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*The relation between mathematics  
education research and teachers'  
professional development*

Proceedings of MADIF 13  
The thirteenth research conference of  
the Swedish Society for Research  
in Mathematics Education  
Växjö, March 29–30, 2022

Editors:  
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Cecilia Kilhamn, Hanna Palmér, Miguel Perez,  
Kerstin Pettersson

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**SMDF** Svensk förening för MatematikDidaktisk Forskning  
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# Preface

The 29<sup>th</sup>–30<sup>th</sup> of March 2022, the Swedish Society for Research in Mathematics Education (SMDF) welcomed everyone interested in scholarly discussions in mathematics education to Linnaeus University and the thirteenth Swedish Mathematics Education Research Conference – MADIF 13. The theme of the conference was *The relation between mathematics education research and teachers' professional development*. Altogether, 107 participants engaged in the conference, which offered 10 paper presentations, 27 short presentations and 3 symposia. This volume constitutes the proceeding of this conference, providing a glimpse of current exciting research in mathematics education in Sweden and in other Nordic countries.

## Acknowledgment

After almost two years of lockdown, due to the pandemic, it was a relief to finally be able to meet scholarly friends face to face and it was a pleasure to see senior researchers mingle among doctoral students and young researchers – all contributing to an open-hearted community in mathematics education. Thus, although this volume acts as a documentation of the written contributions to the conference, hopefully it also acts as a reminder of all the special moments of curious and intensive discussions, of thrilling inspirations as well as of friendly smiles and laughter between participants that permeated the two days in Växjö. These meetings are what a conference is all about. Therefore, the program committee would like to thank all participants of MADIF 13 for making this conference a lovely example of how to exchange research perspectives and thoughts in an open and positive way.

The members of the program committee would also like to express their gratitude to the invited speakers Professor Susanne Prediger, Institute of development and research in mathematics education, Dortmund, and Professor Paul Cobb, Vanderbilt University, Nashville, for framing the conference with their inspiring plenary lectures *Promoting teacher expertise for fostering students' understanding of arithmetic: an example for content-related PD research* and *Investigating what it takes to improve mathematics teaching and learning on a large scale* respectively. The presentations can be found on the website of SMDF ([matematikdidaktik.org](http://matematikdidaktik.org)).

Furthermore, we would like to thank all of you who submitted papers, short presentations, or symposia to MADIF 13. Your contributions resulted in many interesting interactions at the conference as well as in writings presented in this volume. The papers are research reports of original work not previously published; the symposium proposals are documents presenting two or three studies on a common theme which were presented and debated during a symposium. The abstracts of the short presentations describe parts of projects or on-going research that were presented at the conference. At MADIF 13, all presentations were followed by discussions led by allocated discussants. The program committee would like to recognize the fine work done by these scholars: Linda Marie Ahl, Camilla Björklund, Magnus Fahlström, Laura Fainsilber, Peter Frejd, Robert Gunnarsson, Kerstin Larsson, Niclas Larsson, Elisabet Mellroth, Rimma Nyman, Johanna Pejlare, Jöran Petersson, Helena Roos, Ulrika Ryan, Johan Sidenvall, Lovisa Sumpter, Anna Ida Säfström, Jorryt van Bommel, and Jonas Bergman Ärlebäck.

Moreover, we are very thankful for the great work done by our reviewers – some of you also presenting at the conference, others contributing to our community as external reviewers: Mario Sánchez Aguilar, Linda Marie Ahl, Lluís Albarracín, Ewa Bergqvist, Camilla Björklund, Magnus Fahlström, Laura Fainsilber, Peter Frejd, Staffan Frid, Robyn Gandell, Barbro Grevholm, Said Hadjerrouit, Uffe Thomas Jankvist, Maria Johansson, Monica Johansson, Kristina Juter, Jimmy Karlsson, Angelika Kullberg, Lena Landgren, Morten Misfeldt, Reidar Mosvold, Hans Kristian Nilsen, Malin Norberg, Eva Norén, Guri Nortvedt, Rimma Nyman, Peter Nyström, Johanna Pejlare, Johan Prytz, Elin Reikerås, Helena Roos, Ann-Sofi Røj-Lindblad, Frode Rønning, Johan Sidenvall, Leif Bjørn Skorpen, Jeppe Skott, Lovisa Sumpter, Anna Ida Säfström, Anna Teledahl, Anne Tossavainen, Jorryt van Bommel, Hendrik van Stenbrugge, Mirela Vinerean, Pauline Vos, Jonas Bergman Ärlebäck and Magnus Österholm.

The members of the program committee would also like to express their gratitude to the organisers of *Matematikbiennalen 2022* for financially supporting the conference, and the Linnaeus University for providing premises and infrastructure. Your support is a great contribution to the spread of research and thought exchanges in mathematics education in Sweden.

Finally, as the chair of the program committee for MADIF 13, I, Linda Mattsson (Blekinge Institute of Technology), would like to thank the excellent members of the committee: Martin Carlsen (University of Agder, Norway), Johan Häggström (University of Gothenburg), Cecilia Kilhamn (University of Gothenburg), Hanna Palmér (Linnaeus University), Miguel Perez (Linnaeus University), and Kerstin Pettersson (Stockholm University). We have all different competences, and the committee could not have worked this well without any of you. It has been a solid teamwork and it is easy to be the captain when all players do more than anyone could ask for.

## Review process of and reference to proceeding

In a rigorous two-step review process for presentation and publication, all papers were peer-reviewed by two or three researchers. Contributions viewed as short presentations or symposium proposals were reviewed by members of the program committee. The MADIF Proceedings are published by SMDF (skrifter från Svensk Förening för MatematikDidaktisk Forskning), which since 2010 has been designated scientific level 1 in the Norwegian list of autho-rised publication channels available at Norwegian register for Scientific Journals, Series and Publishers ([kanalregister.hkdir.no](http://kanalregister.hkdir.no)). Due to climate considerations SMDF has decided to publish only a digital version of this proceeding. When referring to this volume please write the following.

Mattsson, L., Häggström, J., Carlsen, M., Kilhamn, C., Palmér, H., Perez, M. & Pettersson, K. (Eds.) (2022). *The relation between mathematics education research and teachers' professional development. Proceedings of MADIF13. The thirteenth research seminar of the Swedish Society for Research in Mathematics Education, Växjö, 2022.* SMDF.

All previous proceedings of MADIF, except the very first, are available in printed versions published in the SMDF writing series. They are also electronically available on the SMDF website ([matematikdidaktik.org](http://matematikdidaktik.org)).

The next conference, MADIF 14, will be held at Örebro University in 2024. Then SMDF celebrates 25 years of continuously growing interest in and development of research in mathematics education. No matter if you have a newly awakened interest in this field or if you have participated in one, a few, or all of the previous conferences, we hope to see you in Örebro.

MADIF 1, 1999, Stockholm	MADIF 2, 2000, Göteborg
MADIF 3, 2002, Norrköping	MADIF 4, 2004, Malmö
MADIF 5, 2006, Malmö	MADIF 6, 2008, Stockholm
MADIF 7, 2010, Stockholm	MADIF 8, 2012, Umeå
MADIF 9, 2014, Umeå	MADIF 10, 2016, Karlstad
MADIF 11, 2018, Karlstad	MADIF 12, 2020, Växjö
MADIF 13, 2022, Växjö	

On behalf of the program committee for MADIF 13

Linda Mattsson, chair

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# Students' meanings of inclusion in mathematics – implication for practice

HELENA ROOS

The purpose of this paper is to highlight implications for practice reflecting on the results of a study of students' meanings of inclusion in mathematics education. The main result from the prior study suggest that three Discourses influences students meaning(s) of inclusion: Discourse of mathematics classroom setting, of assessment, and of accessibility in mathematics education. The implication for practice building on these Discourses concerns construction of tests, grading in relation to what students perceive as mathematics, (un)challenge and theme of tasks, a pedagogical stance and tactfulness of the teacher, valuing of students, organization in terms of the use of textbooks, discussions and "going-through", variation in teaching approaches, being in a small group, and how the label of "SEM student" may affect participation and access.

In a study of students' meaning(s) of inclusion in mathematics Roos (2019a) investigated what students in special educational needs in mathematics attributed to inclusion in mathematics learning and teaching and what framed students' meaning(s) of inclusion in mathematics learning and teaching. This paper is a reflective paper about how the results of the study (Roos, 2019a) can implicate practice in mathematics education, both at schools and for teachers' professional development. This implication can help both mathematics education and teachers and teacher education to highlight and enhance inclusive mathematics education. Hence, the overall aim is to highlight how students' meaning(s) of inclusion can implicate practice. The research questions are: how can the results of the prior study implicate practice on school level? How can the result of the prior study implicate teachers' professional development?

## Inclusion

To set the context of this paper there is a need to explain the notion of inclusion. This notion is frequently used in research to highlight education for every student in the classroom, or related notions such as *inclusive pedagogy*. Often,

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inclusion is used to describe an ideological stance or a way of working in mathematics (Roos, 2019a) to provide "a meaningful education for all" (Florian et al., 2017, p. 14). "For all" implies that the focus of inclusive education is not only on low attaining students and the difficulties they encounter but also on issues of diversity to avoid marginalization (Florian et al., 2017). However, at the same time, the notion *for all* affords a gaze on all students' learning, raising contradictions regarding who is seen, heard, and supported. This has been intensely debated in research with a fear of, instead of producing inclusion, actually producing exclusion (e.g. Chronaki, 2018; Pokewitz, 2004). The notion of in(ex)clusion (Valero, 2017) has been used to describe this as an ordering and ranking of "individuals and populations in relation to how much their mathematical achievement indicates their human capital" (Valero, 2017, p. 2). In(ex)clusion can be seen as notion trying to frame the importance to always be careful when planning for inclusion, to reflect on who is seen, heard, and supported.

Inclusive settings and working inclusively can be defined as ways of accommodating all learning differences among students within a classroom and creating opportunities for every student to participate in the education (Barton, 1997). This definition has its origin in the paradigm of special education and the notion of inclusion has often been connected to special education rather than to a democratic education overall (Allan, 2012). The connection to special education is also highlighted in Swedish research (Magnússon et al., 2019). The connection can be seen in the light of the development of the notion of inclusion, from the use in the Salamanca declaration 1994 with a focus on special education and deficits towards its current state focusing education for all. This means that the notion has historically been tightly connected to deficits but has moved towards focusing a democratic education for all. Although the definition of the notion of inclusion has moved, what inclusive education is depends on the situation and context in both policy and culture (Magnússon et al., 2019). Göransson and Nilholm (2014) identified four different types of definitions of inclusive education when investigating how it is used in research literature; the placement of students in special educational needs in mainstream classrooms; social academic needs of students in special educational needs; social academic needs of every student; creating communities. This implies inclusion is interpreted and used differently, from a strong connection to special education to a community issue depending on the context and culture.

If considering the use of the notion of inclusion in mathematics education research, there are several different definitions. Often the notions diversity and equity (e.g. Askew, 2015) are used together with inclusion, which can be seen as indicating an ideological stance (Roos, 2019b) and a community definition. Also, as mentioned above research on inclusion in mathematics education also discusses processes of exclusion (e.g. Chronaki, 2018; Valero, 2017). On the other hand, there also research using words like interventions (e.g. Hart Barnett

& Cleary, 2015) and inclusive classroom (Moorehead & Grillo, 2014) together with inclusion, which can be seen as using inclusion as a notion describing a tool to teach all students in the same classroom. This way of using inclusion indicates a placement definition of inclusion. Hence, mathematics education research on inclusion can be seen as working in two directions, one covering societal issues and having an ideological stance and the other covering classroom and individual issues having a practical stance using inclusion as a tool. These two directions are most often not overlapping in research on inclusion in mathematics, which leaves somewhat of a gap between ideology and practice (Roos, 2019b).

## The prior study

The study which is reflected upon in this paper is situated in the intersection between the research paradigms mathematics education and special education focusing on inclusion from a student perspective. It is a collective case study (Stake, 1995), as it focuses three students in grade 7 and 8 in a public Swedish lower secondary school. The students are all regarded as being in special educational needs in mathematics (SEM) by the teachers, one of them because he is in access to the mathematics presented in the classroom but in need of something else to get access to learning in mathematics, and two of them because they are struggling to get access to the mathematics presented in the classroom. The school has approximately 550 students and 5 classes in each grade from grade 7 (13-year-olds) to grade 9 (16-year-olds). The catchment area is both urban and suburban, and there is cultural as well as social diversity. This school has set out to work inclusively, meaning its aim is to include all students in the ordinary classroom in every subject and to incorporate special education into the ordinary teaching with no fixed special educational groups. The school states that inclusion is a core issue and that everybody is welcome in the classroom where the support will primarily take place by co-teaching between the teachers and special teachers.

The object of the study was the meaning(s) of inclusion in student talk and the data consists of both interviews and observations conducted during one semester. The students were interviewed at least five times each during the semester. The observations took place in the grade 7 and grade 8 classroom where the interviewed students were enrolled. The use of the observations in the research was two folded, firstly, they situated the interview questions to be near the students in time and content, and secondly, they were used in the analysis. This study is discursive and discourse analysis (DA) as described by Gee (2014a, 2014b) is used, both as a theoretical frame and as an analytical tool. This implies the focus is on the students' interactions, both spoken and written. The theoretical notions *Discourse (D)* and *discourse (d)* are used. Discourse

represents a wider context, both social and political, and is constructed upon ways of saying, doing, and being:

If you put language, action, interaction, values, beliefs, symbols, objects, tools, and places together in such a way that other *recognize* you as a particular type of who (identity) engaged in a particular type of what (activity), here and now, then you have pulled of a Discourse.

(Gee, 2014a, p. 52, Gee’s italics).

When looking at discourse (with a small d), it focuses on language in use – the ”stretches of language” we can see in the conversations we investigate (Gee, 2014a, 2014b), meaning the relations between words and sentences and how these relations visualize themes within the conversations.

## Result of prior study

The results of the prior study stems from the discourse analysis made and answer to the following research questions: What meaning(s) do the students ascribe to inclusion in mathematics learning and teaching? And what frames students’ meaning(s) of inclusion in mathematics learning and teaching? These questions helped to show how meaning(s) of inclusion in student talk can be described by three overarching and interrelated Discourses: The Discourse of *mathematics classroom setting*, of *assessment*, and of *accessibility in mathematics education*. Within these Discourses, smaller discourses make issues of meanings of inclusion for the students visible. The relation between the D(d)iscourses is displayed in table 1 below.

