyprus, and grades 6 and 7 for Norway. In Cyprus, only one textbook series is available, produced by the Ministry of Education. The digital form (pdf) of the textbooks can be openly accessed at https://mathd.schools.ac.cy/index.php/el/. In Norway, four commercial textbook series by different publishers are widely used: *Matemagisk* (by Aschehoug), *Matematikk* (by Cappelen Damm), *Multi* (by Gyldendal), and *Volum* (by Fagbokforlaget). For the purposes of data collection, we examined all textbook series, except the last, which has not been published in complete form. Xenofontos read the Cypriot textbooks, and Smestad the Norwegian, each identifying references to real people (historical figures and people still alive). In doing so, we decided to include only direct references to individuals, presenting biographical information, and references to individuals' actions or products. Indirect references were not counted. One example comes from the

Cypriot textbooks (grade 5, part 4, page 57), in which a timeline with various historical events is presented. The timeline includes a point noted as "0 Birth of Jesus Christ". This task was not included in our analysis, as it referred to a widely used timepoint reference, and not Jesus Christ as a historical figure. Another example of an excluded task comes from the Norwegian series *Matematikk* (grade 6, page 237), which referred to a clown called Euclid. Even though we see an allusion to the historical figure the clown was named after, this task was excluded as the reference to the ancient Greek mathematician was indirect.

To analyse each reference, we focus on observable characteristics explicitly mentioned or implied in the tasks. Specifically, our analyses consider variables such as gender, nationality (implicit or explicit), and profession (or activity in which the person is presented). Below, we present two examples, one from Cyprus and one from Norway. In both cases, we include the original texts in Greek and Norwegian, as well as our translations.

	Original text	English translation	Analysis
Cyprus	Να γράψεις σε λεκτική και αναλυτική μορφή τον αριθμό που αναφέρεται σε κάθε πρόταση, όπως στο παράδειγμα. (β) «Ο Μικρός Πρίγκιπας», το διάσημο βιβλίο του Αντουάν Σαιντ-Εξιπερί, ξεπέρασε τα 140 000 000 αντίτυπα σε πωλήσεις παγκοσμίως.	Write the number mentioned in each sentence in verbal and expanded form, like in the example. (b) "The Little Prince", the famous book by Antoine de Saint-Exupéry, exceeded 140 000 000 copies in sales worldwide.	Antoine de Saint-Exupéry Gender: Man Nationality: French (implicit) Profession/activity: Writer Connection to mathematics: None
Norway	I året 1978 vant Grete Waitz, som første norske kvinne, New York maraton. Det er 2468 år etter slaget ved Marathon, som løpet er oppkalt etter. Hvilket år var slaget ved Marathon?[1]	In 1978, Grete Waitz was the first Norwegian woman to win the New York marathon. That was 2468 years after the Battle of Marathon, after which the race was named. What year was the Battle of Marathon?	Grete Waitz Gender: Woman Nationality: Norwegian (explicit) Profession/activity: Athlete Connection to mathematics: None

Table 1: Tasks with real people – Cyprus (Grade 5, part 4, p. 69) and Norway (Matematikk 7, p. 33)

Results and discussion

In total, 30 real people were identified in the Cypriot textbooks and 18 in the Norwegian textbooks. In Norway, seven real people were found in the series Matemagisk, 11 real people in the series

[1] In the Gregorian calendar, there is no year zero. Thus, strictly speaking, there is a mathematical error in the task.

Matematikk and none in the series Multi. In presenting and discussing our findings, we turn to our initial research questions: on demographic characteristics and people's connection to mathematics.

Demographic characteristics

In the Cypriot textbooks, 29 of the 30 identified references concerned men. For 19 of these, the gender is explicitly stated, with the use of a pronoun, an explicit designation "man" or "woman", or by the grammatical gender used. For the remaining ten, there is no direct or indirect indication of gender, which could only be confirmed by an easy search online. The single reference to a woman regards the French cyclist Françoise Hervé. Yet, the presentation of her name with Greek characters and without any grammatical article covers her gender. In the Norwegian textbooks, the gender of all persons is explicitly stated, resulting in the identification of nine men and nine women.

Table 2 shows the nationalities of real people in the textbooks. In the Norwegian series, there are two references to people from Norway, while in the Cypriot series, there is no reference to anyone from Cyprus. Yet, as the Greek-Cypriot educational system is often found to maintain, reproduce, and promote the national(istic) narrative of Cyprus being part of the wider Greek nation (Kemal & Zembylas, 2022), it would be argued that, not least for the textbook authors, the four references to people from Greece suffice. For both Cyprus and Norway, most of the real people are from the rest of Europe.

	Cyprus	Norway
Own country		Norway (2)
Rest of Europe	France (6), Germany (1), Great Britain (3), Greece (4), Ireland (1), Netherlands (1), Prussia (1), Russia (2), Spain (2)	Bulgaria (1), Germany (1), Great Britain (1), Greece (3), Ireland (1), Netherlands (1), Poland (1), Switzerland (1), USSR (1)
Rest of World	Argentina (2), Brazil (1), Colombia (1), Uruguay (2), USA (3)	Jamaica (1), Kufa (now Iraq) (1), USA (3)

Table 2. The nationalities of real people in textbooks.

Table 3 shows the professions of the identified people. Interestingly, most references in both countries concern athletes. We also note that, when it comes to historical figures from antiquity, it is not always easy to use labels the way these are understood today. For example, Pythagoras is called a mathematician, while Plato is called a philosopher. Nevertheless, it should be acknowledged that in Greek Antiquity, the lines between mathematics and philosophy were not as distinct as we commonly perceive them today (Knorr, 1980).

These findings are aligned with prior research indicating that some groups of people are underrepresented in mathematics textbooks, while it may not be easy for several children to identify with the existing representations (e.g., Parise, 2021). In the case of Cyprus, girls may not see connections between themselves and the real people included. Furthermore, regarding the textbooks of both Cyprus and Norway, pupils with ethnic backgrounds outside Europe and those aspiring to be anything other than athletes, artists, or mathematicians may not find themselves represented.

	Cyprus	Norway
Athletes	19	14
Mathematician	6	2
Artist/author	5	0
Philosopher	0	1
King	0	1

Table 3. The professions of real people in textbooks.

People's connections to mathematics

Real people's connections to mathematics are found to take two forms. For most references, there is no direct connection between a real person (or the activity they are involved in) and mathematics. Rather, the tasks use facts about these peoples' lives or professional activities for some mathematical concepts to be explored (see examples in Table 1). This is observed, among others, for all references to athletes. Athletes often use mathematical principles in their training and execution of skills, although it may not be apparent during their performances (Karlis et al., 2021). Nevertheless, while mathematics plays a role in sports, it is not the primary focus as it is in the work of a mathematician. In Figure 1, we see one such example from Norway, for which children are invited to put the lengths of long jumps in ascending order (ordering decimal numbers).

The second form regards tasks in which children are asked to explore the mathematical ideas of or behind people's work. This form is only found in the Cypriot textbooks. We present two such examples in Figure 2 and Figure 3. The former concerns the work of the mathematician Christian Goldbach and his famous conjecture (every even integer greater than two is the sum of two prime numbers). Children are asked to explore the conjecture by writing given even integers as sums of primes. Examples of this type can be seen as attempts to include elements from the history of mathematics in the school subject (Xenofontos & Papadopoulos, 2015). The latter has to do with the work of non-mathematicians, which, however, is based on specific mathematical concepts/ideas. The example in Figure 3 illustrates paintings by the graphic artist Maurits Cornelis Escher, designed with the use of repeating patterns and geometric transformations such as translation, rotation, and reflections. Children are asked to describe how Escher used mathematics to create these paintings.

To revisit the four rationales for contextualising mathematics by Rubel and McCloskey (2021), tasks involving real people in the textbooks of Cyprus and Norway do not appear to ascribe to the rationale of supporting the learning of mathematics by building on pupils' contextual knowledge, the rationale of supporting the teaching of everyday problem solving, nor the rationale for supporting mathematics teaching for social justice. The only rationale for which we can see some relevance is the rationale for motivating pupils to learn mathematics by making connections to their perceived interests. Yet, the limited variation in real people's professions can only be motivating for pupils who have specific interests in sports, mathematics, and/or the visual arts.

29 Her ser du dei seks siste verdsrekordane i lengdehopp for menn.
Here you see the six latest world records in long jump for men.

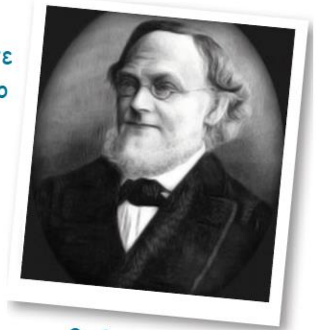
Lengd	Namn	Land	Dato	Stad
8,31 m	Ralph Boston	USA	15.08.1964	Kingston
8,34 m	Ralph Boston	USA	12.09.1964	Los Angeles
8,35 m	Ralph Boston	USA	29.05.1965	Modesto
8,35 m	Igor Ter-Ovanesian	USSR	19.10.1967	Mexico by
8,90 m	Bob Beamon	USA	18.10.1968	Mexico by
8,95 m	Mike Powell	USA	30.08.1991	Tokyo

Teikn ei tallinje. Plasser resultata så nøyaktig som mogleg på tallinja.
Draw a number line. Place the results as exactly as possible on the number line.

Figure 1. Task with numbers to be sorted, from Norway (Matemagisk 6A, p. 60)

Ο μαθηματικός Christian Goldbach υποστήριξε ότι οποιοσδήποτε άρτιος αριθμός, εκτός από το 2, μπορεί να γραφεί ως άθροισμα δύο πρώτων αριθμών. Για παράδειγμα:

The mathematician Christian Goldbach stated that apart from 2, any even number can be written as the sum of two prime numbers. For example:



10 = 3 + 7   16 = 3 + 13   24 = 7 + 17

Να γράψεις τους πιο κάτω άρτιους αριθμούς ως άθροισμα δύο πρώτων αριθμών.

Write the following even numbers as a sum of two prime numbers.

12 = ____________   36 = ____________

18 = ____________   40 = ____________

28 = ____________   50 = ____________

Figure 2. Task involving a mathematician and their work, from Cyprus (Grade 5, part 1, p. 142)

Ο Maurits Cornelis Escher (1898 - 1972 μ.Χ.) ήταν Ολλανδός εικαστικός καλλιτέχνης. Στη γραφική του τέχνη, απεικόνιζε μαθηματικές σχέσεις μεταξύ των σχημάτων, των μορφών και του χώρου.

Να περιγράψεις τον τρόπο δημιουργίας των πιο κάτω σχεδίων στους πίνακες ζωγραφικής του Escher.

Maurits Cornelis Escher (1898 – 1972) was a Dutch visual artist. In his art, he depicted mathematical relationships between shapes, figures, and space.

Describe how Escher created the figures in the paintings below.

Figure 3. Task about an artist and the links between their artistic work and mathematics, from Cyprus (Grade 5, part 4, p. 101)

To us, it is worrying that the links between most real people and mathematics are indirect. The works of mathematicians and visual artists, as presented in the Cypriot textbooks, are a few encouraging exceptions. On the contrary, in both countries, we see the references to athletes as artificial attempts to mathematise real-world situations. In this sense, these examples are quite the opposite of Rubel's and McCloskey's (2021) fourth rationale, that of teaching mathematics for social justice, where mathematics is used as a tool for social change.

Recommendations

Mathematics education research has identified many rationales for contextualising mathematics. School textbooks as educational artefacts have the potential, we reckon, to serve as a platform that brings research and teaching practice together; they need to be up to date with current research developments on the one hand, while they are widely used by teachers on a daily basis, on the other. However, it appears that, in both Cyprus and Norway, the inclusion of real people in the textbooks takes place mostly for motivational reasons, by establishing connections between mathematics and what textbook authors perceive as pupils' interests (Rubel & McCloskey, 2021). Therefore, we conclude this article with some recommendations. Regarding textbook authors, it is vital that the inclusion of real people in the textbooks is reconsidered. Pupils need to be able to make connections between school mathematics, references to real people, and the world beyond school. In doing so, textbook authors are advised to consider, among other things, how different genders, ethnicities, and professions are represented. Teachers, in turn, could modify task context (provided that the mathematical content is not influenced), so that more pupils get opportunities to see themselves represented. Furthermore, it can be worthwhile for teachers to use different contexts based on the same mathematical content and processes, and engage pupils in conversations on the content-context relationship. This way, mathematical tasks can support the learning of the subject by building on pupils' contextual knowledge, support the teaching of everyday problem-solving, as well as provide opportunities for teaching mathematics for social justice (Rubel & McCloskey, 2021). Finally, it is critical that teacher educators support pre- and in-service teachers in developing deep sociopolitical awareness regarding how different groups of people are represented in textbooks and other instructional materials.

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Teacher interventions in group modelling

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As mathematical modelling became part of the curriculum in Norwegian schools, preservice teachers in Norway need to build their modelling teaching competencies, of which being able to support the students during a modelling lesson is an important aspect. Aiming to understand how preservice teachers intervene during group modelling activities, we conducted a case study with two preservice teachers, who implemented the same modelling activity in two 5th-grade classes in practicum. By analysing their interactions with four student groups, we found that their interventions fell into three categories: to establish the modelling context, to diagnose and promote the modelling process and to promote group collaboration. Our findings showed that the preservice teachers could use questions at the meta-level or content-oriented strategic questions to promote the modelling process but needed more support in promoting reflections on the modelling process and assisting group collaboration.

Keywords: Teacher intervention, collaborative mathematical modelling, preservice teachers.

Introduction

Developing the students' modelling competency can help them connect mathematics with the real world and prepare them for participating in societal developments as responsible citizens (Blum, 2011). To successfully develop the students' modelling competency, the students should be allowed to experience complexity and uncertainty in a modelling situation, and more importantly to work on the modelling problem with autonomy (Şahin et al., 2022). But this doesn't mean that the students should be left alone, and in fact, teachers' support is indispensable in modelling (Blum, 2011). Furthermore, researchers have come to a consensus that when guiding students through the modelling process, teachers ought to keep a permanent balance between students' independence and their guidance (Şahin et al., 2022; Borromeo Ferri, 2018; Blum, 2011). However, this guideline is quite abstract, and teachers still find it difficult to decide whether or how to intervene during a modelling lesson in concrete situations (see e.g., Stender & Kaiser, 2015; Şahin et al., 2022).

Tropper et al. (2015) studied five in-service teachers' teacher-student interactions during modelling lessons where 9th-graders worked in pairs and found that content-related support was predominant. In addition, they pointed out that only a small portion of the teachers' support was adaptive, and their expectations and too direct help led to non-adaptive support. In a case study of an 8th-grade in-service teacher, Şahin et al. (2022) found that many of the teacher interventions during the modelling lesson were reactions to students' requests for assistance and did not reach the level of strategic intervention. Like these two studies, earlier research has focused on teachers' interventions related to the modelling process, and few have emphasised group collaboration (Stender & Kaiser, 2015). However, a modelling problem is more appropriate to carry out in groups than individually as it is open to interpretation, idea generation and group discussions (Borromeo Ferri, 2018). Therefore, when teaching mathematical modelling, one also needs to manage group collaborations as "simply placing students in groups and telling them to work together does not in and of itself promote greater understanding or ability to communicate one's mathematical reasoning" (Johnson & Johnson, 1990, p. 104). Teacher interventions are necessary when no students in the group can answer a question, no true dialogue exists, or students have difficulty communicating with each other (Ding et al., 2007).

