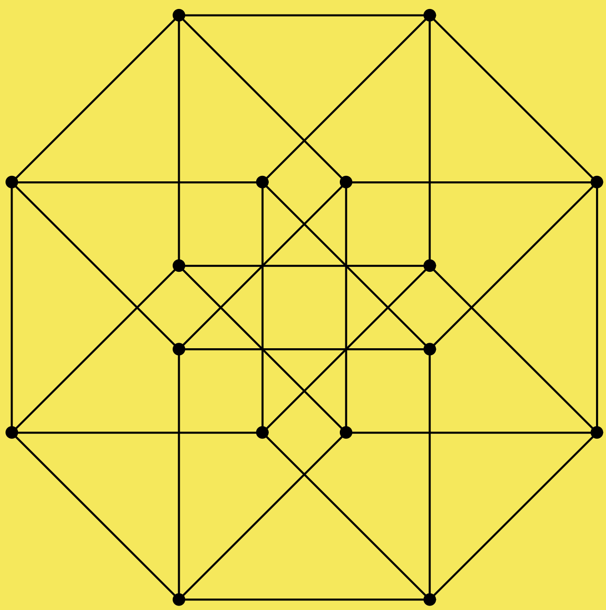
norma 01

- Conceptions of Mathematics -



Edited by

Christer Bergsten *and* **Barbro Grevholm**

Skrifter från *SMDF*, Nr 4
Linköping 2005

*Proceedings of **norma 01***
Third Nordic Conference on Mathematics Education
Kristianstad, Sweden, June 8-12, 2001

norma 01

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The Third Nordic Conference on Mathematics Education – *Norma 01* – was held June 8-12 2001 in Kristianstad, Sweden, with 85 participants from 16 countries of 5 continents. The conference was hosted by Högskolan Kristianstad. Previous conferences in the series were *Norma 94* in Lahti (Finland) and *Norma 98* in Kristiansand (Norway). *Norma 05* will take place in Trondheim (Norway) in September 2005.

This volume, edited by Christer Bergsten and Barbro Grevholm, contains most of the papers presented at *Norma 01*, including workshops and posters.

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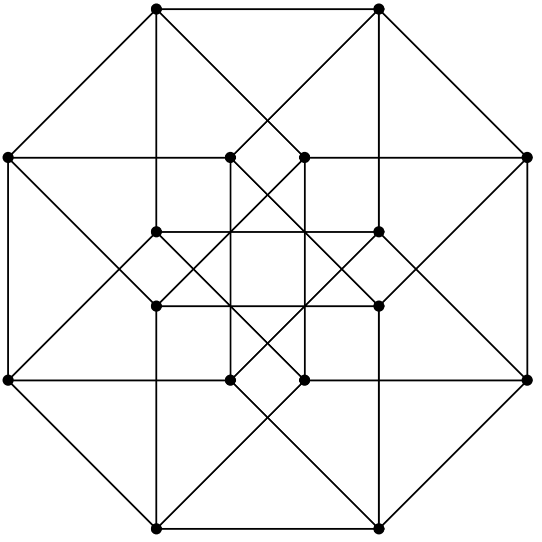

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Preface

The Third Nordic Conference on Mathematics Education – *Norma 01* – was held June 8-12 2001 in Kristianstad, Sweden, with 85 participants from 16 countries of 5 continents. Previous conferences in the series were *Norma 94* in Lahti (Finland) and *Norma 98* in Kristiansand (Norway). *Norma 05* will take place in Trondheim (Norway) in September 2005. This volume contains most of the papers presented at *Norma 01*, including workshops and posters.

The theme of the conference – *Conceptions of Mathematics* – was to be interpreted to include ideas and images that students, teachers, researchers, and society have on the nature, concepts and processes, and teaching and learning of mathematics at all levels. These ideas and images permeate activities, engagement, attitudes, and knowledge and skill development of all individuals and groups engaged in mathematics education, i.e. students, teachers, researchers, and policy makers. Therefore it is important to understand the nature of and relationship between such conceptions and different aspects of the theory and practice of mathematics education. This was also reflected in many of the contributions at the conference, and presented in this volume.

We could feel that the participants did enjoy the early June summer days in Kristianstad, deeply engaged in deep and highly professional and supportive discussions in a friendly and relaxed atmosphere. We want to thank all who contributed to *Norma 01*, by plenary, paper, workshop, and poster presentations, and all who shared their ideas and took part in the discussions. Thanks also to Högskolan Kristianstad (Kristianstad University) for hosting the conference and to *SMDF*, the Swedish Society for Research in Mathematics Education, for publishing this book of proceedings in its book series. A big thanks also to the local organisers for their careful preparation, to the city of Kristianstad for giving a most generous reception, and Gleerups förlag for supplying conference material. And finally – on behalf of the *Norma 01* conference – we express our gratitude for the economic support from Riksbankens Jubileumsfond (The Bank of Sweden Tercentenary Foundation) and from Högskolan Kristianstad that made this conference possible.

Linköping and Lund, July 2005

Christer Bergsten, Barbro Grevholm

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Content

Introducing Pupils to Proof: Is it Needed? Is it Possible?

Maria Alessandra Mariotti
Università di Pisa

Abstract

This paper aims to contribute to the debate on proof that has been conducted in the last years within the community of mathematics educators. I will discuss both the importance of proof, and more generally of theoretical thinking, and the possibility of its introduction in the mathematical curriculum.

A long-term research project, aiming to introduce pupils to proof, has been carried out, involving a number of 9th-10th grade classes. Some results drawn from this project will be presented and some examples discussed, to support general hypotheses concerning the mediating function of particular micro-worlds, in respect to the introduction to theoretical thinking.

Introduction

In the last decade, mathematical proof has been at the core of an active debate in the community of mathematics educators: often claimed to be responsible of pupils' difficulties, but also recognised as a crucial aspect of mathematics activity.

In the recent past the role and the place that proof takes in the mathematical curriculum have often changed. For instance, in the United States, after a period of 'banishment' proof has got a central position in the new Standards (Knuth, 2000).

To give an example, consider the following excerpts drawn form the last version of the Principles and standards for school mathematics published in year 2000.

> Reasoning and Proof as fundamental aspects of mathematics. Reasoning and Proof are not special activities reserved for special times or special topics in the curriculum but should be a natural, ongoing part of classroom discussions, no matter what topic is being studied." (NCTM, 2000, p. 342)

> By the end of the secondary school, student should be able to understand and produce mathematical proofs – arguments consisting of logically rigorous deductions of conclusions from hypotheses – and should appreciate the value of such arguments. […] Reasoning mathe-

> matically is a habit of mind, and like all habits, it must be developed through consistent use of many contexts (NCTM, 2000, p. 56)

Nevertheless, certainly the idea of "proof for all" is not a view most teachers hold, and even where there is a longstanding tradition of including proof in the curriculum (for instance in my country, Italy), the main difficulties encountered have lead many teachers to abandon this practice.

Thus the new trend, in order to become accepted and introduced in school practice must be supported by a deep discussion clarifying two main points:

- why proof is so crucial in the mathematics culture to be worth to be included in school curriculum;
- how it is possible to overcome the difficulties so often described.

An historic and epistemological analysis highlights the role of proof in the evolution and systematisation of mathematics knowledge throughout the centuries.

Mathematics cannot be reduced to theoretical systems, but certainly its theoretical nature constitutes a fundamental component, as clearly expressed by Hilbert and Cohn Vossen in the introduction to their book "Intuitive geometry".

> In mathematics … we find two tendencies present. On the one hand, the tendency towards abstraction seeks to crystallise the logical relations in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency towards intuitive understanding foster a more immediate grasp of the objects, a live rapport with them, so to speak, which stress the correct meaning of their relations. (Hilbert & Cohn – Vossen, 1932)

A twofold nature characterises mathematics: On the one hand intuitive understanding and on the other hand a systematic order within logical relations.

Thus, the theoretical perspective is a crucial aspect in mathematics and from the educational point of view it seems difficult to ignore it. The following question arises: If mathematics is such a complex activity how much is it possible to elude this complexity without loosing its sense?

The theoretical component of mathematics

The theoretical perspective of mathematics has old roots, leading us to the ancient book of Euclid's Elements. That particular way of presenting the 'corpus' of knowledge, the deductive way which characterised since then math exposition and more generally the scientific discourse. Heath in his edition of the Euclid's Elements reports the following passage from Proclo.

> Now it is difficult, in each science, both to select and arrange in due order the elements from which all the rest proceeds, and into which all the rest is resolved. (…) In all these ways Euclid 's system of elements will be found to be superior to the rest. (Heath, 1956, vol. I, pp. 115-116)

The crucial point seems to be the suitable order in which a set of known properties may be expressed and communicated. The problem of the transmission of knowledge was solved by Euclid in a very peculiar way: rather than in terms of “revelation”, the elements have been transmitted according to “logic arguments”.

And one of the salient features of the structure of Euclid’s Elements is certainly its logical structure, the deductive structure, although this definitely differs from the “deductive structure of logic derivation within a formal theory” (Rav, 1999, p. 29). The main difference consists by the fact that, contrary to what happens within a formal theory, in a mathematical proof deduction depends on understanding and on prior assimilation of the meaning of the concepts from which certain properties are to follow logically. It is in this sense that Euclid’s work has to be interpreted: the elements are to be considered as means of fostering understanding of the whole geometry.

The style of rationality introduced by Euclid has become a prototype for all science, and the power of this method may be related to its treatment of truth. A twofold criterion of truth characterises the structure of Euclid’s Elements: evidence, on which principles are based, and consistency, on which the truth of derived knowledge is based. As expressed by Lakatos:

> a truth-injection at the top (a finite conjunction of axioms) – so that truth, flowing down from the top through the safe truth-preserving channels of valid inferences, inundates the whole system. (Lakatos, 1967, p. 33)

In this framework a deep unity relates organisation of knowledge and understanding, making organisation functional to understanding, which become strictly tied to the constraints of acceptability and validation within a scientific community.

A crucial point that I like to stress is that of the twofold aim: on the one hand the need of understanding on the other hand the need of validity, i.e. to be accepted by a community. These two aspects seem to be recognised as characteristic of a theoretical corpus and can be found in most of the discussions about the nature and the function of proof. Let us take for instance the following quotation from a classic paper by Hanna.

Mathematicians accept a new theorem only when some combination of the following holds:

1) They understand the theorem (that is, the concepts embodied in it, its logical antecedents, and its implications) and there is nothing to suggest it is not true;
2) The theorem is significant enough to have implications in one or more branches of mathematics, and thus to warrant a detailed study and analysis;
3) The theorem is consistent with the body of accepted results;

4) The author has an unimpeachable reputation as an expert in the subject of the theorem;
5) There is a convincing mathematical argument for it, rigorous or otherwise, of a type they have encountered before.

(Hanna, 1989, pp. 21-22)

Despite the crucial change of perspective which has characterised the beginning of the last century and led the mathematicians to a radical revision of the idea of truth, the relationship between understanding and acceptability of mathematical statements, has not dramatically changed in the centuries, and still constitutes a characterising element of this discipline.

The slow elaboration of the idea of rigour which had its climax at the end of the XIX century, has a counterpart in the development of the relationship, more and more complex, between two fundamental moments of the production of mathematical knowledge: the formulation of a conjecture, as the core of the production of knowledge, and the systematisation of such knowledge within a theoretical corpus.

This leads one to recognise a deep continuity between the development of knowledge and its systematisation within a theoretical “corpus”, between aspects relevant in the communication process, such as the need of understanding and aspects related to the fact that knowledge is a shared cultural product, such as acceptability.

Different approaches to the problem of proof

Certainly the relationship between the semantic and the theoretical perspective raises the issue of understanding, and in particular the issue of the relationship between proof as an hypothetical-deductive argument and its explaining function. Different opinions are possible, according to the relevance acknowledged to the distance between the semantic level, where the truth (the epistemic value) of a statement is fundamental and the theoretical level, where only the logical validity of an argument is concerned.

If the logical dependence of a statement in respect to axioms and theorems of the theory is considered independent from the epistemic value that one attributes to the propositions in play, the distance may be thought of as a cognitive rupture (Duval, 1991). On the contrary if the independence is refused so is the rupture and the peculiarity of theoretical arguments.

These two points of view are not necessarily opposite (Balacheff, 1987), they simply focus on different crucial aspects of proof: the function of theoretical validation and the function of explanation, strictly related to understanding.

The issue of understanding arises as far as it refers to the links between the meanings involved both in the statement and the arguments. On the one hand, these links may not necessarily be expressed through the structure of logic consequence; on the other hand, when required, it is impossible to formulate and prove the logic link between two statements without any reference to meanings.

In other terms, in spite of the fact that, at the theoretical level, the epistemic value has no relevance it is impossible to conceive a practice of mathematical proof without any reference to the semantic level. From both an epistemological and a cognitive point of view, is seems impossible to make a clear separation between the semantic and the theoretical level, as required by a pure formal perspective on mathematics.

> to expose, or to find, a proof people certainly argue, in various ways, discursive or pictorial, possibly resorting to rhetorical expedients, with all the resources of conversation, but with a special aim ... that of letting the interlocutor see a certain pattern, a series of links connecting chunks of knowledge. (Lolli, 1999)

The explanation function of proof is fundamental, because it provides the support needed for understanding, but this function depends on the semantics of the statements and the truth values given to them. As a consequence, in spite of the fact of its theoretical autonomy, from the cognitive point of view, proof is strictly related to semantic.

In this perspective, interesting studies have been carried out, in the field of mathematics education, aiming to clarify the relationship between the process of producing a conjecture and the proof provided. The notion of *cognitive unity* (Garuti et al. 1996; Mariotti et al., 1997) has been introduced for this purpose; interesting results have already been obtained. Evidence has been found of argumentative processes appearing in the solution of open-ended problems that require both a conjecture and its proof. Thus, producing theorems, that is formulating a conjecture in a statement and producing a proof for this statement, has been suggested as a powerful approach to the idea of proof (Boero et al., 1996).

Nevertheless, as Duval reminds us, the rupture between the two levels (the semantic and the theoretical) may be irretrievable, and the conception of proof as a process of which the basic aim is to affect the epistemic value of a statement, may become a main obstacle to produce and accept a mathematical proof, i.e. an argument consistent with the rules of acceptability of the community of mathematicians[1].

The main point seems to be how to manage this rupture and the consequent possible conflict between the two functions of proof, achieving a flexible thinking moving from one function to the other, from validating to explaining and vice versa.

In their practice mathematicians prove what they call “true” statement, but “truth” is meant relatively to certain theory; it is a *relative truth*, drawn from the hypothetical truth of the stated axioms and from accepting the fact that the

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[1] Only by entering the game of deduction within a theory is it possible to overcome the difficulty of making sense of a proof of an obvious statement.

rules of inference “transform truth into truth”. However, relativity of truth is fundamental: axioms are neither absolute nor factual truths.

> A statement B can be a theorem only relatively to some theory; it is senseless to say that it is a theorem (or a truth) in itself: even a proposition like '2+2=4' is a theorem in a theory A (e.g. some fragment of arithmetic). (Arzarello, 2000)

It seems not spontaneous to reach such a perspective of relative truth, which on the contrary becomes automatic and unconscious for the expert; in this sense, this way of thinking may be considered an educational goal.

However, the complexity of this goal should not be ignored. In fact, the confusion between the two functions (validation and explication) may have serious consequences.

> Ce serait à mes yeux une erreur de caractère épistémologique que de laisser croire aux élèves, par quelque effet jourdain, qu’ils seraient capables de production de preuves mathématiques quand ils n’auraient qu’argumenté. (Balacheff, 1999)

Thus, taking into account the complex relationship between explaining and validating, education cannot ignore the theoretical perspective.

The following sections will be devoted to discuss a proposal concerning the possibility of introducing pupils to theoretical thinking.

A teaching experiment

In the last years a long term research project has been carried out, consisting of a sequence of teaching experiments. A number of experimental classes have been involved, each of them for at least two years, with the constraint that the experiment had to be included in the regular curriculum at the upper secondary school level (9th and 10th grades). The content of the Geometry curriculum was not upset, but the general approach changed dramatically through the use of the software Cabri-géomètre; the software was integrated in the classroom activity, not only as a didactic support, but as an essential part of the teaching/learning process.

The experiment is in the line of "research for innovation" (Bartolini Bussi, 1994), i.e. the experience in the classroom is both a means and a result of the evolution of the research study. That means that, as long as the project developed, different issues arose all of them related to the general research aim: to investigate the feasibility of a teaching approach to theoretical thinking.

Characteristics of the experiment

Classroom activities were organized within the Field of Experience (Boero & al., 1995, p. 153) of Geometrical constructions within the microworld Cabri; the evolution of the field of experience was realized through the social activities of the class, and the core was constituted by *mathematical discussions* (Bartolini Bussi, 1996, 1998), aimed at social construction of knowledge.

According to a Vygotskian perspective, our main hypothesis was that of using the different tools, provided by Cabri as instruments of semiotic mediation (Vygotsky, 1978).

Geometry theory, as imbedded in the Cabri microworld, may be evoked by the observable phenomena and the commands available in the Cabri menu. Figures and commands may be thought of as external signs of Geometry theory, and as such they may become instruments of semiotic mediation as long as they can be used by the teacher in the concrete realization of classroom activity and according to the motive of introducing pupils to theoretical thinking (Mariotti, in press c).

The construction task

The basic element of our experimental project is the idea of construction as a geometrical (theoretical) problem; the core of all the activities proposed to the pupils is the "construction task".

Since antiquity geometrical constructions have had a fundamental theoretical importance (Heath, 1956, p. 124) clearly illustrated by the history of the classic impossible problems, which so much puzzled the Greek geometers.

Actually, the theoretical meaning of geometrical constructions, i. e. the relationship between a geometrical construction and the theorem which validates it, is very complex and certainly, not immediate for students, as clearly described and discussed by Schoenfeld (1985). As the author clearly explains in that case “many of the counterproductive behaviors we see in students are learned as unintended by-products of their mathematics instruction” (p. 374). It seems that the very nature of the construction problem makes it difficult to take a theoretical perspective, as shown in a completely different school context (Mariotti, 1996).

In spite of the long tradition, geometrical constructions have lost their position in the geometry curriculum; in Italy, for instance construction problems have completely disappeared and they can only be found as a topic within Technical Drawing at the junior school level.

The appearance of Dynamic Geometry Softwares has renewed the interest for constructions and the basic role played by construction has been brought on the scene by the instrumental approach related to the use of graphic tools.

In particular, Cabri-géomètre offers a microworld which embodies Euclidean geometry, referring to the classic world of "ruler and compass" constructions. In fact, any Cabri-figure is the result of a construction process, i.e. it is obtained after the repeated use of tools, chosen among those available in the “tool bar”; moreover, the effect of most of the Cabri tools corresponds to the effect of the classic geometric tools, i.e. ruler and compass: a Cabri-figure is obtained intersecting lines, lines and circles, constructing perpendicular or parallel lines or the like.

But the main characteristic which makes Cabri so interesting is the fact that there is the possibility of direct manipulation of its figures and that this manipulation is conceived in terms of the logic system of Euclidean geometry. Cabri-figures possess an intrinsic logic that is the logic of their construction; the elements of a figure are related in a hierarchy of relationships, corresponding to the procedure of construction.

But there is something more. The Cabri environment introduces a specific criterion of validation for the solution of the construction problems: a solution is valid if and only if the figure on the screen is stable under the dragging test.

Thus, the dynamic system of Cabri-figures embodies a system of relationships consistent in the broad system of a geometrical theory; in other terms solving construction problems means accepting not only all the facilities of the software, but also accepting a logic system within which to make sense of them.

For all these reasons, the idea of *geometrical construction within the Cabri environment* may be considered a *key to accessing* theoretical thinking.

In fact, within the Cabri environment the teacher can find specific tools of semiotic mediation to be used in the discussion in order to guide the evolution of personal senses towards the theoretical meaning of a construction.

Pupils' development of a theoretical perspective is based on the process of semiotic mediation, accomplished by the teacher through the use of specific elements of the software.

Semiotic Mediation

As reminded above, Geometry theory (elementary Euclidean geometry) is imbedded in the Cabri microworld and evoked by the observable phenomena and the commands available in the Cabri menu.

According to the Vygotskian theory, figures and commands may be thought of as external signs of Geometry theory, and as such they may become *instruments of semiotic mediation* (Vygotsky, 1978) as long as they can be used by the teacher in the concrete realization of classroom activity and according to the motive of introducing pupils to theoretical thinking (Mariotti, in press a, b).

The sequence of activities developed in a structured manner, where activities within the microworld (construction tasks) alternated with activities of collective discussions where, under the guidance of the teacher pupils constructed a parallel between the world of Cabri constructions and Geometry as a theoretical system. Let us describe how this parallel is conceived.

The construction of a Theory

In Mariotti et al (1997) we referred to "Mathematical Theorem" as the unity of a statement, a proof and a theory of reference. In fact, when a deductive system is concerned, there are two interwoven aspects: the idea of proof and the idea of theoretical system (both local and global theorization may be considered). These two aspects correspond to two levels of difficulties: on the one hand, the idea of validation must be introduced, on the other hand the rules of validation must be stated. The acceptance of validation depends on the acceptance of rules and of the meaning of these rules.

Starting from the basic relationship between Cabri constructions and geometrical theorems we guided pupils to enter the geometrical system, through the solution of construction tasks.

Taking advantage of the flexibility of the Cabri microworld we adapted the menu to our educational objective. At the beginning, an empty menu is presented and the choice of commands discussed, according to specific statements selected as axioms; in this way a double process is started, concerning on the one hand the enlargement of the Cabri menu and on the other hand the enlargement of the theoretical system. New constructions are achieved in the microworld and, in parallel, the corresponding theorems are added to the theory: new elements are introduced by theorems and definitions, new commands are introduced in the menu.

Table 1 contains a short summary of the first part of the sequence, based on the parallel development of Geometry Theory and Cabri environment. On one side one finds the new elements enclosed in the Cabri menu, on the other side one finds the theoretical elements discussed and enclosed in the shared theoretical system.

In this way a main difficulty can be overcome. In fact, when the whole Cabri menu is used, the whole Euclidean geometry is available, thus the complexity of the theoretical system may become too high to be dominated by novices. Because of the richness of the 'geometrical tools' available, it is difficult to state what is given (axioms or 'old' theorems) and what must be proved ('new' theorems). The richness of the environment may emphasize the ambiguity about intuitive facts and theorems and may constitute an obstacle to the choice of correct elements of the deductive chain of a proof. In other terms, there is the risk that pupils will not be able to control the relationship between what is given and what is to be deduced.

On the contrary, in our sequence the system is built up, step by step, slowly increasing its complexity, a complexity which can be manageable by pupils and directly checked through the availability of the tools. In this sense the instrumental aspect of the axioms and the theorems may be experienced by pupils, contributing to constructing the meaning of proof.

Moreover, we aimed to make pupils take part in the construction of a deductive system. Instead of proposing an already-made Euclidean axiomatisation, pupils were directly involved in the construction of both the Cabri menu and the corresponding geometry system. According to our hypothesis, participating in that process is fundamental for the evolution of the meaning of theory.[2]

[2] See also (Mariotti & Fischbein, 1997) about the case of definitions.

Table 1 The parallel between Geometry and Cabri tools

GEOMETRY THEORY	CABRI MENU
• Definition of the primitive elements Criteria of congruence in terms of construction and in the classic way	• Intersection of two objects • (Creation) line, circle, segment, … • (Construction) Report of Segment
• Transport of an angle (Theorem)	• Report of Angle
• Construction of the angle bisector (Theorem)	• (Construction) Angle Bisector
• Def. Perpendicular line as angle bisector of a straight angle • Construction of the Perpendicular line from a point to a given line • $P \in r$ • $P \notin r$ (Theorem)	• (Construction) Perpendicular line
• Definition of parallel lines • Construction of a parallel line to a given line (Theorem) • Axiom of parallel lines $r \parallel s \Leftrightarrow$ internal alternate angles are equal.	• (Construction) Parallel line

Note: What has to be included in the theory and what has to be added to the Cabri menu is always discussed collectively.

A very particular element of the class activity is constituted by the production of a personal notebook; each pupil has his/her own notebook, where all the new achievements in the theory are reported and commented: axioms and definition, new constructions and related theorems, all this material is systematically reported. The teacher periodically revises these notebooks, but most of the time this revision is made as a social activity, where pupils compare their texts and comment on them. This constitutes a fundamental part of the teaching – learning activity. Results coming from the analysis of these reports are discussed in Mariotti (in press b).

Collective discussions

As described above the second basic element of our experiment is the practice of mathematical discussions. Among the classroom activities, collective discussions play an essential part in the teaching/learning process, with specific aims: cognitive (construction of knowledge) and meta-cognitive (construction of attitudes towards learning mathematics).

In the literature, the role of discussion in the introduction of the idea of proof has been analyzed and the difference between an argumentation and a proof has been clearly described.

> ...il y avait une très grande distance cognitive entre le fonctionnement d'un raisonnement qui est centré sur les valeurs épistémique liées à la compréhension du contenu des propositions et le fonctionnement d'un raisonnement centré sur les valeurs épistémique lieées au statut théorique des propositions. (Duval, 1992-93)

Our proposal refers to a specific type of discussion, 'Mathematical Discussion' defined as "a polyphony of articulated voices on a mathematical object" (Bartolini Bussi, 1996).

Mathematical Discussion is neither a simple comparison of different points of view, nor a simple contrast between arguments, its main characteristic is the cognitive dialectics between different personal senses and the general meaning (the terms are used according to Leont'ev, 1981), which is constructed and promoted by the teacher.

In this case, the cognitive dialectics concerns the sense of justification and the general meaning of mathematical proof. The motive of the discussion activity concerns the evolution of the meaning of justification, related to the problem of construction.

The role played by the teacher is fundamental in every mathematical discussion. In this case, the discussion is developed in a special context, Cabri constructions. As discussed above, Cabri offers specific tools of semiotic mediation which can be used by the teacher according to the particular educational goal. Consequently in addition to the standard strategies that are used by the teacher to manage discussions in a whichever context, we have strategies that are specific for the Cabri environment.

An example, concerning the History command and its potential as an instrument of semiotic mediation is fully discussed in (Mariotti & Bartolini Bussi, 1998). The History tool is used by the teacher in order to orientate collective discussion towards the intended mathematical meaning of proof.

When a construction task is considered, taking a theoretical perspective means to interpret the question about the correctness of the drawing, as referred to the geometrical figure that it represents. Actually, it is very difficult, and often impossible, to directly access this theoretical meaning of the construction task; the main point consists of the shift of focus from the drawing to the procedure which produced it.

The specificity of the Cabri environment makes sense of the question about the procedure, rather than about the drawing produced; so that the mediation of the Cabri environment makes this meaning accessible. In particular, the History command, as it represents an external sign of the construction procedure .

When a given construction is under discussion, the teacher shows the history in the "master computer" and recurs to two games:

1) the interpretation game, lead by questioning which could have been the intention or the goal of the author in making such construction; for instance the teacher can ask: why did the authors choose this operation? what is the use for it?
2) the prediction game, lead by questioning which could have been the following step in this construction; for instance the teacher can ask how would you go on from this point?

Both games are possible because the History command provides a decontextualised, depersonalised and detemporalised copy of the construction procedure, which becomes the object of the discourse.

The evolution of the sense of justification

This section is devoted to discuss a few examples illustrating pupils' solution to construction tasks. We aim to give an idea of the evolution of pupils' solution as far as the theoretical perspective is concerned.

The construction of the angle bisector

Let us start with describing the context in which pupils work. As already said, any task is referred to a specific set of tools available in the Cabri environment and to a specific set of properties available in the Geometry theory (see Table 1). At this point of the teaching sequence, besides the primitives of the creation menu, in the construction menu the commands are reduced to include only two commands: the Report of Segment and the Report of Angle. From the theoretical point of view, this situation corresponds to the three criteria of congruence for the triangles, that pupils already have included in their theoretical system and that they know are the only principles they can refer to in their justifications. The following task is presented to the pupils.

Construct the bisector of an angle. Describe and geometrically justify your solution.

This is one of the first construction problems proposed to the pupils; they are grouped in pairs at the computer and they are asked to provide a joint text for the solution.

Students do not experience great difficulties in finding the construction procedure, but carrying out the procedure is only the first step of the task. Difficulties arise when the procedure must be described and justified according to the accepted rules, i. e. referring to the principles stated in the theory.

The following protocols show some examples of solutions that can be expected. At the beginning, not everybody has successfully entered the theoretical perspective; although a general acceptance of the validation by dragging is present, not everybody clearly relates the construction to the Theory available.

Alex and Gio (9th grade)

1° Attempt: We took two points and we made a line pass through them,

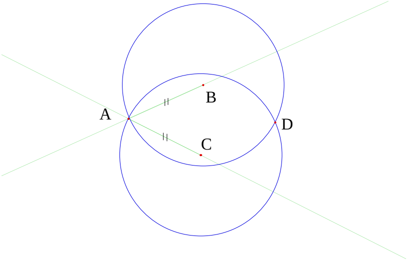

Figure 1a. 1° attempt

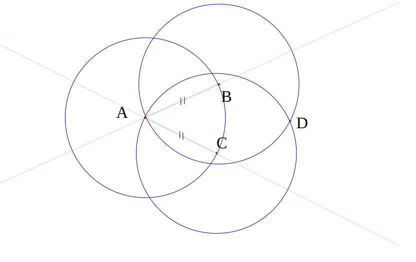

Figure 1b. 2° Attempt

then we took another point C, which does not belong to the first line.
We joined the point which doesn't belong to r1 with a second line, in so doing we determined an angle.
We transferred (ital. *abbiamo riportato*) a segment AB, belonging to r2 and we transferred the same segment on r1 (AB=AC); we drew two circles (centre, point) centre in C and point A and centre in B and point A (puntando in C e apertura AC e puntando in B con apertura AB).
We joined A and D (line through two points). We took the intersection between the circle and the line, but FAILED!

2° attempt: We drew an angle as we did in the first attempt. We drew a circle (centre/point), taking a point belonging to r1.
This circle gave us the segments AB and AC belonging to r1 and r2, which are equal because rays of the same circle. We drew two circles (centre B and C point A) using the intersection of two objects (of the two circles) we found the point D that we joined with A determining the angle bisector.

The solution presents two successive attempts. In the first attempt, something happens in the realization, which leads the construction to fail; in fact, the pupils made the two segments equal "by eye". After the first failure pupils start a new attempt. It is interesting to remark that the first attempt is considered a failure because it does not pass the dragging test; that means that the dragging test has been accepted as means of validation. In the second part, the text of the description is more accurate, as if, after the first failure, the pupils had felt the need of being more attentive; at the same time, a first rudimentary trace of a justification appears.

> This circle gave us two segments AB and CD belonging to r1 and r2, which are equal because they are rays of the same circle.

Although this sentence, just inserted within the description of the procedure, cannot be considered a "proof", it witnesses that the pupils have accepted the request of a justification; this sentence may be considered the germ of a proof, in fact it shows that pupils entered the game of construction, in fact they try to use the validating principles coming from the command used.

The following example shows a more developed solution. The protocol does not present any description of the procedure, but there is a sketch of the Cabri-figure, drawn with ruler and compass.

> *Lorenzo (9th grade)*
> I consider the triangles ABD and ACD. They have the side AD in common and the side AB of the first is equal to the side AC of the second. In fact, if I take the circle with the centre in A and point B, it passes through both B and C. Thus, the sides AB and AC are equal because they can be considered as rays of a circle.

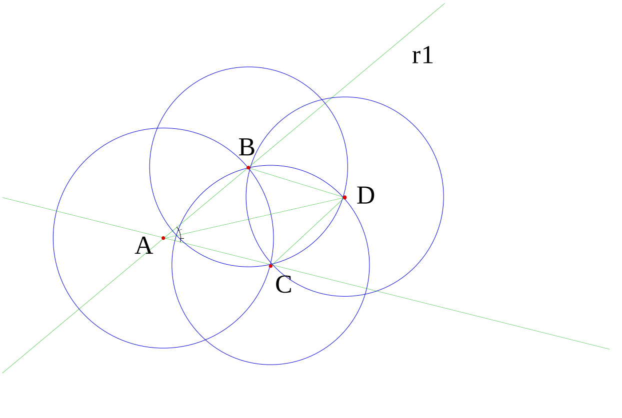

Figure 2. Lorenzo's construction

> If I also point in D with the ray DC, the circle passes through both C and B. Thus, the sides BD and DC are equal for the same reason of the

previous ones.
I discovered that the triangles ABD and ADC have the sides respectively equal; for this reason the 2 triangles are congruent for the 3rd criterion of congruence.
If the two triangles are equal, there is the rule that equal sides are opposite to equal angles.
Thus the angles 1 and 2, which are opposite to equal sides BD and DC, are equal.

This protocol shows a more developed form of 'proof', although the difficulty in selecting the correct hypotheses clearly appears. Such a difficulty is also witnessed by the fact that after the first step, when the equality of two of the sides is correctly derived from the construction, the second step consider the equality of the other sides by considering the circle with centre D and ray DB; actually, the fact that "the circle with centre D and passing through B will pass also through C", is a consequence of the construction of the point D.

It is interesting to remark that, once the construction of the three circles is done, the sequence of the operations used in the construction is non more available. Certainly it disappears in the sketch drawn on the paper, but even in the Cabri-figure, the correct order of the construction cannot be established immediately; when the figure is dragged the mutual relationships among the three circles are preserved and it is impossible to state which circle was drawn first. Only referring to the basic points allows to detect the correct relationship. The configuration is globally clear, but Lorenzo is not able to keep the logic control of the relationships between the elements of the geometrical figure.
Let us consider a last example of solution for this task.

Massimiliano (9th grade)
Prove that the angle bisector "by construction" (in Italian, "per costruzione") is angle bisector "by the equality criterion" (in Italian, "per criterio di uguaglianza").

AB = AC because of circle.
AO in common.
OB = OC because of circle centre C and B.
The two triangles are equal because of the third criterion of equality; (ABO = AOC).
Equal sides are opposite to equal

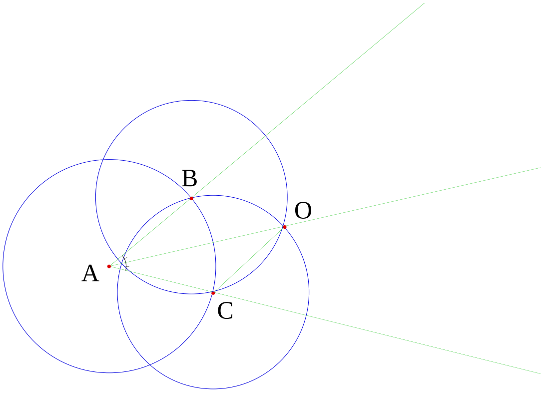

Figure 3. Massimiliano’s construction

> angles, thus OAC = BAO. AO ... is the angle bisector of BAC as we wanted to prove.

The text is synthetic but effective, mainly in expressing the core of the problem. Some expressions are incorrect and the justification is certainly insufficient. For instance, the statement "AB = AC" is justified by an unspecified "circle", instead of saying that the two segments are rays of the same circle with centre A. Nevertheless, Massimiliano shows a good control on the theoretical sense of his solution, in fact, his arguments are based on the construction procedure and find their justification in the axioms assumed, i.e. the criteria of congruence.

The construction of the perpendicular

As the sequence of activities progresses, theory is slowly enlarged; new constructions have become part of the theory. For example, at the point at which students encounter the next task, the theory contains some more elements; after the construction of the angle bisector, perpendicularity was introduced according to the following definition: "line t is perpendicular to line s if and only if t is the angle bisector of a straight angle with the vertex on s". Then the theorem of the isosceles triangle was proved. At this point, the following task was proposed to the students:

> Given a straight line r and a point P on it, construct the perpendicular to r passing through P.

Let us consider the following protocol. It refers to the pair of pupils Alex and Gio, who were not successful in the first task concerning the construction of the angle bisector. A clear improvement is observable.

> *Alex & Gio (9th grade)*
> We took a line r, passing through points A and B, then we took a point C $\notin$ r.
> Then we drew a circle (centre, point) having AC as its radius and then we traced it, entering taking the centre in A. We drew a circle (centre, point) having radius BA and centre B.
> We then determined points C and D with "intersection of two objects" and we joined C and D.
> CD $\perp$ AB
>
> Proof
> Let us consider the triangles ABC and ABD, which are equal for the 3rd criterion → AB in common,

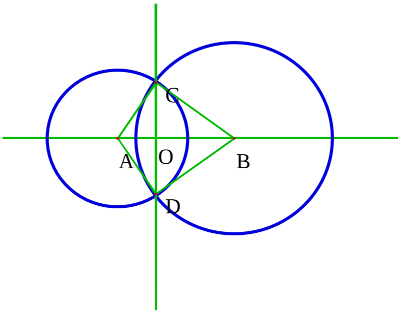

Figure 4. The construction of the perpendicular line

> AC = AD because they are radii of the same circle.
> DB = CB because radii of the same circle.
> Equal angles are opposed to equal sides, therefore the equal angles ∠ ABC and ∠ ABD are opposed to sides CA and AD. Angles CAB and BAD are opposed to sides CB and BD. We know that angles ∠ BCD and ∠ BDC are equal because they are at the basis of an isosceles triangle and also angles ∠ ACD and ∠ ADC are equal.
> Triangles DOA and AOC [*point O is marked on the drawing but it was not explicitly defined*] are equal according to the 2nd criterion, namely equal angles are opposed to equal sides, therefore angles COA and DOA are equal and right angles … angles AOC and BOD are equal because they are vertically opposite angles.

Compare this protocol with the previous one reporting the solution of Alex and Gio to the first construction task; a clear change is evident.

The pupils show a good theoretical control of the figure, i.e. the image on the screen and its construction. The pupils clearly separate the description of the construction and its justification, the justification is correctly referred to the theory available and can be considered an acceptable proof. This is not a common construction, and actually, the use of the two points (A and B) determining the given line r, originates two circles with different rays, this makes the validation more complicated, requiring a delicate analysis of the figure in order to overcome intuitive evidence; nevertheless, the pupils succeed in finding the way of correctly relating the construction process to their argument, referring to the theory.

Conclusions

The discussion, we started with, introduced the main issue of this paper: the claim of educational value of introducing pupils to theoretical thinking.

Proof and theory are a challenge for mathematics education; difficulties and obstacles have been largely described and discussed. The main challenge consists in elaborating specific approaches to theoretical thinking that take into account different modalities of thought and introduce a theoretical perspective without ignoring or neglecting other ways of thinking. The objective is to make pupils achieve a flexible way of thinking, which takes advantage of different type of argument, but maintains the consciousness of their diversity.

Solving a mathematical problem is a very complex task; it is not so important to provide a rigorous proof of its solution, but rather to have an idea of what such a proof should consist of and know that it will be required for a final acceptance.

In this stream, a possible proposal was presented in the previous sections, based on the potentiality offered by a particular microworld. The proposal is centered on the activity of construction, in particular construction in the Cabri environment, and aims at making a theoretical perspective evolve.

Our results are very encouraging. The sense of a construction, deeply rooted in the experience in the Cabri environment, is put into question in the collective discussions. As a consequence, the descriptions of the construction procedure change, improving their clarity through an increasing master of correct terms; on the other hand, the argumentation approach the status of theorems, that is the justifications provided by the pupils assume the form of a proposition and a proof. Pupils show that they achieve a theoretical perspective in the solution of construction problems, and moreover, this seems to provide a key of accessing the general meaning of Theorem.

Certainly, a successful experiment does not solve the problem; difficulties do not disappear by magic. As shown in our classes, argumentation does not evolve in a proof spontaneously, nevertheless this evolution is possible: the specificity of the environment to which argumentation is related and the direct guide of the teacher may determine this evolution. All that requires a long and patient work. In our opinion it is worth making such an effort; mathematics without the flavour of theory looses one of its main cultural values.

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Exploiting Mathematical Opportunities Afforded by Mathematical Tasks: A Case Study of Shifting from Arithmetical to Algebraic Thinking

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Abstract

I am interested in the lived experience of mathematics, whether in doing it, learning it, or teaching it. I am concerned that, in England at least, many teachers' lived experience does not include awareness that I have of potential afforded by tasks for students to encounter mathematical thinking, heuristics, and themes, or to use their own mathematical powers. I conjecture that this is at least partly because they are unaware of significant aspects of mathematical thinking, of major heuristics, and of principal themes. Lacking functional awarenesses, they are not in a position to recognise mathematical potential in tasks, and are therefore not in a position to exploit the thinking that is available. In order to understand this state, I turn my attention to how it is that I recognise the possibilities that I do, in order to support others in doing the same.

Although different conceptions of mathematics and of how children learn mathematics most effectively may be used to account for different states of awareness, I challenge the construct of 'conception' itself as generative, suggesting instead that it is at best participative. I look in detail at just one awareness: how some tasks *could* be seen to be on the boundary of arithmetic and algebra, and therefore *could* be used to promote shifts in children's thinking. I surmise that if teachers are unaware of the value of, the need for, or the nature of such a shift, indeed if they have not themselves made a similar shift securely, become aware *of* that shift, and become aware of how to prompt that shift in others, then they are not in a strong position to stimulate the education of children's awarenesses, and hence cannot be effective even in training children's behaviour, much less in prompting them to educate their awareness.

Conceptions and awareness

It is common to discuss beliefs, attitudes, conceptions, perspectives, etc. as if they were well defined 'things' which people possess in the way that they occupy a body and can think. For example, the question "what do you believe…?" is usually awkward to deal with, yet in the midst of making assertions it is not uncommon to find yourself saying "I believe that…". It takes only a moment's reflection to recognise that beliefs are things we assert when we are challenged to justify some other assertion, and what other people construct in order to account for what they observe in our behaviour (our actions speech, writing, gesture, posture, …). But it does not follow that 'beliefs generate behaviour'. The most we can say is that beliefs are constructions which may appear to be consistent or inconsistent with observed behaviour, bearing in mind that observations are themselves based on the observer's own sensitivities to notice.

Attitudes are constructions of pitch, yaw and roll, that is, orientation in three dimensions, taken metaphorically over into the affective domain. However, just like a body in space, our affect also pitches, yaws, and rolls. I suggest that people do not 'have conceptions or attitudes' in the sense of possessing something. Rather they have a variety of habitual patterns of behaviour which are displayed on different occasions, triggered by different factors, and which can be accounted for locally by saying that they 'have these conceptions or those attitudes'. More precisely, we each have multiple selves. 'Attitudes' are perspectives manifested by different selves (Bennett, 1964; Marton, 1981; Minsky, 1986). But *to express is to over stress*: the act of bringing to expression necessarily overlooks subtle details and stresses some features of what is being attended to over others. Consequently great caution is needed in taking what people say about their beliefs and attitudes at face value.

What to me is much more important is that in each moment we are aware of some things consciously and of other things subconsciously, while being oblivious to many other things, all at the same time. What is significant is the scope and structure of that awareness at each moment, because this is what defines what is possible in the next moment, and this is what surrounds and imbues students, creating possibilities for learning. Furthermore we all have characteristic triggers which alert us to notice certain features or aspects of situations, and to overlook or ignore others. These can usefully be accounted for in terms of multiple selves which take over control of the psyche.

I am trying to be cautious here, because just because someone says something, it does not follow (for me at least) that they always believe and act as if they believe, nor even sometimes believe or sometimes act as if they believe. What is articulated is only a tiny part of the lived experience. I have found that the best way to make contact with lived experience in others is to make contact with it in myself. Once I am sensitised and aware of that sensitivity, I can learn to let go of my own propensities and to use that sensitivity to become aware of what others are experiencing.

In the spirit of phenomenography (Marton, 1981; Marton & Booth, 1997), I record here some of the variation in conceptions of mathematics that I have heard or seen expressed.

Mathematics as (consisting of, consisting mainly of) skills to be mastered (even delivered!);

Mathematics as (consisting of, consisting mainly of) rules to be memorised;

Mathematics as (consisting of, consisting mainly of) topics to take in;

Mathematics as (consisting of, consisting mainly of) opportunity to exercise powers;

Mathematics as (consisting of, consisting mainly of) a social practice;

Mathematics as (consisting of, consisting mainly of) an economic gate-keeper;

Mathematics as (consisting of, consisting mainly of) the language of modelling;

Mathematics as (consisting of, consisting mainly of) social or psychological watershed;

Mathematics as (consisting of, consisting mainly of) a specialised discourse;

Mathematics as (consisting of, consisting mainly of) a personal construction;

Mathematics as (consisting of, consisting mainly of) part of the format of culture-society;

Mathematics as (consisting of, consisting mainly of) a formal game with symbols;

Mathematics as (consisting of, consisting mainly of) training for work;

Mathematics as (consisting of, consisting mainly of) the language in which to articulate the laws of science;

Mathematics as (consisting of, consisting mainly of) the language in which to describe the material world and through which to control it;

Mathematics as (consisting of, consisting mainly of) a domain of exploration.

Mathematics as a necessary evil;

Mathematics as the home of certainty in an uncertain world;

We can also consider each of these from the point of view of a child, of a bemused or frustrated adult, of a mathematician, and of a harassed teacher. Each conception will read differently in those different contexts.

I hope that you may recognise some of them at least. If there are some which resonate less well for you, then it may be worth while exploring those dimensions, because each of these have been maintained as primary or central by one or more thinkers.

Whatever aspects of mathematics are being stressed at a given moment, each mathematics teacher arranges that children are given tasks to do which at least superficially have to do with mathematics. The question I am pursuing here is how one goes about recognising mathematical potential, that is, potential for mathematical thinking, which is presumably the mathematical purpose in engaging in the task. As Christiansen and Walter (1986) observed, activity is what arises when a task is taken on. But of course the task as conceived by its author, the task as set by a teacher, the task as construed by children, the task as tackled, and where assessed, the task outcome as expected by the marker, are often quite different.

Method

My approach to investigating questions is to interrogate my own experience as closely as possible; to locate distinctions which make sense to me, which help me make sense of past and present experience, and which seem to be effective in informing future actions by providing me with additional sensitivity so that I can choose to respond freshly to situations; and to develop and refine task-exercises which seem to provide others with access to those same distinctions, those same sensitivities, so that they too can test out in their past, present, and future experience whether those distinctions are helpful.

My own data are drawn from my experience, but the data I offer you comes from your experience: that which comes to your awareness as you engage in some task-exercises. I begin therefore with some tasks for you to undertake. What you notice, what you distinguish, constitutes the data which I will then address.

Data collection: Task-exercises

I have chosen three domains of tasks to illustrate issues in being aware of mathematical possibilities: arithmetic calculations, arithmogons as a structure exploiting arithmetic, and word problems, all with the potential for leading to algebraic thinking. I invite you, in reflecting upon your experience of the following tasks, to consider how that experience might be different if at the same time, you were also led to articulate different conceptions such as those listed above.

Doing & undoing

> Write down a number which when doubled and added to three, then divided by four, then has five subtracted, gives six.

Where you surprised that the answer was not a whole number? How did you find it? Did you work forwards by trying an example, seeing if it worked, and then adjusting it to make it work? When abstracted as a process independent of particular problems, this approach is the origin of the rules of false position and of double false position, which dominated medieval European mathematics, traces of which can be found in ancient Chinese manuscripts and Egyptian papyri.

Perhaps you found yourself working backwards, undoing the stated calculations (add 5, multiply by 4, subtract 3, divide by 2).

Attention placed on *how* people find answers, will reveal, perhaps on trying several such examples, that each operation can be *undone* by using its reverse or inverse operation.

Pause for a moment and consider the range-of-change possible in such a task. What are the limits beyond which the structure changes and requires a fresh technique in addition to or instead of reversing each operation in turn?

I am confident that you are aware that the specific numbers used are irrelevant, as is the order of the operations or the number of compounded operations. These represent some of the possible range-of-change, the dimensions of variation, as Ulla Runesson called it (this volume, p. 36; see also Marton & Booth, 1997). Attention can be drawn away from specific numbers and a specific sequence of operations, and onto the way in which the undoing reverses the order and reverses the operation. This is of course a central notion in mathematics. Dave Hewitt (OU 1991) uses such sequences to get students at the end of an hour's lesson, having never solved equations before, to be able to solve a general equation of the form

$$((ax + b)/g - d)e + z = h$$

but with more operations and in different orders, the Greek letters for the coefficients being entirely unfamiliar signs to them! Even if they don't go this far, students can at least develop a general rule for answering a whole class of specific questions like the one given but starting with *any* final number. The significance of the task is to appreciate and be led to employ symbols to stand for particular but as yet unspecified numbers. Now try this one.

> Write down a number which when doubled and added to 7 gives a number which leaves a remainder of 3 when divided by 8. Write down such a number which you are confident no-one else will write down! Write down all such numbers!

Using remainders forces us to look for a whole number. Note the device of asking for a particular, a peculiar (e.g. one that you think no-one else will write down, or one which is unusual in some way), and a general. This three-pronged task almost forces students to become aware of and to contemplate the *range-of-change,* the range of possibilities from which they can choose, rather than jumping at the first example that comes to mind. Algebraic thinking depends on awareness of scope of generality, that is, the range-of-change under which some structure stays invariant. This important awareness has to be not just within the experience of teachers, but part of their conscious pedagogic awareness if it is to be employed successfully with pupils. As I developed elsewhere (Mason, 1998), to function effectively in mathematics you need to employ

awareness-in-action consisting of the powers of construal and of acting in the material world which we have from birth and which are manifested by gaining observable mechanical skills, including mathematical routines;

In order to become an expert practitioner (e.g. a mathematician), you need

awareness of awareness-in-action, or *awareness-in-discipline,* which enables articulation and formalisation of awarenesses-in-action, and which is closely linked to one form of shift of attention; this is how disciplines arise.

In his presentation at this conference, I saw Bill Barton as inviting participants to consider alternative awarenesses-in-discipline starting from awarenesses-in-action displayed by south-sea islanders among others, in order to broaden and enrich mathematics beyond that developed in Europe.

To become an effective teacher, you may not need to hone your awareness-in-discipline to the same level of precision as a mathematician, but you need enough in order to develop

awareness of awareness-in-discipline or *awareness-in-counsel*, which is the self-awareness required in order to be sensitive to what others require in order to build their own awarenesses-in-action and in-discipline, that is, to teach (Mason, 1998).

In order to experience the working of such awarenesses, pause for a moment and think about the various ways that the 'doubled and added to seven leaves a remainder of 3 on dividing by 8' task could be altered, varied, and augmented. What is the range-of-change, what are the dimensions of variation within which it remains the same sort of task calling upon the same sorts of powers, and affording encounters with the same sorts of mathematical ideas?

This pausing and becoming aware of the range-of-change, the scope of generality, is an important part of pedagogic preparation. It mirrors for teachers what students need to do, which is to appreciate what an example exemplifies. This means becoming aware of what could be changed without substantially altering those principles, powers, and techniques used to resolve a problem, but possibly extending that thinking. Thus each problem that a student works on is not just a task to be completed but represents a class of tasks or problems. The aim of resolving a problem is to become aware of the method *as* a method, that is, of the particular as representative of a general class, so that the method becomes a formula, an algorithm, or a method. Some students (typically but not exclusively male) are content to get a task finished, while others (typically but not exclusively female) are not content unless they have reached a state in which they think they could solve the problem again in the future. Most students will benefit from being stimulated and enculturated into the practice of 'seeing the general through the particular' by seeing a particular problem as typical of a whole class of problems which can be solved using the same technique. Stimulating and enculturating requires awareness on the part of the teacher.

> Now write down a number which when doubled and added to 7 gives a number which leaves a remainder of 3 when divided by 8 less than the original number.

Suddenly there are only finitely many integer solutions. Has this example altered your awareness of the range-of-change of this task type? It certainly introduces the need for a new technique for resolving problems such as

> 'for what values of y will $x = \frac{4+8y}{y-2}$ have integer solutions?'.

One approach is to perform the synthetic division to reach $x = 8+\frac{20}{y-2}$ which forces y – 2 to be factor of 20 which forces y – 2 to be…

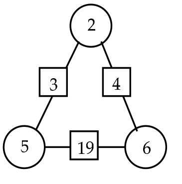
The point is to experience a shift in awareness of the range-of-change possible, in order to feel what it is like for students and for colleagues not to be aware of the same scope of generality that you experience in other situations or tasks. The notion mentioned in passing of attending to *how* you 'do the particular' makes a vital contribution to experiencing and expressing generality (Mason et al, 1985).

Arithmogons

Arithmogons (McIntosh & Quadling, 1975) is the name given to a collection of tasks which have re-surfaced periodically in the UK, the most recent being the new National Numeracy Strategy in the UK (DfEE, 1999, p. 79); see also Mason and Houssart (2000) for some history and some extensions. It is an excellent example of a task structure which has enormous mathematical potential, but this potential can only be actualised if the teacher is aware of possibilities and of ways of activating them.

Colleagues from the UK will recognise the title, if not this particular variant. I have chosen to offer a more sophisticated version first, in order to challenge those who are unimpressed with pupil-level tasks, whereas in the previous domain I built up to the more sophisticated from the simpler.

Arithmogons Variant

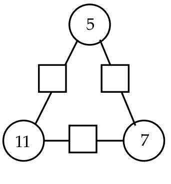


The numbers in the squares on the edges of the triangle are calculated as the product of the vertex numbers on that edge minus their sum.

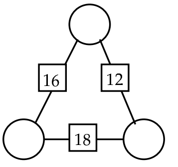
Now can you go backwards? For example, if the edge-numbers are 5, 35, and 59, what choices are there for the vertex numbers?

It is important that you work on this for a bit, in order to see what you notice about how you go about the task. One approach is to resort to algebra, which exploits and expresses structure and enables you to isolate and re-specify a more tractable problem (e.g. one equation in one unknown, or to recognise the structure of the edge numbers being 1 less than the product of 1 less than the vertex numbers). Another approach is to make up your own examples of vertex numbers and then fill in the edge numbers, in order to try to detect structure or pattern through your familiarity with numbers.

Traditional Arithmogons



In the traditional presentation of arithmogons, the initial task can be to insert, for each edge, the *sum* of the vertex numbers at the ends of that edge. This is perfectly straightforward, and typical of tasks which embed repetitive arithmetic operations in some structure. Whole numbers, fractions, negatives, and decimals could all be used as appropriate to the pupils.

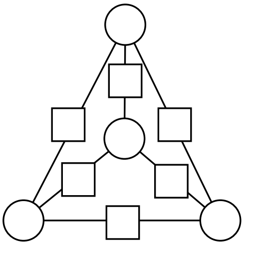


Interesting things happen when the 'doing' of the calculation is reversed: when the boxed numbers are given and the vertex numbers are to be determined.

It is clear that the 'undoing' version of *arithmogons* can be tackled entirely arithmetically, at first by 'guess-check-and-modify', and later by making use of arithmetical structure imposed by the arithmogon triangle.

I see it as perched on the boundary between arithmetic and algebra, for structure can be sensed, conjectured, and used. It can be expressed arithmetically in particular, verbally or symbolically in general. Symbols can be used to locate and express structure (here, the sum of the three edge numbers is twice the sum of the three vertex numbers, and then that this fact enables the vertex numbers to be reconstructed). This formula or algorithm can be discovered by creating several examples for oneself (going from vertex-numbers to edge-numbers), especially by being systematic. It is but a tiny step to move to the general, for example by doing an example without actually doing the arithmetic, or by deciding how to check a proposed answer, and then substituting letters for the proposed numbers to reveal the constraints inherent in the arithmogon structure.

Consider for a moment the range-of-change possibilities within the arithmogon structure which have come to mind as you have been reading.



Some possibilities that have been considered include replacing triangles by quadrilaterals and other polygons (not all polygons have the same effect!), and more complex diagrams could be used such as the one indicated.

Some edges could have one computation while others have another (e.g. sum and product, or GCD and LCM). Entries on the edges (or vertices) could be whole numbers, integers, rationals, reals, or algebraic expressions such as polynomials or rational polynomials. Students could be asked to characterise constraints to be imposed on entries on edges in order to enable a solution to be found belonging to one of these classes.

The sophisticated version I gave at first was intended to prompt you to want to construct some examples for yourself (deciding on vertex-numbers from which to deduce edge numbers) in order to locate some structure or pattern. Notice that if this happens, you are also rehearsing the arithmogon operation (in the sophisticated version, product minus sum), but on your own examples, with your attention directed to structure rather than to 'getting the answer' as would be the case in ordinary exercises. This principle, studied by Dave Hewitt (1994) has multiple applications: if you want students to automate some procedure, give them a task in which they wish to construct and do examples that use that procedure, rather than just giving them multiple questions in which to use that procedure. The idea is to attract their attention away from the doing of the procedure in order to automate it.

To have the requisite awareness of this principle in order to invoke it in different contexts requires an appreciation of the mathematical process of ‘Doing and Undoing’, as well as adopting a compatible didactical perspective concerning the role and purpose of rehearsing procedures.

Reflection

The reason for mentioning *arithmogons* is that its full potential is rarely actualised. Schemes and texts which include it exploit only a small part of the potential explicitly, but many teachers are not sufficiently mathematically aware to recognise the wider potential. What you are aware of as potential in a task is limited partly by your domain of confidence within mathematics, partly by your conception of mathematics, and partly by the pressures under which you are operating. Where teachers have time and support for engaging in mathematical thinking together, there is more chance of the mathematical potential of tasks being actualised.

I am suggesting that these remarks apply not just to arithmogons but to tasks used in classrooms generally: awareness of mathematical themes such as *doing & undoing*, *invariance amidst change*, and *freedom & constraint* can suggest ways of modifying and extending tasks, in order to convert them from routine exercises into vehicles for developing and consolidating mathematical thinking.

Word problems

The use and abuse of word problems as pedagogical instruments over the centuries is a long and complicated tale. I can do no more here than suggest that *word-problems* have been used in mathematics education (and for entertainment) since the earliest of written records. Most word problems can be solved using arithmetic, which is perhaps why traditionally they have drifted out of algebra texts and into arithmetic texts. But they actually lie on the boundary between arithmetic and algebra, for they offer spendid opportunities for experiencing and expressing generalities, achieving a sense of power by obtaining a formula for doing ‘all questions of this type’ or at least an algorithm or method. Thus they offer opportunities for creativity, rather than an extra and hateful burden on students.

Consider as an example these problems taken from William Thompson's *Chambers’ Algebra for Schools* (1898):

> Divide 48 into two parts, such that one-half of the greater added to one-fifth of the less shall be 18. (p. 189)

Pause for a moment and consider the potential range-of-change that you are aware of in this task.
Later in the book we find a collection of exercises stated in the general (an unusual feature to be found in only a few books in different generations).

> Divide n into two parts, so that p times the greater may exceed q times the less by r. (p. 216)

A few pages later we find:

> It is required to divide a into four parts, such that if the first be diminished by m, the second increased by n, b times the third increased by p, and c times the fourth diminished by q, the results shall all be equal. (p. 218)

Now consider the potential range-of-change of these tasks. It may be that these further examples suggest more dimensions of variability than had occurred to you just a moment previously. If so, then you have specific and direct experience of transition from being unaware, to being aware of pedagogic possibilities. But in general it is not sufficient to 'tell' people of an added dimension of variability. Somehow they have to integrate it into their scope of possibilties. This depends on their awareness of mathematical themes, their awareness of pedagogical devices, as well as the local conditions, intentions, constraints, etc.

In his book on algebra, *Arithmetica Universalis*, Isaac Newton (1683; see Whitehead, 1972) posed and solved most problems in general, then in particular, sometimes using the general as a formula, and sometimes using the same method to do the particular. A few authors in each generation since have posed both particular and general versions, but most have been content to pose only the particular. I have always seen word problems as opportunities to generalise, to create my own method, even my own formula, and I have been mystified by the number of authors who provide multiple minor variants of the same problem without any indications that the questions they pose are but representatives from a general class of problems. Indeed many authors go so far as to 'mix up' the question types, presumably so that students gain experience of thinking carefully about each problem (Mason, 1999).

Reflection

Recognising range-of-change, the dimensions of percieved variability connected with a task, is the first step in actualising those possibilities pedagogically. But without those awarenesses, little is possible, and little mathematical thinking is likely. The whole purpose of setting tasks for children is that they use their powers of mathematical thinking, becoming aware of them and developing them. Through this exercise of powers they encounter mathematical concepts and techniques, as well as mathematical themes and heuristics. They even encounter 'themselves' in the form of their propensities and habits, and thereby have the opportunity to develop new ones.

Theoretical reflections

With a limited conception of mathematics, of the possibilities afforded by a mathematical task, the *tranposition didactique* (Chevellard, 1985) is likely to lead a teacher to teach students procedures to 'do the task', such as solving equations, arithmogons, or particular forms of word problems. At its most basic, this means lots of practice, for example, in specific one-operation equations, in forward arithmogons', and in simple word problems, perhaps seen as a context in which to get students to practice arithmetic operations. But the teacher has to provide lots of examples for students 'to do'. Alternatively, and more powerfully, informed by the theme of 'doing and undoing', much more interesting tasks can be generated.

To rehearse addition or some other operation	Forward Arithmetical, synthetical	Teacher needs to provide plenty of worksheets of examples for children
		Children have to work their way through lots of examples
	Backward Empirical, abductive or inductive	Teacher needs only to initiate the backward activity
		Children make up own examples on which to try to solve the backwards problem in particular
To move from rehearsal of operation and empirical to structural	Backward Structural, algebraical, analytical	Teacher needs to initiate backward activity and shift attention to structural or general

Two or three contrasting examples can highlight differences (dimensions of variation), which in turn offer either a disturbance to be accounted for and generalised, or indication of what might be permitted to change and hence what might be invariant. Thus several different contexts may highlight what is essential and common to all members of a class or type of problem.

Seeing potential

As suggested in the case-studies of specific tasks, assuming that a task, no matter how well designed, will inevitably force students to encounter something, or to learn something, is to depend very strongly on a cause-and-effect mechanism whereby students undertake tasks and learn from doing (some construed version of) them. A more appropriate conjecture is that while a task may lead to insight, to encounters with important mathematical themes and heuristics, and to students becoming aware of their own powers in ways which enable them to develop those powers, this developmental aspect is much more likely when the teachers are themselves aware of such possibilities. I *do not* mean that the teacher has to be aware of them in advance, because making a list of possibilities in advance is likely to result in student attention being directed according to teacher desires, without the benefits arising from teachers connecting directly with student experience. I *do* mean that the teacher becomes aware of possibilities as the activity generated by the task unfolds. This awareness is resonated from past experience with rich themes such as doing-and-undoing, invariance-amidst-change, and freedom-and-constraint. The teacher can only intervene, prompt, probe, or guide to the extent that they are aware of possibilities themselves, in the moment. What matters most is what comes to mind in the moment-by-moment unfolding of the activity.

The possibilities which come to mind for a given teacher depend very strongly on features of the particular situation: the preparation, the mood, the class, the room, the time, the specific mathematical topic, how it is taught in texts, etc. They will be attenuated, delimited and defined by the teacher's awareness, which may be articulated in the form of beliefs, conceptions of mathematics, and pedagogical strategies, but none of these drive the others. Rather, they all co-emerge (Varela, Rosch & Thompson, 1991).

I have concentrated here on tasks at the boundary of arithmetic and algebra, because algebra forms a mathematical watershed for a majority of the population. Yet I am confident that many more people could experience the power and pleasure of the use of their undoubted powers to experience and express generality. Indeed I take the view that a lesson without the opportunity to generalise is not a mathematics lesson.

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What Matters in the Mathematics Classroom? Exploring Critical Differences in the Space of Learning

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How is it possible to make sense of and to understand how the classroom, I as a teacher manage and have responsibility for, affects students' learning?

In this presentation I propose a framework for analysing the classroom from the point of view of possibilities and constraints for learning (Marton & Booth, 1997; Bowden & Marton, 1998). I will describe some of the underpinning ideas of this framework but also demonstrate how this has a potential for revealing features of classrooms that are critical for what the learners have possibilities to learn.

When I started my research some ten years ago I had been a teacher for more than 15 years and I had met hundreds of teachers in in-service training and teacher education. I was puzzeled by the fact that, although there seemed to be a consensus about what was meant by good mathematics classrooms, mathematics classrooms were very different. For instance, there seemed to be consensus that a problem solving approach, taking students' knowledge into consideration in mathematics classroom, using small group learning, discussing mathematics, using manipulates, were features of good mathematics teaching. However, from a lot of school visits, I had experienced that classrooms in some respect could be very similar, but yet very different. For instance the same manipulatives could be used, the students could work with the same problem in an interactive discussion in two different classrooms - but yet – and this was an intuitive feeling I had – their classrooms were different in some respect. And, I realised, when examining how good mathematics classrooms were described, that descriptions like "small group work", “problem solving approach" and so on, were merely descriptions of how the classroom was organised and of teaching methods in general.

In the same way, there are descriptions of the role of the "good" teacher. The teacher as a facilitator who proactively supports student learning or the teacher as a coach, are teacher characteristics and actions that are considered as desirable. So, I asked myself: if two classrooms are organised in the same way, if the students are taught with the same teaching methods, solving the same problem etc. and the teacher acts as a coach, can you still say that the students have the same learning opportunities? Or are there other things that matter? Are there other aspects of the mathematics classroom that the learners encounter that matters for learning than organisation, and teaching methods also? And what does the teacher do when she is coaching or facilitating students' learning?

Studying “what matters” in mathematics classrooms – a matter of perspective

Before I continue and try to answer the question “what matters?” I will start with a very brief overview of how effects of mathematics teaching have been studied. It is not bold to claim, and perhaps it is known to everybody in the audience, that there has been a radical reconceptualization of understanding of classroom learning and teaching during the two last decades. Koehler and Grouws (1992) give a solid and close exposition of research of the effects of teaching practise in the ‘Handbook of research on mathematics teaching and learning’ (Grouws, 1992) just as Graham Nuthall (1997) in his article in The International Handbook on Teachers and Teaching from 1997 does. It can be concluded that the teaching – learning process has been studied from various perspectives, with various underlying philosophies and with a range of methodologies. The introduction of new methodologies and disciplines into the study of classroom experience implied a paradigm shift and has resulted in the acknowledgement of the multi-dimensions and multi-layered nature of classroom processes.