Table 1. *The relation between D(d)iscourses in the prior study*

Discourse(s)	discourse(s)
Discourse of assessment	Testing Grades
Discourse of accessibility in mathematics education	Tasks The importance of the teacher (Not) being valued Dislike
Discourse of mathematics education setting	Classroom organization Being in a small group

Hence, in the *Discourse of assessment* all the students somehow talked about assessment and how it influenced inclusion in mathematics negatively in terms of different tests and grades. Regardless of if the students were in struggle to get access to the mathematics presented in the classroom or were in access to the mathematics presented in the classroom, their inclusion in mathematics

was limited by assessments. Also, in the *Discourse of accessibility* the students highlighted the importance to get access to the mathematics education by tasks that helps their mathematical development. They also highlighted the importance of being valued in the classroom as a student in SEM and the importance of having variation in the teaching and learning to not dislike mathematics. In the *Discourse of mathematics education setting* the students struggling to get access to mathematics highlighted the importance of sometimes being able to be in a small group getting instructions from the special teacher. Also, variation in the teaching approaches was something the students talked about as a positive factor for inclusion. In relation to that they all talked about the importance of the teachers, reflection on what and how to present on the white board.

Even though the results are from a collective case study with only three students, the students can be regarded as extreme and critical cases because they are in SEM and are cases within the overall collective case. A claim is that these extreme cases provide with vital information about the meaning(s) of inclusion. These critical cases help us to reflect about the collective case, as what is seen as valid for the collective case, may apply to all cases (Flyvbjerg, 2011). Another claim is that this is an in-depth and an information-rich case to get "a best-case scenario". This because the choice of a school setting out to work inclusively. Accordingly, if students' meaning(s) of inclusion are not explicit in this case, then where are they made explicit?

## Discussion of implications for practice

The previous described study is conducted at one school and is in depth with three students. The study does not claim that the results are valid in every school with every student, but it opens for a possibility of transferability when reflecting on the results. Though, important to consider regarding this possible transfer is limitations in moving from students' interpreted meanings to implications for schools and teachers' professional development. This move and reflection on the results needs to be situated in a practice to show potential implications for that particular practice. This move also needs to be considered in possibilities for teacher development. Here there is a need reflect on the results in relation to teacher practice in mathematics education. This limitation shows that there is a need to be reflexive and situate the results. Though there are merits in this, because when reflecting on a specific practice other local issues from a student perspective that was not visible may appear.

When reflecting on the results of the study of the meaning(s) of inclusion in mathematics education in student talk on an overall level, one can conclude that the overarching issues are the same, yet different when looking into detail. This displays the need to always reflect on the result in relation to the practice and the students at hand. Even though, this study shows some overarching

issues that can be of help when reflecting on how to enhance inclusion for every student (table 2).

Table 2. *Implications for inclusion in mathematics in relation to the discourses in the study*

Discourse(s)	discourse(s)	Implications for practice
Discourse of assessment	Testing	The construction of tests and the demands on the students, in terms of explaining, influences both participation and access.
	Grades	Grading influences what students perceive as mathematics and thereby limits their participation and access.
Discourse of accessibility in mathematics education	Tasks	The (un)challenge in tasks influencing participation and access. The theme of the task influencing participation and access.
	The importance of the teacher	The pedagogical stance and tactfulness of the teacher enhancing or diminishing students' participation.
	(Not) being valued	How the mathematics education values students is of importance for students' participation.
	Dislike	The meaning of mathematics as something boring challenges students' participation.
Discourse of mathematics education setting	Classroom organization	How the organization, in terms of the textbook, discussions and "going-through" <sup>1</sup> frames students' participation and how variation in teaching approaches increases students' participation.
	Being in a small group	How being in a small group enhances or diminishes students' participation in mathematics education; also, how the label of "SEM student" may affect participation and access.

Below the discussion of implications for practice is divided into two sections. The first section describes how the results of the prior study implicate practice on school level, and the second section describes how the result of the prior study implicate teachers' professional development.

### Implications on school level

An explicit implication for practice regarding inclusion is that inclusion is not equivalent with every student being in the same classroom always, the placement definition of inclusion (Göransson & Nilholm, 2014). Although the investigated school set out to work inclusively from an ideological point of view, focusing on the placement with all students in the same classroom, some D(d) discourses show a limitation in students' participation in mathematics education. Thus, the *possibility* to be in a small group outside the classroom is expressed as positive for the students. Consequently, one implication is that the education needs to move beyond seeing inclusive classrooms as a physical room where every student is always present physically to a more dynamic view on

inclusion, which is more situated and related to the students and their prerequisites. This implies that education needs to move from implicitly trying to fit the students into what is considered "normal" of all students towards departing from the opportunities of *every* student. With this stated, there is a need to be mindful and careful so as not to label students in SEM and not create stigma, as highlighted by Civil and Planas (2004). If the construction of a SEM student changed and diversity would be taken as a point of departure in the mathematics education, maybe the D(d)iscourses would change, and as a result, the way school and the society look upon special needs would change. Here it might be helpful for school development to look upon contextual influences that bear on the way schools carry out their education (Ainscow, 2020). Ainscow (2020) suggests four bearing influences: Inclusion and equity as principles; the use of evidence; administration and community involvement. These influences can work both as encouragement and hinders for inclusive education depending on the direction of views upon inclusion and equity at the school at hand.

Another implication for practice is that a student in access to mathematics education also can be in special educational needs since she or he might not be in access to learning in mathematics. This implies a need to acknowledge these SEM students and offer special mathematics education to enhance their participation in mathematics education and access to mathematics learning. Yet another implication for practice is to critically reflect on the organization of the mathematics education, changing it from being governed by a textbook to a more flexible way of organizing, with more variations in how to actually do mathematics when learning. More variations would perhaps change the D(d)iscourses and thereby perhaps the students' meanings of inclusion in mathematics would change as well. Perhaps a way of looking at the mathematics education as an inclusive landscapes of investigation (Skovsmose, 2019) would help to develop the education at the school and have more variations in the teaching. Skovsmose (2019) describes it as a way of facilitating meeting amongst differences promoting inclusion in mathematics. These inclusive landscapes have three major elements: facilitate investigations, accessible to everybody and facilitate collaborations.

### Implications for teachers' professional development

When looking at the Discourse of of accessibility in mathematics education, implication for teachers' professional development is visible how a relational perspective in terms of pedagogical stance and tactfulness of the teacher can enhance or diminish students' participation. This relational perspective has also been shown in another Swedish study made by Ljungblad (2016). This implies that teaching mathematics is so much more than just communicating a mathematical content. It is just a small part of what it means to teach mathematics and promote every students learning. Hence, there is a need in teacher

development to discuss how to have a relational perspective in mathematics education. In relation to this it is of importance to reflect on how the teacher (and the mathematics education) value students. This to be able to enhance students' participation and change the meaning of mathematics as something boring.

Yet another implication is when working with assessments, as a teacher be aware of how it can affect students' participation and access in terms of the construction of tests and the demands on the students when taking tests. For instance, in terms of the demands of depth in explanations (Roos, 2019a). A thing to be aware of in relation to grading is that grading can influence what students perceive as mathematics and thereby limits their participation and access (Roos, 2019a). Hence, there is a need as a teacher to be aware of inequalities involved in assessment in mathematics (Bagger, 2017).

## Conclusions

Looking back at this reflection on implication for practice from the study of students meaning(s) of inclusion, it is striking how complex and challenging teaching mathematics is when considering the individual student(s) need. This depends partly on the prevailing discourses on what mathematics is and how it is supposed to be taught and learned. This challenges the teaching when the students need something else. Partly it depends on the diversity of students, and the mathematics education in the classrooms needs to be able to meet diversity in the education (Askew, 2015). Accordingly, diversity among students demands diversity in mathematics education. This implies that the prevailing discourses on how mathematics is supposed to be taught and learned needs to be challenged. This is not at all an easy task and researchers, as well as mathematics teachers and special teachers in mathematics needs to take on that challenge. Researchers need to highlight important issues and needs regarding inclusive mathematics teaching and collaborate with teachers to interpret them in practice. Mathematics teachers and special teachers need to collaborate at both an organizational level as well as a group and individual level meeting the needs of every student (Roos & Gadler, 2018).

Teachers are expected to be able to handle students' diversity and promote every students' mathematical development. To enhance students' inclusion in mathematics education demands that the teacher knows her or his students, is flexible, has a pedagogical stance and tactfulness (Ljungblad, 2016), and is knowledgeable in mathematics and mathematics education. Also, in relation to assessment it demands that the teacher can take a critical stance and resist the prevailing discourse of assessment and to try to resist the processes and systemic patterns prompting inequalities in assessments (Bagger, 2017) that can sometimes overshadow the mathematics education.

Taking a student perspective on inclusion shows how complex and challenging it is being a student in mathematics. Students are expected to relate to,

understand, and participate in all the Discourses existing at the same time in a single mathematics classroom. This needs to be acknowledged and reflected upon both from a school organization, and teacher perspective.

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## Note

- 1 "Going-through" is used to describe *genomgång* in Swedish. Andrews and Nosrati (2018) point out three instances of what can be considered "going-through": when the teachers inform the students of what to work with, when presenting new models, and when demonstrating solutions to problems the students find difficult.

# Who or how many are missing? Toddlers experiencing numerical meaning in a game

CAMILLA BJÖRKLUND, HANNA PALMÉR AND LENA LANDGREN

In this paper, we present results from an inquiry into how basic number meaning can be taught in preschool through a game. The game was designed in collaboration between preschool teachers from three preschools and researchers in accordance with both theoretical and empirically founded principles. Based on video observations of teacher and child interaction (27 toddlers, 179 video recordings) when playing the game, we elaborate on how the meaning of numbers is made possible to discern and what needs to be differentiated in order to make the meaning of numbers discernible. Results show that non-numerical features play a bridging role for using the game to teach the meaning of numbers.

This paper is part of a larger study<sup>1</sup> in which researchers and teachers collaborate to develop mathematics teaching activities and educational principles for the youngest children in early childhood education (1- to 3-year-olds). There are indeed studies of preschool activities that are supposed to support young children's mathematical learning, taking an educational perspective (Gejard, 2018; Sæbbe, 2019). However, there are few theoretically grounded studies focusing on what learning outcomes are made possible. The field of early mathematics education thus seems to lack discussions about *why* learning is made possible or not through particular activities.

In mathematics education, regardless of the age of the student, a critical question is how to teach the meaning of something that cannot be seen or perceptually experienced, such as numbers. To overcome this fundamental question, representations become necessary for mediating the meaning of the mathematical object. While we can never access mathematics without representations, the representations are not to be confused with the mathematical objects they represent (Duval, 2006). Duval's (2006) critical view on representations is essential: "How can they [learning children] distinguish the represented object from the semiotic representation used if they cannot get access to the mathematical object apart from the semiotic representations?" (p. 107). In early childhood

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mathematics education, and particularly concerning the youngest toddlers, this is at the core of the educational endeavour since mathematical objects such as numbers are truly novel to the children.

This paper concerns the learning of numbers in early years and particularly the role of representations when drawing attention to and opening up for exploration of the meaning of numbers. The object of analysis is a specially designed game focusing on cardinality and numerical relations between small quantities. Two research questions are posed: RQ1) To what extent are numbers made the focus of attention in the designed game? and RQ2) What needs to be differentiated within the game in order to make discernment of the meaning of numbers possible? These questions are answered through analysis of empirical data consisting of 179 video recordings of the specially designed game played by 27 toddlers and their preschool teachers during one year.

## Learning the meaning of numbers

Children encounter numbers early in their lives when communicating with others about quantifiable phenomena. From cognitive science we know that children are able to perceive differences in small quantities in a process called subitizing (see Clements et al., 2019; Wynn, 1998) and to estimate larger amounts such as the approximate number system (Dehaene, 1992; see also Ulrich & Norton, 2019). These are intuitive abilities that support human survival, but they may also be a foundation for learning to reason about changes in quantities with cultural tools such as counting words and symbols. Such reasoning skills are today highly valued, visible even in the mathematics curriculum for 1-year-olds entering their first step in the education system (see *Curriculum for the Preschool lpfö 18*, National Agency for Education, 2019).

Fuson (1992) describes contexts in which children encounter different number meanings: cardinal, ordinal, measure, sequential, symbolic and non-numerical. The meaning of numbers differs between these contexts, and Fuson concludes that it takes many years before children are expected to identify these different meanings. Thus, the learning of numbers is complex; for example, how to interpret numbers as ordinal when seeing a sign on a house but cardinal when reading a price tag and even non-numerical when looking for the number on a bus. However, emphasizing this complexity does not reveal how children learn these different meanings.

How children learn the meaning of numbers is the focus of a study by Björklund et al. (2021) showing that it is necessary for children to discern and attend to the aspects of cardinality, ordinality, part-whole relations and representations in order to understand numbers in a flexible and prosperous way. Depending on the situation in which numbers appear, one or more of these aspects may be foregrounded while others are kept in the background. The first three aspects (cardinality, ordinality, part-whole relations) need to be

discerned in order to determine the number of a set of objects and also in order to reason about changes in quantities and to solve basic arithmetic problems. The fourth aspect, representation, is critical since it is both an aspect of numbers itself and a means to mediate numerical meaning (see also Duval, 2006 above). For example, counting words are verbal symbols that do not in themselves have numerical meaning. Verbal symbols are often the primary representation for pointing out numerical meaning and thus important for learning the specific difference in meaning between, for example, "two" and "three". Research in mathematics education has primarily focused on representations as means to mediate mathematical meaning, but a reasonable conclusion is that in early childhood mathematics education it becomes critical to not only use representations for visualizing mathematical meaning but also to show how a representation in itself is an aspect of mathematical concepts. Recent findings also reveal that young children who make use of several representations (for example, counting words and finger patterns), whether they use them in correct correspondence with a number of objects or not, are more attentive to learning the cardinal meaning of numbers (Gibson et al., 2019). Thus, representations play a significant role in the early learning of numbers and should not be restricted to verbal or graphical signs.