Hofmann and Mercer (2016) also pointed out that there is still vagueness and uncertainty about how teachers can support and sustain discussions in groups.

According to Prediger et al. (2019), situated in a complex class situation as a modelling lesson, teachers need to utilise available resources, structure the lesson, and support students in a modelling situation. In this case study, we investigate two preservice teachers' (PTs') interventions during modelling lessons for 5th-graders. This study aims to gain more empirical knowledge about teacher interventions during a modelling lesson at a classroom level (cf., Şahin et al., 2022), and to support PTs' professional development at the teacher education level (cf. Prediger et al., 2019). Our data analysis was guided by the following research question: What are the characteristics of the preservice teacher interventions in group modelling activities?

Theoretical background

Teaching modelling is a complex task. Borromeo Ferri (2018) divides the competencies needed in teaching mathematical modelling into four dimensions: theoretical dimension, task dimension, instructional dimension, and diagnostic dimension. While the diagnostic dimension requires the ability to see difficulties that arise in the modelling process and evaluate mathematical modelling activities, the instruction dimension covers planning and carrying out modelling lessons as well as how to intervene, support and give feedback during modelling activities. These two dimensions are interrelated as only after diagnosing the students' solutions and their difficulties may the teacher provide adapted support, feedback, or interventions to their students (Şahin et al., 2022).

When discussing teachers' roles in a complex instructional situation during modelling lessons, two concepts are usually used: teacher intervention and scaffolding. These concepts are closely related to each other. Teacher intervention is a wide concept that includes all verbal and non-verbal behaviours of teachers that have influences on the students' work (Stender & Kaiser, 2015), while scaffolding is considered as the support given by a teacher to a student when performing a task that the student might otherwise not be able to accomplish (Tropper et al., 2015). Contingency, fading, and transfer of responsibility are the three important characteristics of scaffolding (Van de Pol et al., 2010). Here contingency is referred to as tailored and adapted support. To ensure the students can independently solve problems in the long term, scaffolding should also be fading over time, that is, scaffolding should be withdrawn gradually and the responsibility for the performance of the task should eventually be transferred to the students. A concrete intervention is recognized as scaffolding if these characteristics are fulfilled (Stender & Kaiser, 2015).

Tropper et al. (2015) studied among other issues the teachers' scaffolding/temporary support for pairs working in modelling lessons according to reasons, areas and intentions of their support. They identified four areas where the support lay: content-related issues, strategic behaviour, affective components, and organisational conditions. Zech (1996) proposed a taxonomy to differentiate teacher interventions according to the intensity of the intervention into five categories: motivational, feedback, general strategic, content-oriented strategic and content-oriented. An utterance like "I am sure you will make it" is in the category of motivational intervention. A feedback example is "You are right. Go on like this." By studying teacher interventions during modelling, researchers found that strategic interventions that provide hints to the students on a meta-level, like "How far have you got?", "What is still missing?", "Does this result fit the real situation?", and prompts that focus on content-oriented strategies, like "Did you see any patterns that you recognise?", "Have you tried to represent this in a different way?", are most effective (Blum, 2011; Antonius et al., 2007). Stender and Kaiser (2015) used Zech's taxonomy to analyse teacher interventions in the different steps of the modelling cycle, and they found that the effectiveness and appropriateness of teacher interventions seem to strongly depend on how these measures are accepted and adopted by the students. However, they identified that the teachers' request to the students to present the state of their work is a powerful and effective strategic intervention as it is a prerequisite for adequate further support from the teacher.

Conducting modelling as a group activity requires that the teachers also support the students' effective group collaboration. Webb (2009) emphasises that teachers have an important role to play in fostering beneficial group dialogue and preventing debilitating processes. To prepare the students for group work, providing instructions on expected behaviours during group work, and asking groups to carry out certain strategies or activities can be useful and effective. He also suggested that teachers need to carefully evaluate group progress before deciding to intervene, and their help should be tied to students' ideas and probe further thinking, in line with Tropper et al. (2015).

Much of the early research on teacher interventions analysed teachers' utterances one by one to categorise them according to reasons, intentions (Tropper et al., 2015) or intensity of intervention (Stender & Kaiser, 2015; Zech, 1996). To zoom out and have a holistic analysis of the modelling lesson, we adapt the didactical tetrahedron (see Figure 1) introduced by Prediger et al. (2019) in a modelling situation. In this study, teachers are PTs and the content is mathematical modelling. Our analysis focuses on how PTs intervene to coordinate the content and the students during their modelling lessons, bearing in mind that these interventions are situated in a bigger context and are connected with other facets of the tetrahedron.

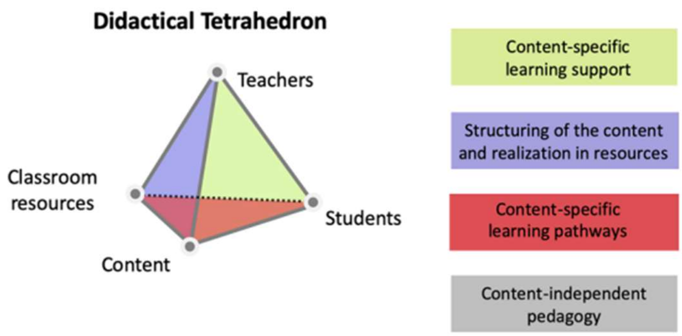

Figure 1: The didactical tetrahedron (Prediger et al., 2019)

Methods: data collection and data analysis

In our teacher education for 1^{st}–7^{th}-grade PTs, mathematical modelling is a theme in an obligatory second-year mathematics course. The PTs learn about mathematical modelling theories, such as the mathematical modelling cycle and worked-out modelling examples. In addition, they need to plan and implement a modelling lesson in their practicum and reflect on the modelling lesson in an assignment. In this study, we followed two PTs in their practicum. These two PTs designed a modelling activity *A Dream Bag of Candies* and carried it out together in two 5^{th}-grade classes in a

Norwegian primary school. Both lessons lasted 45 minutes. In the modelling lessons, the PTs assigned the students to groups of three or four. They started the modelling class by bringing a bag of candies of different types to set up the task *A Dream Bag of Candies*: *You can choose from the list every kind of candy you want, but each group can only have a maximum of 300 grams. Share the candies as fairly as possible in your group.* The PTs chose to weigh the candies in front of the class due to hygiene rules and wrote a list of candies and their weights on the whiteboard before the group started to work on the problem. The PTs' actions and four student groups with consent were video recorded. The recordings were transcribed verbatim (with PT1 and PT2 referring to the two PTs).

We used content analysis (Bengtsson, 2016) as our analysis method. To identify aspects that may be neglected or not be emphasised from earlier research, our analysis units are episodes of interventions. In each episode, there could be a mixture of different types of interventions, for example, the teacher could be motivational in one utterance and strategic in another (cf., Şahin et al., 2022; Zech, 1996). Therefore, we formed overarching categories related to the content and the student group work (Figure 1). By episodes, we mean episodes of the PTs' attendance in front of the groups instead of single utterances. An episode can be a linguistic intervention or a passive observation. It typically begins with a PT's presence in front of a group commencing activities like silent observation, linguistic intervention, or responding to a student group's request for assistance. It ends when they leave the group and transfer the responsibilities for performance to the students, for example, the PT's observation ceased, the interventional action was finished, or the question posed by the student group was answered. The coding was first formed as descriptive codes and then grouped into categories. Figure 2 illustrates these categories, written on the top of the rectangles with their keywords grouped below them.

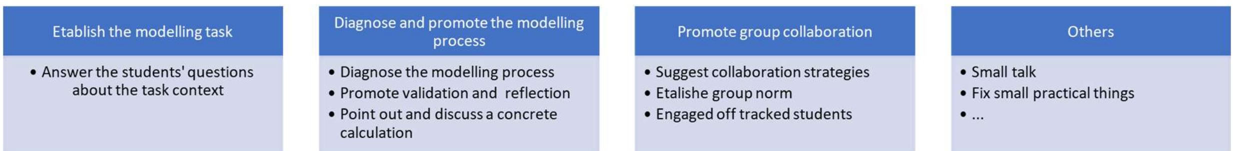

Figure 2: The preservice teachers' support of group work during a modelling activity

We identified a total of 66 linguistic episodes of teacher-student interactions of different lengths of duration.

Results

The teacher interventions fell into three main categories: *to establish the modelling task* (11 episodes), *to diagnose and promote the modelling process* (33 episodes), and *to promote group collaboration* (11 episodes). Other interventions (such as small talk, fixing small practical things etc.) were grouped as *others* (11 episodes). The first two categories are related to the content while the third concerns students as in Figure 1. A big portion of the interventions concerning content went to diagnosis and promotion of the modelling process; while interventions concerning the organisation of students went to the management of group work. We found that the PTs largely managed to use questions at the meta-level or content-oriented strategic questions to assist group modelling, but they met challenges when they tried to promote validation and reflection on the students' solutions and when they tried to promote group collaborations. In the following, we illustrate our findings with examples. We have used fictive names for the students.

Establishing the modelling task

There were 11 episodes where the PTs helped the students understand the context of the modelling task. One example of such episodes is the following: The pupils first discussed among themselves what the modelling task required them to do. Then they became uncertain, needing confirmation or redirection from the PT to proceed.

Irene:	Do we have to write the same (list)?
PT1:	Sorry?
Adam:	Must we write the same, or do we write different things?
PT1:	No, you will make a dream bag together.
Adam:	Oh.
PT1:	So, you need to agree on what you will have in your candy bag.
Noah:	OK… Does everyone want…

The group was unsure about the requirement. PT1 established the task by saying they needed to agree on what candies to choose and form one single candy bag together.

Diagnosing and promoting the modelling process

Under this category, the PTs' interventions aimed to diagnose the modelling process, promote validation and reflection, or point out and discuss a concrete calculation. We identified 14 episodes in which they aimed to locate where the group was in the modelling process. They frequently asked, "Have you found a way to do this?" and "Have you made an agreement on what you are going to have in your dream bag?". In 15 episodes they tried to encourage the students to reflect on and validate their solutions. The following is a typical example.

Odin:	We are done. We think. We took away a Sour Smilingface so that it becomes 210 grams, and it is possible to divide by three.
PT1:	Yes, that is right. Yeah, it is a solution. Could you have done it in another way?
Theo:	We don't want to do it in another way.
PT1:	But can we do it? Is this physically possible?
Theo:	Yeah, certainly.
PT1:	OK? How then?
Theo:	Don't know.

In this case, the students claimed that they were done with the task by suggesting a solution. But PT1 asked if this could be done in another way, without getting a positive response. Then PT1 tried to provoke reflection on the solution in relation to reality by asking whether it was physically doable. However, the students were not willing to think about it.

In some other cases, the PTs asked questions like "Is this fair?", "What is actually fair? The total number of candies or the total weight?", aiming to lead the students to think about fairness in this context. However, most groups answered by simply dividing the weight by the number of persons in the group. It appeared that this reluctance to relate to reality or reflect on the concept of fairness prevailed among other groups. The PTs' promotion of validation and reflection was often met with protest or reluctance.

Promoting group collaboration

We found that the PTs also needed to support group collaboration in 11 episodes. When they observed that some students in a group were off track, they tried to bring them back by either addressing the group by asking how the collaboration was going or engaging one member directly by saying "Don't you want to find out what you are going to have in the bag?". In some episodes, the PTs also tried to suggest collaboration strategies. The following episode is an example. PT2 came to a group and discovered that the group members were working separately.

PT2:	Have you done any division of tasks? You know what I mean?
Levi:	Everyone should have 100 grams.
Chris:	I know what you mean. 100 grams in every bag.
PT2:	You will have one bag. … What about that one makes the list and writes down what you all want to have, and one has to find out how much space you still have? What do you think?

The PT observed that the students were trying to make a list of candies each. Then he asked whether they were thinking about work division inside the group. The students told him their idea that they could have 100 grams in each of their bags. The PT then reacted by first reminding them that they would have one bag altogether. After some discussion, the PT suggested a strategy of collaboration.

In other episodes, the PTs needed to reestablish some group norms. For example, PT2 observed a group and discovered that one of the group members was not active.

PT2:	How is it going?
Tim/Glen:	Bad.
PT2:	Bad?
Tim:	Because Johan doesn't like Bamsemums.
Johan:	No, it is disgusting.
PT2:	Do you remember that we talked about respecting that we have different opinions? Respect that there are people who like things that you may not like. So, we need to be considerate and show respect to that they may get Bamsemums since they like it, and you can get something else.

Here the group members disagreed with what they should have in their candy bag, and the PT came in and tried to remind them of the norms they should follow, that is, to respect each other's opinions. However, after this conversation, the group did not change their way of working. They carried on with their calculations with the one member who stayed inactive.

Discussions and conclusions

To answer the research question, we analysed PTs' interventions during modelling activities in groups. We found that they mainly supported the students in three characteristic ways: to establish the modelling task, to diagnose and promote the modelling process, and to promote group collaboration. The interventions in the first two categories can be seen as content-related and content-oriented strategic interventions as in Zech (1996). Even though these types of interventions are dominant in line with Tropper et al. (2015), we also found that a big portion of interventions from the PTs went to support group collaboration, which is less mentioned in the literature about teacher interventions during modelling activities.

The first category, providing support to the students to establish the modelling context, resonates with the fact that understanding the situation and constructing a situation model is a cognitive barrier for students (Blum, 2011). To identify what modelling processes were carried out in groups, the PTs used many interventions to diagnose group modelling progress. They often intervened by asking questions at a meta-level, such as "How far have you gone?" and "Have you found a way to do this?". Such questions can be seen as a way to get the students to present the state of their work, which is considered to be the most effective according to Stender and Kaiser (2015). Such questions can also ensure that the PTs' support is tied to students' ideas (Webb, 2009).

The PTs encountered challenges when they attempted to encourage the students to validate and reflect on their calculations. Firstly, many student groups equated sharing candies among themselves with dividing the weights by 3 without considering the practical feasibility. Even when the PTs posed questions like "Is this physically possible?", the students were reluctant to interpret their answers in relation to reality. Secondly, the majority of students considered fairness merely as equal weight. The PTs tried to stimulate some reflection by asking, for example, "Can it be fair?" and "Do you like the candies equally much?". However, from the student groups' solutions, it appeared that they did not think these aspects were important. Here the PTs asked questions related to content-oriented strategies as recommended by Blum (2011) and Antonius et al. (2007). However, it seemed that only these questions themselves were not enough to push the students to go further.