This paradigm shift was a reaction to the previously dominated correlational and experimental research paradigm, often characterised as process-product research. In a "typical" process-product study the frequency of particular teacher and student behaviour, such as types of questions asked, length of responses, number of and types of examples used, were noted (Koehler & Grouws, 1992). Student outcomes - in terms of achievement on test scores - were correlated to the frequency of the observed behaviours to determine what behaviour was associated with high performance in learning. This research tradition was much criticised (Brophy & Good, 1986) for not taking the complexity of classrooms and the teachers' - as well as the students' intentions, beliefs and attitudes - into consideration. Recently, as research and data analysis techniques have become more advanced, more aspects of the teaching act have been looked at in detail. But mostly, there has been a stronger - but also a diverse - theoretical foundation for this kind of studies (Koehler & Grouws, 1992).

Although the theoretical point of departure for these studies may be different, one premise is shared; the research on teaching in mathematics should be combined with research on learning (ibid.). However, when approaching the issue of what matters in the mathematics classroom the underpinning philosophical ideas about how students learn have implications for what aspects of the teaching – learning process that is taken into consideration. For instance, the constructivist assumptions about how students learn imply certain desirable teacher actions and hence those are considered to be of concern for "what matters?" Cobb and Bauersfeld and their associates (See e.g. Cobb & Bauersfeld, 1995) have created a framework for approaching the mathematics classroom as an environment for learning. Teaching and learning have been studied from the point of view of how meaning is negotiated and how reflective discourse can promote learning. A cognitivist tradition assuming the importance of linking existing knowledge to new knowledge, focuses on how the teacher provides instruction appropriate for each student (Koehler & Grouws, 1992), whereas, within the socio-cultural tradition, learning implies participating in a community of practice. How this happens in the classroom practise is the object of research (see e.g. Sfard, 1998).

So, it can be concluded that the theoretical perspective on learning is important for approaching the issue of "what matters?" In the same way the particular theoretical framework that I have been using in my studies, has certain assumptions about learning and a particular ontological and epistemological foundation and hence pays attention to particular aspects of the phenomenon.

When I approached my research project with the particular background I mentioned earlier, I aimed at coming "beyond" such aspects like organisation, teaching methods etc. and focus on *what* students had possibilities to learn in the mathematics classroom. The point of departure taken was that what is most fundamental for learning is how the learner experiences, understands, or perceives that which is learnt. And, when examining the classroom as a space for learning, that which is *possible* to experience or understand must be of concern. Since learning always has an object - there is always something learned - how this object of learning is handled in the classroom, was the object of my research. So, I wanted to study the mathematics classroom from the point of view of what was possible to learn.

When the teacher and the learners interact around a topic, an object of learning is constituted. The object of learning is jointly moulded, mostly by linguistic means, in the interaction between the teacher and the students or by the students themselves. This object of learning is an "enacted object of learning" (Marton & Morris, in press); it is the researcher's description of what students encounter in the classroom, what they are afforded to learn.

The pedagogy of variation

To illustrate how the enacted object of learning can be studied and it its significance for learning, I will present two studies of mathematics lessons.

Teaching fractions

The first study is a study of five teachers and their students in grade 6 and 7 in the Swedish comprehensive school (Runesson, 1999). The classes were followed during some 8 consecutive lessons. The teachers were all experienced teachers, with 5 to 25 years of teaching experience, however, with different experience of in-service training in mathematics education. The lessons were audio-taped and transcribed verbatim and in addition field notes were taken by an observer in the classroom. The teachers were interviewed twice, once before the teaching sessions and once after. All five teachers taught fractions or percentages. So, the mathematics content was the same in all lessons.

Several similarities between the lessons were found. For instance, the organisation was very similar in the five classes. These were mixed lessons (i.e. plenary lessons combined with desk work). The students worked more or less organised in pairs or groups during the lessons. The teachers all used manipulatives as teaching aids and there was a "realistic maths" approach in all classes, in that mathematics was contextualised in everyday situations that were supposed to be familiar to students. Three of the classes even had the same textbook. However, there was a clear difference between the classrooms. This difference concerned how that which was taught was handled.

When the teacher tried to make the students understand something or notice something, she brought some aspect of the mathematical content to the fore of the students' attention. I found a difference in what aspects of the content that were focussed on and what aspects that were not focussed on (i.e. they were left out or were taken-for granted). For instance, Kate, one of the teachers, used a piece of elastic, which was partitioned into four equal parts. First it was indicated that each part was a quarter of the whole, and then she stretched the elastic alongside the desk, indicating the length of a quarter of the desk. After that, she stretched it alongside the blackboard and pointed to the quarter of the length of the blackboard. Finally, she compared the extent of a quarter of the blackboard with the extent of quarter of the desk. In this way she pointed out that the relative size of a quarter, is not equal to the absolute size of a quarter. So, this particular aspect of fraction was focussed on. The students were afforded to experience, or you can also say, they had possibilities to learn that the size of a fraction is related to the size of the whole. This aspect was paid attention to only in Kate's teaching. It was left out in the other lessons.

If we look closer at what Kate did, we will find that she varied the whole, she used two different lengths, in this case the length of the blackboard and the length of the desk, but kept the fraction (i.e. a quarter) constant. So, the whole was varied, while the fraction was invariant. In this way she opened for a dimension of variation within the whole. In this case she took two instances, two different wholes, but she could have taken more of course. This implies that she opened a dimension of variation of the whole.

I found that variation played an important role when the object of learning was moulded. *All* teachers used variation when they tried to draw the students' attention to various aspects of the content taught, however they did this differently. In this way different patterns of variation were constituted in the different classes.

I will just give you an illustration of different kinds of patterns of variation that I found; an illustration that is very distinctive, but also perhaps familiar to you. Let us look closer at two lessons. In both lessons a procedure for calculating a/b of c was the topic. Teacher A started with showing a piece of string of 90 cm. "How much is 1/3 of this piece of string?" After the first example she changed the nominator from 1/3 to 2/3. She presented a procedure: "divide 90 by three and multiply by two". Then she took a new example 1/5 of 40, followed by changing the nominator again i.e. 3/5 of 40 and finally 3/5 of 60. The dialogue is presented alongside our analysis below.

Teacher A

T: OK. Here I have a piece of string. It's 90 centimeters (The teacher holds up a piece of string). Three persons shall share that equally. How do you go about with that? Fair share? Tell me, Sylvia.

The teacher introduces the problem. A manipulative aid is used. A strategy for solving the problem (1/3 of 90) is introduced.

S1: Well…divide by three.

T: Yeah, each one will get a third. But let's say, one of them will have some more than the others. The string is still 90 centimeters and I want 2/3. How could we figure that out? 2/3 of a string that is 90 centimeters? Thomas?

The nominator is changed;1/3 is changed to 2/3 (2/3 of 90).

S2: (inaudible)

T: Right. First you figure out the length of a third and then take another one… and together that makes…? What did you say? 60 centimeters? Yes. So, first you have to figure out the length of 1/3. Measure that and then take another one. (The teacher first marking 1/3, then 2/3 of the whole length of the string)

The teacher elucidates the strategy and illustrates with the manipulative aid.
(1/3 of 90=90/3=30)

T: OK. Lets take a look at this piece of string (The teacher is holding up a shorter piece of string). This is only 40 centimeters. I would like to have one fifth of 40 centimeters. (Writing on the blackboard: 1/5 of 40 cm).

A new problem (1/5 of 40) is introduced. A manipulative aid is used.
Written representation

S2: 8 centimeters.

T: Yes, each fifth is 8 centimeters. But let's say we will have 3/5. How do you figure that out? Tell me, Lisa.

3/5 of 40 (the nominator is changed). The teacher asks for an appropriate strategy.

S3: 3 times 8.

T: OK. At first we must figure out how much is 1/5, so you divide 40 by 5, and you'll get 8. And 3 fifths must be three times as much. Three such pieces. That's 24. But let's say that the piece of string is 60 centimeters instead. (Writes 60 cm on the board). One of you should have 3/5, and another one 2/5. How much will the person who gets 3/5 have? ---OK. How do we go about with this? The whole piece of string is 60. I should have 3/5, then I must figure something out first, what...? Martin.

The teacher elucidates the strategy

The whole is changed (3/5 of 60). Written representation

The teacher asks for the appropriate strategy

S4: 5 divided by 60

T: Well; now you said it the other way around - 60 divided by 5. What's that?

S5: 12

T: OK. 12. So now we know that 1/5 is 12. How much is then 3/5?

If we analyse this data as regards what aspects that are focused or lifted up, it is apparent that the strategy for solving the problem (calculating the length of a fractional part of a piece of string), was the focus. This is what the teacher tried to draw the students' attention to. But since only one strategy or procedure was present and hence, not varied, the strategy itself did not make up a dimension of variation in this situation. This particular aspect was focused, but it was kept invariant. On the other hand, the teacher changed the parameters in the problem - after introducing the problem and presenting an appropriate solving strategy, the teacher changed the length of the piece of string as well as the size of the fractional part (1/5 of 40). In the next example, the numerator was changed (3/5 of 40), and finally in the last example, the whole (i.e. the length of the string) was changed. Thus, *the strategy was invariant, while the numbers were varied in a systematic way*. So, in this case *the numbers involved in the problem made up a dimension of variation.*

This could be compared to teacher B. The day before, the students had been working with a particular problem of marking 3/7 of a 7 x 8 squared rectangle. The rectangle was shown on an OHT. The teacher invited the students to come up with different solutions and to justify their strategies. Different solutions were presented and re-described by the teacher.

Teacher B

The rectangle is shown on an OHT. To begin with the teacher asks Lena to tell the class how this could be done.	*A manipulative aid is used*
S1: If you just take 7 squares from the whole, and then take three of those…If you count 'one, two, three' and mark them.	*In each group of seven squares, three are marked*
T: Why?	*The teacher asks for an argument*
S1: Well it is 3/7 of the small pile. And then I continue: one, two, three, four, five, six, seven, go on like that. I always count to seven and mark three of them.	*The pupil is explaining her strategy*
T: Oh yeah. I understand! You counted one, two, three, four, five six, seven and then you mark three of them. And then one, two, three, four, five, six, seven and you mark them. In other words, you do it like this (pointing at the OH) one, two, three, four, five, six, seven, you can mark the last ones like that? How do you go on? In the same way?	*The teacher elucidates Lena's strategy* *A manipulative aid is used*
S1: Yes.	
T: Well did anyone do this differently? Did you all do like that?	*The teacher asks for alternative strategies.*
Ss: No.	
T: Well what about you…Sophie?	
S2: Well, I just divided it into seven parts.	*Another pupil explains her strategy, which is different from the previous one.*
T: OK. You just counted all the squares and divided them into seven. OK, Maria what about you?	*The teacher asks for alternative strategies.*
S3: Well I tried different numbers like that until I got seven parts.	*Yet another pupil explains her strategy, which is different from the previous one.*

Similar to teacher A, in this situation, teacher B focused on the solving strategy. But, while this did not vary in teacher A's lesson, teacher B asked the students to come up with a variation of strategies. During the discussion *the parameters* of the problem were the same, and hence this was *invariant*, whereas the solving *strategy was varied and was presented as a dimension of variation*. It is also worth noticing that the opening of the dimension of variation of solving strategies is a result of the existing variation in how the students solved the problem. The variation in students' solving strategies were made explicit when the teacher asked for alternative strategies, like: " Did you all do like that?" "Did anyone do it differently"? So, by encouraging the students to come up with their own ways of solving the problem, a dimension of variation in how the students experience the problem is opened as well.

However, there is also another dimension of variation that could be identified in this situation. The variation in solving strategies also involves a variation in a semantic interpretation of the operator aspect of fractions. 3/7 of 56 squares for instance, could be interpreted as dividing 56 squares into seven groups and then taking three groups out of the seven groups. (C.f. duplicator/partition-reducer interpretation Behr, Harel, Post & Lesh, 1993). But 3/7 of 56 squares could also be interpreted as arranging the 56 squares into groups of seven and then taking three out of seven in each group (stretcher/shrinker interpretation). Both these semantic interpretations were held among the students and were made explicit when the teacher asked for different strategies. Thus, in this situation, there is also an opening of a *variation in semantic interpretation of the concept.*

From Table 1 it is possible to compare the two lessons in respect to variation and in-variation. There is a distinctive difference between the lessons in respect to the pattern of variation that was constituted in the two classes. So, the pattern of variation that the students were exposed to, or had possibilities to experience, was quite different.

	Teacher A	Teacher B
Solving strategy	Invariant	Varied
Parameters of operation	Varied	Invariant
Representation	Varied	Invariant
Pupil's understanding	Invariant	Varied
Semantic interpretation	Invariant	Varied

Table 1. The space of variation constituted in lesson A and B respectively.

Whether variation was presented simultaneously or in sequence, was another difference found.

In two of the lessons the same aspects of fractions were presented to the class. In one of them, this presentation was done in sequence (i.e. each aspect was presented after one another). First, the part/whole aspect was presented, and then, the division/quotient aspect, and finally the two aspects were brought together as two aspects of a fractional number; the teacher linked the two aspects of fractions by concluding that they corresponded to the same number. In comparison, in the other lesson, the teacher focused on both aspects (the part/whole aspect *and* division/quotient aspect) at the same time during three steps. He first asked the class to mentally work out the solutions of two problems, each pointing to one of the aspects of fractions and resulting in the same number ('4'). Then, he required the students to represent their solutions in terms of mathematical symbols. Finally, he concluded that the same number could be arrived at by methods that reflect the different aspects of fractions. In this way, the relationship between the two aspects was *simultaneously* brought to the fore of the learners' attention (Runesson, 1999, pp. 159, 209).

To summarise: Although the five teachers' were teaching the same topic, the aspects of the topic that were focussed, what was taken-for-granted, that which varied and that which was constant, and what varied simultaneously or appeared in sequence, was different. In this way a pattern of variation and in-variation was identified. So, the space of learning, the space in which learning can happen, was a space of variation.

Learning and variation

Initially I stated that what is most fundamental for learning is how the learner sees, understands or experiences that which is learnt, and that what is learnt could be seen in one way or another. Some 25 years of research with in the phenomenographic tradition (Marton & Booth, 1997; Bowden & Marton, 1998)) have described how that which appears to be the same thing, is experienced differently by the learners. To educators this is of importance, since education aims at developing a certain way of seeing, understanding or experiencing or developing certain capabilities (Note! not necessarily one way of understanding!)

What do I then mean by "experience"? In every situation – like in this one for example - it is possible to pay attention an almost unlimited number of aspects. However, we do not. What happens is that some are paid attention to – others are not. Some are discerned, they come to the fore of our awareness, whereas others are left out, they are un-discerned or taken for granted. A certain way of experiencing takes the simultaneous discernment of certain aspects. So a way of seeing something can be characterized as the aspects that are discerned at a certain point in time. The aspects discerned that define a particular way of seeing something are *critical features* of what is seen in relation to that particular way of seeing.

What is, for instance, required for experiencing the second pie in Figure 1 as partitioned into thirds while the first one is not? What are the critical features for seeing or experiencing something as thirds? To experience the shaded part in the second pie as 1/3 takes the discernment of certain aspects, namely, the discernment of parts – whole, the discernment of number of parts, and the mutual size of the parts. And, in addition, all these must be discerned simultaneously.

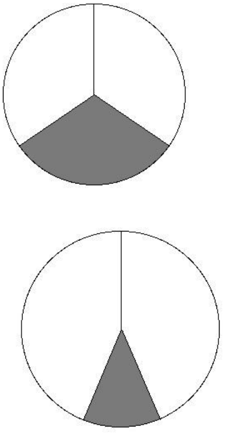

Figure 1. Pies partioned into thirds and three parts respectively

If the mutual size is taken for granted, and thus is un-discerned, the shaded parts will both be experienced as representing 1/3. So, the discernment of the mutual size is critical in this case. You can argue that the simultaneous discernment of these aspects is critical for a certain way of understanding.

However, the discernment of an aspect takes an experienced variation of the aspect in question. It is easy to realise that the discernment of coldness presupposes the experience of heat, that the discernment of colours presupposes an experienced variation of colours; of different colours. The discernment of a particular aspect takes the experience that it could vary. Understanding the many-ness of e.g. “seven” presupposes an experienced variation of other numbers, nine, six, two and so on. Understanding what a square is presupposes the experience of other shapes that are not squares. So, a certain way of seeing implies the experience of patterns of variation.

From these assumptions, I argue that variation, or more precisely, the pattern of variation that is possible for the learners to experience in the learning situation, is of importance for what they have possibilities to learn.

However, it must be noted that it not the variation *per se* that is important, for instance, that the more variation – the better. Instead it is *that* which is varying that is of interest; that which is varying at the same time, that which is focussed on and that which is taken-for granted, is critical for learning.

In the previous examples different patterns of variation were identified. From the assumption taken follows that the students had opportunities to experience different patterns of variation in the different classes and hence, to discern aspects of what was taught differently.

But you may ask: Does it matter? Does it affect students’ learning? In this study I did not study what students had learned. It was only the constraints and possibilities for learning that were examined. However, other similar studies of lessons in economics (Rovio-Johansson, 1999) and in language (Marton &

Morris, in press) where the theoretical framework described previously has been applied, have shown that what pupils learn reflects to a great extent how the content was handled during the lesson.

Teaching and learning velocity graphs

That the presence, but also the absence of variation related to the object of learning is critical for students' learning is also indicated in an ongoing study (Runesson, in progress), which I will say a few words about. A mathematics lesson in grade 8 was videotaped with a particular technique (Clarke, 2001). Three cameras were present in the classroom, one recording the teacher, another the whole class and the third focussing a group of students. The tapes were mixed, so it was possible to see the teacher's actions and the students' actions at the same time. One of the focussed students was interviewed after the lesson. During the interview the integrated video-tape was watched by the students. The interviewer stopped at several occasions and posed questions about what had happened during the lesson.

The topic of the lesson was graphical representation of speed versus time. One of the tasks was to match five different situations (a-e) each describing a moving object changing velocity as a rate of time in eight different graphs (see Figure 2).

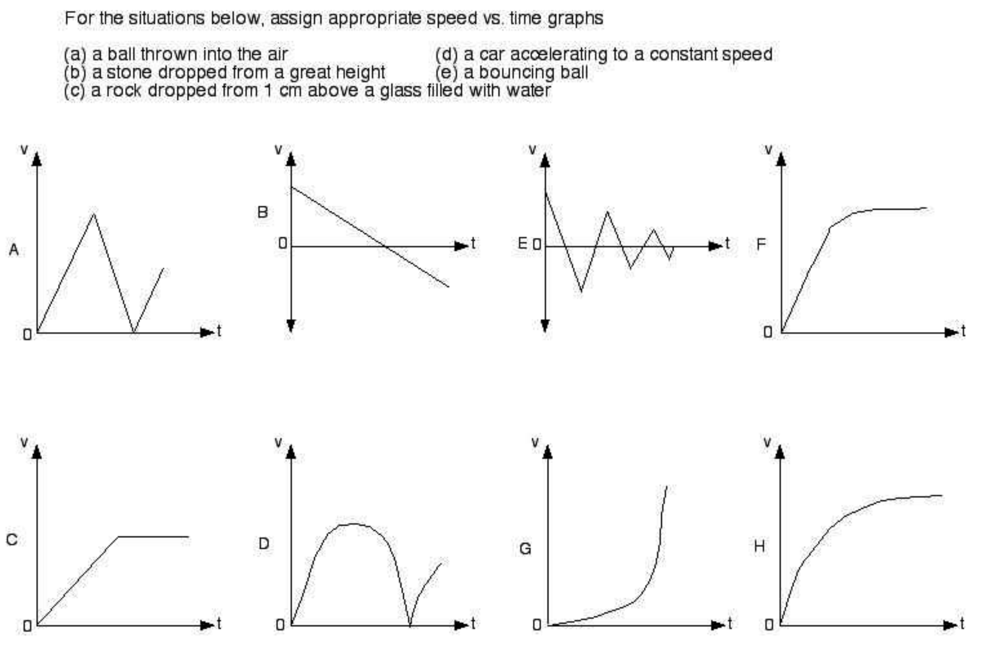

Figure 2. The task presented in the textbook

It could be noted that two of the situations (a and e) implied a bi-directional movement. That is, they did not only include a change of velocity but also a change of direction. Unlike the unidirectional motion of a falling stone for instance, a bouncing ball is bi-directional (i.e. moves in different directions). A bouncing ball changes velocity *and* direction. In that sense this situation is more complex. To two of the students, Laura and Fiona in this study, A and E were indeed the most problematic ones. During the peer work and after some discussion the girls chose graph D as corresponding to "a ball thrown in the air". Graph D was once again chosen by one of the girls when she was interviewed immediately after the lesson. So, neither in the peer interaction nor in the interview, could the girls draw an acceptable v(t) curve, similar to B in Figure 2.

Is it possible to gain understanding about this failure from the point of view of what was possible to learn during the lesson? Were necessary conditions given, in terms of how the object of learning was handled during the lesson, for solving the task? And what was critical in the learning environment for the students' learning?

In this study the way the object of learning was handled by the teacher in the introduction was examined from the point of view of what could possibly be discerned by the students, and I could identify some features, which seem to be essential for their possibility to solve the task.

In the whole class instruction graph B was presented to the students. However it was not chosen as representing a ball thrown into the air, as was the case in the textbook, but a car slowing down, stopping and then reversing in opposite direction with increasing speed. This situation was supposed to illustrate that a graph intersecting the x-axis indicates a change of direction. The following excerpt from the introduction illustrates the discussion:

1. T: OK can you see that the graph is going down all the time? OK, so we've got a situation where we actually have a graph going down like that. Now, the speed, have a look at it here. And have a look at it just a little bit later. What's the speed, is it higher or lower?
2. Laura and Fiona: Lower.
3. T: So, what's the car doing? Slowing down? OK, now watch this carefully. At this particular time (points where the line intersects the x-axis) what's happening?
4. Fiona: It's stopped
5. T: Has it?
6. Fiona (?): Um
7. T: Ok. The car is going down and slowing down and slowing and slowing. Is it going down nice and evenly?
8. Fiona: Yes
9. Laura: (nods head)

It is common that the teacher, like in this case, does not take the same example as in the textbook when introducing the students to a new topic. This could be a way to challenge the learners, to make them apply knowledge to a new situation, or to understand the generality of mathematics. But, how can the learner learn to understand this generality? What is essential for understanding the general nature of a graph, that one particular graph could represent several different situations? In this case, it implies understanding that the same graph (B) could represent both a situation like a car first speeding up, stopping, and then speeding up in the opposite direction (i.e. the example taken by the teacher) *and* a ball thrown into the air, or other situations with change of positive and negative velocity. From a logical point of view, it is reasonable to assume that the understanding of a general principle presupposes the experience of (at least) two examples. To be able to match graph B to the situation with the ball, the learners themselves must be able to see that the car and the ball are *different examples of situations* represented by *the same graph*. Or, put differently, they must experience the example as a dimension of variation.

To experience this as a dimension of variation, a variation in that respect must be opened, either by the teacher or by the learners themselves. As was shown above, in the introduction this did not happen. The teacher used one example only.

When the girls were working with the problem on their own, this did not happen either. Laura and Fiona were considering the task with the bouncing ball (e). It was suggested that it could be either graph D or E. Laura, who suggested D, argued:

> L: Wouldn't be like E, it's not, a ball's not gonna like –

And Fiona filled in:

> F: Go backwards. It's going to go woo-woo (moving hand back and forth once fast two and half times horizontally). It's not going to do that.

In my interpretation, the girls must have experienced that alternative E, with the graph intersecting the x-axis, represented a change in direction. Thus, they have discerned one critical feature of that kind of graph; that the curve indicated positive and negative velocity. However, they took the direction of the movement for-granted – that this kind of graph represents a horizontal change of the direction, only. The girls explicitly stated that the change of the movement of the ball is vertical. It does not "go backwards". My interpretation is that they were referring to the example with the reversing car. Since, the ball does not go backwards, alternative E, must be rejected.

It seems as that the possibility of that the graph could represent *both* a horizontal and a vertical movement, did not appear to the girls. They did not open up for this variation (i.e. that the change of speed could imply a vertical *or* a horizontal movement). The fact that they omitted the possibility that a graph intersecting the x-axis could indicate a vertical as well as a horizontal change of direction, was critical for their solving of the problem, and thus for their learning.

It can be concluded that in this case the absence of variation made constraints and possibilities for learning a certain kind of graphs. Since in the introduction, only one example was taken by the teacher, it was not possible for the students to experience that this particular graph could represent change of positive and negative velocity regardless of whether the movement is horizontal or vertical. That this aspect was left out; or it was not highlighted, neither in the whole class interaction nor in the peer work, seemed to be critical for learning

Concluding remarks

Finally, I have been trying to demonstrate how teachers' pedagogical actions can be described in a way that reveals dimensions of the learning environment that are critical for learning. In practise, of course, there are a lot of things that I as a teacher must take into consideration when planning my teaching. I do not claim that the theoretical framework I have presented here captures the full complexity of the issue "what matters?" What I have been trying to argue for here is that the object of learning must be taken into consideration. If teaching aims at developing certain ways of understanding or certain ways of experiencing, the learning environment must afford the learners to discern aspects, parts, wholes and the relations between them in a certain way; in a way that corresponds to the kind of understanding, experiencing etc. you want to achieve. Aspects are likely to be discerned if they are present to the learner as dimensions of variation. That which is varying is likely to be discerned. So, from the point of view of "what matters" features like presence and absence of variation are of significant importance.

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Mathematical Symbols: Means of Describing or of Constructing Reality?

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Abstract

From an epistemological perspective, every mathematical concept needs certain sign or symbol systems for coding the mathematical knowledge. In primary mathematics teaching these are first of all the arithmetical signs representing the number concept. For endowing these signs with meaning one is in need of certain reference contexts. During processes of communication, students and teacher interactively construct relationships between "signs/symbols" and "objects/reference contexts" that can be analyzed epistemologically. The epistemological analysis of the interaction in a third grade classroom between students and teacher about the correct explanation of numbers beyond 1000 shows a change in the role of the mathematical signs for numbers: First, numbers are explained with the help of concrete, empirical objects and properties, but later one can observe a kind of reversal: Numbers can be constructed autonomously by combining several ciphers. A student constructs the cipher combination 10050 and in the interactive discussion about the meaning of the number behind this cipher combination exemplifies that mathematical symbols are autonomous means for constructing a mathematical reality.

Mathematical symbols: Expression of realistic properties or construction of relations? – The example of numbers in elementary mathematics teaching

There is a widespread belief about the nature of natural numbers in primary teaching which says that they can be naturally explained as amounts of objects to be counted that exist in the children’s world of everyday experiences. The creation of numbers from pre-existing objects in reality is the central foundation for the mathematical number concept. This empirical fundament – numbers as names for objects, or numbers as amounts of several objects – can be exemplified by many number images, real world images with according number sentences in mathematical textbooks for elementary school; this kind of an empirical foundation of numbers is mentioned, proposed and also criticized in didactic literature. Jörg Voigt for instance remarks: “… especially in elementary school, the meanings of symbols (signs) are related to empirical issues (numerals to materials, geometrical terms to the physical space, etc.).” (Voigt, 1994, p. 280).

The empirical links between numbers and objects in the real world could be a helpful start for introducing the number concept; but later, they could also become severe obstacles for developing rich arithmetical strategies of a manifold number concept (Steinbring, 1997). In contrast to the empirical understanding of numbers as *numbers for counting objects* or as *names of sets*, such a conception is fundamentally questioned from a philosophical and epistemological perspective. Paul Benacerraf (1984; compare also Jahnke, Steinbring, & Vogel, 1975, pp. 216ff.) demonstrates by means of a philosophical and logical argumentation that numbers cannot be defined in a universal and definite manner by reduction to objects given unequivocally. The central consequence of his analysis is that numbers can neither be objects nor names for objects. “I therefore argue, … that numbers could not be objects at all; for there is no reason to identify any individual number with any one particular object than with any other (not already known to be a number).” (Benacerraf, 1984, pp. 290/1).

But if numbers are not objects, what else are they?

> To <u>be</u> the number 3 is no more and no less than to be preceded by 2, 1, and possibly 0, and to be followed by 4, 5…… Any object can play the role of 3; that is any object can be the third element in some progression. What is peculiar to 3 is that it defines that role - not being a paradigm of any object which plays it, but by representing the relation that any third member of a progression bears to the rest of the progression. Arithmetics is therefore the science that elaborates the abstract structure that all progressions have in common merely in virtue of being progressions. It is not a science concerned with particular objects - the numbers. (Benacerraf, 1984, p. 291).

Are mathematical concepts such as the number concept the result of empirical qualities which are then described by them, or are they social constructions which constitute a new reality? Reuben Hersh has developed the following interpretation concerning this matter:

> I propose a way of thinking about the reality and existence of mathematics which lets us keep our mathematical objects really existing, really meaningful, without resort to mysticism. The key observation is that in our world there are not two but three main kinds of reality. Mind and matter are familiar. But they do not help with our puzzle, because mathematical objects are not material, and they are not mental, in the sense of being part of anyone's private subjectivity. (Hersh, 1998, p. 13).

Hersh debates a third, a *social existence* for mathematical knowledge.

> We have not two but three choices. Material and mental are wrong. What about social? I claim that social is right. … I surely have five fingers on my left hand, so “five” has a physical meaning. On the other hand, **N** includes some very large numbers, ((2 to a very high power) raised to a very high power) raised to a very high power. It is questionable what physical meaning this big number has. So the natural numbers as describing physical objects are not the same as the natural numbers in pure

> mathematics. The fact that I have five fingers on my left hand is an empirical observation. “Five” in that usage is an adjective. There is no conceptual difficulty there, any more than in saying my fingers are long or short. But five in pure mathematics is less than the big number I just defined, and is relatively prime to it, and so on. It possesses an endless list of properties and relationships, not only in **N**, but also in **R**, in **C** , and beyond. It's part of an abstract theory. As such, it is not a material object, not a mental object, but a shared concept, existing in the social consciousness of mathematicians and others.” (Hersh, 1998, pp. 13/14).

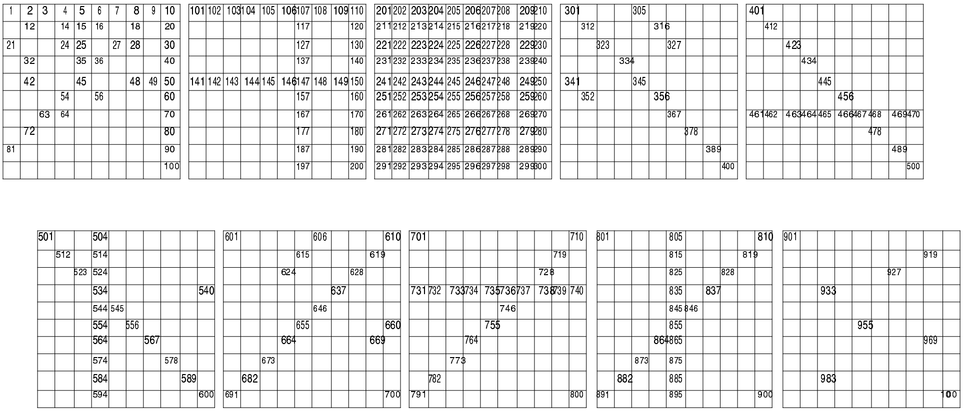
Can such a philosophical and social interpretation be a solid and useful basis for elementary arithmetics teaching at the same time? In the course of developmental processes in history as well as in teaching and learning, changes of perspectives and transitions from an empiristic and objective to a relational and functional foundation of the number concept are always necessary. The philosopher Ernst Cassirer introduces the differentiation between *substance concepts*, referring to objects and to properties of objects, and *functional concepts* which are based on the idea of relationships between elements of knowledge. The difficulty in the relation between substance and function concepts are deeply-rooted and there are often tendencies to hypostatize relational concepts later on.

> But perhaps we can best appreciate the meaning and origin of this way of thinking if we consider that even in scientific knowledge the sharp distinction between thing on the one hand and attribute, state, and relation on the other results only gradually from unremitting intellectual struggles. Here too the boundaries between the 'substantial' and the 'functional' are ever and ever again blurred, so that a semimythical hypostasis of purely functional and relational concepts arises. (Cassirer, 1955, pp. 58/9)

In the following a short teaching episode from mathematics teaching in an early third grade class will be analyzed in order to obtain a better insight into the interactive ways of dealing with the difficult relationship between an empirical and a relational interpretation of the number concept. We shall observe that changes and re-interpretations of arithmetical conceptions and of justifications become inevitable and in which way local fundamental changes from a substance concept to a relational concept are negotiated and performed.

Analysis of an exemplary episode from a primary mathematics classroom

At the beginning of the third grade the number space is extended from one hundred to one thousand. For this purpose the students of this class will use a manifold of different means of visualization and structured diagrams, as for instance the number line, or the thousands book and the 1000 dots field. From their mathematical work in the second grade, the children are already used to cope with such structured diagrams for interpreting and justifying arithmetical relationships and operations. During the mathematical lesson observed here, the children are first of all expected to become familiar with to thousands book to some extent.



In the course of the lesson, the teacher wants the students to read off numbers from the thousands book one after the other in steps of 50 starting with the number 50. Some of these numbers are written in this book in the familiar way of decimal ciphers; others are not written down, but have to be concluded structurally. This exercise, just like many others, intends to support the children‘s first orientation in reading and exploring this new number book.

Summarizing description of the first phase of the episode

The complete episode “And what comes after one thousand?” consists of two larger phases:

1. Counting in the thousands book in steps of 50, and 2. And what comes after 1000?

During the first phase the task proposed by the teacher is exercised; one child after the other names a number in the series of numbers from 50 to 1000 in steps of 50. The teacher writes every number in mathematical notation at the blackboard. Two major problems arise in the beginning: “What is the next number after 50?” Is it the 110, on the next page right on the top in the thousands book , or is it the 100? One student proposed as following numbers: “One hundred ten, two hundred.” And the second problem: “Does 700 follow the number 200?”. Here, Julia goes on in steps of 500. These problems are interactively negotiated and clarified; for instance, Jana shows on the third page how to reach the number 250 starting at 200 and moving in steps of tens. Subsequently the children are able to quickly recite the numbers up to 1000.

61 S.: Seven hundred and fifty.
62 S.: Eight hundred.
63 S.: Eight hundred and fifty.
64 S.: Nine hundred.

65 T.: Nine hundred? And further?
66 S.: Nine hundred and fifty.
67 T.: Phillip!
68 Phillip: Nine hundred and fifty.
69 T.: Fifty, Marc?
70 Marc: One hundred, uhm, one thousand.
71 T.: Yes, and what comes then? After one thousand? Another fifty?

At this point the second phase starts. The emerging problem refers to questions as, “What comes after 1000?”, “How is this number called?”, “What is the name of this number?”, “How is this number written down correctly with ciphers?”.

After the wanted number has been correctly named by one student with the name “One thousand and fifty”, this number is to be written down with ciphers, i.e. as a mathematical symbol. Different proposals are made: Kai proposes the notation “1050”, Marc writes down “1005” and Svenja writes “10050”. When discussing which of these proposals is correct, the student Felix uses the position table for interpreting this number - a suggestion strongly supported by the teacher. A position table is drawn on the blackboard and now the children are asked to write simple numbers bigger than one thousand according to their positions into this table; the positions have to be identified: ones, tens, hundreds, thousands. The numbers written down into this table are the numbers proposed by the children during the earlier discussion and some further numbers: 1050, one thousand, one thousand and five hundred, 1005, one thousand and one and 10050. In the course of explaining the numbers with the position table, the question whether one can represent numbers above one thousand with one or several thousands books arises. In this way it is clarified that one would need one and half a thousands books for the number one thousand and five hundred. The second phase (and in this way the whole episode) closes with the problem of writing Svenja's number 10050 into the position table and to reflect about how many thousands books one would need for this number.

Analysis of two important interaction scenes during the second phase of the episode

The extension of the number space from 100 to 1000 that has been carried out at the beginning of the episode represents a construction of the new numbers in which these numbers are given by the concrete positions - and therefore are based on empirical qualities. The new numbers are for the time being quantities for positions in the thousands book. Accordingly, the mathematical notations 1050, 1005, and 10050 for the verbally named number “one thousand and fifty” represent something like abbreviating names for the quantities. Since three different “mathematical names” for the number word “one thousand and fifty” are proposed, the question arises which mathematical name is the right one. We will see that the role of the mathematical number symbols begins to change with this problem.

In order to be able to decide which of the proposed notations is correct, the teacher now intends to refer to the position table as a means of providing justifications; the students are expected to look for the single positions of the new, big numbers.

111 T.: What hint could you give to Marc, Kai, and Svenja? We already have names for the different ciphers. Here we have a number with three ciphers, and here we have a number with two ciphers, here a number with four ciphers, and there also a number with five ciphers. Uhm?
112 *Teacher points to the numbers at the blackboard: 950, 50, 1050, 1005, 10050.*
113 S.: And with five ciphers.

This implicit reference to the positions of the number might be a reason for Svenja to revise her notation. In this way, she obtains a correct cipher representation; she says she has to change something and continues:

119 Svenja: Then I put there a one and a zero, and then over there the fifty.
121 T.: You think better this way too? Why now?
122 *Teacher points to the 1050.*
123 Svenja: Yes, when I look up there.
124 T.: Where did you look at? … Everybody pay attention, please.
What is, how is this number called?
125 Svenja: Two hundred and fifty.
126 T.: Uhm, and, how did this then help you here in this case?
127 Svenja: That I have to wipe off a number, because there, there are also only two hundred and fifty.

Svenja seems to make a comparison of the syntactic structure of numbers between the correct symbolic notation “250” for the number two hundred and fifty, and the imagined symbolic notation “2050” for this number according to the principle she has used for writing the number one thousand and fifty in mathematical terms as “10050”. Consequently one thousand and fifty also could only be symbolically written as “1050”. Svenja wants to wipe off a number (127).

Then the teacher explicitly introduces the position table after Felix's interpretation of the single ciphers as “ones”, “tens”, “hundreds”, and “thousands”.

131 *Felix goes to the blackboard and points at the different positions of the number 1050.*
132 Felix: Actually, it's really simple, because this is the thousand unit, this the hundred unit, the tens, the ones.
134 *The teacher writes the following table down on the blackboard, above the number 1050:*

T	H	T	O
1	0	5	0

The question whether the notation “10050” for the number a thousand and fifty is correct cannot be answered on the foundation of the prevailing empirical understanding of the new numbers as quantities of positions in the thousands book. Svenja regards the internal structure of the ciphers of the number symbol compared to the notation of the number two hundred and fifty which is already familiar to her; and Felix refers to the characterization by the place value designation of the number ciphers: ones, tens, hundreds, and thousands. With the help of Svenja's and Felix's explanations, the interpretation of the number symbols is beginning to change: Number symbols are no longer the direct result of an empirical quality respectively abbreviated names for quantities, but numbers have their own internal structure and there is a production mechanism for numbers which makes them independent from the given reality.

In the course of this episode, additional numbers bigger than 1000 are entered into the position table, and it is tried to find out how many thousands books one requires for the numbers 1500 and 1050.

The proposal made by Svenja to write the number a thousand and fifty with the mathematical notation 10050 - that has been corrected by Svenja to 1050 - shows that one can produce new numbers independently from a substantial conception, only by the combination of a quantity of number ciphers. Thereby the question which number is hidden behind Svenja's notation 10050 arises. It has to be a number, but which one? At an earlier point of time, Johann already supposed that it was a matter of millions.

106 Johann (whispering): She has written one million and fifty.

At the end of the episode a discussion about the question which number is concealed behind Svenja's symbolic notation 10050 takes place. This number is to be noted in the position table.

216 T.: …… We still have Svenja's number. Can we write it down now? Who writes it down, Felix?
217 *Felix goes to the blackboard and points to the 1 in the number 10050.*
218 Felix.: This is one million.
219 T.: It's better to start with the ones, and write them down on the right position.
220 Felix: Zero ones, five tens, …
221 *Felix writes down in the position table:*
222 Felix: …… and then again zero hundreds and thousands
224 Felix: And then still one million
225 T.: Is this really one million? … Miriam? Jana? … think about how many thousands books we need for this number?

T	H	T	O
1	0	5	0
1	0	0	0
1	5	0	0
1	0	0	5
1	0	0	1
		5	0

At this point the question of the correct naming of the new cipher position above “thousands” comes up; an answer to this question could be supported by the internal structure of the position table as well as by reflections about the number of thousands books necessary to represent this new number. Here the teaching episode closes.

The search for an answer to the question which number is expressed by Svenja's symbolic notation clearly shows that a justification of this number can no longer take place on an empirical basis by concrete qualities; this new number has been produced by the combination of cipher symbols, therefore, this new symbol constructs a new, mathematical reality, and it is not dependent on a given reality. The spoken name of such new numbers as a combination of several ciphers contains conventional aspects - like the name a million for 1000000 - on the one hand; on the other, the meaning of the new numbers results from the internal regulations and the connection of the positions of the single ciphers to each other: An additional position added to the cipher combination represents ten times as much as the position before. One does not require empirical objects for the construction of the new numbers any more, and one does actually not need spoken names for the new numbers constructed in this way: the cipher combinations are sufficient for the symbolization of the numbers, but even this cipher combination - the semiotic representation system - to be equated with the number concept.

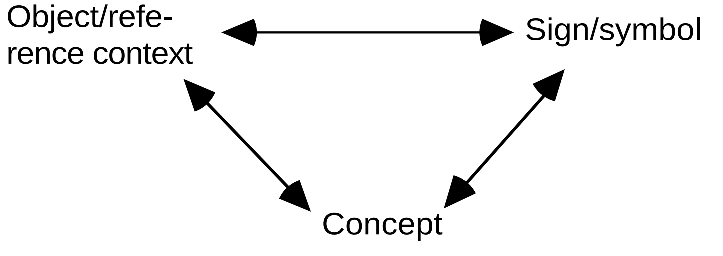
From an empirical towards a theoretical understanding of symbols: An epistemological perspective on the interactive construction of numbers

Mathematical concepts are not empirical things, but represent relations. Raymond Duval explains this position as “the paradoxical character of mathematical knowledge”:

> ... there is an important gap between mathematical knowledge and knowledge in other sciences such as astronomy, physics, biology, or botany. We do not have any perceptive or instrumental access to mathematical objects, even the most elementary, ... We cannot see them, study them through a microscope or take a picture of them. The only way of gaining access to them is using signs, words or symbols, expressions or drawings. But, at the same time, mathematical objects must not be confused with the used semiotic representations. This conflicting requirement makes the specific core of mathematical knowledge. And it begins early with numbers which do not have to be identified with digits and the used numeral systems (binary, decimal). (Duval, 2000, p. 61)

Mathematical knowledge must be represented by signs or symbols which constitute a semiotic system that is of fundamental importance for mathematical activity. This is where the portrayed paradox originates: In order to treat and understand a mathematical concept which is not directly accessible, one is in need of an appropriate symbolic representation system; in order not to mistake this sign system for the mathematical system, and in order to operate meaningfully in this system, the knowledge of the respective mathematical concept is necessary. (cf. Duval, 1993, p. 37f; Steinbring, 1997, 1998a). With regard to this epistemological position, mathematical knowledge is not simply a finished product for the classroom and for learning. The (open) concept-relations make up mathematical knowledge, and these relations are constructed actively by the student only in the social process of teaching and learning. “In these conditions, how can a student learn to distinguish a mathematical object from any particular semiotic representation? And therefore, how can a student learn to recognize a mathematical object through its possible different representations?” (Duval, 2000, p.62).

This relationship between the signs for coding the knowledge and the reference contexts for establishing the meaning of the knowledge can be structured in the epistemological triangle besides (cf. Steinbring, 1989, 1991, 1999).



In the epistemological triangle, the construction of relations between “sign/symbol” and “object/ reference context” via “concept” does not lead to a final, unique definition, but is conceived as a complex mutual interplay. The links between the corners of the epistemological triangle are not defined explicitly and invariably, they rather form a mutually supported, balanced system. In the course of further developing knowledge, the interpretations of sign systems and their accompanying reference contexts will be modified.

The crucial point here is not to consider the reference context (or the object) simply as given beforehand in a definite manner, but to be aware of the fact that this context will change during the process of knowledge development into a *relational* connection. “Things become 'objects' only through the activities of a subject.” (Bauersfeld, 1995, p. 273). According to this fundamental change of the status of the reference context, the production of mathematical meaning in the interplay between the sign system and the reference context can be described as a process in which a relatively familiar situation (the reference context) is put into a relationship with a still new and unfamiliar sign system, and, in this way, the sign system may be endowed partially with meaning by interpreting the sign system analogous to the referential system.

In the following this epistemological triangle shall be used as an instrument of analysis for the development of the meaning of numbers in this described episode; in the course of the analysis further conceptual aspects of this triangle will be clarified.

What are the meanings, representations and notational forms of representations which are negotiated and used in the course of this teaching episode?

What are specific elements of the number concept in this episode, which could be interpreted according to the epistemological triangle as instances for the categories “Sign / Symbol”, “Object / Reference Context”, and “Concept”?

• **Object / reference context**: Numbers are represented in the thousands book (at the beginning of the episode)

The central means of representation is the thousands book, in a way the **Object**; in this frame, the numbers do also exist (partially) in their way of writing ciphers, especially the quantity of the number is represented (steps of tens, fifties, series, number strips, to jump, to go on, half a thousands book, etc.). The thousands book on the one hand is as an empirical object, representing the quantity of the number; on the other hand it can be interpreted as a simple relational structure, displaying relations between numbers.

• **Object / reference context**: Cipher structure of the numbers is represented in the thousands book (later in the episode)

The position table displays the internal, systemic structure of the number-symbols and exceeds the name function and the function of arbitrary (conventionalized) mathematical signs.

• **Sign / symbol:** Numbers as outspoken names (at the beginning of the episode)

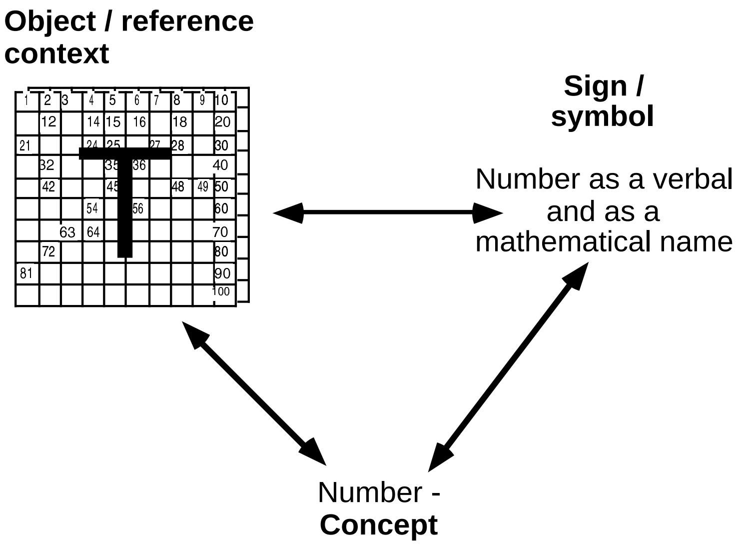
Numbers are named by words of natural language, they have a verbal, outspoken name: two hundred and fifty, eight hundred etc..

• **Sign / symbol:** Numbers as mathematical signs (a little bit later in the episode)

Numbers are written by means of ciphers: 250, 800; also this representation is conceived first of all as a “mathematical” name, an abbreviation for the “verbal” name. With the introduction of the cipher writing the name function of the number name begins to change into a mathematical sign and later into a true symbol!

According to these background considerations it is possible to make different interpretations of the epistemological triangle for the analysis of this teaching episode.

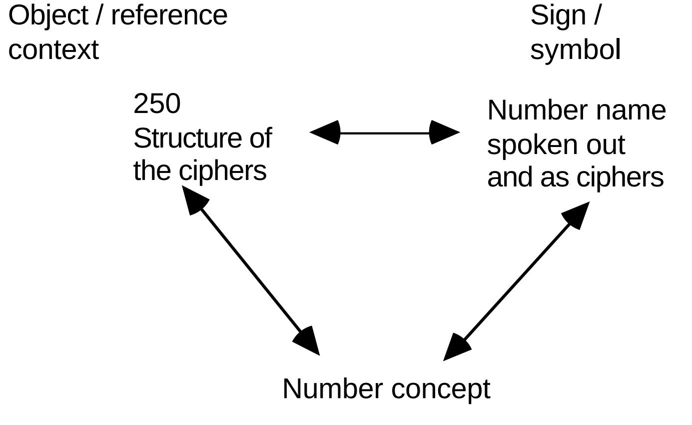
In the beginning of the episode, numbers are understood as “existing” empirical properties in the thousands book and in this way the children count in steps of fifties up to one thousand; one could describe this with the help of the above epistemological triangle: Numbers are names for empirical objects, for quantities, etc.; these names are verbal or mathematical. The numbers are verbally outspoken, written in their cipher representation and pointed at in the thousands book.



By stating the question: “Yes, and what comes then? After one thousand?” (71), the old context is exceeded and broken up. The name “One thousand and fifty” now is to be written down in mathematical notation; the contributions and the critiques of the students clearly show how the verbal name and the mathematical name (the mathematical sign code) differ, and that they are no longer - in principle - identical. The cipher name and the verbal name of the number are definitely different from each other!

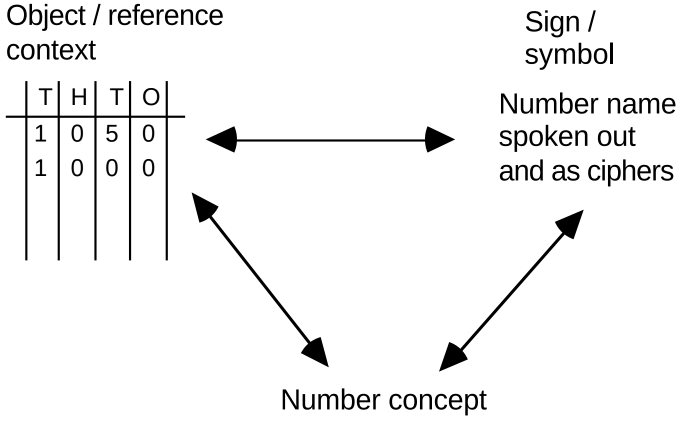
On this basis there are now two different proposals made by the children of how to decide which proposal is correct:

1) Svenja corrects her number and justifies this by a comparison with the number 250; this comparison seems to be based on the syntactical structure of the ciphers; but it is not possible to succinctly follow Svenja's justification. In this way, the interpretation in the epistemological triangle shifts again; now the object is no longer the thousands book but the syntactical structure of the cipher representation.

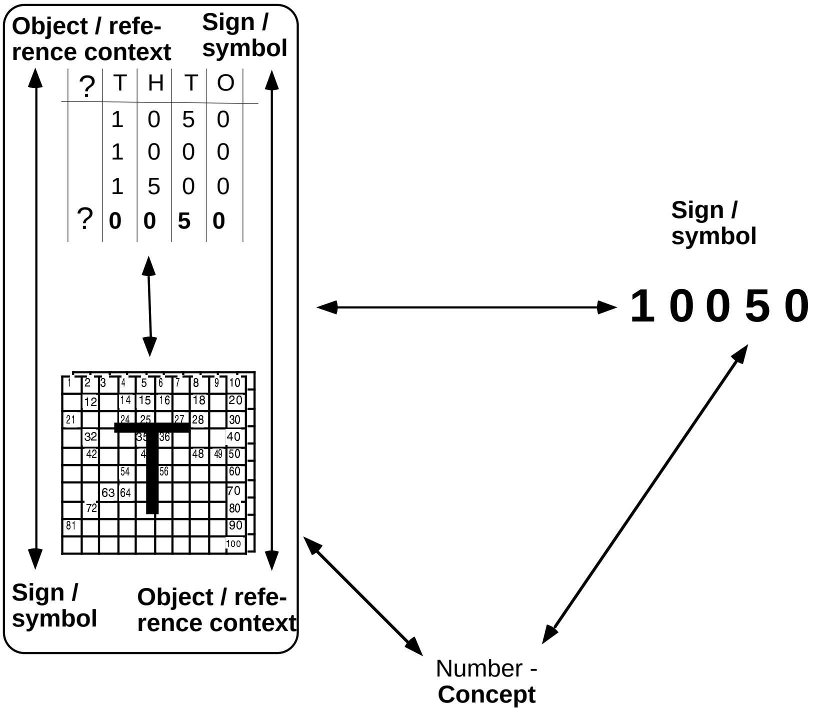


2) Felix remembers the interpretation of the separate ciphers in the cipher-name of a number as “Thousands, hundreds, tens and ones”; the teacher reinforces this by introducing the position table; besides the syntactical structure of the number, this position table emphasizes the internal, systemic connections between the ciphers themselves. With this introduction of the position table (the discrimination between name and cipher) the numbers obtain a new referential context, and once again, the reading of the epistemological triangle is changed.

Here, the reference context does no longer consist of the thousands book and consequently of an empirical, given foundation which the new numbers seemingly can be built up on. The new, changed reference context represents a symbolic, structured system itself. The numbers given by the mathematical symbols, as for example 1005 or 1050, are developed meaningfully by referring to internal structures present in the symbol itself - the positions and their relations to each other.



Especially, the changed interpretations of the epistemological triangle can be observed in the phase during which the number 10050 is to be interpreted with the help of the thousands book and the position table. First, Felix writes down Svenja's number into the known reference context “position table”, starting from the right with the single ciphers and he names the positions he knows: “ones”, “tens”, “hundreds”, and “thousands”. The position following next is still unknown to him, he assumes it might be “millions”.



At this point the rather familiar reference context “position table” partially turns into an unknown symbol system because of the problem of how to interpret the number “10050”; for still having a basis for further reflections and argumentations the teacher offers many thousands books as a new possible familiar reference context by stating the question:

225 T.: Is this really one million? … Miriam? Jana? … think about how many thousands books we need for this number?

One would need 10 thousands books, which could possibly provide a plausible explanation for the name of the new cipher position as “ten thousands”. And this could be an exemplary expression for the important internal relation between different place values that has to be constituted: The »times 10« relation between one place value and the next “above” it. At this point, the teaching episode ends with a conflicting situation: On the one hand, the cipher combination mentioned by Svenja definitely represents a number, but by what means and in what way can this number be understood substantially? Is a reference to an according quantity of external objects necessary and connected to the correct explanation given by the teacher's authority about which quantity of objects this number represents? Or can the meaning of the number given by the cipher combination be inferred from the combination itself?

Concluding remarks: What comes first – mathematical signs or objects?

The interaction during this episode started with an implicit agreement about what the numbers in the given situation are; they are provided by the thousands book and the students are all able - after some difficulties at the beginning - to read off these numbers correctly. The numbers are to be found in the thousands book, they have a name one can speak out, and the children also can express the name of the number from the mathematical number signs in the book.

In the course of this episode three forms of structured contexts are used to explore the relational character of the number "one thousand and fifty". Svenja seems to make use of the internal syntactical structure between the ciphers for determining the correct symbolic notation for "one thousand and fifty" by comparing with the cipher notation of 250; she refers here to a rather abstract relational structure. This procedure then is continued and made more concrete by the introduction of the position table as an already known material. The interpretation for the single ciphers now is given by the positions: "ones", "tens", "hundreds", etc. The sign "5" for the number "five" can be differently interpreted within the structure of the ciphers, according to its position; consequently the internal relational structure of the mathematical sign chain has to be looked at. At last the thousands book is used as a relational reference context (and as a sign / symbol system as well) and not only as an empirical object in which one can directly find the numbers as things, as for instance at the beginning of the episode. In particular, this new function of the thousands book is constituted by using further thousands books for interpreting numbers beyond "one thousand".

At the end, the interactively negotiated and changed interpretation from a rather empirical to a relational conception of numbers nearly shows a kind of paradox: Numbers are not names for things, they are not objects and the number "one thousand and fifty" cannot be identified with the cipher structure "1050". The number "one thousand and fifty" is determined by its relationship to other numbers preceding or following it. In the course of the interactive discovery of this relational structure, this arithmetical progression, with the help of the position table and two thousands books the students are faced with the situation of being neither able to use the position table nor the thousands book as a certain basis for serious argumentation when trying to interpret Svenja's number "10050". Do several thousands books explain the position table, or, vice versa, does the position table explain the structure and use of the thousands books?

Suddenly, a seemingly secure and familiar, concrete basis for the explanation and justification of numbers is lost. But a deep epistemological problem of a mathematical concept comes to the fore. The interactive process more and more reveals the relational structure of the number concept, an "abstract structure, …" and the elements of this structure have no other purpose than to put them into relation with the other elements of the structure (cf. Benacerraf, 1984, p. 291). The change of a substance concept to a function concept (a relational concept) makes the number concept (understood as a relationship between numbers, being an arithmetical progression – in our example in form of the "times 10" relation between neighbored place values) an autonomous entity with regard to the categories of “Object / reference context” and “Sign / symbol system”. Now it is the number concept itself together with the constituting relationship, which allows an exploration of the sign chain “10050” with the help of the symbolic diagrams “position table” and “thousands books”, but not in a way of simple reduction to one of these diagrams.

The role of mathematical symbols for the elementary number concept has radically changed during the interactive discussion in this exemplary teaching episode out of primary mathematics teaching: While the things of the findable experience world - objects to count, here the given positions in the thousands book - were at the beginning the incontestable basis which only made the construction of numbers building up on it possible, namely as symbols to describe given qualities of reality, at the end of the teaching episode, the number symbols became autonomous means of constructing an arithmetic reality; Svenja's cipher-combination 10050 can be regarded as a typical example for the automated construction of numbers, for which a theoretical number understanding has to be developed in the classroom discussion with the children.

In a philosophical analysis Brian Rotman (1993) has pointed out, referring to the historical development of the number zero, by a changing interpretation of the zero from zero as a sign for nothing into the new interpretation of zero as a sign for the absence of other numbers, how through this new role of the number zero the system of cipher combinations gradually became an autonomous production mechanism for the numbers. According to Rotman, the zero becomes a metasign (it refers to other, non-present signs in the position notation) which nevertheless takes part in the production process of the number signs itself, in fact even starts this process and thus loosens itself from a given reality. “In other words, the simple picture of an independent reality of objects providing a pre-existing field of referents for signs conceived after them, in a naming, pointing ostending, or referring relation to them, cannot be sustained. … The result is a reversal of the original movement from object to sign. The signs of the system become creative and autonomous. The things that are ultimately ’real’, that is numbers, … are precisely what the system allows to be presented as such. The system becomes both the source of reality, it articulates what is real, and provides the means of ’describing’ this reality as if it were a domain eternal and prior to itself…” (Rotman, 1993, pp. 27/28).

In the classroom episode dealt with, the role of the zero is treated partially, for example in Svenja's first proposal; especially by the utilization of the position table the role of the zero as a sign for the absence of other number signs is used implicitly, and this role as a metasign leads already in primary teaching to the start of an autonomous production of numbers which results in a loosening from the given existence of concrete objects. This detaching could be demonstrated in an exemplary and situative way in the observed episode. The sign 10050, constructed as a cipher-combination, does not require an explaining object, it emerges by autonomous construction. One fundamental orientation of elementary school mathematics consists in the effort of making the rather abstract mathematical concepts more visual by a foundation in concrete objects and qualities. Our example shows that the meaning of this new number ultimately cannot be reasoned satisfactorily on the basis of given objects and qualities (thousands book or quantities): For an appropriate explanation of the new and big numbers, it is meaningful and necessary to reveal the construction mechanism of these cipher combinations - i.e. the structure between the different positions in the position table - and to make it understood gradually; and with the help of this construction of numbers these number symbols become means of constructing an autonomous mathematical reality which can then also be interpreted into experience reality.

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"Mathematics - that's what I can't do." People's Affective and Social Relationship with Mathematics[i]

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Introduction

The overall purpose of mathematics education research can be described as investigating and forming people's relationship with mathematics. The statement "Mathematics - that's what I can't do." - summarises findings from qualitative and quantitative studies of adults' relationship with mathematics (Cockcroft, 1981; Wedege, 1995; Lindenskov, 1996; Wedege, 1999; Coben, 2000; Evans, 2000). Defeat and selection in school mathematics education may result in this belief in adults. However, the statement is contrary to the math-containing competences that many semi-skilled workers demonstrate in their job and everyday practice.

Differences between informal mathematics (street mathematics, folk mathematics) and school mathematics (that people learn and practice in formal education) have been investigated in a series of studies. This paper argues that these differences are one of the reasons why adults don't recognize the informal mathematics in their everyday life as mathematics.

Competence is a readiness for action and thought based on knowledge, know-how and dispositions learned and incorporated through school and everyday life. Although there is no 'typical' participant in further education, blocks and resistance are two central phenomena when adults learn mathematics. This is connected to how they experience themselves as competent persons without mathematics and that mathematics is not perceived as relevant to their life project. If adult numeracy, as a personal math-containing competence, instead of academic mathematics, is seen as aim in adult training and education programs, then there are some possible consequences for teaching practice. In the development of the first Danish Adult Numeracy Curriculum, basis has been to make mathematics visible in people's everyday practices and in the math-containing competences of adults.

Context and affect

In the 1990's, Diana Coben and Gilian Thumpston conducted a series of research interviews about adults' experiences of mathematics in their lives, both past and present, in order to create what they have called "mathematics life histories". They term the mathematics one can do but which one does not recognise as mathematics "invisible mathematics", after Mary Harris and the ethnomathematician Paulus Gerdes (Coben, 2000). Their work is an important contribution to the understanding of adults' relationship with mathematics, and the following terminological clarifications of the two terms 'context' and 'affect' match their findings.

Researchers in mathematics education use the term 'context' in two meanings and I propose a terminological distinction. The one is context representing reality in tasks, word problems, examples, textbooks, teaching materials etc. (e.g. everyday life situations evoked in a problem-solving task). I call this type *task context*. The other fundamental meaning has to do with context for learning, using and knowing mathematics (e.g. school, everyday life, workplace), or context of mathematics education (e.g. educational system, educational policies). I call this type *situation context* (Wedege, 1999).

I would like to stress that this is just a terminological clarification, not a theoretical/conceptual definition of ‘context’. My distinction is only meant to shake things up and I will use a situation from my favourite, the Danish cartoonist Claus Deleuran, to illustrate the terminology. Here the task context is "digging ditches" but the situation context is a mathematics classroom. When the situation context is a building site the task is in fact "digging ditches" - not doing calculations although the worker is doing a lot of calculations.

In a mathematics classroom:

The teacher says,
- "Problem number 123"
- 3 men have to dig a ditch that is 1.20 m wide and .85m deep. If together they can dig a 3 X 3m (cubic meter) ditch in 30 minutes, how long will the ditch be after 8 hours of digging. ”
- Figure that one out!

The pupil thinks,
- Hmm, if we figure a good shovelful is 3 litres, then 3000 times 1 divided by 3 equals 1000 shovelfuls divided among 3 men. That's 33 1/3 divided by 30 minutes. That's a little more than 11 shovelfuls per man per minute. That's the same as one shovelful per 5 3/4 seconds. I wonder if they can keep that tempo all day?

The pupil says,
- Is it on an hourly wage or by contract? And how many breaks do they get?

The teacher says,
- Søren!! That's too much!
- Why can't you just do your calculations like everyone else?
- Are you just being a smart alec or what?

(Søren is thrown out the door.)

My second terminological clarification concerns "affect". According to McLeod's review (1992), beliefs, attitudes, and emotions are used to describe a wide range of affective responses to mathematics. These three terms are not easy to distinguish but they vary in the stability: beliefs and attitudes are generally stable, but emotions may change rapidly. They also vary in level of intensity of the affects that they describe, increasing in intensity from "cold" beliefs about mathematics to "cool" attitudes related to liking or disliking mathematics to "hot" emotional reactions to the frustrations of solving non-routine problems. McLeod also distinguishes beliefs, attitudes, and emotions in the degree to which cognition plays a role, and in the time they take to develop. This is not the only analytical description in different dimensions of the affective area in mathematics education research (cf. Evans, 2000, pp. 43-45) but I find these three dimensions both operational and cognitively satisfactory.

Thus I have adopted McLeod's characterisation of the affective and understand *affect* as comprising three dimensions: beliefs, attitudes and emotions where *beliefs* include self-perception(e.g. "Mathematics - that's what I can't do."), aspects of 'identity' (e.g. "We - the semi-skilled workers - not using mathematics versus "the others" using mathematics), and confidence; and *attitudes* (e.g. maths anxiety) are more stable than *emotions* (e.g. panic).

Invisible mathematics

Coben and Thumpston have used the term "mathematics life histories" to describe adults' accounts of their mathematical experiences throughout life - both those that are explicitly mathematical (such as being taught subtraction at school, or working out a budget as an adult) as well as those in which mathematics is implicit (such as knitting or judging distances when driving). They used qualitative research techniques involving semi-structured interviews which they recorded on audio tape.

Almost all the interviewees remarked the importance of mathematics and success in math examinations. On the other hand it appears that once people have succeeded in applying a piece of mathematics, it becomes 'non-mathematics' or 'common-sense'. Thus they never perceive themselves as successful: mathematics is always what they cannot do. Some themes emerged from their research, two of which they designated as follows:

- *the door* - marked 'Mathematics', locked or unlocked, through which one has to go to enter or progress within a chosen line of work or study. This image was often used, reflecting the frequency with which mathematics tests are used to filter entry into training and employment.
- *invisible mathematics* - the mathematics one can do, which one does not think of as mathematics - also known as common sense. (Coben, 2000, pp. 54-55)

Coben and Thumpston claim that this invisible mathematics may have limiting effects on the individuals concerned and perhaps more widely, on conceptions of mathematics in society in general:

Firstly, for the individuals concerned, 'mathematics' is rendered unattainable. It becomes by definition, what they cannot do. Secondly, the individuals' negative self-image as someone who is unable to do mathematics may impact on their confidence as learners, since mathematical ability is widely considered as index of general intellectual ability. Thirdly, in society at large, the image of mathematics as difficult, only for the selected few, is maintained rather than challenged.