## Theoretical frame for teaching the meaning of numbers

The study was conducted in accordance with variation theory of learning (VT) (Marton, 2015). According to VT, when a child discerns what s/he has not previously been aware of, this changes the child's way of experiencing the phenomenon; the child has learnt a new or more nuanced meaning of the phenomenon. When teaching, the teacher therefore needs to understand what the child has not yet "seen" and thus what teaching should afford the child to discern. However, even when taking part in the same teaching activity, children tend to learn different things because they enter the activity with different experiences of the learning object. These different experiences impact what they "see" and how they interpret what the teacher is trying to mediate. Thus, the children also leave the teaching situation with different experiences of the learning object. In this teaching-learning process, then, it is critical for the teacher to identify what the child has not yet discerned and to understand how to enable the child to discern what is necessary in order to change his/her way of experiencing the object of learning.

A variation theory approach to knowledge, teaching and learning helps us understand why children experience a phenomenon (numbers) differently. Children's encounters with numbers are indeed novel in the early years, but the children do have *some* experiences of numbers. These might be intuitive (e.g., perceived differences between sets of objects) or semiotic (use of counting words in certain situations). When the child encounters contrasting meanings

of numbers, meanings that differ from the child's understanding, new ways of "seeing" numbers may occur. The core principle of teaching is thereby, according to VT (Marton, 2015), to deliberately choose what aspect to make possible for the child to discern and design situations making that aspect discernible. This is done by varying what is intended to be discerned, for example, representations, and keeping other aspects, such as the number to be represented, invariant. If instead the cardinality of a set is to be discerned, it needs to be contrasted against another set with a different cardinality while other aspects such as representations are kept invariant (e.g., comparing two red balls with three red balls). Designing teaching activities in preschool that facilitate this learning is a delicate task, however. Research on toddlers (Palmér & Björklund, 2019) has shown that before focusing on the learning object, it becomes important to first make sure that the toddlers are able to discern critical structural elements of the activity, such as rules of a game, turn-taking or accepted level of initiatives. If not, there is a risk that shared attention to an intended learning object may not be achieved, with implications for expected learning outcomes.

## Methods

The above-described theoretical foundation provides a framework for how to design teaching. However, it is empirically challenging to accomplish teaching in the dynamic context of early childhood education and thus also to study how learning is facilitated among young children. How children experience a phenomenon, in our case numbers, has to be interpreted through observations of their interaction with numbers. In this study, two researchers and three preschool teachers developed teaching activities during three semesters. The activities were tried out and evaluated in authentic preschool environments in an iterative process, with a total of 27 toddlers (age between 12 and 27 months at the start of the study). The activities were documented with video, used for assessment and further development but also for thorough analyses of the children's learning opportunities. One of the activities is used as data for analysis in this paper.

### Design of the game

With the aim of facilitating toddlers' learning of the meaning of numbers, we took as our departure point "Kim's game". This choice of game was based on observations in the three preschools where the children showed enjoyment of games where they themselves and/or objects were hidden. Using activities from observations in authentic settings provided external and ecological validity, making the results transferable to the educational setting (Cohen et al., 2018). In the traditional game, a set of (different) items is shown to the participants, for instance, on a tray. The items are then covered with a cloth and some of them are removed, unseen by the participants. When the cloth is taken away, the participants



Figure 1. *Three conditions of the game: First using different items to introduce the rules of the game (all kinds of features vary), then using the same kind of items only differing in colour (a particular feature varies) and, in the last step, using identical items (only the numbers vary).*

are to tell which items are missing. In the study, we redesigned the game in accordance with the following principles (see Palmér & Björklund, 2019): P1) the children need to understand the rules of the game to be able to participate, P2) the game needs to initiate interaction and engagement in order to enable shared attention to a specific learning object, and P3) numbers (within the subitizing range) are made possible to differentiate as the items are presented in ways that make necessary aspects of numbers discernible (see VT described above, Marton, 2015). Kim's game was redesigned in accordance with these three principles, ending up in a three-step model. The framing of the game was kept invariant but the props were altered to enable critical conditions for the principles to come through (see figure 1).

#### *Theoretical principles embedded in the game*

The three conditions of the game were designed to allow the children to become acquainted with the game so that they knew what the rules were and what was expected from the participants (P1, P2). Through a theoretical lens of VT, the three conditions, and particularly their ordering, should give the children opportunities to discern certain aspects of numbers (P3). *The first condition* (different items) appeals to the familiarity of the items that the children recognize, while many aspects may vary between and within the items (different in all kinds of ways). In an explorative sense, the children can associate the items to certain characters' names, songs related to the figures, or any other familiar context. *The second condition* also appeals to the child's familiarity with the items but is now restricted to one kind of item. The features of the items are limited in that the general appearance and size are kept invariant and only the colours differ between the items. The difference between conditions 1 and 2 is that in the former all kinds of features vary while in the latter there is only one feature that distinguishes the items and, consequently, the child's attention is drawn to this particular feature. Since colour is one of the basic concepts preschool children are likely to encounter in daily communication with caregivers and

teachers, this is also a familiar feature that most children know the word for (or as observed in our study, often supported with sign language). In both of these conditions of the game it is possible to enumerate the items as the numerical aspect is one possible feature of the game.

Enumerating the items becomes necessary only in *the third condition* where all features of the items are the same, which directs attention to them as a composite set. It then becomes relevant to represent them as parts (one or two) constituting a composite set (a whole of three) in words or gestures when completing the task of figuring out how many are hidden.

In conclusion, by gradually narrowing the features of the items that are likely to be foregrounded in the children's awareness while playing the game, the three conditions should, theoretically, support children in attending to numbers since the three-step model allows numbers to be discerned as a significant feature.

## Data and analysis

The data for analysis consists of 179 video recordings of the three teachers playing the game with the 27 toddlers. The game was played both individually and with peers, sometimes playing through only one condition but mostly through two or all three conditions. This was an expected irregularity because the data was collected in naturalistic teaching settings. The recordings were analysed in a two-step process.

In the first step, each recording was classified according to: 1) the three conditions (figure 1) of the game, 2) how the child completed the game (not participating, partly participating, completing the task), and 3) how the child responded to the task in the game (no response, tries to reach the hidden items, marks the spot on the table where the hidden item was taken from, verbalizes the hidden item with feature or name, uses counting words or finger patterns). In each video recording there were most often several responses identified. This first analysis revealed to what extent numbers or some aspect of numbers was foregrounded in the child's awareness when playing the game.

In the second step, the analysis concerned only the recordings where numbers were expressed through counting words or finger patterns found in condition 2 and/or condition 3, a sample of 14 recordings<sup>2</sup>. This analysis focused on how counting words were used and interpreted, thus at the centre of this analysis was not merely the use of words or gestures per se but *what variation in meaning* the words or gestures expressed.

## Results

The first analysis revealed a pattern in responses that gave support to the theoretical grounding of the activity, thus answering to what extent numbers were made the focus of attention in the designed game (RQ1). This, in relation to the child's tendency to complete the task, tells us how the designed game

Table 1. *Response frequency in the three conditions embedded in the game*

Classification of responses	Condition 1	Condition 2	Condition 3
A. No response	19 (29%)	16 (22%)	24 (20%)
B. Reaching for the item	1 (1.5%)	12 (16%)	27 (23%)
C. Marking the spot on the table	2 (3%)	10 (13%)	16 (13%)
D. Naming the hidden item	42 (65%)	20 (27%)	13 (11%)
E. Using counting words or fingers	1 (1.5%)	16 (22%)	40 (33%)
Total	65 (100%)	74 (100%)	120 (100%)

contributes to the potential for learning about numbers. Table 1 shows that in conditions 1 and 2, where the hidden items can be described in terms of features or names, the children tend to use the "shortest way" and interpret the game as if its goal is to find *which item* is hidden. Thus, the numerical aspect of the game is rarely coming through. The more different the features of the items are, the more likely the children are to name them (see table 1, D). In line with the VT principles, when visual features are kept invariant (condition 3), numbers come through as a possible feature to attend to in order to complete the task. This is shown in the toddlers using counting words or fingers (E) to a larger extent than in the other conditions. Marking the spot on the table from where the hidden object was taken (C) may be an indication of the toddler experiencing that the game centres around a group of objects but does not induce responses indicating that the toddler is able to single out any specific relation between what is hidden and what is visible. When toddlers are reaching for the item (B), the intention seems rather to be to find the missing object, as in a hide-and-seek game, which is also familiar to most preschoolers. Response A is quite evenly found across conditions, while responses B to E show distinctions between conditions.

As mentioned, the second analysis focused only on the recordings where numbers were expressed through counting words or finger use (57 documented responses in category E distributed among 14 recordings). Table 2 shows that, among five of these recordings (b, c, e, f and k), counting words or fingers were used in both conditions 2 and 3, but most common were different ways of responding to the task in the different conditions. This indicates that there are variations in ways of experiencing numbers also when the use of counting words or fingers is observed. Such variations may be related to the toddlers'

Table 2. *Overview of 14 recordings of toddlers' use of counting words or fingers in two conditions*

	a	b	c	d	e	f	g	h	i	j	k	l	m	n
Cond. 2	B	E	E	D/E	E	E	E	B/E	B/E	E	E	E	D	D
Cond. 3	B/E	E	E	E	E	E	A	B/E	B	D/E	E	A	E	E

opportunities to discern the intended meaning of numbers and are therefore of interest in this educational study.

Below follows a qualitative analysis of what meaning of numbers these observations contain and what then needs to be differentiated for the game to successfully facilitate toddlers' learning of numbers' meaning (RQ2). The analysis revealed four categories of description.

#### *Numbers are words in a random sequence*

Some toddlers are observed to use counting words but without apparent connection to the items on the table or the hidden ones. A typical response to the task of finding out how many are hidden is "one two three four". The teacher might then ask the toddler to count the visible items again and the toddler responds "one two three four six" (obs. g).

#### *Numbers as words ordered in a sequence*

When one visible item is added, extending a set, some toddlers are observed saying the next word in the counting sequence. Enumerating is indeed one way of determining quantity, requiring discernment of both cardinality and ordinality. However, our observations show that in many cases there is a close connection to the physical items, and particularly actions on the items (taking out one item at a time from under the cloth), which indicates that the cardinality of the counting words is not necessarily discerned. For example, the teacher asks how many are hidden, and when the toddler does not respond, she asks how many the toddler wants her to take out. The toddler says "one", she takes one item out, asks if the toddler wants more, and when the toddler answers "two", she takes another item out saying, "now you have two, do you want more?", and the toddler says "three". Thus, the toddler is not expressing cardinal meaning in his use of counting words (obs. d).

#### *Numbers as labels of a visible set*

Some toddlers are observed to instantly say "two" or show two fingers simultaneously as a finger pattern when the cloth is removed, indicating their experiencing the set of items in a cardinal sense and how to represent the number. However, when asked to determine how many are hidden, the hidden items are labelled with identifiable features such as colour. For example, when the cloth is removed to reveal one blue bear, the toddler says "a bear", takes the bear in her hand, kisses it and puts it back again. The teacher asks how many are hidden in the cloth and the toddler says "one bear, one green bear and one red bear" (obs. n).

#### *Numbers are related*

Some toddlers are observed to use numbers in a way that indicates their experiencing numbers as generalized representations of sets of items that are related:

If there are two visible, one is hidden; if there is one visible, more than one is hidden. For instance, one toddler sees one blue bear when the cloth is removed, saying "only one" and shows his index finger. The teacher confirms "now there's one there, how many do I have?" The toddler hesitates for a moment, then says "eh, two" and unfolds his index and long finger (obs. b). This way of experiencing numbers, as describing a relation between sets, is observed also when the toddler uses finger patterns instead of counting words, indicating that the numerical relation is discerned.

## Conclusion

Based on the observations of toddlers' ways of experiencing numbers (see categories above) we conclude that different ways of experiencing numbers influence the game's potential for their further learning of the meaning of numbers in terms of discerned aspects (see Marton, 2015). *First*, if toddlers are not discerning the cardinality of numbers, they turn to labelling items such as "mommy, daddy" or in terms of colours (often expressed in condition 2). When they encounter the task in condition 3, the toddlers find labels to help them differentiate not how many items but which items are hidden, for instance, labelling items as "a bear". This further indicates their experiencing the items as single items (non-cardinal, one bear and another bear), since mainly non-numerical features are foregrounded, also making it harder to discern them as composed sets.

*Second*, even though toddlers are able to correctly label a set of visible items with a counting word (indicating that they are experiencing some meaning of cardinality in the counting words), they are labelling hidden items with names such as "daddy". This helps them keep their attention on the hidden item, but the numerical aspect is kept in the background. However, in order to find out the hidden *number* of items, the relation between the visible and hidden set has to be discerned simultaneously. Thus, to solve the task, toddlers are observed to attend to different features (both non-numerical and numerical) that help them discern necessary clues of this relation. They may, for instance, name the items with a generalized label "bears", which seems to help them discern bears as constituting a composed set, but also mixing features "three bears and one green" (obs. m). The quote reflects an awareness of the total number of bears as a cardinal number, but the reference to "one green" reflects that the set of three is not seen as a composition of related parts; the hidden bear is described by other features. These two conclusions point to the game: even though it is theoretically well designed to bring numerical aspects of numbers to the fore, it is challenging to the toddlers, and they often foreground non-numerical features while seeming to ignore the numerical feature.

*Third*, a critical issue becomes which set is asked for. When numerical aspects are discerned by the toddler, it becomes necessary to differentiate other

aspects as well. In the game, the focus is directed towards the original set of three items; when the cloth is removed, it reveals that part of this set remains on the table and another part is hidden. This becomes critical in condition 3, where no visible features are available to help the toddler discern the composition of the sets, which demands an awareness that there is a numerical relation to be explored. The teacher's directed questions and actions to direct attention to what part of the set is asked for then becomes significant for the toddler's possibility to discern the particular part-whole relation.

In sum, the ways toddlers experience numbers' meaning does not necessarily include any numerical aspects that would allow reasoning about relations between sets (number of visible–hidden items), as in toddlers' experiencing numbers as words in a random or ordered sequence, or as labels of a visible set. Condition 3 (identical items) will therefore not facilitate learning of number relations if the child has not discerned aspects that are necessary for making use of the game's designed condition. This becomes critical if Kim's game is to have the best potential to facilitate the learning of numerical relations (in terms of part-whole relations). However, the non-numerical features do seem to bridge the toddlers' discernment of items as constituting a composite set with an emerging sense of number relations. This indicates that non-numerical features play a significant role for toddlers' possibilities to discern the critical numerical aspects, which is why representations (particularly the features the representations bring forward) seem to be key in this kind of teaching.