The third category identified was supporting group collaboration. The PTs tried to help the students who were off track to come back to the group work by asking, for example, "Don't you want to find out what you are going to have in the bag?" to engage them in group conversations. The identification of this category confirms that the groups require support when the groups do not have true dialogues (Ding et al., 2007). We also found that to enhance group collaboration, the PTs tried to establish group norms such as respecting each other's opinions, and they tried to provide suggestions for collaboration such as work division. These attempts agreed with the suggestions of Webb (2009), who recommends that to prepare the students for collaborative work, the teachers can describe behaviours that are expected during group work. It seemed that the PTs had certain strategies to manage group work. However, we could observe that sometimes the groups did not work together effectively. Therefore, providing PTs with more strategies for promoting group collaboration can be beneficial in a modelling activity. Such strategies are situated on the edge of the didactical tetrahedron of Prediger et al. (2019) and can be both content-related and general.

In sum, we have provided a detailed analysis of an empirical example of two PTs' interventions during their first trial of teaching modelling activities. Our results add to empirical knowledge on PT interventions and emphasise that supporting group collaboration is an inevitable area during collaborative modelling. They can also be informative for teacher education programs (cf., Prediger et al., 2019). Further research is needed on how to support teacher interventions during collaborative modelling to promote reflection and validation in the modelling process and group collaboration.

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Developing Creativity in steM Education

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Abstract: This paper aims to answer a question: how can we understand and develop students' creativity in steM education? This question drives us firstly to develop a new knowledge conversation model that indicates how creativity, learning, and mathematical knowledge interplay with each other; secondly, a case study on steM education is carried out with students and teachers in a seventh-grade class in a Danish school. Student workshops were organized and recorded that focused on learning mathematics and solving real-life steM problems. The findings reveal:1) students became actively to participate in creative problem-solving process; 2) the mathematics knowlege become tacit level in practice of solving problems. This indicates teachers should pay more attention to students' explicit learning experience in developing creativity in steM education.

Keywords: Creativity, learning, mathematics knowledge, steM education, problem-solving

1. Introduction

Creativity has been recognized as one of key learning skills that students should master to face future challenges in the twenty-first century. In general, creativity is the ability to generate novel ideas, raise new questions, and come up with solutions to ill-defined problems (Zhou, 2012). In mathematics education, efforts in creativity studies have shown the interests in talent education (Grégoire, 2016), classroom culture (Czarnocha, 2014), imagination and divergent thinking (Karwowski, Jansowska & Szwajkowshi, 2017), problem-solving and teacher education (Kandemir & Gür, 2007; 2009).

In this paper, we highlight that mathematics knowledge has a very important role in everyday life, especially in context of problem-solving where creativity usually happens. It is well known that problem solving is a crucial part of mathematics learning; it is a means to sharpen reasoning that is careful, logical, critical, analytical, and creative (Hasibusan, Saragih & Amry, 2019). To solve a problem requires creativity; problems are sources as well as goals for developing creativity. In this sense, we focus on 'everyday creativity' in mathematics, which concerns every student can be creative in responds to daily learning tasks in a novel and meaningful way (Zhou, 2015). Moreover, we view both 'creativity' and 'mathematics learning' as contextual activities that underpin the significance of social approaches to develop creativity in mathematics education.

We also address a notion of 'steM' education; the capital letter M means that there is specific focus on mathematics. Instead of an 'add-on', mathematics stands on a fundamental position to develop integrated STEM education. Experiencing real life applications and modelling real world situations mathematically is central to integrating mathematics with science, technology and engineering that serves for solving complex problems. STEM education is an interdisciplinary and applied approach that is coupled with real-world, problem-based learning; the four subjects cannot and should not be taught in isolation (Maass, Geiger, Ariza & Goos, 2019). Accordingly, 'steM' education, or the issue on ‘how to learn and teach M in and for STEM’ has been increasingly discussed in studies on pedagogical practice (Fitzallen, 2015)

Following above lines, this paper focuses on a question: how can we understand and develop students’ creativity in steM education? In answering this question, this paper will firstly develop a theoretical framework that bridges studies on creativity, learning, and mathematics knowledge in one framework; secondly a case study will be discussed that leads to a shown that draws a part of project LabSTEM at University of Southern Denmark (SDU); lastly a discussion will be explored with findings that also leads to a conclusion with implications for future creative steM education.

2. Creativity, Learning and Mathematics Knowledge

2.1 Creativity and Learning Mathematics

Learning can refer to the outcomes of the learning process that take place in the individual, the mental processes that take place in the individual that can lead to changes or outcomes, or the interaction processes between individuals in social environment (Illeris, 2007). According to Sfard (1998), we can use two metaphors to describe learning - the acquisition metaphor and the participation metaphor. Learning in mathematics is often regarded as an act of sense-making that is socially constructed (Shoenfeld, 2016). Learners are thus seen as transforming as well as being transformed when participating in communities of practice; knowledge is not a fixed and stable commodity, but rather co-constructed by people in interaction (Zhou, 2020).

Mathematics education reflects the idea of mathematics as an activity aimed at making sense if students are to come to understand and use mathematics in meaningful ways. As Zhou (2015) described, when we learn something new, we are making new connections between ideas and making sense of them for ourselves, and in this sense, creativity helps to shape new opportunities to learn new knowledge. Learning to the extent that it also involves actively building a network of interrelated ideas, discovering or rediscovering concepts and principles, is itself a creative act. In mathematics class, when students learn new knowledge, they build on each other’s ideas to reach an understanding that is not available to any of them initially, and they must enter critical and constructive negotiations of each other’s suggestions (Zhou, 2020). At the same time, they need to share and evaluate well-grounded arguments and counterarguments through collective talk (Craft, 2007). During such a process of knowledge exchanges, the interplay of creativity and learning happens. Accordingly, we can reach a conclusion that creativity and learning go hand in hand; creativity is embedded in the learning process and meanwhile influences and is influenced by a learning environment (Zhou, 2020).

2.2 A Knowledge Conversation Model

Creativity helps students to shape new directions of learning new knowledge in solving mathematics problems. In the context of solving real-life problems, students are usually encouraged to work collaboratively in a team. As Fitzallen (2015) described, problem-solving draws on concepts and procedures from mathematics and science; this also needs to be incorporated in the teamwork and design methodology of engineering using appropriate technology. Gaining new knowledge cannot occur without some understanding of what already exists (Zhou, 2020). This guides us to discuss on knowledge types including tacit knowledge, explicit knowledge, individual knowledge and collective knowledge, transformation between different types of knowledge, and influences of creativity on the transformation, which forms a knowledge conversation model (Zhou, 2020) (Figure 1).

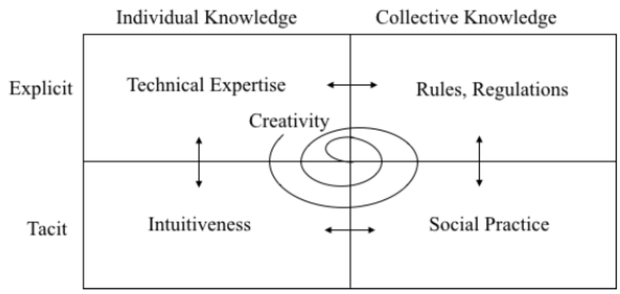

Figure 1: A knowledge conversation model on creativity and learning

The model was built upon the previous efforts (Baumard, 1999; Zhou, 2020) on discussion of a knowledge approach to creativity and learning process in contexts of collaboration, problem-solving and STEM education (Zhou, 2020). The model involves four types of knowledge:

a) Knowledge that is explicit and individual provides techniques and technical experience that allow us to counter nets and traps. In a mathematical context, this could refer to an individual's personal knowledge and skills within specific techniques or methods. For example, it may include specific knowledge of effective methods for solving different types of equations or integrals. It may also include more theoretical insights like individual understanding of theoretical concepts and methods that can be discussed with others and applied to analyze and solve mathematical problems.
b) Knowledge that is collective and explicit leads us to achieve profound knowledge of a terrain, the environment, rules, regulations, and laws. Apparently, this includes specific mathematical rules, regulations, and laws governing various concepts and operations within the domain. This encompasses our shared knowledge of foundational theories and principles within a specific mathematical domain that forms the basis for a comprehensive understanding and collective awareness of the mathematical landscape including the structures, patterns, and relationships inherent in the subject. Thus, this also involves our common understanding, approaches and strategies for addressing mathematical challenges and complexities.
c) Knowledge that is tacit and collective is of the unspoken and invisible structure but with common understanding among a group of people in social practice. This may refer to a shared, implicit understanding within a community engaged in mathematical activities. This might include collective and unspoken insights into intuitive problem-solving approaches that are not explicitly articulated but are commonly understood within the group. Examples such as unwritten norms, conventions, and practices within the mathematical community that are collectively understood and followed, contributing to a shared mathematical culture. This also involves our shared practices, habits, or techniques that are not explicitly codified but are collectively embraced and applied by members of the mathematical community.
d) Knowledge that is tacit and individual is intuitiveness that is deeply rooted individual's experience, schemata, beliefs, and perceptions in the worldview that is taken for granted. This might refer to an individual's intuitive understanding of mathematical concepts based on

personal experiences and internalized cognitive structures. This includes individual cognitive frameworks or mental models developed through prior mathematical experiences that shape how the person perceives and approaches new mathematical challenges. The individual's subjective beliefs and perspectives about mathematical concepts are involved, which influence their intuitive grasp of mathematical ideas. This means the overall perspective through which an individual interprets mathematical concepts, often is taken for granted and deeply ingrained in their thinking.

As indicated by the model, we can think that creativity in the learning process is dynamic but not linear and contains uninterrupted knowledge conversations between different knowledge types. This shows creativity as a 'spiral' growing from the junction of the four types of knowledge (Zhou, 2020). This model calls students to develop creativity for learning mathematics in depth. Students should be able not only to 'do mathematics tacitly' in practice with action-oriented ways but also to 'communicate mathematics explicitly' with others. As Tambychik and Meerach (2010) argued, conceptual understanding and procedural knowledge are essential to skills in problem-solving (Geary 2004). These skills should be supported by cognitive systems that control focus and interference in information processing. Apart from that, language and visual-spatial skills are also important to interpret and to manipulate information effectively in the working memory (Zhou, 2020).

3. A Case Study

3.1 Background of the Case

The case study is drawn from a part of the project LabSTEM (2020-2022) at SDU, Denmark. An overall objective of the project was to develop innovative steM education. A total of 26 STEM activities have been developed. The primary activities included research-based establishment of STEM laboratories in Danish primary and lower secondary schools where teachers met in workshops and developed their own integrated STEM activities. LabSTEM encourages teachers to explore meaningful and effective pedagogical practice and hold on principles such as Problem-Based Learning, and Inquiry-Based Learning.

3.2 Activities and Workshops

The activities and workshops in this case aimed at fostering the students' understanding, learning, and using mathematics in solving real-life steM problems. The activities were designed as a field day nearby a hydroelectric plant, for students in a seventh-grade class, by three teachers (all teach mathematics, but one also teaches the physics and chemistry). Table 1 shows details of the activities and workshops.

Activity	Content	Details of Tasks
1	Group Discussion	Question: Should we dig more artificial water channels to increase water flow to the hydroelectric plant, thereby generating more green energy, although this will affect the biodiversity of the area?
2	A Photo Safari	Take pictures and explain different parts of the power plant

3	Four Workshops	1) Identify inquiry: how much water can be transported from the top to the bottom of the tank and what is the water velocity? (4 students per group) 2) Measure the water flow rate out from the hydroelectric power plant, 3) Calculate how much water the water supply pipe can hold, and 4) Find the volume of the water tank above the pipe.
4	Open Activity	Build your own hydroelectric plant: How can you build the most efficient hydroelectric power plant using different materials?

Table 1: An overview of Activities

In Table 1, the third activity comprised of four various workshops conducted throughout the hydroelectric power plant. In this context, students were organized into groups, each engaging in a specific activity at a time. The first workshop involved students transporting water from the top of the reservoir to the turbines as swiftly as possible. Students were required to pour the water into a cone-shaped bucket and subsequently calculate the amount of water transported, followed by determining the velocity of the water. The data was then to be compared with the actual rate at which the water descended through the pipes of the hydroelectric power plant. In the second workshop, students were tasked with calculating the volume of water in the top-reservoir. Initially, students were required to measure the length and width of the reservoir. Subsequently, they lowered a stone into the water and measured the depth. Once these measurements were obtained, they were to calculate the rate at which the reservoir could be emptied, given the knowledge that 6,000 liter of water flowed through the pipe every second. In the third workshop, students were tasked with calculating the speed of water as it exits the hydroelectric power plant. This was achieved by throwing various small objects and floating cream into the water and measuring the time it took to travel a specific distance. The students were required to determine the average of multiple measurements. Subsequently, this data was compared to the speed of the water as it descends through the pipe towards the turbine. In the fourth workshop, students were tasked with measuring the volume of the pipe itself, which was approximately 80 meters long. This necessitated measuring the length, followed by the circumference. From these measurements, they calculated the diameter/radius and subsequently determined the volume of the pipe.

The fourth activity marked the culmination of the day, intending for students to apply their acquired knowledge about hydropower plants. The activity commenced with an introduction, setting a scenario where students were collectively summoned as some of the finest engineers in the area. Their task was to collaboratively devise an efficient hydropower plant. The teacher explained that she and other educators would act as judges, evaluating and determining which group had created the most effective and innovative hydropower plant. Students were provided with a variety of materials for construction, including different types of pipes (long, short, thick, thin), tape, various funnels, plugs, and more, scattered on the ground.

3.3 Data Collection and Analysis

One of co-author of this paper collected the data by observation and video recording. The entire field day at the hydroelectric power plant was recorded on video with a handheld camera. Given that several of the exercises occurred in different workshops, a group was selected that was video filmed throughout the different activities. During the final evaluation exercise, videorecording was done in slightly different groups to observe various approaches to solving the task. Besides this, the researcher also taken notes during her observation.

The video recorded data is used as the primary data. As Margaret (2002) suggested, when we seek to study real people in real situations, doing real activities, video recorded data can provide us with comprehensive contextual data. It also allows us to experience an event repeatedly by playing it back. With each repeated viewing, we can change our focus somewhat and see things we had not seen at the time of taping or on previous viewings. Replaying the event also allows us more time to contemplate, deliberate, and ponder the data before drawing conclusions, and hence serves to ward off premature interpretation of the data (Erickson, 1992). The field note taking can help to mark some specific focuses or reminders for analysing video recorded data; accordingly, notes of observation were taken as a supplement of data collection.

We employ a framework of video data analysis (Nasssauer & Legewie, 2021), which focuses on situational dynamics and behaviours to understand how people act and interact, and which consequences situational dynamics have for social outcomes. The applications above show that by observing a person's movements, fields of vision, uses of space, interactions, exchanges of glances and gestures, facial expressions, and body postures, it is possible to decipher the syntax of situational dynamics. From replaying the video, we observe how students engage in creative problem-solving process, how they use mathematics knowledge, how they interact with each other, and how they show their outcomes. By taking the knowledge conversation as the theoretical foundation and focusing on research question in this paper, we explore findings and discussions in the following.