Eileen is one of their interviewees. She is 39 years old and studying for a psychology degree. Having come to recognise her own 'invisible mathematics' as an adult, Eileen puts it:

> If somebody says "I can't do maths", I think what they are saying is "I can't do that part of it", they are not saying "I can't add up or take away, I can't work out how much my mortgage is going to be, or I can't work out how much I've got left". What they are saying is "I can't do that part of it" but *that's what they call maths* and I realise that was what I was doing. (Coben, 2000, p. 55 - my italics)

Some people do not recognise what they can do as mathematics unless it is in the form of a standard algorithm or formula. For most of the interviewees, 'proper mathematics' seemed to consist mainly of standard algorithms in arithmetic. This narrow conception of mathematics is compounded by the widely-held view that there is only one standard algorithm for each operation - usually the one the person was taught in school (p. 56). In everyday life people use their own methods, developed by the individual or handed down through a community. From the English Cockcroft Report we know that adults often get a guilty conscience because their methods are different from the 'correct methods' they learned at school (Cockcroft, 1982). Coben and Thumpston have this example: one of the male interviewees talked about a problem he had of marking out an athletics field for young children, reducing the standard adult track and throwing pitch markings:

> He had converted the running track but was having difficulty with the curved markings for the throwing events. How could he find the correct formula that would allow him to mark the pitch? He knew he could do it by using a rope and pegs - but as he would not be able to write down the calculation in a suitably 'mathematical' form *he felt that this was not 'doing mathematics'*. (Coben, 2000, pp. 56 - my italics)

Another of their interviewees, May, a woman of 79, expressed it like this when talking about doing do-it-yourself jobs around the house:

> You measure, put up shelves, you measure distance, size, and the backets, where they go - that all involves general maths. To me, though,

> *that's just common sense*. […] You don't think of [it] as being maths. (Coben, 2000, p. 57, - my italics)

From my own research, I have similar examples. In order to identify and describe mathematics in semi-skilled job functions and to analyse how mathematics knowledge at work is interwoven with vocational qualifications, I have investigated selected firms within four lines of industry: building and construction, the commercial and clerical area, the metal industry, and transport (Wedege, 2000). At a large electronics factory, I observed a semi-skilled worker with many years of experience in production. She is now working in the quality control. When I interviewed her after the observation asking questions about the mathematics that I found in her work, she said: " … that's just the logic of battery hens." In this situation context, I think that common sense is seen as instinct or intuition as opposed to that which has to be learned or as self-evident as opposed to serious knowledge.

We may find another example of this belief in a 'mathematics life history' interview that I made with my mother, Ruth, five years ago. It was a narrative interview, not a structured interview. Although she knew that I was interested in the mathematics in her life she didn't speak of herself as someone doing mathematics. Only once during the whole interview, she linked 'mathematics' with her everyday competence. Ruth is an active woman of 75. Bridge is her main interest and every summer she organises tournaments.

> T: How do you plan a tournament, for example with 9 tables?
>
> R: Oh, that's quite difficult. 5 tables are a minimum. You have to make a table plan and plan how you are going to move around. 5 tables, that's 10 pairs, and they have to play 9 rounds if everyone is to play against everyone. 3 games at each table, that is 27 games, and that's enough for one evening. But if there are 6 tables, that's an even number, then the cards have to be put over to tables 3 and 4. They are not played. That's called a relay - in order to make everything work out. Tables 1 and 6 share cards.
>
> T: What does that mean?
>
> R: If we play 3, maybe 4, rounds, then table 1 plays first. ... A tournament with 6 tables is called a Howell tournament. 7 tables and it's a Mitchell. (...) At the tables you sit east-west and north-south. The tables are numbered. When the guests arrive each pair gets a number: table 2, north-south, or table 4, east-west. North-south remains at the tables. East-west go to a table with a higher number. The cards go to a table with a lower number. It's all calculated very *mathematically*, so that everyone plays with everyone else and nobody plays the same cards.
>
> (Wedege, 1999, p. 221 - my italics)

Earlier in the interview, when Ruth talked about leading a bridge tournament she said, "It's also a question about arithmetic". It was only later when I asked her to explain the principles for planning a tournament, she said, "It's all calculated very mathematically".

This was the first and only time during the whole interview that Ruth linked her own competence in everyday life with mathematics. But her first reaction to the question was that this was difficult. The way she explained to an ignorant person like me the way a tournament leader plans is evidence of classical mathematical thinking when solving problems: she starts with a simple example in order to explain what is more complicated.

The belief in adults ("Mathematics that's what I can't do") is contrary to the math-containing competences that many adults demonstrate in their job and everyday practice, and the statement must be analysed from three inter-related perspectives: (1) the adult's mathematical skills, (2) the adult's conception of mathematics, and (3) the adult's self-perception in relation to mathematics (belief).

Competence as a readiness for action and thought – Numeracy

In English-speaking countries, ‘numeracy’ is constructed as an analytical concept for certain basic skills and understandings in mathematics, which people need in various situations in their daily life. The term ‘numeracy’ is often used as a parallel to the concept of ‘literacy’. Danish does not yet have a single expression corresponding to the term 'numeracy'. Nevertheless, Lena Lindenskov and I have chosen to use the noun *numeralitet.* A term now adopted by the Ministry of Education.

'Numeralitet' (numeracy) describes a math-containing everyday competence that everyone, in principle, needs in any given society at any given time:

- Numeracy consists of functional mathematical skills and understanding that in principle all people need to have.
- Numeracy changes in time and space along with social change and technological development (Lindenskov & Wedege, 2001, p. 5)

It is this “in principle” that makes possible a general evaluation (as in the big international surveys) and the developing of general courses in numeracy. All adults who participate in a numeracy course will, in fact, have their own perspectives (why am I here?), their own backgrounds and needs (what am I going to learn?) and their own strategies (what am I learning?).

We have developed an operative model for the study of adult numeracy. It has four dimensions, which are

- *Media* (a) written information and communication (b) oral information and communication, c) concrete materials, d) time and e) processes.
- *Context* - in the meaning of situation context - (a) working life, (b) family life, (c) educational context, (d) social life, and (e) leisure.
- *Personal intention* (a) to inform/be informed, (b) to construe, (c) to evaluate, (d) to understand, (e) to practice, etc.

- *Skills & Understanding* - Dealing with and sense of (a) quantity and numbers, (b) dimension and form, (c) patterns and relations, (d) data and chance, (e) change, (f) models.

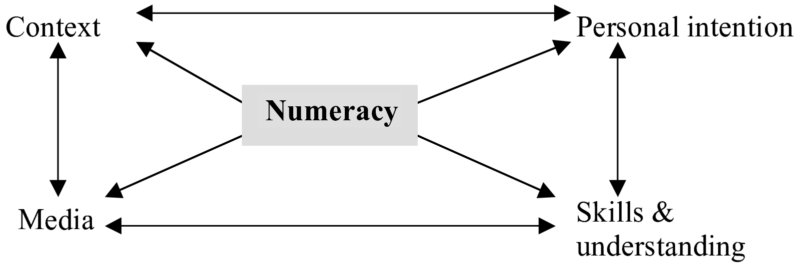

Figure 1. Four dimensions of numeracy

So far the operative model has been productive. Concurrently with the use of the working model for numeracy in empirical studies and in educational planning, we have clarified and developed the division and exemplification of the four dimensions (Lindenskov & Wedege, 2001).

From the beginning of 2000, Lena Lindenskovs and my own educational engagement has been focussed on basic adult mathematics education. We were asked by the Ministry of Education to develop an adult numeracy curriculum and a new teacher education.

The aim of the education is that adults develop their *numeracy*, as described above. The content is described as a dynamic interplay between a series of activities, various types of data and media, as well as selected mathematical concepts and operations. We found the inspiration to these activities (counting, localising, measuring, designing, playing, explaining) in Alan Bishop's cross-cultural studies of mathematical components in everyday activity (Bishop, 1988).

Mathematics in workplace vs Mathematics in school

"Do you use mathematics in your work?" Although many adults use numbers and formulas in their daily life, "No" is the most common answer to this question (Harris, 1991; Wedege, 1999, 2000). Mathematics is interwoven in technology - in technique, work organization and qualifications. However, modern computer technique hides the use of mathematics in the software, and mathematics as a visible tool disappears in many workplace routines. But that isn't the only reason for the negative answer. The adults just don't connect the every day activity with mathematics which most of them associate to the school subject or the discipline.

Differences between informal mathematics (street mathematics, folk mathematics) and school mathematics (that people learn and practice in formal education) have been investigated in a series of studies. A working hypothesis in my investigations has been that there are systematic differences between mathematics at the workplace (or numeracy) and in traditional mathematics instruction.

This statement is developed and documented on the basis of my own and others research (e.g. Harris, 1991; Hoyles, 1991; FitzSimons, 2000; Wedege 2000ab). The well known activity 'solving tasks' serves as an example:

In traditional mathematics instruction, reality is a pretext to use mathematical ideas and techniques. The task constitutes a central element and structures the teaching. The task is primarily used to practice skills (use of algorithms and concepts) and to test skills and understandings. Thus, the task is often solved by the individual student and it might be conceived as cheating to hand in a joint solution. The task is formulated by the teacher, the textbook or the program. The task has one correct solution and many wrong solutions. (Accuracy in the school and tolerance at the workplace are two different things.) Solving tasks has no practical meaning: the results are not used for anything except, maybe, solving more tasks. In the so-called 'problems' the task context is often practical problems, but the aim is to find the correct result by using the correct algorithm not to solve the practical problem.

At the workplace, reality requires the use of mathematical ideas and techniques. The 'tasks' result from solving a working task where the numbers are to be found or constructed with the relevant units of measurement (hours; kg; mm). It is the working tasks and functions in a given technological context which controls and structures the process, not the 'task'. Some of these tasks look like school tasks (the procedure is given in the work instruction) but the experienced worker has his/her own routines, methods of measurement and calculation. Circumstances in the production might cause deviations from the instruction or that the number of random samples in the quality control is raised or reduced. It is characteristic that tasks are solved in different ways and that different procedures and solutions might be OK. At the workplace solving tasks is a joint matter: you have to collaborate, not compete. Solving tasks has always practical consequences: a product, a working plan, distribution of products, a price etc. (Wedege, 2000a).

Personal dispositions

Blocks and resistance are two central phenomena when adults learn mathematics. Although one cannot say that there is 'typical' participant in further education, a significant number of the participants have negative perceptions of themselves, the institutions and mathematics. Many adults who start on vocational education are surprised that the programme includes teaching in mathematics. One of their reactions may be resistance in the learning situation which has to do with the fact that they have experienced themselves as competent persons without mathematics and that mathematics has not been perceived as relevant to their life project (Wedege, 1995, Lindenskov, 1996). This belief stems from their experience in various communities of practice (work, family, leisure), where basic arithmetical skills have perhaps been sufficient to cope with the challenges, or where mathematics has been hidden in techniques and technologies. Or, in other words, in communities with different practices - as contexts for knowing and learning mathematics.

We know that some adults change their attitude to mathematics during a training course while others fail to do so. For some people, this means something for their image of themselves and their life project, for others not. I would claim that these differences cannot be explained solely within a situation context that consists of the teaching and the participants' current situation and their perspectives for learning or not learning. In our analyses of the level of identity in the adults' experience with mathematics we find personal dispositions which generate actions and attitudes in interaction with personal understanding and skills. With inspiration from a search model for subjectivity viewed in the perspective of qualification (Andersen et al., 1996), we can describe participants' experience on three analytical levels:

The level of skills	Specific skills in arithmetic and mathematics.
The level of understanding	General mathematical knowledge, e.g. understanding and dealing with the theory-practice relation, and conception of mathematics.
The level of identity	A mixture of incorporated skills and understanding (mathematical thinking, tacit knowledge) *and* self-concept, attitudes, emotions and motives.

The basic level of experience can be summed up in adults' relationship to mathematics, mathematics in the world surrounding us and ourselves; one could say, subjective dispositions to mathematics which generate actions and attitudes in interaction with personal understanding and skills. These observations have lead me to *'habitus'*, the sociological concept defined and employed by Pierre Bourdieu as systems of durable, transposable dispositions as principles of the generating and structuring of practices and representations (Wedege, 1999, 2000a). People's habitus is incorporated in the life they have lived up to the present. Habitus (as a system of dispositions) contributes to the social world being recreated or changed from time to time when there is disagreement between the people's habitus and the social world. The dispositions which constitute habitus are durable. This means that although they are tenacious, they are not permanent (Bourdieu, 1980).

Jean Lave's theory of situated learning offers a theoretical framework to describe and analyse adults learning mathematics through teaching and everyday practice. But in her theory where communities of practice function as situation context for learning, a concept is missing which makes it possible at once to understand the subjective and objective conditions of the learners' dispositions for learning, or in other words, the objectively determined subjective conditions for learning. Lave is "taking learning to be a matter of changing participation in ongoing, changing social practice." (Lave, 1997). Her theory fits perfectly with the idea of lifelong learning, but within her theoretical framework the situation context is limited and it isn't possible to take account for the adult's mathematics life history in the analysis.

Social relationship with mathematics

In the introduction, I stated that the overall purpose of mathematics education research might be described as investigating and forming people's relationship with mathematics. The inspiration to this formulation is found in Roland Fischer's article concerning Mathematics as a mean and a system (1993). I claim that the core issue in mathematics education research is people's learning - or not learning - mathematics. And that the learning process has three analytical dimensions: a cognitive, an affective and a social dimension. In the research field, we are interested in "people's cognitive, affective and social relationship with mathematics."

But what is people's social relationship with mathematics? The following statement by a former graphic designer illustrates at one and the same time a feeling of impotence in the face of technological progress, and a belief of mathematics as power: "Mathematics is not democratic. Mathematics is evil. It has caused unemployment in my trade." The development of technology and the pace at which it takes place are experienced as being unavoidable, as something we are helplessly confronting. In this situation context, mathematics may become personified and perceived as power (Wedege, 1995). Another example of what I would call "people's social relationship with mathematics" may be found in an interview survey with 45 participants in Adult Vocational Training conducted by Lena Lindenskov. Here many of the interviewees have a perception of mathematics as "We" - the semi-skilled workers not using mathematics versus "the others" - the engineers, the foremen etc. using mathematics. (Lindenskov, 1996)

While people's social relationship with mathematics has not been on Bourdieu's agenda in any of his empirical work, there are several reasons why it seems fruitful to attempt to combine habitus as an analytical concept with the theory of situated learning when working with adults and mathematics (Wedege, 1999, 2000a). The dialectic between cognitive, social and affective dimensions in the learning process makes inter-disciplinarity a must in research and practice of adults knowing and learning mathematics.

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[i] A version of this presentation has after the conference been published in Wedege (2002).

Assessment of Younger Children's Mathematical Knowledge

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Abstract

In this paper we account for the ways in which teachers assess and describe pupils' work. We also consider what concepts pupils display of numbers in fraction and decimal form. In our work with The National Test for school year 5 we continuously collect data from pupils and teachers in all of Sweden and we use these for our analyses.

Background

We work in the PRIM-group, a research group at the Stockholm Institute of Education, which has been commissioned by the National Agency for Education to produce national tests in mathematics.

A new Swedish national curriculum for compulsory basic school and for the upper secondary school went into effect in 1994. In addition, there is a nationally defined syllabus for each subject. The compulsory basic school syllabi indicate the purposes, contents and goals for teaching in each subject. The goals are of two kinds: goals to aim for and goals to be attained.

At the end of school year 5, national tests are held in Swedish, English and Mathematics. These tests are not compulsory. The main purpose of the tests is to help the teachers in assessing whether the pupils have reached the demands of the curriculum and syllabi. The tests also have a diagnostic purpose. The teachers are supposed to assess the test holistically and use no points. It is important to analyse how the pupils solve the problems and examine the quality of their work to find their strengths and weaknesses in mathematics. The teachers are then supposed to consider both the assessment of the pupils' work in the subject test as well as the overall assessment of the pupils' mathematical knowledge. To describe each pupil's mathematical knowledge the teachers can use the Competency Profile. Our hope is that teachers, with the help of the profile, can gain a more balanced picture of the pupils' knowledge in mathematics.

Teachers' assessment

In this section of the paper we describe how teachers perform assessment in mathematics and what conceptions they have about assessment in mathematics.

Our questions are:

In what way and with what quality do teachers assess and describe pupils' work?
What conceptions do teachers have about assessment in mathematics?

1 (2)

Competency profile in mathematics year 5 for ____________________

The purpose of the competency profile is to show your *overall assessment* (assessment of Subject Test 5 + other work) of the pupil's merits and weaknesses in mathematics.

Competency areas	Teacher's comments	Attainment of objectives for year 5
		Satisfactory
Concrete problems in the immediate environment		
Arithmetic		
basic number perception and number concepts for natural numbers calculating in natural numbers: mental calculation, written calculating methods		
understanding and being able to use calculating methods calculating with a calculator		
numerical patterns, unknown numbers number perception – simple numbers in the form of fractions and decimals		
Geometry		
comparing, estimating, measuring: length, areas, volumes, angles, mass		
basic spatial perceptions geometrical figures and patterns scale		
time and time differences		
Statistics		
diagrams, tables, measures of central tendency		

We have analysed 200 competency profiles and the teachers' assessment of some tasks in 200 pupils' works. These data has been examined both qualitatively and quantitatively. We have analysed the teachers' answers to a questionnaire as well. These inquiries then form a picture of the conceptions of teachers' assessment of younger pupils' knowledge in mathematics.

Competency profile

The competency profile is divided into 8 competency areas corresponding to the goals, which the pupils are to have attained by the end of the fifth school year. The teachers are supposed to show their assessment by placing crosses on lines. There is one line for each competency area. They can also verbalise the assessment by writing in spaces beside the lines.

- In 66 % of the profiles the teachers do not use or use to a very little extent the spaces beside the lines to comment their assessment of the pupils' knowledge in words.
- In 88 % of the profiles the teachers show their assessment by putting crosses on 6 or more lines.

The profiles in which the teachers have used the spaces to comment are analysed.

- In 31 % of the profiles, the teachers have only written about the pupil's weaknesses, for example, what knowledge the pupil does not have.
- In 17 % the teachers have only written about the pupil's strengths.

According to the teachers, most of the pupils attain all goals. When we compare this with the two statements above, we find that teachers, when writing comments, tend to focus on negative opinions.

The test contains only some of the goals to be attained. The teachers can use their own diagnosis or tests from earlier years to assess the areas missing in the test. Our research group analyse pupils' work and also assess the pupils' knowledge in competency areas.

- The teachers' assessments of the goals that are tested correspond with our assessment of the same goals.
- The teachers overestimate the pupils' competencies in the mathematical areas, which are missing in the test.

Teachers' assessments of pupils' work

The test provides assessment instructions for the teachers. These instructions focus on each goal to be attained and those areas of the competency profile to which the sub-tests correspond. The instructions include examples of acceptable answers, assessment comments and authentic pupil solutions. The teachers' assessments of one sub-test are analysed. This part contains different kinds of tasks.

- In 67 % of the pupils' work, the teachers *have* followed the assessment instructions for all tasks.
- In 23 % the teachers *have* followed the assessment instructions for all but one task.
- In 11 % the teachers *have not* followed the assessment instructions for two or more tasks.

When it comes to mistakes made by the teachers, we find that the teachers are not accurate enough and that they fail to see that a solution can be considered correct even if the answer is incorrect. Furthermore, some teachers make an assessment that is difficult to understand.

Teachers' answers to the questionnaire

The test includes a questionnaire to which more than 2000 teachers sent in their answers. We have analysed all teachers' answers to some of the questions.

- Most teachers have an overall positive opinion about the test. They like the tasks, the structure of the test and the competency profile. Some teachers dislike that the test, especially the group tasks, takes a lot of time to carry out and to assess.
- Most teachers think that they get enough help from the assessment instructions.
- When using the test many teachers change their opinion about a few pupils' performances. (Alm & Björklund, 2000)

Conclusions

We find that most teachers have an acceptable competence in correcting pupils' work and assessing pupils' knowledge in competency areas *when* they get help from an assessment tool such as a National Test. Most teachers do not, however, verbalise their assessment. When they do, they tend to focus on weaknesses.

There are some teachers who make mistakes in the assessment that can become a disadvantage for some pupils. One group is pupils who have serious misconceptions without the teacher noticing. Another group is pupils who make errors that are not serious but to which the teacher attaches too great an importance.

Analysis of pupils' work

In this section we describe what we can learn by the analysis of pupils' work with the Subject Test in Mathematics for year 5.

An important element in qualitatively assessing pupils' knowledge is to analyse, through various tasks and situations, how the pupils work with and master an area of mathematics. We examine whether the pupils have tried to find a solution to the task, how they have understood the task, which concepts they have displayed and in which way the pupils have dealt with the task. The pupils can work on tasks in many different ways. We have seen that those who arrive at correct results may have used different strategies, such as ones that are dependent on the context, or that are more general. The pupils who arrive at incorrect results may display errors, which are more *temporary*, that is, they are not to be found systematically but are of a more random character. There are also errors, which are *systematic*, that is, they occur consistently. This often indicates deficiencies in grasping concepts. The reasons for giving an incorrect result are many. The pupils can also have misunderstood the task, have understood the task but have used a wrong method or have understood the task and have used an adequate method but have made a miscalculation (Pettersson, 1990).

Concepts and misconceptions

There are different components to be assessed in mathematics. We look at knowledge of facts, knowledge of skills, knowledge of concepts, “higher order skills” and the pupils’ conceptions about mathematics and of their own learning. One of our questions is:

What concepts of numbers in fraction and decimal form do pupils display and how do they explain their thoughts?

We have found that some pupils have difficulties with numbers in fraction and decimal form and can’t reach the goal which the pupils are to have attained by the end of the fifth school year: “Pupils should have a basic understanding of numbers, covering natural numbers and simple numbers in fraction and decimal form.” (Skolverket, 2000)

Numbers in fraction form, “part/whole”

In the test of year 2000 (Skolverket, 2000) there are two tasks about fractions. The first is: “Amir is in school for a quarter of a day and Linda for a third of a day. Who has the longest school day? Write down how you worked out your answer.” Seventy-five percent of the children solve this task correctly. The most common way they use to show their understanding is by pictures (38 %), then by calculations (29 %) and also by explanations with words (6 %). The most common misconception, by 8 % of the pupils, is that 1/4 is greater than 1/3 because “ 4 is greater than 3” or “4 is more than 3”. The pupils look at the denominator as a whole number. (Alm & Björklund, 2000)

The other task is: “On a plate in the classroom there are apples, bananas and oranges. Half of the fruits consist of apples, a quarter of the fruits consists of bananas and the rest of the fruits consists of oranges. Make a decision of how many apples, bananas and oranges there might be on the plate. Feel free to draw a picture. Feel free to write more suggestions”.

An acceptable solution is given by 58 % of the pupils. Perhaps it depends on that the task is open-ended, and that the pupils have to “mathematise” and use several thoughts. The pupils try, however, to solve it to a higher extent than the other task.

The analyses show that for almost all pupils the concept “half of” is known. The most common error made by 15 % of the pupils, is that the pupils draw a picture divided into three parts, one half and two quarters. They understand what a half and a quarter is, as “part of a whole”, but they do not understand what “a part of an amount” is. A common misconception is that they think that a quarter is the same as the number of four. They draw four bananas (5 %). Perhaps the pupils sometimes may understand what a quarter is, when they suggest the bananas to be a quarter of the numbers of apples, not of all fruits.

Numbers in decimal form

Sometimes one diagnostic task can reveal a lot about the pupil’s understanding, especially if the task is followed by a question where the pupil is asked to explain why he/she answers in the way he/she does. There are some tasks of that kind in the tests. In one task pupils are asked to write results from a 60 m dash in class 5 in the right order and begin with the time for the pupil who won. Only 40 % of the pupils give an acceptable answer.

The most common misconception about decimals is that the pupils think, for example, that 10.12 is greater than 10.2 because “12 is more than 2”. They (58 %) look at the numbers after the point as whole numbers. It probably depends on that they have experience with money and length before teaching about numbers in decimal form and have made the interpretation that the decimals in 2.50 SEK and in 1.35 m, are whole numbers but followed by another unit than SEK and metres (Brekke, 1995).

Connections between pupils’ understanding of numbers in fraction and decimal form and division

Our next question is:

What connections can be found between pupils’ understanding of numbers in fraction and decimal form and division?

An investigation of the results to the test for year 2000, shows that only 4.5 % of the 200 children have an acceptable answer to the task with numbers in decimal form but no acceptable answer to the tasks with numbers in fraction form. It is, accordingly, very unusual that someone has an understanding of numbers in decimal form but not of numbers in fraction form. Pupils more often solve both of the tasks of fractions than only one and those who manage to solve the task with numbers in decimal form, also manage to solve one or two tasks with numbers in fraction form (89 %). In addition, 25 % of the pupils neither manage to solve numbers in decimal form nor numbers in fraction form.

An investigation has also been done in four classes at the same school. The pupils worked with the tasks that we have discussed and also with four tasks of division from earlier National tests. Our findings are:

The pupils manage best with division (76 %), then with numbers in fraction form (72 %) and do least well with numbers in decimal form (31 %). This corresponds with the thoughts of researchers, such as Sfard, who discusses concepts to be understood as an operation and as a structure. She maintains that pupils have operational understanding before structural understanding (Sfard, 1991).

Conclusions

Our investigations show that pupils must have a good understanding of the concept “part/whole” to manage very well with numbers in decimal form. Teachers may consider this when planning the teaching. It is important to have good concepts of numbers in decimal form because of their use in everyday life. An idea is to work with numbers in fraction and decimal form at the same time after an initial period of basic learning of the concept “part of”, starting from the understanding that the pupils already have (Engström, 1997). In our analyses we can see the growth of mathematical understanding of a concept. The pupils are on their way from a more primitive to a deeper understanding.

In the teachers’ guide to the National Test, we give examples of pupils’ understanding. We hope that when teachers discover what kind of strategies, concepts and misconceptions pupils have, they change their teaching to be of a more diagnostic nature. The teachers must create cognitive conflicts, that is, teaching in a way that misconceptions and unclear concepts are to be revealed. Cognitive conflicts must be solved by discussions and reflections and the concepts must be put in new situations (Brekke, 1995). We want to see assessment and teaching deeply connected and we hope that the National Test will contribute to better teaching and learning with focus on concepts, number sense and structural understanding.

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Adopting a Different Point of View on Problem Solving

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Abstract

The wide gap between researchers and practitioners in mathematics education is a big problem in Hungary as in many other countries. In a developing democracy it is interesting to investigate how to reach the mathematics teachers, how to convince them about the necessity of the change of their point of view on teaching problem solving. Some experiences will be presented taken from our pre-service and in-service teacher training courses: the one-sided preference of the use of symbolic representations leads to shortcomings in students.

Introductory remarks

Mogens Niss emphasized in his plenary lecture at the ICME-9 congress in 2000, the widening gap between theory and practice, between researchers and teachers in mathematics education:

> The course for concern lies in the fact that it is widening. There are very good explanations for this fact, but for the health and welfare of our field, we have to do our utmost to find ways to reduce the gap as much as possible. (Niss, 2000)

But how to solve this problem, how to reach the mathematics teachers in the schools, is still an open question, and surely without an easy answer. We try to analyse the problem in the Hungarian context.

Think globally, act locally

The role of teachers in mathematics education – The international perspective

Jeremy Kilpatrick emphasizes the importance of the teachers' role:

> …we have often overlooked the symmetry of roles between teacher and student. We tend to think of the teacher as causing learning and the student being the one whose role it is to produce and manage mathematical knowledge. The teacher, however, is also doing mathematics and has an equally important role in the production and management of that knowledge. We need to begin examining more closely the teacher's role in doing mathematics in the classroom. (Kilpatrick, 1999, p. 57)

John Mason states:

> My guiding principle is that I cannot change others. But what I can do is worth at changing myself, and in that way, I have found that others may begin to work at changing themselves. For example, trying to "tell" teachers what they "should do" on the basis of research never works unless the teachers understand and appreciate the basis of the alternative practice and unless it fits with their own experience and appears to them as an attractive alternative. (Mason, 1998)

Another important opinion, in our view, belongs to Shlomo Vinner:

> … for many people doing mathematics is activated by the ritual schema. In other words, consciously or unconsciously, many people behave in mathematical contexts (mathematics classes, homework assignments and examinations) as if they take part in a ritual the elements of which they do not understand. /…/ …there are two essential conflicting elements in the human psychology which are active in the domain of teaching and learning mathematics: the need for *meaning* and the *ritual schema*. (Vinner, 2000)

The local context – some characteristics of Hungarian mathematics education
Jeremy Kilpatrick states the following:

> The mathematics content may look much the same around the globe, but how that content is taught and what role it plays in the society's educational agenda, are set locally. (Kilpatrick, 1999, p. 61)

> Problems that researchers may have some trouble communicating about within a country become magnified when they are communicating across national borders. (Kilpatrick, 1999, p. 62)

Before I am going to speak about Hungarian experiences, it is necessary to introduce some main characteristics of our mathematics education. For a long time Hungary was a strongly centralised, authorised society. The teachers were used to fulfil the prescriptions and norms (one curriculum for all, one textbook series, etc.). Since 1990 we try to build a democracy, and the teachers have great autonomy (local curriculum, several textbook series, there are no school inspectors, it is not compulsory to participate in in-service teacher training courses). Prescriptions and norms versus great freedom – these are the opposites experienced by most Hungarian teachers. But we now have a new problem, i.e. how to reach, in this freedom, the teachers.

At an in-service teacher training course I asked the participants, i.e. 60 mathematics teachers, to fill in a questionnaire about the problems of teaching indirect proofs. Only one teacher sent it back, answering the questions.

I made interviews with teachers on the same topic. One teacher said:

> *We have very little time for such things, because we must prepare our students for the different exams and competitions. Our work will be evaluated by the school direction according to the achievements of our*

> *students on these exams and competitions. Every student needs to solve tasks, problems taken from certain task collections, as much as possible. The indirect proofs hardly have any importance. Maybe it is a good research topic for a mathematics didactician, who needs to write a dissertation, to get a PhD degree.*

To understand the text above, I shall describe the Hungarian final exam-system for mathematics at the end of secondary school. The Ministry of Education chooses the exam problems from a book containing about 4000 tasks, problems covering the teaching material in secondary school. This book is at the hand of the secondary school students from grade 9. The teachers are training their students to solve the problems taken from this book.

Two examples illustrate the Hungarian situation:

Analysing the reasons for the rejection of the use of concrete, visual representations in Hungarian mathematics education, a PhD student of mine – an experienced mathematics teacher – expressed the general opinion like this: the officials responsible for mathematics education should call the attention of the teachers for the importance of the use of concrete representations.

At an in-service teacher training course meeting, we discussed the role and use of open problems and problem fields in mathematics education. A participant teacher asked me: Please, publish a task collection containing such problems for each topic.

Although I have emphasised many times that the teachers should construct open problems and problem fields from the closed problems taken from the task collections, my participants remained at the *TSG model*:

T – Tell me, how…
S – Show me, how…
G – Give me concrete examples for each topic!

"There is only one way to communicate mathematics" – The dominant role of the use of symbolic representations

In the remaining part of my paper, I will restrict myself to one problem, i.e. the dominant role of the use of symbolic representations in mathematics education in Hungary. Csíkos (2000) investigated the opinions of approximately 4500 students (grades 5, 7, 9, and 11) and their teachers about arguments and proofs given on a questionnaire (the examples were taken according to the Harel-Sowder system). Without going into the details, from our point of view it is interesting that the senseless symbol manipulations were overestimated by the students and their teachers. On the other hand, the underestimation of empirical proofs (investigating concrete examples) was typical. In Hungarian mathematics education there is a tendency to use symbolic representations as early as possible and as much as possible. As an example, introducing the balance method for solving equations very often happens in grade 5. The strong belief in the symbolic representations among mathematics teachers is typical in Hungary. At the upper secondary level the principle "There is only one way to communicate mathematics" is characteristic.

In the pre-service and in-service teacher training courses, I often give the following problem:

Solve the following equation in the set of real numbers: $|x-1|+|x+2|=7$

Most of the Hungarian mathematics teachers and university mathematics student teachers prefer the symbolic solution, the so called "interval method", using the formal definition of the absolute value.

If $x \leq -2$ then $|x-1| = -x+1$ and $|x+2| = -x-2$

If $-2 \leq x < 1$ then …

If $x \geq 1$ then …

Only very rarely we meet graphical solutions. Drawing the graphs of the functions $y=|x-1|$ and $y=7-|x+2|$ in a common Cartesian coordinate system, and projecting the intersection points on the *x*-axis, we get the solutions.

The first definition for the absolute value of a number is in grade 5 the following: *The absolute value of a number is the distance on the number line of the number from the zero*. Nobody uses it for solving the equation above. Using this definition, $|x-1|$ means the distance of the number x from 1 on the number line, and $|x+2|$ means the distance of the number x from -2 on the number line. Our task is to find the number x so that the sums of the two distances above will be 7. It is not difficult to find those points on the number line. The number line is well known for the pupils, and the work with it easy because of concrete actions on it.

An example where the exclusive use of symbolic representations leads to an incomplete solution

Consider the following problem.

How many solutions does the system of equations have depending on the value of the parameter a? $\begin{cases} x^2 - y^2 = 0 \\ (x-a)^2 + y^2 = 1 \end{cases}$

Algebraic solution

Expressing $y^2 = x^2$ from the first equation and substituting into the second equation, we get $(x-a)^2 + x^2 = 1$ and then $2x^2 - 2ax + a^2 - 1 = 0$. Depending on the value of the discriminant of this equation, it may have 0, 1, or 2 solutions. Here the discriminant is $D = 2 - a^2$, leading to the following cases:

If $D < 0$, i.e. $a^2 > 2$ ($a > \sqrt{2}$ or $a < -\sqrt{2}$), then the equation has no solution.

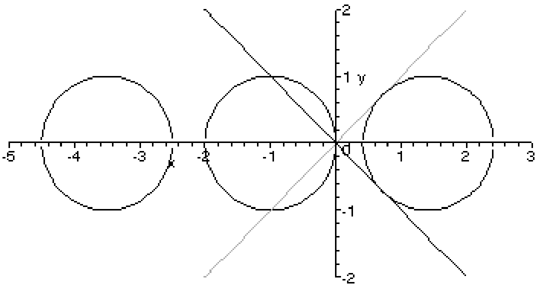
If $D = 0$, i.e. $a^2 = 2$ ($a = \sqrt{2}$ or $a = -\sqrt{2}$), then the equation has one solution for *x*. Substituting this into the first equation we get two solutions for *y*, giving two solutions for the system of equations.

If $D > 0$, i.e. $a^2 < 2$ ($-\sqrt{2} < a < -\sqrt{2}$), then the equation has two solutions for *x*, giving four solutions for the system of equations. But there is a special case: if the numerator in the formula for solving quadratic equations takes the value zero, i.e. one of the solutions for *x* is zero, and the corresponding value for *y* is also zero. In this case our system has three solutions:

$a \pm \sqrt{2 - a^2} = 0 \Rightarrow 2a^2 = 2 \Rightarrow a = \pm 1$.

Geometrical solution

Let us consider the equations as equations of curves in a Cartesian coordinate system. The first equation then represents a pair of straight lines, i.e. $y = x$ and $y = -x$, and the second equation a unit circle with centre at $(a, 0)$.



Moving the circle horizontally we may see the possible cases. Far away form the origin, the circle does not touch the lines and there are no solutions. There are two cases when the circle touches the lines, giving two solutions. When the circle is intersecting the lines but not passing the origin, there are four solutions, when also passing the origin three solutions. The values of *a* corresponding to the different cases we can determine with simple geometrical considerations, using the fact that the lines $y = \pm x$ make an angle of $45°$with the *x*-axis.

Important here is that the pupils may “see” the whole solution process.

Experiences with the problem

At my university courses I usually give this problem to my mathematics student teachers (years 3 and 4). Until now every student solved the problem algebraically. From our point of view, what is interesting is that nobody has found the case with three solutions. The *ritual schema* in the sense of Vinner is: If you have a quadratic equation, at $D < 0$ there is no solution, at $D = 0$ there is one solution, and at $D > 0$ there are two solutions. In our problem, because of $y^2 = x^2$, the system of equations has 0, 2, or 4 solutions. The case $x = 0$ ($y = 0$) is out of this ritual schema and needs special investigations of the numerator. The schema must be expanded in this case.

A secondary school mathematics teacher, who attended one of my in-service teacher training courses, said to me that “I have a very good mathematics group in grade 11. They are perfect in solving algebraic problems. I am very curious how my students will solve this problem.” No surprise, her students tried to solve the system of equations algebraically, and nobody noticed the case with 3 solutions. The teacher was surprised of this result, but she and her students were ready to accept the necessity of the use of visual representations also for algebraic problems. A similar positive effect I could observe at other in-service courses.

But we are far from the full success. Although my university students agree on the importance of the use of visual representations, most of them do not use them in their work. I have met a mathematics student teacher who expressed her opinion clearly: “I hate graphical solutions!”

In the Hungarian mathematical periodical for secondary school students (KÖMAL), the following problem was posed as a competition problem.

For what value of the parameter p does the equation $x+1=\sqrt{px}$ *have exactly one solution?*

The competition is meant for the talented pupils in secondary school mathematics. Out of the 481 solutions that came to the editors of the periodical, only 57 were correct (algebraic solutions). In the journal the correct solution presented was an algebraic solution. Only in small letters the following remark was enclosed: “The solution of the problem will be clear, and we may avoid the pitfalls, if we apply the unworthily neglected graphical method” (the graphs were enclosed).

A direct consequence is that teachers train their students for the algebraic methods, because these are the officially accepted ones. But they may often fail!

Concluding remarks

To overcome the exclusive use of the symbolic representations of an algebraic ritual scheme, it is necessary to confront the teachers with the shortcomings of their students. Parallel to it, it is desirable to show geometrical solutions, with the aim that the students will also use this method when appropriate.

If we modify our problem above so that we are interested also in finding the roots, both visual and algebraic methods are desirable. To determine the four roots is easier using algebraic considerations. Eisenberg (1994) states that “…whenever and wherever possible, both visual and analytical modes of representations must be used in the mathematical classroom.”

For those who know the school reality it is clear how difficult a task it is to educate the students for a flexible use of these two methods.

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Development of Social Skills during the Lessons of Mathematics in Compulsory School

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Abstract

This paper will describe the content of the social skills of the students, a model of development of social skills and criteria that are used in the lessons of mathematics in compulsory school.

Latvia as a candidate state for European Union pays special attention to education to prepare its citizens to live in the European Union. It means they have to be ready to work in a team, to communicate with different people, to work with different technologies, to know foreign languages. They have to be able psychologically and practically to compete in the labour market – as well as to enrich their knowledge and to develop their skills during their entire lives.

For this reason, in the National standards of compulsory Education (1998) the formation of social skills is pointed out as one of the main factors in the whole study process. It concerns mathematics as well.

In Latvia the knowledge of the subjects were of great importance during the Soviet times, when the teaching was dominating in the whole study process. Now, when we have started the way to pupil's orientated approach, the formation and the development of social skills have become of great importance. The teachers and students gradually are moving to a learning society.

Social skills and abilities are necessary for everybody in the process of social interaction. Social skills are acquired just in the communication process – pupil with a pupil, pupil with the adults and environment, where the pupil lives.

Usually it is possible to point out communicative and organisational skills. It is said in the White Paper (1996) of the Commission of European Union that social skills include the skills of mutual communication, as well as corresponding skills allowing to take up responsibility, skills to work and co-operate in the team, creative attitude towards the work and striving for quality.

The acquisition of social skills is of great importance at the age of teenager, when the assessment of his/her activities by the surrounding people and fellows is so significant in the building process of the self-assessment and personality.

The research was carried out during four years. The gained results allowed us to define the content, the model of the acquisition and the criteria of the social skills of the teenagers. The model is depicted in the table 1 (see Appendix).

In this table in the file *content* the certain parts of skills and features are pointed out, that form the content of social skills. The author has tried to choose the most essential skills that are necessary to acquire at this age.

The study forms that are used in the process of the acquisition of the content of the concrete social skills are marked in the second file *organisational forms of the studies*.

It is seen, that it is difficult, to point out one study form, that could form only one certain skill.

Skilful use of any organisational study form during the mathematics lessons permits to develop a chain of social skills. Skills do not exist separately, acquiring one skill, the other skill is being promoted or developed.

The active study process favours the acquisition of social skills, because the organised studies form different models of co-operation and possibilities of communication. In this aspect the choice of interactive study forms during the lessons are of great importance.

The social skills of the teenager are developing also according to his/her study aims, values, needs. They are in mutual interaction with the use of interactive study forms and methods. All the factors mentioned above influence the acquisition of the social skills.

It is stressed by the Norwegian colleague Sven Eric Fjed that the pupils in good schools have stable contacts with class and school mates, that communication serves as a precondition of a good school as organisation.

Per Dalin and Hans Ginter Rolf stress that the existence of new and active, more responsible pupils in school is of great importance if we wish to use the maximum resources that are available in education.

The activity of the pupil must be discussed as a pedagogical phenomenon. The scientist M. Skatkin points out that the development of the child's personality takes place in the study process. The child is involved in many-sided relations with fellow teenagers, group or school mates, adults and the environment. These relations are closely connected with ethical feeling, that influences the formation of motives, feelings, habits. They, in return, determine the content and dimension of social skills.

The content of social skills include the following skills:

- to listen to another person,
- to observe certain regulations,
- to work in team,
- to organise personal time,
- to assess oneself and others,
- mutual respect,
- common understanding and tolerance.

Different authors have focussed differently on certain social skills. For example:

R.Bergen and R.V.Henderson – social skills link with the biological stage of the children.
P. Kutnick – as cognitive styles, self-concepts.
K.J. Tillman – as models of relations of subject and society.
Gofman and Blumer – as common interaction of individualisation and socialisation.
L. Kolberg – as child's moral and ethical development.
D. Liegeniece – as mutual communication.

During the research in Latvia the pupils considered that the skills to co-operate and mutual understanding were very essential in the everyday life. Therefore they stressed that greater attention should be paid to obtaining these skills in the study process. The author gave students the possibility to choose the study forms and different activities during the lessons. They took part in the planning of the lesson and in the analysis of the achieved results.

The research results, different views of theoreticians (Kutnicky, Davidov, Liegeniece, Nelsen, Bozovicha, Bukov, Shpona, Maslo, Kolb, Rodger, Vygotsky, Piaget, Skinner etc.) and the results of the international comparative research (TIMSS, 1999) about the motivation of the learning and the importance of formation of social skills, promoted the author to work out the model of obtaining social skills during the lessons of mathematics (figure 1).

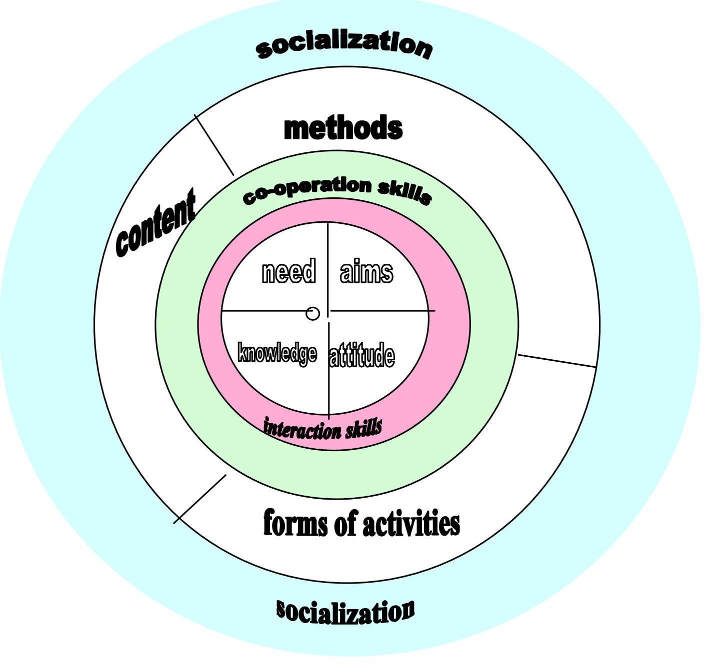

Figure 1. The model of obtaining social skills during the study process.

Aims, needs, values and ways of co-operation determine the quality of the obtaining of social skills. It is necessary to underline that the model will be a success only, if there will be mutual understanding and co-operation of pupils and teachers, pupils and pupils.

Criteria of evaluation of obtaining social skills are characterised in table 2. Each of the criteria can be valued in the three levels: high, medium and low.

Obtaining social skills	**Criteria**			
	Aims		consciously forwarded	+1
			adopted	0
			pressed	-1
	Needs		consciously forwarded	+1
			adopted	0
			pressed	-1
	Values	Self value (“I”)	positive	+1
			neutral	0
			negative	-1
		Society (others around me)	positive	+1
			neutral	0
			negative	-1
		Knowledge	positive	+1
			neutral	0
			negative	-1
		State	positive	+1
			neutral	0
			negative	-1
		Nature	positive	+1
			neutral	0
			negative	-1
	Co-operation skills	Skills to work individually	yes	+1
			partly	0
			no	-1
		Skills to work in a small group	yes	+1
			partly	0
			no	-1
		Skills to work in a big group	yes	+1
			partly	0
			no	-1

Table 2. Criteria for evaluation of the obtaining of social skills.

Questioning the teenagers (81 pupils, form7) the author got the evaluation of the acquisition of social skills (figure 2).

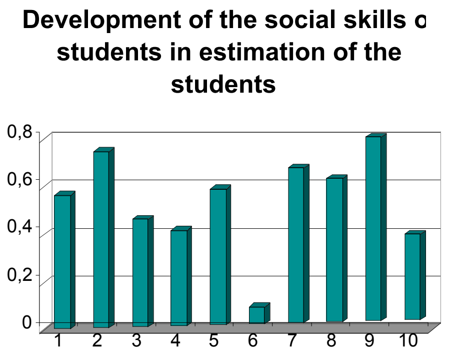

Figure 2. 1-aims; 2-needs; 3-values: self value; 4-values: society; 5-values: knowledge; 6-values: state; 7-values: nature; 8-co-operation skills: skills to work individually; 9-co-operation skills: skills to work in a small group; 10-co-operation skills: skills to work in a big group

The level of the obtaining of social skills according to self-assessment results is rather high. The aims are conscious not only accepted, the needs are understandable, the level of the acquisition of the co-operative skills is different:

- the acquired skills in the bigger dimension refer to work individually or in the small group,
- the skill to work in the big group is more partial.

Knowledge is recognised as positive values, especially attitude to the state – more neutral than positive. Questioning the teachers (46 respondents) the author got the evaluation of social skills that differs only a bit from the pupils self-assessment results (figure 3, axes as in figure 2).

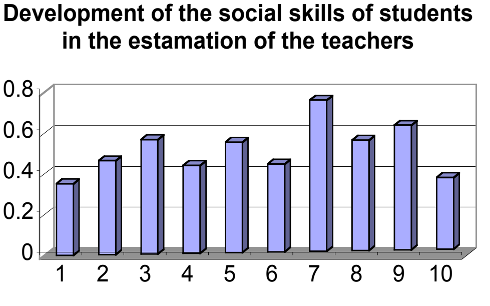

Figure 3

According to the results of the evaluation of pupils' skills by teachers, the level of the acquisition of the skills is sufficiently higher. According to teachers' answers the aims and needs are more accepted than put forward personally.

The attitude towards the values is different:

- towards state and society – more neutral than positive,
- towards nature, knowledge, self- value more positive than neutral.

According to co-operative skills:

- pupils have skills to work individually and in small groups,
  pupils have partially skills to work in a big group.

As a result of the research, it is possible to draw the following **conclusions**:

The teachers have to give pupils the possibilities to be active during the study process to help the pupils to reach their aims, needs, to develop values and social skills, it means, that the pupil must become the active user of the offered possibilities.

Skills to work in a big group are possible to form in longer period, therefore the low level of the acquisition of these skills require more active work with the pupils and more time as well.

Sometimes teachers do not understand pupils fully, they do not understand their aims, needs and values. It means, that mutual understanding is partial and imperfect (incomplete) and this is one obstacle in the way to obtaining social skills.

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Table 1

The model of the content of social skills

Organi-sational form of study work	**Content of social skills**										
	Mutual respect	**Under-standing**	**Tole-rance**	**Skill to listen to another person**	**Skill to express over-view**	**Skill to co-operate**	**Skill to observe regula-tions**	**Skill to work in team**	**Skill to organise personal time**	**Skill of self-assess-ment**	**Skill to assess other person**
Indivi-dual work					✓		✓		✓	✓	
Frontal work	✓	✓	✓		✓		✓		✓	✓	✓
Peer work	✓	✓	✓	✓	✓	✓	✓		✓	✓	✓
Group work	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Project	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

Mathematics and Language: Divergence or Convergence?

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Abstract

This paper presents two opposing conceptions of mathematics and its development. The first is the taken-for-granted idea that mathematics has a universal quality underpinning a variety of expressions and applications. The second is that mathematics began in diversity, and retains the potential to be diverse, although strong forces make its development convergent. Evidence for the second conception can be found in the history of the subject, within mathematics itself, within linguistics, and within the emerging field of ethnomathematics. Some examples of the resulting limitations are given. These conceptions are linked to wider philosophical issues and are applied to the field of mathematics education.

Introduction

In New Zealand in the 1980s there was a renewed call for all subjects, including mathematics, to be taught in the indigenous Maori language. I was lucky enough to be involved over a 12-year period as a Maori mathematical discourse was consciously developed. Eventually Maori vocabulary and grammar was developed so that mathematics could be taught in this language up to the end of secondary school. This vocabulary development is written up elsewhere (Barton, Fairhall & Trinick, 1995a, 1995b, 1998) as a successful project, and yet several of us who were closely involved were left feeling uncomfortable. It felt as though we had created a Trojan Horse which allowed English conceptions to infiltrate the Maori language – but we could not see how this had happened.

Eventually we came across our first piece of evidence for the Trojan Horse hypothesis: linguistic analysis shows that pre-European Maori treated numbers as verbs — not as adjectives as we do in English, nor as nouns as we do in mathematics (Barton, 1999a). I now know that most Polynesian languages are like this, and so also are some North American First Nation languages (Denny, 1986), and some African languages (Watson, 1990). The Maori language has become like English over time: with new vocabulary, with changes to syntax, and there have been changes in concepts and the ethos of the language. Further aspects of change in Maori mathematical language have been uncovered by Trinick (1999).

The diverse expression belief

What happened in Maori mathematical discourse is a reflection of what has been happening throughout the history of mathematics. As mathematics has become an academic discipline, one way of talking mathematically has developed. This is the result of the Diverse Expression/Convergent Assumption belief: the idea that, although we may describe quantity, space or relationships differently, although we may use them in different ways, write them using different symbols, what we are ultimately talking about is one and the same thing. In the end, mathematics is universal. The implication is that it makes no difference which language we use to express mathematics.

Many people may resist the use of the word "converging". They might point out that, far from converging, mathematics is growing: more is being written, new areas of mathematics are emerging, more and more aspects of our world are being explained in mathematical terms. However this is a tree-like growth. The roots converge into one trunk that sprouts more branches of the same wood. Mathematical ideas sprout more ideas of the same kind. Establishing the wider generality of existing mathematical ideas does not imply that these ideas are the only ones which can be used to explain our world, nor even that they are the best ideas. Indeed their very applicability and utility may reduce the impetus to consider other possible ideas.

The Diverse Expression/Convergent Assumption belief is not Platonism, although Platonism is a form of this belief. In Platonism the convergence has already occurred – it has occurred in some "ideal" world. But the Diverse Expression/Convergent Assumption belief is also a component of most modern philosophies of mathematics: neo-Platonism, empiricism, quasi-empiricism – Maddy (1990), Lakatos (1976), Tymoczko (1986). All these involve the convergence of mathematics. The recent cognitivist theories (Dehaene, 1997; Lakoff & Nuñez, 1997) are even more strongly convergent with ideas of our brains being "hard-wired" for number.

Am I challenging all mathematical philosophers? There is a difference between what is happening in mathematics, and mathematics itself. Mathematics is converging, this is happening. The changes in Maori language demonstrate the power of social forces to universalise and stamp out diversity. The history of mathematics involves merging a wide diversity of ideas, illustrated clearly in Joseph's writing. Not only are mathematical ideas subsumed into one grand mathematics (Joseph, 1992), but the fundamentally different ways of doing mathematics are conventionally seen to be the same in essence (Joseph, 1994). Because mathematics is, historically, convergent, does not mean that it must be this way, that mathematics is universal. Can we conceive of mathematical development otherwise? What happens if we reject the belief that mathematics should converge?

The diverse assumption

The identification of a deep-seated difference in the expression of quantity in pre-European Maori, and in other languages, has made me look more carefully at the way different languages, especially indigenous ones, express ideas of quantity, space and relationships (Barton, 1999b). As this quest turns up more and more conceptions which are familiar, and as I read about the new, exciting discoveries of other ethnomathematicians, I have come to feel that, for mathematics, the appropriate stance is a Diverse Assumption/Convergent Expression belief. This is the belief that mathematics has a diversity of origins, and retains multiple potentialities, but that communicative, socialising, political and historical forces make both its development and its expression convergent. Let us look at some of the evidence.

The ethnomathematical evidence includes: different structures and logics amongst kinship relations of different people (Cooke, 1990); alternative spatial systems of weaving and indigenous design, some of which are incommensurable with conventional mathematical systems (Barton, 1995); the systems of Polynesian navigation (Turnbull, 1991); Inca quipu (Ascher & Ascher, 1981); the symbolisations of Lusona drawings (Gerdes, 1991); Mayan mathematics; and the Japanese Wasan. The contributions of these to mathematics have been lost, or dismissed, or changed. Mathematically significant parts have been co-opted into the large programme mathematics, under the rubric of one great universal language and truth. In this process they have been stripped of culturally distinguishing characteristics, rendering their divergent potential sterile. The mechanism of the stripping process has parallels with what happens when languages meet (Brenzinger, 2001).

Further evidence for divergence in mathematics can be found in the history of mathematics. Conventional historiography purports to show a single converging stream of development, but little emphasis is ever placed on unresolved divergent branches: standard and non-standard analysis; Bayesian and frequentist probability; set theoretic and category theory foundations; not to mention the alternative basic axioms proposed within set theory.

Mathematics is now widely recognised as fundamentally a human social activity. Hermann Weyl has said:

> The question of the ultimate foundations and the ultimate meaning of mathematics remains open: we do not know in what direction it will find its final solution or even whether a final objective answer can be expected at all. ‘Mathematizing’ may well be a creative activity of man, like language or music, of primary originality, whose historical decisions defy complete objective rationalisation.

Consider some other aspects of human endeavour by which we make sense of our world: language, music, art. The attempt to show the existence of a universal human grammar is foundering, as did the search for a universal language before it. And no-one has suggested that all music or all art is of the one kind, or from the same source. Therefore why should we think that mathematics is the single universal created from human experience?

The implication of all this evidence is the diverse assumption, in other words mathematics can diverge in many directions, but only one potentiality of mathematics is being developed.

The influence of language

One of the mechanisms by which we are losing the other potentialities of mathematics is through increasing reliance on only two or three languages to do mathematics. English is increasingly becoming the language of research mathematics. The pressure for international communication is the *raison d'etre* for official languages at conferences, and has, *de facto*, made English into a *lingua franca* for mathematicians. Many mathematicians shrug their shoulders at these developments: *Que sera, sera*.. However, as has been demonstrated above with four different languages, we often use foreign expressions to convey accurately ideas which cannot be expressed in English. This implies the existence of general concepts which are language specific: do mathematical concepts also get lost or changed when expressed in a language of international communication?

Another linguistic limitation to mathematics has emerged from the IT revolution. Some people are currently working on the basic mathematical units by which ALL web-based mathematics will be done. The reason is that it is not easy to transfer mathematical expressions between various mathematical environments (e.g. Matlab, Mathematica), so providing a common base in which all these environments will run provides for intercommunication. This is no doubt useful, however mathematicians do not seem to realise that they are limiting mathematics by this process, cutting off potentially different ways of conceptualising mathematics. When we realise that more and more research mathematics is done using computers, the dangers become increasingly more acute.

In areas where there is a recognition of the need to use and preserve minority languages, mathematics is often the one subject to be taught in English. The common perception of mathematics as that subject which is beyond culture leads to an assumption that it does not matter in which language it is taught. However there is good linguistic evidence that this is not the case. Recent work has shown that different languages carry different ways of conceiving even those very basic aspects of the mathematical world: number and space. Australian aborigines dominant formatting of local space is by use of global north, south, east, west directions (Harris, 1991); Navajo languages speak of geometric 'objects' as actions (Pinxten, van Dooren & Soberon, 1987); and, as has been noted above, Polynesian and some Inuit languages express numbers as verbs (Barton, 1999b). The tendency of Indo-European languages to nominalise concepts reinforces the tendency in Western philosophy towards an object ontology. It is by no means a universal philosophical or linguistic habit, for example in the Basque language Euskera what we might consider objects are perhaps better understood as states (Barton & Frank, 2001). What would have happened if mathematics had developed through these conceptions? We will never know, of course, but it is interesting to play thought experiments. Mathematical representation of space tends to use single origins – in line with Western philosophy's Ego-Other orientation: What sort of geometric system would have evolved from describing space with multiple origins? Much scientific effort went into the development of timekeepers so that navigation could be based on a latitude/longitude system: What would have happened if this effort was put into developing Polynesian path navigation concepts such as analysing wave-swell patterns? What sort of systems would have evolved from thinking of numbers and shapes as actions instead of objects?

Metaphors for mathematics

One of the effects of cultural investigations in mathematics is to widen the ambit of what is meant by the term mathematics. Rather than thinking of the mathematics which is known world over through formal education, we need to expand our vision to include any form of quantitative, relational or spatial systems. Bishop (1988) makes a distinction between Mathematics and mathematics that has some of these characteristics.

How are we to understand this wider vision? Above it is suggested that the growth in mathematics is a tree-like growth. Roots converge to one trunk that then sprouts, but only more branches of the same wood. The ideas in this paper suggest it is more appropriate to think of mathematics as a forest in which different trees might grow. Different trees have many similarities, and, as mathematicians, we tend to focus on their likenesses. As the forest evolves a particular type of tree (the English-language one) has dominated to the exclusion of the others, or, possibly, in a strange type of cross-pollination, these trees take on a few minor characteristics of other trees.

I find a more useful metaphor for mathematics in its wider meaning is provided by thinking of boats on a harbour. Each type of boat (each mathematics) gives the sailor a different type of experience of the world of the harbour. A fisherman in a fishing boat comes to see the harbour as a network of positions, places where it is good to catch certain types of fish. A passenger on a ferry comes to see the harbour as a network of possible paths, trips that might be taken. Speed boats, yachts, large passenger or cargo ships, windsurfers, kayakers all have their own reality. The ferry can travel on the harbour under conditions too rough for the fishing boat. The fishing boat can go to rocky places where the ferry cannot navigate. It is the same world, but it is a different understanding. Neither is the truth.

One more metaphor for the Diverse Assumption/Convergent Expression belief is a lithograph by M C Escher. In this image, sets of two different coloured birds are each defined by the spaces between the birds of the other set, although in some parts of the picture only one type of bird is recognisable. The birds can represent two different mathematical "spaces". In most areas the two birds define each other: it is clear that the same "space" is being described, but it is being done in two different ways. But there are some parts of the world which are only definable using the white bird, and other parts which are only definable using the dark bird.

It does not matter whether one of the birds is more detailed than the other. The advances in academic mathematics can be seen as putting more and more detail into one type of bird. No matter how much we do that, we will never make sense of the space defined by the other type of bird. I am worried that, if we lose the ability to understand the conceptions behind indigenous languages, then we will lose forever the possibility of understanding those other aspects of our world.

Educational implications

The import of this paper is predominantly aimed at mathematics. What mathematical developments have been missed because the history of mathematics has gone down one track and not the other? Can these other lines of possible development be recovered? What mathematical developments have been missed because particular cultural conceptions of quantity, relationships and space have come to dominate? Can other cultural conceptions lead to previously unthought-of mathematics?

However the different view presented here does have implications for mathematics education. The obvious first implication is that bi- or multi-lingualism in mathematics should be encouraged. This applies both at the level of children learning mathematics at school, but also at the level of research mathematics. It is hypothesised that doing mathematics in one language may lead to directions in research different from those if it is done in another language. The practicality of international communication notwithstanding, opportunities to follow mathematical thought in all possible modes should be taken.

Another implication which is not so obvious is that “raw” mathematisation should be encouraged. Young children first coming to mathematics are likely to be less bound by the predetermined conventions of the subject. Encouraging students to mathematise their familiar conceptions of quantity, relationships and space will promote diverse and creative thinking. It is exactly this attitude to mathematics which needs to be fostered for the health of the subject. The view of mathematics proposed in this paper implies that there are many unrealised opportunities for mathematics in the very basic formulations of the subject.

In the pedagogical arena, the implications are both political and curricular. This paper presents a case for increased sensitivity to the difficulties of particular groups, especially those whose natural language is widely different from Indo-European languages. It has implications for the way in which the mathematical gate-keeping in our society is cultural biased. With increasing technology the consequences of such a bias are increasingly drastic in social terms.

With respect to curriculum, there is an important consideration for people from those cultures which experience cultural estrangement when studying mathematics which has been developed through a different world view. Overcoming this estrangement is no easy task, but acknowledging the problem is essential. Such acknowledgment must be given by teachers, but also in the curriculum. One attempt at this is described by Lipka (1994) talking of an Alaskan programme:

> The pressure behind developing a Yup'ik mathematics is three-fold:
>
> 1) to show students that mathematics is socially constructed;
> 2) to engage students in a process of constructing a system of mathematics based on their cultural knowledge;
> 3) to connect students' knowledge of "their mathematics" through comparisons and bridges to other aboriginal and Western system.

In other words, access to the conventional, widespread field known as 'mathematics' must come through the world-view in which it is expressed. If your world-view is different from this, then it is first necessary to understand the role of your own world-view in making sense of quantity, relationships and space, so that you can appreciate another one.

Such an educational task seems to place an added burden on anyone who is starting from a different world-view than that of conventional mathematics. This is true, but there are two important points to be made. Cummins (1986) has produced evidence that bilingual learners, provided they are fluent in both languages, have a cognitive advantage in any educational task. I interpret this to mean that the sort of knowing which results from having two (or more) world-views is a deeper, more aware, sort of knowing than that which results from having only one. Hence people learning mathematics from a different world-view have to do more, but they reach a different, deeper understanding.

The second point is that mathematics learners from the same the world-view as that of conventional mathematics also have an added task if they wish to reach this deeper level of understanding. It is a feature of many education systems, especially mono-lingual English-speaking ones, that such a different level of understanding is not even recognised. It behoves us as mathematics teachers to create this awareness in our students. I think of this as putting more emphasis on mathematics as a humanity than on mathematics as a science - and particularly to avoid teaching mathematics as an unquestioned series of results and techniques. At the very least it means that we have a duty as mathematics educators to teach something *about* mathematics, not just to focus on mathematical methods and results.

Conclusion

So, mathematics has diverse origins, and it is converging even though there are new research fields, more PhD's, more books, more applications. Mathematics may be growing, but this is a tree-like growth. The roots converge into one trunk that sprouts only more branches of the same wood. Mathematics has had (and continues to have) a diversity of origins based in different cultural environments, but communicative, socialising, political and historical forces make both its development and its expression convergent. D'Ambrosio, of course, has been calling for recognition of diverse epistemology for a long time (1987, p74):

> ... we face a need for alternative epistemologies if we want to explain alternative forms of knowledge. Although derived from the same natural reality, these knowledges are structured differently.

Simply because one knowledge system has been subsuming parts of others and has had enormous amounts of time and energy put into its development, this does not make it the only, nor the correctly structured, knowledge. We can acknowledge its power, its beauty, and its apparent generality, but we must also acknowledge its destructiveness (both of our world and of other knowledges), its inconsistencies, and the gaps in its pervasiveness. Other ways of doing mathematics were – possibly still are – available.

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School Algebra: Manipulations of Empty Symbols on a Piece of Paper?

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Summary
The paper is based on students' written responses to selected items in a large-scale assessment project in Norway. Findings related to students' knowledge of the symbolic notations in algebra are discussed. The findings are related to the Norwegian tradition to focus extensively on executing rules for symbolic manipulations.

The paper is based on a large-scale assessment project from grade 5 to 11, where assessment is used as a basis to aid conceptual development. Two of the objectives of the project are to develop:

- integrated test- and in-service training packages that can be used by teachers as part of their assessment practice
- a collection of test-instruments of diagnostic character, which can be used as a starting point for teaching practice within various parts of the subject matter.

Sets of diagnostic test items for various parts of the mathematics curriculum, with the intention to cover most of the key concepts of school mathematics are developed. Materials for teachers are produced for each set of test items. The main focus of the materials is to report and discuss the extent of conceptual obstacles identified by these items. The items are developed to enquire into different aspects of the particular concept in question.

In selecting items for the project the main intention was to inform teachers about several well-known conceptual obstacles in students' algebra learning. The research perspective had a second priority. Data reported in this paper are based on written responses to items from 1805 grade 6 students from 100 classrooms. The corresponding figures were 1953 students, 91 classrooms (g8) and 1957 students, 90 classrooms (g10). Average ages were: 11.5, 13.5 and 15.5.

Introduction
There is a large body of research literature about the teaching and learning of algebra. It documents students' difficulties in grasping fundamental aspects of the notations used. This involves how to write simple expressions and equations containing variables, numerals, operation signs and brackets. (e.g. Booth, 1984; Herscovics, & Linchevski, 1996; Janvier, 1996; Kieran, 1981; Küchemann, 1981; MacGregor & Stacey, 1996; Stacey & MacGregor, 1996 and 1999.). One aim of the KIM project was to make teachers aware of learning outcomes of an extensive emphasis on symbolic manipulation in the teaching of algebra. A national survey of Norwegian students' understanding of different aspects of

algebra was needed to promote a change in teaching. The teaching activities linked to the national survey are designed to help students to close the cognitive gap between arithmetic and algebra (Herscovics, & Linchevski, 1996).