## Discussion

In this paper, we set out to investigate how the meaning of numbers was experienced in a designed game and what had to be differentiated in order to make discernment of numbers' meaning possible. Duval's (2006) view on representations for mathematics learning has inspired our investigation and brought to light the importance of taking children's perspectives as the outset when analysing learning opportunities. Together with a VT approach (Marton, 2015) to understanding how learning of mathematical objects such as numbers may be facilitated, our analyses may shed some light on the educational dilemma that Duval raised. Representations mediate meanings that are not evidently discernible to all learners, but through an ordered sequence of reducing irrelevant features, numerical aspects may be discerned by young children.

The inquiry presented in this paper is both theoretical in that we aimed to contribute knowledge about how mathematical meaning emerges (expanding descriptions of context-related meaning of numbers as presented, for example, by Fuson, 1992) and empirical in our attempt to show *how* this is facilitated, on theoretical grounds, in an authentic preschool activity. In conclusion, our study shows that preschool activities may facilitate directed attention to numbers, but different conditions in terms of selection of representations afford this attention

to various degrees, emphasizing the need to account for the learners' ways of experiencing numbers and the task at hand.

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## Notes

- 1 The DUTTA project, financed by the Swedish Institute for Educational Research (Grant no. 2018-00014)
- 2 A single response in condition 1 was not included in the further analysis.



# Lower secondary school students' gendered conceptions about mathematics and related careers

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Sweden is considered as one of the leading countries regarding equity work, but at the same time a country with a highly segregated labour market. This is true for graduate education as well, especially in mathematics, but not at secondary level. Previous studies have concluded that mathematics is considered a male domain, but little is known about mathematics and related careers. Here, the focus is on lower secondary students, and an online questionnaire was used. The results show that little gender stereotyping was made by both groups, but there were some nuances in the replies. Boys more often thought that female teachers and mums would reply that it is more important for boys to study mathematics in order to get a good job. Regarding who has the best requisites to study mathematics, the main response was "Both". The qualitative replies signal a relatively advanced understanding of gender as a social construct.

Sweden is a country recognised internationally for its gender equality work (Weiner, 2005) and usually score high on international measures of gender equality. For instance, in 2019, Sweden was ranked first in EU (Eige, 2019). At the same time, national and international research indicate that stereotyping is still an issue.

Negative stereotypes about girls' and women's abilities in mathematics and science persist despite girls' and women's considerable gains in participation and performance in these areas during the last few decades. Two stereotypes are prevalent: girls are not as good as boys in math, and scientific work is better suited to boys and men. (Hill et al., 2010, p. 38)

This is, for instance, present in teaching when studying teachers' interaction with students in science class, male teachers tend to favour boys in their interaction (Eliasson et al., 2016). Also, Sweden has gender segregated occupation, one of the highest in EU (Keisu et al., 2021) and, segregated university education (Sumpter & Sumpter, 2021). At the same time, at secondary level only a few vocational programmes are gender segregated and most programmes including

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the Natural science programme has a division within the 40/60 division (SCB, 2019). It appears that there is a shift between secondary school level and further education, and Brandell (2008) has concluded that gender work in Sweden has stagnated regarding mathematics, a conclusion still valid given the gender-equality paradox found in STEM education (Stoet & Geary, 2018).

Continuing with gender stereotyping, recent studies show that views/conceptions/ attitudes can be complex allowing contradictory elements (e.g. Sumpter, 2012), where students can express a very developed understanding of gender (e.g. Frid et al., 2021) and teachers saying "gender is not an issue" (Gannerud, 2009). There is a tension which has been concluded as an area of "limited consensus" (Forgasz et al., 2014, p. 371). With respect to this tension, especially regarding the segregated labour market, the aim of the present study is to study lower secondary school students' conceptions about who might have stereotypical views about girls, boys, and mathematics. The research questions are: (1) Which groups have, according to 15 year olds, stereotypical views about mathematics and professions?; (2) Which groups have stereotypical views about who has the best requisites to work in professions using mathematics?; and, (3) How and in what ways have this changed over time?

## Background

Gender is here considered a social construction, more than just a consequence of a biological sex (Acker, 2012). This includes seeing gender as a pattern of social relations, and that definitions of women and men depend on the context and definitions can be changed (Connell, 2019). The patterns are under constant negotiations and gender is thereby a dynamic process (Acker, 2012). As a theoretical framing, gender is here divided into the four different aspects (Bjerrum Nielsen, 2003): structural, symbolic, personal, and interactional gender. Structural gender cover social structures alongside with other factors such as class and ethnicity. One example of such a study that falls under this aspect is Mozahem et al. (2021) looking at age and gender as factors to understand how self-efficacy is developed, and the findings are discussed using theories about social roles. The second aspect is symbolic gender which appears in the shape of symbols and discourses (Bjerrum Nielsen, 2003). These symbols are part of a norm, hence providing information about what is considered normal and what is deviant. Symbols are bidirectional: it can be that an object or an abstract concept that is considered male or female, such as the idea of mathematics as a male domain (e.g. Brandell & Staberg, 2008), but it could also be about how men and women are perceived such as the "the hard working female" and "the male genius" (Leslie et al., 2015). Since it is a dynamic process (Connell, 2019). Personal gender is how individuals perceive the structure with its symbols (Bjerrum Nielsen, 2003). The following quote illustrates the experience of not fitting in to the created norm.

An advantage of being male would be to have been more encouraged to pursue a career in mathematics/engineering/technology. I would also have fitted in at high school better than I did – my Years 9 and 10 were spent on an all-girls campus where it was supremely uncool to be good at maths and science (Leder, 2010, p. 453).

This quote illustrates how gender symbolism and personal gender can be inter-related and seen as a interchange (Bjerrum Nielsen, 2017). Symbols have been shown to be powerful: the symbols mentioned above, "the hard working female" (e.g. Hermione Granger) and "the male genius" (e.g. Sherlock Holmes), are considered the main reason for gender imbalance at university level, functioning as an explanation for success (Leslie et al., 2015). Another study indicates that both boys and girls at Natural science programme in Sweden stereotype girls as being insecure in mathematics, but when asked from a personal view, this was not repeated where boys more often answered that they were not sure (Sumpter, 2012). The last aspect described by Bjerrum Nielsen (2003) is interactional gender. It covers the interactions between individuals within the structure and its symbols. Here, we are interested in how individuals perceive themselves in the structure (i.e. personal gender) and symbols including stereotyping (i.e. symbolic gender).

Looking closer at mathematics education in Sweden, at lower levels and secondary levels, there are small differences regarding participation and in grades (Brandell 2008; Brandell & Staberg, 2008). Regarding participation, the main differences appear to be at graduate level, and mathematics stands out together with IT compared to other subjects (Sumpter, 2012). There are only a few studies focusing on gender stereotyping and professions related to mathematics (e.g. Hill et al., 2010), and in a relatively recent one, where 784 individuals from nine different countries were asked "Who are more suited to be a scientist?", most countries, such as Canada, Israel, and Singapore, had over 50% of the respondents going for the option "Same" (Forgasz et al., 2014). Only two countries, China and United Arab Emirates (UAE) had the main response "Boys". Such results do not support the conclusion of mathematics as a male domain (e.g. Brandell & Staberg, 2008).

## Methods

The methods section contains a description of how the instrument were designed, using a questionnaire that was revised, and some theoretical underpinnings of this revision. Then, there is a short description on how data was generated and methods of analysis.

### Design of the instrument

The data was collected using an instrument that builds upon a well-known questionnaire that was designed to study individual's attitudes about gender and

mathematics (e.g. Gómez-Chacón et al., 2014). The first attempt on translating and piloting the questionnaire indicated several limitations although following "good practices" (Nortvedt & Sumpter, 2017). The results were about both intercultural and intracultural differences, including feedback such as "you can't ask question like this". The decision was to reconstruct the questionnaire so it could function in a Nordic context. As a step towards this revision, a literature review was made. It showed that most prior research treat gender as a cultural-neutral construct and do not consider cultural dimensions (Sumpter & Nortvedt, 2018). This meant, for instance, that the respondents very seldom were able to demonstrate knowledge about gender beyond the classic male-female dichotomy or express any nuances or an awareness about gender stereotyping, which falls under gender symbolism (e.g. Bjerrum Nielsen, 2003). As a theoretical tool for the revision, the choice was to apply Clarke's (2013) seven dilemmas: (1) Cultural-specificity of cross-cultural codes; (2) Inclusive vs Distinctive; (3) Evaluative criteria; (4) Form vs Function; (5) Linguistic preclusion; (6) Omission; and, (7) Disconnection. It resulted in several revisions, where one solution was to use vignettes (Nortvedt & Sumpter, 2018). For Question 2a, the vignette was: "Traditionally, one has said that it is important to study mathematics to get a 'good' job. What a 'good' job is has not been defined and there can be many different conceptions what it is". By adding such a vignette, the context enables the respondent to express perceived gender stereotyping from others whilst expressing a personal attitude that might differ. The question posed to the respondents were: What do you think the different groups would answer to the question "For whom is it most important to study mathematics to get a 'good' job, girls or boys?". The response alternatives consisted of a matrix where one dimension had the alternatives Most important for girls – Most important for boys – Equally important for both groups – It is not about gender – I'm not sure, and the other dimension had the groups Girls in grade 9 – Boys in grade 9 – Dads – Mums – Male teachers – Female teachers – Girls in general – Boys in general – You. This would allow the respondents to answer both from a personal view but also signal that other groups might think differently (e.g. Sumpter & Sumpter, 2021). Question 2b was a follow up question asking if this has changed over time allowing the respondents to write a comment if they wanted to. Question 3 had the same set up, but with the focus on who has the best requisites about professions where one use mathematics. The pilot study indicated that the questionnaire did allow respondents to demonstrate their awareness of a range of culturally rooted differences in attitudes towards boys' and girls' abilities to learn mathematics (Nortvedt & Sumpter, 2017). The results from question 1a and 1b, then focusing on the attributed symbols regarding who is considered best in mathematics, have been presented in an earlier paper (Frid et al., 2021).

## Data collection and methods of analysis

The data was generated by asking lower secondary school students (grade 9; age 15;  $n=241$ ) from seven schools in different locations in Sweden (north/south; rural/town/city). Since we followed the Ethics rules provided by Swedish Research Council, meant that students who had not turned 15 before December 2019 could not participate. According to Statistics Sweden, it should be around 6% of the population which is equivalent to two students per class. The questionnaire was made in Survey tool provided by X University which included both safe treatment of data and anonymous replies. Given that online surveys have less response rate (Fan & Yan, 2010), the second author used personal contacts to find participating schools. The quantitative data was analysed using statistical analysis, chi-squared test, of the replies used stated gender (boy/girl) as a factor ( $n=222$ ) with the aim to see where girls' replies differ from boys. The descriptive statistics were generated through the Survey tool. Reliability measured by Cronbach's Alpha coefficient for this scale was .909. The qualitative responses were analysed using inductive thematic analysis (e.g. Braun & Clarke, 2006), and then compared to previous research as a second step. Thematic analysis, in short, meant that we searched for similarities and differences in the written replies, gathering similar statements using a coding scheme. One example of code could be "symbols has changed" and the overarching theme was called "change over time". The themes were then compared to make sure that they were not overlapping.

## Results

The results are presented with first looking at the results about symbols about for whom it is most important to study mathematics, and how in what ways this might have changed over time. Then, the focus is on related professions to mathematics and which groups might have stereotypical views about this. Note that each category might have small variations in the number of respondents with the results that percentages could differs slightly. The main responses in each category are presented in bold.

As we can see in table 1, most students opt for the response "Equally important" for all groups including "You". The analysis showed that girls' and boys' responses differ statistically significant regarding three groups. The first two groups, mums and female teachers, can be seen as stereotyping, and it is boys that state that these groups to a larger extent say it is more important for boys. Girls, on the other hand, think that female teachers would say that this is not about gender. When looking at responses related to the aspect "personal gender", you, the main difference is that girls more often say it is more important for girls to study mathematics in order to get a "good" job. The next step was to study if this has changed over time (see table 2).

Table 1. Responses to "For whom is it most important to study mathematics to get a 'good' job, girls or boys?", n(%)

		Most important for girls	Most important for boys	Equally important	Not about gender	I'm unsure	p
Girls in grade 9	Girls	24(21.6)	5(4.5)	<b>60(54.1)</b>	18(16.2)	4(3.6)	>0.05
	Boys	13(11.9)	8(7.3)	<b>56(51.4)</b>	21(19.3)	11(10.1)	
Boys in grade 9	Girls	13(12.1)	19(17.8)	<b>51(47.7)</b>	18(16.8)	6(5.6)	>0.05
	Boys	9(8.4)	19(17.8)	<b>50(46.7)</b>	20(18.7)	9(8.4)	
Dads	Girls	7(6.5)	15(13.9)	<b>63(58.3)</b>	20(18.5)	3(2.8)	>0.05
	Boys	5(4.8)	18(17.1)	<b>54(51.4)</b>	18(17.1)	10(9.5)	
Mums	Girls	8(7.4)	2(1.9)	<b>68(63.0)</b>	26(24.1)	4(3.7)	<0.05
	Boys	6(5.7)	13(12.4)	<b>54(51.4)</b>	23(21.0)	9(8.6)	
Male teachers	Girls	7(6.6)	7(6.6)	<b>60(56.6)</b>	29(27.4)	3(2.8)	>0.05
	Boys	4(3.8)	11(10.5)	<b>59(56.2)</b>	22(21.0)	9(8.6)	
Female teachers	Girls	9(8.6)	2(1.9)	<b>61(58.1)</b>	31(20.5)	2(1.9)	<0.05
	Boys	7(6.6)	9(8.5)	<b>59(55.7)</b>	23(21.7)	8(7.5)	
Girls in general	Girls	24(22.4)	3(2.8)	<b>51(47.7)</b>	24(22.4)	5(4.7)	>0.05
	Boys	16(15.1)	11(10.4)	<b>47(44.3)</b>	21(19.8)	11(10.4)	
Boys in general	Girls	12(11.2)	16(15.0)	<b>52(48.6)</b>	19(17.8)	8(7.5)	>0.05
	Boys	8(7.5)	24(22.6)	<b>43(40.6)</b>	21(19.3)	10(9.4)	
You	Girls	16(14.8)	2(1.9)	<b>58(53.7)</b>	30(27.8)	2(1.9)	<0.05
	Boys	3(2.9)	8(7.6)	<b>55(52.4)</b>	28(26.7)	11(10.5)	

Table 2. Responses to "Has this changed over time?", n(%)

	Yes	No	I'm unsure	p
Girls	<b>85 (75.9)</b>	9 (8.0)	18 (16.1)	<0.05
Boys	<b>56 (50.9)</b>	22 (20.0)	32 (29.1)	

Table 2 show that there the responses are statistically significant different: girls more often reply that this has changed over time. Looking at the qualitative replies, two main themes were identified. The first one is about the process of gender symbols and gender structure has changed:

Girl [3]: Before, professions were more divided into what was considered female and male, but now it is not as much of that.