4. Findings and Discussions

4.1 Active Participation in Creative Problem-Solving Process

The data shows that after introducing the tasks in open activity, students started to try out various approaches to solve the problems. Some of them started from scratch, some attempted to create a narrower diameter at the end of the pipe to generate higher pressure, while others aimed for a nearly vertical orientation of their pipes. The students were also very actively to assign individual tasks in group work and test ideas between each other. They promptly engaged in group discussions to strategize the best approach, and all groups quickly commenced the building process. Some opted for thick pipes, some selected long pipes, and some positioned themselves atop small hills.

This indicates the challenges of the task stimulate students' intrinsic motivation to seek for the solutions that drove them to actively participate to process of creative problem-solving. As Zhou (2015) suggested, intrinsic motivation is one of key factor in creativity development; it can be driven by an interest or enjoyment in the task itself. The individuals who undertake a task for its own sake are intrinsically motivated the intrinsically motivated individuals are more likely to expend energy exploring the problem and finding creative solutions. The open-ended questions or solving real-life problems are emphasized as the characteristics of a task that can simulate creativity. Problems can take various forms in practice, such as failure to perform, situations in need of immediate attention or improvement, a need to find better or new ways to do things, unexplained phenomena or observations, or gaps in information and knowledge. These are all triggers of intrinsic motivation, creative thinking, and deep thinking. At the same time, when the students finished all the workshops and started to work on the open activity, the Problem-Based Learning approach provided a collaborative knowledge building and self-directed learning environment. This enabled students to translate their individual knowledge and experience into participation in group creativity development with common goals, shared objectives, and supports of peers.

4.2 Mathematics Becomes Silent in Practice

The activities aimed at facilitating students to apply mathematics knowledge into practice. However, the data shows even though in students were taught explicitly with mathematics knowledge in workshops, they had difficulty in discussing and conceptualizing mathematics in the process of working on the open activity. It was also found that teachers lacked attention in the facilitation to improve students' explicit understanding when mathematics was used in practice.

This issue can be seen from some dialogues between students and teachers. For example, when the students finish their tasks, they concluded by presenting their designs. During the show off, each group must explain the rationale behind their design. A student explained, 'we chose a small hole at the bottom, so that when the water travels from a thick pipe with a large diameter to a thin one with a smaller diameter at the bottom, the water shoots out. We have also opted to apply pressure to the pipe by closing the hole first, filling the pipe with water, and then when we open it up, the water is pushed out. We've also worked on the angle of the pipe, and we can't make it steeper than it is now since we also have a relatively long pipe, about 4 meters.' The teacher asked, 'Are you saying this because you wanted it to be steeper?' The student responded, 'Yes, we tried to keep the pipe at a larger angle, but it works better the smaller the angle is-if that makes sense.'

Therefore, this case shows that mathematics became tacit knowledge in solving a real-life problem. However, mathematics knowledge and knowing mathematics cannot be separated from situations where they are used or where they take place. The poor conceptualization on mathematics may lead the students to lose the meaning of learning and social recognition in creative group work, because students had trouble in defining 'the value of the knowledge I am using', and difficulty in defining 'what I do through my creative collaboration could influence how it is regarded or valued by others'. According to Shaughnessy (2013), STEM education must involve significant mathematics for students. Otherwise, the M in STEM is silent. If we are going to promote STEM education, as mathematics teachers, we must make the mathematics transparent and explicit. We cannot just assume that everyone will 'see' the mathematics that is involved in a particular problem. The M in STEM will remain silent for our colleagues unless we clearly shine the light on the mathematics.

5. Conclusion

A creativity perspective to rethink mathematics learning and steM education addresses the importance of 'dialogic approach' to improve steM education. Mathematics teachers play important roles in facilitating creative problem-solving process. Facilitation is a subtle skill. It involves knowing when an appropriate question should be asked, when the students are going off track, and when the problem-based learning process is stalled. In other words, setting student learning in problem-solving contexts is no guarantee of successful learning, but the stimulation of interactions between students towards correct directions is a prerequisite, steM education thus places high demands on explicit facilitation skills of the teachers in order to hear stronger voices of mathematics in the future.

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Learning to elicit and promote reasoning and proving: pre-service teachers' accounts

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A growing body of research reports on opportunities for learning to teach that can arise when working with approximations of practice in teacher education. However, there is a call for studies investigating pre-service teachers' (PSTs') views on participating in approximations of practice, and this study is based on interviews with PSTs who have participated in role-plays of interactions focusing on reasoning and proving. The interviews were analysed through communities of practice theory, revealing that a community of practice was formed through the PSTs' participation in role-plays. Based on the characteristics of the community, opportunities for learning to plan and enact discussions that centre student thinking were identified. However, challenges concerning reasoning and proving were also found.

Keywords: Communities of practice, teaching practices, preservice teachers, validity

Introduction

An important task of mathematics teacher education is preparing PSTs to lead high-quality mathematics instruction. Here, PSTs need to learn to value and build on students' contributions in teaching, support student participation, and maintain high academic rigour (McDonald et al., 2013) by pressing for justification and explanation when responding to students' contributions (Lampert et al., 2013). This view on mathematics instruction builds on the assumption that all students will develop their reasoning skills (Forzani, 2014), making reasoning and proving, defined as processes for developing and justifying mathematical claims (Jeannotte & Kieran, 2017), a central part of teaching (Stylianides & Stylianides, 2017). However, building on and developing students' reasoning is complex work. Research suggests a practice-based approach to teacher education to support PSTs in learning the complex work of teaching and developing students' mathematical thinking (Forzani, 2014). In practice-based approaches, teacher education is centred around specific core practices of teaching. That is, methods for learning to investigate and enact meaningful teacher practices that promote students' deep learning (e.g., Lampert et al., 2013; McDonald et al., 2013).

A pedagogy for implementing core practices in teacher education is through the learning cycle, which consists of investigating the practice, approximating the practice through planning and enacting, and analysing the enactment (McDonald et al., 2013). Through approximations of practice, PSTs are provided opportunities to enact core practices without the full complexity of authentic teaching (Grossman et al., 2009). Previous research on the learning cycle shows promising results. For example, novice teachers are provided the opportunity to learn to pursue students' mathematical understanding, eliciting ideas (Lampert et al., 2013), eliciting explanations, connecting strategies, and using representations (Ghousseini, 2017). Specifically addressing mathematical reasoning, Kazemi et al. (2021) found that approximations supported teachers in learning to attend to and develop students' mathematical argumentation. However, Buchbinder and McCrone (2020) found that PSTs faced challenges in enacting a precise language for reasoning and proving in their lessons despite being successful in their planning. Given PSTs' challenges with learning to enact teaching that emphasises reasoning and proving, more research on interventions is needed in teacher education (Stylianides, Stylianides, & Weber, 2017). Moreover, research on the opinions of PSTs participating in approximations of practice is needed in the literature on practice-based approaches to teacher education (Matsumoto-Royo & Ramírez-Montoya, 2021).

To provide PSTs opportunities to learn to elicit and promote students' reasoning and proving in mathematical discussions, lessons designed around the learning cycle, using role-plays as an approximation of practice, were implemented in a mathematics teacher education course for primary school PSTs. I investigate the PSTs' learning based on interviews conducted following the lessons, taking a social view on learning by describing learning to teach as participation in communities of practice (Wenger, 1998). The research question is: *how do PSTs characterise the community of practice that emerged in cycles of investigating, planning, enacting, and reflecting on role-plays of interactions aiming to elicit and promote students' reasoning and proving?* Based on the results, I discuss opportunities and challenges for learning to elicit and promote reasoning and proving.

Theoretical framework

To account for the situated character of learning to teach within learning cycles, I adopt a perspective of learning as a development of participation with others in communities of practice (Wenger, 1998), where learning occurs over time in interaction with the specific context (McDonald et al., 2013). Learning is characterised as a social process through four components: community, practice, identity, and meaning (Wenger, 1998). These components are mutually defining, and learning can be approached by putting any of these into focus (Wenger, 1998). I investigate learning from the community formed as PSTs participated in the role-plays of interactions aiming to elicit and promote students' reasoning and proving.

The relations of practice and community can be described through three dimensions: mutual engagement, joint enterprise, and shared repertoire (Wenger, 1998). First, through mutual engagement, PSTs can participate in role-plays and the practices that matter. Engagement positions participants to contribute with overlapping or complementary competence, for example, related to proving. Second, a joint enterprise is what the community strives towards, either explicitly stated or not. The enterprise results from the PSTs' collective negotiation as they participate in role-plays and define what matters and what does not. It also gives participants opportunities to be concerned about what goes on around them. That the enterprise is joint does not mean that everyone agrees on the same thing but that it is communally negotiated. Since the enterprise is negotiated in the community, it cannot be fully determined by an outside mandate. However, one cannot deny the enterprise's position within a broader system of communities, such as those of policymakers for teacher education programs and research on teacher education. Last, a shared repertoire results from the community's pursuit of a joint enterprise through mutual engagement. The repertoire can contain words, artifacts, routines, and concepts reflecting a history of mutual engagement in the community.

Locality is another aspect of community that is relevant in the context of teacher education. In teacher education, one can argue that a specific course is the source for negotiating a community of practice. However, the course is part of larger constellations of other communities of practice, e.g., the teacher education program, the discipline subjects taught, and the teaching profession. The constellations are too abstract to investigate negotiation of meaning but can serve as a useful lens to investigate the interconnectedness of practices (Wenger, 1998). Continuity across communities inside constellations can be due to the transfer of repertoire and enterprises (Wenger, 1998). For example, a teaching technique worked on in teacher education courses can be used when PSTs teach in school placement.

Methods

To investigate the community of practice formed through PSTs' participation in role-plays of interactions aiming to elicit and promote students' reasoning and proving, I use a phenomenological approach (Creswell & Poth, 2018), investigating the PSTs' experiences of engaging in the learning cycles. Together with researchers on a larger project about reasoning and proving, I designed lessons that used role-plays as an approximation to practice. The role-plays were implemented in the first mathematics teacher education course for first-year PSTs of grades 1-7. The lessons were implemented in an intervention consisting of three iterations of four lessons in three different classes. Before the intervention, the PSTs were introduced to reasoning and proving by analysing students' justifications and proving a statement themselves. The PSTs also investigated a transcript of a classroom dialog utilising moves to support students' reasoning (Ellis et al., 2019). We did not prepare the PSTs for playing the roles of teachers and students before the intervention, nor did the teacher educators guide the PSTs during their enactments of the role-plays. Thus, the role-plays were enactments of hypothetical scenarios based on the PSTs' planning and improvisations.

Each lesson consisted of a cycle of four phases inspired by the learning cycles (McDonald et al., 2013). First, the teacher educator introduced a case from a classroom containing a task given to students and the students' work on the task. Second, the PSTs were prompted to plan an interaction with the students from the case, aiming to elicit and promote students' reasoning and proving through an interaction with a goal set by the teacher educator. Third, the PSTs enacted the interactions in groups or whole-class role-plays, with one PST (or sometimes two or a small group of PSTs) enacting the role of the teacher, some PSTs playing the roles of the students, and some PSTs observing. Last, the PSTs were prompted to reflect on the enactments. After each iteration of four lessons, I interviewed some of the PSTs. Based on a short questionnaire the PSTs answered during the lessons, I asked PSTs with different opinions to participate in the interview, aiming to investigate a range of perspectives. In total, 12 PSTs were interviewed, and these interviews constitute the data for this study. The interviews were semi-structured, asking open-ended questions about the PSTs' experiences of participating in the role-plays. Each interview played out differently, as the follow-up questions were based on the PSTs' responses. The interviews lasted 10-15 minutes.

The method of analysis was thematic (Bryman, 2016). First, the interviews were analysed inductively, coding statement by statement in vivo to avoid losing the PSTs' voices and staying close to data. Next, I grouped similar statements after themes. These groups reflected different experiences. Across these groups, the theoretical perspective of communities of practice guided my analysis of mutual engagement, joint enterprise, and shared repertoire. The mutual engagement emerged through what the PSTs said enabled or hindered their engagement regarding the organisation of the lessons and their group work. I interpreted joint enterprise based on what the PSTs expressed that they had worked thoroughly with during the lessons and what they conceived as important and less important. The shared repertoire emerged as concepts, tools, and routines the PSTs explicitly said they had used or learned in the role-plays. I also analysed the community's locality through the PSTs' mentions of interconnected communities.

Results

I present the results in terms of mutual engagement, joint enterprise, shared repertoire, and locality.

Mutual engagement

All PSTs emphasised that they enjoyed the experience of participating in the cycles, and some stated that they would like to enact role-plays in other courses as well. Particularly, the PSTs said that their engagement was enabled through group work because it allowed them to discuss mathematical problems, plan discussions, and try different questions. Through participating in several learning cycles, the PSTs stated that the enactments felt more realistic when they experimented with more spontaneous questions, contrary to the first enactment, where they had planned the questions literally. One PST even said that when she played the teacher, she really felt like a teacher, focusing on the students and forgetting about those who observed. The PSTs also emphasised that the enactments made them committed to planning, which opened for discussions they think they would not have had otherwise, as exemplified by Camilla's statement:

> Camilla: I think it provoked more discussions than what we would have had if not. So, when you, sort of, enter a role, you have to, well, I was never the teacher, but as a student, I had to try to understand the student responses. (…) So, personally, I don't think I would have examined [the student work] so closely if it was only a group work. But with the following enactments (…) I had to understand it.

Another feature of the lessons that the PSTs put forward as enabling their engagement was the different roles of playing a teacher, student, or observer in the enactments. They reported on the usefulness of playing the teacher and suggested that the teacher educator could organise the enactments such that all PSTs played the teacher at least once. They emphasised the importance of observing to learn how they could support students to find the answers themselves.

Several of the PSTs also mentioned some features of the lessons that hindered their engagement. The most central one was the whole-class enactments, which the PSTs said were intimidating. They suggested solutions to this problem and pointed to the teacher educator's role in making them feel more prepared. They emphasised that the teacher educator had a complementary role in the community and should support the PSTs by discussing the conjectures, proofs, student solutions, and the different teacher questions before the enactments. The PSTs also said that the enactments in smaller groups felt safer and suggested enacting the role-plays in small groups before the whole-class enactments.

Joint enterprise

The negotiated enterprise reported on by the PSTs was *planning and enacting role-plays of discussions that emphasise students' thinking*, expressed explicitly as intentions and implicitly through what they found important and challenging. The explicit intentions were to make sense of the mathematics, plan questions, predict responses, value different student responses, build the discussions on the students' contributions, and think through the goal of the discussion and the moves and drawings that can support the students in reaching the goal. The PSTs expressed their view on work with reasoning and proving as making sense of mathematical operations and strategies. To centre student thinking, the PSTs emphasised that it is important to value students' responses, to build on their contributions, and to be open to many different responses, as seen in Siri's statement below.

> Siri: No reasoning is right or wrong, it is possible to reason in very many different ways, and it is very important that we as teachers don't have a method, that we only think about one particular method that we want the students to learn.