Issues in the transition from arithmetic to algebra

The priority of multiplication and division

The intention of the items of problem 1: *Write the correct number to go in each of the boxes:* 1b: $\square \cdot 2 + 4 = 12$, 1c: $3 + 2 \cdot \square = 15$, and 1d: $25 - 2 \cdot \square = 17$ is to focus on students awareness of the priority of multiplication compared to addition or subtraction. Respectively 61, 77 and 88 percent of the students answered item 1b correctly. The most common wrong answer was 2. These answers are probably a result of first adding 2 and 4, and next multiply 6 by 2. Table 1 illustrates this strategy is much more prevailing in 1c. One additional reason for the large increase of incorrect answers could be the tendency to perform calculations from left to right. The answer 5 indicates that these students add 3 and 2 and write the result in the empty box.

Item 1c	Grade 6	Grade 8	Grade 10
6 correct	13	17	33
3	67	73	63
5	8	3	1

Table 1. Percentages of correct, and most common incorrect responses. Item 1c.

The idea of priority amongst operations is even less transparent in an algebraic setting. It is my opinion that this has to be specifically dealt with in arithmetic to be applied in an algebraic setting. That students do not manage this convention for numerical operation is presumably because it is difficult to understand, but rather that this is usually not specifically dealt with in arithmetic teaching.

Students' use of symbols

Lack of closure and evaluate a specific unknown

Item 4b is taken from Küchemann (1981): *If $e+f = 8$ then $e+f+g =$ ___.* Matching the two expressions can solve this item. Table 2 demonstrates that to operate with *g* as a *specific unknown* caused problems for the majority of our students. Basically the students tried to solve this problem of symbolizing in two different ways. One was to join the number 8 and the specific unknown *g* and the other were to evaluate *g* in different ways.

Item 4b	Grade 6	Grade 8	Grade 10	CSMS
8 + g or g + 8 (correct)	4	8	38	41
8g, 8xg or similar (joining)	4	1	27	3
9	21	16	3	6
12	18	16	7	26
Other numbers evaluated	21	22	8	-

Table 2. Percentages of correct and most common incorrect responses. Item 4b.

The CSMS study involved fourteen year olds, and would on average be half a year older than our eight grade students. It should also be noticed that the formalistic teaching of algebra starts in grade 8. Note that CSMS students performed better than our grade 10 students. In our sample there is a large, and increasing, proportion of "joiners" as student grow older. We also notice that there are different "popular" ways of evaluating g between the two samples. Apparently, near the end of compulsory school, grade 10, students stop evaluating variables, but the part of students who have problem with the lack of closure of algebraic expressions are growing. This is an opposite change by age compared to CSMS results shown in table 2. Similar lines of thinking illustrated by the responses to item 4b are also observed for other items used in of the KIM project, for example items 3b: *"Add 2 and n+5"* and 3c: *"Add 4 and 3n"*. Respectively 4, 9 and 37 percent of the students answered item 3b correctly. The corresponding figures for item 3c were 4, 7 and 46. Two types of joining, avoiding lack of closure, were observed in item 3b: $2n + 5$, joining 2 to n, and $7n$ or $n7$, adding the numbers and join the variable. The percentages of joiners on item 3c were respectively 63, 69 and 39, compared to 31% in the CSMS study.

Simplifying expressions

The test contained eleven expressions to be simplified for grade 8 and 10. We will first consider the following items, 8b: $a + 4 + a - 4$; 8c: $x + y - x + y$ and 8d: $(a + b) + (a - b)$. The facility level of these items was respectively 33, 12, and 8 % for grade 8, raising till respectively 79, 57, and 40 % for grade 10. A relatively small numbers of "joiners" were observed for all of these items, just below 10 % for grade 8 and around 4 % for grade 10. Notice that the careful professional progression from item 8b to item 8d has a large effect on the responses. The most common incorrect answers to item c was 0, $0x + 0y$, $0xy$ or similar, respectively 21 % and 16 %. They probably treat $x + y$ as *one* object. It is more difficult to explain the reason for the response 0 or $0a$ (12 and 5 %).

The power notation is also brought into play in grade 10 students' responses to the items above as well as to item 8e: $3a - (b + a)$, presumably because this content is currently taught at this grade level. In addition the conjugate equality pops up in many students' responses to item 8d, as illustrated in figure 1 below.

d) $(a+b)+(a-b) =$
$a \cdot a - a \cdot b + b \cdot a - b \cdot b =$
$a^2 - ab + ba - b^2 =$
$a^2 - b^2$

e) $3a - (b+a) =$
$3a \cdot -b + 3a \cdot a =$
$3a \cdot b - 3a \cdot a =$
$3ab - 3a^2$

Figure 1. Example of wrong application of the conjugate equality.

The power notation is employed by between 6 and 9% of the grade 10 students on items 8b to 8e. The conjugate equality is employed by 15% on item 8d.

The most difficult item to simplify was item 8j: $\frac{4x+2}{8x}$. We found more than 40 structurally different responses to this item, for example the difference between numerator and denominator, or only between the parts of the numerator that contained the unknown. Only 2% in grade 10 gave a correct response. Some answered 0.75 (or equivalent), dropping *x* and cancel, others 0.75*x*, dropping *x*, cancel and then introducing *x* again. Others operated as illustrated in figure 2:

j) $\frac{\overset{1}{\cancel{4x}}+2}{\underset{2}{\cancel{8x}}} = \ldots\frac{3}{2}\ldots$

Figure 2. Example of simplifying of expressions.

Other students cancel 4*x* by 8*x* to get 0.5 and then add 2 to give the solution 2.5, while others again introduce the *x* again and write 2.5*x*. Another alternative is:

$\frac{\overset{2}{\cancel{4}}x+2}{4\cancel{8}x} = \frac{2x+2}{4x} \quad \ldots 0.5x+2$

Figure 3. Example of simplifying of expressions.

$4x + 2$, $-4\,x + 2$ and $\frac{2}{4x}$ are all examples of subtractions between nominator and denominator. Two per cent of the students managed to cancel “everything” and answered 0, and seven per cent, nearly four times as many, managed to find an incorrect solution, cancelled as illustrated in figure 4.

$\frac{\cancel{2\cdot2\cdot x}+2}{\cancel{2\cdot2\cdot2\cdot x}} \quad \frac{2}{2} = 1$

Figure 4. Example of simplifying of expressions.

Letter used as a generalised number

Problem 13 below is used to discuss students’ comprehension of how letters are used to represent a generalised number. The format is probably unfamiliar to many students, but still gives valuable information of typical interpretations.

Item 13: Tick the box for the correct answer:

a. $a + b + c = c + a + b$

☐ It is always true ☐ It is never true ☐ It is sometimes true

Explain the reason for your answer.

The same questions were asked to the following identities:

b. $4 + x = 4 + y$

c. $2a + 3 = 2a - 3$

d. $\mathrm{l + m + n = l + p + n}$

Table 3 shows the distributions of the ticks (correct ticks are indicated by *).

	Non response	Always true	Never true	Sometimes true
a. Grade 8	12	*39	12	33
a. Grade 10	7	*65	9	18
b. Grade 8	14	11	23	*31
b. Grade 10	7	4	54	*34
c. Grade 8	15	9	*60	14
c. Grade 10	9	4	*72	14
d. Grade 10	12	5	49	*33

Table 3. Percentage distribution of students' responses to problem 13.

Students who ticked but did not explain were also registered in table 3. Grade 8 students gave fewer explanations than the older students. Among those who ticked for the correct alternative it was respectively 31 %, 58 % and 43 % who gave no reason for their choice for the three items in grade 8. The corresponding percentages for the four items in grade 10 were 18 %, 23 %, 29 % and 25 %. The percentages of no explanations amongst those students who ticked for a wrong alternative are much higher than those above. We notice also that the two items (b and d) where the identity *could* be correct under certain conditions are more difficult. It is conspicuous that the alternative *never true* is more attractive than the correct alternative in these cases. One reasonable explanation is that students have much less experience with the equality sign as a symbol of equivalence than as a symbol separating a calculation task from its solution. A further discussion of this will follow below when analysing types of student explanations.

The most common correct explanation to item 13 a was that the order of addition is indifferent. Two thirds of the students who made a correct tick gave such reasons. But there were also explanations that indicate diffuse ideas. For example: "*It is the same, but usually they comes in alphabetical order. When we add it is the same which factor comes first*". (Direct translations) Some used one or two numerical examples, implying that these are general, such as: "*The answer is always the same because if you add 1 + 2 + 3 = 6, and turn it around you get 3 + 2 + 1 = 6.*" Others refer to that the letters are the same, which could indicate that they interpret the letters as objects.

The most common incorrect explanations are based on that the order of the variables is changed, for example: It is always true because *"It is correct that the letters should be separated, but they have to come in alphabetical order a + b + c"*. Other students made connections between variables and reading, it is never true because *"a + b + c is not the same as. Example: bil(car) is not the same as hus(house)."* Others again conceive letters as objects. Figure 5 shows an example (in Norwegian). It is never true: *"because a+b+c = abc and c+b+a = cba these are totally different things."*

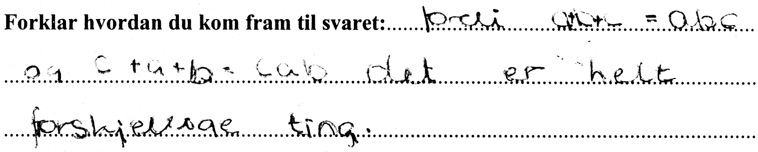
Forklar hvordan du kom fram til svaret: fordi a+b+c = abc og c+b+a = cba det er helt forskjellige ting.

Figure 5. Example of student's response to item 13a. (In Norwegian)

For all the items of problem 13 we find that many students conceive the equal sign as an operator, a signal for a numerical operation or to perform a procedure. The "task" is to the left of the equal sign, and the solution to the right. It is never true because *"When it says a + b + c the answer is abc.* Amongst the students who ticked for *sometimes true*, more than two third gave no reason for their choices. Those who gave reasons refer to the order of the letters involved.

Item b was more difficult. A considerably larger part of the grade 10 students believe that this equivalence can never be true, than those who understand that it can be fulfilled under certain conditions. The most common correct type of explanations refer to *x* and *y* as general expressions for numbers that *may* have the same value. Explanations such as *"It depends of the values of x and y"*, *"If they have the same value it is true, if not it is never true."* are typical correct explanations. One student wrote: *"x and y have usually different values, but they may also be the same by chance in an equation"*. It was respectively 9 and 19 % of the students who gave these types of explanations. Around 70 % of those who ticked the correct box *and* wrote an explanation applied this type of reasoning.

The most common reasons among those who ticked for *never true* are related to the idea that *x* and *y* **have to** represent different numerical values **because** they are symbolised by different letters. Answers such as *"x and y are different in any case"*, *"4 + x and 4 + y cannot be the same because x and y never stand for the same number"* and *the values of x and y are not equal in the <u>same</u> task"*. Students' who believe variables to represent objects arrive at the same conclusion, for example *"x is one word/thing and y is another word/thing"*.

As for item a, we find several examples of explanations that reflect the interpretation that the identity represents a calculation task to the left and it's solution to the right of the equal sign. For example: *"The solution to the task cannot be true because 4 + x becomes 4x and <u>not</u> 4 + y"*, *"Because one has to* *have a y in the task to get a y in the solution"* and *"Where does this y come from? One cannot suddenly bring it in when it wasn't mentioned in the task"*. On the other hand, the explanation *"Because it is an equation, and equations are always equal on both sides,"* indicates a correct interpretation of the equal sign, but the reasoning is associated with the symbolisation rather than with numerical values of these symbols. 11 % and 4 % ticked for *Always true*. Around half of these students explained their reasoning. Most of them argue: since it is indifferent which letter one chose to represent the variable, the identity will always be valid.

It is easier to establish a correct content knowledge to item c than to the other items. For grade eight, the differences between c and the other items are especially large. One reason could be that it is possible to apply other strategies that are not applicable in the previous items since these involved more than one variable. It is a large proportion of the students that tick one of the alternatives. Respectively 43 and 29 % of those who ticked for *never true* did not explain their reasons. The following line of reasoning is linked to the outcomes of an addition and a subtraction *"2a is the same number on both sides and cannot add something on one side and subtract on the other side to keep it the same"*. Some students, 26 and 41 % of all students who wrote an explanation, focused only on the signs of the equality for example, "+ *and – are not the same"*. In a few cases, 1 % and 4 %, based their argumentation on one or more examples. Few students who ticked a wrong box explained their choice. The only error type that could be traced to a systematic line of reasoning can be illustrated by the following response: *"When carry the numbers across = you have to change sign"*, referring to a procedure for solving equations. This line of reasoning was used both by those who ticked *always true* and those who ticked *sometimes true.*
Item d was given only to tenth graders. It has the same structure as item b. Even though it is possible to use the same argumentation as in item b, one would expect that this item should be more complicated, both because of the number of variables and that the identity does not contain an explicit number. Table 3 shows that roughly the same number of students ticks for the correct option. Likewise it is about same number of students who tick for *never true*. An analysis of students individual responses to these items showed 83 % of those who ticked *never true* for item d did the same for item b. Similar, 77 % of those who ticked *sometimes true*, did the same for item b. It should be noted that many of these students do not explain their choices, but respectively 85 % and 86 % of those students who explained their ticks for either *never true* and *sometimes true* used similar reasoning on both items, which indicates consequent thinking.

Conclusion

A general conclusion from the analysis of our data is that the majority of students apply arbitrary, but to some degree consequent, procedures in transforming algebraic expressions. These are usually linked to bits of partial understanding of arithmetic, but also procedures in other fields of mathematics. Their procedures become in this way isolated and therefore difficult to apply in unfamiliar situations. It is my opinion that the best way to overcome this is to strengthen the links between arithmetic and algebra.

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Applying Didactic Varieties in Geometry

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Abstract

The recognition of problem solving in geometry has provided a wide analysis by considering various approaches. Some of them concern clarifying in a general way its meaning as an activity, as a study of solving strategies and others in relation to their classification. A learning-teaching proposal based on problem solving through the building of a didactic model called Mathematical Didactics Varieties (VDM) is presented in this study. Our classification is inserted in the Chilean Educational reform framework[1] *in terms of applying didactic varieties in geometry, differentiating types of problems, registers and evaluating its solving at secondary level. To that effect, a learning experience on the subject "similarities of plane figures" based on the incorporation of the model proposed was devised. This qualitative research in geometry related a didactic view of mathematics in the teaching of geometry concepts and problem resolution, specifically that concerning the area of similarities in the development of the ability of problem solving.*

Background

The importance of problem solving as one of the main aspects in the learning process in the field of mathematics is widely recognised and is still a main concern for educators and researchers in the field of mathematics education. The importance now given to problem solving and evaluation in the field of mathematics has originated several proposals for the teaching of this area. For this reason, the work of Schoenfeld (1988) plays an important role in implementing strategies for the problem-solving process.

Besides this, there are other interpretations regarding problem solving itself, which consider it a goal, a process, and a basic skill. When considering problem solving as a goal, the need to learn how to solve problems becomes essential, since it is assumed that this ability will help the students to cope with various types of problems. In the second case, what really matters is the method, the process, and the strategy, the last of which should be the focus of the curriculum in mathematics. As for the third interpretation, problem solving can be presented as a basic skill together with other skills; this would make it necessary to distinguish content from problems, to highlight possible classifications that could be made, or to identify the different methods needed to solve the problem.

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[1] Project Fondecyt – Chile, N° 1990558, 1999-2001, of which this article is a partial result.

In an attempt to modify the mathematical contents so as to consider the needs of a society immersed in the twenty-first century, problem solving must be recognised as the main reason to study mathematics. Thus, it should be regarded as the process of applying previously acquired knowledge to new and unknown situations. The question that immediately arises than is: Do the student's acquired skills enable him/her to solve mathematical problems? If problem solving is to become the main focus in the teaching of mathematics, it should also become the main focus of its evaluation.

One didactic variety of mathematics is a situation of learning associated to mathematics that is built with problems situations, type of problems and registers of expression. This model centres on a problem situation which acts as distinguished variable and establishes other variables which support it. The distinguished variable is the mathematics framework and the associated variables are the context and registers of expressions. There are several mathematics frameworks and several contents and registers of expressions for only one mathematical concept. The registers are expression forms and can be graphic, symbolic, algebraic, tabulations, and natural language.

This model relates registers of expressions as representation forms and types of mathematical problems. The types of problems have been considered according to their nature as routine and non-routine, and according to their context as real, realistic, fantasy and purely mathematical.

Non-routine problems refer to those whose answer and a previously established procedure is not known by the student. Example: Explain why two triangles with two proportional sides and the same angles are similar.

Real context problems: A context is real if it is effectually produced in the reality and the student is involved in this. Example: Get the measure of a window in your classroom and determine what length and width should be right to be similar to others whose sides are the double.

Realistic context problems: A context is realistic if it is able to be really produced. It deals with a simulation of the reality or of a part of reality. Example: A pine tree at eleven in the morning of a certain day casts a shadow of 6.5 m. Next to it there is a booth 2.8 m high which projects a 0.70 m shadow. What is the height of the pine tree?

Fantasy context problems: A context is fantasy if it is the product of imagination not founded in reality. Example: At Osorno airport an Unidentified Flying Object (UFO) is seen. Its characteristics are similar to a triangle whose sides are 0.6 cm, 0.8 cm, and 1.0 cm. For investigation purposes another similar object is built on earth whose smallest side is 3 m. What is the ratio of similarity?

Purely mathematics context problems: A context is purely mathematics if it refers exclusively to mathematical objects such as numbers, relations and arithmetical operations, diagrams, etc. Example: The sides of a triangle are 0.3 dm, 0.4 dm, and 0.5 dm. The shortest side of another similar triangle is 0.15 dm. Find the other sides of this triangle.

In general the solving of a problem of a real, realistic or fantasy context needs the mathematization of the given situation, that is, its translation into a mathematical language. Since this deals with a problem, the process of mathematization should demand a certain searching from part of the student who is working with the problem. If he can mathematize the situation in an almost automatic way and without any effort, then it does not have to do with a problem of context, but rather with an exercise of mathematization.

Methodology of research

A didactic experience to articulate the mathematical concepts regarding a specific teaching unit and the solving of types of problems for secondary school (i.e. grades 9-12 in Chile) was devised applying mathematical didactic varieties. The tasks were geometry problems and the unit dealt with similarities of plane figures. The contents were structures as indicated by the Chilean Educational Reform for secondary school, specifically a middle second grade school from a scientific-humanist class whose ages ranged from 15 to 18 years.

The study of participating observation was carried out in 2001 during 4 weeks of through 3 weekly meetings of 90 minutes each, and four pairs of students to be observed during the course of the teaching-learning experience.

The teaching strategies in the classroom were active, being focused on the students' actions. The students worked in groups with teaching materials made by the researchers, and structured according to the classifying of non-routine problem context, and differentiating registers of expressions. The teaching material consisted of problems, situations and questions in relation to the similarity of plane figures, criteria of similarity, scale drawings in various contexts, Thales' theorem on proportional lines, and aureus ratio.

Data collection was made by observations in classes, individual interviews in meetings, and an attitude questionnaire with open answers. All the meetings were recorded and 8 protocols were produced.

Development of the experience

The application of the test based on the solving of types of problems was followed by application of an attitude questionnaire to the four pairs of students under observation. Students start interacting. They do not know how to convert certain measurement units and make comments about not remembering how to calculate the area of a rectangle:

Fransisca: Does anybody remember how to calculate the area?

They make comparisons between them but differ in the measurements. They attempt the required conversions with a certain difficulty. Andres and Miguel stand out concerning the precision of the reproduction but in general they take more time to reach the increment, only one of them makes a mistake when using the scale. The presence of the tape recorder distracts them and inhibit to some extent their mutual relationship and with the teacher.

They wonder how long the maquette front is, and how high it is, how many times the building size has been reduced. They make the comment: can we draw the height considering the other front? I do not understand the last question, maybe we must convert it to meter. Miguel explains the existing relation between the drawing measurements and the measurements in reality to the rest of the students being observed.

An activity in a fantasy context refers to a fictitious character from the book *Gulliver's travels*, where the *Lilliput* country is described as a world in miniature made at scale. During its solving they talk to each other:

Pamela: The Lilliput is the little one.
Miguel: It is made at scale.
Karmi: I have got the measures, it is 1.3, that is, the giant is 11 times bigger than the king.
Andres: I counted the Lilliputs, they are 9.
Carlos: The giant eats 11 times more than the king.
Karmi: This exercise is easy.
Francisca: I need to have an idea of how much this would measure.
Pamela: When we measure the map, do we have to do it in a straight line?
Miguel: We have to pass from km into cm.

They show difficulties in establishing the ratio of the similarity between the figures, and in solving the problems of a real context. They comment: this is very difficult and I will never finish it, it would be better to have only one drawing.
Even when they show their dislike for so many problems about the same thing, we see the extent of appropriation of concepts as a reason of similarity and similarity of figures. For the Thales' theorem problems of purely mathematical context are proposed for their solving, this leading to an evident advanced degree.

There is a significant mastering in solving routine problems, and they show they are motivated to answer them. But most of the students have difficulties to solve non-routine problems. To the problem "Two triangles have two proportional sides and the same angle, is this enough to assert that they are similar? Justify your answer.", four of them answer affirmatively but they do not justify their answer.

Finally, the test with which the research started was given to the whole class.

Results

Categorization of the results

Based on the data obtained, the class observations, literally transcribed at the end of each working session, and the pre- and post attitude questionnaire, all the information was classified in order to find the convergence. The idea was to obtain a corpus of data that allowed a more systematic analysis of each situation, leading to the formation of categories from the similarities in order to maintain internal homogeneity, or the differences related to external heterogeneity, trying to establish clear and coherent criteria for the classification and ordering of the information obtained. Similarities found for the three pairs of students observed are detailed below.

- *The solving*

Regarding the solving development

The main types of problems undertaken and really solved by the students were those routine ones of realistic, fantasy and purely mathematical types, excepting some real problems. The types of problems considered as non-routine ones did not reach an adequate development of solving, although they were tackled.

Regarding the degree of difficulty of the problems

At the beginning, the students showed a high degree of difficulty in determined routine and non-routine problems. The degree of difficulty in the non-routine ones is shown through all the application of the experience.

Regarding the knowledge used

The geometry knowledge applied by the students at the beginning was not enough to answer the situations and problems set up during the experience. Subsequently, with the application of the teaching material, the knowledge became sufficient to solve the situations and problems proposed. Even when the students show a concern about so may problems about the same thing, we may confirm the degree of appropriation of concepts in relation to a similarity of plane figures.

Regarding the algorithmic processes

This is an outstanding situation during the development of the experience, since the students show a greater management in the comprehension of concept aspects in mathematics than in the application of algorithmic processes of solving.

- *Of teaching*

Regarding the work methodology

The students establish a significant difference between the lectures considered as traditional by them and this form of teaching through problems, this resulting in something interesting to them because it is entertaining and didactic, and participate since they have the possibility of making group work.

Regarding the material

From the beginning of the experience the students valued the teaching material being used in a positive way, since it was made taking into account a sequence of activities, most of them with additional graphic representations designed according to learning situations, setting out types of problems and geometry concepts being involved.

- *Of the mathematical concepts*

Regarding the meaning of similarity of plane figures

From the analysis of the answers given by the students during the interviews and through the development of the 48 proposed activities, we proved similitude in the meaning they give to the concept of similarity, since they are able to show their comprehension through examples of types of specific problems made by themselves.

- *Of the achievement being reached*

The application of the tests at the beginning and at the end of the experience showed that the students got percentages of learning achievement which ranged between 0% and 15.6% at the beginning and between 31.2% and 93.7% at the end. We should underline that out of the four pairs of students under observation, 7 students recorded an achievement percentage over 62%.

Conclusions

Based on the analysis of the application of this didactic strategy, we can identify a substantive increase in the development of the ability of problem solving. This could be verified in the level of achievement of the students during class work with solving types of mathematical problems in geometry.

In order to reach specifically the concept of similarity of plane figures, this research in the area of geometry considered a conception of mathematics teaching that established a relationship between the teaching of general geometric concepts and the solving of problems, with the support of teaching material fundamentally based on types of problems categorised according to their nature and their context.

Based on the scores obtained by the students in the tests, interview interpretations and attitude questionnaires, we may conclude that there is a significant increase of the development of the ability for solving types of geometry problems, particularly those of routine problems of fantasy, purely mathematical and realistic context. They showed a moderate ability in the area of real and non-routine problems, that is, those in which their action is required and those whose solving procedure is unknown.

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Snowboard, Skateboard, and Understanding of Mathematics

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Abstract

Students who participate in snowboard and skateboard activities, perform better than other students at some mathematics items. This could be due to their experience from spare time activities. They perform lower than other students at a mathematics test with items from TIMSS and TIMSS-Repeat. Their mathematics marks are also below those of other students.

Geometry is grasping space…that space in which the child lives, breathes and moves. The space that the child must learn to know, explore, conquer, in order to live, breathe, and move better in it.
(Freudenthal, quoted in Clements & Battista, 1992, p. 420)

Do skaters perform better than other students at some mathematics?

By the term *"skaters"* I mean students from 10th grade who participate weekly or more often in snowboard or skateboard activities. These students have some specific experience from participating in snowboard and skateboard activities in their spare time. I presume they have some specific experience from navigating *"through space and through constructions of shapes"* (OECD, 1999, p. 49). Further, according to OECD/PISA, our learning about space and shape include *"We must be aware of how we see things and why we see them as we do"*. The American scientist and philosopher Norwood Russel Hanson claims *"Seeing is not only the having of a visual experience; it is also the way in which the visual experience is had"* (Hanson, 1958, p. 15). Inspired by the late Wittgenstein, Russel Hanson carefully chose a similar language. I have chosen to use Hanson's theories about observation and his language as well, as a tool for a possible explanation of why the skaters performed better than other students in some mathematical items. My study (Fyhn, 2000) focused on possible connections between participating in activities outside school and performance in mathematics.

"*There is more to seeing than meets the eyeball*" (Norwood Russel Hanson)

Russel Hanson imagined Tycho Brahe and Johannes Kepler watching the sunrise together. Most likely they would make the same observation, but their interpretations would differ: Kepler would regard the sun as fixed, it was the earth that moved. According to Brahe the earth would be fixed and all other celestial bodies moved around it. Russel Hanson asked: "Do Kepler and Tycho see the same thing in the east at dawn?” They watched the same visual object. They could make identical drawings but different interpretations. “*To interpret is to think, to do something; seeing is an experiential state.*” (Hanson, 1958, p. 11)

See as and *see that* – Russel Hanson's terminology

At first our observations are registered. Then the brain finds what knowledge we have about them. In this way brain and eye cooperate. An s-shaped mark in a steep white winter mountain can be *seen as* a snowboard track, even if there is no snowboarder present.

We can *see* a winter mountain *as* an attractive snowboard arena, *as* a cold and frightening place or *as* a lot of unstructured snow and rocks. A Dane, who has never been outside Denmark, will probably not have the necessary knowledge to *see* such a mountain *as* an attractive snowboard arena. When you shift from *seeing* something *as* “something” to “something else”, organizing of what is seen, changes. Seeing is a state of experience: If a skater and a teacher watch the same skateboarder performing a trick, they will not necessarily see the same thing. They will organize what they see in different ways.

Seeing that threads knowledge into our seeing; it saves us from re-identifying everything that meets our eye. The gap between pictures and language locates the logical function of *seeing that*. A sentence does not show that the mountain is covered with snow, but a sentence can state that the mountain is covered with snow. “*Only by showing how picturing and speaking are different can one suggest how “seeing that” may bring them together; and brought together they must be if observations are to be significant and noteworthy.*“ (Hanson, 1958, p. 25)

From 360 to 900

In the skaters' terminology ordinary jumps are named such as three-sixty, five-forty, seven-twenty and nine-hundred. The code is: three-sixty is a jump where the board is rotated 360° horizontally. The skaters are familiar with using these numbers as names for the jumps.

In Extreme-games in San Diego the summer of 1997, snowboard was one of the disciplines. Peter Line won the competition by performing a jump described as "a perfect nine hundred" by the Eurosport commentator. I have watched this video tape several times, but still I have a problem in seeing Line rotating two and a half times, and neither have I automated that 900 is 2,5 times 360. A trained eye will immediately see if a skater performs seven-twenty or nine-hundred.

Many skaters do not know there is 360° round a circle. They have been told that the name of this jump is three-sixty, and they exercise in performing the jump.

An example from mesospace activity

According to Berthelot and Salin, (1998, p. 72), the students' daily life interactions take place in three different spaces: *"Students' natural knowledge of space is strongly structured into three main representations: microspace (corresponding to the usual prehension relations), mesospace (corresponding to the usual domestic spatial interactions) and macrospace (corresponding to unknown city, maritime or rural spaces…)"*

In these terms, the skaters' daily activities on their boards usually take place in mesospace. When they work with geometry problems at school, they use their microspace representation instead of geometrical knowledge from their own experiences in mesospace. Berthelot and Salin (1998, p. 73), claim that *"..we have good reasons to expect that one of the main sources of learning difficulties in geometry is the previous treatment of geometrical figures on paper during elementary school"*.

One afternoon I asked one of the skaters at my school: "How do you see the difference between a seven-twenty and a nine-hundred?" The cool sixteen year old that did not care much about school, answered: "It's easy. You just count inside yourself how many times he does three-sixty". I was expected to understand that it was easy (!) to count that 2,5 times 360 would be 900 at the same time as my eyes should watch an object rotating with great speed. "But how can you see the difference between one-hundred-and-eighty and three-sixty?" I asked. "You just see it", he could not give any further explanation. His friend, who was nearby, continued: "You see how he is landing. If it is one-hundred-and-eighty, then his back is this way in the landing. If it is three-sixty then his back is the other way when he is landing." The boys showed me, but they were not convinced if the mathematics teacher did understand.

Do the skaters and the teacher see the same thing?

These skaters have learned to interpret what they see. They watch a jump and *see that* the skater does a seven-twenty or *that* he does a nine-hundred. I can only *see that* he is jumping and rotating with a snowboard (or skateboard). These boys and I can watch the same thing, but we do have different interpretations of what we see. I assume the boys' interpretation to be that the jumper rotates twice, then begins the third rotation and lands with his back "the wrong way". That is what they call nine-hundred. I am an inexperienced watcher of snowboard jumps, thus my organizing of what I see is not good, and I interpret what I see as a rotating jump. My formal geometrical concepts are more developed than those of the boys (I hope), but I am not able to connect these concepts to what I see. The boys have a weak understanding of the angle concept and are not sure what kind of angle 360° is. But they know exactly what they see.

We, who are teachers of mathematics, are here challenged to make the youths find connection between their knowledge from their spare time and what goes on in the mathematics lessons.

The skater, who characterizes the difference between the two jumps seven-twenty and nine-hundred, is able to separate representing and referring, arranging and characterizing (Hansson, 1958). He is able to separate picturing and language-using. This skater will probably not be able to tell by himself that he does so, but unlike me he is able to characterize what he sees. This characterizing can shorten his way to deeper understanding of the angle concept.

The test and the results

I set up a mathematics test with items from TIMSS and TIMSS-Repeat. 638 Norwegian students at 10th grade performed the test the autumn of 1998. In addition to the test, the students answered questions on gender and on their participating in activities in their spare time. Close to nine percent of the students showed to be skaters.

	Marks*			Test score			
Skater/ nonskater	Mean	N	Deviation	Mean	N	Deviation	Percent of test correctly answered
nonskater	**3,18**	490	0,81	**17,5**	507	5,25	**58**
skater	**2,89**	127	0,77	**16,1**	131	5,25	**54**
Total	**3,12**	617	0,81	**17,2**	638	5,27	**57**

Table 1. Mean values of marks and test scores for nonskaters and skaters.
* Norwegian marks; S=5, M=4, G=3, Ng=2, Lg=1

Recognizing a rotated object

Two of the items primarily tested the students' understanding of rotation in a plane as shown i figure 1. The item text said that a half-turn about a given point in the plane should be applied to a given figure. The students had to tell which one out of five alternatives that showed the result of a half-turn.

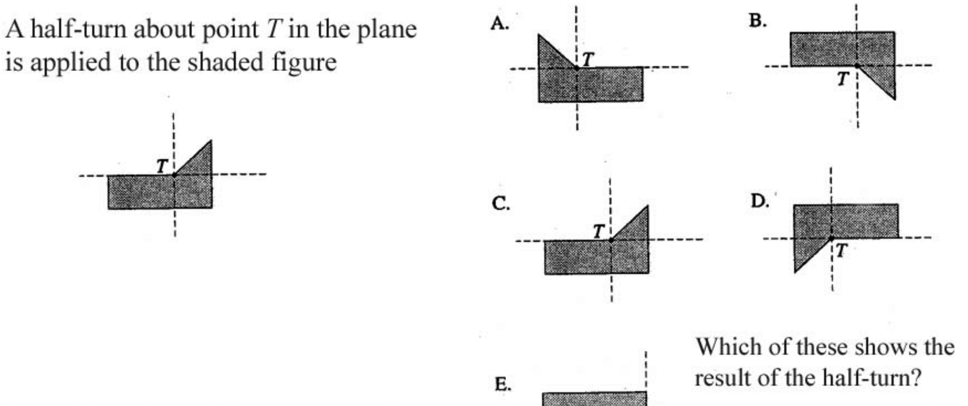

Figure 1. The half-turn issue from TIMSS

In a quite similar item the rotated figure was a triangle, as shown in figure 2.

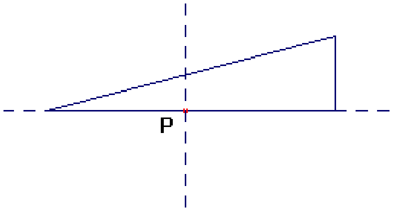

Figure 2. The rotated object from the TIMSS-Repeat issue

These two items did not test much of the students' formal mathematics knowledge. The item with the irregular hexagon was the easier one, as shown in table 2. This can be due to coincidence, but it also can be explained the following way: Students at the van Hiele visual level will recognize figures as visual gestalts (Clements & Battista, 1992). They will probably be able to recognize the rotated hexagon because of its characteristic corner. To them the rotated triangle will just appear as a horizontal slim object in the five alternative figures.

The triangle issue was one of the skater items in the test. According to Russel Hanson, I claim that the skaters do not necessarily *see* the triangle *as* representing the class of triangles (Clements & Battista, 1992). Many of them will be at the visual or descriptive level. By performing physical activity in mesospace (Berthelot & Salin, 1998), many of the skaters have learned to recognize objects visually; they are able to *see that* the correct figure must be the rotated triangle.

The skaters have a lot of experience from rotation in mesospace, and they have learned to organize what they see. School geometry is primarily taking place in microspace, and the skaters get limited possibilities to connect their experiences with school geometry. From their own physical activities the skaters have learned to recognize objects visually. We, who are teachers, do not usually have these experiences. Our interpretations of the items are the result of a mapping where a figure is rotated round a point.

Percent of skaters and of nonskaters who chose answers D and E	Hexagon item			Triangle item		
	N	D - correct answer %	Answer D or E %	N	D - correct answer %	Answer D or E %
skaters	91	53	**77**	40	53	**56**
nonskaters	352	60	**70**	155	39	**46**
total	443	58	**72**	195	42	**48**

Table 2. Percent of skaters and of nonskaters who chose answers D and E

Some students chose wrong answers

The students could choose between five different solutions to each of these two items. Students who were familiar with rotation and not familiar with the formal language: rotation “about a given point in the plane”, could risk choosing alternative E in the issues.

A larger amount of skaters than nonskaters chose alternatives D or E in these items, as shown in table 2. The skaters seem to have understood the half-turn, but not necessarily the more formal “about a given point”. A half turn is part of the daily language in their board activities; according to Hanson's terminology they *see that* the correct answer must be D or E.

A dropped rubber ball

The number one skater item was about a rubber ball being dropped. In the item the ball rebounds to one-third the height it drops, and the students are asked to find the total distance traveled by the time it hits the ground the third time. In another skater item the rubber ball rebounds to half the height it drops.

My thesis is founded in the following way: The skaters have an inner understanding of the ball’s moves, because they are familiar with the moves of a skater in a half-pipe or in a vert. The halfpipe and the vert have similar u-formed shape, as shown in figure 3. The halfpipe is made of snow and the vert is wooden. The skater starts on top of the vert, slides down to the bottom, moves up again and so on. The rubber ball starts at a certain height, falls to the ground, moves up again and so on. I presume the skaters can *see* the ball’s moves *as* the moves of a skater in a vert or in a halfpipe. Other youths are familiar with dropping balls, but that is an outer-understanding. Usually people are not philosophizing on what it is like being a passenger on a rubber ball.

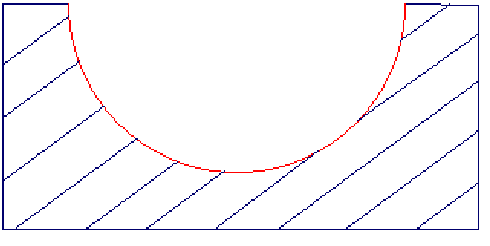

Figure 3.
Halfpipe or vert

If you watch a skater doing vert exercises on TV, you can hear the rhythm of the board’s wheels against the wooden vert. The sound differs when the board is rolling up and down. If the skater is jumping when he or she reaches the top of the vert, the sound is rhythmically erupted. The skater must use his or her own muscles to get the board up to the same height as it came from. If you drop a rubber ball, you can hear the rhythm of the ball as well. My thesis is that more skaters than other students seem to be familiar with the pattern of the ball’s move because of the skaters' personal physical experience. They *see* a moving ball *as* a model of a well-known pattern of movements. This pattern they have experienced from their activity in mesospace (Berthelot and Salin, 1998). Thus many skaters *see that* the ball will move 2,5 times between top and bottom. Most teachers lack experience from halfpipes and verts and will probably have trouble connecting school mathematics to this kind of knowledge.

Footsteps

Skaters can be seen sliding along the street by standing on the skateboard with one foot, pushing themselves forward by the other foot. Before you go performing tricks on the board, you must be able to do these moves. I claim the skaters *see that* a few long steps and a lot of short steps lead to the same result.

One single skater and about ten percent of the nonskaters chose one of the wrong answers to an item in this context. In the item the lengths of four girls' footsteps were given. The students were asked to tell who would use most steps to walk from one end of a doorway to the other. I presume the skaters scored high at this item is because of their experience from physical activities in mesospace.

An item with similar mathematical content took place in a different context. The students were informed that three different cars used accordingly 3, 3.5 and 4 hours to drive a given distance. Then students were asked which one of the cars was the fastest. 63% of the skaters and 67% of the nonskaters chose the correct answer. The car context does not seem to be a success in this item.

Which came first

There is one possibility that I cannot exclude. Maybe one important reason why some students become skaters, is because of some skills that make them able to see and understand different movements, able to better understand geometry.

Maybe there are different entrances to understanding mathematics. We know by history that Pythagoras was a musician who focused on numbers and structures. Plato, on the other hand, was an athlete who focused on geometry. His academy was founded next to an athletics ground, and physical activity was a regular part of his students work.

Summary

In my test I found that the skaters chose correct answers to some mathematics items at a lower frequency than the nonskaters. The skaters performed a mathematics test consisting of items from TIMSS and TIMSS-Repeat. More skaters than nonskaters answered some items in the test correctly. These results could be due to coincidence, but they also can be explained by the skaters' physical experience in mesospace; in Russel Hanson's terminology seeing is a state of experience.

My thesis is that the skaters' high score at some of the items in the test is due to the fact that they have learned some mathematics from their spare time activity- mathematics that they have not learned in the mathematics lessons at school - mathematics that in *"PISA-words"* is useful for constructive, concerned, and reflective skaters.

Maybe the old Norwegian proverb "What you loose in your head, you have got in your feet" needs to be rewritten into "What you've got in your feet, you can get into your head, too."

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Concept Maps as a Tool in Research on Student Teachers' Learning in Mathematics and Mathematics Education

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Introduction

Longstanding work with mathematics in-service and pre-service teacher education has stimulated my curiosity in how teacher students develop concepts in mathematics and mathematics education. In 1996 a new teacher education programme started in Kristianstad University for prospective teachers in mathematics and science for school years 4-9. I had the opportunity to lead the work with creating the mathematics courses for this programme (Grevholm, 1998). In doing this I built on all my future knowledge and experience from teacher education. As a consequence I was eager to follow the development of the education and its outcomes. I decided to carry out a longitudinal research study and in this try to focus on teacher students' conceptual development, as I had noticed how important this is for the learning. Many other researchers have studied the conceptual development in mathematics (for references see below) and pointed at its importance. In Sweden no such study was conducted earlier.

It is complicated to do research on students' conceptual development as the concepts an individual holds are not open to direct observation. They can be studied only indirectly through actions, statements or answers given by the individual student. Thus I started out to collect all data that could possibly help me to observe students' conceptual development. Soon after I had started the data collection, Joseph Novak came to visit our department. When he heard about my study he tried to convince me that it should be productive to use concept maps in the research. At the beginning I hesitated. From the examples I saw of biological concepts it was obvious that objects and events could be studied and observations formulated in knowledge propositions and representted in concept maps. You could for example study a plant and describe the development. But I did not find it possible to look at 'objects and events', when it came to mathematics. All mathematical concepts are abstract. It took me some time to reflect more closely on the differences between concepts in science and in mathematics. After a while I came to the conclusion that objects in mathematics can mean the mathematical objects like numbers, shapes, equations, functions, and expressions and so on. And events can be seen as the operations, processes, constructions or actions we take with these objects. After this interpretation it became clear to me how useful the concept maps could be also in my study. Based on experiences from the study, the use of concept maps as a tool for research on development of concepts is explored and discussed in this paper.

The aim of this paper

In this paper I want to explore the use of concept maps in research. I will try to answer the questions:

- Can concept maps as a tool in research contribute to our understanding of students' conceptual development in mathematics?
- In what ways can concept maps be useful?

Theoretical background

Theories in mathematics education deal with phenomena such as meaningful learning versus rote learning, conceptual knowledge or procedural knowledge, mathematical phenomena seen as procedures or objects, and conceptual change and development as an important part of learning (Ausubel, 1963, 2001; Hiebert and Lefevre, 1986; Sfard, 1991; Tall, 1994; Tall & Vinner, 1981). In several of these theories mathematical concepts and the development of concepts are crucial. Both my own experience as a teacher educator and the reading about studies on concept development convinced me of the importance of focussing on this phenomenon.

Knowledge construction is a complex product of the human capacity to build meaning, cultural context, and evolutionary changes in relevant knowledge structures and tools for acquiring new knowledge according to Novak (1993). Novak claims that concepts play a central role in both psychology of learning and in epistemology. In his Human constructivism Novak (1993) builds on Ausubel's assimilation theory of learning to describe the process by which humans engage in meaningful learning. Two key ideas in assimilation theory are *progressive differentiation* and *integrative reconciliation*. Novak explains that as new concepts are linked nonarbitrarily to an individual's cognitive structure progressive differentiation occurs. The integrative reconciliation occurs when groups of concepts are seen in new relationships.

Hiebert and Lefevre (1986) devoted much interest to the discussion of conceptual and procedural knowledge. Conceptual knowledge is equated with connected networks. Conceptual knowledge is knowledge that is rich in relationships. Procedural knowledge is a sequence of actions. Sfard’s reification theory (1991) concerns mathematical phenomena seen as processes or as objects. Tall (1994) discussed process and concept in mathematics and also introduced the term *procept* as a combination of the two. In an often quoted paper Tall and Vinner (1981) discussed *concept image* and *concept definition*. They note that many concepts are not formally defined at all but we learn to recognise them by experience and usage in appropriate contexts. After some time the concept may be refined in its meaning and interpreted with increasing subtlety. They continue:

> Usually in this process the concept is given a symbol or name which enables it to be communicated and aids in its mental manipulation. But the total cognitive structure which colours the meaning of the concept is far greater than the evocation of a single symbol. It is more than any mental picture, be it pictorial, symbolic or otherwise. During the mental processes of recalling and manipulating a concept, many associated processes are brought into play consciously and unconsciously affecting usage and meaning (p. 152).

At this stage they introduce the term concept image to describe the total cognitive structure that is associated with the concept. The concept image is personal and changes when the person meets new stimuli and matures. Tall and Vinner (1981) also make a difference between the formal concept definition (accepted by the mathematical community) and the personal concept definition (the words the student uses for his own explanation).

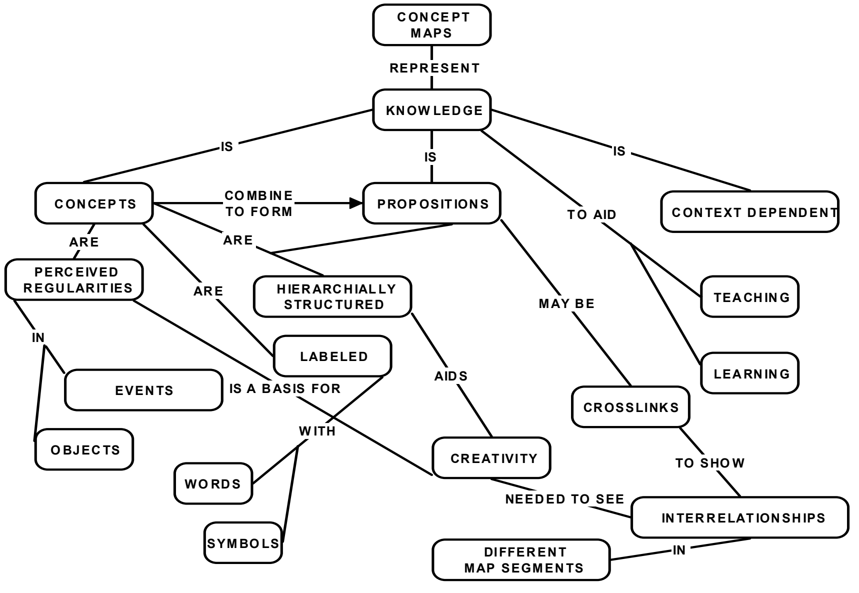
Other researchers have discussed the meaning and development of concepts. Vollrath (1994) claimed that didactical discussions sooner or later end up in the problem of what understanding a concept means. He claimed that the student reaches stages of understanding and that there is no final understanding. Ausubel (1963) contrasts meaningful learning to rote learning, where meaningful learning results in the creation and assimilation of new knowledge structures. In many of the theories there seems to be a continuum from lower quality learning to higher quality learning, where higher quality often includes concept development. The theories can be seen as different ways to model the quality of learning and how it evolves.

What is a concept map?

Joseph Novak (Novak & Gowin, 1984) has introduced *concept maps* as a cognitive tool and as a research tool. He first used concept maps as a tool for researchers to catch the main content of answers in interviews. In his case the researcher drew the maps in order to give a concentrated representation of what the interviewee answered (Novak, 1998). The map is used for data reduction and concentration of content. From his map below (Novak, 1998) can be seen how he defines a concept and what he means by a concept map. In Novak's maps it is important that the map is built of knowledge propositions. The nodes that contain concepts should be connected by linking words to form propositions, which represent knowledge sentences. Normally the map is also hierarchical. The map can be seen as a picture or image that the learner chooses to draw from what he experiences as the mental representation of his knowledge.

In research literature many different sorts of concept maps have been introduced (see for example Williams, 1998). A concept map differs from for example a mind map, which is a looser construction and does not necessarily show how the learner wants to draw his knowledge representation. A spider web map has no hierarchical character. In this paper I use Novak's definition of concept and concept map.

Novak (1998) describes a concept map in this drawing of a concept map:



Research using concept maps as a tool

Williams (1998) used concept maps to assess the conceptual knowledge of function. She studied concept maps drawn by students and professors of mathematics and compared them. Her dissertation was based on that work and she claims that "Concept maps are a direct method of looking at the organization and structure of an individual's knowledge..." (p. 414). This strong claim can be questioned and she modifies herself in the conclusions. There she states (p. 420): "The degree to which concept maps describe a person's mental representations is, of course, impossible to know." But her final conclusions are important:

> The analysis also provided information about students' understanding that is not readily gained from traditional paper-and-pen tests. Concept maps therefore, provide important information about conceptual understanding and can play a useful role in the mathematics researcher's repertoire of tools. (p. 420)

Novak and his colleagues used concept maps in many studies and argue strongly for their potential in research and in learning (Novak, 1985, 1993, 1998).

In her master's research, conducted at Samoa University, Afamasaga-Fuatai (1998) used concept maps. Her research shows that students found concept maps useful in their learning and understanding of mathematics. It helped in systematic analysis of a topic for the interconnections between relevant concepts and procedures, and facilitated problem solving.

Peuckert and Fischler (1999) used concept maps constructed by students to elicit their conceptions and maps constructed by researchers to summarize all statements made within each interview including students' maps.

There are many other studies available using concept maps in research, but space restrictions for this paper prevents me from saying more about them here. For references see Williams (1998), who found many studies using concept maps, mainly in science didactics but also in mathematics didactics. Also McGowen and Tall (1999) refer to a number of such studies.

The use of concept maps in my research study

There are many ways to use concept maps and Novak has written about the use of maps as a tool for learning and research (1985, 1998). As a tool in research I first used it in an a priori analysis (Artigue, 2002) of the expected learning of students in their course. For example I drew a map of the concept *fractions*, where I tried to include all the important features about fractions that I consider crucial in the course the student teachers were going to take. The map has 25 nodes and 30 links and it is not possible to show it here for space reasons. Later, after the students had answered a questionnaire about the course in number theory, I used a concept map in the analysis for data reduction on the answers about fractions. In the map I drew all alternative answers given by the students and the links they proposed. In comparing my a priori map with the map constructed by the students' answers I could see what parts of the expected exposed learning that had taken place and not.

Another way I used the maps was to let students express their view of a concept. I used this repeatedly over time to follow the conceptual development. Below I will show some examples of the data it produced and discuss what results one can get from it. In order to do that I need to say something short about the study in which the maps were used.

The study in Kristianstad

The method used in the study drawn upon here is mainly qualitative investigation of data from different types of documentation of students’ cognitive development during a teacher preparation program. Concept maps are used as a tool both for analysing the content of the teacher education to find the fundamental concepts, to investigate students’ answers in questionnaires and interviews, and for the students to express a picture of their current concept structure.

The overarching questions posed are phrased like this: How are the studies of mathematics and mathematics education influencing student teachers’ development of concepts in these areas? How do student teachers’ perceptions of and attitudes to mathematics change during the education? What impact does the development of concepts have for the learning outcome and for the students’ perception of their own learning?

The studied group consisted of 48 student teachers studying to become compulsory school teachers in mathematics and science for school years 4-9. I have reported on this study elsewhere (Grevholm, 1999, 2000ab, in press, 2002, 2003ab, 2004) and here I am only going to discuss the use of concept maps as a tool in research.

The problem in focus in this paper
Students' concepts are not open to direct study by the researcher. They have to be observed in an indirect way and often only in fragments. Some researchers argue that concepts should be studied through their appearance in students' actions. This is however time-consuming and a difficult process. Here questionnaires were the first attempt to get an image of students' conceptions, followed by interviews based on the answers given. The impression was that far too little of what students carry in their heads about the concept was revealed in this way. I was convinced that students could reveal more to me about their concept image. At this stage concept maps were introduced as the answer form for students. As will be shown below a much richer material was retrieved in this way and substantial knowledge about how students express their mental structures through maps became available. By having students draw maps at several times with long intervals the development over time of the structures could be studied.

With the examples below I want to illustrate that if concept maps are used as a tool for research, the findings differ in a positive way from results from questionnaires and interviews.

Examples of collected data
In the investigation an example of the outcome of the questionnaires could look like this. To the question 'What do you mean by a function?' Lina, one of the students, answered before and after the course in function theory (calculus):

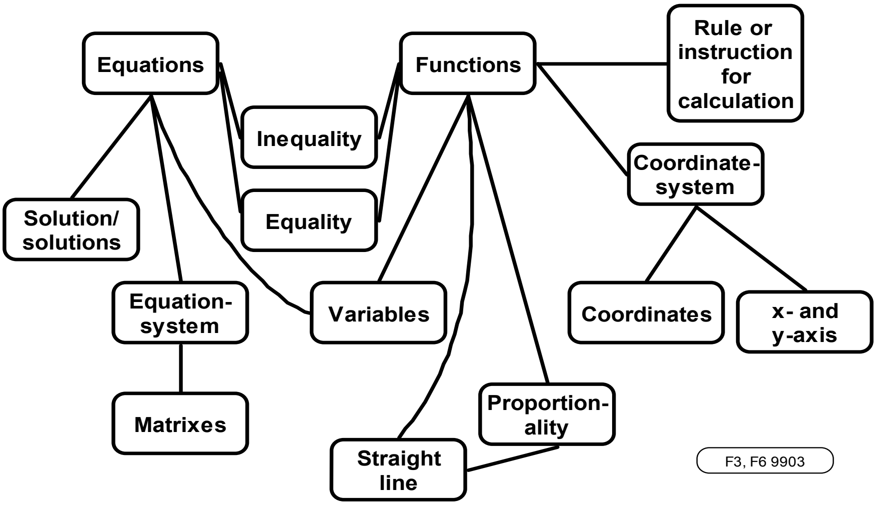
1) – y is depending on how big x is. There is an infinite number of answers as you can vary x.
2) – for example y=kx+m. y is here a function of x. So y is depending on the x-value. You can illustrate a function graphically.

At both occasions Lina holds on to the idea that y is depending on x. In the first answer she talks about answers to the function, which may indicate that she perceives each calculation of the y-value as an answer to a problem. She cannot see a function as an object (Sfard, 1991). In the second answer she gives an example, the simplest possible function she has worked with, the linear function, although in a general form. She also adds the information that one can illustrate a function graphically. In the second answer she actually reveals more than in the first answer.

Still both these answers give very little information about the mental representation or concept image (Tall & Vinner, 1981) she has of the concept function. I was convinced that the student could show me more of her knowledge structures than these short sentences. At this stage of the research study I decided to use concept maps as a form for students' answers. The drawing of concept maps was already familiar to the students from other subjects in their education.

Still I was aware of the fact that it is very demanding to try to draw a concept map of your own knowledge.

At the end of the course in calculus (five weeks in the sixth term of the 4.5 years long education programme) Lina together with one fellow student drew this map of the concept function. The task given was to individually draw one map of function and one of equation. As can be seen the students did not follow the instructions. It is not an individual map and it is a map of both equation and function in the same picture.

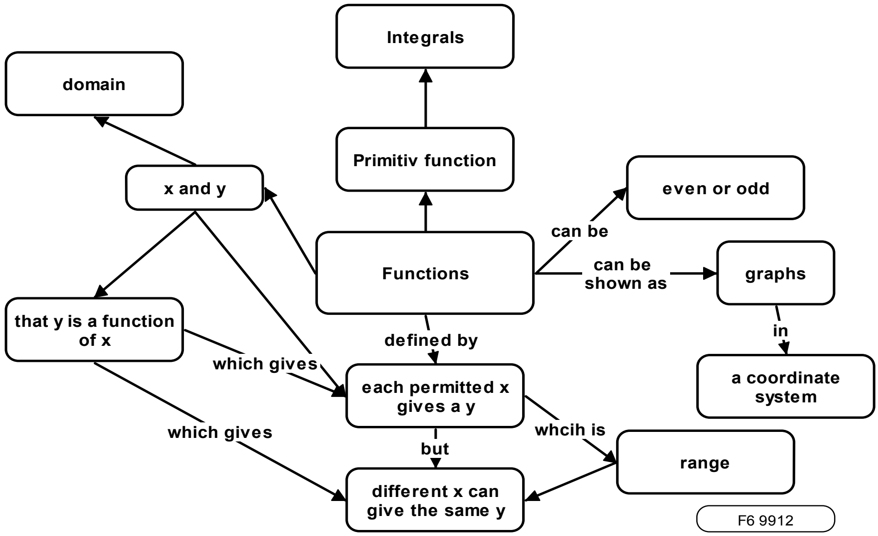


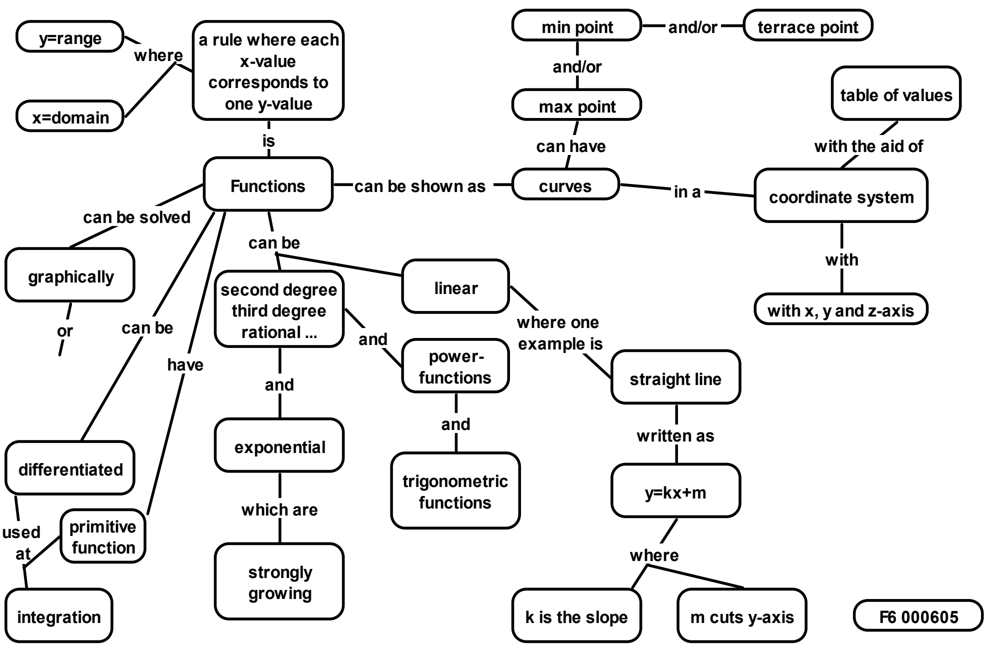
This map is consistent with Lina's answers in the questionnaires but contains more. She mentions proportionality and rule or instruction for the calculation.

Nine months after the first map was drawn I met Lina again for an interview and she had been asked to draw a second map without looking at the first one (which resided with me). In the meantime Lina had studied other subjects than mathematics and she had not worked with her mathematics in organised studies at all. In spite of this it is obvious that her second map is richer than the first one. It contains more concepts and more propositions.

She has removed the concepts straight line and proportionality and has added on domain, range, the properties even or odd, graph, primitive function and integral. In adding the properties she shows that progressive differentiation in her concept picture has taken place (Novak, 1993). She removes variables and co-ordinates and writes *x* and *y* instead. In the first map she talks about a rule of instruction for calculation. In the second map she gives a definition instead. She also explains that the same *y*-value can be related to different *x*-values. Still there are several unclear links in her map. Why does she connect domain and range in different ways? Why are *x* and *y* not connected to the box 'a coordinate system'?

Although she did not study mathematics from March 99 to December 99, changes in her concept map have taken place. What the map shows is probably knowledge that has been learnt in a meaningful way (Ausubel, 1963). Otherwise it would have been forgotten and not retrievable after such a long time. Below are the second and third maps drawn by Lina.





The third map was drawn six months after the second one, again without access to the first and second ones and without Lina having had any mathematics studies in the meantime. Again the third map is still richer than the two earlier ones.

In the third map she has linked range and domain to the definition of function in a better structured way than before. This is an example of integrative reconciliation in the concept structure (Novak, 1993). Instead of talking about graphs she now mentions curves and gives a number of possible properties for them. She adds table of values and links it to coordinate system and to this node she also adds a third axis, the *z*-axis. She returns to linear function, which she had in the first map (and excluded in the second) as straight line and explains how it can be written as y=kx+m. She also explains the meaning of k and m. Another example of progressive differentiation is that she in addition to linear function also mentions other function classes as polynomial, rational, power, exponential and trigonometric functions, and so on. Williams (1998) noted that the experts in her study used a grouping that referred to classes of common types of functions, mentioning terms as exponential, polynomial, trigonometric and logarithmic. Thus here Lina's map has a feature that is typical for experts' maps.

Lina holds on to the nodes primitive function and integration and adds differentiated. One link seems to be not so well expressed: 'Functions can be solved graphically or....' It is not clear what she means here. It can be a mix up with solutions of equations but it can also be that she is thinking of problem solving with the aid of the graph of the function. This last proposition is an example of the student's lack of professional language, which many of the maps illustrate (Grevholm, 2004). While her second map has twelve nodes the third one has 25, more than twice as many. It strongly illustrates the progressive differentiation her function concept has undergone.

The maps were drawn over a period of 15 months where the student had no teaching of mathematics. But the maps show that great changes happen nevertheless. It seems as if the conceptual structure, the concept image, that the student is able to recall is getting richer as time goes by. One can of course argue that she is learning through repeated drawing of maps. This argument does not hold as can be seen by giving students the same mathematical problem again. There is normally no or little improvement in results even if the student has solved the problem once before. And the students did not keep the map that was once drawn and so could not rehearse it before drawing a new one. Peuckert and Fischler (1999) conclude that students' concept maps to a great extent contain coherently used propositions and stand for those conceptions that are stable in different contexts. They continue: "The way concept maps are used for investigation within a set of other methods and as a tool for reconstruction and analysis of conceptions seems to be applicable to other contexts of research."

The example I show here is a typical one. The kind of development over time shown in the example of Lina is not special in any way. The maps are very individual, each student has her way of drawing and is true to the model and design. The language in the maps reveals much about the student's ability to use the concepts involved in discussions (Grevholm, 2004). Lina's three concept maps illustrates Vollrath's claim (1994) that "the student reaches stages of understanding and that there is no final understanding". The conceptual structure undergoes changes over time and is dynamical and time-dependent. The maps indicate that we are dealing with a slow development and maybe as researchers we are sometimes to eager and do not wait for the concept development to take place and for the student to reach different stages?

Why are concepts maps rewarding as research tools?

What the researcher can learn about students' concept development from answers in a questionnaire and from the drawing of concept maps seems to be different. The verbal answers give short, often one-dimensional answers while the concept maps tends to give richer answers with more content and several dimensions of the concept. Students are vague and not enough specific when they try to explain verbally how they understand a concept.

The concept maps seem to reveal some properties of the concept development that are of interest. What are the advantages of using concept maps as answer form for students? From the example it is clear that the maps give the student better opportunities to express her concept image. It obviously invites to more multidimensional answers than a sentence which in its form is linear. The written answer does not open for hierarchy or additional lines of thought in the same way as the map. Knowledge that students express through a concept map seems to be lasting.

The way I used maps to make an a priory analysis of intended learning and then use another map to express students' answers in propositions and compare them has not been found in any other research report. To study concept development over time maps have been used by several researchers. McGowen and Tall (1999) traced students cognitive development throughout a mathematics course by the use of concept maps at intervals during the course. They drew schematic diagrams of the maps of each student in order to see how students build maps by keeping some old elements, reorganising and introducing new elements. The results show that high achieving students "can show a level of flexible thinking building rich collages on anchoring concepts that develop in sophistication and power. The low achievers however reveal few stable concepts with cognitive collages that have few stable elements and leave the student with increasingly desperate efforts to use learned routines in inflexible and often inappropriate ways" (p. 287). Their findings are consistent with what I have shown. Also Peuckert & Fischler (1999) used maps over time to follow conceptual development and point to the value of the method. Thus it is obvious that concept maps can be used as a tool in research and in different ways as has been described here.

Concluding remarks

Obviously the problems that drive the research on student teachers' conceptual development derive from my experience as a mathematics teacher educator and originate from a desire to better understand the process during teacher education and to improve teacher education in mathematics. Can this be achieved if we know more about concept development? Can students experience more meaningful learning if we use new knowledge on concept development? Novak (1993) writes: "What remains to be demonstrated are the positive results that will occur in schools or other educational settings when the best that we know about human constructivism is applied widely. To my knowledge no school comes close to wide-scale use of such practices, even though there are no financial or human constraints that preclude this."

Can teacher educators design better learning situations for students when they know more about the cognitive development of students? Improvement of our knowledge on student teachers' development of concepts during the education might contribute in a constructive way to the redesign of teacher education. At least in Kristianstad University the learning of the teacher educators from my research study resulted in a development project which was highly appreciated by the students and the teacher educators (Grevholm, 2003b; Grevholm & Holmberg, 2004).

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A Case Study of Two Students' Belief Systems and Goal Systems in a Conflict over Teaching Methods

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Abstract

This paper elaborates on goal systems and belief systems and aims at providing an analytical tool for understanding change in beliefs and attitudes. The theoretical framework is applied and tested in a case study of two students who reacted differently in a situation when teaching was not conducted in the way they had been used to. Conditions for change will be discussed.

Introduction

Motive for many education researchers is change. How should we develop our educational system? How can we change teaching in schools? How can we help students learn more? And how can we change students' beliefs? Beliefs as obstacles for change have been discussed in (Pehkonen, 1999). Since 1996 I have been trying to understand how students' attitudes and beliefs change, and how their teacher can initiate and direct such changes. My approach has been to focus on a small group of students, and to try to understand, in depth, their beliefs and attitudes and the changes that take place (e.g. Hannula, 1997; 1998a, 1998b, 2000). Through those case-studies it became evident that emotions have a central role in the process of change. Furthermore, as emotions relate to goal-directed behaviour, motivation became an issue of importance. In a nutshell: what students want, has a strong influence on their experiences - and what they experience influences their beliefs. In this paper I shall analyse two students' beliefs and behaviour in a classroom and try to understand why they experienced the same classroom differently and why their reactions to teaching were different.

The theoretical background will combine belief systems with motivation and goal structures. Connections between these two systems will be explored. After this theoretical background the case studies of Anna and Eva will be analysed.