Boy [3]: Since technology has developed, more jobs are opened up that both boys and girls want to have.

The other theme is about gender structures and how it can have consequences, resulting in a segregated labour market:

Girl [4]: I think it is more important for girls since in general, it is more difficult for them to get a good job since girls on average get less in salary than boys.

Here, the reply signal that if one is working towards a change, it is more important for girls to study mathematics. There is no implication that girls or boys are better or worse at the subject, it is about the structure (less salary) and what that

Table 3. Responses to "Who has the best requisites to work in professions using mathematics, girls or boys?", n(%)

		Girls	Boys	Equally good requisites	Not about gender	I'm unsure	p
Girls in grade 9	Girls	15(13.4)	23(20.5)	<b>41(36.6)</b>	23(20.5)	10(8.9)	>0.05
	Boys	17(15.6)	29(26.6)	<b>37(33.9)</b>	19(17.4)	7(6.4)	
Boys in grade 9	Girls	11(10.2)	31(28.7)	<b>44(40.7)</b>	14(13.0)	8(7.4)	>0.05
	Boys	8(7.5)	<b>40(37.4)</b>	31(29.0)	19(17.8)	9(8.4)	
Dads	Girls	7(6.5)	31(28.7)	<b>48(44.4)</b>	12(11.1)	10(9.3)	>0.05
	Boys	7(6.6)	33(31.1)	<b>37(34.0)</b>	23(21.7)	6(5.7)	
Mums	Girls	8(7.4)	13(12.1)	<b>55(51.4)</b>	22(20.6)	9(8.4)	>0.05
	Boys	12(11.3)	19(17.9)	<b>41(38.7)</b>	28(26.4)	6(5.7)	
Male teachers	Girls	6(5.6)	18(16.3)	<b>55(51.4)</b>	21(19.6)	7(6.5)	>0.05
	Boys	6(5.8)	18(17.5)	<b>52(50.5)</b>	21(20.4)	6(5.8)	
Female teachers	Girls	5(4.6)	12(11.1)	<b>62(57.4)</b>	22(20.4)	7(6.5)	>0.05
	Boys	9(8.6)	13(12.4)	<b>52(49.5)</b>	25(23.8)	6(5.7)	
Girls in general	Girls	15(14.2)	21(19.3)	<b>37(34.0)</b>	23(21.7)	10(9.4)	>0.05
	Boys	20(19.0)	26(24.8)	<b>29(27.6)</b>	19(18.1)	11(10.5)	
Boys in general	Girls	7(6.6)	30(28.3)	<b>40(37.7)</b>	19(17.9)	10(9.4)	>0.05
	Boys	12(11.4)	<b>37(35.2)</b>	28(26.7)	19(18.1)	9(8.6)	
You	Girls	4(3.7)	19(17.8)	<b>42(39.3)</b>	31(29.0)	11(10.3)	>0.05
	Boys	7(6.8)	21(20.4)	<b>33(32.0)</b>	29(28.2)	13(12.6)	

Table 4. Responses to "Has this changed over time?", n(%)

	Yes	No	I'm unsure	p
Girls	<b>51 (47.2)</b>	18 (16.7)	39 (36.1)	>0.05
Boys	<b>48 (44.4)</b>	23 (21.3)	37 (34.3)	

entails (difficulty getting a certain profession). Neither of these themes explains why girls and boys replies differ.

The next results are about how the students perceive if and how different groups might have stereotypical views about who has the best requisites to work in professions using mathematics, see table 3.

Table 3 shows that none of the categories have responses that are statistically significant different. But there are some patterns within the responses. One that is worth lifting is that only 3.7% of the girls say that girls have the best requisites compared to 20.7% of the boys about boys having the best conditions. Also, the stereotyping made by boys towards the groups Boys in grade 9 and Boys in general, is not repeated when replying as "You". When asked if this has changed over time, the majority replied "Yes":

As we can see in table 4, the responses are not statistically significant different. The qualitative replies were gathered into two themes: those who have witnessed a change and those who thinks that there has been no change.

Girl [5]: It is more common with women working in those professions, and because of that the conditions for girls/ women who wants to work in those types of business have been better.

- Boy [5]: Before, boys had the norm on their side to get more advanced professions but now, this has stabilised.
- Girl [6]: I don't think this has changed over time since the norm is that guys should work with such professions and then, they automatically get better prerequisites since they are right for the job.

These replies illustrate an awareness of gender structures with including norms and the dynamic process of gender. The motivation given by Girl [6] show how symbolism inter-relate with other aspects of gender. Combined, there is an awareness that some things have changed but still gender inequality exists in our society.

## Discussion

This study focused on lower secondary school students' conceptions about mathematics and in particular how they perceived how different group would stereotype mathematics and related professions. The main result the majority of girls and boys reply that they think that most groups (e.g. parents, teachers) do not hold stereotypical views regarding who have the best requisites to work professions using mathematics. These results do not support the traditional view that mathematics is a male domain (e.g. Brandell & Staberg, 2008). Regarding who are most suited to work in a profession, the most common response was that both groups were equally suited. This is similar result as Forsgasz et al. (2014), and the explanation of the segregated higher education and labour market (Keisu et al., 2021; Sumpter & Sumpter, 2021) lays elsewhere. If we want to understand why girls do not continue with mathematics, we need to look at beyond compulsory schooling. This is a suggestion for further research.

There are some micro-level results. One is the perceived difference between boys and girls regarding stereotyping whether it is more important for one gender to study mathematics. The qualitative analysis generated two themes that neither fully explains why there is a difference. When trying to understand the themes using the theoretical underpinnings of gender (e.g. Acker, 2012; Connell, 2019; Bjerrum Nielssen, 2003), one notice how aware the students appear to be of gender as a social construct, this independent if they state there has been a change or not. The students talk about dynamic processes, using norms and example of gender structure as a base for their motivations. In that sense, gender work has not stagnated (c.f. Brandell, 2008). When comparing their awareness with the teachers in Gannerud's (2009) study, an interesting implication arrives: either the students are more informed than teachers, or teachers do involve gender in their teaching. Such a study, comparing two different groups, would be an appropriate second step of research. Also, when teaching about gender issues in school, students might have a more advanced view of gender compared to their teachers. This is something that should be acknowledged in teacher education.

Another result is that boys sometimes tend to stereotype boys as a group which could be seen as an indication of "mathematics as a male domain", which is not replicated when asked from an individual perspective, as personal gender. This could be seen as an example of intra-cultural tensions (e.g. Clarke, 2013; Frid et al., 2021; Nortvedt & Sumpter, 2018). This is a more refined description compared to "limited consensus" (Forgasz, et al., 2014, p. 371). One possible explanation is that the updated instrument does allow different views to be expressed, including nuances, compared to earlier studies (e.g. Brandell & Staberg, 2008; Forgasz, et al., 2014). We therefore suggest a second study on upper secondary school students from different programmes to see if the idea of mathematics as a male domain is (still) present.

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# Dividing cookies: What do students discern?

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This paper is on problem posing with 6-year-old students in Swedish preschool class. First, 77 students worked on a problem-solving task where they had to divide cookies equally amongst themselves. After that, the students were asked to pose a similar task to a friend. The focus of this paper is on similarities and differences between the initial problem-solving task and the tasks posed by the students. Almost all of the posed tasks were classified as mathematical tasks where the majority dealt with the same mathematical content (division) as in the initial problem-solving task. However, there were several tasks where other mathematics was needed to solve the task, as well as tasks where the context differed. One implication is that previous experiences and the circumstances in which problem posing is introduced seem to have an impact on the tasks posed by the students.

The context of this paper is a longitudinal educational design research study in which problem solving and problem posing are used as a starting point for teaching mathematics with six-years-olds in Sweden. In Sweden, as well as in many other countries, problem solving is emphasized in the syllabus, aiming to educate students to become competent problem solvers (National Agency for Education, 2019). According to Niss and Højgaard (2019), problem posing is also to be considered a part of the problem-solving competence, implying that to become competent problem solvers, students should encounter tasks where they both solve and pose problems. However, there is more research on problem solving than on problem posing (Cai & Hwang, 2020), which is why we know more about students' ability to solve problem-solving tasks than their ability to pose (problem-solving) tasks. This is especially true for younger students, which is why the rationale for this study is to deepen our knowledge on young students' problem-posing abilities.

The empirical material in this paper is from one design cycle within the longitudinal educational design research study, in which the students worked on both problem solving and problem posing. Seventy-seven students from four classes first worked on a problem-solving task with the mathematical content

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*division*. After that, the students were asked to pose a *similar* task to a friend. The focus of this paper is on how the young students interpret *similar*, thus on similarities and differences between the initial problem-solving task and the tasks posed by the students. More specifically, the following question will be elaborated on: When asked to pose a similar task to a friend, what aspects of the initial problem-solving task are visible in the tasks posed by the students?

## Problem posing

By incorporating problem posing as part of problem solving, students' problem-posing as well as problem-solving skills can be developed (Ellerton et al., 2015; Palmér & van Bommel, 2020). Posing tasks is cognitively demanding (Cai et al., 2020) and offers challenges through the "low-floor but high ceiling opportunities" (Zhang & Cai, 2021, p. 962). Further, when posing tasks, students get an opportunity to operate on different mathematical content that may develop their understanding (Brown & Walter, 2004).

Problem posing can be seen as an instructional activity and differs from asking questions as part of regular classroom discourse as well as from asking clarifying questions (Cai et al., 2020). The posing activity can take place before, during or after problem solving, where the students either are asked to generate a new task without a model to follow or to reformulate a given task (Silver, 1994).

In this study, the problem posing takes part after problem solving and the students are asked to pose a *similar* task which can be understood as reformulation of a given task (Zhang & Cai, 2021). This approach, where students are asked to pose a task relating to a given stimulus or a given situation, is sometimes referred as problem posing in a semi-structured situation (Stoyonova & Ellerton, 1996). Posing a similar task involves reflections on the original task and therefore reformulation can inform about the students' interpretation of what the original problem-solving task was really about (Carrillo & Cruz, 2016; Palmér & van Bommel, 2020). When reformulating, students can be asked to pose a task with a structure or a content similar to that of the original problem-solving task, or to pose a task that can be solved by using a similar method (Carrillo & Cruz, 2016). The students in the study presented here were not given any such explicit directions on how *similar* was to be interpreted.

## Young students and division

Understanding division means understanding the relationship between the number of parts and the size of each part, which pre-schoolers often practise successfully in everyday situations where objects are to be divided equally. For instance, when sharing something between themselves, they are usually very careful that everyone gets the same amount or the same number. Such equal sharing is then the basis for both division and fraction, as both of these

are about relationships between parts and the whole. Most often, 3- and 4-year-olds perceive cardinality as wholes but have more difficulty reasoning about subsets (Dehaene, 1997). A basis for understanding division is understanding that all subsets must be equal or consist of the same number of objects. When working on sharing in everyday situations, at early ages, one-to-one distribution is often used for distribution, supported by knowledge in addition and/or subtraction (Parmar, 2003).

A division like  $12/4$  can be contextualized in different ways, resulting in different ways to obtain the answer, in different kinds of division. The task "divide 12 cakes evenly over 4 bags; how many will there be in each bag?" describes a partitive division, where the 12 cakes (dividend) are divided over a given number of bags (divisor) to obtain the number of cakes in each bag (the quotient). However, the task can be posed as a quotative problem: "12 cakes have to be put in bags of 4, how many bags do you need?" The numbers of the dividend, divisor and the quotient are the same, the task is still  $12/4$ , but now the numbers represent other parts in the problem (table 1).

Table 1. *Dividend, Divisor and Quotient in  $12/4 = 3$*

	Dividend	Divisor	Quotient
$12/4=3$	12	4	3
Divide 12 cakes evenly over 4 bags	12 cakes	4 bags	3 cakes in each bag
12 cakes have to be put in bags of 4	12 cakes	4 cakes in each bag	3 bags are needed

Seventy years ago, research on division for young students (2nd grade elementary school) was conducted (Gunderson, 1955) describing differences in how young students worked on partitive and quotative division. Gunderson found that the students managed quotative division to a higher degree than partitive division. However, other research has suggested the opposite: that partitive division is easier for students as it follows the everyday experience of sharing and dividing things evenly and equally (Ching & Wu, 2021; Frydman & Bryant, 1988). In a study by Palmér (2008), 3-year-olds worked on partitive division. They had no problems when working on tasks like  $9/3$  but when faced with tasks like  $9/4$  they struggled with how to divide equally, as an equal amount or equal number.

## Method

Educational design research encompasses the design and implementation of teaching as part of the research, with the aim of developing theories as well as new forms of instruction (Anderson & Shattuck, 2012). Thus, educational

design research intends to be of value for both research and practice (Bakker, 2018). To develop theories that inform and guide the practice of teaching and learning, the studies are conducted in an iterative cyclic process of designing and testing interventions situated within an educational context. Each design cycle includes preparing for teaching, implementing the teaching, and finally, conducting a retrospective analysis of the teaching and learning (Cobb & Gravemeijer, 2008).

As mentioned, the empirical material in this paper is from one design cycle within the longitudinal educational design research study in which four pre-school classes (6-year-old students) were involved. Based on several years of collaboration, the teachers in these classes were familiar with problem solving and problem posing, as well as the concept of educational design research and the aim of the study (Palmér & van Bommel, 2021). The teachers are educated as preschool teachers, which implies that they have completed a three-year university course in preschool teacher education. In line with the ethical guidelines provided by the Swedish Research Council (2017), the guardians of the students were given information about the study and consented to the participation of their child. All students participated in the lessons given by their regular teacher. After the lesson, documentation was gathered if consent was obtained.