Several PSTs expressed the importance of planning and predicting student responses and said they will do this in school placement. They further emphasised that thinking through the drawings that can support the students is important. The PSTs also talked about the importance of asking good questions, for example, as described by Amund:

> Amund: It is useful to try to understand the student responses and how they have been thinking, and then try to plan how you can make them reflect independently so that they get it right, without just telling them how they should do it.

The implicit intentions were identified based on what the PSTs found important (but perhaps challenging). The PSTs stated that they liked to be challenged to lead a discussion, that they found it important to enact the discussions, and that the enactments made them more prepared to handle difficult situations in teaching later. In the utterance below, Eva makes a point of enacting role-plays in contrast to more traditional teacher education.

> Eva: I think it was nice to have the opportunity to enact situations that can occur in school placement. It is one thing to read the curriculum and solve mathematical tasks, but it is something completely different to get unexpected answers from the students and take them into consideration.

Although the PSTs stated the importance of planning and enacting, they also emphasised that it is challenging. In particular, they expressed that predicting student responses and playing the student is challenging because they do not know the students and their knowledge. Another challenge mentioned by the PSTs was related to reasoning and proving. The PSTs said it was challenging to know what qualified as a proof and when the discussion reached a proof. It was also difficult to communicate about reasoning with students in a manner that young students could understand. The challenges reported by the PSTs indicate effort and, thus, contribute to the joint enterprise.

Shared repertoire

The shared repertoire consisted of concepts, tools, and routines. The concept of reasoning and proving was part of the PSTs' shared repertoire. They emphasised that students should explain how they have calculated, be able to change their thinking and answers, and generalise strategies to be used in more than one task. One example of how the PSTs described reasoning and proving is given below.

> Ella: Mathematical reasoning is, well, it is to come to an answer and know what is correct and what is incorrect, and to be able to change the answer if it is incorrect, and to be able to go a bit further into the thoughts and ideas that are behind the answer, and to go into (…) why, why have you thought this way, how can you help that student or the group further in their thinking.

Moreover, two artifacts provided as planning tools in the intervention, a set of moves for eliciting and promoting mathematical reasoning and a framework for mathematical reasoning, were part of the shared repertoire. The PSTs said they used them to think about the different questions a teacher can ask, understand what the teacher does when observing others' teaching, and examine students' mathematical reasoning. They further mentioned that these artifacts will be useful in school placement. Finally, the routine eliciting students' thinking was a part of the shared repertoire. The PSTs emphasised that they learnt about the different answers students can give to a problem and to make student thinking central in the discussions. Several PSTs claimed that the students' thinking can be elicited in discussions, even if they have the wrong answer.

Locality

The last step of the analysis provided information about the locality of the community of practice in a constellation of communities as perceived by the PSTs. In relation to the learning cycles, the PSTs brought up other teacher education courses, mathematics as a discipline, and school placements. First, the PSTs emphasised the importance of understanding student thinking in all subjects and that they learnt from seeing different ways of thinking that would help them in their assignments. Second, concerning mathematics as a discipline, almost all PSTs said that they found reasoning and proving difficult, and their ways of talking about it confirmed the difficulties of proving in mathematics contrary to explaining how you think. Last, connections between the role-plays and school placement were highlighted by all the PSTs. Planning questions and predicting student responses are routines that the PSTs said they would bring into school placement. Further, several PSTs said they would focus on centring student thinking in future practice.

Discussion

The community of practice, as reported on by the PSTs, was characterised by mutual engagement through collaboration, complementary roles, and accountability, a joint enterprise of planning and enacting role-plays of interactions that emphasise student thinking, and a shared repertoire of teacher moves and theory as tools, a definition of mathematical reasoning as a concept, and the routine of understanding and building on students' answers. In the following, I discuss opportunities and challenges for learning to elicit and promote reasoning and proving provided by the lesson design.

The features of the community of practice that the PSTs reported on show that the lesson design provided several opportunities for learning to elicit and promote reasoning and proving, which adds to the research on approximations of practice. Group work provided opportunities for mutual engagement through the complementary and overlapping roles, which enabled opportunities for making sense of mathematics and students' thinking, planning questions, and predicting students' responses. Moreover, the enactments that followed the planning made the PSTs accountable during planning and thus enabled PSTs' engagement. Playing the teacher in multiple enactments was also emphasised by the PSTs as an opportunity to practice eliciting student thinking. Thus, implementing learning cycles is a feature of the lessons that provide opportunities for learning.

The joint enterprise of planning and enacting role-plays of interactions that emphasise students' thinking, as reported by the PSTs, reveals that the PSTs had opportunities to learn to centre students' thinking through planning and enacting in approximations of practice, similar to Ghousseini's (2017) and Lampert et al.'s (2013) results. Moreover, the PSTs reported on opportunities to learn to elicit students' mathematical explanations and ideas and steer the interaction towards the mathematical goal (in line with Ghousseini (2017) and Lampert et al. (2013)). By focusing on eliciting students' explanations, the PSTs reported on opportunities to learn to elicit students' reasoning and proving. The shared repertoire also reflects opportunities for learning about teacher moves and theory through tools and opportunities to learn to elicit students' thinking as a routine. The interconnectedness of the community of the role-plays and the school placements emphasised by the PSTs shows that the use of role-plays gave PSTs opportunities to learn teaching routines that they can bring into school.

Across the characteristics of the reported community of practice, I can also identify some challenges related to learning to elicit and promote reasoning and proving. First, the PSTs reported on features of the lessons that hindered their engagement. One was that it was challenging to play the role of the student if they did not know enough about the student's thinking. Next, the PSTs emphasised that whole-class enactments were intimidating and hindered their engagement. Last, I identify a challenge concerning reasoning and proving based on the PSTs' reported joint enterprise and shared repertoire. The way the PSTs talked about reasoning and proving, as exemplified by Ella in the results section, defines proving as explaining why one has chosen a strategy, which does not align with the definition introduced by the teacher educators (see Jeannotte & Kieran, 2017), emphasising proving as explaining why a strategy is *valid*. Furthermore, reasoning and proving was not an explicit part of the joint enterprise. The challenges related to proving in this study indicate that the PSTs put less emphasis on explaining why than what we intended when we designed the lessons, that the PSTs themselves were uncertain about what counts as valid proof, and that proving became less central in the role-plays than intended in the design. These challenges add to the challenges identified by Buchbinder and McCrone (2020).

The identified challenges should be addressed in further designs of lessons using role-plays as an approximation of practice. First, PSTs should be supported in playing students. For example, detailed records of the students' thinking can support the PSTs in getting to know the ideas behind the students' utterances. Another support could be providing the PSTs with some lines or ideas they can contribute with during the enactments. Next, the challenge of intimidating whole-class enactments can be addressed by enacting the discussions in small groups before the whole-class enactments. The PSTs also suggested that the teacher educator could provide more support before and during the enactments. Finally, challenges related to reasoning and proving can be addressed by working more specifically with proof before introducing role-plays centred around proving, as suggested by Buchbinder and McCrone (2020). The lessons I report on in this study were some of the first lessons in mathematics education that the PSTs took part in, and the complexity of managing mathematical interactions *and* emphasising a mathematically complex topic like reasoning and proving can be too complex too early. However, as the PSTs acknowledged challenges related to reasoning and proving themselves, the lesson design probably gave the PSTs opportunities to increase their awareness of what it takes to enact interactions that elicit and promote students' reasoning and proving.

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The future of research on programming in mathematics education

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In many countries, programming is an explicit part of the mathematics curriculum, and the research field of programming in mathematics education has expanded in recent years. In this paper we use the Nordic mathematics education research community as a guide for new insights into how future research can deepen our understanding of integrating programming into mathematics education. We invited participants of the NORMA 24 conference to join a Working Group on the topic. In group discussions, the approximately 30 participants reflected on a case from a programming activity in a Norwegian primary grade mathematics lesson. Additionally, they reflected on challenges and possibilities associated with integrating programming in mathematics education independently of the case. The group discussions were analysed in successive steps, where the first inductive analysis and discussions among the authors framed the next step of the analysis. Three suggestions for future research were identified: (1) Reasons for integrating programming in mathematics education, (2) Designing tasks that integrate programming in mathematics education, and (3) Pedagogical practices for teaching programming in mathematics.

Keywords: Programming in school, mathematics education, research methods

Introduction and background

The integration of programming and mathematics offers challenges as well as possibilities. Some have argued that programming activities can foster mathematical competencies like reasoning and argumentation (Kaufmann & Stenseth, 2021) and shift classroom dynamics from teacher-led to student-centred learning (Bray & Tangney, 2017). However, the connection between mathematics and programming is not straightforward. Many studies focus primarily on programming or computational thinking, often neglecting specific mathematical topics, particularly in primary education (Hickmott et al., 2018; Holo et al., 2022). Furthermore, even if there is a potential for students to learn mathematics from programming activities, there are also studies that report negative results (Forsström & Kaufmann, 2018; Laurent et al., 2022). For instance, Zhong & Xia (2020) argue that using robotics in mathematics education can increase students' cognitive load and distract them from mathematical problem-solving as they focus on the novelty of robotics. Research integrating mathematics and programming often consists of small-scale case studies led by researchers, raising questions about their long-term impacts and effectiveness when implemented in typical classroom settings (Bray & Tangney, 2017).

While the integration of programming in mathematics poses significant challenges, programming has been incorporated into mathematics curricula in many countries over the past decade (Balanskat & Engelhardt, 2015), at times with poorly defined implementation strategies. One of the issues is the lack of clarity around the reasons for integrating programming into mathematics. This is, however, a discussion concerning school mathematics in general, and not only programming. “Reasons for teaching mathematics are typically not explicit, well defined and articulated, let alone agreed upon and stated in public” (Niss, 1996, p. 12). The practical implementation of programming is therefore mostly left to teachers and school resource providers. For teachers who are uncertain about programming, one may hope that they get support from textbooks, given that textbooks and curriculum material are found to be influential in changing teachers’ teaching in times of curricular change (Rezat et al., 2021). Unfortunately, this is not always the case. Nyman et al. (2024) finds a weak relationship between mathematics and programming in Swedish textbooks. However, the national curricula are different, and a comparative analysis of Swedish and Danish curriculum resources found substantial differences in the programming and computational thinking concepts and mathematical concepts of the tasks in the two countries (Elicer et al., 2023).

According to Kilhamn et al. (2021b), teachers faced several challenges: limited personal experience with programming, difficulties aligning programming tasks with mathematics, and concerns that the time required for programming could detract from core mathematics content. Additionally, teachers may have different reasons for integrating programming, and Kilhamn et al. (2021a) found that Swedish teachers’ arguments for implementing programming in mathematics education can be divided into four categories: (1) programming is a powerful tool, (2) it increases engagement, (3) it develops computational thinking, and (4) it provides learning of mathematics. Different reasons will favour different activities in the classrooms. It is important to recognice that programming is also new in teacher education programmes. Thus, it remains unclear whether the formal integration of programming into the school curriculum has effectively led to changes in mathematics classrooms.

A methodology for teaching programming that is often used in schools is PRIMM (predict, run, investigate, modify, make) (Sentance et al., 2019). The PRIMM method, which builds on the Use-modify-create model (Lee et al., 2011), is a guide for teachers on how they can structure their lessons. These models, however, are developed to teach programming, not necessarily mathematics. It is likely that these frameworks can be used in a mathematics class. There is, however, a need for further research targeting mathematics education specifically. Relatively minor actions taken by the teacher may shift the focus of the students to mathematics when working with programming tasks (Berge et al., 2024). However, teachers often lack the pedagogical and didactic competence to combine programming with mathematics (Kravik et al., 2022).

The discussion above clearly demonstrates the importance of exploring the potential of programming in mathematics education and determining how it can be effectively exploited to ensure lasting benefits. The Tenth Nordic Conference on Mathematics Education (NORMA 24) offered a chance to discuss these topics at a Working Group with researchers and teacher educators interested in programming and mathematics education research in the Nordic context. The topic of the Working Group was: How can research enhance our understanding of integrating programming into mathematics education? The data utilised in this paper comprise audio recordings of the group discussions and the digital boards from the Working Group. The research question for this study is: *What future research directions do Nordic researchers and teacher educators reflect on to deepen our understanding of integrating programming into mathematics education?*

Method

The aim of the method is to use the Nordic mathematics education research community as a guide for new insights into how future research can deepen our understanding of integrating programming into mathematics education. Approximately 30 researchers and teacher educators from Norway, Sweden, Denmark, Estonia, and Finland, who participated in the NORMA conference in June 2024, were voluntarily recruited to the Working Group through its listing in the conference programme.

The participants were first introduced to a case from a primary school in Norway. In this case, primary school students participated in mathematics instruction through a programming activity using Lego Spike robots. Students worked in pairs solving a task in four parts. Using a pre-built robot, the students were asked to programme the robot to stop at a known finish line. Secondly, they were asked to generalise their strategy and programme the robot to stop at an unknown distance. In the third part, they were given a programme including a loop and variables printed on paper, and were asked what the programme would do. In the last task, they were asked to modify the programme to measure the distance to an unknown finish line. The data included video recordings of students' activities and interactions, as well as screen captures of their coding on tablets. Two sequences (~4 minutes each) were shown to the Working Group participants. The first video sequence showed a pair of students working on the first task. The second sequence showed another group working on the fourth task. A transcript of the dialogues in the sequences was handed out.

After the presentation of the case, the participants in the Working Group were divided into five groups of 4-6 participants each. The group discussions were audio-recorded with the participants' informed consent. The recordings were automatically transcribed[1] and translated into English for the two groups who discussed in Scandinavian languages. First, the groups were asked to discuss what they found interesting about the case, also from a research perspective. In this process, the roles of the teacher, programming, and mathematics in the case were discussed. The groups were asked to share what they found interesting about the case on a digital board, and in a short plenary discussion. Subsequently, the discussions were guided towards future perspectives on research. The Working Group participants were asked to discuss and provide answers on the digital board to the following questions, independently from the mentioned case: (A) Compare and share your experiences, challenges, and possibilities, regarding programming in mathematics education. (B) What research questions and designs can advance our understanding of how programming should be integrated into mathematics education?

The group work concluded with short plenary presentations where the groups highlighted key points from their discussions. The authors engaged in the group discussions by monitoring one group each. They participated by answering questions and providing comments to guide the discussions towards the questions raised. The analysis process started here, and after the workshop, the authors met to discuss their impressions of content from the different group discussions and the concluding plenary

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[1] Automatic speech-to-text tool based on the Whisper service, a software that is based on AI models that run on local servers.

discussion. The analysis process consisted of successive steps where the first analysis and discussions framed the next step of analysis and discussions.

The analysis continued by one of the authors performing an inductive content analysis of the transcriptions from the group discussions, aiming to find new directions for research. The results of the analysis were discussed with all authors in several rounds and were found to correspond to the themes highlighted by the groups on the two digital boards. From this inductive and iterative process, there was a negotiation towards consensus on three initial themes for future research.