Belief systems

There is no general agreement on how to define or characterize beliefs or beliefs systems (Furinghetti & Pehkonen, 1999). Therefore it is necessary to define how beliefs are understood in this paper. The reader should be aware, that other researchers might use same terminology with other meanings behind words. In present view belief systems are divided into three kinds of elements: beliefs, values, and emotions. Beliefs are purely cognitive, the personal knowledge concerning objects (e.g. mathematics), agents (e.g. self), and events (e.g. failure). An important aspect of beliefs are expectations that one has in different situations. Values are also a cognitive element, but of different kind. Values are the subjective evaluations of different objects, agents, and events. Whereas beliefs have a truth value, values are essentially normative and can not be true or false. Emotions are the 'affective colouring' of different objects, agents, and events. Objects, agents, and events always associate to emotions, which, however, can be of low intensity or completely neutral. Note, however, that there are also situational emotions that do not relate to belief systems directly. Instead, they regulate goal-directed behaviour. Associated emotions are automations of situational emotions; they are faster but less adaptable to situational variation. The complex issue of emotions is elaborated more deeply elsewhere (Hannula, submitted).

Goal systems

Motivation is the answer to the question why people do what they do. In the literature (e.g. Ryan & Deci, 2000) one important approach to motivation has been to distinguish between intrinsic and extrinsic motivation. Another approach to motivation has been to distinguish three motivational orientations in educational settings: mastery orientation, performance orientation, and avoidance orientation (e.g. Linnenbrink & Pintrich, 2000).

In this paper motivation is conceptualised through a structure of needs, goals and means (Shah & Kruglanski, 2000). Needs are seen as stable psychological constructs, such as autonomy (a need to self-determine own actions) and social needs. Actions can be seen as means to fulfil needs. As part of child's development a complex network of goals and sub-goals evolves between needs and means. Goals may serve multiple needs, and the same goal may serve multiple needs. Furthermore, goals may be in a conflict, i.e. reaching one goal could prevent one from reaching another goal. The relationship between goals and sub-goals is similar to the relationship between needs and goals. There may be several layers of sub-goals, but, in the end, there are means that one sees as leading through sub-goals and goals to the fulfilment of needs. In some cases the connection between needs and means may be quite simple. For example, hunger (a need) can be fulfilled by eating (a mean).

In the context of mathematics education I will look at two kinds of needs: 1) student's need for autonomy, and 2) students social needs (Figure 1). The need for autonomy can be served by mainly two goals: understanding and performance. Understanding mathematics gives power to learn mathematics more independently. Furthermore, mathematical thinking can be a powerful tool also outside mathematics class. Performance in mathematics, on the other hand, is required for many career choices.

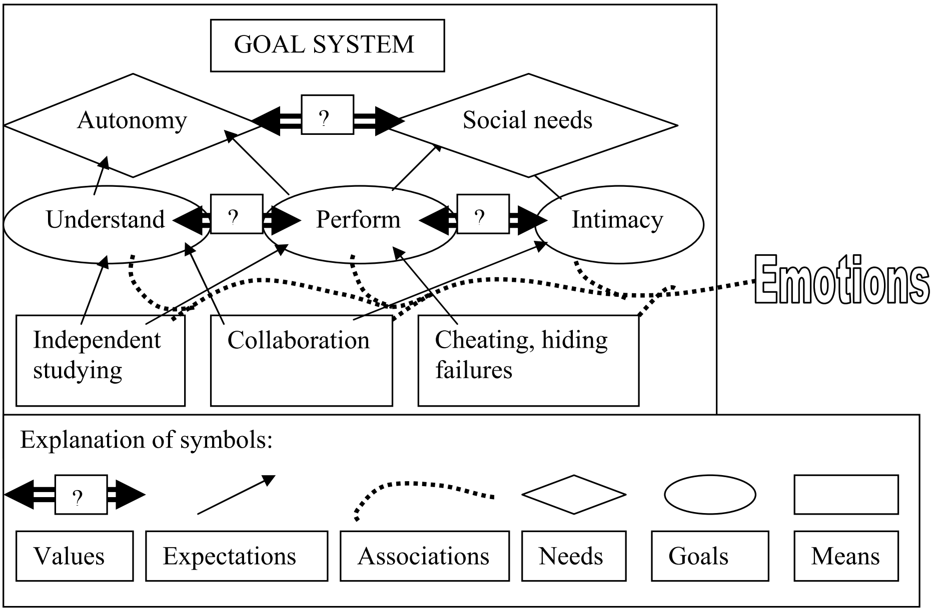

Figure 1. Relationships between goal system and belief system in the context of mathematics education

Social needs in mathematics class are served mainly by two goals: performance and intimacy. Performance in mathematics is one way to gain status in the class; it is a proof of smartness. Hence, low achievers often try to attribute their failures to another, more acceptable cause, such as lack of effort. Social needs can be served also through intimacy. Intimacy in mathematics classroom means collaboration with teacher or peers in the spirit of empathy and understanding. This intimacy may take place around mathematical ideas, but off-task socialising may serve the goal equally well.

Students' different goals in mathematics class lead them to apply different means. Goal of performance may lead to more surface strategies for learning than the goal of understanding. Social 'power game' may also impair group work, while goals of intimacy and understanding may promote productive collaboration. In (Hannula, 2001) there are examples of how students' different goals influence their co-operative problem solving process.

There are several connections between goal systems and belief systems. The most fundamental connection to my understanding is the values one gives for different needs. From these values other values are derived. People have personal beliefs (expectations) about which goals are accessible, which means will lead to which goals, and which goals serve their needs. Situational emotions have an important role in regulating human behaviour towards desired goals. However, the automatic, associated emotions that are part of the belief system, may prevent flexible development of goal structure. For example, if the use of own methods was not accepted in primary school, those might have become associated with negative emotions. Consequently, it would be unpleasant for the student to start developing own methods later.

Methodology

The present paper part is of a research project focused on the development of Finnish lower secondary school pupils' beliefs about, and attitudes towards mathematics (grades 7 to 9) (Pehkonen, 1999). This study is a qualitative one, done with ethnographic approach. I was a participating observer, working in a school as a teacher. The setting could be called 'researcher as a teacher'. This is also action research, because I had a deliberate intention to promote certain beliefs and attitudes in my class. Mainly two data gathering methods, field notes and interviews, were used. Quotes from field notes are marked with a date at the end (14.10.). Eight pupils were interviewed in two groups in December (I1) and the two focus students of this study again in January (I2).

This article will focus on the stories of Anna and Eva. Anna and Eva did equally well in tests, but their experiences in the class were different. I have chosen which parts of the material to exclude and which to include, but I hope that the voice you hear is of those pupils, not mine. I try to interpret the pupils' stories with the presented model. Elsewhere (Hannula, 1998b) I have presented stories of this situation in more detail from three points of view: my own as a teacher, Anna’s, and Eva’s. Then the framework used to analyse the episodes was also different.

Context of events

I began to teach mathematics for this class at seventh grade, which is the first grade at lower secondary school. I tried to imply a gender inclusive teaching and guidelines for teaching were group-work and discovery learning. I also wanted to create a truly democratic classroom climate and restrict the role of the teacher to facilitating students' work. Thus, the teaching was different from students' expectations.

In the beginning things went fine. Later on, it turned out that pupils had difficulties, too. There became more and more criticism in the classroom. The peak of the criticism was probably, when pupils and their parents complained to the headmaster and we had an open discussion in the class about my teaching (25.11.). Reflecting my teaching I had to confess, that the implementation of ideas was not properly done. As a young teacher I had had an idealistic picture of the pupils' skills and motivation. After some changes in teaching the atmosphere become better. Some pupils, however, kept a strongly critical approach, which caused problems for the whole class.

The case of Anna

Anna, the first focus student, describes her experiences as follows:

> At first I was so irritated. Well, I mean, at the very beginning it was really nice, since from the first mathematics lesson I noticed, that somehow, at lower grade it was so strict, it was somehow such a relief that I no longer needed to strain in math class. [ ] And then, after a while, some couple of weeks, it started to annoy me that we didn't learn anything. Well, it must have depended from my own attitude most, and I began to feel pissed off, and I abused you and I abused all the others and I only used bad language in the class, and I didn't get anything done. [ ] So I changed my attitude. I thought that it doesn't help anything, that you shall be our teacher anyway and so on. So then I changed my attitude, and decided to study more myself. (I2)

Field notes confirm the process. I had at least two notes on Anna and Helena solving problems in good collaboration (21.8. and 9.9) and one note (13.11.), where Anna's group didn't do the assigned task. Later, she actually began to defend me against the criticism in the class. When Eva is complaining that she doesn't learn, Anna replies to her: "*It depends a bit on ones own attitude too*". *(10.1.)*

To understand better her behaviour, let's look at her experiences in the primary school:

> Our teacher was quite demanding, so that she almost all the time had surprise tests, with awfully difficult tasks, and hardly anyone could solve those. And otherwise, too, that even if you got a ten {highest} in all tests, she wouldn't give you more than an eight in school report if you don't keep your hand up to almost all tasks and be otherwise active, too. [ ] But it was so dry, somehow, the teaching in primary school [ ]. We always went exactly according to the book [ ]. First it was taught in theory all that, and then there wasn't much else, no project works or anything like that. That left me somehow awful traumas, sort of. The teacher newer even asked who would like to do some task on the board, she just commanded. [ ] The lessons caused awful traumas, 'coz one always strained somehow awfully. (I2)

We can now summarise Anna's story. She believed that the teacher ought to control learning She expected learning of mathematics to be dry, difficult, and strenuous. When this was not the case, she felt a relief. Later, she became angry, because she was not approaching her learning goal. However, she did not expect change in teaching, and, furthermore, she felt guilt for her misbehaviour in the class. Thus she decided to change her 'attitude' and so she took a more active role in her own learning.

The case of Eva

Eva was the other focus student. Her experiences were quite different:

> [W]hen we began or when we came to school, I did try to participate in the very beginning, [ ] I asked you some advice and you walked away

> and said, "Ask your group". I didn't know them. For sure I dare ask them. And I thought that I am the only one not to understand anything. [ ] So it just stayed that way, and I had to be silent because I was so stupid, 'coz I don't understand it and all the others do. And so began my attitude. So, if you had helped me better in the beginning, I could be somewhat more eager in mathematics.

For Eva, her social context had a strong influence on how she experienced class:

> I try to [study well] at the beginning of every lesson, but then I am disturbed so it annoys me. Well, I don't know, 'coz at the beginning of every lesson I try. Today I went to sit alone. Then Paula comes next to me. And then Ursula comes there close and I can't be calm. I just can't be. (I2)

> In that group I haven't got a right to say anything. They think that whatever I say, I'm wrong. Then in Julia's group I might be right. It {laughter} isn't necessary right, but I have a chance to say something... (I2)

> That gave me ardour. It was really fun to do the pair work. (I2)

When we take a look at her primary school experiences, we can feel the safeness she remembered from those days:

> And I remember the one lesson, when there were girls from our class, and we just chatted and told whether we shall marry some rich man. {Laughter} [ ] We had a teacher [ ], who taught so well, that everyone got excellent marks. (I1)

However, her view of mathematics learning was narrow:

> I want normal mathematics. [] So that first we do the new thing for the lesson. And then that is asked from everyone in a row. Then you give pages and everyone counts and asks the teacher if something is wrong. (I1)

> [Better in mathematics are those] who have a good memory. [] They have followed the lesson more carefully. (I2)

Eva's prior experiences from home were unpleasant:

> ... I mean my mother doesn't... Mother doesn't... Well, mother doesn't. [ ] If she starts to help me and I don't understand something, then [she would say] "Why can't you now understand this! Yak yak yak!" (I2)

Summarising Eva’s story, we can say that her view of learning was to memorise what the teacher has said. She is uncertain and anxious about peer reactions. She was ashamed not to understand, and tried to hide it. Later she joined other students in blaming the teacher.

Conclusions

When we compare these two students, we see similarities and differences in their belief systems and goal systems. Originally, they both had similar beliefs about teacher as the controller of learning. Furthermore, Anna expected learning to be dry and difficult whereas Eva thought that it would be easy. Hence, their initial reactions were different. This led to different ‘affective colourings’ of mathematics lessons. Of these two students, Anna had somewhat higher self-confidence (expectations about self) in mathematics and learning goal was more important for her. Furthermore, Eva associated lack of understanding with shame.

We must recognize that the presented data and even the full data available for the researcher will always leave room for multiple interpretations. However, we shall now try to conclude from the theoretical framework and this case study some possible conditions for a change in behaviour. First, there must be a goal (learning mathematics) that is not accessible by old means (listen to the teacher). Other means (self-directed learning) must be available. One must believe that the goal is accessible by the new means (self-confidence). One possible obstacle is emotional associations with the new means (ashamed of need for help). Another possible obstacle is conflicting goals (behave like friends), which has higher value for the individual.

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From Vincenzo Viviani to Paul Erdös

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In our school practice it is very important to work out mathematical problem fields for secondary school pupils. We have to raise the pupils' interest in the classroom too. Our main viewpoint is to enrich the teaching of mathematics with historical aspects. The history of mathematics confirms human values (success, pleasure, self realisation, social aspects, cultural values). We want to acquaint our pupils with the Curriculum Vitae of famous mathematical people (Viviani, Erdös, Mordell) and with their mathematical works and results.

We focused our teaching on heuristic work of the pupils themsleves, we formulated the posed problems as open problems (What happens, if …? What can you formulate? What is your conjecture? How can you vary the conditions, the concepts, the way of the proof?). So pupils can participate in creation of their own mathematical ideas. We have chosen our topic from the geometry of triangles: Viviani's theorem and its generalizations.

Viviani's theorem

For a point $\boldsymbol{P}$ inside an equilateral triangle $\boldsymbol{ABC}$ the sum of the lengths of the perpendiculars $\boldsymbol{d_1}$, $\boldsymbol{d_2}$, $\boldsymbol{d_3}$ from the point $\boldsymbol{P}$ to the sides is equal to the altitude $\boldsymbol{h}$.
We formulated Viviani's theorem in an open form:

> ***Problem 1***
> Let be given an equilateral triangle $\boldsymbol{ABC}$ and a point $\boldsymbol{P}$ inside triangle $\boldsymbol{ABC}$. What is the sum s_p of the distances from the point $\boldsymbol{P}$ to the three sides? ($s_p = d_1+d_2+d_3$).

At first we allowed to the pupils to draw, to measure, to construct their conjecture and after that to prove it theoretically. We hoped that the pupils would recognize:

a) the value of s_p (i.e. s_p is constant, or $s_p = h = 1/2\, a \sqrt{3}$, where h denotes the length of the altitude of the equilateral triangle $\boldsymbol{ABC}$, and a is the length of its side)

b) the sum s_p is independent from the position of the interior point $\boldsymbol{P}$.

Our pupils choose two ways for solving Problem 1.

Way 1

The base of the proof was, that we can write the area of the equilateral triangle $\boldsymbol{ABC}$ as a sum of the areas of the triangle $\boldsymbol{ABP}$, triangle $\boldsymbol{ACP}$, triangle $\boldsymbol{BCP}$. Some pupils made a qualitative proof, they showed $s_p = h$, the other pupils made a quantitative proof, they showed, that $s_p = 1/2\ a\sqrt{3}$. (Figure 1) I found that the pupils (aged 14-16) prefer the quantitative version of the proof.

Way 2
We draw parallel lines to the sides ***AC, AB, CB*** through the point ***P***. We get 3 parallelograms and 3 small triangles. We can give the altitude of the triangle ***ABC*** by the help of the altitudes of the small triangles. (Figure 2)

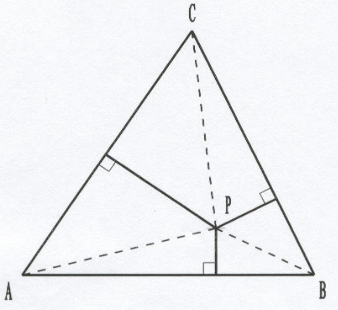

Figure 1

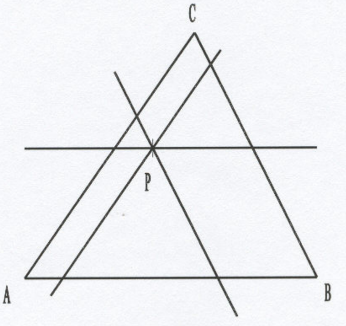

Figure 2

We started with this wellknown property of the equilateral triangle and we varied

- the placement of the point ***P*** (inside, outside, on a side, on the extension of a side of the equilateral triangle ***ABC***)
- the shape of the triangle (equilateral, isosceles, scalene).

We generalized

- the shape of a regular triangle (regural *n*-gon)
- the dimension of the equilateral and scalene triangle, we stepped out of the plane and went over to space (regular tetrahedron, generalized inequality of Erdös-Mordell)

The scale of the problems was very wide, from simple exercises (aged 13-14) through contest problems (aged 14-18) to scientific theorems (theorem of Erdös-Mordell, Barrow's theorem, inequality of Kazarinoff (aged 16-19).

Viviani's theorem is convenient for derivation of a problem field, to built new concepts, to generalize theorems and to investigate converse problems too. There are easier questions for investigation in classroom, but there are harder problems aimed at advanced pupils. Our next question was:

What happens if in the equilateral triangle ***ABC*** the point ***P*** is not located inside the triangle? We got different options for the locating of the point ***P*** and formulated new problems.

Problem 2
What happens if in the equilateral triangle ***ABC*** the point ***P*** is on a side of the triangle ***ABC,*** with the sum of the distances ***P*** to the two sides?

It was obvious, that $s_p = d_1 + d_2 = h$. Both of the two ways are good for proving.

Problem 3

What happens if in the equilateral triangle ***ABC*** the point ***P*** lies on the extension of a side of triangle ***ABC***?

Their conjecture was that $s_p=d_2-d_1$ or $s_p=d_1-d_2$. The result surprised them, but we could unify the two values by the help of absolute value ($s_p=h=|d_1-d_2|$) then we summarized that $h=k_1d_1+k_2\,d_2$, where $k_1=1$ and $k_2=-1$ or $k_1=-1$ and $k_2=1$.

Problem 4

Let be given an equilateral triangle ***ABC*** and a point ***P*** in the exterior of the triangle ***ABC***. How can we give the length of the altitude h by the help of the distances from the point ***P*** to the three sidelines?

Solving Problem 4 seemed to be a little bit complicated. They found different locations of the point ***P*** and got different results for the linear combinations $k_1d_1+k_2d_2+k_3d_3=h$ $(k_1,k_2,k_3=\pm 1)$. (Figure 3, Figure 4)

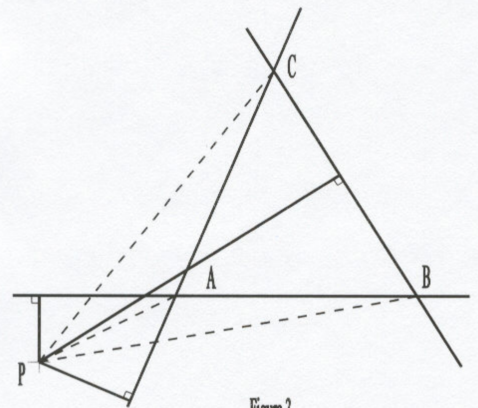

Figure 3

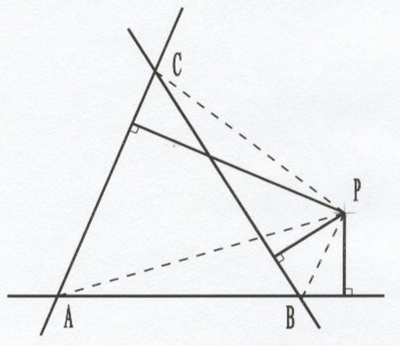

Figure 4

We made the proof by the help of the Way l, with counting the areas of the proper triangles. The pupils recognized that the coefficients k_1,k_2,k_3 are equal to $\pm$ 1, depending on the location of the point ***P***, but at first they did not know which of the distances will be positive or negative. Their conjecture was that the determining factor is whether the foot of the perpendicular intersects a side of a triangle or whether it intersects the extension of the side. But it was easy to show that this conjecture is not true.

By looking at the figures and rethinking the various cases they found that the sign of the distance in the linear combination is based on the relative position of the side to which its perpendicular is drawn and the vertex opposite that side (barycentric coordinates).

We could summarize:

Let triangle $A_1A_2A_3$ be an equilateral triangle with altitude $\boldsymbol{h}$. Denote by $\boldsymbol{a_i}$ the side opposite vertex $\boldsymbol{V_i}$ (i = 1,2,3). If $\boldsymbol{P}$ is any point in the plane of the triangle $A_1A_2A_3$ with d_i (i=1,2,3) equal to the perpendicular distance $\boldsymbol{P}$ to $\boldsymbol{a_i}$ (i=1,2,3), or its extension, then $k_1d_1+k_2d_2+k_3d_3=h$, where k_i=1 if $\boldsymbol{P}$ and $\boldsymbol{A_i}$ are on the same side of the line containing $\boldsymbol{a_i}$, or if $\boldsymbol{P}$ is on $\boldsymbol{a_i}$, k_i= -1 if $\boldsymbol{P}$ and $\boldsymbol{A_i}$ are on opposite sides of the line containing $\boldsymbol{a_i}$ (i=1,2,3).

If we follow the Way 2 in the proof of Viviani's theorem we get a new result.

> ***Problem 5***
> Let be given an equilateral triangle $\boldsymbol{ABC}$ and a point $\boldsymbol{P}$ inside of triangle $\boldsymbol{ABC}$. If we draw parallels through the point $\boldsymbol{P}$ to the sides, then these parallels cut out 3 segments from the triangle $\boldsymbol{ABC}$. Is the sum of these 3 segments constant? *($d_1+d_2+d_3=2a$).*

The segments make an angle of 60° to the proper side of the equilateral triangle. With the variation of the angle we get the Problem 6.

> ***Problem 6***
> Let be given an equilateral triangle $\boldsymbol{ABC}$ and a point $\boldsymbol{P}$ inside triangle $\boldsymbol{ABC}$. We draw segments through the interior point $\boldsymbol{P}$. These segments make the same angle with the proper sides of the triangle. What can we say about the sum of these segments? (s=constant)

From this point on, we made our investigations in another direction. We generalised the shape of the equilateral triangle, we generalized Problem 2.

> ***Problem 7***
> What can we say about the sum s_p, if the point $\boldsymbol{P}$ is on the base of the isosceles triangle? (d_1+d_2=*constant* = *2 a*, where $\boldsymbol{a}$ is the length of the two equal sides).

Here we raised the question of the converse theorem: Which of the discussed problems have a true converse theorem? Can we invert Viviani's theorem? The answer is yes. Viviani's theorem has a converse theorem, and we can prove it different ways. Problem 7 is also reversible. We continued our investigations in the direction of the regular n-gons.

> ***Problem 8***
> Let be given a regular n-gon (n≥4) and a point $\boldsymbol{P}$ inside of it. What can we say about the sum s_p of the distances from the point $\boldsymbol{P}$ to the sides of the n-gons?

The first step was to change the form of the regular triangle to regular 4-gon (square), regular 5-gon, regular 6-gon (etc.), regular n-gons (n≥4). For solving these problems we need the results of the Problems 1-4. We raised the question of converse theorem, but it was easy to find, e.g. in the case of the convex 4-gon, a counter-example: If we take a rectangle $\boldsymbol{ABCD}$ ($d_{AB} = d_{BC}$), we immediately can see that for arbitrary interior point $\boldsymbol{P}$ of the rectangle $\boldsymbol{ABCD}$ the sum of the distances from the point $\boldsymbol{P}$ to the sides of the rectangle is constant and it is equal to the sum of the lengths of its two different sides. At the normal high school level we can extend Viviani's theorem to space and we can formulate a similar theorem for regular tetrahedron.

Problem 9

Let be given a regular tetrahedron ***ABCD*** and a point ***P*** inside it. Find the sum s_p of the distances from the point ***P*** to the four faces. Is the sum s_p independent from the position of the interior point ***P***?

The proof of the Problem 9 is similar to Way 1 in the plane. We use the fact that the volume of the regular tetrahedron is equal to the sum of the volumes of the tetrahedra ***PABC, PABD, PACD, PCDB***, and the value s_p is constant and equal to the length of the tetrahedron's altitude (s_p=h).

We formulated the converse theorem too. The pupils were convinced that it will be true. Their concept was false, the converse theorem of Problem 9., is not true. From the sum $d_1+d_2+d_3+d_4=h=3V/A^2$ follows only the equality of the areas of triangle ***ABC***, triangle ***ABD***, triangle ***ACD***, triangle ***BCD***, i.e. the tetrahedron ***ABCD*** is not regular, only its faces have equal areas. (equifaced tetrahedron).

The other parts of the problem field of Viviani's theorem are difficult. At the advanced high school level or at the University in courses of elementary mathematics we can deal with the theorem of Erdös-Mordell, Barrow's theorem (scalene triangle), with the generalization of Problem 9. to the outside point ***P*** and with the generalized inequality of Erdös-Mordell for the tetrahedron.

Problem 10 (inequality of Erdös-Mordell)

If ***P*** is any point inside or on the boundary of a triangle ***ABC***, and if d_1, d_2, d_3 are the distances from point ***P*** to the sides of the triangle, then $d_{PA}+d_{PB}+d_{PC}\geq 2(d_1+d_2+d_3)$, with equality of and only if triangle ***ABC*** is equilateral and the point ***P*** is its circumcenter.

The first proof of the Erdös-Mordell inequality was published in the Hungarian KöMaL (1935) by Professor Mordell. In 1937 The American Monthly published two proofs by Professor Mordell and by D.R. Barrow, but neither proof was elementary. More recently D.K. Kazarinoff found an elementary proof, which is based upon the idea of reflection. Barrow proved a generalized form of the Erd_s-Mordell inequality, which follows from Barrow's theorem as a special case.

Problem 11 (Barrow's theorem)

Let the point ***P*** be an arbitrary interior point of the triangle ***ABC***. Prove that $d_{PA}+d_{PB}+d_{PC}\geq 2(d_{PA'}+d_{PB'}+d_{PC'})$, where *A', B', C'* are the intersection point of the bisectors of the angles ***BPC, CPA, ABC*** with the sides ***BC, CA, AB*** of the triangle ***ABC***.

The proofs of these problems we find in several Hungarian books. The proofs require only elementary knowledge, but the methods are very miscellaneous (principle of reflection, use of the congruent transformations, computation of areas, trigonometric connections).

The Erdös - Mordell inequality has also a generalization.

Problem 12

Let the point ***P*** be an arbitrary interior point of the tetrahedron ***ABCD***. Prove that
$d_{PA}+d_{PB}+d_{PC}+d_{PD}> 2\sqrt{2}(d_{PA'}+d_{PB'}+d_{PC'}+d_{PD'})$, where A', B', C', D' are the perpendicular projections of the point ***P*** to the planes ***BCD, CDA, DAB, ABC.***

Problem 12 is a generalization of Problem 10. We found that the value of the coefficient in the plane was 2, so we expect that the value of the coefficient in the space will be 3. But this is not true. D. K. Kazarinoff constructed an orthogonal tetrahedron with the coefficient $2\sqrt{2}$.

Problem 13

The tetrahedron ***ABCD*** is equifaced and the point ***P*** is an interior point of the tetrahedron ***ABCD***. Prove that $d_{PA}+d_{PB}+d_{PC}+d_{PD}\geq 3(d_1+d_2+d_3+d_4)$, where d_i (i=1,2,3,4) are the distances of ***P*** to the faces of the tetrahedron. In the case of the equality the tetrahedron ***ABCD*** is regular and the point ***P*** is the centre of its circumscribed sphere.

Contests problems

Viviani's theorem and its variants are popular at high school contests. I collected some of them (aged 13-18). If we prepare our pupils for the contests, we have to solve these problems with them.

Problem 14

Let be given an equilateral triangle and a point ***P*** inside of triangle ***ABC.*** We denote the feet of perpendiculars from the point ***P*** to the sides ***AB, BC, CA*** by ***D, E, F***. Prove that the value of the fraction $\frac{PD+PE+PF}{BC+AC+AB}$ is independent of the situation of the point ***P***. (aged 14)

Problem 15

The length of the sides of an equilateral triangle is 5. Let us draw parallels through the interior point ***P*** to the sides. For which point / points ***P*** will the sum of these parallel segments be maximal? (aged 13)

Problem 16

There is given an equilateral triangle and a point ***P*** inside of the triangle. The feet of the perpendiculars to the sides divide each side into two segments. Prove that the sum of the non-joint 3 segments is independent of the location of the point ***P.*** (aged 15-16)

Problem 17

The point ***P*** is an interior point of the equilateral triangle ***ABC.*** The feet of the perpendiculars from point ***P*** to the sides ***BC, CA, AB*** are A_1, B_1, C_1.

Prove that the sum of the areas of the triangles APC_1, BPA_1, CPB_1 is independent of the location of the interior point ***P.*** (aged 16-17)

Problem 18

The point ***P*** is an interior point of a regular tetrahedron. From this point ***P*** we draw perpendiculars to the plane of the base-face of the regular tetrahedron. These perpendiculars intersect the plane of the face-sides at the point ***X, Y, Z.*** Prove that the sum $d_{PX}+d_{PY}+d_{PZ}$ is independent of the interior point's location. (aged 17-18)

Which are the benefits of using historical problems?

- We can show the continuity of mathematical concepts and processes over past centuries
- We motivate the learning process in the classroom, because our pupils deal with problems which centuries ago were objects of investigation. These problems allow the pupils to touch ancient and recent past.
- Pupils connect mathematics to various cultures and other intellectual developments in sciences
- We need to bring biographies into the mathematics classroom. The life story of mathematical people often encourages talented pupils and fill them with emotions. They think and believe that if they can solve problems posed by famous mathematicians in their youth - as Paul Erdös in Hungary - may be later they will become such great mathematicians as P. Erdös was. It is nice for them to realize that they are part of history. Some mathematicians have interesting lives (f.e. V. Viviani, G. Galilei, P. Erdös) and it is good to know interesting things.
- We often can learn from the mathematical mistakes and the unsolved problems of the past.

I think it is necessary to share my knowledge about the panorama and people of mathematics with my pupils and students at the university. Besides the mathematical discussion of the selected problem field we can deal with other mathematical results of Viviani (tangent to a cycloid, four windows problem, so called Florentine problem, trisection of an angle by using an equilateral hyperbola, Viviani's curve), or we can analyse the history of Galilei's time (physics, philosophy, inquisition).

In connection with the activity of P. Erdös we can investigate other nice problems posed by Erdös, or problems from the old series of the Hungarian High School Mathematics and Physics Journal (KöMaL). May be we read some book about the "travelling ambassador of mathematics" or we look at a videotape dealing with him.

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Different Aspects of Mathematical Knowledge Assessed in National Tests in Secondary Schools

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Abstract

When you work with assessment in mathematics you find that mathematic is a subject of many dimensions. Besides the mathematical content you can look at the matematical process and students' level of achievement. In this paper we describe why and how we developed an Assessment matrix, which focuses on different aspects of knowledge based on the syllibi in mathematics. The matrix is a part of the assessment/scoring guide for more extensive items in the national tests in mathematics.

Background

In Sweden there is no external examination when the students leave secondary or upper secondary school. The teachers do all the assessing and grading. Grades are awarded on a three-grade scale from the eighth year of schooling onwards. The grades are Pass, Pass with distinction and Pass with special distinction. In the upper secondary school the grade Fail is added and the students in upper secondary school receive a grade in every course. The grading is criterion referenced and goal related; i.e. the grades relate students' knowledge and achievements to the goals set in the syllabus. To support the teachers' awarding of grades we have national tests in mathematics at the end of year 9 and at the end of every course in the upper secondary school. The purpose of having national tests is also to create grounds for assessment that is as unified as possible across the country.

The PRIM-group at the Stockholm Institute of Education in Sweden is a research group that conducts its research within the area of Assessment of knowledge. The PRIM-group has been commissioned by the National Agency for Education to construct national tests in Mathematics. Presently we are developing a series of various assessment and evaluation instruments within the mathematical area as well as in other areas.

The starting point for the construction of a test is the view of knowledge in the curriculum, the view of the subject in the syllabus and the criteria of the different grades. The emphasis should lie on understanding, analysis of the entire solution process, critical examination of the results, as well as the ability to draw conclusions, which according to the curriculum are more important than isolated training of skills.

The test set is made up of several different parts in order to give the students an opportunity to show as many sides of their competencies in mathematics as possible. We try to make our tests as balanced as possible with different kinds of items in a variety of contexts and a range of response formats. This paper however only discusses how we assess/mark more extensive items. In all national tests there is at least one more extensive item that sometimes is based on an authentic context.

This item is distinguished by that it assesses, more than other items, the students' ability to do independent work, be creative, show the ability to systematise, form mathematical reasoning, create mathematical models, formulate and test assumptions, as well as draw conclusions. It shall be possible to assess the students' complete solution on different qualitative levels. A demand for these items is to give all students the opportunity to start on a solution, but at the same time the item shall be so challenging that a solution can show quality on the highest-grade level. The intention is that the students should use 30–50 minutes to work with this item.

This kind of items was first used was in the national tests in 1995 and in the assessment guidelines the teachers were told to make a holistic assessment of the students' work. To help the teacher to assess and grade there were descriptions of students' work at different achievement level as well as authentic students' work assessed and graded by researchers and a group of teachers.

On the next page we present an item from the national test for course A in the upper secondary school. Course A, which is compulsory, is the first mathematics course in upper secondary school. This item was used in May 2000, just a few weeks before the opening of the Öresund bridge and the information about the fares was given in many Swedish newspapers.

Text for next page:[1]

❖ Prepare the work by *doing calculations and/or diagrams*, which can help you to quickly suggest the cheapest alternative for different car travellers for example:

- the traveller who for pleasure goes across only occasionally
- the shopping traveller, who regularly goes across for shopping
- the commuter who lives in Malmö but has his/her work on the other side of Öresund.

❖ Explore also how many journeys a month or per six months you have to do to make the Öresundspendler agreement profitable.

[1] ©Skolverket

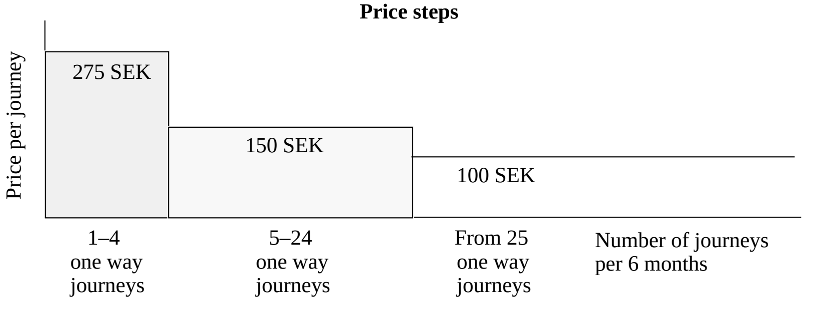
By car across the Öresund bridge

It will cost money to go by car across the Öresund bridge. The traveller may choose between several different alternatives to pay the toll fee. These have been created to fit all types of travellers.

Information about car fares across the Öresund bridge

Normal price
The normal price for a one way journey by private car is 275 SEK (Swedish crowns) but there are two different possibilities to get a lower price.

Öresundsbonus
With an Öresund bonus agreement it gets cheaper and cheaper after the first four one way journeys.



The agreement is valid for an unlimited number of journeys during *a period of six months*. At the end of the period one starts again and pays the normal price for the first four journeys.

Öresundspendlare
The Öresundspendlare may sign *a monthly agreement*. It costs 4 080 SEK for a private car per month and includes up to 50 one way journeys.

Nothing extra is charged for signing an agreement. The deposit one pays for the “Brobizz” (the electronic identity card) is refunded when the card is returned.

Exercises
Imagine, that you have got a holiday work in an information stall at the Öresund bridge. There you are to help travellers to interpret the information (see page 2) and you must also be able to suggest the cheapest payment alternative.

The development of the assessment matrix

We have a long tradition of teacher-assessed national tests in Sweden and the teachers are used to detailed assessment/scoring guides. The holistic way of assessing students' performance turned out to be difficult for the teachers and some of them found it subjective and not fair to their students. According to research, performance-based items are less reliably marked if they are marked holistically than if they have a structured analytic marking guide (Gipps, 1994).

We started looking for other assessment models than holistic ones and found among others analytic scoring scales. Analytic scoring is particularly useful in assessing students' problem-solving efforts. When you use analytic scoring you look at the different phases in problem solving such as understanding the problem, planning the solution and getting an answer. An analytic scoring scale also includes specific criteria for awarding partial credit for each phase.

Communication, which is very important in our grading criteria, is not included in analytic scoring. In the book Mathematics assessment and evaluation edited by Thomas Romberg, we found another way of assessing problem solving in Vermont's work with assessing portfolios (Romberg, 1992). In this assessment they use four different aspects of problem-solving skills and three aspects of communication skills. The aspects are organised in a matrix with descriptions of four different quality levels for each aspect. This matrix was the starting point for the development of our matrix.

We organised the different aspects of the problem-solving process according to our syllabi in mathematics and the grading criteria. The problem-solving skills were divided into comprehension and accomplishment. The communication skills were divided into mathematical language and clarity of presentation. The quality levels were described in three levels according to the criteria for different grades. One of the purposes of the assessment matrix (se next page) was also that it would be possible to use it independently of mathematical content and in different courses.

Tests must be scored fairly and in a way the students understand (Gipps, 1994). The purpose of the matrix is also to show the students the different aspects of knowledge that can be assessed as well as to describe the different qualitative levels within each knowledge aspect. The students as well as their teachers can find the assessment matrix at our homepage (www.lhs.se/prim/) along with authentic students' work assessed with the support of the matrix. See further Kjellström and Pettersson (1995) and Kjellström (1996, 1999, 2000) for our developmental work.

Assessment matrix

Problem solving capability

Comprehension and method

The assessment concerns: To what degree the student shows an understanding of the problem. What strategy/method the student chooses to solve the problem? To what extent the student reflects on, and analyses the chosen strategy and the result. The quality of the student's conclusions. What concepts and generalisations does the student use?

Accomplishment

The assessment concerns: How complete and how well the student works through the chosen method, makes necessary calculations and motivates the working.

Communication capability

Mathematical language and/or representation

The assessment concerns: How well the student uses mathematical language and representation (symbolic language, graphs, illustrations, tables and diagrams).

Clarity of presentation

The assessment concerns: How clear, distinct and complete the work of the student is. To what extent the solution is possible to follow.

	Qualitative levels		
Comprehension and method	Shows some understanding of the problem, chooses a strategy, which functions only partially.	Understands the problem almost completely, chooses a strategy which functions and shows some reflective thinking.	Understands the problem, chooses if possible a general strategy and analyses one's own solution.
Accomplishment	Works through only parts of the problem or shows weaknesses in procedures and methods.	Shows knowledge about methods but may make minor mistakes.	Uses relevant methods correctly.
Mathematical language and/or representation	Poor and occasionally wrong.	Acceptable but with some shortages.	Correct and appropriate.
Clarity of Presentation	Possible to follow in parts or includes only parts of the problem.	Mostly clear and distinct but might be meagre.	Well structured, complete and clear.

Further development of the assessment matrix

After the first test, in which the Assessment matrix was used, we evaluated the teachers' opinion in a teacher questionnaire. Half of the teachers answered that the matrix facilitates the assessment and that it was fairer to the students than holistic assessment. Many teachers thought, however, that the assessment was time-consuming but half of the teachers answered that it was worth the effort.

In the national test 2000 for year 9 we had an oral part where the students worked in groups. To assess this part we developed a new matrix, which focused on understanding, language and participation. The oral Assessment matrix was well accepted among the teachers, but they thought that we ought to make a special matrix for each item.

During 2000 the syllabuses and the criterion for the different grades were revised and we changed the matrix according to them. We learned from the oral part that it was easier with only three aspects and that the teachers wanted special matrixes for each item.

We always pre-test all items in our national tests. For the tests in 2000/2001 we have developed special matrixes for each more extensive item. The first step of the process to develop a special matrix for an item involves an in-depth analysis of the students' work. We started with the general matrix and with the help of this analysis we made item-specific descriptions in the matrix. Together with the matrix we also published authentic student work assessed and graded by researchers and a group of teachers.

Recently we received the result from the first test with this item-specific matrix and almost all teachers thought that it was worth the effort and that it facilitated the assessing/marking.

We have a course for teacher students 'Assessment of knowledge. How to do it in mathematics?' (5 credits) where our students try a variety of assessment models in different situations and with pupils of different ages. Many of them used assessment matrixes to assess also small children's performance in mathematics.

Related assessment of processes and actions in mathematics

In our work looking for different ways of assessing more extensive items we have found also other projects. OECD/PISA (Programme for International Student Assessment) is a survey of students' skills and knowledge as they approach the end of compulsory education (age 15). In PISA three broad dimensions have been identified. These are processes, mathematical content and context. The process is divided in different aspects i.e. mathematical thinking, argumentation, modelling, problem posing and solving, representation, symbols and formalism and communication. In order to describe levels of mathematical competency, PISA organises processes into three competency classes.[2]

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[2] See web page www.pisa.oecd.org

Similarly in a project called Balanced Assessment in Mathematics, the Harvard Group (http://balancedassessment.gse.harvard.edu) presents a model of assessing students' work in three dimensions; objects, actions and the quality of the students' work.

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How do Students Express their Own Learning from Studies in Mathematics and Didactics?

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Abstract

Is it possible to use the student's written statements and reflections about learning from lessons as assessment? Is it possible for a student to monitor his own learning and express it effectively in written terms? The students in my three courses of Mathematics and Didactics supply examples from the area of the questions. The paper is about three levels of writing and reading problem during half a year.

A traditional way of assessing mathematical knowledge is by written tests. These tests evaluate a student's perception of the course, what knowledge has been taught and ultimately what the student has learnt. The tests are made up of a number of problems and questions, which the student has to work through to show knowledge and understanding of the topic.

Another type of evaluation and assessment of mathematical knowledge is now emerging in the way of a constructivist approach to learning, moving away from the traditional Swedish curriculum's behaviourist way of teaching. Didactics courses, for pre-service as well as in-service teachers, suggest ideas of trying methods of testing and understanding both conceptual and procedural knowledge in the classroom.

It is often difficult for both the teacher and the learner to find a model of effectively communicating the taught concepts. The student can document what he has learnt in the way of a written test, but this may not be easily understood when being read by the tutor.

A portfolio model is one example of a "modern" way of teaching, which includes evaluation and assessment. The student takes his information from lessons, homework and tests and puts them into a personal folder where he also stores personal accounts both of lessons and his own learning.

Another example is a "logbook" which offers an ongoing communication between teacher and student. The student writes his thoughts regularly about concepts and any personal difficulties from the lesson along with any questions about mathematical problems. The teacher then gives comments and feedback by writing in his logbook.

In both cases, many problems are evident for the teacher when they try to assess the learning and knowledge of the student from his writing.

- Even if the student has a good perception of what he has learnt and fully understands the lessons, his communication and written skills may not effectively convey this to the teacher,
- The student can write in such a way as to influence the teacher, communicating in such a way that he knows would be appreciated, so therefore:
- How is it possible for the teacher to more effectively gauge the knowledge and learning of the student as expressed in his paper?

In summary, how can it be possible for a student to monitor his own learning and express it effectively in written terms?

Stating the issue

It is difficult for a teacher to understand how and what the students have learnt from a lesson. So I want to get examples from my student group: Could it be possible to use the student’s written statements and reflections about learning from lessons as assessment? How can I interpret the written statements as an assessment of the course?

In the mathematical courses the skills level for each is to be set to one of three levels. This paper will strive to discuss criteria for effectively assessing mathematics skills as well as didactics skills.

Three consecutive studies

The first year of education for school years 4-9 and high school pre-service teachers in mathematics and physics at the teacher education programme in Sweden where the present study was conducted, contains five mathematics courses that are linked to didactics. The work effort for these courses corresponds to five weeks of full time studies. The courses in Arithmetic & Didactics, Algebra & Didactics and Geometry & Didactics have been in focus here when studying how the students express their learning.

The sample group

The group members are aged between 20 and 50, with the average age being 36. Some of the students have studied one year to complement their mathematical competence to the level of high school science programme. Others have studied corresponding courses at municipal adult education before entering teacher education. Some of the students have not studied for the last 8 years, claiming to have forgotten a majority of their learning during that time, whereas others have been teachers themselves and currently have children going to school.

A group of 16 students have been followed for all three courses. During the study nobody left the course. In the third course, on geometry, 30 students, including the16-group, gave their notes of learning. In all 16 students have been followed for the courses, and their comments evaluated. After every second lesson the students were requested to document what they have learnt in the

Mathematics and Didactics lessons. This material forms the basis and data of the study.

An ordinary lesson

An ordinary lesson in the three courses includes following parts with connection to the literature in the course:

- Problem solving part. The problem is prepared at home or within groups. Solving the problem with mathematics methods and discussing the solution and alternative solutions make this situation connected to mathematics and didactics.
- Every student must be a seminar leader at least once in a course. Everybody in the whole group must read a chapter of the textbook "Algebra for all" or "Billstein" in the next lesson. Questions are written from chapter and given to the seminar leader.
- School in reality. Mathematics methods to learn from books and from reality. Leader: Student or teacher
- New concepts and problems in mathematics, led by the teacher.

Study 1

The students were asked to fill out a form directly after a lesson. The questions posed were consistent throughout the study period.

a) What I have learnt / What I have not understood in Mathematics
b) What I have learnt / What I have associated in Didactics

When the student responses were read, their notes from four lessons were categorised in order to appropriately answer the fundamental questions originally set.

What about learning from Mathematics and what is Didactics in the lessons?

The notes from every lesson could be divided in one of six categories:

1. M: a mathematical reference from the lesson
2. D: a didactical reference to the lesson
3. R: reflections, e.g. about a working situation
4. P: quite personal reflections (e.g. something difficult)
5. -: No answer
6. 0:absent

You can compare every lesson in a mathematical and a didactical view and see the result:

- There are more M and D in the mathematical field in the beginning of the course
- "No answers" are a little more frequent in Didactics
- More "reflection" at the end of the course both in Mathematics and Didactics

- To compare student 1 with student 5 and 16 there is a big difference of the number of M and Ds shown. What do students write regarding Mathematics and Didactics lessons?

The examples from the notes of some students can explain the four categories from the four algebra lessons:

M: “I have learnt to solve an equation numerically.
R: “To solve an equation was very hard for many students.”

The same lesson gives connection to Didactics through:

M: “The concept of equality symbol”
D: “To speak, to listen, to write to read. Important! Interpreting.
D: “Different dividing of groups makes different solutions”

From the next three lessons are these M-sentences taken:

M: “When has the proof come to an end?”
M: “I know the factor theorem = I understand it”
P: “Complex Numbers, so difficult, now it is easier.”

And looking at the lessons, as above, you can read with Didactics focus:

D: “It is important to change between different forms of expressions.”
R: “To have fun is not the same as not hard working.”
P: “I have improved in keeping silent and let others having a chance.”

Study 2

Evaluating notes about learning from the first four lessons formed part of the fifth lesson. The intention was that after discussing these in “4-groups” the students could perhaps write more about their own personal learning to be assessed. The students then documented their responses for the next two lessons. The question was then raised:

Could any differences be recognised compared to the earlier answers given?

Now the student was required to be conscious of describing what they learn from the lesson and not describing the lesson itself.

In the next lesson different notes were gathered from the first study and evaluated further. The aim was to read through the documentation together with the student and describe which notes represented a concept of learning in Mathematics or Didactics. The students discussed this topic in various groups and it was agreed that this process was hard work.

Below are selected notes from this working process:

I learnt that I must take the square of both sides of the equality sign in an equation to get the roots. The concept of the equality sign is difficult to understand for the pupil. I have learnt there are many way of thinking False roots What I thought was evident could I skip You must have time to think To fold an A4 and see one side is square root two of the other Algebraic, Geometric and Numeric solutions are useful to know I have learnt to take everything in pieces I feel that I can translate problem and express them better now To develop an algebraic language from simple to the hard Begin to handle with + and – when dividing polynomials I have learnt the factor theorem with understanding I am not sure of division of polynomials

The examples led to discussion in groups of 4 where everyone was faced with the following questions:

Which examples can reveal a learning of Mathematics or Didactics?

The aim here was to influence the students into thinking about themselves: IF they learn, WHAT they learn and perhaps also HOW they learn.

Did the students change the writing of learning?

The answer is no, it is still impossible to separate the lesson from the learning from lesson. At this point, it is still difficult to understand what the student has learnt from what they have written.

The next step was to request the students to document a situation where they actually learnt Mathematics from the course, and the timescale given for this was one week.

Study 3

In the Geometry course a new model for questioning was suggested: The students were asked to describe the situation where they learnt something from the course and also describe what they learnt. The students were asked to "Describe a moment of clarity from the course and describe the situation in which it occurred". The objective here was to establish an understanding of a student's learning from a situation connected to a classroom situation during the whole course, instead of one particular lesson within the course.

At the end of the Geometry course a new model for questioning was suggested. The students must describe the situation where they learnt something from the course. They must also describe what they have learnt. The objective was to be able to understand the student's learning better from a situation connected to the classroom during the whole course instead of one lesson of the course. Therefore the following three points were restructured in the research.

- time for reflecting (a week instead of directly after lesson)
- period of reflecting (a whole course instead of one lesson
- describing the situation where and what they learnt instead of describing only the information that was learnt

The question which to give insight into this issue was: “Describe a moment of clarity from the course and describe the situation in which it occurred”.

Before analysing the reflections of the students, first is shown one report of a particular student. Her thoughts are exactly related to the issues in question. Following that, in the table below, is a summary of different categories of situations of learning. But first the reflections of one student on his own learning in a situation:

> “Here is a small summary in Didactics of things that I have learnt in the course, situations where they have been learnt and in which way I can use this knowledge. As time is restricted, I have concentrated specifically on the latter part of the course.”

What have I learnt?	Which was the situation when I learnt this?	In which way can I use this?
It’s important to take pupil and the work of pupil seriously.	When Anna expressed anger and disappointment about not getting to summarise chapter 11 in Billstein when she had worked a whole day with it.	I shall be observant how my pupils understand my priorities. What can be important for me can be unimportant for them and vice versa
It’s important to give time to understanding the pupil solutions.	When I went up to the black-board to help Johan in proving without thinking of his way of thinking	In a “real” situation where I have the “control” I can take the time I need to understand the problem
To be flexible and adjust the plan to reality	The situation in our hetero-geneous group was changed.	I am going to think about this from the start of planning a course
A question can be more effective than accusing and nagging	When my question ”How do you treat ‘pigging back’ in a system without assessment?”, I got response from both teacher and student.	I am reflecting alternative ways to reach my goals. You can al-ways get better response if you allow it to come from yourself than press your opinions to others.
The goal of teaching must always be distinct and the result and asses-sing too. Besides I must be ready to argument for this.		I will plan my work and my teaching so good that I can be responsible for it and argument.
What reflections does Eva-Stina want?	I didn’t care of that and instead I made a summary of my own that I can use.	Now I have a system for myself that I can use in different con-nections.

The students' writing is now more alive than in Study 1 and Study 2 and the categories where they relate their new knowledge are from real situations in different lessons. Both mathematical knowledge and didactical knowledge is evident.

Shown in the table below are different expressions from their learning and what they have learnt. It is possible, at the same time, to connect the student writing of learning to a certain subject in a lesson. It is certainly more interesting to see the variety of expressions than the frequency of them.

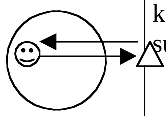
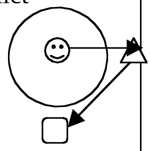
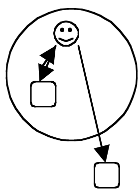
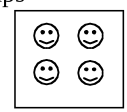
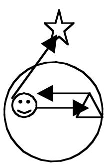
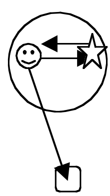
The golden ratio	Cabri and the Geoboard
1. When Anna showed her model to see the Golden ratio and solve its value…In reality you must hang up your knowledge to something 2. Golden ratio – exactly before I should go to the blackboard and show the golden ratio, I was sitting in my desk sweating as I didn't know how it worked. 30 s before my turn it said "pling" and every piece found its place. This experience I never shall forget.	1. When we were working with Cabri-program you asked me why the triangle did not change its area. When I changed the shape at a distinct line then I got a moment of clarity when I saw that the altitude and the base didn't change in spite of the triangle goes to infinity. 2. An extra bonus was when I visited my school and pupils were working with TI-92. They were sitting there quite confused both teacher and pupils. And I could help them. I felt satisfied! 3. I now see fantastic possibilities for me as a teacher in using geoboard in school.

Among the situations that are related to a learning situation are "The golden ratio, Cabri and the geoboard", being all of fictional characters in mathematical education. These subjects themselves are constructive in many ways and the situations invite to co-operation in the classroom for problem solving.

To learn from working with port folio is expressed in the following way:

1. "All the time I'm studying my own learning by using portfolio. But I have still no knowledge in structuring. I will work to know is in future because it interesting for me. Not only for me myself I want this knowledge in my profession and pass the idea to the children."
2. "It's a kind of art to build a portfolio. I have plenty of material but I am interested in ordering it in a way, which is my way."

To see the process of the three studies in a model

		A problem situation giving a conflict for the student		
First study	He gets an image of the situation in his head		He react by thinking if he knows the subject or not	He writes about himself through the conflict 
Second study	He compares his own writing in a lesson situation with the others in other situations but the same lesson		Discussion groups 	
Third study		He must find his own situation were he learnt Mathematics or Didactics		
	This he learnt 	**Now he has chosen a problem situation of his own**. Therefore when he solves **his** problem he also knows from where it comes and how to express it	External expression from what he learnt.	

The result of the third study is more about writing about learning than the two earlier studies. But it is still difficult to see understanding of the concepts of the courses. And how can I know that the student can use their knowledge? That question is now more evident in assessment in one of three levels of the student.

Semiotics in Education

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Abstract

On the particular topic, the graph of the function $y = kx + b$, a conception of a proof is concidered. A few Lithuanian textbooks are analyzed. Graphical representation of a proof and other innovations are suggested. For analysis and design of texts Greimas's approach of semiotics is applied.

Problem description

Proofs are disappearing in school mathematics. Students must more and more trust the truths, which are presented in textbooks without sufficient substantiation. Later on when they enter universities they have a lot of problems, especially freshmen. They are depressed by the word "proof" only. In this paper I consider the conception of a proof. I present the current situation (Bagdoniene et al., 2000) and two historical references (Kiseliovas, 1959; Busilas & Balutis, 1934) about the graph of a function $y = kx + b$ in Lithuanian textbooks. I would like on this particular topic to revive the conception of a proof (Kudzma, in press). For reaching these goals semiotics (Greimas, 1970, 1979) is applied. A.J. Greimas (1917-1992), Lithuanian by nationality, wrote his most important works in Paris and was the maitre of the well-known Paris school semiotics. It is very impressive that semiotics laws are universal and can be applied for analyzing any text (poem, tale, story, picture, film, lecture, advertising clip, etc.). Semiotics is useful for design, too.

The function $y = kx$

Let us look at the contemporary Lithuanian textbook for the 9th grade (Bagdoniene et al., 2000, p. 22): "Let us draw the graph of a function $y = 2x$. Choosing a few values of an argument make the table of values of the function. Let us put these points (x, y) on the coordinate plane. After joining them we see that all these points lie on the straight line, which passes the origin of coordinates. In a similar way we can convince ourselves that for any value k the graph of a function $f(x) = kx$ is a straight line, which passes the origin of the coordinate plane".

If we look back to the history we will find that similar texts were reproduced in textbooks for almost 40 years. But in Kiseliov (1959, p. 34) we can find the strict proof of the statement: "The graph of the direct proportionality ($y = kx$) is the straight line, which passes the origin of the coordinates and the point $(1, k)$."

Comparing these two texts we can notice the difference in formulations. The first one is not completely correct – a straight line, which passes the origin of coordinates is not uniquely determined and not related with the coefficient k. Kudzmiene and Kudzma (2001) pointed out that if the proof is omitted then the mistake in the formulation does appear. If we take a few points for plotting the graph then all points are equal, except

(0, 0), and it is difficult to distinguish the second point (1, k) (why this but not another one?).

A few words about Lithuanian textbooks need to be made. After reestablishing independence in 1990 Lithuanian authors started writing textbooks for secondary school. This is the first generation of them. The last one is for the 9^{th} grade (Bagdoniene et al, 2000). During the period 1940 – 1990 there were textbooks unified for all Soviet Union (except Estonia). We had to translate them from Russian only. But there were national textbooks in 1918 – 1940. It was very interesting to find that Busilas & Balutis (1934, p. 132) proved the assertion about the graph of direct proportionality $y = kx$ and used the correct formulation.

Semiotics – introduction

According to Greimas' (1970, 1979) semiotics theory a text is organized and therefore can be analyzed in three levels:

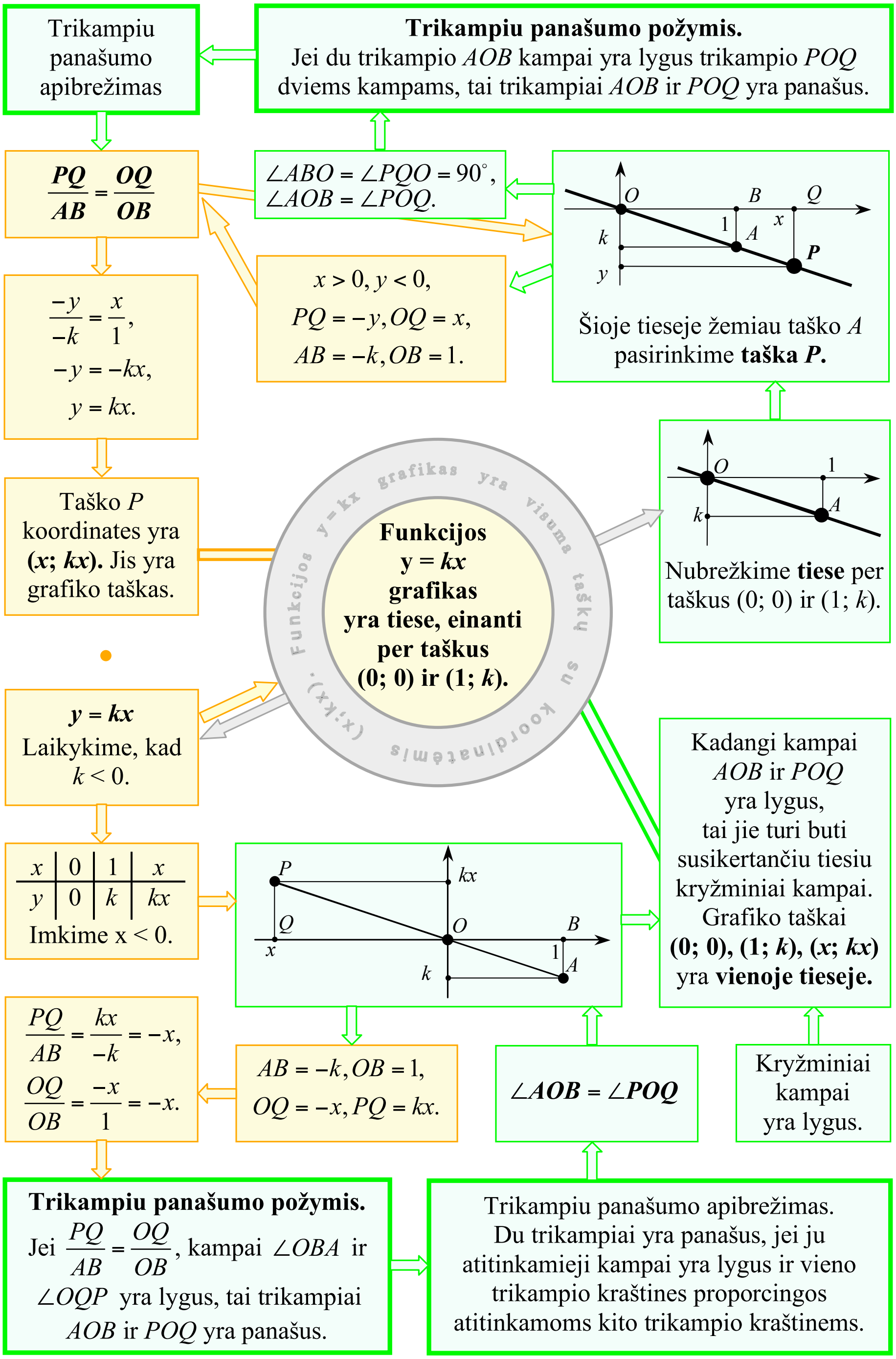
1) discursive,
2) narrative,
3) semiotic or logical-semantic.

The interplay among these levels builds the frame of the theory. The key point of the third level is, so called, semiotic square. This square governs the narrative level. Theory distinguishes four phases of narrative level: manipulation, competence, performance, sanction. Actants, like, addresser, addressee, subject (hero), anti-subject (rival, villain), object of value, helper, etc., are the notions of narrative grammar. Narrative level has clearly expressed anthropomorphic character. Profound analysis of tales and myths stimulated development of semiotics, at least Greimas' semiotics. From another side, tales have very powerful didactical charge. Why not to use this tremendous didactical experience in mathematics education? Ernest (1997) considered parallel between myths and a mathematical proof, too.

Discourse analysis

Let's try to analyze the discourse of the selected page below.

1) The proof is represented graphically. Visualization is a global tendency today. Graphics is attractive for youth.
2) The proof (path of a hero) is of a circular form. There is a very natural reason for that - we start from a function and finish with the same functional relation. To start from and to finish with the same condition or relation is quite common in mathematics, especially in proving theorems about necessary and sufficient conditions. Ernest (1997) also noticed a circular nature of a proof, but in different sense.
3) The main result (object of value) is placed in the center. This is logical consequence of a circular representation of the proof. The first glance at the page must catch the most important thing.



A selected page (Kudzma, to appear)

4) The definition of a graph of a function girds up the center. Somebody or something (rival) must protect the object of value. This protection is realized as a shield with the definition. I want to stress the role of the definition as a shield and to make it visible.
5) Two arrows cross the shield and reach the center – object of value. The proof consists of two parts and when a part ends an arrow reaches the center.
6) Theorems, definitions, which are used in the proof (helpers), are placed in corners. There are several arguments for such design:
   i) Corner elements form the frame of construction;
   ii) Auxiliary theorems (helpers) in the corners represent forces which do not allow run out of the circle during the proof and press to the center where the desirable result is situated;
   iii) Readers must not seek and look at the page No. n for reference;
   iv) Readers visually see how theorems are being used;
   v) Repetition of auxiliary results several times strengthens remembering.
7) Algebra is separated from geometry by putting algebra in the left side and geometry in the right side of the page and by painting corresponding blocs in different colors - geometry in green (like earth) and algebra in yellow-brown (like sand from Arabic deserts). The separation of algebra from geometry was the first serious step in designing of such a page.

Narrative analysis

Narrative grammar says that the story starts from the manipulation phase: somebody (addresser) formulates a problem and finds addressee (subject or hero) who wants and is able to solve it. The second phase is competence: subject (hero) is put into the situation, which permits him to act. Mostly in textbooks this subject-hero is "we", but it depends on discourse. The first two narrative phases are not so important in this paper and I'll dwell on two last ones. I prefer to use terminology from tales (hero) to pure semiotics (subject) in the analysis of performance.

As usual, the hero must pass three tests for the victory. The first test is to find the graph of the function $y = ½\ x$. For the first we (hero) take few values of an argument x, find values of the function, then put points on the coordinate plane. One can see and guess that all these points lie on the same line. But it should be proved. Hero always needs a helper. At this time the theorem about equality of triangles helps to prove what is asked. But in general, the first attempt is not successful. The next time hero invites a stronger helper – the theorem about similarity of triangles. Now he proves that the obtained line is the graph of a function. The second test is to find the graph of a function $y = kx$. Hero has an experience, the reliable helper and passes the test again. The third test is the most difficult - to find the graph of the function $y = kx + b$. The hero carries out the task without any problems.

Let us compare the stories mentioned above. In Bagdoniene et al. (2000) truthfulness of the statement is left to "seeing" and believing. Thinking is excluded. The very first sentence in Kiseliov's (1959) proof - "Let us prove that the graph of a function $y = kx$ is a straight line" is not didactically substantiated. Where does a straight line come from? Both "experimental" part (getting the line) and the proof that this line is the graph of a function do exist in Busilas & Balutis (1934). I follow this approach.

From the first sight a repetition of three times of very similar proofs could be boring. The logical scheme is the same in all three cases but there are slight and important differences concerning signs of coefficients and arguments. These differences have influence into the graphical and textual representation of a page. Studying all cases helps to form a complete picture of a problem. The last one, didactical remark – *Repetition est mater studiorum*.

In the sanction phase the addresser must say very important phrases:

1) The function $y = kx + b$ is called linear, because its graph is a straight line.
2) From now the straight line which is the graph of the function $y = kx + b$ will be called simply "(straight) line $y = kx + b$".

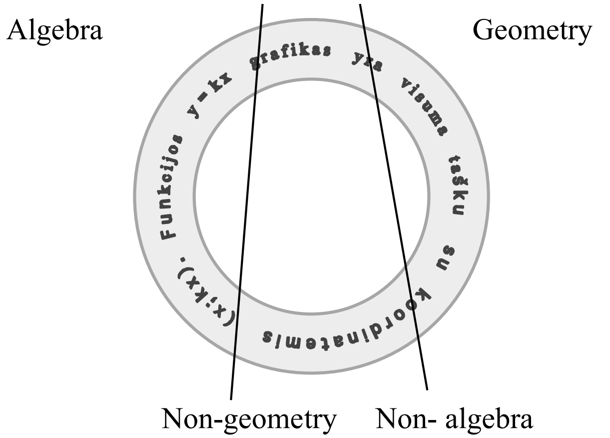
Such sentences do exist in Kiseliov (1959, p. 40) but I miss them in a great number of textbooks. It was interesting (not for the first time!) to find the following text in Busilas & Balutis (1934, p. 216): "The function $y = kx + b$ is called the first order function because an argument x has the first power. It is called linear too, because its graph is the straight line". People almost a century ago called things with right names! Semiotics gives a very simple explanation – sanction follows after performance. If there is no sufficient performance (proof) then there is no need for any sanction. Omitting proves we loose understanding of right names. At the end of narrative analysis I wouldd like to emphasize one thing: A function $y = kx + b$ is not linear until it is not related with a line.