### The problem-solving and the problem-posing lessons

Two lessons were conducted within the design cycle. The two lessons were planned in collaboration between the researchers and the teachers on Zoom (because of the covid situation) and then documented in writing by the researchers and sent to the teachers. In the first lesson the focus was on problem solving. The students were divided into groups of three students in each group. Each group received 15 biscuits in clay, where the first task was to divide the biscuits equally between them ( $15/3$ , partitive division). One group at a time worked on the task. Then the teacher said that she also wanted to take part, whereby the students were asked to divide the 15 biscuits equally between four ( $15/4$ , partitive division).

In the second lesson, the focus was on problem posing. The students were reminded of the task they had worked on in the previous lesson and were asked to pose a *similar* task to a friend. This time they worked individually. The tasks posed by the students were documented on paper and the teacher helped with the writing if needed. The teachers also took notes. There were 77 students posing tasks but as one student posed two tasks, 78 tasks were analysed.

### Analysis

The analysis focused on the paper-and-pen work from the students when asked to pose a similar task to a friend. When analysing reformulated tasks posed by students, context, structure and content are of interest (Carrillo & Cruz, 2016). Classification of posed tasks based on such aspects enables a focus on

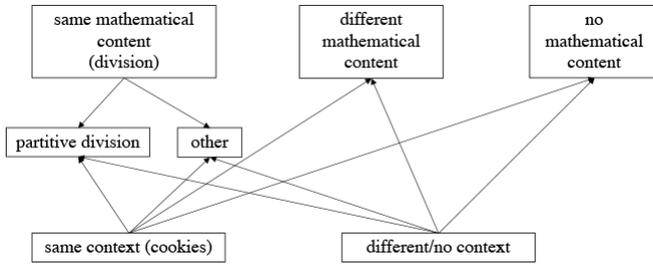


Figure 1. Classification scheme used to analyse the students’ tasks

the connection between the posed tasks and the mathematical topic in the initial task, the structure of the tasks (alike or different questions) and the context of the tasks (same or different context). Based on a classification scheme developed by Carrillo and Cruz (2016), the scheme used in this study was inductive, developed to cover the different tasks posed by the students (figure 1). First, the scheme focuses on the *content* and the tasks are divided into three groups: (1) tasks based on the same mathematical content (division) as the initial task, (2) tasks based on a different mathematical content to the initial task and (3) tasks without mathematical content. Then, the *structure* of the question in the division tasks is analysed: same as in the initial task (partitive division), other division questions, different from the initial task. Finally, an additional classification is made identifying the *context* in the posed tasks: whether the context is identical to the context in the initial task (cookies), or whether a different context (or no context at all) is offered in the task. This last classification is made for all the posed tasks.

## Results

In this section, we will first describe the results regarding the content of the tasks posed by the students, after which the results regarding the context will be dealt with. Table 2 shows a summary of the classification of the 78 tasks posed by the students. This classification is based on the steps described in figure 1. Table 2 will be used for the upcoming analyses.

Table 2. Result of classification of the students’ tasks

	Same mathematical content (division)		Different mathematical content	No mathematical content	Total
	Partitive division	Other			
Same context (cookies)	18	8	8	2	36
Different/no context	19	2	16	5	42
Total	37	10	24	7	78
		71			

### Focusing on the content of the posed tasks

Of the 78 posed tasks, 71 were classified as mathematical tasks and 7 as non-mathematical tasks. Some of the non-mathematical tasks did have some mathematical elements; for example, the question posed to the picture on the left in figure 2 was *Colour the shapes*, where typical mathematical shapes were used, in this case a circle, a triangle and a square. However, no mathematics is needed to solve the task. Another example of a non-mathematical task is the picture on the right in figure 2, with the posed question *What letter is it?*

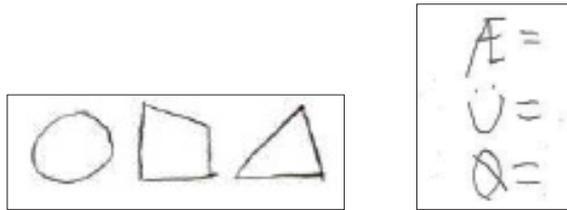


Figure 2. Example of two non-mathematical tasks

Of the 71 tasks where mathematics was needed to solve the task, 47 tasks dealt with division, as in the initial problem-solving task. In the remaining 24 tasks, other mathematics was needed to solve the task. One such content was addition; for example, the question posed to the picture in figure 3 was *How many cookies will there be if you put all cookies together?* Another example of a task on addition was *You have nine rings and get two more, how many are there in total?*

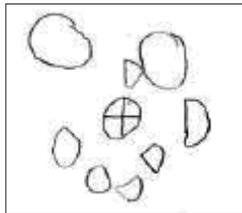


Figure 3. Example of a posed task with the mathematical content "addition"

Multiplication, *How many cookies do you need if all three children will get two each?*, counting, *Count the number of dots on the cookies, patterns, How does the pattern continue?* (figure 4), and subtraction, *200–109, what is the answer?* are some examples of other mathematical content used.

Of the 47 tasks on division, most tasks were on partitive division as in the initial problem-solving task. Here the quotient could be a natural number, such as *Two children will divide 28 liquorice or 12 pieces of dough to be divided by four persons*, but also a rational number, such as *Four chocolate bars have to be divided by three persons*. In most posed tasks it was inferred that the division

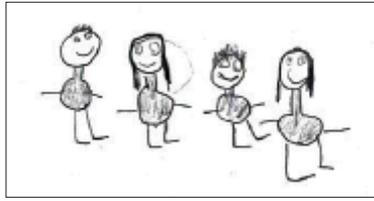


Figure 4. Example of a posed task with the mathematical content "patterns"

should result in equally "much" and some students stated this explicitly in their posed questions. Some of the students, however, focused on equally "many" as in the task in figure 5, *Divide all the cookies so each girl gets equally many.*

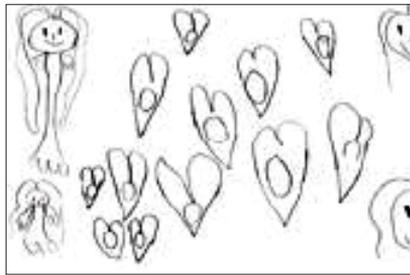


Figure 5. Example of a posed task focusing on equally many

In some of the posed tasks, the dividend was a whole number but represented as a rational number. One example is Figure 6 below, where the posed task is to divide four whole cookies and four half cookies between two persons.

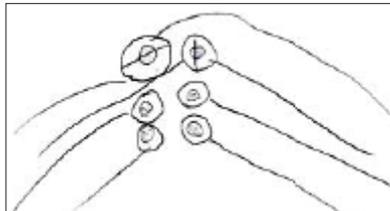


Figure 6. Example of  $6/2$ .  $(4 + 4 \cdot (1/2))/2$

Ten of the posed tasks with the content division rephrased the question in such a way that it differed from the initial task. In one of those posed tasks, 15 cookies were to be divided between a dad, a mum and a child. The cookies were of different size and division came with a rule that dad should have the biggest cookies, mum should have the medium-sized cookies and the child was to get the smallest ones. Other tasks within this category used quotative division; for example, *How many quarters will you have if you divide three cookies?* or a combination

of partitive and quotative division, *How many halves will each person get if they divide two whole cookies between the two of them?* Yet another example was *Two old men will divide 15 cookies, but they divide each cookie, so how many half cookies will they get each?*

### Focusing on the context of the posed tasks

Cookies were present in 36 of the 78 tasks. This context was used in both posed mathematical (see for instance figures 5 and 6 above) and non-mathematical tasks, for example *Find the divided cookie*.

The context used in the posed mathematical tasks (71) differed. In 34 of these tasks "cookies" were used to operate on. Most other contexts were comparable to cookies, such as sweets or pie. One student explicitly stated to the teacher, *You know, you can actually divide other things as well and draw four ice-creams to divide amongst two persons*.

In some of the posed tasks other objects were used, resulting in divisions where division in fractions could be seen as problematic. One of the students seemed to recognize this and changed the context of his posed task. The task started with five "Dracula-teeth" to divide between two persons, but then the student changed his picture, added a tooth and stated, "so, now both can have three". Another student drew a bathtub and stated that both children in the bathtub should have equally much water. This student then changed and posed another task.

Some of the tasks did not use any context at all, for example *200–109, what is the answer*. Also, in tasks in which students were to *draw equally many*, no specific context was used.

## Discussion and implications

The aim with educational design research is to develop theories that inform and guide the practice of teaching and learning. As mentioned in the introduction, we know less about young students' ability to pose (problem-solving) tasks than their ability to solve problem-solving tasks (Cai & Hwang, 2020). In the design cycle presented in this paper, problem posing took part after problem solving and the students were asked to pose a *similar* task that can be understood as a reformulation of a given task (Zhang & Cai, 2021). The question focused on is what aspects of the initial problem-solving task are visible in the tasks posed by the students? The interest in this question is two-fold, partly relating to the ability of these young students to pose tasks and partly to their reflections on the original task. The tasks posed by the students inform us about the students' interpretation of what the original problem-solving task was really about (Carrillo & Cruz, 2016; Palmér & van Bommel, 2020).

As presented in the results, only 7 of the 78 posed tasks were non-mathematical tasks. Thus, the students seemed to interpret problem solving as a mathematical activity. This differs from a previous study by Di Martino (2019) where students in kindergarten and 1st grade (5–6-year-olds) most often did not refer to mathematical problems when being asked about problem solving. In Di Martino's study, the students referred to everyday life problems. An important difference between Di Martino's study and the study presented here is that problem posing in our study was closely connected to problem solving. However, in the study by Di Martino (2019) students in grade 3 and 5 were studied as well, and they had the same conditions as the students in kindergarten and 1st grade. Only grade 3 and 5 posed mathematical tasks. Thus, not age but previous experiences and the circumstances in which problem posing is introduced seem to have an impact on the tasks posed by the students.

Regardless of how the students interpret *similar*, their posed tasks show their experiences of mathematics. There were several posed tasks on mathematical content that is fairly uncommon in preschool class, for example multiplication. Another example is shown in figure 6, where the posed task is to divide four whole cookies and four half cookies between two persons – thus, a more complex variation of  $6/2: (4 + 4 \cdot (1/2)) / 2$ .

In this study, 47 of the 71 mathematical tasks (where mathematics was needed to solve the task) dealt with division, as in the initial problem-solving task. Thus, the content of the original task was discerned by a majority of the students. For the remaining 24 tasks, other mathematics was needed to solve the task. This does not, however, imply that these students did not discern division as the content of the initial task, rather it may imply that *similar* is interpreted differently by different students. What aspect of the initial task is to be similar in the posed task? As the students were not told how to interpret *similar*, some students may have interpreted *similar* as a task about cookies, others as a task on division and yet others as a mathematical task in general. Thus, for example, if a teacher wants to know how the students experience division, this needs to be made explicit when introducing problem posing. Or, if a teacher wants to know what the students themselves discern in the problem-solving activity, the problem posing ought to be as open as in this design cycle. Another pedagogical recommendation when working with open problem solving, as in this design cycle, is to explore the diversity of the posed tasks with the students – what do they discern as similar and different between the tasks posed by their classmates and the initial task?

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# Mathematics teachers and the role of physical environment

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The physical environment (PE) affects the teaching and learning in school. Research is conclusive that different characteristics of PE can be enabling or hindering learning activities. Still, we need to know more about the role of PE in mathematics education to utilize what a good PE can offer and to avoid the hindering situations. The aim of this study is to characterize the different roles PE play in relation to the teacher, the student, the learning content, and their interactions. For this, mathematics teachers' stories about their experiences of PE in teaching are analysed. The results show that teachers often try to prevent disturbances or distractions from insufficiencies in PE. The results also suggest that aspects of classroom PE, such as classroom layout, sustain classroom norms whereas other elements in PE can be an aid in breaking norms.

The physical environment (PE) affects the teaching and learning in school (Earthman & Lemasters, 2009; Tanner, 2009; Woolner et al., 2007). When examining mathematics education research, PE could either have an enabling or hindering role in the mathematics classroom (Fahlström, 2017; Fahlström & Sumpter, 2018). The enabling role can be found when the conditions in PE are sufficient and satisfactory for the educational activity. In less successful cases, the teaching can be hindered by poor building status or insufficient classroom size, which eventually affect wellbeing and learning outcomes (Earthman & Lemasters, 2009; Tanner, 2009; Woolner et al., 2007). In these cases, the insufficiencies in PE are predominantly known preconditions, however not fully controllable for the teacher (Fahlström & Sumpter, 2018). Hence, we need to know more about the role of PE in mathematics education in order to avoid PE from hindering the learning activity and preferably utilize the benefits that come from good PE conditions.

Studies regarding the physical learning environment in schools have different objectives in focus. One such objective is to look for evidence linking technical aspects and other specific criteria of school PE to wellbeing, behaviours, and large-scale learning outcomes, for instance measured by standardized tests (Tanner, 2009; Uline & Tschannen-Moran, 2008; Woolner et al., 2007).

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Another type is research that starts out in ideas about specific teaching practices and examines whether the physical classroom settings promote or hinder such teaching practices (Cleveland & Fisher, 2014). In relation to this category there is research examining whether alterations made to school buildings and classrooms have had the desired effects on teaching practices. Most of these studies conclude that involving teachers and students in the process of building new, or altering existing, physical school environments is crucial for long-lasting changes to the classroom practices (Cleveland & Fisher, 2014; Uline et al., 2010; Woolner et al., 2012; Woolner et al., 2018). There are also studies that focus on teachers' views on school PE and these studies often picture school designers, school authority, and related research on the receiving end of their research (Barrett & Zhang, 2012; Earthman & Lemasters, 2009). The present study falls into this category, where mathematics teachers' stories about their experiences of PE in different teaching situations will be studied. PE is defined as the collection of physical objects that we can touch and physical phenomena that we can sense to this end.

The aim of this study is to characterize the different roles PE play in the stories of different teaching situations told by the participating mathematics teachers. A role is defined as the purpose or influence of someone or something in a particular situation. The identified roles that PE play will be characterized in relation to the actors, the teacher, the student, the learning content, and the pairwise interactions between them. The characterizations will also include the direction of influence or purpose in the role, if the role is enabling or hindering, and whether the role is active before, during, or after the described teaching situation.

## Background

In a previous study, the didactical triangle (see figure 1) was extended to include PE as a fourth actor together with the three original actors the teacher, the student, and the content. With the content being mathematics, the extension could be used as an analytical tool (see figure 2) which allowed for the investigation of the role of PE in mathematics education (Fahlström & Sumpter, 2018).