In the next step of the analysis, one author coded utterances concerning the integration of programming in mathematics education with descriptive sentences, inspired by Tjora's (2019) stepwise-deductive induction. For example, the utterance: "So I would be curious about hearing some of the justifications for actually doing this from a mathematics education point of view. What is it that we believe it can do for students learning?" was coded: "Want justifications for integrating programming from a mathematics education point of view." Utterances about the case were not coded unless they had a potential for insights about programming in mathematics education on a more general level. Some utterances were given separate codes to capture all aspects of the utterance. The codes were related to the three initial themes for future research. All codes fit well into the themes, and no new theme for future research was found. The results of the analysis were discussed with the other authors, resulting in only small adjustments in the coding.

Results

The three directions for future research are presented as headings, and the codes within these themes are presented in tables under each of the headings.

Theme 1: Reasons for integrating programming in mathematics education

On the digital board, Group 4 pointed out different aims for teaching programming in mathematics education:

> Digital board: - teaching students programming in mathEd [mathematics education]
> - teaching students mathematics through programming environments
> - teaching students about how technology such as AI works through mathematics

Group 4 further suggested that an understanding of the reasons for integrating programming in mathematics education can advance our understanding of how programming should be integrated:

> Digital board: What is our aim/goal of teaching programming in mathematics education? (…) The criteria of whether our approaches and efforts are successful highly depends on what we want to achieve.

As shown by the codes in Table 1, the groups discussed different aims and integrations of programming and their possible effects. Three of the groups highlighted the necessity for justifications from the perspective of mathematics education. For example, "What can it [programming] do for children's understanding of mathematics" (Gr2) and "I would be curious about hearing some of the justifications for actually doing this from a mathematics educator's point of view" (Gr4). The latter group further highlighted that the initiative for programming in mathematics has been imposed top-down into mathematics education. "It's more like the industry demands new sets

Table 1: Codes within Theme 1

Code	Count
Mathematics/computer science, different disciplines hard to balance	7
Examples where programming is mathematics	6
Want justifications from a mathematics education point of view	5
It is difficult to learn programming and mathematics at the same time	3
Teachers lack experience	3
Commercial influence through programming materials in school	2
Instrumental genesis is important to learn mathematics	2
Opinion that programming is mathematics	2
Reasons for not integrating programming	2
Research designs to study the integration of programming in mathematics	2
Justifications for integrating programming differ by countries	2
Different reasons to integrate programming in mathematics	1
Justifications are not considering what benefits mathematics education	1
Reasons for taking programming in and out of the curriculum	1
The 1990s failed attempt at integrating programming in school	1
The reason for integrating programming is important for the assessment	1
Identify overlaps between programming and school mathematics	1
Total	43

of skills" (Gr4). Two of the groups mentioned the role of the private sector that sells educational materials for programming, while two groups mentioned the teachers' (in some countries) lack of experience: "They don't know how to programme, and then they don't know how to connect programming with mathematics" (Gr5). The potential and pitfalls of integrating the coding culture in school mathematics were highlighted by Group 2 as a potential research question. They wrote on the digital board:

Digital board: There is an interesting and complex culture clash between maker/coding thinking and school mathematics. There is potential in relating these cultures, but also huge potential for mistakes.

Group 4 suggested that one of the reasons to integrate programming in mathematics education is to learn programming. Three of the groups mentioned that the teaching of programming and mathematics should not be done at the same time, for example: "It would be nice for the teacher and student if we could teach this [programming] in motivating situations where you don't at the same time work on a mathematical aim" (Gr2). Two of the groups mentioned instrumental genesis related to programming in general, where we must gain "the understanding of how this new programme or technology works before we use it in the mathematics context" (Gr3). Four of the groups discussed, based on their experiences, that programming/computational thinking and mathematics seem to be too different and are therefore difficult to balance, either with the result that "we set some things together that don't fit" (Gr2), or they do not seem to "connect with an equal balancing" (Gr4). Two of the groups mentioned reasons for not teaching mathematics through programming. "It's not helping us do anything better or faster" (Gr3), and according to Group 4 traditional teachers claim that in school mathematics "everything works on pen and paper and we are not going to change anything" (Gr4).

Two of the groups discussed that programming *is* mathematics and did not make a clear distinction between them. “I think we need to understand what kind of mathematics it is” (Gr5). The other group compared it to how “some would argue statistics is not mathematics. Still, it is in the mathematics curriculum and nobody kind of questions it nowadays” (Gr4). The latter group did, however, also discuss reasons for fading out programming from the mathematics curriculum.

Three of the groups discussed the potential for teaching mathematics through programming. This was listed on the digital board as one of the reasons to integrate programming in mathematics education. Two of them mentioned potential topics, like “in geometry you can actually learn lots about mathematics through programming” (Gr1), and “in probability theory and statistics it’s very useful, and you can build a few lines of code to do the simulations that you could not do in practice” (Gr3). The latter group summarised that “[t]here are some elements [mathematical content areas] that perhaps are closer related to programming rather than others. The issue is to find out where it’s most meaningful to integrate it.” The third group focused on how mathematicians use programming: “You have both instances where programming is the specific and analysis is the general, and where analysis is the specific and programming is the general. […] In many contexts in school mathematics, I believe it could work really well” (Gr5).

Theme 2: Designing tasks that integrate programming in mathematics

The participants discussed task design in considerable detail by emphasising the importance of tasks and task goals, suggesting closer cooperation between researchers and teachers, and highlighting a more active role of teachers in research processes and lesson studies. Some participants argued that awareness of task goals is essential, and a suggestion presented by Group 1 on the digital board was to have

> Digital board: […] a combined focus on task design and learning potential with programming. It would be nice to have tried planning a best possible test, where the researchers do their best planning how to use a digital tool, what kind of tasks to use in order to take advantage of the potential that lies in the digital tool.

This group also discussed potential research questions, such as “which opportunities and challenges could be identified in activities including robotics and other digital tools?” The group discussed

Table 2: Codes within Theme 2

Code	Count
Explore tasks to learn both programming and mathematics	7
Involve teachers in designing classroom activities	5
Design tasks to be applied by teachers	5
Explore best practice integration of programming for learning mathematics	4
Find out how we can meaningfully integrate programming in mathematics education	3
Pros and cons of using robots and coding toys to learn mathematics	2
Task design principles for students to learn mathematics	2
Commercial programming materials not well suited for learning mathematics	1
Identify factors that make a task unsuccessful	1
Pre-service teachers are not understanding tasks due to syntax	1
Student response when working on tasks on a screen	1
Study students’ learning in mathematics with and without programming	1
Total	33

the consequences of not involving the teacher in the design process, and argued that it is harder to implement an activity that is designed by someone else. It was pointed out that "[t]he researchers should collaborate with teachers to develop activities that foster learning of mathematics using the equipment which is available at their schools." Group 3 also discussed the design process, but from the perspective of teacher educators. It was pointed out that we need research which provides guidelines for the designing process, to help teacher educators assist (pre-service) teachers in implementing programming in mathematics, and to point out the potential of programming connected to the learning of mathematics: "There are some elements [mathematical content areas] that perhaps are closer related to programming rather than others. The issue is to find out where it's most meaningful to integrate it" (Gr3) and the research should assist the (pre-service) teachers in identifying these topics.

Group 5 discussed the different mathematical content areas included in programming activities. They suggested observing classroom activities to explore "the connection between mathematics and programming, or how strong it is, or if it's there or if it's not." Group 4 discussed the hidden mathematics when working with programming and robotics, that the students and teachers need help to discover and how to grasp mathematical concepts through programming. It was pointed out how challenging it is to find or create good activities, and argued that this is what needs to be identified through research: "How do we develop suitable activities?" This group also raised the question: "What are the characteristics of a design [of a programming activity] which provides learning of mathematics?" (Gr2).

The research design to explore best practices in integrating programming for learning mathematics was also discussed:

> Group 2: If the aim [of the research] is to find out, or to identify good strategies and activities I believe an iterative process where the teacher is involved in analysing, watching through the video recordings together with the researchers discussing the activities.

The importance of well-designed tasks that integrate programming and mathematics is emphasised, and that this is challenging for teachers:

> Group 1: I'm thinking about designs that can advance our understanding. I think teachers are really in need of good designs that can work, that they can use with their students where one avoids many of the bugs [...] That they get some help and access to some good designs that can simply help them.

Regarding research design, several participants from different groups called for closer collaboration between researchers and teachers:

> Group 2: If it is design-based research, then it is one of the functions where there should be collaboration between the teacher and the researcher, and it should be done in several cycles, trying out and making small changes.

Theme 3: Pedagogical practices for teaching programming in mathematics

The case included two videos showing individual groups of students working on different tasks, with the teacher circulating between groups. When working on the case, three of the groups highlighted the role of the teacher on the digital board.

Table 3: Codes within Theme 3

Code	Count
Useful pedagogical methods for learning programming and mathematics (e.g. PRIMM)	4
It is important to accommodate students' reasoning	2
It is important to mathematise the students' experiences	2
Teachers' role in learning mathematics from programming activities	1
Total	9

Digital board: We discussed the potentials for mathematics teaching in this situation. In the situation we follow very much the students - but we are actually interested in the teacher because it is he that can make these activities into mathematics. So how are the mathematical aspects of the activities domesticated [sic]?

Digital board: […] students only to a limited extent draw on and engage with mathematics in their approach to solving the task. We wonder whether a more mathematical framing of the task could have changed this.

Digital board: Taking the teacher perspective could be interesting. The teacher played a quite active role. The students' work is characterised by being a bit searching, trial and error. (Gr3)

When we asked the participants to discuss what they found interesting about the case, the groups showed interest in the role of the teacher, how it affects how the students work with the task, and how they relate the programming of the robot to the mathematical content.

As shown in Table 3, four different codes are found in the discussion on research questions and designs. One of the codes is the importance of mathematising experiences, which includes this: "But it doesn't help much if the students don't get the opportunity to see the mathematics that is involved. Perhaps an important component of this is that the teacher may point it out" (Gr1). In addition, two other groups mentioned the importance of providing questions for reasoning: "If you then pose the question, why is this? I would say then you are probably on the way to the mathematics. And then you have to look at the code, you have to look at how does this code correspond to the definition of a circle, let's say" (Gr5).

Pedagogical practices were not explicitly written on the digital board, however, one of the groups highlighted a research design where the teacher participates in developing content and pedagogical practices in teaching activities:

Digital board: Working together with the teacher(s) in an iterative design process. Particularly discussing what the aim is and if the lesson/task achieves this aim or not.

One of the participants in Group 1, who studied students' learning in programming activities that should facilitate mathematics learning, also suggested that research on the mathematics teacher's role in such activities is needed. Two of the groups also discussed how pedagogical frameworks like PRIMM (predict, run, investigate, modify, make) are useful for supporting students' thinking and mathematising, for example: "Read the code and analyse it… students should talk about it, and you discuss it in class" (Gr3).

Discussion

The focus of this study was to identify research questions and designs to improve the integration of programming in mathematics education. The groups' suggestions for future research align with recent literature reviews on programming and computational thinking in mathematics education (Holo et al., 2022; Forsström & Kaufmann, 2018). The participants highlighted similar challenges that teachers face, as found by Kilhamn et al. (2021a), such as limited programming experience, difficulty connecting programming activities to mathematics, and concerns that programming can detract attention from the mathematical content. They suggested that research should actively involve teachers to enhance the relevance and applicability of programming in mathematics. To build on this idea, and based on earlier studies, we propose three types of studies that should directly involve teachers in this work:

1. Including teachers in clarifying the goals of programming in mathematics education

Programming has been integrated into the mathematics curriculum in several countries, making it the teachers' responsibility to teach it, develop relevant tasks, and understand how and when to integrate it effectively. This is challenging, as most teachers lack a background in programming or knowledge about its integration. Most groups pointed out the pivotal role of the teacher in the classroom, suggesting that they should therefore be central in connecting programming activities and mathematics.

Participants noted a lack of clear goals for programming integration in mathematics curricula. As Kilhamn et al. (2021a) found, teachers often struggle to understand programming's purpose in mathematics education. These discussions reflect the need, as stated by Niss (1996), for explicit reasoning behind curricular changes, as unclear purposes can leave teachers uncertain about how to effectively implement and align new content areas like programming within established subjects. Future research could engage teachers in discussions that clarify these goals, linking programming objectives with broader mathematical thinking skills and computational goals. Integration goals, such as using programming to deepen understanding of specific mathematical concepts, apply computational thinking to problem-solving, and create mathematical models, could offer practical ways for teachers to see how programming can be linked with other goals in the mathematics curriculum.

2. Task design and evaluation of materials together with teachers

Given the significant role that textbooks and other instructional materials play in shaping teachers' practices (Rezat et al., 2021), conducting in-depth analyses of these materials with teachers would be valuable. Kilhamn et al. (2021b) developed a framework for analysing tasks based on the relationship between programming and mathematics, which should guide future analyses of textbooks and other resources to foster a more integrated approach. Comparative studies of curriculum resources across countries, such as the work by Elicer et al. (2023), highlight differences in programming integration, suggesting the need for cross-national research to identify best practices and develop resources that better align programming tasks with mathematical goals.

The participants emphasised the need for research focused on task design in cooperation with teachers, refining tasks to integrate programming in mathematics education. Tasks that focus on different aspects of integrating programming in mathematics, and what mathematics is in the era of programming, should be explored. Future studies could, for example, explore how programming can provide real-world applications, and connect abstract mathematical concepts with tangible outcomes.

Previous studies suggest that programming activities, including those involving robots, have the potential to bridge abstract mathematics with practical applications, thereby enriching students' understanding and engagement (Zhong & Xia, 2020). This approach highlights the importance of rethinking the purpose of integrating programming in mathematics education to fully leverage its educational benefits.

3. Developing pedagogical practices and guidelines together with teachers

While previous research has highlighted several potentials of integrating programming in mathematics education, it remains unclear how to effectively utilise these potentials in the classroom. The groups discussed this issue, emphasising the need to identify best practices. It was suggested that researchers should engage in studies with teachers to collaboratively plan and identify best practices. This approach can encompass both task design and pedagogical practices. By adopting a design-research methodology, researchers and teachers can develop, test, and refine effective strategies for integrating programming into mathematics education. Holo et al. (2022) also advocated for additional research conducted in collaboration with teachers, following their review of empirical studies on programming in mathematics. Participants emphasised the need for adaptable strategies and instructional supports that enable teachers to integrate programming in ways that connect naturally with mathematics content. PRIMM may be one such strategy (Sentance et al., 2019). It should, however, be adapted to the mathematical content. The integration of programming reflects a transformed role for teachers, balancing guidance in programming skills with the reinforcement of mathematical objectives. Smaller adaptations and teacher actions may change this balance (Berge et al., 2024). Collaborative research with teachers was highlighted as key to developing practical approaches that clarify when and how programming can effectively support mathematical understanding.

Based on this study and earlier research, the integration of programming into mathematics education is not without challenges, and teachers often face significant pressures as they navigate these complexities (Kravik et al., 2022). Teachers should not be left to handle these demands alone. A proposed idea is that teachers should be more involved in developing tasks and teaching practices, as well as in formulating future research questions based on the challenges they encounter in their classrooms. In this context, the lesson study approach is emphasised as a method where teachers can collectively discuss and formulate research questions relevant to their practices. Through this approach, the teachers can observe, learn, and provide feedback on each other's teaching practices, and collaboratively develop new methods and programming activities that are suitable for mathematics classrooms.