Semiotic analysis

First of all it requires finding the main opposition in the text. It is between algebra and geometry and even expressed graphically. Algebra and geometry are two different means to investigate nature and two different languages at the same time. The problem is to create strict correspondence between objects of algebra and geometry. It is very easy to form the semiotic square:

algebra	geometry
non-geometry	non-algebra

Algebra and geometry are joined by something from intersection of non-algebra and non-geometry (general feature in semiotics). Very interesting words (objects) belong to this intersection - graph, abscissa, ordinate, coordinate(s), etc. They are functions from algebraic objects to geometric or visa versa. If I represent the definition of a graph or formulation of a theorem about the graph in a circle then algebra is in the left, geometry in the right and words "graph", "coordinates" find their place in the middle:



This phenomenon might depend on the language. Such order of words – “The function’s $y = kx$ graph is the set of points with coordinates (x, kx)” is a natural order in Lithuanian but not in English. It is possible to reformulate the definition in the following way: “A function $y = kx$ has its graph as a set of points with coordinates (x, kx)”. This form of the sentence can be represented in the circle.

Final remarks

I think that the topic about the graph of a function $y = kx + b$ is very important. It forms the foundation for further studies of Calculus (Mathematical Analysis). Descartes gave us very powerful tool, the method of coordinates, to connect algebra with geometry and we must use it with corresponding respect. On this particular topic I tried to show how semiotics could be applied for analyzing and designing texts and developing the conception of a proof. I hope that semiotics is a quite reliable helper.

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The Current Sate of Variations in the “Lived Space” of Mathematics Learning

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Abstract

Recent studies show that Hong Kong students have a narrow conception of mathematics. Why are they so? One of the possible causes is that they have a narrow lived space of mathematics learning. The present research reviewed the types of problems Grade 7 and Grade 9 students did in class, at home and in examination. The results indicate that the students were exposed to a variety of problem types, but the level of contextualisation was low, level of open-endedness was low. Most questions required students to apply rules and routine procedures.

Introduction

Despite the outstanding performance of Asian students in general and Hong Kong students in particular in international comparisons in mathematics (Beaton et al., 1996), there had been increasing evidence that these students may not be exceptionally good in non-routine problems (Cai, 1999). That might be attributed to students’ conceptions of mathematics. When students conceive mathematics as an absolute truth or as a set of rules for manipulating symbols, they would tend to treat doing mathematics as the memorisation of a series of steps of tackling questions and learning mathematics as a transmission of knowledge from the teacher to the student (Clay & Kolb, 1983; McLeod, 1992). This was what we have found in various research studies conducted in Hong Kong (Lam, Wong & Wong, 1999; Wong, Lam & Wong, 1998).

A study in Hong Kong also indicates that students’ conception of mathematics is shaped by classroom experience (Wong, 2000; Wong, Marton, Wong, & Lam, in preparation). If most problems given to students lack variations, possess a unique answer and allows only one way of tackling them, it will not be surprising that students see mathematics as a set of rules, the task of solving mathematical problems is to search for these rules and mathematics learning is to have these rules transmitted from the teacher.

The research reported here focuses on reviewing the current state of the problem-solving environment in the mathematics classroom, the ‘lived space’ of mathematics learning. We aim at investigating the variation of mathematical problems according to the levels and ability of secondary school students by analysing the problems given to them.

Sampling and procedure

The focus of the present study was on Grade 7 & 9 students. A topic in algebra and another topic in geometry were chosen in each of grade levels according to the current Mathematics syllabus developed by the official curriculum development agency in Hong Kong. Algebra and geometry were chosen because number sense and spatial sense are two corner stones of school mathematics. The topics we have chosen are:

Formulae, open sentences and simple equations (Grade 7)

Fifteen teachers were invited to provide the researchers with the homework and mathematics test papers of these topics they used in their schools.

A total of 1,557 mathematics problems (of which 1,200 were homework and 357 were test items) from 15 schools (5 of high mathematics standards, 4 medium and 6 low) were collected. All the mathematical problems were grouped and analysed according to different criteria: (a) problem types, (b) the extent of contextualisation of the problems, (c) openness at the given information, (d) openness at the goal and solving process and (e) the levels of expected performance.

Stratified analysis of the above were performed according to the groupings of (i) academic standards of the school, (ii) occasions at which the problems are given (homework or test), (iii) grade level and (iv) topic (i.e. algebra or geometry).

Framework for analysis

Problems used in teaching and assessment

With initial analysis, problem types of each topic were identified. For example, for the topic "Formulae, open sentences and simple equations", we have identified:

(i) Rewrite mathematically
(ii) Substitute into well-known formulae
(iii) Substitute into "home-made" formulae
(iv) Perform algebraic manipulations
(v) Set up equations
(vi) Solve equation involving one to two techniques
(vii) Solve equation involving more than two techniques

The problems were classified according to whether they were: (i) situated in a rich context, (ii) put in a context but the data can be readily obtained, or (iii) posted in a purely symbolic setting. Furthermore, the problems were analysed to see whether they contained (i) redundant data, (ii) an exact number of necessary data, or (iii) missing data. Also, they were classified according to these conditions: (i) having a single answer and a single solution, (ii) having a single answer but with multiple solutions, and (iii) other types of open-endedness.

The levels of expected performance were analysed using the TIMSS (The Third International Mathematics and Science Study) framework (Robitaille et al., 1993). The framework is composed of the following levels:

(a) Knowing
(b) Using routine procedures
(c) Investigating and problem solving
(d) Mathematical reasoning
(e) Communicating

Results

Overall picture

The different problem types in the various topics reveal that we had a variety of problem types (ranging from 7 to 10) across the four topics. Table 1 is an example. However, the categories of the skills involved by their problems revealed that most of them only required students to perform some routine procedures, similar to what had been taught in the examples of the textbooks.

Problem Type	**%**
Find volume of blocks without given cross-sectional areas	24
Find area of irregular shapes, without supplementary lines	21
Find area of regular shapes	16
Find length/area directly	10
Find length, but need to find the area or volume first	10
Find area of irregular shapes, with supplementary lines	10
Find volume of blocks with given cross-sectional areas	5
Application	4
Find volume of blocks without a uniform cross-section (one have to divide the blocks into smaller blocks first)	1
	100

Table 1. Problem types of Topic # 1.5 and their relative percentages (in decreasing order of percentages)

Results also reveal that 85% of the problems were posted in a purely symbolic setting. Although we found some problems (12%) that were put in a contextual situation, the numerical data could be readily obtained. Only 3% of the problems were set in a rich context.

As for the openness of the problems, exact information was given in 99% of the problems and 98% of them does not allow multiple solutions. As regards the level of expected performance, although considerable variations were found across the problems, 79% of the problems required students to carry out routine procedures only (3% involves “knowing”, 10% involves “investigating and problem solving” whereas 8% involves “mathematical reasoning).

Variations across schools of different academic standards

When we compared the problem types used in schools with different academic standards, the situations resembled the overall picture (when the entire problem pool was considered). However, we noticed that some of these problems types were absent if we consider the schools separately. In other words, some schools did not provide certain types of problems that other schools gave. This reflects a certain extent of curriculum tailoring in schools.

Results also revealed that the problems given in schools of better academic standards had slightly greater extent of contextualisation than schools with lower standards. For schools with high academic standard, 86% of the problem was purely symbolic, for medium standard, we have 89% and for low standard, 78%.

Since the problems given to students across the schools lacked openness in the given information, it is obvious that the situation would be the same among schools of different academic standards. The percentages of problems providing exact information for schools with high, medium and low academic standards were 99%, 100% and 99% (rounded off) respectively. The case is similar for openness at the “goal” and “process” stages. There was little openness regardless of the academic standards of schools (the corresponding percentages were 96%, 99% and 99%).

As for levels of expected performance, slightly more problems of higher levels of expected performance were provided by schools of better academic standards. For instance, the percentages of problems that involves “investigating and problem solving” in schools with high, medium and low academic standards were 16%, 8% and 5% respectively.

Variations between homework and test

The problem types of homework and tests were compared. It was found that the range of problem types given in tests was a bit narrower than that given in homework. Also, the emphasis of the types of problems in tests differed quite obviously with the emphasis in homework. The types of problems in tests were more evenly spread.

There were much similarity between the problem types given in homework and tests (symbolic: homework – 83%, test – 86%; exact information: both homework and test were 99%, unique solution: homework – 98%, test – 97%) except possibility the expected performance of the problems. For homework, 82% required the use of routine procedures and the percentage was only 71%. In tests, 14% of the problems required investigation and problem solving whereas the percentage for homework was only 8%.

Variations across grade levels

When we compared the problems given to Secondary 1 and Secondary 3, it was found that the higher the grade level, the more the problems being purely symbolic (94% for Secondary 3 and 78% for Secondary 1). Although there was little openness among the problems in general, some problems with redundant data were found in Secondary 1 (even though the amount was very small, around

2% of the problems in Secondary 1) compared to the whole sample. Little openness was found at the goal and solving process too. However, we had a small portion of problems that could be solved with more than one solution. Around 4% of the problems could be solved with multiple solutions in Secondary 1, while only 1% of such kind of problems was found in Secondary 3.

When we compared the level of expected performance with reference to the grade level at which the problems were given, we found that problems requiring mathematical reasoning were found in Secondary 3 (all from the Topic # 3.4.1 and 3.4.2, and most of them involving "prove …", "show …" and "give reasons") but none in Secondary 1. Furthermore, 11% of the Secondary 1 problems required investigating and problem solving.

Other observations

The analysis also reveals a number of interesting points. First, the textbook was found to be the main source of mathematics problems given to students. To a certain extent, it reflects the textbook dependence of classroom teaching. The lack of variation further worsened since most of the schools picked out mathematical problems from the same textbook[#]. While some teachers in the present study reflected that the choice of mathematics problems was decided during mathematics subject panel meetings in schools, in many cases, teachers just followed the convenient way of circling a number of problems in the textbooks for the class work and assigning the rest as homework. One commonly adopted practice was also identified: it is to ask students to do odd-numbered problem during the lesson and leave the even-numbered problems for homework assignments. It was also observed that more exercises were given in the algebraic topics than the geometrical topics.

A number of "uncommon" practices, which were probably performed to exercise curriculum tailoring, were also found. For example, while there was a teacher who gave some challenging problems for more able students, there was another teacher who broke down problems into smaller parts in order to help students understand the problems.

Discussion

Though only four topics were involved in the study, the analysis of these one thousand and five hundred problems provided by the teachers gave a clear picture of the "lived space" of student learning in mathematics. Obviously, the lack of variation of the problems, though sort of expected, is disappointing. The convergence of problems was shown in two dimensions. First, there was little open-endedness among the individual problems given to students. Second, the types of problems given to students were quite unified across different schools. The major reason is that most teachers relied on the same "data bank," i.e. the

[#] Textboks in Hong Kong are developed by commercial publishers. Teachers and schools choose from the market. At present, a particular textbook is occupying over 90% of the market.

most popular textbooks in Hong Kong. The use of other sources like overseas textbooks, self-created worksheets, articles from periodicals was rare. The expected performance of the problems given was also low, whether in homework or in tests. The situation was a bit improved when students moved from Grade 7 to 9. Moreover, more openness and more problems with higher expectations in the area of geometry were also found.

In fact, the same phenomenon has been repeatedly found in other studies. In the most recent TIMSS-V Study (The Third International Mathematics and Science Video Study), it was found that the teaching styles and problems posed to students are very much the same across schools with different academic standards. Catering for individual difference is lacking since teaching is conformed to textbooks and to examination formats (Tseng, 2000). This is in fact echoed in the present study.

Naturally, mathematics problems constitute the major part of students' experience with mathematics. If we agree that variation is conducive to the development of mathematical understanding, obviously the scenario depicted in this study is not encouraging. The lack of variation may attribute to the general narrow conception of mathematics among the students (Wong, Lam & Wong, 1998). Furthermore, the lack of variation across schools reflects the absence of curriculum tailoring to cater for individual differences, which is an essential issue in universal education (Wong et al., 1999). Although the lack of variation could be a result of standardisation due to an examination orientation (Kong & Wong, 1998), we must be aware that variation of problems can bring about impact among the students and lead to betterment of learning (Marton, & Booth, 1997). A teacher taking part in the present study wrote the following remark: "I once gripped a test from a school of higher band and gave it to my students. The students all failed with very low scores and they blamed me. I then went through the test with my students. Eventually they realized that the problems were not as difficult as they thought and they had higher efficacy in tackling those problems." His'her remark vividly shows how variations can bring about a learning environment favourable to conceptual understanding.

Students did hundreds of thousands of mathematics problems during their long course of schooling. These problems should constitute a rich lived space of experience for them. However, in places like Hong Kong where students are required to do so many exercises the crux of the matter seems to be quality rather than quantity. Though one may say that numerical computation and direct application of formulas are involved in most problems and the domination of them among the strategies employed by the students is no surprise, we see in the present study that the involvement of higher order thinking like reasoning is far from the desirable level. Would it be a waste of energy if students are drilled with numerical computations repeatedly in their course of study? Practice is essential in mathematics learning, repetitive learning can bring about transcendence to a higher level of understanding only if systematic variation is introduced (Richardson, 1999). How ancient Chinese learn could shed light on this (Wong, 1998). It is hoped that by the introduction of a varied and gradually opened up lived space, students could arrive at a rich and structured outcome space of mathematics learning.

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Conceptions of Mathematics in Different Ability and Achievement Groups among 7th Grade Students

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Abstract

This report deals with 7th grade pupils' conceptions of mathematics, its learning and teaching. The report focuses on the identification and comparison of views expressed by pupil groups of different mathematical ability and achievement. The analysis is based on the results of the ability tests, subject tests and a questionnaire conducted among the 7th grade pupils of Estonian schools.

Introduction

Pupils' conceptions of mathematics and themselves as learners of mathematics are of fundamental importance in their learning of and performance in mathematics. Previous research has shown that differences in pupils' conceptions are much bigger between countries than within one country. This implies that each country has its own teaching style and teaching culture, which do not change very easily. The biggest differences in the conceptions of pupils of different countries are found in the understanding of the role of the teacher. For example, Estonian pupils are more teacher-centred (Pehkonen, 1996). Compared to their counterparts in other countries (such as Sweden, Finland, Hungary, and Germany), they prefer a teacher who explains well, helps them immediately and always tells them exactly what they ought to do to solve a mathematical problem. Previous research on pupils' conceptions has also shown no significant differences between the conceptions of girls and boys. Some mathematics educators (A. Schoenfeld, E. Silver) have pointed out that pupils' conceptions of mathematics may prove to be an impediment to the solving of non-routine problems and to effective learning. However, the solving of such problems is important with respect to pupils' ability and creativity.

The purpose of our study was to answer the following question: How is the student's conception of mathematics teaching connected with his or her ability and academic achievement?

We also tried to answer the following subquestions:

- What characterises the ways of mathematics learning of more talented and less talented pupils?
- In what way do the conceptions differ between students achieving better results and those achieving poorer results?

Data gathering and tests

This paper summarises year 1 results of a [three-year] research project aimed at analysing the progress made by and the changes in the conceptions of mathematics teaching and mathematics learning of a cohort of pupils proceeding from grade 7 to grade 9.

In year 1 of the project (1998), an ability test (*Potential*), three achievement tests (*Numbers, Algebra, Shape & Space)* and a questionnaire including 33 questions about the teaching and learning of mathematics were carried out among the 7th grade pupils in 14 schools in Estonia. The ability and achievement tests used were prepared within the framework of the Kassel-Exeter (*Kassex)* project (1994-1998). The general coordinator of the project was *Centre for Innovation in Mathematics Teaching* – CIMT of the University of Exeter (Anon. a & b, 1994). The questionnaire was prepared by E. Pehkonen and B. Zimmermann and, like the *Kassex* project tests, has been used in several countries.

The initial test *(Potential)* consisted of 26 questions and the subject tests of 50 questions each. The answers were graded on a dichotomous scale as follows: “1” - true, “0” - false. Thus, the highest possible score was 26 points on the initial test and 50 points on each of the subject tests.

The questionnaire (responses according to the following pattern: “-2” – I strongly disagree; “-1” – I disagree; "0" - I do not know; “1” – I agree; “2” – I strongly agree) addresses four different conceptual domains: *Conceptions of mathematics (*questions of this type in the questionnaire are below marked with a C)*; Conceptions of the way of doing mathematics* (D)*; Conceptions of mathematics learning* (L) *and Conceptions of mathematics teaching* (T).

The number of pupils who completed at least one test or the questionnaire was 414. Of these, 198 (47.8%) were girls and 216 (52.2%) were boys. The average age of the respondents was 13.5 years. Those pupils who did not complete all the tests and the questionnaire were left out of the following analysis.

The average results of the tests are presented in Table 1. Since the *Kassex* project required the tests *Numbers, Algebra and Shape & Space* to be applicable to the testing of the same pupils for three consecutive school years (ages 13+, 14+ and 15+), the average test results for this age group (13+, Grade7) are, for natural reasons, relatively low.

Test	N	Possible maximum	Minimum	Maximum	Mean	Std. Deviation
Potential	396	26	2	24,0	12,7	3,6
Numbers	399	50	7	40,0	24,1	6,7
Algebra	396	50	3	24,0	11,2	4,2
Shape and Space	395	50	0	29,0	12,4	5,1

Table 1. Test results

We assessed the reliability of the tests using two methods: as internal consistency by Cronbach Alpha, which, due to the dichotomous assessment of the test results, is here equivalent to the Kuder-Richardson Formula 20 (KR20), and by finding the Guttman's lower bound for the true reliability. It appeared that the reliability of our tests is at least satisfactory, remaining within the range of 0.68 (*Potential* and *Algebra*) to 0.86 (*Numbers)* for Cronbach α and 0.68 *(Shape and Space)* to 0.87 *(Numbers)* for Guttman's coefficient.

For grouping the pupils by their *mathematical ability***,** we divided them into three groups on the basis of the results of the ability test *(Potential)* as follows: *Low ability group* (17% of the pupils, total score 2 - 9 points), *Medium ability group* (68%, 10 - 16 p) and *High ability group* (15%, 17 - 24 p).

For grouping the pupils by their *mathematical achievement,* we used the arithmetic mean of the results of the three subject (achievement) tests (*Numbers, Algebra, Shape and Space)*. Since the arithmetic mean ranged from 0.0 … 30.0 points, we divided the pupils into three achievement groups as follows: *Low achievement group* (13% of the pupils tested, the mean being 0.0 – 10.0 points; *Medium achievement group* (69%, mean 10.1 ... 20.0) and *High achievement group* (18%, mean 20.1 ... 30.0). The following presentation of some of the results of our research on the pupils' conceptions includes only the high and low ability and achievement groups, leaving aside the respective medium groups.

The conceptions of the pupils in different ability and achievement groups

It appeared that in most cases (79% of all 33 statements) it was impossible to prove the existence of differences between the conceptions of the pupils of the two extreme ability groups and the two extreme achievement groups at the significance level of $p < 0.05$. This is evidenced by the statistically significant correlations (for Pearson's r and Spearman's ρ the p<0.000) and by the similarity of the respective divisions (χ^2–test, p<0.999) between the evaluations of pupils of different groups along the means of points in the questionnaire.

A. Conceptual domain – *the content of mathematics*

This conceptual domain was measured by the following 9 items:
Good mathematics teaching includes:

C1 doing calculations by heart
C2 mechanical calculations
C3 drawing figures (e.g. triangles)
C4 doing word problems
C5 using calculators
C6 different topics … taught … separately
C7 problems … have practical applications
C8 calculating areas and volumes
C9 constructing concrete objects

There are no significant differences (t-test, p<0.05) in the conceptions of *the content of mathematics* between the *low and high ability* groups. Both groups consider calculating to be of the highest importance. The importance of calculating with the help of a calculator is significantly lower than calculating by heart or in writing both for the high-ability and low-ability pupils. The use of a calculator is considered necessary or highly necessary by only 50% of the low-ability and 51% of the high-ability students. In this respect, the conceptions of Estonian pupils have undergone considerable changes: in 1990, calculating with the help of a calculator was the most preferred method (Lepmann, 2000). Surprisingly, there are also no differences of opinion with regard to non-routine tasks: 76% of the low-ability students and 77% of the high-ability students think that textual tasks (C4) are an integral part of mathematics teaching.

In the achievement groups, statistically relevant differences are only evidenced in the responses to three questions related to calculating skills (C1, C2, C8). The responses reveal that calculating skills with their different aspects are considered significantly more important by the pupils in the high achievement group than those in the low achievement group (Figure 1). Strikingly, the conceptions on this domain of the high ability and the high achievement groups coincide fully – of the highest importance is the skill of calculation by heart while the skills related to the construction of concrete objects are ranked lowest. The respective opinions on this domain of the low ability and the low achievement groups, however, are less coincidental ($r=0.71$, $p>0.05$ and $\rho=0.63$, $p>0.05$). Nevertheless, in both cases the pupils attribute the lowest importance to the statements related to geometry. An analysis of the responses to items C3 and C9 given by the low group pupils reveals that in both instances these students are slightly more in favour of enactive representation than iconic representation. Among high-ability pupils, it is on the contrary.

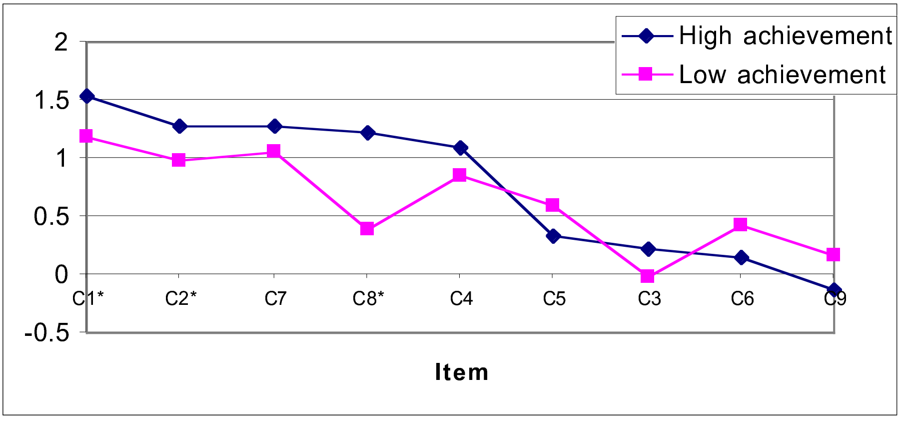

Figure 1

B. Conceptual domain *– the way of doing mathematics*
This conceptual domain was measured by the following 5 items:

D1 right answer… more important than the way

D2 everything … expressed … exactly

D3 there is … procedure … to exactly follow

D4 everything … reasoned exactly

D5 there is … more than one way of solving

In this domain, no significant differences could be observed in the conceptions of the extreme groups. It is only worth mentioning that both extremes considered exact reasoning (D4) to be of the highest importance. All groups of pupils liked solving the same task in different ways. Encouraging from the perspective of promoting constructivist approach is the fact that all the groups responded negatively to Statement D1: the right answer is more important than the way to get it. However, the low-ability students were more in favour of the statement that everything ought to be expressed as exactly as possible (D2). Probably, they understand the material presented in this fashion better.

***C. Conceptual domain** – mathematics learning*

This conceptual domain was measured by the following 8 items:

L1 all pupils understand

L2 much will be learned by heart

L3 as much repetition as possible

L4 only … talented pupils can solve it

L5 learning is not always fun

L6 it demands much effort

L7 as much practice as possible

L8 all the material … will be understood

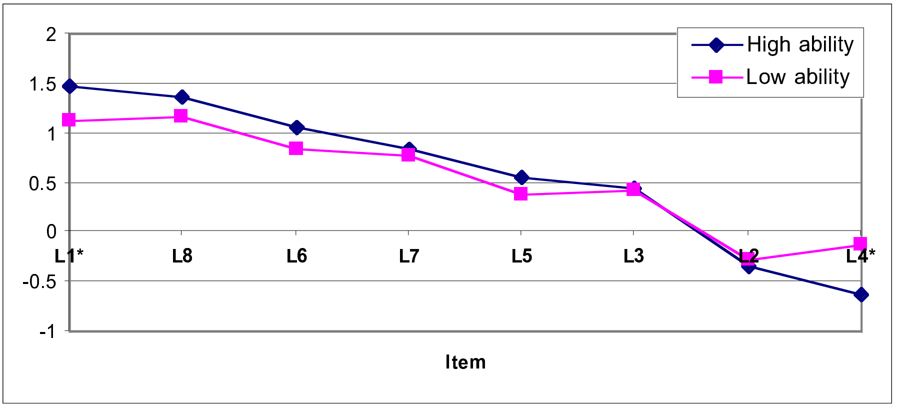

Figure 2

With respect to mathematics learning, both ability groups (Figure 2) consider understanding the material (L1, L8) to be of the highest importance. Interestingly, learning by heart is opposed by the low-ability pupils in the same way as by the high-ability pupils. A closer analysis (Figure 3) reveals that:

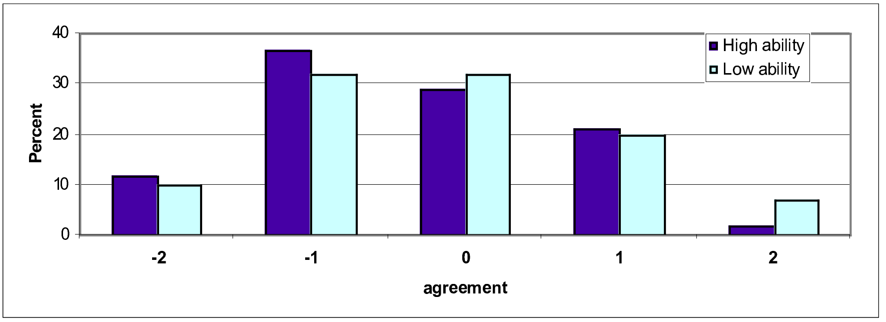

Figure 3

both of the groups include more than 40% of those who are opposed to learning by heart, and more than 20% of those who consider it necessary. However, there are significant differences in the attitude of the pupil groups towards the relative importance of ability. Namely, both the low ability and the low achievement groups (Figure 4) tend to agree with statement L4 - that only talented pupils can solve most of the tasks. Nevertheless, the average opinion on this statement stays fairly close to neutral in both cases, with 38% of the low-achievement pupils and 32% of the low-ability pupils having responded to this questions with the option "0"- “I do not know”. A tendency can be observed that the pupils of the highest ability believe more than those of low ability that all the pupils are able to learn mathematics and that success in learning is possible if effort is made. They have a somewhat higher degree of self-perceived ability and desire to learn mathematics (L5, L6). The low-ability pupils, however, are more disposed to give up.

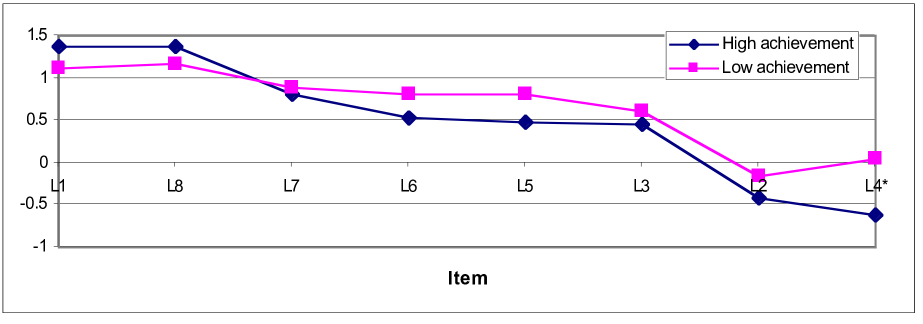

Figure 4

***D. Conceptual domain –** mathematics teaching*

T1 pupils … guess and ponder

T2 right answer quickly

T3 strict discipline

T4 pupils put … own questions

T5 teachers help … when in difficulty

T6 learning games

T7 teachers explain every stage

T8 pupils solve … independently

T9 working in small groups

T10 teachers tell exactly what to do

T11 pupils can work according to abilities

In this domain, the opinions of different ability groups differ the most (Figure 5). The greatest differences for the entire questionnaire are manifested in the responses to statement T11: high-ability pupils are considerably more willing to agree to the view that everyone should be able to work according to his or her ability. Thus, it is the more able pupils that need a greater differentiation of teaching. Surprisingly, the views of the two extreme achievement groups on statement T11 are identical, whereas those of the two extreme ability groups who differ significantly in this respect. Both the ability and the achievement groups approve of process-orientated teaching, but their conceptions concerning some aspects of this process are different. The high-ability and the high-achievement pupils appreciate significantly more the fact that pupils are allowed to (try to) solve problems as independently as possible (T8); they want to make their own guesses at the solution of a problem (T1). At the same time, the low-ability and the low-achievement pupils are more expectant to receive help from the teacher (T7, T10).

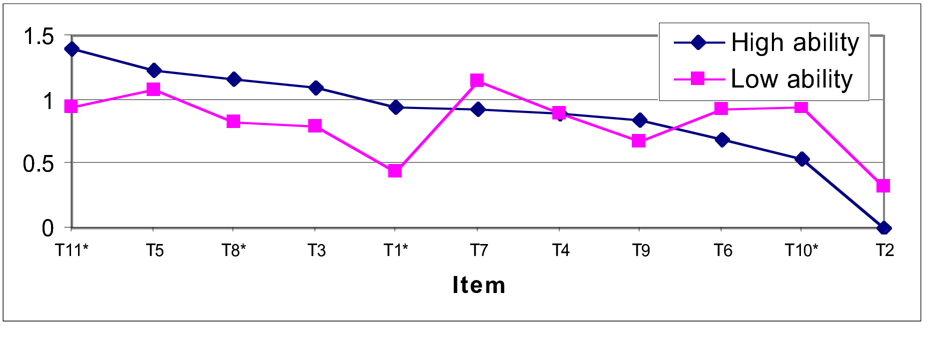

Figure 5

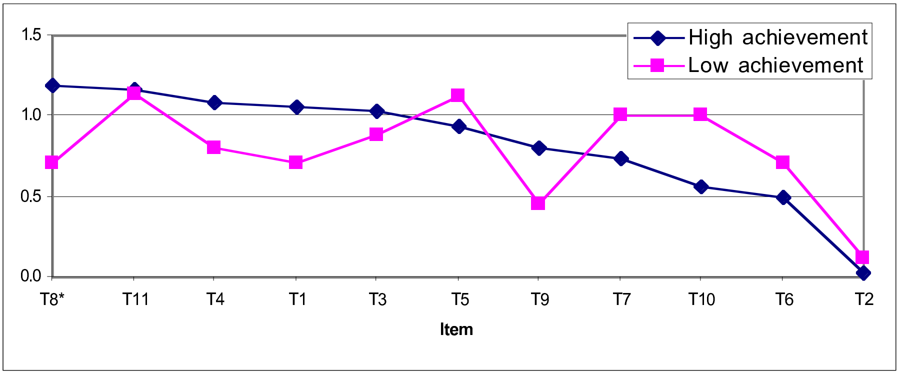

Figure 6

Conclusions and Discussion

The findings of the study support the view laid out in the introduction that pupils' conceptions of mathematics and the learning of mathematics are generally correlated with not so much their individual ability and achievements as the general paradigm of mathematics teaching prevalent in a given country. However, certain differences based on achievement and ability may still be observed among the pupil groups examined in our study.

The study revealed that high-ability pupils have considerably greater faith in achieving success in mathematics learning than low-ability pupils. Compared to other pupils, high-ability students are considerably more desirous of each pupil being able to work according to his or her ability. They want to develop their ability and are ready to do more work in the name of success. Low-ability pupils, however, are more disposed to give up than pupils with high ability.

The greatest differences between the students of low and high ability lie in the determination of a pupil's independence and activity by the pupil himself. Low-ability pupils are considerably more disposed to receiving explanations from the teacher while high-ability students desire to work independently. Nevertheless, several studies indicate that all students must be actively involved in their learning. Under such organization of the learning process, students are able to achieve higher grade point averages than under traditional learning methods (Alper et al., 1997). As students work a task, the teacher must give them freedom to solve the problem in their way rather than in some predetermined way. However, the teacher's duty is to organise independent work in such a fashion that each pupil (more able and less able alike) receives instructions from the teacher to the extent necessary for him or her.

Understanding the subject is considered equally important both by less able and more able pupils. This finding was also confirmed by an analysis of the part of the questionnaire where free answers were allowed. Sometimes, however, teachers do not believe that each pupil is able to understand the material. For instance, C. Römer writes that in the opinion of a pre-service teacher only 50% of the students have the chance to learn something in mathematics lessons by understanding. The rest have to learn by heart only to survive his time at school (Römer, 1997). Nevertheless, the pupils' responses demonstrate how important it is for the teacher to look for a suitable way for less able pupils also to be able to understand the subject they learn. For instance, one of the low-ability students in our study claims that *mathematics can be very interesting if I understand everything.*

The conceptions of the extreme achievement groups broadly coincide with those of the extreme ability groups. Significant differences were manifested in two domains: a) mathematics teaching and b) the content of mathematics.

Regarding the ways of mathematics teaching there were no significant differences in the opinions of the extreme achievement groups whereas the extreme ability groups evidenced statistically significant differences in four of the eleven items. All these items were related to the teaching methods encouraging independent learning; in this respect, the high-ability students preferred teaching methods granting them greater freedom. This may imply that teaching methods presupposing freedom of activity are particularly necessary for mathematically talented pupils. Pupils whose mathematical achievement is based not only on innate ability but also other factors are less enthusiastic about teaching methods that give them a free hand.

Regarding the content of mathematics, however, the only significant differences were those between the two achievement groups. They concerned the importance of calculation skills in mathematics. The high achievement group considered these skills an important feature of good mathematics teaching. The establishment of the type of high-achievement pupils who consider the above feature particularly important would require further research.

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Algebraic Knowledge in Upper Secondary School

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Abstract

In August 1998 we started a longitudinal study of how students in upper secondary school develop their knowledge of algebra. The main source of inspiration for this study was an animated debate that was going on in the end of 1997, on beginner university students' skills – or rather lack of skills – in mathematics. We have now followed about 100 students at the science programme of Klippans Gymnasieskola during their three years in upper secondary school. In this paper we give a summary of the study and present some preliminary results.

Background

It is a well-known fact in most countries for researchers as well as teachers, that the learning and teaching of algebra is difficult. Many students – perhaps most of them – find it hard to learn algebra and the outcome of the teaching is often depressing. On the other hand proficiency in algebra is a critical filter for successful studies in mathematics, science, engineering etc. and most failures on undergraduate level in these careers are probably due to lack of algebraic ability. Consequently there is a great interest in improving the learning and teaching of algebra. Even if we still wait for a general and testable theory in the field there are some results of interest for classroom practice. However, much work remains to be done, and it will certainly take a long time to understand thoroughly how students learn algebra (Kieran, 1992; Bednarz et al., 1996; Bergsten et al., 1997).

Algebra in this context is traditional school algebra. Typical topics are simplification of algebraic expressions, linear and quadratic equations, simple systems of equations and simple functions. One way to structure school algebra is to use the three different phases of the algebraic cycle (cf. e.g. Bell, 1996, p. 185):

- translation from a problem situation to an algebraic expression (representing)
- transformation and simplification of an algebraic expressions (manipulating)
- interpretation of an algebraic expression (interpreting).

In traditional algebra teaching much emphasis has been on transformations, but we think all the three phases are equally important. One reason for this is modern technology. Computers and calculators can manipulate algebraic expressions, but they can't translate and interpret them.

So far results from this study have been presented in four reports (Persson & Wennström, 1999, 2000a, 2000b, 2000c). As the students have now (June 2001) left upper secondary school, the acquisition of data is completed. Some further analysis of the material collected remains. Results and a summary of the whole study will be published in a fifth and final report. This is expected to be available in the end of this year.

Questions of interest

Our perspective is that of the practitioner – the teacher in the classroom. Therefore our main concern is how to improve our students' learning of algebra and our own teaching practice. Of course a lot of different questions can be posed in this context. We have focused on the following ones:

- What skills of algebra are necessary today and which of them are most important? What by hand skills must the students know and how certain must they be in performing them? What should comprise a minimal curriculum?
- How will modern technology affect the learning and teaching of algebra? What algebra should be learnt, when relatively cheap calculators that can perform symbolic calculations are available? How can we use the technology to improve the understanding of algebra?
- What factors facilitate the learning of algebra?
- What factors obstruct the learning of algebra?
- What skills of comprehensive school mathematics are most important in learning algebra?
- What factors make some lower-ability students – having initially many difficulties in learning mathematics – succeed in the end?

Methods

The study is mainly a qualitative one although some quantitative results will be given. To collect data the following methods have been used: tests, questionnaires and other written material, interviews and observations of the students. As we have been teaching about half the number of students ourselves, it has been possible to observe these students almost daily.

By testing at intervals we have been able to monitor the development of the students' knowledge of algebra. The first test was given when the students began upper secondary school in August 1998. They also answered a questionnaire and some (about 20) were interviewed. The purpose was to classify their skills and attitudes at the beginning of upper secondary school (Persson & Wennström, 1999).

The second test and questionnaire was given a year later in the beginning of September 1999. A lot of other written material from the students' first year was also collected. Using all this material we could see if and in what way the students' algebraic knowledge had changed during the first year (Persson & Wennström, 2000a, 2000b).

In the end of the second year the students were given a third test (Persson & Wennström, 2000c) and a fourth test was given in the end of their final year. The main purpose of these two tests was to study what knowledge of algebra the students retain.

Some methodological problems should be noted:

- The amount of data collected is huge. How are we supposed to select and structure the material in order to get the most interesting and relevant information? How to avoid bias?
- As we have been teaching some of the students, we have been both observers and participants. How does this affect our objectivity? All observers are more or less subjective, but how do we minimise the subjectivity?
- Our study is performed in a certain environment with a limited group of students. How general are our findings?

These problems are not easily resolved. All experimental educational research is facing them and it is always necessary to stress that the findings of such research never have the same reliability as e.g. those in physics (cf. Wheeler, 1996, p.147). One way – maybe the only one – to test the plausibility of one's findings is to compare them with those of others – scholars as well as practitioners.

Results

To give the reader an idea of the tests used in the study we present here the fourth and final test. Some results with comments are given and comparisons are made with the other tests. Some facts should be stressed:

- The students' learning situation has not been favourable. The curriculum of 1994 has been too ambitious. Too much material (in algebra, trigonometry, calculus etc.) has been covered in too short a time. For most of the students this has resulted in superficial and procedural learning without a deeper understanding of many of the concepts discussed. Both knowledge and skills have been affected negatively by the shortage of time. (In a new curriculum more time will be allocated to mathematics.)
- Because of practical and administrative problems it has not been possible for all the students to perform the test. 74 out of 92 have done it. We think this is enough to give a true picture of the situation. Besides we have a lot of other material to check the reliability of our findings.
- In a longitudinal study too many tests, questionnaires etc. may wear out the students and as time goes by they become less motivated to participate in the study. This has not been a great problem in our study. Most of our students have been very co-operative and made a real effort. Nevertheless we have encountered some small problems of motivation on the last two tests

The test consisted of 20 questions. It was a paper and pencil test. Calculators were not permitted. On 17 of the questions (1a-8) an answer should be given and on 3 (9a-c) a solution. The figures given below (in brackets) are the number of correct answers/solutions in percent.

1. *Solve the following equations*

   a) $4x - 15 = 75 - x$ (72%)

   b) $\frac{x}{5} - 6 = 14$ (86%)

   c) $x^2 + 3 = 7$ (70%)

   d) $(x-3)(2x+1) = 0$ (28%)

Most wrong answers on question 1a and 1b are slips. Almost all students know how to solve these types of equations. On the first test there were similar equations e.g. $4x - 15 = 75$ (71%) and $\frac{x}{5} + 6 = 14$ (81%). The conclusion is that students know how to solve simple equations already when they leave comprehensive school and that they maintain and improve this ability in upper secondary school.

Question 1c shows that most students know the meaning of a solution of an equation and can find its root by trial and error. Comparison with a similar question on the first test $2^x - 1 = 7$ (51%) indicates some progress.

The results on question 1d are a bit disappointing. Many students multiply the parenthesis and then they use a formula to solve the quadratic equation. They get $1.25 \pm \sqrt{1.25^2 + 1.5}$. But without a calculator they get stuck. On this question most students show more instrumental skills than a real understanding of equations.

2. *Simplify the following expressions*

   a) $3(4-3x) + 4(3-4x)$ (77%)

   b) $3(4+3x) - 4(3-4x)$ (88%)

   c) $(2x-5)(3x+4)$ (84%)

   d) $\frac{2x^2 - 8}{x - 2}$ (22%)

If we compare question 2a-c with similar ones on the first two tests (figures in brackets are results of the first and second test),

$10x + 3(4-3x) + 8$ (54%, 88%)

$10x - 3(4-3x) - 8$ (42%, 77%)

$(2x-5)(3x+4)$ (19%, 79%)

we observe that the improvement in transforming simple algebraic expressions the students achieved during their first year in upper secondary school is retained. This supports the opinion Ekenstam and Greger expressed as early as 1987:

> The customary way of teaching elementary algebra has to be reassessed: algebraic transformations and manipulations, described at the beginning of this article seem to be introduced too early and too quickly. (Ekenstam & Greger, 1987, p. 312)

Our study strongly indicates that most of algebraic transformations and simplifications could wait until upper secondary school. The students should meet algebra in compulsory school but other aspects of it as understanding of variables and the use of letters in algebraic expressions should be stressed.

The very depressing result on question 2d was expected. When the students start upper secondary school most of them lack both understanding of rational numbers and skills in using them. As there hasn't been enough time to learn this properly in upper secondary school it has been more and less pointless to work with rational expressions.

3. *What can you say about c if $c+d=10$ and $c<d$?* (61%)
4. *Which is larger 2n or n + 2.? Explain!* (41%)

These two questions test the interpretations of letters on the highest levels according to the classification by Kücheman - letter as generalised number and letter as variable (Kücheman, 1981). Most wrong answers also indicate some understanding e.g. “$0 \le c \le 4.9$ *or* $1 \le c \le 4$” on question 3 and “*if* $n > 1.5$ $2n > n+2$” on question 4. The conclusion is that most students have reached the highest levels according to Kücheman's scheme.

5. *At a school there are 9 times as many students as teachers. Find a formula between the number of students S and number of teachers T.* (59%)

This is the classical Student Professor problem (cf. e.g. Kieran, 1992, p. 393). About 30% of the students give the wrong answer T=9S. That is something to reflect on.

6. *How many stars are there in*
   a) pic. 5 ? *(92%)*
   b) pic. 100 ? *(66%)*
   c) pic. n ? *(66%)*

* *	* * *	* * * *	* * * * *
	* * *	* * * *	* * * * *
		* * * *	* * * * *
			* * * * *
pic.1	pic.2	pic.3	pic.4

That almost 70% of the students can give a correct general formula is encouraging. Compared to a similar question on the first test it is an improvement.

7. *From a wire of 12-cm length a piece of x cm is cut off. From this piece of x cm a circle is formed and from the remaining piece of the wire a quadrate.*
   a) Find the radius of the circle expressed in x. (35%)
   b) Find the length of the side of the quadrate expressed in x. (69%)

It is a bit surprising that twice as many give the correct answer on question 7b than 7a. A general reflection – supported by results of other tests and many observations – is that the ability to translate from a problem situation to an algebraic expression could be improved.

8. *Apples cost a kr each and pears b kr each. What does the expression 3a + 5b mean?* (92%)

The result of this question shows that almost all students can interpret simple algebraic expressions. Other material supports this conclusion.

9. *Solve the following equations.*

   a) $2x^2 + x - 3 = 0$ (12%)

   b) $4(x-1)^2 - (2x-1)^2 = 39$ (31%)

   c) $\frac{3}{x-2} = \frac{2}{3}$ (43%)

The result of question 9a may seem very depressing. Yet the students know how to solve a quadratic equation. Many of them get the expression $-0.25 \pm \sqrt{1.5625}$, but without a calculator they can't go on (cf. question 1d). The difficulty is avoided if one uses fractions instead of decimal numbers, but as our students haven't much practice to use fractions they avoid this.

The mistakes on question 9b are mostly small ones. For example, $12x + 3 = 39$ gives $12x = 42$ and $4(x^2 + 2x + 1) - (4x^2 - 4x + 1) = 39$ gives $4x^2 + 8x + 8 - 4x^2 + 4x - 1 = 39$. There are minor errors of plus and minus. Most students know how to simplify expressions of this kind. They have an understanding, but lack the certainty to perform by hand calculations correctly. More drill could be a remedy. But is this certainty important today, when we have calculators and computers that can perform symbolic calculations? Can't the time available be used in a better way e.g. to improve the students' understanding of mathematical concepts and their ability to solve problems?

Summary

Let us sum up the major findings of our study so far:

- What makes some lower-ability students succeed is a very complicated interplay of cognitive and affective factors. However, one very important factor for success – perhaps the most important one - is that both the student <u>and</u> the teacher believe it is possible. Another is that the student is allowed to start from what she/he knows and not from what she/he is supposed to know.
- Important skills of comprehensive school mathematics are good number sense, understanding of variables and the use of letters in algebraic expressions. This knowledge is more important than being able to manipulate algebraic expressions. Often simplifications and transformations are introduced too early and too quickly. The students don't understand what they are doing and the answers are often haphazard.
- Affective factors as motivation and self-confidence are very important in learning algebra.

- Acquiring knowledge of algebra often occurs in leaps. There are thresholds to be passed.
- It is absolutely necessary to analyse the student's mistakes thoroughly to be able to find the right remedy. If e.g. the student makes mistakes in manipulating algebraic expression and the errors are mainly due to difficulties in handling negative numbers, it is pointless and might even be harmful to practice more manipulations. It is not possible to go on until the student masters negative numbers.

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On the Aim of Service Mathematics at Universities in Changing Circumstances

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Abstract

In universities the teaching of service mathematics often encounters negative attitudes as if it were redundant in the age of computers and, moreover, mathematics is not easy to learn and understand, especially if the starting level is not high. Here some objectives for learning mathematics, apart from that of gaining general mathematical knowledge, are formulated and some ways of achieving them are described. It seems that if a student espouses the concept of lifelong study, then through the formulated objectives and effective study methods it is possible to attain a result of greater generality: the student should be able to continue to handle mathematical materials and help himself in applications that may arise.

Introduction

The part of mathematics used in applied sciences and therefore taught at universities to students of numerous specialties had remained relatively stable for a long time in all countries, including Estonia. However, the opportunities of the past decade to lead our own political and economic life have led us to a need to reform our educational life as well – to give sense to the nature of education under the new conditions and to make any corresponding changes in the contents and methods of what was being taught. Grounded on long-time experience in teaching mathematics at a tertiary institution inclined towards applied science, being aware of the basic mathematical topics required in applications, and having observed how the knowledge acquired by pupils at school as well as their readiness for acquisition of new mathematical knowledge has been changing, we have had to conclude that the gap between the needs and the possibilities was widening. A lot of work has already been done in Estonia on the rearrangements of curricula and course systems in schools of general education as well as universities, but it is not clear at all, due to the inertia inherent in education, whether these changes are working in the desired direction. In the hope of generating ideas conducive to progress, let us note some observations and make some suggestions based on experience.

In a computerised world it is important to educate the public that mathematics is not merely computation, but a special system of knowledge developed to help mankind answer questions it has faced (McComas, Almazroa and Clough, 1998). It serves that same basic purpose today. This is especially true from the viewpoint of applied sciences, which it serves as the language of problem solving, a language that is itself in need of study (Lozinskaja, 1999) and further development for its own internal requirements. Without doubt, only interested mathematical enthusiasts work in this field, and here we will not deal with specific problems of their preparedness for this task, but rather direct our comments for consideration to fellow educators who work in schools of general education or with tertiary students of applied subjects.

Objectives for learning mathematics

The purpose of teaching mathematics in general education is the familiarisation of the student with basic mathematical concepts, operations and facts, all of which set the ground for immediate application of such knowledge as well as for the development of logical thinking and a scientific outlook. No doubt this purpose will remain. However, use of a wider context, i.e. reference to life around us from which to seek and find mathematical problems of suitable difficulty capable of solution, might facilitate acquisition of this knowledge. At every stage, in the ideal case, one wants to expect that the student develops a world outlook conducive to a more favourable attitude on the part of the general public towards mathematics and its study (Kahn, 1999). This should certainly be an aim in universities. Ever decreasing time availabilities at all stages of the educational road create a difficult situation for first-year tertiary students and the staff working with them, due to a continuous decline of basic mathematical knowledge of the entrants. It is therefore more imperative for universities to define their objectives to which they aspire with any particular mathematical study course and which take account of both the knowledge level of the student as well as his preparedness for serious work, and yet allow him to succeed in meeting future challenges of life. We have in mind the following objectives:

1) ability to see and enunciate the problem;
2) ability to formalise the problem;
3) ability to search for a method of solution;
4) ability to assess the truth value of the solution and to interpret the result.

It seems that this set of goals helps young people to acquire the concept of lifelong study and gives them experience in effectual study methods.

The opportunity for this lies within any kind of mathematics course. The teacher needs to take advantage of it and seek ways of making use of it even when the study programme follows traditional lines. For instance, he might try to achieve the result that, by the end of the course, the student should have acquired the skill to formulate and interpret a problem, and to find a method of solution at least with the aid of literature. This presupposes familiarity with theoretical material of a certain level and proper ability to read mathematical texts. The outcome of the solution should not be the goal in itself. Rather, the goal should be the ability to assess the truth value of the answer, and a skill to use or interpret it. Ways leading in that direction have been examined by Kahn (1999) and by Mousley (1999). At the Estonian Agricultural University we have some practical experience with these goals which essentially rests on the idea of discarding the usual view of mathematics as consisting by and large of mathematical analysis, algebra and geometry, the existence of whose inter-connections is not often given sufficient attention (Riives, 1999).

Our experience

Our one-year course of higher mathematics begins with chapters on linear algebra, vector algebra and analytical geometry. This provides an opportunity to revise and systematise the knowledge gained in high school and to bring it to a level from which one can advance either through additional mathematics or specialty courses. As the material is already largely known to our students, according to current high school curricula, it is possible to devote major attention to precisely the goals mentioned above. Often this presupposes, on the part of the students, a work attitude different from that which was manifest before – continual intensive independent work, to which they are led by home assignments on every integral topic treated. Difficulties that arise can be solved by way of individual counsellings, which can also provide an impression of how well the material has been understood. While questions on linear algebra are often easily done by algorithm and therefore the methods of solution are standard and, with proper equipment, capable of being handled by calculators or computers, problems in vector algebra and analytical geometry are different in that it is possible to refer to the use of handy material and to the importance of interpretation when seeking a method of solution.

Besides other existing reference material, we have employed special tables of the elements of theory. These point out the differences and similarities of corresponding operations and concepts, and draw useful comparisons. In dealing with topics in geometry, it is inevitable that one will visualise concepts and problems in terms of models, other handy devices or drawings. Seeing that problems in analytical geometry require application of methods of linear and vector algebra, and elements of vector algebra will render the solution of a problem very compact and elegant, we have here an opportunity to refer to mathematics as an integral discipline, in which familiarity with one part will help with another part. Also, in this discipline, the formulation of a problem may depend on the point of view. For instance, solution of a system of linear equations is usually given in algebraic terms, but in geometric terms the same problem means determination of the intersection of hyperplanes. This example allows us to speak of a linear equation as one determining a hyperplane or to view a hyperplane as the set of points whose coordinates satisfy a linear equation. No doubt it is possible to come up with many such relationships. The interpretation to be used would depend, on the one hand, on the particular problem under consideration and, on the other hand, on the capabilities and psychology of the working group. First and foremost, however, it would depend on whether the dominant way of thinking is analytical or visual. Which representation is to be preferred, in what part and to what degree, would depend on the emotional intelligence of the teacher, on the degree he knows and considers his students, their level of proficiency, their abilities and interests. The latter should rapidly become evident during counselling on individual difficulties brought to light by assignments.

The main part of a course on higher mathematics is usually devoted to traditional mathematical analysis – differentiation and integration of functions of one or several variables, topics already partly familiar from high school – as well as applications to solving differential equations and in the theory of infinite series. These latter endeavours frequently occur in the private studies and further work of university students. In these areas also, especially in clarification of basic concepts, a geometric approach is very useful. Most students of applied special subjects will arrive at a proper comprehension of a variable and its limit only when, alongside with a mathematically correct definition, they are given a geometric interpretation in which every detail in the definition is viewed in terms of the motion of a point on the real axis. When the need arises to clarify relations between extremal and stationary points of functions of several variables, formal definitions are often insufficient, though it is not difficult to learn them by heart. Considerable excitement and even surprise can be generated in students by treatment of these concepts in terms of examples involving graphs of elliptic and hyperbolic paraboloids. The exciting geometric features of the latter, when first mentioned, have even led to objections, which could be dispelled with the aid of thread models. It is particularly noteworthy that this could not be achieved by rigorous analytical presentation. With reference to the possibilities of use of such surfaces in civil engineering, one has gone a long way towards seeing mathematics as an exciting field of knowledge.

In many topics, in order to systematise the search for a method of solution, use may be made of a special syllabus-based questionnaire, which could lead to a considered choice. For instance, faced with a task of integration, we usually begin by determining the class to which the integrand belongs, after which it becomes evident whether the integral can be found directly from tables or reduced to a table integral by algebraic or trigonometric simplifications or whether a substitution or an integration by parts will supply the solution. The sequence within the questionnaire will eliminate arbitrary attempts at solution to which students are prone when the nature of the task has remained obscure. An ordered arrangement of various differential equations will serve the same purpose. It will lead to a method of solution based knowingly on facts derived from theory, in terms of their types. By analogous procedures it is possible to deal with all the topics in our programme. Such a search for suitable methods of attack could provide enjoyment to the teacher as well.

Conclusions

At the end of the study course it has been satisfying to observe that many students have sensed their ability in mathematics, have acquired a measure of experience in working with mathematical materials, and have learned to see possibilities of application in their areas of specialty. Not of least importance are the positive formal results at completion of the course. To sum up, the set goals are in many instances achievable in the manner described. Our conversations with students to test their knowledge help them to assimilate the material and to appreciate connections between different parts of the course. They could see the course as being an integral whole and also gain experience in precise thinking and expressing their thoughts properly. Often they would confess that the proposed work style helped them master the subject with understanding. It seems that through formulated objectives we can attain a general result: the student should continue to be able to handle mathematical materials and help himself in applications that may arise.

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Design of the System of Genetic Teaching of Algebra at Universities

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For teaching on the basis of the genetic approach, we offer to construct didactical system of study of a mathematical discipline (a part of a mathematical course, important concept or system of concepts) consisting of two parts 1) preliminary analysis of the arrangement of the contents, of didactical means and 2) concrete design of the process of teaching.

The preliminary analysis consists of two stages: 1) genetic elaboration of the subject matter and 2) analysis of the arrangement of a material and possibilities of using various ways of representation and effect on students. The genetic elaboration of a subject matter consists of the analysis of the subject from four points of view:

a) historical;
b) logical;
c) psychological;
d) socio-cultural.

The historical analysis frequently encounters with large complexities because of insufficient knowledge of the history of the origin and development of many branches of modern mathematics included in university curricula, inaccessibility of the literature on the given subjects. Therefore, it is necessary to conduct research of the history both of appropriate areas of modern mathematics, of their inclusion in the university curricula, to study educational literature, text- and problem-books, the history of the teaching of modern mathematics. As more or less accessible sources for the teachers and students the monographs and other scientific works – books and articles, books on the history of mathematics and mathematics education, manuals and encyclopaedias can serve. Very important is also to study original works of great mathematicians, classical textbooks, popular scientific literature, journal and magazine articles. The purpose of the historical analysis is to reveal paths of the origination of scientific knowledge underlying the educational material; to find out, what problems have generated need for this knowledge, what were the real obstacles in the process of the construction of this knowledge.

In designing the system of genetic teaching very important is to develop problem situations on the basis of historical and epistemological analysis of a theme.

The major aspect of logical organisation of an educational material consists in organising a material so that to reveal the necessity of the construction and of development of concepts and ideas. It is necessary to arrange problem situations or tasks, for which the important concepts or ideas, which should be studied, would serve as the best solutions. It is necessary to analyse those problems of knowledge, for which the considered concepts and ideas serve as the necessary solutions. For this purpose, both historical analysis and epistemological considerations, and special search for appropriate problem situations and tasks can help.

In our view, for the logical organisation of a system of concepts and propositions of a theme, of the teaching unit of a mathematical discipline, one should carefully analyse the logical structure of such system, for example, required, for example, for the construction of a concept or for the statement of a proposition. We will name the results of such analysis a logical genealogy of a concept or a proposition. In the university mathematics, especially in higher algebra, such genealogies may be rather complicated (see fig. 1).

Clearly, such complicated structure of concepts and statements, needed for understanding the theorems of large difficulty, requires well-designed activities for successful learning.

Therefore, very important is also the psychological aspect of the genetic approach to the teaching of mathematical disciplines.

The psychological analysis includes determination of the experience and the level of thinking abilities of the students (whether they can learn concepts, ideas and constructions of the appropriate abstraction level?), possible difficulties caused by the beliefs of the students on mathematical activities (for example, the students can bear from school views on mathematics as mere calculations aimed at the search of (usually unique) correct answers with the help of ready instructions etc.). The psychological analysis has also the purpose to plan a structure of the activities of the students on mastering concepts, ideas, algorithms, to plan their actions and operations, and also to find out necessary transformations of objects of study.

When studying university algebra courses, the students usually are encountered with sequentially growing steps of abstraction - with a «ladder of abstractions».

Stolyar (1986, pp. 58-60) has revealed 5 levels of thinking in the field of algebra and has noted, that “the traditional school teaching of algebra does not rise above the third level, and in the logical ordering of properties of operations even this level is not reached completely”. The following is the description of the third, fourth and fifth levels according to Stolyar (ibid., p. 59):

> On the 3-d a level the passage from concrete numbers expressed in digits, to abstract symbolic expressions designating concrete numbers only in determined interpretations of the symbols is carried out. At this level the logical ordering of properties is carried out “locally.

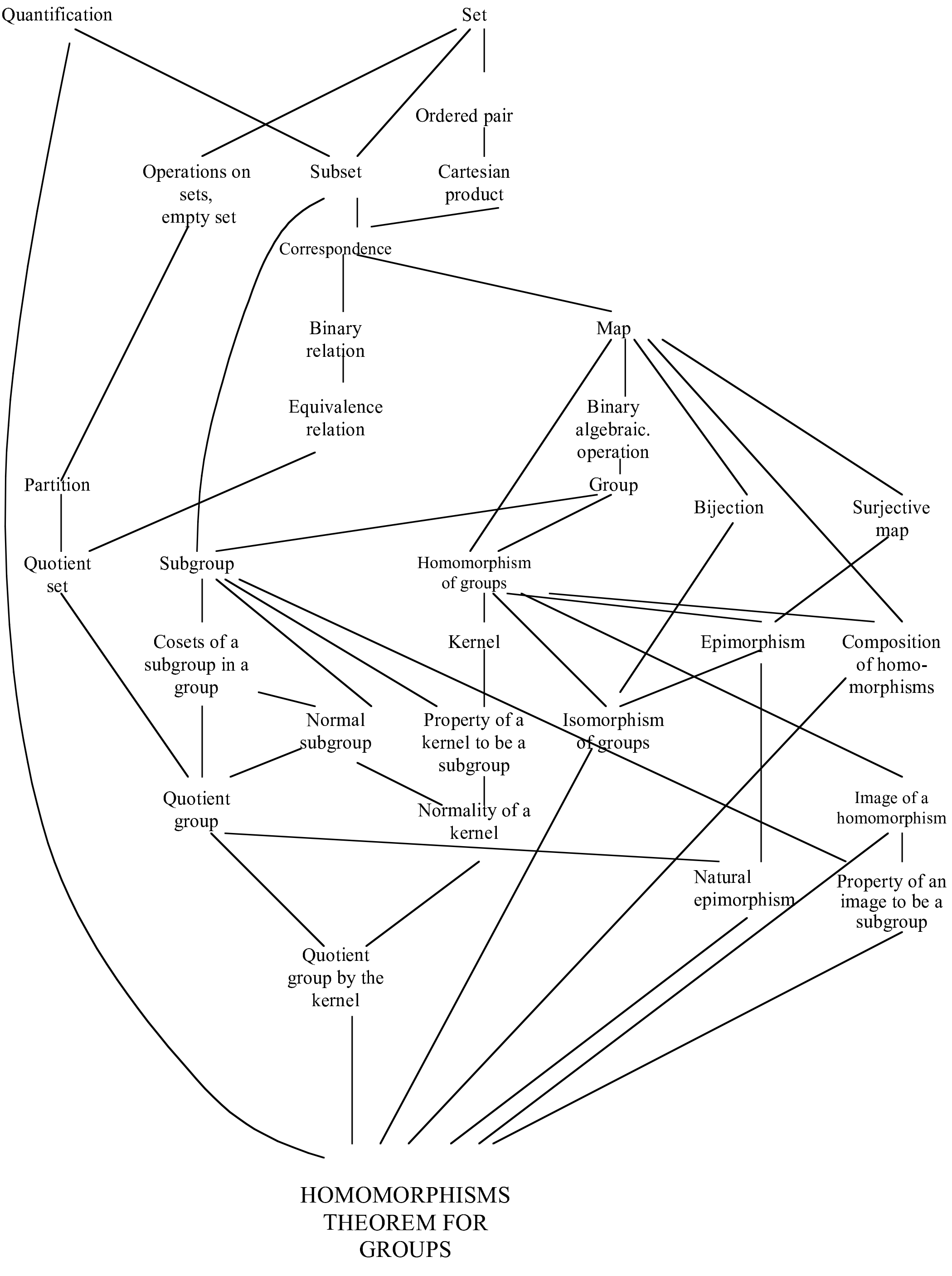

Figure 1. Logical genealogy of the homomorphism theorem for groups.

On the 4-th level the possibility of a deductive construction of the entire algebra in the given concrete interpretation is become clear. Here the letters designating mathematical objects are used as variable names for numbers from some given set (natural, integer, rational or real numbers), and the operations have a usual sense.

At last, on the 5-th level distraction from the concrete nature of mathematical objects, from the concrete meaning of operations takes place. Algebra is being built as an abstract deductive system independent of any interpretations. At this level, the passage from known concrete models to the abstract theory and further to other models is carried out, the possibility of existence of various algebras derived formally by properties of operations is accomplished".

Thus, to the 5-th level the deductive study of groups, rings, serially ordered sets etc. corresponds. The highest degree of abstraction here is the study of general algebraic systems with various many-placed operations.

To the 4-th level corresponds, for example, a systematic and deductive study of the sets of natural numbers or integers. Therefore, taking into account, that in school teaching even the 3-rd level is not completely reached, it would be certainly a big mistake to omit in pedagogical institutes the 4-th level (systematic study of an elementary number theory) and immediately pass to the deductive study of groups, rings and even of general universal algebras (as is done in a textbook by Kulikov, 1979). Therefore, systematic study of the elementary number theory can serve as a good sample of the construction of a deductive theory for preparation for the further construction of the axiomatic theories.

Stolyar built his classification of levels from the point of view of teaching school algebra. In our view, development of algebra as a science in the last decades (after the World War II, under the influence of works of Eilenberg and MacLane, 1945, and Maltsev, 1973) allows to distinguish one more higher, the 6-th level of algebraic thinking - we will name it the *level of algebraic categories*. At this level the entire classes of algebraic systems together with homomorphisms of these systems - varieties of universal algebras, categories of algebraic and other structures (for example, topological spaces, sets and other objects) are considered. Thus, the abstraction from concrete operations in these structures and from the nature of homomorphisms and generally of maps takes place; morphisms between objects of categories are considered simply as arrows subject to axioms of categories – for example, the associativitiy law for the composition. Moreover, the functors between categories – certain maps compatible with the laws of the composition of morphisms, and natural transformations of functors are considered.

Note that Piaget in the last years of his life was interested in the theory of categories as the highest level of abstraction in the development of algebra (Piaget & Garcia, 1989).

The teaching of algebra at this level (theory of categories and varieties of universal algebras) is not included into the obligatory curricula even of leading universities and happens only on special courses. But, nevertheless, the presence of this level demands that the students should master algebraic concepts in obligatory courses in a sufficient degree for understanding the algebraic ideas on the highest level of abstraction.

Essential in teaching algebra and number theory in pedagogical institutes are the 4-th and 5-th levels in the classification of Stolyar. First of all, the 4-th level (which is already beyond the school curricula) should be reached. Therefore, during the first introduction of the definition of group in the beginning of the algebra course, one should not immediately begin the full deductive treatment of the axiomatic theory of groups. Only after the experience of the study at the 4-th level of thinking in the field of algebra, namely of the study of the elements of number theory, it is possible to consider a deductive inference of the most simple constructions and statements of the group theory, and the systematic account of complicated sections of the theory should be postponed to a later time, after studying at the 4-th level of such themes as complex numbers and arithmetical vector spaces.

Piaget who developed the classification of levels for thinking in the fields of geometry and algebra (“intra”, “inter” and “trans”), noted that it is possible to distinguish sublevels inside each level (Piaget & Garcia, 1989).

According to the theory of Leontyev (1981), actions on learning concepts, as well as any actions, consist of operations, which are almost unconscious or completely unconscious. These operations are essentially «contracted» actions with the concepts of the previous level of abstraction. As Kholodnaya (1997) noted, «a contraction is immediate reorganisation of the complete set of all available … Knowledge about the given concept and transformation of that set into a generalised cognitive structure».