The idea to extend the didactical triangle was inspired by an extension with a socio-cultural perspective added to the didactical triangle (Rezat & Sträßer,

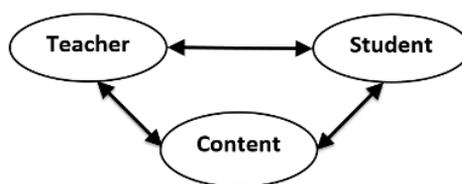


Figure 1. *The didactical triangle illustrating its actors and their interactions*

2012). Another inspiration was the problematizing of the didactical triangle by Schoenfeld (2012). In this paper a broadened view on the classroom activities related in the didactical triangle is suggested and one such view is to reflect on the classroom as a place for doing mathematics (Schoenfeld, 2012). Place is a keyword here, and aspects of the classroom as a place for doing mathematics is parallel to the roles PE can play in the teaching situations described in the mathematics teachers' stories investigated in this study.

In Fahlström and Sumpter (2018) it was suggested that the extension (see figure 2) of the didactical triangle could be used as a theoretical model to talk about, as well as to analyse, the role of PE in mathematics education, other subjects, and education in general. PE is placed in the middle in this extension where the teacher, the student, and the content remain at the implied vertices of the triangle. The pairwise interactions between the three original actors are indicated with dashed double-pointed arrows. The possible roles of PE in relation to these three actors and their pairwise interactions are indicated with solid line double-pointed arrows (see figure 2).

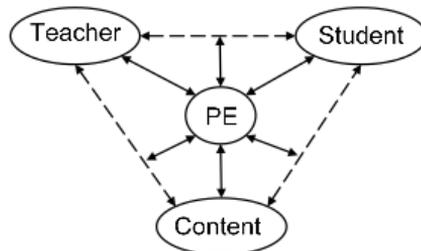


Figure 2. *Extension of the didactical triangle with PE as a fourth actor*

Research concerning PE and mathematics education will be presented here and related to the actors and relations in the theoretical model (see figure 2). The connection to mathematics as the content can be immediate, such as to a classroom activity like mathematical problem solving. The connection can also be indirect, as in cases where research has shown that students learning outcomes in mathematics, English, and other learning subjects are affected by the status and design of the school building (Tanner, 2009; Uline & Tschannen-Moran, 2008). Good school building status can enhance the quest for academic achievement and the forming of a student identity. The building can also inspire students' freedom to move around and explore as well as provide a secure feeling and sense of belonging (Uline & Tschannen-Moran, 2008; Uline et al., 2009). In the case of teachers, a good school building status can promote development of professional identity and contentment together with increased involvement, commitment, sense of security and raised expectations. Student-teacher interactions are also shaped in a positive way by a good school building status (Uline et al., 2009).

In the opposite case with, for instance insufficient space or crowded areas, the PE status can obstruct or limit the interactions between students and teachers. A classroom can be perceived as having insufficient space due to technical equipment occupying considerable space in the classroom (Uline et al., 2010). In addition, a crowded or narrow school building can make students feel claustrophobic, distracted, and cause a sense of inequity (Uline et al., 2010). Poor school building status can also have a hindering influence on teachers. This influence can be to trigger compensation, extra effort, and innovation to improve (Johnston, 2012; Uline et al., 2010). The school building design and classroom layout shape learning activities such as group work and reading as well as time on task in mathematics and other learning subjects (Uline et al., 2009). Individual task solving in mathematics, mainly in silence, is more sensitive to noise disturbances than other, non-silent activities (Fahlström, 2017). Teachers consider and adjust content during the planning and preparation before teaching depending on classroom layout, configuration, and status (Johnston, 2012; Soygenis & Erktin, 2010). In mathematical problem solving, where students stand up and use vertical black or whiteboards to write on, more engaged and inclusive student behaviour can be observed. Students can see solutions or parts of solutions written on the different vertical writing surfaces. Students can also hear mathematical communication from other students and new student-content interactions evolve (Johnston, 2012; Liljedahl, 2016). The geometrical aspects of the classroom, school building, and the surroundings can serve as mathematical content in the classroom and increase the students' awareness of the built physical environment and surroundings (Soygenis & Erktin, 2010).

Recognition of the significance of PE during the educational activity can also be found in mathematics education research where PE is not primarily targeted. One such example is where student workspace and room organization are two out of nine focus elements of mathematics teaching practices (Liljedahl, 2016). Another such example is where the teacher's and the students' spatial positions and the classroom seating arrangement are included in a proposed model for social classroom climate (Kuzle & Glasnović Gracin, 2021).

## Method

The aim of this study is to characterize the different roles PE play in the stories of different teaching situations told by the participating mathematics teachers. It was decided to analyse the data used in Fahlström (2017) with a different method. The 2017 study focused on what factors and effects could be identified in mathematics teachers' conception of PE in teaching. The model developed in Fahlström and Sumpter (2018) will be used as an analytical tool in this study (see figure 2). This choice of method of analysis enables the characterization of the identified roles of PE in relation to the teacher, the student, the learning content,

and the pairwise interactions between them. The data set was generated through semi-structured interviews where the participating mathematics teachers were asked to describe different teaching situations related to elements in PE. The seven participating mathematics teachers taught middle school, lower secondary, or upper secondary school. They worked as mathematics teachers in different parts of Sweden, and their teaching experience ranged from one year to 18 years. During the interviews, the teachers were asked to describe teaching situations, both with and without, disturbance or distractions from elements in PE according to their interpretation. Follow-up questions were posed for clarification at times, for instance, regarding who was affected in a described situation. The interviews were recorded and lasted between 30 minutes and 60 minutes. The interview recordings were transcribed verbatim (Fahlström, 2017).

The first step of the analysis applied in the present study consisted of identifying and coding keywords in the transcripts relating to physical objects that we can touch or physical phenomena that we can sense according to the definition of PE used here. In the following step, the text surrounding the coded keywords was examined for words relating to the actors the teacher, the student(s), or the content. The content can be explicit such as mathematics or other subject content but also implicit such as task or textbook. The indications of the teacher or students can also be explicit or implicit in the interview transcripts. When the actors had been identified, the situation could be classified as belonging to one of the six relations indicated with double-pointed solid line arrows in the analytical tool. The next step was to determine the direction of influence or purpose in the identified role, i.e., identifying who or what influences or has a purpose on who or what. This direction is represented by one of the two ends of the solid line double-pointed arrows in the analytical tool (see figure 2). The roles in the coded situations were then categorized as enabling or hindering and whether they are active before, during, or after the described teaching situations. Several of the situations described by the different mathematics teachers are explicitly not related to mathematics teaching and therefore excluded from the results. Situations related, although not exclusive to mathematics teaching, are included though.

## Results

One coded role in a situation from each relational category is shown in this first part of the result section. In the role coding, (see table 1) PE – T means an influence from an element in PE towards the teacher T. An interaction between students S and content C with an influence towards an element in PE is coded as SC – PE.

In the following part, the remaining situations relating to mathematics teaching are summarized and presented together in one of the six relational

Table 1. *Examples of role coding with influence categorization*

Situation	Role of PE
Technical equipment working flawlessly reduces stress and the number of interrupts for the teacher	PE – T enabling before and during the educational activity
Students put their mobile phones away to avoid distractions or disruptions	S – PE enabling during the educational activity
Students' mobile phone good aid in math class	PE – C enabling during the educational activity
It is easier for the teacher to keep the order if the classroom has proper sound insulation	PE – ST enabling during the educational activity
Students make noise when getting learning materials – the teacher might avoid using them	SC – PE hindering during the educational activity
Teacher carry/transport learning materials to and from the classroom	TC – PE enabling during and hindering (tiresome) before and after the educational activity

categories with an indication of direction of influence when feasible. Transcript excerpts are shown to illustrate one situation from each relational category. It should be noted that some of the transcripts contain more than one role of PE. The excerpts were translated from Swedish to English. During the translation some in-complete sentences were completed and adjusted to avoid ambiguity and to increase comprehension.

### Teachers and PE

In the direction towards teachers, we find that teachers appreciate flexibility in the classroom organization, effortless access to teaching materials, and group rooms in the vicinity of the main classroom (see transcript below). Proper board lighting, air ventilation, and a classroom adapted to the teaching subject with satisfying acoustics are also something appreciated by the teachers.

Teacher: Of course, it helps to have small group rooms that are soundproofed adjacent to the classroom. Someone [a student who missed a previous mathematics test] may need to sit and take a re-test.

In the direction towards PE, we find that teachers often take actions to improve the PE properties. For instance, reorganize seating, tidy if messy, arrange phone storage for students' mobiles, bring extra wall hangings for acoustical reasons, and open classroom windows to let fresh air into the classroom.

### Students and PE

In the direction towards students, we find that tidy, uniform, and ergonomic classroom furniture increase student comfort and have a calming effect on students. The arranged phone storage, where students can put away their mobiles, helps avoiding disruptions. Privacy window tints can prevent visual distractions

from outside the classroom windows. For some students listening to music on earphones can help them concentrate on the task (see transcript below). Rocker chairs can also be a good aid for some students.

Teacher: Sometimes we also use them [students' mobiles] for students who need to have ... for concentration. When we have ... it's time to work [solve routine tasks individually in students' mathematics textbooks], then they usually get to sit down and listen to music [on earphones from a mobile] ... those who need it. They concentrate better then.

## Content and PE

In this category PE is represented by technical equipment dedicated to subject content. Students' mobiles can be used purposely in mathematics class when specific mathematics applications are installed. Video projectors are often used to show subject content in mathematics, science, and other teaching subjects (see transcript below).

Teacher: And this machine [pointing upwards] is something that, not the least in secondary school science classes, I have used a lot. I'm referring to the video projector in the ceiling.

## PE and student-teacher interaction

In the direction from PE, we find that classroom discussions and the general classroom social climate benefits from proper classroom acoustics, good air quality, and lighting. Extra sound proofing for students with hearing impairments is beneficial for everybody in the same classroom (see transcript below). Good furniture high and ample space between student seats promote student-teacher interaction at each student's seat in mathematics class.

Teacher: In classes where I have students with hearing impairments in different ways, they are very disturbed by the chair scraping when someone else changes their position or moves. Now we are testing tennis balls on the chairs in two classrooms.

Interv.: On the feet?

Teacher: On the feet of the chairs. It has become a completely different sound environment in fact.

Interv.: Does it also benefit those who are not hearing impaired?

Teacher: It benefits everyone.

## PE and student-content interaction

In the direction from PE, we find that mini whiteboards can be used by students to write on and also to show written content to other students and the teacher

during mathematics class (see transcript below). There is often mathematical content displayed on the classroom main whiteboard that students can interact with as well. In the latter case it is important with proper directed board light. Students often take photos with their mobiles of the mathematical content written on the whiteboard for later recap.

Teacher: Because I work a little with ... now I do not have one here to show you ... these small tablets. You know ... whiteboards in mini format. So that they [the students] can write themselves. And then you need to look at what your friend has written.

### PE and teacher-content interaction

The situations from the interviews in this category are indirect interaction with the content when teachers decide which teaching materials and textbooks to carry or transport to and from the mathematics classroom (see transcript below).

Teacher: Yes, but oh, now we need this [learning material or book] ... now I'm going to go and look in ... it does not work like that ... I am not so flexible that ... oh ... that I needed that now huh, but yes, but wait I'm going to go and get it.

## Discussion

Several of the situations presented in the result section are, regarded individually, not unique to mathematics classrooms. The situations in the results, regarded as a whole, are relevant to mathematics education though since they stem from stories of mathematics teaching told by mathematics teachers.

The aim of this study was to characterize the different roles PE play in the stories of different teaching situations told by the participating mathematics teachers. These characterizations were enabled by using the model developed in Fahlström and Sumpter (2018) as an analytical tool. The results are presented in the relations indicated in the model (see figure 2). In the first relation between the teacher and PE, we find specific examples of elements in PE that are enabling according to the teachers. Such examples are technical equipment working flawlessly, a classroom adapted to the teaching subject, and a roomy classroom with good air quality. In this relation, we also find teachers who make an extra effort to improve elements of PE. For example, tidy if messy or bring wall hangings for acoustical reasons. In the relation between students and PE the teacher stories predominantly contain arrangements in PE with the intent to prevent students from being distracted. For instance, some students listen to music on earphones to help them focus and to prevent them from distracting other students. These results show that teachers often devote a considerable part of their energy and attention to compensating for or preventing disturbances or distractions from insufficiencies in PE. Comparable findings can be found in

previous research (Johnston, 2012; Uline et al., 2010). These extra or overhead efforts made by teachers are something worth addressing as a problem in itself, but also important to consider when planning and carrying out professional development programmes involving mathematics teachers' teaching practices. An implication of the present study is that these overhead efforts can be recognised and included in what is regarded as teaching practices and increase the potential of successful outcomes from implemented development programmes. The implication also involves interventions in the category that mainly rely on changing classroom PE, as in Cleveland and Fisher (2014), with the aim of changing teaching practices. In essence, this can be formulated as when looking to change teaching practices, always consider PE but avoid relying solely on PE for changes to take place. One example for such undertakings are the nine focus elements of mathematics teaching practices in Liljedahl's (2016) framework.

Results in the relation between PE and content contain examples of technical equipment dedicated to subject content, for instance, mobile phone applications suitable for practising specific mathematical content or video projectors to display content. Students' mobile phones are involved in several PE roles. One is mentioned above with an enabling role of mathematics applications. Another enabling role is when students use their mobile to take photos of mathematical content on the whiteboard for later recap. A third enabling role is when some students listen to music on earphones from their mobile to help them concentrate during individual mathematical task solving. A fourth role is a potential hindering role where students get distracted from their mobiles. This is often avoided by the teacher who collects the students' mobiles before class. There are no examples related to mobile phones in the previous research presented in the background. No examples of where aspects of the school building were used as mathematical content were found in the stories told by the mathematics teachers in this study. This is a suggestion for further studies since Soygenis and Erktin (2010) reported an increase in students' awareness of the built physical environment when used as mathematical content in the classroom. Efforts in this direction could also address issues reported in this study related to teaching materials. Issues such as when teachers have to transport or carry materials to and from the classroom and that the situation can be noisy when the students bring the materials to their seats. If aspects of the school building serve as content it may reduce the need for some teaching materials.