Concluding remarks

In summary, the discussions held at the NORMA 24 Working Group have pinpointed three primary directions for future research. Firstly, we see a need to explore and investigate the various reasons for integrating programming in mathematics education. A critical concern was the perceived gap between mathematics and programming, an issue addressed by the majority of the groups. We acknowledge the challenges in quantifying if, and to what extent, the inclusion of programming in mathematics impacts student learning. The complexity of effect measurements is intensified by the fact that certain mathematical content areas may benefit more than others. The justification for teaching programming may also be linked to other potential positive outcomes. In our technologically advanced society, where computers and algorithms are ubiquitous, possessing programming skills is deemed essential, even if this does not directly enhance mathematical learning. Secondly, many participants expressed the need for the development of appropriate learning activities explicitly designed for the integration of programming in mathematics. There was a call among the participants for close collaboration between teachers and researchers during this design and development process. Lastly, there should be an increased emphasis on exploring various pedagogical practices for teaching programming in mathematics. Although a range of practices has been proposed in different studies, further experimentation and testing are required to better understand how and when the different practices should be utilised.

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Comparative Analysis of Computational Thinking in Mathematics Education: A Nordic Discussion

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This working group report examines the integration of Computational Thinking (CT) in mathematics education across Nordic countries, highlighting variations due to national policies and educational structures. We explain the approaches in Denmark, Sweden, Norway, Finland, Iceland, and Estonia and discuss how to balance mathematical learning with CT. Furthermore, we propose to use a praxeological framework to scaffold the comparison. The findings underscore the need for further research to support successful CT integration in mathematics curricula across diverse educational contexts.

Computational Thinking in the school systems

The concept of computational thinking (CT) has gained prominence in mathematics education research and curriculum policies in recent years (Bocconi et al. 2022). There are many definitions of CT that each highlight different aspects, but a common denominator is that they refer to the thought processes involved in formulating problems so their solutions can be represented as computational steps and algorithms (see e.g. Wing, 2006). Integrating CT into mathematics classrooms is a valuable approach to enhancing learning experiences (Misfeldt & Ejsing-Duun 2015). However, the manner of this integration and its implications vary across countries. This discussion group sought to explore these variations, specifically focusing on the Nordic countries.

A significant challenge in comparing CT integration across countries is the influence of diverse national school policies. These policies, sometimes opaque or inconsistent, affect how CT is implemented and understood in various educational settings (Tamborg, 2022; Bocconi et al. 2022). Furthermore, while the foundations of CT are based on both mathematics and computer science, there is an important question regarding student perception. How do students interpret these concepts, especially when they might not differentiate between the two disciplines? This can lead to challenges such as terminological conflations, since terms like 'problem-solving' sometimes hold different meanings and implications in mathematics compared to computer sciences or other situations where certain symbols, while seemingly similar, might not align across disciplines (such as the % which refers percentage in mathematics and modulo operator in computer science) (Bråting & Kilhamn, 2021).

In the discussion group, participants examined the relationship between how CT was presented in curriculum policies and its representation in educational research. The goal was to understand the challenges and opportunities this presented for the effective integration of CT in mathematics education. Participants were invited to contribute with their insights and experiences as they collectively navigated the complexities of CT in the global educational landscape. In particular they were encouraged to bring short presentations on local situations in different school systems. These presentations, along with inputs from the participants, were used to discuss questions about how and why integration between programming and mathematics was done in primary and secondary schools in different countries in the Nordic region.

Workshop scope

The workshop began with a round of presentations on the local situations regarding CT and mathematics in primary and lower secondary schools. Following this, participants discussed several focal points for comparison, drawing on investigations and frameworks such as those by Tamborg et al. (2023) and Helenius and Misfeldt (2021). Among other things, these focal points regarded what content from the two disciplines to connect and in what interdisciplinary configuration. Based on these inputs, the group collectively addressed questions about what it means to integrate CT into mathematics teaching, how this integration varies across different Nordic school systems, and how factors like national culture and specific implementation decisions influenced the outcomes of integrating CT into mathematics. This was done through a combination of small group work and plenary discussion.

The situations in Sweden and Denmark

The initial presentations showed the different routes that Denmark and Sweden had taken to the integration of digital competence and technology comprehension (TC).

Although there is potential for meaningful interdisciplinary connections between mathematics and CT, evidence of successful integration remains limited. While many theoretical frameworks suggest how integration should work (e.g. Weintrop et. Al. 2016; Kallia et al. 2021), there are few concrete examples where it demonstrably does. A key challenge lies in conceptualizing mathematics and CT as complementary disciplines, striking a balance between their integration, interaction, and respective weights in the curriculum. Although CT is not explicitly mentioned in the Swedish curriculum and only is a sub-component of the much wider Danish Technology Comprehension (TC) curriculum, the curricula revisions in the two contexts address content related to CT.

Insights from a Danish pilot project on Technology Comprehension (TC) (2018-2021), implemented within existing subjects, reveal significant challenges. In Denmark, TC was implemented in a pilot project that tested TC asa subject in its own right and as integrated in existing subject, here among mathematics (Børne- og Undervisningsministeriet, 2018). The integration into mathematics happened by adding new parallel TC competencies, knowledge and skills to the existing mathematical content . Two-thirds of participating teachers found it necessary to override existing subject content to incorporate TC (Børne- og undervisningsministeriet, 2020). There was a clear demand for authentic teaching materials rooted in real-world contexts, though, notably, several schools applied to extend the pilot, signalling sustained interest. The current direction for TC in Denmark involves embedding it within existing subjects, yet without clear guidelines on what, how, or why this should happen. This situation underscores the need to actively listen to and address the experiences and requests of schools, fostering a proactive, rather than reactive, approach that moves beyond awaiting policy decisions. Furthermore, research on issues such as techno-critical mathematics education (Jankvist et al. 2023) highlights the importance of embedding TC authentically within mathematics education, enriching students' understanding of both disciplines in a real-world context.

In the 2022 Swedish curriculum revision concerned the broader concept of digital competence, which all subjects were revised to address, but which implied different modification in the indidual subjects. In mathematics, programming became embedded within the core content of algebra (Skolverket, 2018). This begins as early as Grades 1-3, where students learn about constructing, describing, and following unambiguous step-by-step instructions as a foundation for programming, including the use of symbols. By Grades 4-6, students advance to creating and applying algorithms in visual programming environments, while Grades 7-9 introduce algorithm development, testing, and refinement in both visual and text-based programming environments.

The Swedish national curriculum of 2018, closely mirrored by the 2022 version, left much open to interpretation and required teachers to define its practical meaning. In essence, terms like "stepwise instructions" and "algorithms" were introduced, alongside requirements for using various programming environments. However, teachers across the country found themselves struggling — what exactly were they supposed to teach, and what were students supposed to learn? With limited guidance and many teachers lacking a background in programming, this posed significant challenges.

The other Nordic Countries

Countries participating in the discussion	Programming included in K-9?	Programming: Disciplinary affinity and relation to specific topics
Denmark	Tested since 2018. Forward (from 2025) it is an elective in grade 8. And a topic to be dealt with in other topics.	Several disciplines including mathematic, and as its own topic.
Sweden	Yes from 2018	Related to mathematics and algebra
Norway	Yes from 2020	Related to mathematics and problem solving
Finland	Yes	Related to mathematics and later an individual topic
Estonia	Programming is offered as an elective	No relation to mathematics – moving away from ICT in mathematics education towards paper and pencil math.
Iceland	No real decisions but a lot of stakeholders are pushing to make programming part of the curriculum.	Both related to mathematics and as an independent topic.

Table 1: An overview of the different countries approaches to programming in relation to school mathematics

Addressing programming through the lens of praxeology

When comparing and discussing the different ways that programming has been introduced into schools and especially into mathematics education, we adopted the notion of praxeology. A praxeology is a way of expressing expertise in terms of both knowing about and area and knowing what to do. It is described by Chevallard (2019), is composed of a *praxis block* (know-how) and a *logos block* (know-why). For programming in mathematics, the praxis block could include practical skills like data handling, statistics, proofs, modelling, sequencing, and coding in digital environments, which encompasses tasks such as tinkering, remixing, and algorithmic techniques. The logos block, on the other hand, refers to understanding the reasoning behind these tasks—why specific techniques are used for particular tasks, and why programming relies on logically constructed, unambiguous steps to produce reliable and repeatable outcomes.

One of the problems that we see all over the Nordic countries is that in the context of programming in the curriculum, this praxeology is not directly and thoroughly provided to teachers (Helenius & Misfeldt, 2021). Consequently, as programming was transposed into the school setting, teachers needed to adapt the knowledge of both praxis (techniques and practical tasks) and logos (the underlying purpose and reasoning). However, without a clear epistemological reference model to guide this adaptation, teachers were left with significant interpretive work, needing to translate and contextualize programming knowledge for mathematics education independently.

Also, the praxeological lens allow us to see how different ambitions concerning the integration between programming and mathematics can look. Some teachers see programming as a natural part of mathematics. Others see programming as related to digital literacy and not really related to mathematics (Kilhamn & Rolandsson, 2021).

Concluding discussion

In conclusion, integrating programming as a tool for learning mathematics is a complex endeavor with both potentials and significant challenges. While the idea of using programming to enrich mathematical understanding is appealing, practical implementation often sees mathematics itself sidelined in favor of computational skills. Teaching programming within mathematics requires substantial instructional time, a high level of teacher expertise, and a nuanced understanding of what students learn through such activities. Providing students with pre-made programs to tinker with, rather than starting from scratch, may offer a more feasible approach to introducing computational thinking (CT) without overwhelming the mathematical focus. But this suggestion is rather speculative and more research is needed tied specifically to the educational systems and choices made in the different countries (Helenius & Misfeldt 2021).

From a CT perspective, initial experiments with programming in mathematics have shown promise. However, translating these experiments into genuine mathematical learning remains elusive. Effective integration calls for teaching approaches rooted in tinkering and exploration rather than beginning with coding from a blank slate. Designing tasks that genuinely support mathematical learning through programming is possible but challenging, often requiring specialized expertise that teachers alone may not have.

Reflecting on these challenges, the vision of using programming to teach mathematics is more complex in practice than in theory. While engaging for students, programming within mathematics raises unresolved questions about its core purpose and the boundaries of mathematical learning. Stakeholders across educational and policy spectrums continue to debate the role of programming in mathematics, prompting ongoing discussions about what constitutes the essence of mathematics education and whether it should include programming.

As this field evolves, there is a clear need for empirical research to understand better the difficulties encountered and successes achieved in various contexts. Shared insights and evaluations from different countries would provide valuable guidance for future efforts.

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What is equal participation in mathematics education research?

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This paper reports on a working group that focused on what equal participation in mathematics education research could be. Through methodological imagination, a group of researchers created images of equal participation – images that through the collaborative work became a story that conveyed insights about, for example, power dynamics between participants, challenges and opportunities the group saw in creating equal participation, what we as researchers in this field need to consider about what equal participation in mathematics education research means. We conclude by telling other stories, alternatives to those created in the workshop, and discuss different approaches to equal participation in mathematics education research.

Keywords: Equal participation, Participatory research, Critical mathematics education.

Introduction

This paper reports and builds on a workshop that aimed to explore what equal participation in mathematics education research could be. We invited colleagues who participated in NORMA24 to discuss and share their experiences of equal participation in collaborations between researchers and teachers/prospective teachers. Our intention with the workshop activity was to work and think together about equal participation; to explore what it could be and how it can be enacted in research projects. We engaged our colleagues on the question of how to create opportunities for teachers/prospective teachers and researchers to meet on equal terms, where questions, initiatives, negotiations, and resistance are welcomed in research projects. In this report, we problematise threats/challenges/trouble in collaborations between researchers and teachers/prospective teachers, and the responsibilities of all participants.

The workshop built on two previous studies that explored how participation and relationships between researchers and participants in research are associated with ethical challenges (Skog, 2024; Nordin et al., under review). An ethical challenge concerns who it is taking the initiative to conduct the research. Most likely, a researcher approaches the teacher/prospective teacher for collaboration because it is probably the researchers who have a particular interest. A dilemma then arises concerning participants' roles and the power distribution in the project. Especially in projects where equal participation is sought or explicitly articulated. Power is understood as discursive, relational, and multi-layered, and occurs within restricted systems or rituals that determine what different individuals can say and do (Foucault, 1971). What is judged as equal in research projects could hence be articulated differently depending on the different divisions of power among the participants. When the researcher has the initiative, the teacher can be empowered to participate on equal terms; however, this may not always be possible or perhaps not even desirable (e.g., Nordin et al., under review). Teachers need to negotiate what the time needs to be spent on. They may consequently want to participate, but not fully engage, which could also be seen as equal participation. If teachers are to initiate research cooperation, they need an arena in which to meet researchers and explore possibilities for research. However, inviting teachers/prospective teachers to take initiative sometimes challenges the research and generates disturbing dilemmas. Skog (2014) describes a situation where a prospective teacher questions the intention of her project. An ethical position was then for her to include this questioning as an important part of the study. Consciously including participants' questions and initiatives and allowing studies to be formed through mutual trust and consideration between researchers and participants enables mutual vulnerability in the research (Skog, 2024).

Intentions to enable equal participation in scientific studies where researchers and teachers/prospective teachers cooperate can have different foci. Sometimes, equal power is sought in some or all parts of the process, such as in the articulation of the research problem, in the data generation, in the coding and analysis, or in the communication of results. Skog (2024) problematises the intention to take full advantage of participants' initiatives. Other times, the intention is to give participants' power and opportunity to engage in the design of the study in an equal manner and to take initiative. In Nordin et al. (under review), the participating teacher resists the expectation that teachers can be engaged and take initiative, and the cooperation takes other forms. In the same way, ethical awareness in the research process requires reflexivity in the endeavour of analysing and interpreting the data (Hammersley & Atkinson, 2007). We argue that reflexivity is important when collaborating with teachers/prospective teachers because feelings of power imbalance can raise questions about transparency and trust among the people involved (Skog, 2014). We therefore propose alternative forms of interpreting and understanding mathematics education research practices, which means that we engage with critical awareness and visibility through the whole process (e.g., Valero, 2009). For instance, in critical mathematics education where equal participation is the goal, a partnership between teachers and researchers is desirable (e.g., Atweh, 2004; Murata, 2011). Even where challenges are explored (e.g., Schuck, 2013; Vithal, 2004), many academic papers present as success stories with no problematisation of challenges (e.g., Andersson & Valero, 2016; Eriksson et al., 2021; Hedefalk et al., 2024); although there are exceptions (e.g., Eriksson 2018). When, as researchers in the field of mathematics education, we refrain from telling and publishing other stories, where equality in participation is problematic, or even impossible, we may miss opportunities to imagine the nuances of what equality in research projects might mean. We may also miss different solutions to how collaborations between teachers and researchers could be designed differently to promote participation. In Nordin et al. (under review), a model for the process of such a research design is suggested. The model (see Figure 1) is a modification of Skovsmose and Borba's (2004) model of the process of designing a mathematics lesson. The model can be used by participants in a research collaboration to talk about and design the nature of their cooperation. But, in the same way as a lesson activity rarely comes out exactly as planned, collaboration also has both an imagined and an arranged version of the outcome and these are never identical. The model below offers a language for this process where the discrepancy between the arranged and the imagined situation is included.