The theories of Dubinsky (1991) and Sfard (1991) are close to the Soviet conceptions of actions and operations as contracted actions in mathematics teaching.

In our view, for reaching a contraction of an action with algebraic objects into (automatic) intellectual operation it is necessary, after sufficient training with of this action, to include it in another action, connected with the construction of objects of the next step of abstraction.

One more purpose of the psychological analysis of the subject matter is finding out the ways of the development of motivation of learning.

The socio-cultural analysis has a purpose to establish connections of the subject with natural sciences, engineering and economical problems, with elements of culture, history, public life, to reveal, whenever possible, non-mathematical roots of mathematical knowledge and paths of its application outside of mathematics.

During the second part of analysis, considering the succession of study, it is necessary, in accordance with the principle of concentrism (Safuanov, 1999), to find out, on the one hand, which earlier studied concepts and ideas should be repeated, deepened and included in new connections during the given stage, and, on the other hand, which elements studied at the given stage, anticipate important concepts and ideas, which will be studied more completely, become clearer later, to check, whether there are possibilities of such repetitions and anticipations.

The principle of multiple effect requires also the finding out possibilities of multiple representation of concepts studied, of use of active, iconic and verbal-symbolical modes of transmission of information, of other means of effect on students (the style of the discourse, emotional issues, elements of unexpectedness and humour).

After two stages of analysis, it is necessary to implement the project of the process of study of an educational material. We divide the process of study into four stages.

1) Construction of a problem situation

In the genetic teaching, we search for the most natural paths of the genesis of processes of thinking and cognition.

According to the activity approach to the process of teaching, usually "the initial moment of the mental process is the problem situation … This problem situation involves the person in the thinking process; the thinking process is always directed to the solution of a problem" (Rubinshtein, 1989, p. 369). Therefore, the main purpose of the teacher is to construct a problem situation. The necessity of the construction of a problem situation was underlined by many prominent educators – by constructivists (creation of "disequilibrium") and representatives of the "French didactique" ("didactic engineering", directed on the creation of the didactical situations, on determination of the "epistemological obstacles") as well.

2) Statement of new naturally arising questions

According to the theory of the activity approach to teaching, "the arising of a questions is the first sign of the beginning work of the thinking and the first step to understanding … Every solved problem generates a lot of new problems; the more a man knows, the better he realises what more he should know" (Rubinshtein, 1989, pp. 374-375). Therefore, it is important, after the solution of the initial problem situation, to constantly consider new, naturally arising questions. It was well understood by Izvolsky (1924) in his version of the genetic approach. Thus, in the design of the process of study of a subject the statement of new, naturally arising, questions is necessary.

Actually, both stages – construction of a problem situation and the statement of new, naturally arising questions – are aimed at the same purpose - to help students in the independent mastering of a concept. Therefore it is necessary to organise a construction of problem situations and also statement of new, naturally arising questions in such way that in a certain moment of time (we will name such moment "the hour of truth") the students could, independently or with the minimal help of the teacher, discover the new concept for themselves. It is similar to the moment of the selection in a subject of "the initial universal relation", leading to the theoretical generalisation in the theory of learning activity of Davydov (1986, p. 148), and also to the act of reflective abstraction (as the of interior co-ordination of operations of the subject in a scheme) in the theory of Piaget (Dubinsky, 1991), and also to a moment of a *reification* (Sfard, 1991). Such organisation of teaching frequently may be quite difficult and not always completely possible. For this reason we admit appropriate help from by the teacher.

3) Logical organisation of an educational material
Here, after the problem situation has been dealt with, the paths of its solution, various aspects and natural arisen questions have been discussed, the appropriate motivation has been reached, the construction of the elements of the theory - precise definitions, statements (axioms and theorems), conclusions takes place. At this stage deductive reasoning plays the great role. In our approach this stage may be rather long in time and even, in the accordance with the principle of concentrism (Safuanov, 1999), divided in several stages – "coils of a spiral".

4) Development of applications and algorithms
After the logical organisation of mathematical objects of a studied theory, it is possible to consider various interesting and useful applications of the theory in practice and in mathematics itself. According to the principle of multiple effect (Safuanov, 1999), it is necessary to solve the sufficient number of exercises on the variations of signs of concepts, on the inclusion of concepts in new connections and contexts, on various transformations of mathematical objects under study.

During all stages of study of the teaching unit or theme it is important to help the students to develop their own language for expression of the reasonings and ideas. For this purpose each proposition (definition or statement) should be stated (at lectures and in textbooks), whenever possible, in various languages: logical-symbolical and verbal (this suggestion complies also with the principle of multiple effect).

It is necessary also to give the students the exercises on development of mental operations (analysis, synthesis, generalisation, comparison, analogy, abstraction and concretisation). For example, the exercises on extraction of conclusions from theoretical positions will be useful. Such exercises promote development of abilities of the synthetic reasoning.

Finally, it is very important to encourage reflection in minds of students, i. e. the ability to realise the foundations of their own activities, reasonings and conclusions, to be aware of the structure of their thinking process.

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Affect and Cognition – Two Poles of a Learning Process

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Abstract

The aim of this paper is to analyse the relationship between affect and cognition on the level of the individual learner. The concept of "affect logic" which combines the psychoanalysis of Freud with the genetic epistemology of Piaget, were employed. Within this concept, the psyche is understood as a complex hierarchical structure consisting of affective-cognitive schemata. These are the result of maturation and learning processes which are based on assimilatory/accomodatory interactions with reality. If we regard affect and cognition as two inseparable componets of a mental unit, we are in a position to understand why we can know anything about our feelings.

Introduction

Learning - especially learning of mathematics - was long considered to be only a problem of cognition. The restriction to cognition was helpful in constructing complex cognitive models of the learning process, but neglected the important role of affects. McLeod (1992) emphasised the importance of affect as follows:

> Affective issues play a central role in mathematics learning and instruction. When teachers talk about their mathematics classes, they seem just as likely to mention their students' enthusiasm or hostility towards mathematics as to report their cognitive achievements. (p. 575)

Concerning research on affects and learning of mathematics, McLeod's central demand is the linking of affective and cognitive components in models of the learning process and he also suggests that beliefs (about mathematics, self, mathematics teaching and the social context), attitudes and emotions should be important factors in research on the affective domain in mathematics education. McLeod (1992) distinguishes the following dimensions of variations in affect: Intensity, directions (positive vs. negative) and stability.

Beliefs and attitudes are seen as "cool" and "stable" whereas emotional reactions are "hot" and "unstable".

Many investigations show an interrelation between affects (beliefs, attitudes and emotions) and performance in mathematics. There are also some models that explain the influence of factors that lead to beliefs, attitudes and emotions, but the origin of the psychic structure in the background is not part of these models.

Affect logic - a concept combining affect and cognition

The Swiss psychiatrist, L. Ciompi, developed a concept that combines Freud's psychoanalysis and Piaget's genetic epistemology on the basis of system theory (1982, 1988, 1991, 1999). The following is a brief summary of the most important aspects of the affect logic:

* The psyche is seen as a unit. This means affect and cognition, feeling and thinking are inseparably combined, though dissimilar in nature.

* The psyche is understood as a complex hierarchical structure consisting of affective- cognitive schemata. These affective-cognitive schemata are the result of maturation and learning processes based on assimilatory/ accommodatory interactions with reality.

* The affective-cognitive schemata are condensed to affective-cognitive reference systems that form an individual's "world view" and control acting and thinking.

* The affective-cognitive reference system is structured by both the affective and the cognitive components. Therefore, access is possible through both components, but the access is often not complete because certain parts of the system are unconscious.

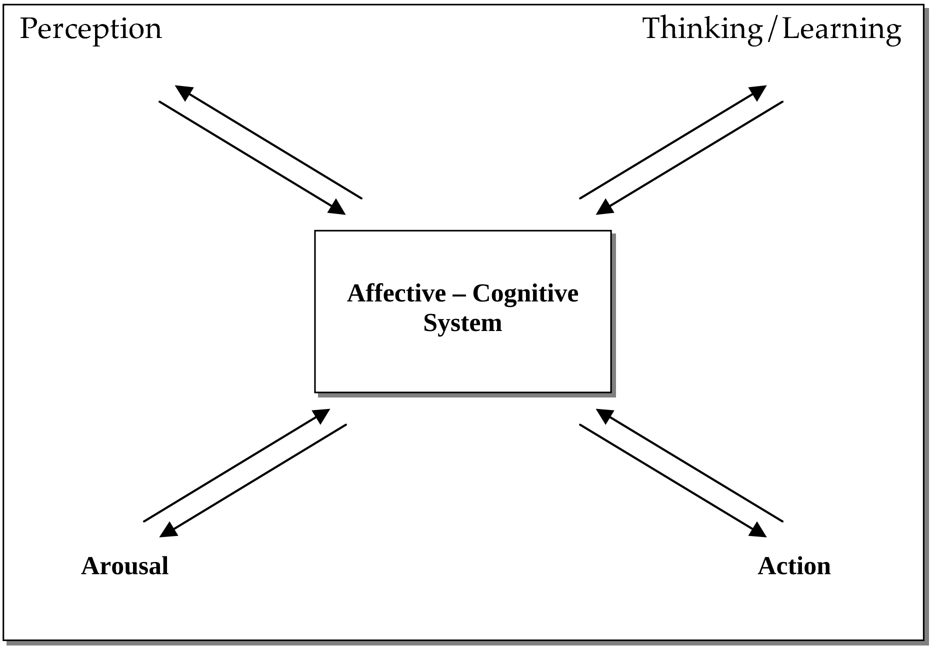
Remarks on the concept of affect logic

For Piaget (Piaget, 1995, p25), affect is only an energy supplier:

> The energetic of behaviour arises from the affectivity, whereas the structure comes from the cognitive functions. (Translation W.S.)

Ciompi's concept extended the influence of affects to general effects, all of which have affects in thinking as well as special effects consisting of the basic feelings (interest, anger, fear, sadness and joy (Ciompi, 1999). In general, affects influence cognition like operators. They provide the energy through which cognitive processes are motivated or hindered. They control attention and memory processes, and influence the hierarchy of cognitive schemata. On the other hand, Ciompi speaks of special "logics" of the basic feelings (for instance, "fear logic" or "logic of joy or anger"). The term "logic" expresses the view that the kind of thinking is different if a special feeling dominates. The graphic below (an adaptation of Case et. al., 1988), illustrates the influence of affects.

The arrows in both directions mean that the affective-cognitive system influences the perception, body reactions, actions, thinking and learning, and that all these, in turn, also influence the affective-cognitive system. To study these, we go back to Piaget's concept of equilibrium, assimilation and accommodation. The central idea is as follows. On the one hand, all systems make an effort to attain and remain in equilibrium with their environment; on the other hand, systems must interact with their environment, but these interactions can disturb the equilibrium. To restore the equilibrium, system-specific reactions are required. These system-specific reactions are "assimilation" and "accommodation".



Piaget distinguishes a number of forms of assimilation. The simplest only needs a simple application of a scheme in the cognitive system. Each new successful application of a scheme extends the application field of the scheme and leads, therefore, to generalisation. But in this simple case, no change of the scheme is required in turn.

The second form of assimilation is the so-called "reciprocal assimilation". This kind of assimilation leads to co-ordination of subsystems or to the integration of a subsystem into a more general system. Piaget (1976, p. 14) formulates these properties of assimilation processes in a postulate:

> Postulate 1: Every assimilation scheme has the tendency to grow, i.e. to assimilate elements that are distinguishable to its nature. (Activity is necessary for a subject.)

Accommodation is the second system-specific type of reaction of a system to environmental requirements. An accommodation process is necessary if a problem is not solvable by assimilation, and requires an alteration of the system. Piaget summarises the relation between assimilation and accommodation in a second postulate:

> Postulate 2: Every assimilation scheme is forced to accommodate the scheme on the specificity of elements. These elements are now assimilated. This postulate requires the necessity of an equilibrium between accommodation and assimilation.

The last part of the postulate means that the goal of all assimilation/accommodation processes is a new, more stable equilibrium.

The concept of affect logic transfers Piaget's system-specific reactions (assimilation and accommodation) to the development of the affective part of an affective-cognitive scheme. If Postulate 1 sees activity as necessary for each human, activity must be seen as emotionally valuable. The basic feeling of "interest" provides, in principle, the positive energy required for discovering new things. In evolutionary terms, humans are open to discovering their environment, to acquiring new experiences - i.e., to learning. The basic feeling of "interest" also motivates learning processes. This readiness, which exists in principle, to discover the "world", to learn new things, can be disturbed by negative experiences in relation to earlier learning processes. In these cases, negative feelings slow down new learning processes and can turn into an obstacle for further progress.

For Piaget, the simplest form of assimilation is a process that only requires the application of an existing cognitive scheme to a new situation. Repeated application leads, in the cognitive case, to stabilisation of the cognitive scheme. The person thereby acquires a routine for solving a special problem. In the affective case, the origin of a scheme as the product of a successful problem solution process is connected with positive feelings. Successful thinking processes and learning processes are delightful. But repeated application of a scheme leads to "emotional neutralisation" of the positive feeling, in some cases to negative feelings. We know, from our experiences with everyday life actions, that these actions are often routines, and that these routines are combined with a low level of emotionality.

On the other hand, we ought to note that every successful application of a scheme to a new situation leads to an extension of the application field, and that the scheme is generalised as a consequence.

If we consider the situation in the case of reciprocal assimilation (the second form of assimilation) from the affective perspective, we find a much more complicated situation. The integration of subschemata into a more general scheme, or the assimilation of a scheme into a more general scheme, is a very complex cognitive as well as affective process. This process is closely connected to the problem of context-related learning and school learning, and is founded on the question of the relation between the special and the general. This is currently being intensively discussed, especially in the case of mathematics learning. Here, the clash is between mathematics as a special tool to be applied only in a special field, and mathematics as a general tool for many problem situations (Lave/ Wenger, 1991; Evans, 1999).

From the affective perspective, a process as difficult as reciprocal assimilation needs strong support from the basic feeling of "interest". But the process is very sensitive to failures, and can, following unsuccessful results, lead to negative feelings.

In my opinion, the process of abstraction and generalisation is one of the most difficult steps in a mathematical learning process. It is one of the origins of negative beliefs and attitudes in relation to mathematics.

Beliefs and affect logic

Belief is a very important concept in the affective-cognitive field, and is used by many researchers. (For a survey of the different aspects of the meanings of mathematical beliefs, see Pehkonen and Törner, 1996.) One of the open questions posed by Pehkonen and Törner in their summary is:

> How do pupils' mathematical beliefs develop under school instruction, and which are the most influential factors in this development?

Furinghetti and Pehkonen (2000) describe the function of beliefs in the following way:

> (a) beliefs form a background system regulating our perceptions, thinking and actions; and therefore, (b) beliefs act as indicators for teaching and learning. Moreover, (c) beliefs can be seen as an inertial force that may work against change, and as a consequence, (d) beliefs have a forecasting character.

In the following, I will try to analyse these functions of beliefs from the perspective of affect logic. The affect logic postulates the existence of an affective-cognitive reference system consisting of a hierarchically structured system of affective-cognitive schemata. This affective-cognitive reference system forms, at each instant, the "world view" of an individual in respect of a special content. It controls the individual's perception, thought, learning and action. The development of this system and its affective-cognitive schemata is the result of maturation of the acting and learning processes based on assimilation and accommodation processes. We ought to note that the processes occur in the subconscious, and that an individual is conscious of only a part of their outcomes. Research into beliefs uses methods such as questionnaires and interviews, methods that only have access to parts of the affective-cognitive system that are available through introspection. Pehkonen and Törner distinguish between conception and primitive beliefs:

> Conceptions could be understood as conscious beliefs, and thus differ from so-called primitive beliefs, which are often unconscious. In the case of conceptions, we believe that the cognitive component is stressed, whereas the affective component is emphasised in primitive beliefs.

We must take into account the fact that we have no insight into the processes at the level of neurons and neuronal systems. At this level, all processes are subconscious. Our research methods provide us with knowledge of all the facts of the mental system that we are conscious of. Therefore, conceptions are beliefs for which we have rational arguments. The arguments explain the background and development of the conceptions in a rational way. For primitive beliefs, we don't have a rational explanation. We are partly conscious of them; they affect our acting and thinking, but their origin is unknown.

Summarising these reflections, beliefs are the conscious window to our affective-cognitive reference system. They contain our knowledge of the mental structure that regulates our perception, thought and action, and are therefore our indicators for learning and teaching.

An affective-cognitive reference system is the result of many assimilation and accommodation processes. In part 3, I discussed the development of routines. Routines are the result of many successful applications of a mental system. Every successful application stabilises the mental scheme. On the other hand, unsuccessful learning experiences lead to a non-stabilised cognitive scheme and to a negative emotional part. If there are many unsuccessful events in connection with a special content, then on the one hand, an unstable cognitive component, and on the other hand, a stable negative affective component of an affective-cognitive scheme, arises. In this case, the affective component leads to a refusal of all operations that are combined with this content. This is because of the fact that affects are very important for assessment. They are part of our assessment system (Roth, 1996), and the assessment system has a great influence on all thinking and acting processes.

Summarising these ideas, the affective component of the affective-cognitive reference system has a stabilising effect on the system, which may be negative or positive. Beliefs, in being the conscious expression of an affective-cognitive system, mirror this reaction of a system. Therefore, beliefs are able to provide information pertaining to the inertial force of an affective-cognitive system. The affective component often works against change if this change leads to more insecurity and uncertainty. On the one hand, the stability of affective-cognitive schemata as a consequence of development is responsible for the forecasting character of beliefs; on the other hand, this stability works against change. We ought to note that only stabilised systems lead to similar reactions in similar situations. Only stabilised systems are suitable as a basis for forecasting certain behaviour.

Social processes and beliefs

In the first sections, I tried to explain the mechanisms within the individuals that lead to beliefs. I did not discuss the influence of social processes. But all the processes take place within a society, within a culture.

> I start from the point of view that people are products of the multiple cultural and social situations in which we are born, grow up and develop. These include gender, ethnicity, class, sexual identity, religion, local community, etc. (Lerman, 2000)

If the social conditions are constitutive of learning processes, the concept of affect logic must be open to the frame conditions of social cognition. The research concept of social cognition is a constructivist concept and therefore assimilation and accommodation processes are part of this concept. In the following, we discuss the conditions of the affect development in a socio-cultural environment.

Characteristic of a social group is often its meaning and value system, as well as special rules for feeling - so called "feeling rules".

> Feeling rules' determine not only the expression of feeling, but also that which one is allowed to feel, or which should be felt in a society, in social position or subgroup. (Ulich & Kapfhammer, 1991).

The process of becoming a member of a social group is called "socialisation". The goal of this process is to "learn" the common, shared meanings, values and feelings of the group. The result of a socialisation process is a sample of meanings, values and feelings that are common to all members of a group and are used by all members in the same way. For all members of a group, this meaning, value and feeling system is an unproblematic background that guides the interpretation of special situations.

From the viewpoint of affect logic, people find it valuable to be member of a group, and therefore they have interest in learning meanings, values and feelings of this group.

Fennema (1989) developed a model to explain gender differences. In this model, external social influences have an effect on affects, and on mediated learning activities, and therefore on mathematics performance and on participation on mathematics courses. Evans (2000) generalises this model.

A difficult problem today is the fact that people belong to more than one social group (family, social class, school, job, ...) and are included in the "discourse practice" of all these groups. Mathematics can be part of different discourse practices, each with its own meanings and feelings of what constitutes structurally equal mathematical methods and terms. For instance, calculation methods are used in everyday life situations, in occupational situations, in democratic processes, in school task, etc. Furthermore, in many applications, mathematical algorithms are part of the context, and are often not seen as mathematics because the uniqueness of a situation allows an algorithm to be adapted to the particular need, without the user ever requiring the algorithm in an abstract form. Therefore, many people do not see the connection between school mathematics - with its general algorithms - and mathematics used in everyday life and occupational situations. On the borderline between structurally equal and contextually different mathematical applications, conflict situations arise that lead to negative affects towards mathematics. These affects are often specific to an entire group, and lead to commonly shared feelings towards mathematics. In particular, the group's understanding of mathematics can be very stabilising for this feeling. For stabilised emotional schemata, people often construct cognitive reasons to rationalise such feelings. Beliefs and attitudes, being a conscious part of affective-cognitive schemata, mirror in special way these stable emotions, as well as their rational foundation.

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Computer Represented Spatial Geometry

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Computer represention of spatial geometry is to become an emancipated mode of media specific representations in addition to the traditional ones. In this contribution some types of current educational software (Tool, Tutor, Internet) for spatial geometry in secondary education will be discussed. Included in this discussion is the design of a tutorial program of a new kind and its quantitative evaluation. This contribution closes with a perspective view of learning geometry in Cyber Space.

Introduction

Experience teaches us continually that representing spatial geometric facts on paper, at the blackboard or in the form of physical models is laborious and often unsuccessful even if certain techniques of representation or design have been practised. The flat representation of a spatial figure doesn't have any spatial depth; it is statically and hardly correctable; it cannot be "manipulated"; it can only be inadequately adapted to a learning and teaching process or a process of exploration etc. These weaknesses of conventional media favour the lesson time for plane geometry in comparison with the lesson time for spatial geometry (in the teaching curricula we mostly find only these topics that can be well created with the traditional media in the lesson)! Therefore the computer representation (including video representation) for spatial geometry is added to paper and pencil and print media representations and physical representations like physical models of solids. However, this causes new interface problems.

The diagram 1 illustrates the relation of the three modes of media specific representation with their corresponding interfaces S_i , S_i‘, i=1,2,3.

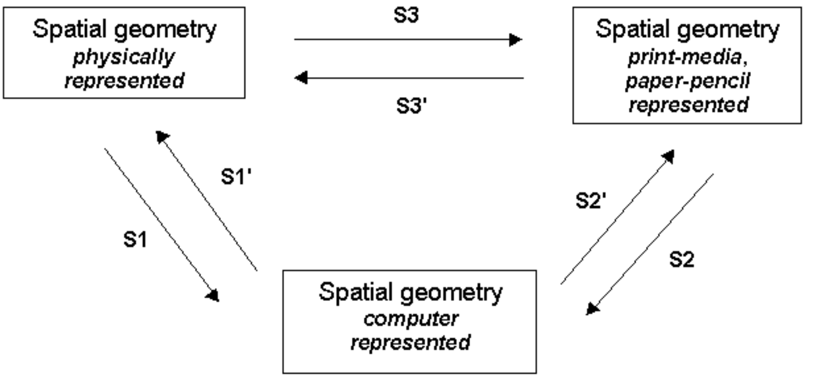

Diagram 1. Media specific representations

Comments about corresponding interfaces:

S_1/S_2: If the solid represented in a drawing or provided as a physical model is available but not already as a digital model in the computer tool, how can it be implemented? If necessary we must express and calculate the corners of the solid in question in a three-dimensional coordinate system (a laser beam scanning of material objects doesn't seem to be a practicable digitalization method in school geometry).

S_1^l: Geometrical screen representations can be printed and doc umented on paper in simple ways.

S_2^l: How do we get from a spatial object on the screen only accessible to the visual perception of a physical object which can also be referred to haptically – but not considering the perception possibilities of the Cyber-Space here? The practicable solution consists in the generation of solid nets on the screen, being able to be printed and then folding them up as surface models. However, this solution of this interface problem does not work for the sphere and its parts etc. - The methods of Solid-Imaging are still not available for school.

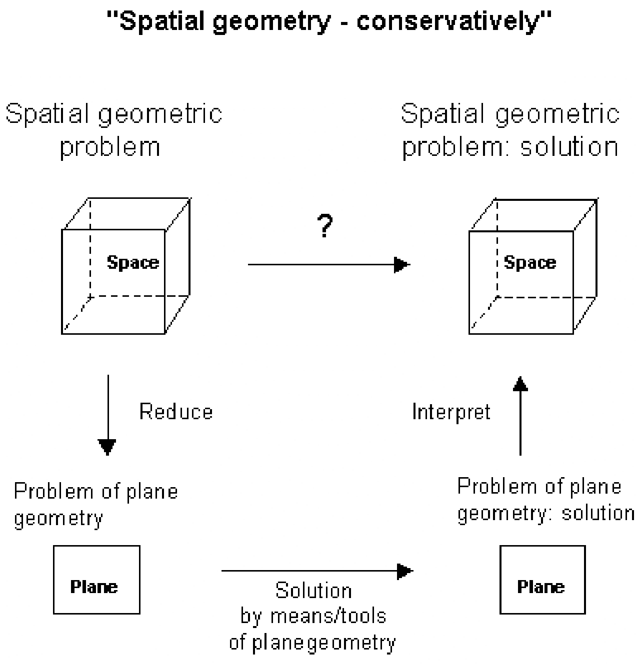

Diagram 2

Educational software for (synthetic) spatial geometry

Computer graphical tools for spatial geometry in middle schools

In general, we can solve spatial geometric problems in the conventional learning environment only by means of the solution of corresponding problems of plane geometry developed by methods of descriptive geometry (Diagram 2).

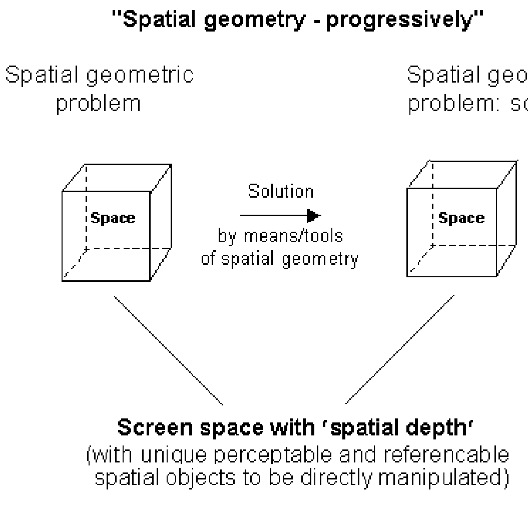

Diagram 3

With the computer use we have the possibility to produce and to represent spatial geometric configurations on the screen with virtual spatial depth and to manipulate these configurations directly (Diagram 3). This simplifies solving spatial geometric problems considerably and avoids the "traditional detour".

Two software developmental lines for computer graphical tools become apparent. On the one hand, tools are developed that are essentially restricted to the representation and processing of solids, and on the other hand, tools that allow essentially spatial construction and visualization like dynamic geometry systems for the plane.

To date the latter tools have been completed only for Macintosh (3D-Geometer, Klemenz, 1994/99) –with the deficit of the clear perceivability of spatial objects in depth (e.g. the relational position of the sphere and a straight line: all cases must be visually perceptible!). MiniGeometer, derived from 3D-Geometer, is a Java-applet for interactive construction in spatial geometry (http://geosoft.ch); it doesn't contain any improvement of the above mentioned problem of perception, and has a quite complex user interface which is a limitation of its use for middle and early secondary education. A hopeful development, the Macintosh program Cabri-Géomètre 3D, has not yet been completed.

Which essential requirements must now a computer graphical tool fulfil to be used for learning and teaching synthetic spatial geometry in secondary education that mostly deals with geometric solids?

- As a visualization tool it allows among other things the possibility to look at the standard solids of school geometry and in addition the solids derived from these solids as if one would have these solids as edge-, surface- or full solid-model in one's "hand". (This is managed by Virtual Sphere Device, this means that a referencable virtual sphere circumscribed around any solid can be arbitrarily moved with the mouse.)
- It makes possible the transition from only visually perceptible solids on the screen to the haptic perception of them. (This can be done by printing of solid nets and then folding them up.)
- As a measuring tool it allows the study of metric properties of a solid in various ways and allows the investigation of the true form of objects which are on or in a solid.
- As a construction tool it allows a flexible creation of solids by dissection, composition, deformation etc.

In addition it must be possible to draw figures into and onto the solids to make them carriers for further geometric information. Therefore the Windows program KOERPERGEOMETRIE (Schumann et al., 1999), which essentially fulfils the above mentioned requirements can provide:

- the demonstration of solid geometric facts
- the support of the knowledge of spatial shapes, the constructive representation, the calculation and the production of geometric solids
- the development and the training of spatial ability (here: ability to imagine spatial objects and relations between spatial objects)
- the experimental finding of knowledge (discovery of geometric statements, production of new solids etc.)
- the reinforcements of working creatively by spatial geometric exploration (e.g. finding the solution to open problems).

Schumann (2001) gives an introduction to spatial geometry instruction using such tools. The three-dimensional module of the tool Shape Up! developed by Sunburst/USA is an edutainment tool only for visualization.

Tutorial software for spatial geometry – an example

Tutorial software for synthetic spatial geometry has not yet been sufficiently developed. But there is not only a need for such development, there must be a quantitative and qualitative evaluation of their efficiency. We confine ourself here to a single example: the computer exercise of estimating solids.

Estimating volume of solids and surface size of solids is a neglected topic in the geometry lesson in spite of its geometrical and practical respective applicational meaning. In the use of conventional media there is a lack of creating corresponding learning environments which in addition would not fullfil certain demands for feedback and self-control functions. This is a typical deficit, which can be eliminated with the help of computer use.

The following questions arise:

- How must a computerized learning environment be created so that the ability of estimating geometrical solids exercised in it can effectively be applied to estimating physical solid models? (It is of no use, if the students can only estimate solids effectively in the virtual space on the computer screen!)
- How does the ability of estimating geometrical solids exercised in a computerized learning environment influence the estimation of physical solid models?

These questions are only specifications of the following fundamental research questions which the geometry didactics must be answering in future:

- How must the "virtual space" be arranged so that the abilities or skills acquired in it can effectively be used in "real space“?
- How do abilities or skills acquired in the "virtual space" influence their application in "real space"?

Regarding these considerations a tutorial program was developed formatively (Schumann / Alavidze, 1999). The task of this program entitled ESTIMATE! is the exercise of estimating the surface and volume measurement of simple convex solids and practising corresponding spatial abilities. The following kinds of problem can be selected in the main menu:

Measurement estimation: Estimating the surface of a given solid. Estimating the volume of a given solid.

Solid adaptation: Adapting a solid to a given surface size. Adapting a solid to a given volume.[1]

Rotations of the solids to be controlled by the user by means of the Virtual Sphere Device help to visualize them. An implemented estimation value is answered by the system with the blinking of the corresponding solid. The problems could be selected within type of problems according to different levels of performance. The results are graphically and historically represented.The result representation enables the user to recognize her/his estimation errors and their tendancy. The following presetting parameters can be adjusted: type of solid (cube, cuboid; prism, pyramid, pyramid frustum– respectively with regular or not regular bases; cylinder, cone, cone frustum, sphere), number of problems, number of trials, size of tolerance, change of the unit dimension. Under use of the corresponding parameter settings there can be defined up to five difficulty levels. The dimensioning of solids is always executed stochastically.

In the context of a quantitative empirical investigation among others the follo-wing question was answered for volume estimation: What is the effect of

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[1] The latter two kinds of problems could hardly be posed in a traditional learning environment.

computer exercised volume estimation using ESTI-MATE! to the volume estimation of some physical surface solid models made from cardboard?

The control-group design was selected as a suitable experimental design accor-ding to conditions like students‘ motivation and concentration.

The volume estimation performance of the students improves significantly by about 30%. In addition, the estimation performances of the computer exercised students deviate to a lesser extent. A clear improvement in the ability to estimate the volume of physical models of solids is reached by the exercise with ESTIMATE!.

Spatial Geometry and the Internet

Today there is no more doubt for the efficiency of the Internet for the distribution and appropriation of information as well as for the communication about information in the context of cognitive education. It is to highlight the access to information with the advantage of its current state independent from place, time and person. As a worldwide language standard the English language gains increasing acceptance.

The internet representation of spatial geometry is still weakly developed – as far as content and software ergonomics are concerned. Most sites only contain information and (dynamic) visualization of spatial geometrical facts. But the conditioning information is still far remote from a systematization such as in a context sensitive system "Mediothek Mathematik" (Klett, 1998) for synthetic geometry in secondary education. Corresponding tools and tutorials are just at the beginning of their development. Nevertheless, the problems of an adequate implementation and processing of input of the learners by tutorial software remain unsolved. Despite the inhomogeneous and uncontrollable growth of information there are a lot of materials to supply online and offline teaching and learning spatial geometry. It puts the challenging task to install a selection of sites about spatial geometry in the national curricular context to be updated permanently because of the dynamic extention of the Internet in order to support the teachers in their school work and in their further education. (By corresponding materials globally offered in the Internet something like an international core curriculum on spatial geometry could crystallize itself; this can help to recognize and to overcome deficits in content and media of national curricula.) Besides the development of such Internet materials tasks of a programming technical manner are the portation of educational software already evaluated successfully to a platform independent standard (at present this is limited by the facilities of Java as a software developmental tool) and the task of adaptation of suitable foreign software to the corresponding cultural context.

In the following we select the search cue "Polyhedra" as an example relevant for the spatial geometry lesson in secondary education.We have chosen this topic, because neither the printed pictorial material nor any naturally restricted collection of physical solid models has the time and place independent possibilities of information and dynamic visualization like the Internet. Among the first 100 about 20 000 (!) sites found by FIREBALL we select “Polyhedra Interactive“ (http://polytopes.wolfram.com) of Research Wolfram as an highly informative site, which is not indicated in the meta site “Links and topics related to Polyhedra“ (http://www.links2go.com/topic/Polyhedra). – Regarding the didactic functionality of such sites: the teacher can demonstrate and explain the beauty, variety and the proporties of the forms of specific polyhedra; the students could hardly explore that by themselves because they are inundated by the information available. In addition there is the danger of a brief perception of the computer graphical phenomena without their theoretical reasoning and without their haptic learnability.

Final Comment

What might computer supported learning of spatial geometry look like in the near future?

Learning of spatial geometry through virtual realities: The use of today's spatial geometry programs still separates students and computer systems: The student can only indirectly communicate with the computer for example using a 'mouse'; therefore her/his kinesthetic feelings and experiences are very restricted; he/she executes options and watches the (spatial) result on the planar screen; the spatial interpretation can only be improved by stereographic representations and the use of red-green spectacles. The development of so-called virtual realities partially overcomes the limits between the student and computer system: By means of suitable interfaces it is possible that the student has the (illusory) sensation to move and act with her/his whole body inside a simulated three dimensional world –for example carrying out virtual operations on objects of virtual reality.

The following scenario for future geometry learning is conceivable: The geometry learner proceeds as a Cybernaut in a three-dimensional geometrical world, for example in one for investigating polyhedra. She/he goes for a walk among the solids, looks at them from the worm’s-eye or the bird's-eye view, climbs on the solids around, feels the pointed corners and the sharp edges, she/he slips down the slippery solid faces, she/he penetrates the solids and views them from inside; she/he moves the solids, combines them, folding them down, changes their size, deforms them arbitrarily and carries out operations on them, for example section operations or she/he takes the role of a solid and for example 'experiences' rolling as a solid etc.

What effect will such a computer represented spatial geometry have on the 'imagination' of the students? How will this change their relationship to the real 3-dimensional world surrounding?

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Why Belief Research Raises the Right Question but Provides the Wrong Type of Answer

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Abstract

Belief research has tried to develop understandings of the relationship between teachers' beliefs about mathematics and mathematics teaching and learning on the one hand and the classroom practices on the other. A dominant implicit premise has been that beliefs may serve as an explanatory principle for practice, i.e. that in the case of apparent compatibility between the two there nothing else to explain, while in other situations mediating factors are called upon to explain the discrepancy. Based on the results of an empirical study of three novice teachers' beliefs and practices I question this premise and briefly present a different approach to belief research.

Over the last decade *belief research* has grown big in mathematics education. It has been part of the research agenda at least for the last 20 years, but it has gained further momentum in the 1990s. In 1996 Törner and Pehkonen identified 764 titles in an incomplete list of the field, and from conference proceedings and journals it is evident that it still attracts a lot of attention. Further, the interest in one aspect of belief research may even have increased further, namely the one of teachers' meta-mathematical understandings (including those of the related teaching-learning processes), of how these beliefs are developed, and of how they may influence the classroom practices.

These questions have been addressed theoretically and empirically with a wide range of quantitative and qualitative research methods, the former often in terms of factor or cluster analyses, the latter using open-ended questionnaires, semi-structured interviews, and classroom observations. In this paper my main intention is to challenge what appears to be a basic rationale or premise underlying many of both the quantitative and the qualitative studies in the field: that teacher beliefs are to be seen as a factor that does or at least should explain the practices of the mathematics classroom. In this paper I shall challenge this premise. However, before doing so I shall describe some of the background of the growing interest in the field and question some of its substantive findings.

Fuelling belief research: the forced autonomy of mathematics teachers

The increased interest in the teacher's meta-mathematical understandings follows from a change in the role envisaged for the teacher in current reform documents and journal articles (for a summary of the theoretical underpinnings of the reform see Skott, 2000, chapter 2). The reform is epistemologically framed by both constructivism (von Glaserfeld, 1995a, 1995b, 1996) and socio-cultural theory (Vygotsky, 1986; Wertsch, 1985), and mathematically it is inspired by fallibilism and by emphases on the social constitution of mathematical knowledge (Lakatos, 1976; Davis & Hersh, 1981). Consequently mathematics classrooms are envisaged to develop into small communities of mathematical practice in which the individual students' learning and the microculture of the classroom are seen reflexively related (Cobb, 2000; Cobb and Yackel, 1998). The teacher is expected to play an essential role for the emergence of these communities, as (s)he is expected to develop and flexibly use a wide range of different tasks in order both to encourage the students' involvement in mathematical processes of experimenting, investigating, generalising, formalising, etc. and to support their conceptual understanding and procedural competence on the way. This requires the teacher for instance to support the emergence of an atmosphere in which the mathematical contributions of individuals and of groups of students are valued; to interpret these contributions and make them become an accepted part of the public domain in the classroom; and to pick out mathematically and pedagogically significant aspects of these different contributions and make them part of the mathematical discourse of the classroom. This role of the teacher was succinctly phrased in the draft of *Standards 2000* (NCTM, 1998):

> Curricular frameworks and guides, instructional materials, and lesson plans are only the first elements needed to help students learn important mathematics well. Teachers must balance purposeful, planned classroom teaching with the ongoing decision-making that can lead the teacher and the class into unanticipated territory from an effective mathematical and pedagogical knowledge base. (p. 33)

In this situation teachers have come to be seen not only as important contributors to educational reform, but also as potential obstacles to change. In particular the beliefs about mathematics and its teaching and learning have become a focal point in the literature, as these beliefs are expected to significantly influence the ways in which the teachers cope with the situation of forced autonomy. Consequently current reform initiatives in mathematics education have fuelled the interest in the part of belief research linking teachers' school mathematical priorities to the classroom practices. Three main questions have been addressed in this connection:

- What are teachers' beliefs about mathematics and its teaching and learning? This question deals with the relative emphases on mathematical processes and products; with the teacher's perception of his/ her own role what role in the classroom as explicators of knowledge or unobtrusive facilitators of learning; and with students as receivers of information or as constructors of knowledge.

- How may these beliefs change or develop? Generally perceived to be relatively stable and resistant to change beliefs need to be challenged in order to develop. This question is concerned with how this development may come about, for instance through student teachers' reflective activity in pre- or in-service programmes conceived in line with current reform initiatives (Cooney et al., 1998).
- do teachers beliefs about mathematics and its teaching and learning play for the ways in which mathematics classrooms develop?

An implicit premise of belief research

The most dominant explicit or implicit answers to the last of the three questions mentioned above are in the affirmative. Indeed, if this were not the case it is difficult to explain why the first two questions were to attract more than minimal attention. However, these affirmative answers differ greatly from very direct and causal descriptions claiming that the teacher's espoused views of mathematics determine both the classroom practices and the students' learning (Schoenfeld, 1992; and to a lesser extent Ernest, 1991), over an insistence on an unspecified reciprocal relationship between the two (e.g. Thompson, 1992), to a claim that there is no relationship between the beliefs espoused in research interviews and the practices of the mathematics classroom and that none should be expected, as beliefs are situated much in the same sense as cognition (Hoyles, 1992). The claim in this latter position is not there is no positive correlation between teacher beliefs and the classroom practices, but that the relevant beliefs are those held in the mathematics classroom, and that these are seen as qualitatively different from those held in other situations.

In a large part of belief research, then, the general affirmative answer to the last of the above questions frames the understandings developed in these studies themselves, even to the extent that beliefs come to serve as an explanatory principle in relation to practice[1]. This means that in these studies there seems to be no attempt to look beyond teacher beliefs when interpreting what happens in the mathematics classroom. More specifically, if - from an observer's perspective - there is apparent compatibility between the beliefs espoused for instance in research interviews and the classroom practices, there is little more to explain. If no such compatibility is found an argument for the apparent lack of impact of beliefs is made (i) by referring to a school culture that in the particular case dominates belief enactment; (ii) with reference to a highly individualistic and often condemning explanation of teacher inconsistency; (iii) by capitalising on the conceptual and methodological problems inherent in the very notion of beliefs, as the classroom practices are judged as dependent on implicit beliefs residing at other levels of consciousness, than the ones described in research

[1] The notion of explanatory principle is borrowed from Bauersfeld (1998) who uses it to describe the role attributed to culture in education.

interviews or questionnaires. Hoyles' conjecture about the situatedness of beliefs may be seen as one version of this.

In other terms, a dominant premise of belief research is that beliefs are and should be the main influence on the classroom, and although the classroom practices may be mediated by external or internal constraints, they are indeed *the teacher's* practices: The teacher's beliefs - conscious or not and explicit or not - are thought to be directly related to the learning opportunities that unfold.

Challenging the substantive results of previous studies

I have previously questioned the substantive conclusions of previous studies that claim a direct relationship between teacher beliefs and classroom practices (Skott, 2000; 2001). I did so on the basis of an empirical study of three novice teachers in the Danish *folkeskole*, the municipal school for children in grades 1 to 10. The three teachers were selected for the study, because they all presented visions of school mathematics (school mathematics images or SMIs) that strongly resemble current reform initiatives in mathematics education. In questionnaires and research interviews immediately before and after their graduation from college they described the students' activity in terms of investigations and experimentation; they conceived mathematics as a way of approaching and posing problems; and they presented their visions of teaching in terms that reflected intentions of being unobtrusively supportive in relation to student learning. In short, the SMIs of these teachers were strongly inspired by the reform, and they all seemed confident that they could enact the reform intentions in their prospective classrooms.

In the case of all three teachers, the classroom interactions often developed in ways that resembled certain aspects of their school mathematical priorities. However, there were also episodes in each classroom in which the teacher's contributions to the interactions appeared at odds with his or her SMIs or in which (s)he was apparently tempted to make such contributions. The teachers were asked to comment on video recordings of some of these episodes. When doing so they sometimes referred to mathematical insecurity on their own part as the reason why the interaction developed the way it did. In these episodes they were primarily involved in attempts to manifest professional and mathematical authority, and consequently contributed to the interactions in ways that seemed counterproductive to student learning and at odds with their SMIs. In other instances the teachers were more concerned with building students' self-confidence by ensuring that they - the students - provided an acceptable solution to a textbook task than with supporting their mathematical learning. As a result they got involved in funnelling types of interaction that in effect depleted the tasks in question of its mathematical contents for the students in question. In yet another type of situation the teachers' activity was primarily directed towards managing the classroom in a stressful situation in which many different (groups of) students simultaneously called for help. In these situations they often became much more explicit in their assistance to the students in order to speed up the process of helping them.

An important characteristic of the episodes that challenged the enactment of the teacher's school mathematical priorities, the *critical incidents of practice* (CIPs), is the simultaneous existence of multiple motives of the teacher's activity (Skott, 2001). In each of them the intention of facilitating mathematical learning is submerged by the emergence of other energising elements (Leont'ev, 1979) of the teacher's activity beyond the teaching of mathematics. The motives of the teacher's activity, then, should not be as seen as pre-determined by his or her school mathematical priorities. Rather they must be understood as entities that may be transformed or even emerge in and as a result of his or her interactions with the students. Consequently the role of the SMIs is not to control the teacher's activity. Rather it becomes an underlying propensity that may play a part as one possible element contributing to his or her interpretive efforts in relation to the situation at hand. These interpretations sometimes lead to the emergence of other motives of the teacher's activity than facilitating the students' mathematical learning, motives that in turn direct the teacher's contributions to the interaction.

Questioning a dominant methodological approach in belief research

The above conclusions not only challenge the *results* of previous studies in belief research. They also question the implicit *premise* that beliefs may serve as an explanatory principle for teacher actions. In the episodes referred to above the classroom practices evolved in and as a result of the instantaneous interpretations on the part of both teacher and students of their mutual intentions and expectations. This supports Bauersfeld's (1988) suggestion to view the classroom from the perspective that "it is a jointly emerging 'reality' rather than a systematic proceeding produced or caused by independent subjects' actions" (p. 29), and it questions whether the classroom should be seen as a field for the *teacher's* practice in the possessive sense of that term. This does not necessarily question the existence of sets of beliefs that - at least in the short term - are relatively stable across contexts. It does, however, indicate that the contextual embeddedness of teaching - in the local interactionist sense of context - challenges the extent to which the intention of facilitating the students' mathematical learning remains the dominant motive of the teacher's activity. This means that the social *interactions* of the mathematics classroom have to be perceived exactly as interactions, i.e. as processes "that *form* human conduct instead of being merely a means or a setting for the expression or release of human conduct" (Blumer, 1969, p. 8; italics in original).

Referring to the use made of beliefs as an explanatory principle in relation to classroom practice (cf. above) there is as much to explain, when there is apparent compatibility between beliefs and practice, as when there is none. It is of obvious importance to address the questions of when and how the classroom interactions allow for the teacher's activity to be directed at facilitating mathematical learning and to be influenced by his or her school mathematical priorities. Further, if there is apparent lack of congruence between espoused priorities and observed practices, there may be no need to refer to beliefs residing at other levels of consciousness than those that are expressed in research interviews. Nor does reference to external constraints on belief enactment in the form of a school culture in and by itself explain an apparent discrepancy between beliefs and practice. If a broader school culture is expected to play a role, an account is needed for how the mutual expectations of teachers and students are influenced by this culture, i.e. how it is re-enacted in the classroom. Only then may such a reference contribute to an understanding of how the culturally derived expectations come to play a role in the formation of the motives of the activity of both teachers and students.

Finally, the above study indicates that the teacher instantaneously manoeuvres in relation to a multiplicity of different tasks rather than merely to one of facilitating mathematical learning. This is inherent in the very notion of CIPs and its defining concept of multiple motives of teacher activity. In other terms, the teacher's activity is related to the sense they make of the situation at hand, and it momentarily focuses on some motive, that (s)he tries to pursue. This means that inconsistency is an observer's perspective that does not do justice to the complexity of the teacher's tasks. For classroom research - and for belief research in particular - this means that it should be conducted with the understanding that teachers' and students' activities do make sense; - or phrased more bluntly in the case of teachers: *teachers cannot be inconsistent*. This, of course, does not mean that teachers necessarily pursue the different motives in what appears to be the most efficient manner, nor that they - even according to their own priorities - strike an appropriate balance between them at any one time. Teacher consistency is a local and instantaneous phenomenon and as such implies that the teacher's activity should be viewed as his or her attempt to relate sensibly to a multitude of different and possibly subjectively incompatible aspects of the situation at hand.

Conclusions

The study referred to in this paper deals with a main question in belief research, i.e. the question of what roles the teacher's beliefs about mathematics and mathematics teaching and learning play for the learning opportunities that unfold in mathematics classrooms. I have claimed the motives of the teacher's activity emerge in the classroom interactions, and that therefore his or her school mathematical priorities may not be so significant as previous studies have suggested. The more general claim is that we need to adopt an interactionist perspective on mathematics classrooms and approach these from the perspective that teachers are never inconsistent. In other terms, I have found it necessary to challenge what appears to be a dominant *premise* of teacher related belief research in order to address the main question of the field. This premise is that beliefs may serve as an explanatory principle in relation to practice. By doing so the study also questions the usefulness of *types of answers* found in much previous research done in the field. In short the argument is that the main question of the belief-practice relationship should still be addressed, but that the dominant perspective needs to be changed if significant answers to this question are to be provided.

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Rehumanising Mathematics to Make Mathematics a Human Right: An Application of Curriculum Architecture and Postmodern Counter Research

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This paper addresses two questions: 1. How can curriculum architecture design a rehumanised precalculus curriculum? 2. What happens in the classroom when this curriculum is implemented? Brief answers are: 1. A dehumanised mathematics curriculum disrespects the concept's history by e.g. presenting linear and exponential change as examples of functions, thus transforming mathematics to 'metamatics'. Uncovering hidden eventuality, curriculum architecture could rehumanise linear and exponential change by presenting them as stories about uniting constant unit- and per-numbers. 2. When implemented the teachers will have big problems leaving their traditional routines, only third time lucky. Most students will show positive reactions, and many will change from dropouts to dropins.

How can curriculum architecture design a rehumanised pre-calculus curriculum?

Curriculum architecture is a core ingredient of postmodern counter research within education. To the question: 'What is postmodern counter research?', a summary answer could be: Postmodern counter research is producing counter examples to qualitative necessity claims. Postmodern counter research is a middle position between structuralism and post-structuralism. To structuralism the world has a structure that can be represented and echoed in language. Post-structuralism denies this, and points out that unaware of a phrasing's hidden eventuality, the free modern individual becomes enslaved and clientified by ruling echo-discourses: echo-phrasing is freezing, re-phrasing is freeing (Foucault, 1970, 1972). Postmodern counter research is based upon the 'pencil-dilemma': Placed between a ruler and a dictionary, a pencil can point to its own length but not its own term. The pencil possesses a necessity enabling it to number itself, but not to name itself. Hence numbering follows from necessity, and naming follows from eventuality, contingency, i.e. from a choice, that might have been otherwise. Modern qualitative research thus becomes problematic, but regains its meaning as a postmodern counter research, accepting the numbering of nature, but producing counter examples to echo-phrasings within ruling discourses, thus guarding the line between necessity and eventuality by checking for h idden eventuality in claimed necessity.

To the question 'What is curriculum architecture?', a summary answer could be: A curriculum is an example of an authoritarian discourse (Pinar et al., 1995).

This ruling discourse is checked for echo-phrasings and hidden eventuality by postmodern counter research using curriculum architecture. There is a widening gap between theory and practice within mathematics education (Niss, 2000). Curriculum architecture is addressing this problem by offering a practice based research method avoiding the ‘London Syndrome’: ‘move all local universities to London, since local knowledge doesn’t matter anyhow’. Postmodern counter research accepts the principle of situated knowledge (Lave et al., 1992): If knowledge is local, then the locals should be allowed to develop local knowledge by bringing micro- or macro curriculum design into the classroom. Curriculum architecture offers to teachers and students a creative alternative to just being textbook echoes, and a possibility to perform postmodern counter research, thus closing the theory-practise gap. Such research reports will not make claims, convincing about necessity, but suggestions, inspiring to look for other examples of hidden eventuality.

Curriculum architecture presupposes a national flexible core curriculum, i.e. a curriculum that draws a line between necessity and eventuality in the curriculum by assigning its authority to elements with no or low degree of eventuality. The degree of eventuality increases from zero to high when moving from nature’s necessities through social practices and institutionalised discourses to personal opinions.

To the question ‘What is a dehumanised curriculum?’, a summary answer could be: A dehumanised curriculum is a curriculum presenting a social constructed concept as a universal concept, e.g. a structuralist curriculum telling about the world from above, thus providing the students with meaningless top-down unknown-unknown relations. This forces the students to construct their own meaning unassisted by the teacher, who is only present as a textbook echo. The lack of meaning implies negative feelings towards the subject. This no-meaning problem is not solved by a constructivist approach, which is still based upon structuralism, and which deprives the teacher of the possibility of story-telling.

To the question ‘What is a rehumanised curriculum?’, a summary answer could be: A rehumanised curriculum is a curriculum presenting concept as social constructions, e.g. a nominalist curriculum telling about the world from below, thus providing the students with unknown-known relations to extend their self-stories. A rehumanised curriculum respects the biological necessity, that the holes in human heads are for food an stories, and that stories must be relevant answers to the selfstory-builder questions ‘tell me something I don’t know, about something I know’. And a rehumanised curriculum is feeding all three brains: the human brain with meaning, the reptile brain with routines, and the mammal brain with positive feelings.

To the question ‘What is a dehumanised mathematics curriculum?’, a summary answer could be: A dehumanised mathematics is disrespecting the nature of mathematics as a grammar of the number language by talking about mathematics as applicable to the world, and is not respecting mathematics’ roots as a historical social construction based upon natural necessities giving birth to social practices and stories. In short, a dehumanisation transforms mathematics to 'metamatics'. An example of a dehumanised mathematics curriculum is a structuralist curriculum presenting abstract concepts as an example of more abstract concepts, as e.g. 'A function is an example of a relation between two sets, that assigns to each element in one set an element in another set'. A pre-calculus curriculum is dehumanised if it phrases linear and exponential change as linear and exponential functions, i.e. as examples of functions. Since the function concept is younger than calculus it cannot be part of a rehumanised pre-calculus curriculum.

To the question 'What is a rehumanised mathematics curriculum?', a summary answer could be: A rehumanised mathematics is respecting the close relationship between the word language and the number language by considering mathematics a grammar of the number language, and is respecting mathematics' roots as a historical social construction based upon natural necessities giving birth to social practices and stories. In short, a rehumanisation transforms 'metamatics' to mathematics. Thus a rehumanised mathematics curriculum is following the principle of low eventuality by placing its authority with the necessity of multiplicity, which gives birth to the practices of bundling and rebundling, and stacking and restacking, and to stories about the total, found by counting or calculating. A rehumanised precalculus curriculum thus might originate from a practise of bundling and uniting constant $-numbers or constant %-numbers, leading to questions like '100$ plus 7 days @ 5$/day total ?$' and '100$ plus 7 days @ 5%/day total ?$', plus the same questions with the question mark placed elsewhere. And eventually leading to linear and exponential growth calculations, but not to functions. Historically functions emerged after calculus originating from a practise of bundling and uniting variable per-numbers, leading to questions like '100m plus 7 seconds @ 5m/s increasing to 6 m/s total ?m.' (Tarp, 1999)

What happens in the classroom when a rehumanised pre-calculus curriculum is implemented?

To the question 'What happens to the teachers?', a summary answer could be: Most teachers are reluctant to try alternatives, even as a micro-curriculum. Interested teachers are discursively willing but practically unable to change their routines, only third time lucky. Thus an implementation of a rehumanised mathematics curriculum presupposes an intensive training program changing both the teacher's discursive and practical conscience (Giddens, 1984) of mathematics, and changing the teacher's success expectation from short term to long term.

To the question 'What happens to the students?' a summary answer could be: Most students will show positive reactions. Many students will change from dropouts to dropins. Some students fear, that a rehumanisation will make mathematics so easy it might be forbidden: 'my former teacher said mathematics must be difficult'. All in all, a rehumanised nominalistic bottom-up mathematics told from below has the potential to make mathematics a human right. So the question is: Is there is a social will to admit mathematics as a social construction?

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Mathematical Thinking of Low Attaining Students

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Abstract

This paper develops the idea that the mathematical thinking of low attaining students in the secondary phase can usefully be seen in terms of proficiencies, rather than deficiencies. Evidence is offered towards an approach based on recognising that low attaining students exhibit conceptual mathematical thinking.

Difficulties created by a negative view of low attaining students

Teachers and authors tend to talk of low attaining students and under-achievers in terms of what they lack or what they cannot do, thus dwelling on negative aspects of academic performance. Haylock (1991) is in most respects a very positive writer about low attaining students, but in case studies his positive comments are usually about behaviour and low level skills, rarely about features of mathematical thinking which might accompany such skills. Denvir, Stolz and Brown (1982) list statements which teachers made about low attaining students: there are a vast number, all of them negative.

Support offered in UK classrooms is often of a kind which simplifies the mathematics until it becomes a sequence of small smooth steps which can be easily traversed. The supporting adult will "take the pupil through the chain of reasoning" and the learner merely fills in the gaps with arithmetical answers, or low-level recall of facts and so on. Achievement in such situations is identified by getting to the end of the work, completing something successfully, filling in the required partial answers, and maintaining some concentration throughout. The student may feel genuine success because, in the terms in which mathematical success has been presented to them for their interpretation, they have indeed achieved (Bergqvist, 1990). But path-smoothing is unlikely to lead to significant learning on its own, since the strategy is to deliberately reduce a problem to what the learner can do already. The learning, presumably, is then deciding what to do and stringing the steps together - but the helper has already done this for the learner! Furthermore, the view is reinforced that if the learner does nothing for long enough, the helper or the teacher will provide the appropriate task-transformations (Bauersfeld, 1988). There is no possibility that leaps could or should be made, since the view is that a low-attaining learner can not make such leaps and might lose confidence through being expected to do so. Teachers may prioritise confidence rather than learning.

For secondary teachers and students, there are two main ways in which prior underachievement presents a daunting task. Firstly, accumulated past failure leads to expectation of future failure on all sides; secondly, there may be utter confusion about arithmetic. It is tempting for secondary schools to try a piece-meal approach, concentrating on remediating what cannot yet be done rather than building on what can be done. This brings students into renewed contact with the site of their previous failures while offering them very difficult learning tasks – to learn, recall and use facts and procedures which are unconnected to anything about which the learner feels secure and which depend on remembered words and instructions – a very expensive use of the brain (Butterworth, 1998). They conceive of mathematics as being fragmentary, illogical, difficult and alienating. It is also very hard for teachers to organise individual attention to weaknesses in the way required by this approach.

A cognitively-guided approach based on individual constructions

Daniels and Anghileri (1995), offer a detailed research-based report of the difficulties inherent in mathematics, the way it is represented and the way it is taught, as well as the debilitating effects of accumulated failure. Their report encourages a move away from labelling the learner (e.g. as low attaining student, under achiever etc.) and towards structuring teaching to enable individual learners to achieve, given the problems they have encountered and the meanings they have constructed in the past. Fennema *et al* (1993) took a similar approach. Teachers read research results relating to students' informal mathematical methods and structured their teaching in response to their students' constructions of mathematical meaning. These approaches required detailed diagnosis of the mathematics of individual students and the support of researchers and fieldworkers whose knowledge of teaching, mathematics and research may be beyond the resources of many schools.

An activity-based approach

Another approach, reported in Boaler (1997), is to present mathematics in a sequence of activities which are structured similarly to the kinds of task students meet elsewhere in their school and outside lives, amenable to generic problem-solving techniques. Low attaining learners in a school which used this approach performed significantly better in their final examinations than similar students in a school which used a more formal, technique-based approach. Teachers believed that students' inability to recall and use what had been taught to them in primary schools would be overshadowed by their ability to develop, adopt and use suitable techniques to make progress with their current task. But their results, although better than in a comparable school, were still low compared to their cohort. A further problem with this approach is that *ad hoc* mathematical methods do not provide a foundation for mathematics at higher levels of abstraction. *Everyday* thinking skills were exploited in mathematics lessons, (which is more useful than the pretence that mathematics lessons support everyday thinking), but *mathematical* thinking was not developed except where it coincided with everyday thinking.

What is special about mathematical thinking?

In reviewing contributions to describing mathematical thinking, it is found that some of the descriptions concentrate on problem-solving heuristics (e.g. Schoenfeld, 1984) while others relate to the development of conceptual understanding in mathematics (Tall, 1991).

Ahmed (1987) and Boaler (1997) give evidence of how low attaining students can often do well in problem-solving, activity-based and open-ended situations. Ahmed's project showed how such teaching styles could fundamentally influence the attainment of students, simultaneously raising expectations and making achievement in mathematics seem worthwhile. But we lack evidence of ways of thinking which would enable low attaining students to develop the abstract conceptual understanding. Tall (1991) says that those who succeed are those who, *without being taught*, can reflect on processes, change representations, abstract entities from them, manipulate these and hence gain an image of concept, while overcoming obstacles to gain a conventional understanding and acceptance. Students for whom the usual logical sequence of presentation of formal mathematical products matches their own cognitive development are in a highly advantaged position and make the matching of conceptual development and formal, generally-accepted mathematics more likely.

The reader may believe that these ideas are irrelevant for low attaining students, but Haylock (1991, p. 24) reports how one student (Ben) voluntarily produces and uses a pictorial representation of division of whole numbers which he then manipulates in order to get a correct answer. This shows the ability of a low attaining student to create and use an appropriate visual image, that is, a kind of concept image which is manipulated until a conventional result is obtained. Since symbolisation and transformation are two mathematical actions, and since this student performed them voluntarily, we could say that he was doing some mathematical thinking.

A mathematical view of learning mathematics

There is a current tendency to describe learning in socio-cultural terms, but alongside this is a tradition of using mathematics, its own structures and practices, to think about how it might be learnt. Freudenthal (1971) claimed mathematics as an activity involving organising and mathematising, not as a body of knowledge. This view would release low attainers from needing to KNOW more things, which keep slipping away, and replace these with opportunities to DO mathematics in the kinds of ways described by Boaler and Ahmed. Vergnaud (1997) says that what is needed to describe progress towards conceptual understanding in mathematics is within the mathematics itself, as part of it. He and Tall, from different perspectives, both appear to support the notion that students need to experience personal conceptual progress towards understanding, and that this journey is describable in terms which are already mathematical, rather than social.

A characteristic of Ahmed's (1987, p.72) approach is that it emphasises *proficiency* rather than *deficiency*. He concentrates on general learning skills such as curiosity, willingness to work, problem-solving skills and rightly points out that all students display these in some contexts, but not perhaps in all contexts. In this paper I go beyond this and describe abilities of some low attaining students which are features of *mathematical* thinking, such as the pattern use suggested above.

For this purpose mathematics is seen as sequences of: objects and their properties; classes of objects with their associated properties; generalisations about classes; abstractions and relations which become objects for more complex levels of activity. All of these can be represented by a variety of symbols which can, like abstractions and relations, be further manipulated in their own right. Drawing on these views, I look for evidence that students can exemplify and counter-exemplify in ways which do more than imitate what a teacher has offered, since this implies some level of generalisation and some knowledge about classes; can change and manipulate representations; can develop and use images of concepts; can abstract by reflecting on processes; can perhaps work with abstractions and relations, which is generally a feature of more advanced study (but see Harries, 2001).

What is not-so-special about mathematical thinking?

It is noticeable that some of these specific learning behaviours associated with mathematical thinking turn out to be ordinary ways of thinking and problem-solving playing a specially important part in mathematics. For instance, it is common to exemplify in order to illustrate complex descriptions; it is common to switch from one representation to another, such as when one draws a map to accompany verbal directions (see the story of Ben above).

Examples of low attaining secondary students thinking mathematically

A small research project was carried out in a small class of very low attaining students in secondary school, year 9. It has been reported in more detail elsewhere (Watson, 2000). The object was to observe, identify and record incidents in which low-attaining students appeared to be exhibiting high-level mathematical thinking skills, as described above. The mathematical contexts, which were chosen by the usual teacher, were very simple for students of their age, and the usual teaching style was to simplify tasks and give plenty of path-smoothing support. I also acted as a support teacher and occasionally took the class, focusing on using approaches which gave responsibility to the students and offered something for them to discuss, informed by my work in Watson and Mason (1998). I was interested in the ways of thinking which the students displayed, given their history of extreme under-achievement in mathematics. The following incidents give some idea of the range of manifestations of mathematical thinking, which took place during the observed lessons.

Giving an example/counter example

Almira had been asked to round 83 to the nearest ten. She replied '80, but if it had been 87 it would have been closer to 90'. Not only was she showing that she knew this particular answer, but that she also recognised the principle being used and could give an example of the another kind of possible question and answer. She had a sense of the classes involved and she recognised and chose exemplification as a way to communicate mathematics.

Choosing to work with structure and relations

Students had been using flow diagrams to calculate outputs of compound functions. They were asked to make up some hard examples of their own for the whole class to do. Most students' idea of complexity was to use more operations and bigger numbers; this is a common response (see, for example, Ellerton, 1986) but Boris suggested constructing one in which the operations and output are known and the input has to be found. Andrew then gave one in which input and output were given but the last operation was missing. These two students were working with the relations rather than the numbers and operations. They saw the structure of the problem as something they could vary.

Reflecting on processes

Students were suggesting factorisations of 144 by responding to "If 144 is the answer to a multiplication, what was the question?" John said "12 by 12", which was the only answer available to them from the posters in the classroom! Dan immediately said "and 24 by 6". There were puzzled faces, so the teacher asked "how do you think Dan got this?" Eventually Raj said "by doubling and halving, you could have 48 by 3". Whether this showed understanding or was something Dan had learnt in the past I do not know, but Raj was using the example to abstract a principle which could be applied again, and doing so successfully.

Using symbolic representation

June was drawing a square in the normal orientation on a coordinate grid and noting the coordinates. I do not know if she would have reflected on the coordinates if she had not been prompted to do so, but once prompted she was able immediately to say which components of the coordinates were equal. She was asked if this pattern would always be true and replied "yes". She then explained why it worked, and was asked if she could say it in algebra. Although the expressions she developed depended on specified coordinates at one vertex they were in other respects generalisations of such squares. She had been able to make a transition, on the basis of one example and application of knowledge of squares, from a specific case to a more general symbolic representation, and to justify this.

A proficiency agenda for low attaining secondary mathematics students
These incidents are not generalisable across the class, nor are they being offered as conclusive proof that all low attaining students can engage in mathematical thinking in all mathematical contexts. Nevertheless, every student at least once during the study exhibited behaviour which showed ability to think in ways which are usually described as mathematical thinking.

Butterworth (e.g. 1998) and other neuro-psychologists put forward the view that only a few aspects of mathematics (particularly arithmetic with small numbers) depend on systems which are 'hard-wired' into the brain; the rest comes from how we make sense of our experiences in mathematics. In other words, weaknesses in mathematics are unlikely to be innate, except in certain pathological cases. All the students in this study showed that they were able to make mathematical sense from mathematical experience. All were able to participate in a learning environment which was not based on simple step-by-step procedures, but which expected conjecture, exemplification, generalisation, reflection on pattern and other aspects of advanced mathematical activity.