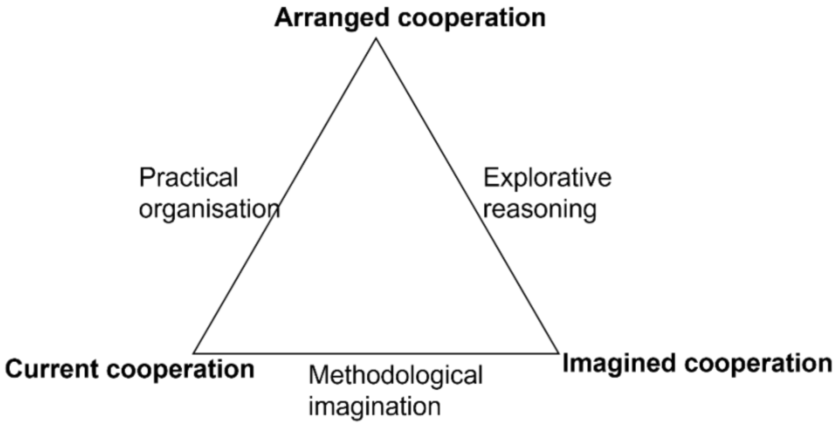

Figure 1: Model of the process of designing cooperation in research (Nordin et al., under review).

Ethical/methodological concerns in research might reveal power dynamics that bring new challenges to the fore. Involving experienced researchers in dialogues about research and data generation offers opportunities for imagination grounded in real experiences and ideas about how research processes can evolve. Following the model from Nordin et al. (under review), we engaged our colleagues in a methodological imagination of what equal participation could be like. The aim of the working group, and this paper, was consequently not to solve a specific problem related to equal participation in research collaborations. The aim was instead to explore what could characterise equality between participants in research collaborations.

The workshop

The working group was arranged as a workshop where the participants were invited to share experiences and discuss how to create opportunities for research collaboration on equal terms. We offered visual artefacts, inspired by Godfrey (2020), to enable participation through improvisation and creativity. These artefacts – coloured circular pieces of paper for the participants to write/draw on, and post-it notes – provided visual representations of the outcome of the discussions as we urged/invited the participants to prioritise what they found to be most important. The orders of importance they suggested allowed us as initiators to ask more questions, and to invite them to argue for why something was considered to have more weight than something else. The visual representations, together with participants' notes form the basis of this report.

There were 10 workshop participants, including the researchers, all active mathematics teacher education researchers in the Nordic countries. Because we aimed to report on the outcome of the workshop by drawing on the data we had collaboratively produced, we needed to ask the participants for informed consent. They received information about the data we wanted to produce with them and about their right to withdraw their participation at any time. They were also assured of their right to remain anonymous. All participants in the workshop chose to sign the consent. There were no records from the conference of who participated in the working group because each session was open to everyone.

The working group was organised into two groups. The participants were given prompts to engage with through cycles of reflection and methodological imagination. After a short introduction to the research problem, based on the introduction of this report, we invited the participants to think of their own experiences of trying to achieve equal participation (including both successes and failures). Before talking to each other, we asked that we reflect on ethical dilemmas and experiences of equal participation from our own research practice, with the question: "What are your experiences from research or developmental projects that you have been involved in?" After the individual reflection, the participants shared their dilemmas in groups. In this cycle they were asked to, collaboratively but based on individual experiences, write on post-it notes the challenges and opportunities associated with aiming for equal participation. Thereafter, in plenary, we shared our challenges and opportunities. We then created a joint visual representation by placing the post-it notes on the wall in groups because we found some similar themes (see example in Figure 2).

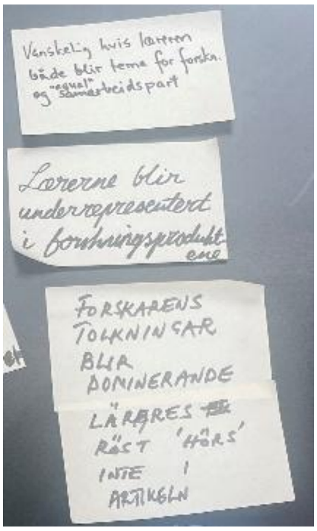

Figure 2. Post-it notes, grouped, because everyone said something about the risk of participating teachers/prospective teachers being under-represented and the researcher being dominant in research collaboration.

The next cycle was based on the discussed dilemmas, and the groups were asked to build further on the visual representation that was about how to create opportunities for equal participation and the opportunities they saw to do so. It was important that we created imaginations so that questions, initiatives, negotiations, and resistance from participating teachers/prospective teachers were seen as resources in a project. To illustrate these conversations, the groups were asked to write down five aspects that, from their experience, were important for promoting equality in teacher/prospective teacher-researcher collaboration. The five aspects were to be ranked with the most important in the largest circle (see example in Figure 3).

Figure 3. Important aspects of aiming for equal participation ranked with the most important in the largest circle (From left to right: Everyone is a participant, common goals, respect, trust, equally distributed resources).

In the next cycle, we created a new joint visual representation with all circles while thinking together about how various aspects could be related to each other, and under what circumstances. To close, each participant got a few minutes to think about their own research in relation to the methodological imaginations we created during the workshop.

Creating the story

The process of creating a story as methodological imagination (Nordin et al., under review) that holds the elements of ethical awareness and reflexivity (Hammersley & Atkinson, 2007) and critical awareness and visibility (Valero, 2009), requires closeness to the empiric, but also trust and creativity because we have to fully rely on what emerges from conversations among colleagues. We invited everyone to share their thoughts, oral or written, in the workshop. As initiators striving for full transparency, we carefully communicated our reflections during the whole process which began in the initial planning of the workshop and will continue beyond the submission of the final version of this report. Closeness to the empiric implies an interest in what the individual participants expressed concerning certain aspects, and carefully adhering to the collaboratively produced data in writing up the story. Trust amongst us in the small workshop community was thus a condition for creativity in conversations and in creating our common visual representations.

Closeness to the empirical, trust, and creativity are also core aspects of the writing this report. We took as our point of departure single notes from participants, common pictures that we created while recalling challenges and opportunities, and the coloured circles that showed what the participants chose to prioritise. The visual representations and the process in the conversations, together with our reflections over time, made it possible to take both micro- and macro perspectives on the research context, and are hence what we used as building blocks for the story of thinking together as colleagues. The story we tell in the following section is not a blueprint of the sequence of events. Rather, it is an attempt to give a sense of the conversations that we as initiators had (sometimes presented as quotes), both about the context we provided, and how we thought about equal participation while analysing the material produced during the workshop.

Thinking together about equal participation

While in the process of exploring how to report on the workshop and how to make sense of the process of thinking together about equal participation, we met colleagues who took part in the workshop and who expressed interest in updates about the process. Their experiences from the workshop seemed to have made an impression on them, and their responses indicated that we had communicated our aim clearly: To invite everyone in the workshop to share with us how they imagine situations of equal participation in research. Our colleagues' curiosity also indicated that they felt listened to as participants in the specific situation, and still do, three months after the workshop. The engagement in research-related issues was one of the reasons we wanted to further explore what we could learn together with experienced and curious/generous colleagues. Another reason was the endeavour of collaboratively identifying core elements of what it means to elaborate on transparent, open, and equitable research settings. It is not possible to attain this in solitude. Cooperation requires peers to engage/involve/create together. In inviting colleagues to think together with us, we are allowed to share ideas and experiences, to rethink how we 'do' research, and to challenge common taken-for-granted truths. Below are the immediate reactions of two of the working group organisers:

- I am happy that all the participants engaged in the questions and wanted to participate. I wonder what would have been the result if they all (or if some of them) had passively sat and waited for us to articulate how we thought about this. Then, the whole approach would have failed.
- Yes, so interesting that they all wanted to share. Yes, some talked more than others, but they all took the opportunity to think about their own experiences of challenging data generation.

We found ourselves in the process of working out how to communicate the outcome of the workshop. Being transparent had to involve sharing as much information as possible from the workshop while simultaneously presenting what we saw, both holistically and in detail. Several questions arose from these experiences, which affected how we reported back about the workshop: What does it mean to initiate cooperative imagination about equal participation in research? How were the conversations influenced by the group members' previous experiences? Whose experiences count in such a setting? Can we rely on everyone having a voice, even in this specific context?

In the workshop, we asked the participants to share their notes on challenges. From these notes we could see how awareness of power relations was articulated. What was prevalent, were notes about being aware of the different roles in a teacher/researcher collaboration where the teacher becomes both an object of the research and a co-operator. In this situation, the teacher might take the role of student, while the researcher becomes the teacher, even though that is not the intention of any of the parts of the research. The power dynamics hence become predetermined and in the long run these roles may lead to the researcher's interpretations being dominant, communicating the research with a loud and clear voice, while the teacher's interpretation and voice become silenced. This may occur even though, given the context, it is the teacher who is the knowledgeable party. The attempt to articulate a common object of study can be challenging because researchers and teachers/prospective teachers might view the 'same' thing very differently. The questions we asked ourselves and discussed in the workshop were: Is it a problem if researchers and teachers view equal participation differently?

Is equal participation something to strive for or should we just accept that one can experience equal participation even if ‘objectively’ the parties do not have equal status? Is equal participation, per definition, the 'same' kind of participation or can a sense of having equal power be the crux? What if we look at opportunities to empower all participants in a research context?

There are definitely connections between challenges and opportunities because different roles and experiences could nurture conversations around a common object of research. The insider-outsider perspective, where the teacher is more knowledgeable in a certain context, enables nuanced reflections around the different roles. The opportunity to create non-traditional reports on research also comes with non-traditional research approaches where participants' responsibilities have equal weight. These approaches can generate both challenges and opportunities. So, to raise awareness about where we see the 'problem' arise and who 'owns' it is key.

- So how can we understand this?

We placed the circles and post-it notes on the table at the centre of our attention. Two obvious arrangements arose, and we had to take these as our points of departure: One group ordered the circles foregrounding the following aspects: *Object, awareness of different experiences, development of collaborative partnerships, material conditions, humour and curiosity*. The other group created the following order of priority: *Everyone is a participant, common goals, respect, trust, resources equally distributed*. We began to move the circles around on the table.

- There must be connections. What if we make a thematisation? If we do, then the highly ranked *object* and *common goals* connect, and on an equal level we find *everyone is a participant* and *awareness of different experiences*. *Respect and development of collaborative partnerships* then connect with *trust*. *Material conditions* and *equally distributed resources* are important for both groups, but not the highest priority. *Humour and curiosity* are left [clearly] isolated.

Figure 4. Thematic arrangement of priorities of what to consider when aiming for equal participation.

The next step was to explore how the challenges and opportunities we identified at the beginning of the workshop related to the priorities illustrated by the circles. Hence the question: Are there connections between what is important for promoting equality in teacher/researcher collaboration and challenges/opportunities in this collaboration? Figure 5 shows one attempt to organise these connections. Aspects that involved the people in research clearly revealed both challenges and opportunities and brought power relations to the fore. Respect, trust, and developing collaborative partnership might hence involve a stronger and louder researcher's voice and them taking responsibility for the interpretations, while teachers become unequal participants – merely a means for research.

Figure 5. Thematic arrangement of priorities in connection with challenges and opportunities.

It was interesting that the aspect that got the least attention – humour and curiosity – was covered extensively in the conversation. This aspect, as we see it, enables creative and non-traditional forms of communication in research and has the potential to contribute to strengthening participation in the field. Arranging the different aspects on one board allowed us to make visible the opportunities for seeing teachers/prospective teachers and researchers from an insider-outsider perspective, having different experiences and roles. And these are strong reasons for building equal research collaborations.

We so much wanted to continue the discussion with our colleagues and have a follow-up workshop where we could develop and deepen our imagined methodologies and write them up together. But this was not possible; so, in the following section, we present some alternatives to what we have sketched above.

Discussion

We thematised the visual representations and our story is based on these. However, other thematisations are possible and it would be possible to tell other stories. As authors of this report, we have the privilege to choose, and we have aimed for transparency and inclusion of different perspectives through sharing the material that was produced during the workshop. We all agreed on the connections we created with the visuals, and differences of opinion were more like fluctuations on the surface than deep waves in our methodological imagination. What was missing in our conversation was an alternative story. We therefore decided to take the opportunity in this discussion to tell other stories. Stories based on the challenges described in the workshop, and sometimes opposite to what was privileged in the workshop as good research practice.

The first story we want to tell concerns the roles involved in conducting research with teachers/prospective teachers. The researcher is here seen as the one who knows how teaching should be conducted and the teacher has the opportunity to learn from the expert, as a student taught by a teacher. This is an alternative story, contrary to our methodological imaginings. We wonder if research has been conducted that looks at such power relations involved where the participants experience the situation as equal. In the workshop, teachers'/prospective teachers' participation was foregrounded as privileged and desirable and we interpret that as a situation to aim for. One possible scenario of a seemingly unequal situation might show that teachers desire something other than that which researchers imagine, for instance, to learn from experts or to participate when time allows. In any case, the researchers need to be attentive and responsive to the participating teachers' contributions, so that their stories are given equal weight in the research. One possible way for participants to be attentive to each other could be to imagine the collaboration together. Such imaginations could, by extension, create space for equal participation, meaning that different participants are allowed to choose their roles and participate in ways that their personality, workload, and competence allow, while still being regarded as equal participants. Experiencing equality in research is hence negotiable and is appropriately interpreted by the people involved.

The second story about conducting research with teachers allows us to imagine a context characterised as a serious endeavour that should follow rigorous methods. Here, there is no room for imagination and creativity. We need to be certain about what we can claim, without doubts and questions. This is an alternative story to our methodological imagining where curiosity, humour, and collaborative imagination were central. Researching with humour and curiosity could be seen as harmful to the seriousness of research. On the other hand, it could also be seen as a necessity for the active engagement of all participants – for the 'glow' and 'glue' in a collaboration.

The participants in the workshop were diverse in age, experience, and background; all active researchers in the Nordic countries. The opportunity to imagine together in the workshop – not knowing, but imagining – brought our dreams and ideals as well as our fears to the table. We think that not having to be responsible for the validity and reliability of what we imagined may also have facilitated a more honest discussion. In imagining, we could meet and any ideas were welcomed. In a more rigorous setting, some of us might have been silenced merely by the perceived seriousness of research.

Finally, we argue that communication about methodological imagination, as we staged it in this workshop, is not another example of a success story (as exemplified in Andersson & Valero, 2016; Eriksson et al., 2021; Hedefalk et al., 2024); instead, it is a contribution to developing research collaboration that is transparent and formed through mutual trust, and that enables mutual vulnerability in the research (Skog, 2024). We hope to inspire colleagues to engage in new forms of collaboration, to imagine novel methodologies, and to communicate research creatively, inspired by curiosity and a sense of humour.

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