The simplicity of the mathematical contexts needs to be considered. Can it really be said that these students were doing mathematical thinking, when the contexts were so simple? Are the stories above, and in Watson (2000), only illustrations of a borderline between everyday thinking and mathematical thinking which some students successfully transcend, but which these students might never transcend? One could say that what characterises low attaining students is their inability to make this transition, but it is unlikely that a deficiency agenda which focuses on remediation and repetition, common practices everywhere, will provide insight into this area.

A *proficiency* agenda would not simply dwell on the positive aspects of behaviour, motivation or attitude, although those would play a part; rather, it would recognise and emphasise the thinking skills which students exhibit and offer opportunity for these to be used to learn mainstream curriculum mathematical concepts. They might then conceive of mathematics as being accessible, as a subject which they can think about, and as an arena for personal satisfaction.

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Two Dimensions of the Conception of Mathematics in Tertiary Education

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Abstract

This paper seeks to provide a systematic analysis of some central aspects of the way in which mathematics is and may be conceived as a field of knowledge to be taught and learned at the tertiary level with more or less specific professional aims. The paper examines the connection and possible coherence between these conceptions in the contexts of training scientists and teachers, in particular as it occurs in the design of undergraduate curricula.

Background

There is a well-known crisis in the recruitment and operation of educational programs in mathematics and mathematics based disciplines in most of the Western world (Jensen et. al., 1998). Partially as a result of this, there is an increasing shortage of qualified mathematics teachers in several countries (e.g. UVM, 1999). Although this paper will not address these issues directly, they are certainly behind the widespread feeling that the basic definitions of mathematics as a field of study must be somehow modernised to meet the challenge of a changing academic and societal environment.

Mathematics is of course not the only subject for which such a need is felt. Other fields of study - in particular the pure sciences and the humanities, which, like mathematics, do not evolve as a function of their direct relation to one particular profession - are under a similar pressure to redefine themselves. The notion of *core curriculum* is increasingly used to convey the idea that in many fields of knowledge, not all elements are equally essential or indispensable; but certain concepts, methods and insights are crucial and permeate the whole field (UVM, 2000). In fact, the identification of such fundamental elements of a field of knowledge is increasingly necessary to address reform issues in an educational landscape with several (at least partially) contradicting trends; in particular, to navigate in the well-known (but still more acute) conflict between the roles of education as preparation for professional life and as a space for individual realisation and development. To prevent reforms from being not only radical but also destructive, it is an important task for mathematicians and mathematics educators to develop a clearer and more complete understanding of the nature of a 'core curriculum' in mathematics.

This task is not to be construed only at a national level. In fact the current attempts towards structural homogeneity in European higher education (as evidenced e.g. by the Bologna Declaration) will only succeed if they are accompanied by common analytical conceptions of subject matter knowledge, not least in major and universal areas like mathematics. This paper constitutes a preliminary attempt to address this need, and, more modestly, to point out some major inconsistencies and false dilemmas in the common conceptions underlying the curriculum debate.

Basic categories for the discussion

Knowledge, in any area, concerns both *contents* (roughly, what is known) and *competencies* (roughly, abilities to put content knowledge to use). An elementary point is that the two are interrelated, in fact inseparable. And a part of our task is to sharpen both of these rough notions in the context of university level mathematics[1].

There, study units are typically specified by a list of 'broad content', with items like 'Second order linear differential equations with constant coefficients'. However, such items are not very telling of what content is actually learned (or meant to be learned), e.g. which definitions, examples and results are concerned. Also, mathematical concepts and results are typically related in complex structures of meaning and dependence; mathematical content knowledge is as much about these *relational structures* as about single topics. As concerns *competencies*, some indications may be given in an official description of the 'goals' of the course, but the evaluation practice may often be a better guide. In the example, the latter may simply imply that the students are expected to be able to recognise and solve equations of the given type; or, that they are also expected to be able to explain a general or specific method; or, leaving the arena of strictly mathematical competencies, that they are expected to know and be able to handle certain cases of extra-mathematical applications. Certainly, the actual practice of the course and its evaluation could give much more detailed information about these and other 'broad competencies'. However, we may still know little of their quality, e.g. *stability* (independence on notation etc.), *durability* and *relatedness* with other competencies.

One inescapable fact remains: we cannot, in actual practice, relate directly to content knowledge and competencies. We cannot dissect a student's brain in order to produce a picture of his content knowledge, nor can any test display the full range of his competencies; we may only get (partial) information on the two through instances of his *performance*. In fact, even a student's performance can be said to be partly inaccessible, as it may include thinking processes that are not evidenced in instances of *discourse* (written, oral, figurative etc.). Discourse, indeed, is all that can be observed in teaching and evaluation practices. It may, however, still be worthwhile to maintain the more general performance focus in a

[1] A more thorough treatment of the linguistic background and terminology may be found in (Winsløw, 2000).

discussion of core curricula, with the understanding that only its articulated part is immediately observable. For instance, the problems of stability and durability are more easily discussed at this level, discourse being always situated in time and place.

Thus, one dimension of our discussion of core curricula is spanned between the rough notions of content and performance. However, in most contexts of higher education, mathematics occurs not only *in se* but also as a means to describe non-mathematical phenomena. Within this *external* aspect of mathematical knowledge, its content (concepts, results etc.) typically becomes *tools* and *methods* for such a description, while performance is concerned with applying these in the context of the description. In this case, the performed discourse will typically be a mixture of mathematical discourse (cf. Winsløw, 2000) and that of one or more other disciplines. Part of the target competency will then be the ability to handle this mixture. More generally, what is often termed *meta-knowledge* belongs to the external dimension. Even for segments or programs of mathematics education that are strictly confined to pure mathematics, it may not be wise to ignore the more philosophical (such as foundational) aspects of this second dimension.

Based on these preliminaries, I propose the following rough guide for the discussion of core curricula at tertiary level[2]:

	Content	**Performance**
Internal	Structure of mathematical concepts and results	Understanding and producing mathematical discourse
External	Range of mathematical methods, and their scope	Understanding and producing mathematical models

Here, the term understanding refers to receptive (as opposed to productive) performance, as in reading and making sense of an exercise; but, like content knowledge, receptive performance can only be observed indirectly through 'productive' performance, e.g. producing a solution to the exercise.

[2] The above model is clearly inspired by linguistics. In the context of learning a foreign language *X*, we have the following analogue:

	Content focus	**Performance focus**
Internal	Lexical inventory, phonetic and grammatical rules of *X*	Understanding and producing correct phrases in *X*
External	Cultural and social aspects of the usage of *X*	Communicating using *X* in relevant contexts

The main theoretical point of this model is that its four fields are mutually dependent. It can be said to relate two main debates in mathematics education: the object-process discussion (e.g. Sfard, 1991) and the role of mathematical knowledge in society (e.g. Niss in Biehler, 1994). For instance, even (in fact, especially) if the official goal of a segment of education can be said to fall entirely in the lower right box of the scheme, it is clearly necessary to take *all* aspects into account.

For the rest of this paper, we shall explore some more pragmatic uses of the model in relation to the problems mentioned in the first section.

Two crucial dilemmas

Undergraduate programs in mathematics are usually constructed with the aim of building the 'basic knowledge' of students; here is a British description (Burn et. al., 1988, p.102), which is probably widely representative of the current situation:

> Our first year courses generally consist of things that have <u>always</u> been there, things <u>we</u> know about, things we feel a student <u>ought</u> to know, and things they will <u>need</u> to do our option courses later on. (…) Our objectives may be administrative (bringing them up to the same levels), mathematical (knowledge and approach) or functional (employment, future courses).

In view of the model of the second section, it is clear that the focus in the first years of such a program is very much on *basic content knowledge*, which is considered necessary for a variety of reasons. Now, many university mathematicians do see this as a problem. The above quote was in fact part of the agenda for a working group at a conference on undergraduate education, setting out to address questions such as

> Should more attention be paid to developing skills of problem solving rather than the ability to reproduce material from memory?
>
> How can we encourage and reward thought and innovation rather than knowledge or imitation? (ibid., p. 102-103)

Here, a shift towards the performance level is evident; indirectly, the unease and experienced problems with the current situation are the main message. The recommendations of the group are summarised in a list of objectives for first year courses, which can all be said to fall in the other three boxes of the model, including the meta-level and modelling aspects. At other conferences in the same series, subjects like 'geometry at A-level' and 'rigour and proof' were more specifically considered, with similar *apparent* outcomes: a desire for a shift towards objectives that emphasise independent discursive and modelling skills, while reducing the role of content to illustrative examples. With 'apparent', I want to say that this may not quite reflect the actual wishes and certainly not the practices of the participants. Indeed, in much of their writing, the need for a 'content core' is implicit, but nowhere explicit. Few mathematicians would consider it immaterial *what* mathematics is used to train the said skills. For instance, this could mean that no new mathematics (beyond school arithmetic and algebra) would ever need to be introduced. Several studies (e.g. Defranco, 1996) have shown that one may develop content knowledge very far alongside with a persistent low level of performance e.g. in problem solving. The opposite would be just as dissatisfactory: few of us would approve of an undergraduate curriculum stripped of substantial new content.

The situation is often quite different within university programs in mathematics partially or exclusively aimed at students in other fields than pure mathematics. Here, courses "designed to provide the students with the mathematics necessary for work at degree level in their main subject" (Burn et. al, 1988, p.181), and they tend to be heavily focused on content in the form of 'tools and methods'. Tests are often in the form of written exams where success is determined by the ability to solve certain standard types of problems (e.g. $ay''+by'+cy=f(x)$ with a, b, c and f varying from test to test). However, teachers in such programs often feel that more concrete applications (to the relevant field(s)) should be included, both as examples and as tasks for independent student work; the following, from a 1978 working group of people involved with service teaching to biology students, seems to be a common stand: "We believe it to be very important that examination questions should not merely test mathematical techniques but should also involve the modelling of real phenomena" (op. cit., 184). In the context of mathematical modelling, the following was expressed by a similar group at the 1979 conference: "We have the strongly held opinion that applications should use only mathematical methods which the students already know (…) it has been suggested that there should be as much as a two year gap between learning and use" (op. cit., 195). In practice, students are often given a 'tough start' with much rote learning of topics and techniques that will only be applied later, but many teachers feel applications should be more immediate. Again, we see an apparent strong tendency to value independent performance over contents. Although the two quotes are taken from different contexts, they combine to point to a dilemma that is quite similar to what we found in the setting of pure mathematics: we do want to develop the students' competencies for independent performance, but we are also committed to extend their domain of contents beyond secondary school level.

The third main bulk of undergraduate mathematics education is *teacher education*. In many countries, future teachers enrol in undergraduate programs of the first kind considered, and then specialise in education. In other countries (including my own), there are separate programs or even institutions meant to train future teachers. Especially in the first case, the content-performance dilemma already explained will arise for a different reason: by nature, the undergraduate curriculum is beyond what the teacher will have to teach at primary and secondary level, and it may often not be clear for students how the more advanced material can be used as 'background knowledge' for teaching. A striking and widely noticed study by Liping Ma (1999) has documented that teachers may have passed several undergraduate courses in mathematics and still be unable to explain (or even produce) solutions to elementary tasks like the division of two simple fractions. Clearly, a teacher must be required to have a high level of discursive performance within the content frame to which his teaching pertains. In fact, a mature mastery is necessary but not sufficient:

> Because teachers must be able to work with content for students in its growing, not finished state, they must be able to do something perverse: work backward from mature and compressed understanding of the content to unpack its constituent elements (Ball and Bass, 2000, p. 98)

But even more acute is the tension in teacher training between the professional needs of mastering *internal* and *external* aspects of mathematics. This is particularly so when teacher training is exclusively focused on preparing teachers for a school curriculum in which the official goals are very much oriented towards 'everyday applications', where the content frame may be reduced to (applied) arithmetic. A part of the most applauded recent Scandinavian work on teacher training seems to tacitly accept this as a consequence of the noble effort to promote values like 'democracy' and 'critical citizenship'. More generally, the focus on the meta-mathematical part of externally oriented performance tends to relegate the content issue (internal as well as external) to a question of supporting illustrative examples. It is a truism that external aspects of mathematics, including real-life models and meta-mathematical issues, cannot be seriously tackled without a solid basis within the relevant internal parts, and it is widely argued that the latter must to some extent precede the former in teaching (e.g. Burn (quote above); Sierpinska, 1995; Sfard, in press). The tension in teacher education arises in part from a lack of seeing the two as interrelated.

The challenge of information technology

Both in its academic and professional contexts, mathematics as an activity is increasingly dependent on information technology. The use of computer based tools must be clearly taken into account when specifying the target *performance* of students in mathematics and related disciplines. Only in the internal content focused part of mathematics education is it pragmatically feasible to disregard this, but it is often still attempted throughout educational programmes (in Denmark, this occurs in all three main contexts of the third section). There is a growing realisation that it may be pragmatically desirable to incorporate information technology to some extent also in the internal content focused part of mathematics education (e.g. Winsløw, in prep.), which in many cases will just amount to an acknowledgement that mathematics related technological competency (e.g. the ability to use a CAS) is not innate and must be grounded in a partially separate structural knowledge. But it is also important to realise that such an inclusion will partially change the pedagogical conceptions and curricular priorities of conventional mathematical content. This may produce desired as well as less desired effects. Two of the main effects observed in practice are

- *Usiskin's problem*: that work with computers tend to encourage inductive work with special cases ('examples'), rather than deductive work with general structure ('rules'), cf. (Usiskin, in Biehler et al., 1994, p. 325), and
- *Dreyfus' potential*: the possibility of using computers for routine computations and visualisations, so that students' time and energy can be focused on more challenging and perhaps (theoretically) advanced activities, cf. (Dreyfus, in Biehler et al., 1994, p. 205).

Whether the former or the latter type of effect is observed, curricula may have to stress matters of content and forms of performance that are deemed to be important but are not immediately within reach of current computer based tools.

Towards coherence and balance

From the above, it is clear that mathematics education at tertiary level is facing acute challenges, although *not the same* in the three domains of education considered in the third section. These can usually be considered separately at the more advanced levels where students will have decided between pure mathematics, other scientific fields, and education. From a curriculum perspective, the most interesting and problematic is their cohabitation at the undergraduate level. This cohabitation may be desirable for more than the usual practical reasons (financial, postponing students' choice of specialisation). To maintain or create a common 'core curriculum' in undergraduate mathematics has the virtue of reflecting the partial dependency of professions – not least between teaching at various levels within the educational system – and to facilitate their cooperation. We shall now use the model of the second section to examine this delicate task of balancing[3] the needs of different students and orientations, while making the single unit both independently meaningful and coherent with other units.

First, the model suggests that we must consider *all four aspects together*. Simply specifying a list of contents (internally or externally oriented, or both) is not sufficient, and as such lists tend to be long, the result will often be a level and a quality of performance that do not satisfy anyone. Attempts to specify only performance (abstracted from contents) - sometimes promoted under the name of 'general competencies' - are equally insufficient for actual curricular purposes, as such specifications merely reflect idealised views e.g. of what it takes to be a full-fledged mathematician (or, more generally, scientist) or what one would hope to find with 'good students'. Performance is always *relative to contents*. Likewise, the two are rarely sought to be exclusively internal or external. If we are to create coherent and balanced undergraduate curricula, we must first create a picture of the content-based forms of performance that are required or desired[4],

[3] The frequent use here of the balance metaphor is much inspired by the insights of (Sfard, in press).

[4] In practice, it will often be impossible to distinguish here. But the nature and rationales of such demands will have to be addressed in the 'second run' (cf. the next paragraph).

including their stability and durability. Then we must look at how they are mutually related, both at the level of contents and in terms of the more general aspects of performance, so that the actual study units may be organised with a maximum of coherence and mutual support. This is where the 'general competencies' - such as the ability to use appropriate computer based tools - have their place. However, the task is by no means finished here, as we shall see, but it represents an important 'first run' of the model.

The second point for our task is that the model suggests crucial aspects of the conditions that our curriculum may succeed. This includes, of course, pre-conditions: what are the content knowledge and performance level of our freshmen? It concerns also our experience with the pace and depth with which the two may be expanded. And it requires an overall view of the nature and mutual dependence of various professions and institutions. Holding this together with what the 'first run' produced, we are likely to be faced with a need to make reductions through negotiations and compromise. As long as the overall coherence and balance are thoroughly considered, this is a sound and certainly inevitable part of the process. In particular, it may be a great opportunity to make use of recent studies in tertiary mathematics education and, indeed, to engage a broader range of efforts in this field.

Finally, the model seems particularly geared towards the obvious (but often neglected) demands concerning accountability: any curriculum must be equipped with clear, testable goals of performance related to explicit and specific domains of contents. At university level, we have in particular to facilitate the variety of directions that may be taken by students (at graduate level, professionally, across institutions and national borders). Even this is only half of the balance; no matter how efficient the program may be in meeting these needs, it must also be sufficiently diverse and challenging to meet the needs for personal fulfilment of a sufficient number of our students. Including this in 'testability' means that the curriculum must be not only thoroughly devised, but also continuously revised.

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The Shaping of the Lived Space of Mathematics Learning

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Abstract

It is well-known that students' conception of mathematics is closely related with their problem solving behaviour. This conception is shaped by a number of factors too, in particular, by the space of learning they "live" in. In particular, the teachers' conception of mathematics could attribute to the shaping of this lived space. These inter-relationships are the foci of a series of studies that the author is involved in during these years. In this particular study, methods parallel to the student studies were used to investigate the teachers' conception of mathematics. Interesting results were obtained.

Introduction

Numerous studies have revealed that beliefs about mathematics as a discipline, beliefs about mathematics learning, beliefs about mathematics teaching, and beliefs about the self situated in a social context in which mathematics is taught and learned are closely related to the students' motivation to learn and their performance in the subject (Cobb, 1985; Crawford, Gordon, Nicholas & Prosser, 1998; McLeod, 1992; Pehkonen & Törner, 1998; Underhill, 1988). Indeed, students' beliefs are the key to understanding their actions (Wittrock, 1986), and students' failures to solve mathematical problems are directly attributable to their less powerful beliefs about the nature of mathematics and mathematics problem solving (Schoenfeld, 1983). Naturally, the conceptions of mathematics among students and their approaches to solve mathematics problems are shaped by the "space" they live in.

The research team (other members: Chi-Chung Lam and Ka-Ming Wong) started off the investigation of the lived space of mathematics learning in 1996 in which twenty-nine students were confronted with ten hypothetical situations in which they were asked to judge whether "doing mathematics" was involved in each case. Most of the situations were taken from Kouba and McDonald (1991). Results revealed that students associated mathematics with its terminology and content, and that mathematics was often perceived as a set of rules. Wider aspects of mathematics such as visual sense and decision making were only seen as tangential to mathematics. In particular, they were not perceived as "calculable." However, students did recognise mathematics as closely related to thinking (Wong, Lam, & Wong, 1998).

In 1997, 9 classes (around 35 students each) of each of grades 3, 6, 7 and 9 were asked to tackle to a set of mathematical problems. Each set comprised 2 computational problems, 2-4 words problems and 4 open-ended questions. Two students from each class ($2 \times 9 \times 4 = 72$ students) were then asked how they approached these problems. The original hypothesis was, a narrow conception of mathematics (as an absolute truth, say) is associated with surface approaches to tackling mathematical problems and a broad conception is associated with deep approaches (Marton & Säljö, 1976). Consistent with what was found in previous research, students repeatedly showed in this study a conception of mathematics being an absolute truth where there is always a routine to solve problems in mathematics. The task of mathematics problem solving is thus the search of such routines. In order to search for these rules, they look for clues embedded in the questions including the given information, what is being asked, the context (which topic does it lie in) and the format of the question (Wong, 2000; Wong, Marton, Wong, & Lam, in preparation).

It is clear that students' conception of mathematics is shaped by classroom experience. In a study conducted in 1999, it was found that in Hong Kong, most problems given to students lack variations, possess a unique answer and allows only one way of tackling them (Lam, 2001; Wong & Lam, in preparation). It is thus not surprising that students see mathematics as a set of rules, the task of solving mathematical problems is to search for these rules and mathematics learning is to have these rules transmitted from the teacher. Thus, it is natural to turn our attention to the investigation of conception of mathematics among teachers.

Method

Twelve secondary school mathematics teachers in Hong Kong and fifteen secondary school mathematics teachers in Changchun were confronted with the same set of hypothetical situations that were used among students. Some samples of them are "One day it rained heavily. Alan was sitting in a car then and looked at the rain through the window", "Siu Ping loves to play with dogs. So, he often runs over to Siu Wan's house to see her dog" and "One day the classmate sitting next to you took out a ruler and measured his/her desk". We asked them what would be their reactions if students have different ways of responding (whether taking them as doing or not doing mathematics) to these situations. In addition to these, we confronted the teachers with some quotations of mathematicians like

(a) "Mathematics has nothing to do with logic" (K. Kodaira)
(b) "The moving power of mathematical invention is not reasoning but imagination" (A. DeMorgan)

The interviews were transcribed and content-analysed.

Results

Mathematics is a subject of number and shapes

In judging whether a certain situations involves doing mathematics, one of the major factors of consideration was whether it concerns number and shapes. This is similar to what we found among students. For instance, in responding to the question "One day it rained heavily. Alan was sitting in a car then and looked at the rain through the window", a teacher said, "apparently, it is not mathematics, but since it involves quantity, it is in fact mathematics" (C-A1-4). Though we may have conflicting responses to the same question, number and shape are still the major criteria. For instance, for "Siu Ping loves to play with dogs. So, he often runs over to Siu Wan's house to see her dog", a teacher did not take it as doing mathematics, "as it does not involve number or shape or any relationship between the two" (C-D1-1) but another teacher thought that it is mathematics, "as going to Siu Wan's house can demonstrate a travel graph, in which there is a lot of mathematics" (C-A3-1). We get similar responses from Hong Kong teachers, e.g. "Number is mathematics" (H-CKC-1) and "Those involving number must be math" (H-LCW-2).

Mathematics is closely related to manipulation

Again, similar to what was found among students that mathematics is a subject of "calculables", many teachers judged by whether the situation involves manipulations. Just like what some Hong Kong teachers said, "Mathematics should be calculable, mere observation is not mathematics" (C-CKC-2, C-CKC-5)[1]. This is clear from the following responses that concerns the question "One day the classmate sitting next to you took out a ruler and measured his/her desk":

S. I think this is doing math. Since it is not likely that one can get the length of the desk by just measuring once. The student may have to make a number of measures and then have them added up.
I. How about if s/he can get the length in one measure ?
S. Then, this is not (doing) mathematics since s/he did not calculate. (C-D2-4)

Along this line of thought, calculations with machines are not regarded as doing mathematics. As it was said that "(Using calculators) is not doing mathematics,… since it does not come from the brain of the student" (C-D2-5).

Mathematics is precise and rigorous

This is particularly salient among teachers in Mainland China. Many teachers took that estimation is not mathematics. It is just a kind of "intuition obtained from daily life experience" (C-C3-1, C-D2-T6). A teacher even said that "it is mathematics if one can get a precise estimation. Otherwise, it is not" (C- A3-T3).

Rigor and precision are repeatedly stressed. This is clear from the following

[1] C-XXX indicates responses from participants in Mainland China and H-XXX indicates responses from participants from Hong Kong.

responses:

> “Mathematics is different from language. No matter how one handles, there is only one solution. There is rigor and uniqueness. … Mathematics is rigorous, the way of handling it is governed by (what is laid down in the) textbooks, and there is only one correct answer” (C-A1-10).
>
> “Mathematics is rigorous. It is the exercise of the mind. … Just like whether you can speak rigorously, it depends on your mathematical training” (C-B1-5).

Not many such responses were found among Hong Kong teachers though they did mention that mathematics involves logic (H-LTY-5, H-CKC-8).

Mathematics is beautiful

Another aspect more salient among teachers in Mainland China was that mathematics is beautiful. Some of them pointed out the joy when a problem was eventually solved and such an “Aha, gotcha” is the beauty of mathematics (C-C3-3, C-C4-5). Obviously some also mentioned that the beauty of different shapes, including symmetry (C-D2-12). Simplicity and precision are other aspect of the mathematical beauty as mentioned by the teachers (C-C2-8, C-C3-7), “Simple language is used to describe a complicated situation, I think this is mathematics” (C-C1-11). Though Hong Kong teachers did not talk about the beauty of mathematics, one of them did point out that mathematics is a “cultural activity” (H-LTY-14).

Mathematics is applicable

The conception that mathematics is “closely related with daily life” (H-LTY-1, H-CKC-7) and thus has extensive applications is repeatedly expressed among teachers in both places. “(There is) much mathematics content around us” (H-CKC-9), “(Mathematics) deals with realistic problems with symbols, formulas and diagrams” (H-LWK-8), “Mathematics originates from the realistic situation and applies back to it, it is the abstraction of daily life practices” (C-B1-4), “Mathematics is not isolated, without application in the society, mathematics would not exist” (C-A3-11) were some of their responses. Certainly we see some teachers who put the idea into the extreme:

> “Pure mathematics is an abstract task… just like the Goldbach conjecture, it is pure mathematics. But to me, it has no value in real, it is a waste of time” (C-B2-6).

This point of view seems to be particularly found among teachers in Mainland China. Here we get another similar response, “mathematics can only be legitimised by whether the task is useful” (C-C3-5).

In the interview, we found that teachers are more aware of the relationship between mathematics and real life applications these years due to such an emphasis in the new syllabus, however, many reflected that they still lack such illustrations in the textbooks.

Mathematics involves thinking

Mathematics involves thinking is again unanimously agreed with the teachers in both places. We get comments like “Mathematics is for training logically thinking” (C-C4-4), “Mathematics trains people, other subjects train people too, but mathematics trains the brain of the people” (C-A1-9), “Developing one’s thinking mode is special in mathematics. This is different with other subjects. Without mathematical concepts, there is no training of thinking” (C-A1-12), “Mathematics should involve thinking” (H-LWK-3), “Mathematics is a thinking exercise” (H-LWK-9), “Mathematics involves reasoning” (H-LCK-5) and “Mathematics is conceptual” (H-LWK-5).

Some Hong Kong teachers even used this as a criterion to judge whether it is mathematics or not. Though an action may involve number and mathematical contents, one is not doing mathematics when thinking is not involved. This is clear from the response that “Though mathematics is a subject of numbers and shapes, thinking must involve in between” (H-LCK-11). The same respondent also mentioned that “Mathematics is computation with thinking (reasoning)” (H-LCK-3). Doing mathematics is to “reason with a theory” (H-LWK-4) and “Mechanical calculation without understanding is not math” (H-LWK-11).

The role of problem solving was mentioned among Hong Kong teachers too. “Mathematics is problem solving” (H-CKC-6), “Mathematics involves the handling of problems” (H-LWK-6), “Mathematics helps to match, rearrange and organise” (H-LWK-12) and “Mathematics is a process linking up input information and output solution” (H-LCK-10) were found in their responses. A teacher also pointed out that mathematics activities include the recording, analysis, understanding, interpretation and presentation of data. (H-LTY-9, H-LTY-10, H-LTY-12, H-LTY-15).

A wider perspective

Some Hong Kong teachers could offer a wider perspective and see both sides of the story. For instance, one of them said that “Not everything that involves number is mathematics” (H-LCK-2). Furthermore, “the working steps is not mathematics, it is just a way of presentation. Real mathematics involve thinking procedure rather” (H-LCK-7). The same teacher elaborated, “Though mathematics not confined to ‘calculables’ but it is not wild guess either. Mathematics involves reasonable ways of estimation” (H-LCK-1). Also, “Something is regarded as doing math or not does not depend on the formalities but whether one has mathematics awareness in the process of doing” (H-LCK-6). Though we can find another Hong Kong teacher who holds a relatively narrow conception of mathematics, stating that “Mathematics is what is found in textbooks” (H-THS-1).

The responses of some teachers are more refined, for instance, “Mathematics involves logic but logic is not mathematics” (H-SYF-15), “mathematics is computation with reasoning” (H-SYF-17), and “computation is a way to train thinking” (H-YPS-37).

Discussion

Though some held a wider perspective, the conceptions of mathematics among the teachers basically resemble those of the students. Nevertheless, it was found that the conception of mathematics among the teachers is broader, among which, “mathematics involves thinking” was unanimously agreed. Other facets of mathematics, as reflected by the teachers, include “Mathematics is a subject of number and shapes”, “Mathematics is closely related to manipulation”, “Mathematics is precise and rigorous”, “Mathematics is beautiful” and “Mathematics is applicable”. Also, the apparent dilemma of mathematics being a structured set of knowledge and being widely applicable in daily life, was raised by the teachers in Changchun. Some Hong Kong teacher could offer a wider perspective of mathematics too.

Inevitably, the conception of mathematics among students is both an antecedent and outcome of mathematics learning. If we see the “lived space” of mathematics learning as one that is shaped by the teachers, the teachers’ conception of mathematics may directly influence the students’ conception of mathematics. This in turn will affect students’ problem solving abilities and other learning outcomes of mathematics. In fact, in a recent analysis of the mathematics problems given to Hong Kong students, an overwhelming portion of them are close-ended, demand only low level cognitive skills and are stereotyped (Lam, 2001; Wong & Lam, in preparation). It is hypothesised that, by the systematic introduction of variation, with the widening of the “lived space” of mathematics learning, students could become more capable mathematics problem solvers (Runesson, 1999; Wong, Marton, Wong, & Lam, in preparation).

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Postmodern Rehumanised Mathematics in Teacher Education: A Third Alternative to Modern and Constructivist Mathematics

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Abstract

This talk will present postmodern mathematics as a third alternative to modern mathematics, both set-based and constructivist. Modern mathematics is based upon Platonic structuralist thinking regarding mathematics as universal metaphysical structure above that is echoed in the physical world below. Postmodern mathematics is based upon a post-structuralist thinking regarding mathematical concepts as cultural constructed names for social practices. This talk focuses upon agriculture showing how mathematical concepts grow up from all over agriculture. The talk was given to students at the Vilnius Pedagogical University, and the students' reactions are reported.

Mathematics education: Two fundamental questions and four answers

There are two fundamental questions within concept education as e.g. math:

- How does concepts come into the world - from above or from below?
- How does concepts come into the students - from outside or from inside?

Modern set-based mathematics answers "from above, and from outside" presenting an abstract concept as an example of a more abstract concept: "A function *is an example of* a relation between two sets, that …". The students hear this as "bublibub is an example of bablibab", i.e. a statement without meaning.

The lack of meaning forces students to construct their own meaning e.g. by using a metaphor, that "carries over" meaning from the same abstraction level: "A function <u>is like</u> a machine processing numbers". Constructivist mathematics thus answers "from above, and from inside" seeing concepts as being constructed by the individual student through activity and communication.

Postmodern mathematics answers "from below, and from outside" presenting an abstract concept as a name for a less abstract concept: "A function *is a name for* a calculation with a variable quantity". The students hear this statement as "bublibub is a name for a calculation" thus obtaining meaning from below, s o they will not have to construct their own meaning.

Practise learning e.g. through apprenticeship answers "from below, and from inside". The apprentices construct meaning from observing and participating in the practise: "A function *is for example* 2+x or 2·s, but not 2+3".

This talk will focus upon how postmodern mathematics gives meaning to mathematics by building mathematics from below as a cultural construction.

Agriculture

In agriculture humans are taking over part of the production by cultivating fields and breeding animals. Agriculture faces three important questions: How to divide the land? How to divide its products? How to distinguish between degrees of many? The first question creates the stories of Geometry, meaning “earth-measuring” in Greek. The second question creates the stories of Algebra, meaning “reuniting” in Arabic. The third question creates number names. In agriculture humans are engaged in practices as sowing, harvesting, bundling and stacking, and rebundling and restacking. The practice of rebundling creates multiplication and division, and the practice of restacking creates addition and subtraction.

In the field: Naming many

After the harvest the products are united into bundles, and bundles can have different sizes. The first ten bundle sizes get different names (one, two, ..., ten) and different symbols (1, 2, ..., 9, D). In most cultures ten is considered a full bundle, so after ten two countings takes place: the one counting full-bundles and the other counting unbundled: 35 means three full-bundles and five unbundled. Zero 0 is introduced to account for absence: 60 means six full-bundles and no unbundled. A full-bundle of full-bundles is called a hundred-bundle C. And a full-bundle of hundred-bundles is called a thousand-bundle M. A full-bundle of thousand-bundles is not named, neither is a full-bundle of full-bundles of thousand-bundles, but a full-bundle of full-bundles of full-bundles of thousand-bundles is called a million.

© © ©	The total is 3 times an apple	T = 3 x © or T = 3 x a
• • •	The total is 3 1s	T = 3 x 1
•••	The total is 1 3s	T = 1 x 3
••• ••• • • • •	The total is 2 3s and 4 1s	T = 2 x 3 & 4 x 1
D •••••••••• • • • •	The total is 2 tens and 4 1s	T = 2 x D & 4 x 1 or T = 24
D D	The total is 2 tens and no 1s	T = 2 x D & 0 x 1 or T = 20
C DDDDDDDDDD DDD •	The total is 2 hundreds and 3 tens and 1 1s	T = 2 x C & 3 x D & 1 x 1 or T = 231
W ••••••• • • • •	The total is 2 7s and 4 1s, or The total is 2 weeks and 4 days	T = 2 x 7 & 4 x 1 or T = 2 x W & 4 x d

Special bundles have special names:
1 x week = 7 x days, 1 x hour = 60 x minutes, 1 x meter = 100 x cm etc.
“T = 3 x 1” is called a total-story, a T-calculation story or a T-equation.

Naming Parts of Bundles
We use fractions to tell, that only part of a bundle has been filled.

••·	The total is 2 of 3 3s	T = 2/3 x 3
••···	The total is 2 of 5 5s	T = 2/5 x 5
••········	The total is 2 of ten tens	T = 2/10 x 10 or T = 0.2 x 10

Rebundling
From 2s to 1s: multiplication (taking away 1s)

•• •• •• ••	The total is 4 2s is ? 1s	T = 4 x 2 = ? x 1
• • • • ?		T = 4 x 2 = 4·2 x 1 = 8 x 1 *Notice: 4 x 2 = 4·2*

From 1s to 2s: division (taking away 2s)

• • • • • • • •	The total is 8 1s is ? 2s	T = 8·1 = ?·2
•• •• ?		T = 8/2·2 = 4·2

From 2s to 3s: multiplication and division (taking away first 1s then 3s)

•• •• •• ••	The total is 4 2s is ? 3s	T = 4·2 = ?·3
••• ?		T = 4·2/3·3 = 8/3·3 = 2·3 & 2·1 = = 2 2/3·3

The rebundling rule
Rebundling can be done manually by rearranging and counting. And rebundling can be done mentally by calculating, thus predicting the result of a rebundling before it is carried out.
The ability to predict through calculations is the heart of (postmodern) mathematics.
The “rebundling rule” tells how to calculate a rebundling from 1s to e.g. 2s:

8 = 8/2·2 or T = T/b·b

The rebundling rule shows the meaning of the Arabic word algebra, reunite: First 8 is divided into 2s (8/2), then 8 is reunited (8 = 8/2·2).

Rebundling into special bundles
From weeks to days

W W W	The total is 3 weeks is ? days	T = 3·w = ?·d (1·w = 7·d)
ddddd ?	*1 week is 7 days*	T = 3·7·d = 21·d

From days to weeks

dddddddddddddd ddddd	The total is 18 days is ? weeks	T = 18·d = ?·w (1·w = 7·d)
W W ?	*1 week is 7 days*	T = 18/7·7·d = 2·w & 4·d, or T = 2 4/7·w

At the farm: Stacking like bundles
Back at the farm we stack like bundles, unless we use parts of bundles.

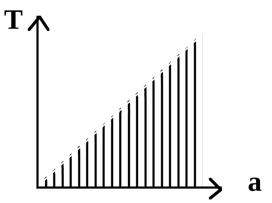
••• ••• • • – ••• • ••• •	The total is 2 3s and 2 1s – a 2-stack of 3s and a 2-stack of 1s	T = 2·3 & 2·1 – T = 2·3 & 2·1
••• ••• • • – ••· ••• •••	The total is 2 3s and 2 1s – a 3-stack of 3s	T = 2·3 & 2·1 – T = 2 2/3·3

Paying parts to the king and to the bishop
The king and the bishop does not work in the field, they provide protection paid for by parts.

••••••••••••••• 500 $\boxed{9}$	The total is 4500 1s The part is 2 9-parts of the total Rebundle the total into 9s!	T = 4500/9·9 = 500·9 = 9·500 1 9-part is 500 = 4500/9 P = 2 9-parts = 2·500 = 1000
••••••••••••••• 45 $\boxed{100}$	The total is 4500 1s The part is 20 100-parts (20%) of the total Rebundle the total into 100s!	T = 4500/100·100 = 45·100 1 100-part is 45 = 4500/100 P = 20 100-parts = 20% P = 20·45 = 900

Coding and decoding
A total of 2 150s can be coded as T = 2·150 = 2·a
Decoding can be guessing by filling out tables or by drawing stacks in diagrams:

a	T = 2·a
0	T = 2·0 = 0
1	T = 2·1 = 2
2	T = 2·2 = 4
3	T = 2·3 = 6
4	T = 2·4 = 8
5	T = 2·5 = 10



Decoding can also be solving the equation T = 2·a, if we know what T is (e.g. 300 or 460 or 720)

a	T = 2·a	
a=?	T = 2·a = 300 = 300/2·2 = 150·2	**a = 150**
a=?	T = 2·a = 460 = 460/2·2 = 230·2	**a = 230**
a=?	T = 2·a = 720 =	**a =**

In the shop: Bying and selling

A shop takes care of trade, i.e. buying and selling e.g. 3 digits numbers, where mr. C trades hundreds, mr. D trades tens, and mr. 1 trades ones.

<u>Buying:</u>	*mr.C mr.D mr.1*		*mr.C*	*mr.D*	*mr.1*
T1	= 4·100 & 6·10 & 3·1	=	4	6	3
B	= 2·100 & 7·10 & 5·1	=	2	7	5
T2 = T1+B	= (4+2)·100 & (6+7)·10 & (3+5)·1				
Full House	= 6·100 & 13·10 & 8·1	=	6	13	8
Restack!	= (6+1)·100 & (13-10)·10 & 8·1	=	6+1	3	8
	= 7·100 & 3·10 & 8·1	**=**	**7**	**3**	**8**
<u>Selling:</u>					
T2	= 7·100 & 3·10 & 8·1	=	7	3	8
S	= 2·100 & 7·10 & 5·1	=	2	7	5
T3 = T2–S	= (7-2)·100 & (3-7)·10 & (8-5)·1				
Empty House	= 5·100 & -4·10 & 3·1	=	5	-4	3
Restack!	= (5-1)·100 & (10-4)·10 & 3·1	=	5-1	10-4	3
	= 4·100 & 6·10 & 3·1	**=**	**4**	**6**	**3**

Multiple packages can be bought (or sold):

B1 = 5·B	= 5·(1·100 & 6·10 & 9·1)	=	5·(1	6	9)
	= 5·1·100 & 5·6·10 & 5·9·1	=	5·1	5·6	5·9
Full House!	= 5·100 & 30·10 & 45·1	=	5	30	45
	= (5+3)·100 & (30-30+4)·10 & (45-40)·1	=	5+3	30-30+4	45-40
	= 8·100 & 4·10 & 5·1	**=**	**8**	**4**	**5**

A package can also be split:

B1 = B/5	= (7·100 & 6·10 & 9·1)/5	=	(7	6	9)/5
	= ((7-2)100 & (20+6-1)·10 & (10+9)·1)/5	=	(7-2	20+6-1	10+9)/5
	= (5·100 & 25·10 & 15·1 & 4·1)/5	=	(5	25	15+4)/5
	= (5/5)·100 & (25/5)·10 & (15/5)·1 & 4/5·1				
	= 1·100 & 5·10 & 3·1 & 4/5·1	**=**	**1**	**5**	**3 4/5**

The restacking rule

Restacking can be done manually by rearranging and counting. And restacking can be done mentally by calculating, thus predicting the result of a restacking before it is carried out.

The ability to predict through calculations is the heart of (postmodern) mathematics.

The “restacking rule” tells how to split e.g. an 8-stack into an e.g. 2-stack & another stack:

8 = 8–2+2 or T = T–b+b

Also the restacking rule shows the meaning of the Arabic word algebra, reunite: First 8 is split into 2 and something (8–2), then 8 is reunited (8 = 8–2+2).
So rebundling creates multiplication and division, and restacking creates addition and subtraction:

Rebundle 8 into 2s by taking away 2s:	8 = 8/2·2= 4·2	T = T/2·2
Restack 8 into 2 & something by taking away 2:	8 = 8–2+2= 6+2	T = T–2+2

Coding and decoding in the shop
Writing bills can lead to coding and decoding and to solving equations
A bill typically consists of a subtotal (e.g. 80$) and an added purchase:

The Total of 80 $ and 5 days @ 6$/day is 110 $:	T = 80 + 5·6 = 110

This bill can be coded in different ways:

The Total of 80 $ and 5 days @ 6 $/day is ? $:	T = 80 + 5·6 = ?
The Total of ? $ and 5 days @ 6 $/day is 110 $:	T = b + 5·6 = 110
The Total of 80 $ and ? days @ 6 $/day is 110 $:	T = 80 + n·6 = 110
The Total of 80 $ and 5 days @ ? $/day is 110 $:	T = 80 + 5·a = 110
The Total of b $ and n days @ a $/day is 110 $:	T = b + n·a

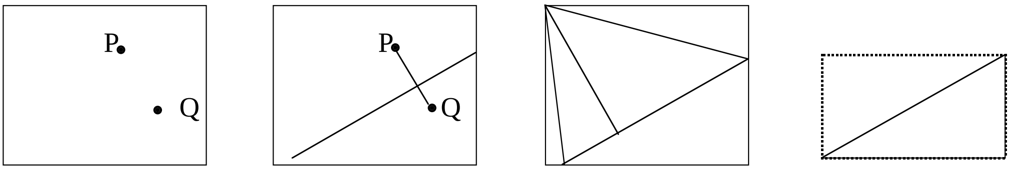
Decoding 110 = 80 + 5·a:

Coded bill:	110 = 80 + 5·a	
Restacking 110 into 80 & something:	110-80+80 = 80 + 5·a	
	30	= 5·a
Rebundling 30 into 5s:	30/5·5	= 5·a
	6·5	= 5·a
Decoded result:	**6**	**= a**

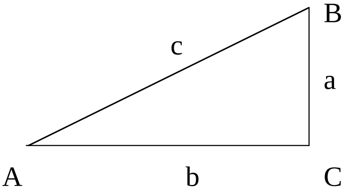
Later decoding becomes solving a linear equation 110 = 80 + 5·a by means of an equation-scheme:

Unknown:	a = ?	T	= b + a·n	Equation
Known:	T = 110	T–b+b	= b + a·n	Restacking T into b & something
	b = 80	T–b	= a·n	
	n = 6	(T–b)/n·n	= a·n	Rebundling T–b into ns
		(T–b)/n	= a	
		(110–80)/6	= a	
		5	= a	

Geometry: Dividing the land



A given piece of land is to be divided between to settlements P and Q so that each has the same distance to the border. This leads to words as points, lines, perpendicular bisectors, polygons and triangles. And to right triangles being the half of a rectangle, divided by a diagonal.



The Greeks
A + B = 90
$a^2 + b^2 = c^2$

The Arabs
A + B = 90
sinA = a/c
cosA = b/c
(tanA = a/b)

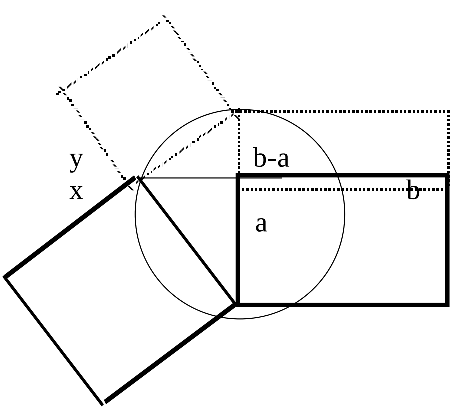
A triangle has six pieces: Three angles and three sides. In a right triangle we know the right angle. In order to draw a right triangle we need additional information about a side and about a third piece. The last three pieces can then be measured, or calculated if we have three equations.

The Greeks failed since they only found two equations: The sum of the angles are 180 degrees. And the Pythagorean Theorem $a^2 + b^2 = c^2$.

The Arabs succeeded in finding two additional equations to the Greek ones by introducing a double measurement of the sides: an outside measurement in "meters", and an inside measurement in diagonals, thus rebundling the short sides in diagonals: a = a/c·c = sinA·c and b = b/c·c = cosA·c.

Rebundling and restacking areas

Rectangular areas can be rebundled and restacked into another shape having the same area. Rebundling and restacking into squares as below will lead to the Pythagorean Theorem: $a \cdot b = x^2$, $(b-a) \cdot b = y^2$, so $b^2 = x^2 + y^2$



Student reactions

After the talk 32 of the teacher students handed in a questionnaire asking them to express their attitude on a scale from very negative to very positive:

0: very neg., 1: neg., 2: a little neg., 3: neutral, 4: a little pos., 5: pos., 6: very pos.
The numbers report the mean, median and the mode and its frequency.

	in primary school (1-4)	in basic school (5-10)	in secondary school (11-12)	in teacher education
Only modern math should be taught	3,6 4 5 (8)	3,9 4 3 (11)	3,9 4 4 (11)	3,5 4 4 (8)
Only postmodern math should be taught	2,3 2 1 (9)	2,7 3 1,3 (8)	3,0 3 3 (13)	3,8 4 3,4,5 (7)
Both should be taught	3,1 3 5 (8)	3,6 4 3,4 (8)	4,2 4 4 (13)	5,4 6 6 (17)

Bundling leads to Total stories as T = 3*4 & 2*1	3,8 4 4(15)
Describing parts of bundles leads to fractions as T = 2/3*3	4,0 4 4(13)
Describing parts of bundles leads to decimals as T = 2/10*10 = 0.2*10	3,9 4 4(12)
Rebundling leads to the rebundle-rule T = T/b*b	4,3 5 5(14)
Coding and decoding can lead to solving an equation as 300 = 2*a	4,5 5 5(17)
Buying leads to addition T =T1+B = (2*w & 4*d)+(3*w & 5*d) = (2+3)*w & (4+5)*d	4,0 4 4(11)
Selling leads to subtraction T3=T2-S=(6*w & 2*d)-(2*w & 3*d) = (6-2)*w & (2-3)*d	4,4 4 4(14)
Restacking leads to the restacking-rule T = T – b + b	4,4 5 5(14)
Writing bills leads to coding and decoding and solving the equation T= b+a*n	4,4 4 4(16)
In the field dividing land leads to points, lines, polygons, triangles and right-angled triangles	5,0 5 5(13)
Right-angled triangles lead to Greek failure with only two of three equations and to Arabic success with 3 equations	4,4 4 4(11)
Rebundling rectangular land into squares leads to the Pythagorean theorem	4,3 4 4(11)

Conclusion

Teaching only modern mathematics in school and in teacher education got a partial positive reaction. Teaching only postmodern mathematics in school got a partial negative reaction, and a partial positive reaction in teacher education. Teaching both modern and postmodern mathematics in school got a partial positive reaction, and a very positive reaction in teacher education.

As to specific topics especially rebundling, restacking, coding & decoding equations and geometry through dividing land got a positive reaction.

Encouraged by this the authors have started to develop a talk on post-modern rehumanised mathematics in the renaissance and in modern industrial culture, thus offering an alternative postmodern approach to secondary mathematics.

Building the Mathematical Confidence of Pre-Service Teachers

Linda Hall
USA

In primary, elementary, and special education teacher preparation programs many students lack confidence in their mathematical abilities. This translates into a lack of success in course work and a less than positive attitude toward mathematics. The pre-service teacher will fail to learn important concepts, strategies, and teaching methods necessary to successfully guide their students in their acquisition of mathematical knowledge. And, in most cases, these teachers will slight the time given for mathematics during the school day. The result is critical since not only are we graduating future teachers with marginal mathematical knowledge and poor attitudes toward mathematics but they are failing to instill in their students' numeracy, the beauty of mathematics, and a love of mathematical learning.

To counteract this critical situation, it is imperative for those in teacher education to view critically the course structure for these pre-service teachers. To require courses with advanced mathematical rigor that encourages failure rather than success is to further alienate these students from mathematics. To require courses that fail to increase numeracy, to instill a quest for mathematical knowledge and to model best teaching practices is to shortchange our children. Building the mathematical confidence of these future teachers is critical to their attitudes toward and success in teaching mathematics.

Many of these pre-service teachers enter teacher education programs with math anxiety, ranging from mild to severe cases. In a class of 27 students enrolled in a mathematics course structured for primary education, just under 90% of the students expressed a lack of confidence in their mathematical ability, anxiety when faced with mathematical problems, and concern about their lack of knowledge in mathematics. To provide an environment that would allow these students to increase their mathematical confidence while increasing their mathematical knowledge necessitated changes in course delivery.

Three things were initiated immediately: no formal or written "tests", a classroom atmosphere that encouraged attempting all mathematical problems and situations without fear of failure or ridicule, and the formation of groups. In announcing that no formal or written "tests" would be given there was an audible sigh in the class. Students later indicated that the knowledge they would not be taking tests relieved the anxiety they experienced when preparing for and taking tests and gave them the freedom to learn. This did not relieve them from evaluation - students were required to research assessment and evaluation practices and to assist in developing an assessment for their course grade and a self-evaluation instrument.

In developing a classroom atmosphere where students were encouraged to attempt all mathematical problems a class ethos was established. No one would be criticized or ridiculed if they did not succeed, respect would be given to all students, and recognizing and correcting mistakes was as important as being correct originally. This also meshed with the formation of the groups. Students were encouraged to work together on problems, to share their thinking, and to assist other group and/or class members who might be having difficulty in understanding concepts or problems. The students were expected to accept responsibility as a group.

Most of these students had come from a background of rote learning, individual work product, and grades based on homework and tests. They were now exposed to a classroom where they were expected to work in groups, share information and thinking, and produce work that would reflect their learning over the entire course. In evaluations, students reflected on their discomfort initially when asked to work as groups. One student indicated that she thought it would be difficult but found it to be a very positive experience. She reflected on the patience of other group members when she did not understand something, how they would explain it and make sure she understood it thoroughly before moving on. They also had to move past the perception of "cheating." After working with Pascal's triangle in class students were asked to find other patterns. One student returned with information obtained through internet sites and was concerned that it would be considered cheating. She had researched the problem, found information that increased her mathematical knowledge, and shared that information with the class in a way that indicated her understanding of the pattern. The class was enriched by her contribution and she realized that what she had accomplished was not cheating but learning.

It was important to model the teaching methods that the students would be expected to use in their classroom. Every class started with at least 15 minutes of mental math. A few problems would be put on the overhead projector for the students to solve mentally. This again was difficult for many students who wanted to revert to paper and pencil for easy problems. When all students had an opportunity to complete the problems students were asked to volunteer an answer. After several answers were given, correct or incorrect, students were asked to explain how they arrived at the answer. Initially students were hesitant to explain, fearing criticism for their approach, their thinking, or their answer. When it became evident that criticism was not forthcoming, students became more willing to share their thinking, even when they had reached an erroneous conclusion. Students indicated in their evaluations how important it had been to hear others explain their thinking, to see different approaches to the same problem, and then to try these themselves.

Mathematical concepts were also learned through problem solving and activity based explorations, similar to the activities they would be using in their own classrooms. These activities allowed the students to explore mathematical concepts, increase their mathematical knowledge, and learn to work cooperatively within their group,

Students were required to keep a journal throughout the class. They were encouraged to look at mathematical insights, insights about teaching and learning, things that contradicted their belief system, insights about themselves as a learner, things they found interesting, frustrating, challenging, exciting, or aha's. These journals helped students reflect on their learning and experiences with mathematics.

For their final assessment students compiled a comprehensive portfolio. In addition to all assignments from the course they selected pieces that included: their most important work, a selection that helped complete the picture of them as a learner/teacher of mathematics, a reflective piece, a letter synthesizing their "big ideas" from the course, a statement of their philosophy of mathematics education, and a reflection on what they learned from other students in the class and members of their group. The last item in the portfolio was a self-assessment that looked at all the goals of the course. Not only did this provide the students with a comprehensive record of all their work in the course, it required them to evaluate their own learning.

Though this approach differs greatly from the majority of mathematics classes required for pre-service teachers it was evident from evaluations that students learned mathematics, became confident in their mathematical ability, and in their ability to teach mathematics. These students have been given a taste and their thirst has not been quenched. They will continue their mathematical learning.

An Inductive Approach to Conceptual Development in the Area of Functions

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Introduction

Concepts and notation in connection with functions often turn out to be very abstract for many pupils and students. At the same time, it is possible to reach a certain level of computational skill sufficient to solve many exercises without possessing a basic understanding of the concepts. Solving equations, solving max/min problems, computing derivatives and integrals serve as examples of this situation.

We will present the way we have let our students in teacher education work with the concepts of differentiation and integration.

Our didactical ideas

According to the national plans for teacher education our students shall '*acquire insight into the basic ideas behind differentiation and integration'*. Some of our students have studied mathematics for three years in secondary school, and they have worked quite a lot with these topics. However, many of them do not have a genuine understanding of the concepts, but they have developed a mechanical mastering which makes them capable of solving problems by manipulating algebraic expressions. These students will not be very motivated to go deeper into the understanding of the concepts as long as their algebraic skills are sufficient to solve problems. On the other hand, most of the students have studied mathematics just one year in secondary school, and they have barely touched the topics differentiation and integration before. For them it sounds very advanced, far from their everyday life, and therefore both frightening and irrelevant. Their fear is also due to the fact that they think that some of their fellow students 'know a lot about it'.

To strengthen the motivation for a deeper understanding for all groups, and to smooth out the differences due to previous experience, our students have been given problems to work with where they themselves to some extent have participated in creating the context, and where algebraic skills have neither been sufficient nor necessary to solve the problems. In this way we have generated a situation where the students have started to discuss with each other and with the teachers in a meaningful way. Here we see traces of what Skovsmose (1998) refers to as a 'landscape of investigation' (Danish: 'undersøkelseslandskap'). It has been vital for us to move away from an arena where the teacher is in the centre – where the teacher is the one to decide what are sensible questions, and then to pose these questions to the students.

Our motive has been to create nearness to the concepts. The students should actively construct the concepts by making them meaningful to themselves. Through discussion, the understanding of the concepts will be more robust. A basis for our work is a constructivist view on learning based on the idea that each individual builds his/her own knowledge. In addition we will underline the importance of being active. The students gain knowledge through activity, and it is important that this takes place in an interaction between fellow students and teachers. We have tried to take the students to a 'landscape' where they become curious and want to investigate mathematical coherence.

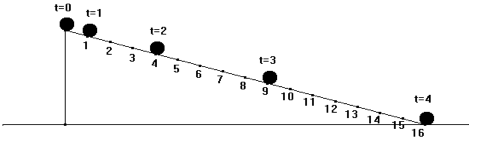
Examples with experiences and comments

Our program starts with developing a qualitative understanding of phenomena that can be described by differentiation and integration. We preferred to use the concepts 'rate of change' and 'summation' instead of the words differentiation and integration. Gradually we develop quantitative calculations that require that one goes to the core of the concept, but at no point we develop algebraic methods (formal differentiation and integration). The main scope of our program is to give a good basis for the understanding of the mathematical concepts, and at the same time we want to show how these concepts can be found in the society around us.

Example 1. How to measure the speed

If a ball rolls down an inclined plane, its speed will gradually increase. By measuring we find that it has covered 1 meter in 1 second, 4 meters in 2 seconds, 9 meters in 3 second and 16 meters in 4 seconds. The problems that were given in connection with this were the following:

- Discuss how you could measure the speed of the ball at the various points based on the information in the figure below.
- Discuss how you could find the speed more accurately by making more observations.



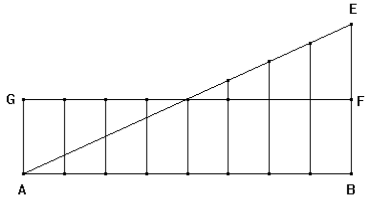
For the first problem we got several different answers. To get the speed for $t = 2$, say, some calculated the average speed from 0 up to 2, some from 1 to 2, and some even from 1 to 3. When they got different answers, they started to discuss the reason for this, and also to discuss what would be the most correct answer in the sense which answer would be closest to the actual speed, and why this was so. For the second problem, the idea of measuring over shorter intervals came up in a natural way. They also developed the idea that it would be wise to measure a little bit before and a little bit after the point we are interested in since the speed is increasing, and therefore *"it will become more accurate if we take one piece where the speed is too small, and one piece where it is too large"*.

Example 2. How to find the distance knowing the speed

You are driving a car, and your speed increases steadily from 0 km/h to 60 km/h for 6 minutes. After that you have constant speed 60 km/h for 12 minutes, then your speed again increases steadily from 60 km/h to 80 km/h for 6 minutes. Finally you drive with constant speed 80 km/h for 36 minutes.

- What is the total distance you have covered?
- Draw in a coordinate system a graph describing the speed as a function of time
- Compute the area of the region bounded by the *x*-axis, *y*-axis and the graph you have drawn. Comment on the result.

Before working with this example, we had worked with 'the constant speed situation', so the idea of '*distance equals speed times time* ($s = v \cdot t$)' was familiar, and so was the geometric interpretation of the distance as the area of a rectangle in this case. But what about the situation when the speed was not constant, and the graph was no longer a horizontal line? In the discussion following this example, the idea of finding the average speed in the intervals where the speed was increasing, came up. Intuitively, they argued that the average speed in such an interval had to be midway between the speed in the beginning and the speed at the end. Now the average speed could be introduced in the graph as a horizontal line (GF in the figure below), the $s = v \cdot t$ formula could be used, and the distance represented by the area of a rectangle was again brought back into the picture.



Comparing the area of the rectangle ABFG and the triangle ABE, they found that they were equal. Therefore, the distance could also be interpreted as the area under the actual (non constant) speed curve, because as one of the students said: *"What we have extra until we are half way is just as much as we have too little on the last half."*

We find it of interest to mention that these students now made the same observation as Galilei did more than 350 years ago (*Two New Sciences, 1638*), here in a quotation from Gårding (1983).

> The time it takes a body with uniform acceleration to move a certain distance equals the time it takes for the same body to move the same distance with a uniform velocity equal to half the sum of the least and the largest velocities in the uniformly accelerated motion.

The value of the approach in a wider perspective

We think that this approach has a value in two main respects. First, we must bear in mind that these students will be primary school teachers. Therefore they will not face the need for teaching formal differentiation and integration in their future profession. However, these concepts are found in our everyday life, and it is relevant to work with the ideas also on primary school level. An algebraic approach, in particular combined with poor understanding of the concepts themselves, will be of little or no value in transforming the ideas to primary school level. Second, some of these students will go further in mathematics, and then formal calculus will be a natural next step. We believe that the basis given through this approach will be a good one since applications of calculus in problem solving require a solid understanding of the concept.

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Using the Internet for Assessment

Linda Hall
USA

One of the most time-consuming tasks for any educator is the construction and correcting of tests. With the emphasis on standards, accountability, and standardized testing, classroom assessments become even more important. Textbook related tests or pre-constructed tests do not always align to state or local standards and benchmarks. How does the teacher or administrator determine with any degree of accuracy how students are progressing, whether they understand concepts required by standards, and whether the questions really align to the standards and assess that concept?

Internet based assessment programs may revolutionize how assessment is conducted in education. Though this technology could be considered "first generation" it is advancing rapidly. These programs vary in the product and support they provide to teachers, and administrators. Some allow a teacher to select the standards she/he would like to test, select questions that align to these standards, and construct the test. Students take the test on-line, it is graded on-line, and the results are provided to the teacher. Reports can be generated that track the progress of an individual student throughout the year, indicate possible deficiencies in student knowledge, pinpoint gaps in curriculum, show test results according to selected demographics, and allow administrators to view district progress on standards. These comprehensive programs give the teachers and administrators a wealth of information that allows them to truly assess the progress and knowledge of their students.

Other sites only provide questions for use as paper and pencil tests. Many times these questions have been submitted by teachers with no alignment to national or state standards. There are also sites that provide assessment opportunities on the internet at specific times under controlled conditions.

With internet based assessment increasing, it is imperative for educators to be aware of this rapidly advancing technology and become familiar with some of the many sites that offer these programs.

Teaching Algebra at Lower Secondary School

Tiiu Kaljas
Estonia

Teaching mathematics at the lower secondary school level comprises mostly of teaching basic arithmetic, geometry and algebra. The goal of the teacher is to present basics to the student in an interesting and understandable manner. Learning of algebra requires relatively higher levels of comprehension and abstraction. Therefore the challenge for the teacher is to teach algebra in a manner that would not lose the students' motivation and comprehension.

For the past 50 years, Estonian teachers have relied on a teacher-oriented method of presenting facts, as opposed to encouraging student initiative to research their own solutions to problems and answers to questions:

- It is easier for a teacher to pontificate from a standard class program than it is to encourage student initiative;
- It is easier to evaluate each individual student with a standardized test as opposed to evaluating the success or shortcomings of individual initiatives.

With the new millennium, we need to rethink our teacher-oriented teaching methods in our schools and encourage individual student initiatives and interaction. Even good teachers can fall into a route, tending to stick to their familiar methods as opposed to customizing their teaching to the students.

With the introduction of information technology often being an enigma to many of today's teachers, it is the younger teachers who are open to new forms of education. Younger teachers are quicker to accept the idea of a student focused educational system and more willing to accept the added responsibilities and obligations of student based education.

They are beginning to introduce new methods while older teachers keep to continuity. Until the new educational focus becomes generally accepted, teachers today focus on and make the most of subject based teaching. Teachers today can rely more on multimedia visual aids to enhance comprehension. Teachers can utilize more integrated methods for understanding, i.e. practical applications to reinforce theorems.

Every concept in school algebra should be presented in four different forms: verbally, symbolically, by figures, and visually.

Literature

Maletsky, E.M. (1996). Handling, seeing, and thinking experiences in mathematics. In *The art of problem solving* (pp. 173-198). Thousand Oaks, CA: Corwin.

The Problem Field of Viviani's Theorem

Tünde Kantor
Hungary

	generalized inequality of Erdös-Mordell for the tetrahedron	
	↑	
generalization to the out-side point P: $k_1d_1+k_2d_2+k_3d_3+k_4d_4=$ const. $k_1, k_2, k_3, k_4=\pm1$	similar theorem for the regular tetrahedron $d_1+d_2+d_3+d_4=$const.$=h$	
	↑	
Barrow's theorem (scalene triangle)	**VIVIANI'S theorem (regular triangle)** $d_1+d_2+d_3=$const.$=h$	theorem of Erdös-Mordell (scalene triangle)
	converse theorem	
	↓	
P is on the base of the isosceles triangle $d_1+d_2=$const.$=2a$ converse theorem	P is on the side of the equilateral triangle $d_1+d_2=$const$=2a$	P lies on the extension of a side of the equilateral triangle $k_1d_1+k_2d_2=$const. $k_1, k_2=\pm1$
	↓	
	P is an exterior point of the equilateral triangle $k_1d_1+k_2d_2+k_3d_3=$const. $k_1, k_2, k_3=\pm1$	
	↓	
in an equilateral triangle we draw parallels through the interior point P to the sides $d_1+d_2+d_3=$const.	in an equilateral triangle we draw segments through the interior point P. The segments make the same angle with the proper sides of the triangle. $d_1+d_2+d_3=$const.	generalization to regular n-gons $k_1d_1+k_2d_2+\ldots+k_nd_n=$const. $k_1, k_2,\ldots, k_n=\pm1$

A Master Program in Rehumanised Mathematics?

Allan Tarp
Danish University of Education

Target group: Graduates within humanities wanting to become number language teachers**.**

Background: A decreasing enrolment in mathematics and mathematics based education within science and technology, and a coming mathematics teacher shortage.

Philosophy: A rehumanised mathematics curriculum is following the principle of high necessity by placing its authority with the necessities of nature that gave birth to mathematics. First to multiplicity giving birth to the practices of bundling and rebundling, and stacking and restacking, and to stories about the total, found by counting or calculating. Later to the four necessities of nature that physics is built upon: Mass, charge and extension in time and space.

Content: The quantitative literature of different eras

Module 1. The antique dominated by agriculture and local trade
Cultural background: From hunting and gathering to agriculture and local trade
1. Phrasing and quantifying the necessity of multiplicity. Different number systems
2. Phrasing and quantifying the dividing up of earth. Greek and Arabic geometry
3. Phrasing and quantifying the dividing up and reuniting of numbers. Arabic algebra

Module 2. The renaissance dominated by mining and global trade
Cultural background: From local trade to silver based global trade
1. Quantifying the mine. Levers, wheels and pulleys
2. Quantifying money. Bookkeeping and interest calculations
3. Quantifying local motion and the necessity of time. The local laws of falling bodies
4. Quantifying figures. Perspective and co-ordinate geometry

Module 3. Early modernity dominated by steam power
Cultural background: From His incalculable will to nature's calculable will
1. Quantifying the necessity of space. Vectors
2. Quantifying macro motion. The global laws of falling bodies
3. Quantifying games and chance. Probability and statistics

Module 4. Late modernity dominated by electronic technology

Cultural background: From manpower to cyborgs

1. Quantifying micro motion. From calculable atoms to incalculable electrons
2. Automating numbers. Electronic number processing
3. Automating calculations. Systems: linear, dynamical, risk, chaotic

Language Perspective: Qualities and quantities. Word language and number language. Language and meta language. Mathematics as a grammar of the number language. Modern structuralism and Bourbakism dehumanising the number language Genres of quantitative modelling: fact, fiction and fiddle. The question of representing the world in language.

Learning Perspective: Biological brain forms: Reptile-, mammal- and human brains. The two main questions of learning: How do concepts come into the world, from above or from below? How do concepts come into the students, from outside or from inside? The corresponding learning theories: Structuralism and nominalism, transfer and constructivism.

Organisation: Each module consists of a number of meetings and assignments, and a terminal project. Distance education will be an option. The language and the learning perspective are integrated into the modules.

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