Many of the situations described in the interviews illustrate a classroom practice with a one-to-many communication pattern between the teacher and the students. For instance, when the teacher shows content on a screen with a video projector or writes content on the main whiteboard. Such situations are often followed by individual student work. For example, the situation when some students can listen to music when it is time to work or the appreciation of good furniture height and ample space between student seats for one-to-one

teacher student interaction at each student's seat. This pattern of practice is often the norm and sustained by how classroom PE is configured. A common classroom configuration is with the main whiteboard, projector screen, and teacher position at the front of the classroom and the students seated in rows facing forward. Forward or the front, being at one short end of a rectangular classroom. There are also examples in the results indicating another type of classroom practice, or deviation from the norm practice. These examples include an enabling role of PE. For instance, teachers that appreciate flexibility in the classroom and reorganize the seating. The appreciated flexibility in the classroom organization indicates that something needs to be changed before certain activities can be initiated. This also indicates that it takes an effort to do the change from norm organization and therefore the teacher might avoid activities that needs a change of classroom organization. Another example with an enabling role of PE is when mini whiteboards are used. In this situation, students can write content on their mini whiteboard so that other students and the teacher can see what is written from a distance. In this case, PE in the form of mini whiteboards enables student-content interaction, student-student interaction, and student-teacher interaction. Results from this study indicates that certain aspects of classroom PE can be sustaining classroom norms and other elements in PE can be a help in breaking norms. If we want more of the type of classroom activities that are rich of verbal communication between students and students interacting with tangible learning materials, we need a PE that allows and enables it.

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# Professional development as a means for implementing mathematics education innovations: results from a systematic review

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To get an overview of the characteristics of the studies in mathematics education research that explicitly state that they deal with implementation, we have conducted a systematic review. In this paper, we report on a subset of the identified studies from the review, dealing with large-scale professional development for teachers. For the subset of the 11 identified papers, we ask the question: What designs are used to support teachers to adopt new ideas in their practice and what dimensions of scaling are considered in the studies? To articulate design and dimensions of scaling, we draw on theoretical constructs from both mathematics education research and more general implementation research. Results indicate that the choice of facilitating strategy impacts the dimensions of scaling considered in the implementation.

Implementation research, by our working definition, is the systematic inquiry of innovations enacted in controlled settings or in ordinary practice, the factors that influence innovation enactment, and relationships between innovations, influential factors, and outcomes.

(Century & Cassata, p. 181)

That implementation research (IR) in mathematics education research (MER) has gained momentum during the past few years is beyond any doubt. Since 2017, a thematic working group (TWG 23) at CERME has been dedicated to the topic. In 2021, a new journal – *Implementation and Replication Studies in Mathematics Education* (IRME) – was launched by the well-established Dutch publishing house, Brill. In addition, in 2021 a special issue of ZDM was dedicated to the topic of implementation research in mathematics education (Koichu et al., 2021). Ongoing discussions in relation to IR in MER concern,

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for example, the use of theoretical constructs from outside the field of MER (e.g. health science, economics, etc.) versus those available inside of MER; what we should take *implementability* to mean in relation to IR in MER; to what extent IR should mainly address large scale studies; etc. (Jankvist et al., 2021). Yet, it seems to us that to engage in these discussions on an enlightened basis, a natural starting point is to get an overview of both the numerosity and type of studies in the MER literature which specifically addresses "implementation". We have taken on this task by conducting a systematic literature review of the field of implementation research on educational reforms in mathematics education.

In all educational reforms, teachers are viewed as agents of change and thus are expected to play a key role in changing schools and classrooms (Prawat, 1992). In this paper, we focus on a smaller subset of papers from the systematic review, reporting on professional development (PD) programs for teachers.<sup>1</sup> While almost all papers in our review address some kind of PD for teachers, our sample consist of papers that foreground the PD and also discusses scaling of the PD. Papers touching on PD that foreground a curriculum reform or new curriculum materials are not included in the sample. Furthermore, papers that do not report on any approach to scaling are also not included in the sample.

We are interested in what kind of different designs for facilitating change in teacher practice that are used, and what kinds of scaling with the aim to create lasting change that are planned for, in the implementation projects reported on in our sample. The question under investigation still needs definitions of some notions before it can be formulated coherently. Therefore, the specific research question will be presented after the section "IR theoretical constructs applied".

## Review methodology

Conducting a systematic review on implementation research involved a few delicate considerations on our behalf, not least since a large portion of the research studies in MER may be considered studies addressing implementation. We settled on two inclusion criteria to avoid a too large number of papers. Firstly, we limited the review to include papers that clearly stated to be dealing with some kind of implementation. Secondly, we limited the review to only consider studies published in the top twenty quality-ranked MER journals following the recent journal categorization by Williams and Leatham (2017).

We conducted the literature searches in ERIC (EBSCO) searching for manuscripts with *implement\** in the title and/or abstract, journal by journal of our top 20 samples (Williams & Leatham, 2017). The advantage of doing the entire search in one database is that it is easy to collect the results in one folder. To ensure that no article had been overlooked, we repeated the search *implement\** in the title and/or abstract on each journal's website. We found 1,093 peer-reviewed articles fitting the search criteria. We used the software *Covidence* to manage our literature review.

Each paper was screened by two reviewers. The screening was made in two steps. First, we screened the title and abstract. In cases where we were hesitant, e.g. because the abstract did not provide sufficient information, we chose to forward the paper to full-text screening. A total of 138 papers were forwarded to full-text screening. In the full-text screening, papers were included if they were in line with Century and Cassata's (2016) definition of IR:

[...] the systematic inquiry of innovations enacted in controlled settings or in ordinary practice, the factors that influence innovation enactment, and relationships between innovations, influential factors, and outcomes.

(Century & Cassata, p. 181)

As evident from this quote, another central term in IR is that of *innovation*. Innovation refers to the practical implementation of ideas resulting from research that involve a change (e.g. in behavior or practice) for the individuals enacting them (Century & Cassata, 2016).

Of the 138 papers, 95 remained after the full-text screening. To obtain smaller and more manageable units, these were categorized as: Instructional sequences on mathematical concepts, and/or competencies, (19); Curriculum materials (21); Professional development (PD) projects (25); and Curriculum reform (30). There are no clear cuts between the categories. An instructional sequence may stem from a new curriculum material that is implemented through a PD project due to curriculum reform. The category foregrounded in each of the 95 papers was decisive for how the categorization was done.

The data extraction from the papers included general information on the author(s), title, purpose statement(s), country where the study was conducted, research question(s), methods, target group, and results. The specific information about the implementations included what kind of innovation from mathematics education the study concerned, specific or general goals of short term or long term, phase of the implementation studied, stakeholders responsible for the implementation, and identified factors of influence for the outcomes of the implementation.

From the subset of the 25 papers concerning professional development programs, we report on the 10 empirical papers from the subset, discussing design for scaling. We also included 1 theoretical paper that, in line with our focus, discusses support for teachers' long-term use of research-based instruction in large-scale projects (Cobb & Jackson, 2015). In the next section, we will elaborate on design for implementing instructional change through PD programs and define long-term goals as well as four dimensions of scaling.

## IR theoretical constructs applied

While the form and content of PD programs come in different shapes, the unifying goal is to implement innovations that increase students' knowledge

through improved teaching practice. We use Kennedy's (2016) classification of PD designs, *prescription*, *strategies*, *insight*, and *body of knowledge*, to categorize the form of PD programs in our review. *Prescription* refers to the implementation of scripted instructions for teachers to follow, for changing practice. *Strategies* refer to "toolboxes" from which teachers can choose different strategies to address specific goals for practice, e.g. a problem-solving strategy. *Insight* rests on the idea that increased knowledge exemplified by teaching practices can give teachers tools to change practice. *Body of knowledge* carries the idea that if teachers gain more knowledge about mathematics, didactics, and pedagogy (e.g. MKT, PCK) they will be able to plan and implement better teaching. Body of knowledge often consist of regular university courses or lectures. The different designs for helping teachers enact new ideas within their own ongoing systems of practice set different requirements for how innovations may survive over time. Regardless of the choice of facilitating design for creating change in teaching practice, it is necessary to plan for how the innovation shall survive over time, if one has long-term goals for the implementation.

*Long-term goals* refer to innovations that intend to change the nature of mathematics teaching practice in a sustained way. For example, as a result of alarms from international tests, politicians may plan for increasing the mathematics teachers' general content knowledge at scale and/or state-wide curriculum reforms to be implemented with long-term goals. On the other hand, small-scale and pilot studies with the goal to try out innovations over a given time span provide valuable insights for future action, implementation studies with long-term goals may need to consider dimensions of scaling (Coburn, 2003).

Our definition of scaling follows Coburn's (2003) notions of *depth*, *sustainability*, *spread*, and *shift in reform ownership*. *Depth* refers to change in classroom practice that goes beyond a shift in teaching resources and the introduction of specific activities. Coburn argues that scaling in depth includes a shift in teachers' beliefs, norms for communication, and pedagogical practices. *Sustainability* concerns the scaffolding that is left to maintain the vitality of the innovation after the support of the reform leaders is withdrawn from the organization. When Coburn considers *spread*, she, in addition to scaling to other schools and classrooms, also includes spread within the organization. Finally, Coburn adds the dimension of a *shift in reform ownership* to the notion of scale. When reform is launched, the ideas and activities are owned by the creators of the reform. According to Coburn, the authority to scale the implementation needs to shift to the districts, schools, and teachers. Only then can scaling in-depth, sustainability, and spread be maintained.

We are now ready to present our research question. Following Kennedy's (2016) designs for PD and Coburn's (2003) dimensions for scaling, we ask: *What designs are used to support teachers to adopt new ideas in their practice and what dimensions of scaling are considered in the studies?*

## Results

In table 1, we summarize the answers to the posed question above for the 10 empirical papers in our sample. Due to space limitations, it is not possible to summarize the research design for each study. Instead we provide information about the target group to give some idea of the context in which the innovation was implemented. We follow up by describing what kind of scaling dimension of PD that is in play in terms of each of the four facilitating designs. We close the result section by reflecting on scale from the view put forward in the theoretical paper in our sample.

Table 1. *Characteristics of PD-programs*

Facilitating Strategy for PD	Characteristics of Scale	Author(s)	Target population(s)	Innovation
Prescription	Depth Spread	(Clements et al., 2011)	Preschool teachers	A research-based curriculum "the building blocks" carries the prescription of learning trajectories for teaching number sense and geometry.
Prescription	Depth Spread	(Corcoran, 2018)	In-service teachers in K-5	Teaching for conceptual understanding and procedural fluency.
Strategies	Depth Sustainability Spread A shift in reformed ownership	(Clark-Wilson & Hoyles, 2019)	In-service teachers in lower secondary school	Algebraic patterns and expressions, linear functions, and geometric similarity.
Strategies	Depth Spread	(Swan, 2007)	In-service teachers with post 16 students	Reformed teaching
Insight	Depth Spread	(Ferrini-Mundy et al., 2007)	In-service teachers in grades K-8	Mathematical knowledge for teaching (MKT), curricular coherence, and learning trajectories.
Insight	Depth Sustainability Spread	(Higgins & Parsons, 2011)	In-service teachers in primary and middle school	Numeracy
Insight	Depth Spread	(Jankvist & Niss, 2015)	Upper secondary school teachers with a master's degree	Conceptual knowledge, modeling and, reasoning and proof.
Insight	Depth Sustainability Spread A shift in reformed ownership	(Prediger et al., 2019)	In-service middle school teachers	Basic conceptual understanding, research-based materials, and community-based collaboration.
Body of knowledge	Spread	(Buchholtz & Kaiser, 2013)	Prospective teachers for secondary school	Mathematical content knowledge (MCK) and mathematical pedagogical content knowledge (MPCK).
Body of knowledge	Depth Spread	(Gainsburg, 2013)	Prospective K-12 teachers	Reformed teaching

## Scaling in programs using prescription

Scaling in-depth, in terms of fidelity to the program theory, is discussed in both PD programs using prescription as a facilitating design for change in practice (Clements et al., 2011; Corcoran, 2018). Both studies also scale by spread to many schools. The sample from Clements et al. (2011) comes from 10 school districts, 42 schools, and 106 classrooms. The study is a scaling up of the program *Building blocks*. Corcoran's sample includes 11 schools from one school district, but the program, the *ORIGO Stepping stone*, is used by more than 100 school districts with more than 450,000 students. Scaling by sustainability and shift in reform ownership are not discussed in these studies.

## Scaling in programs using strategies

The PD-program *Cornerstone maths* addresses all dimensions of scaling (Clark-Wilson & Hoyles, 2019). Sustainability of the innovation and spread within the school is catered for by a web-based professional development toolkit, to maintain scaling beyond the timeline of the funded project. The design of the toolkit aims to provoke a rethinking of mathematics and challenge existing beliefs. Altogether, the design aims to facilitate a shift in reform ownership. Scaling in depth by challenging teachers' beliefs is also one of the goals with the task-based PD presented by Swan (2007). By pull-out workshops for facilitators from different schools, and a toolbox of tasks to use in practice, the program aims to spread to new schools and within schools.

## Scaling in programs using insight

Scaling by depth is a component of all programs in our subset using the facilitating design insight, namely *PROMSE* (Ferrini-Mundy et al., 2007), *Numeracy development project* (Higgins & Parsons, 2011), *Maths counsellor* (Jankvist & Niss, 2015), and *Mastering math* (Prediger et al., 2007). The core of the programs is that the new knowledge should provide insights that give participants a changed, or deepened, view of what mathematics teaching is. All four programs use facilitators for spread within schools. Either the program aims at educating facilitators (Ferrini-Mundy et al., 2007; Jankvist & Niss, 2015) or use external facilitators employed to assist teachers in practice (Higgins & Parsons, 2011; Prediger et al., 2007).

The report from the Mastering maths program has been designed for sustainability and a shift in reform ownership. Design for scaling up with sustainability is reported from the Numeracy development project (Higgins & Parsons, 2011), but not a shift in reform ownership. Sustainability and shift in reform ownership are not discussed in the studies by Ferrini-Mundy et al. (2007) and Jankvist and Niss (2015).

## Scaling in programs using a body of knowledge

Innovations implemented in university courses for prospective teachers have the body of knowledge as design for facilitating change in practice, i.e., the studies by Buchholtz and Kaiser (2013) and Gainsburg (2013). Scaling, as the spread of innovations, goes through the graduated prospective teachers into their classrooms. Depth, e.g. changed beliefs on what mathematics teaching should be, is discussed as an important factor for the spread in the study by Gainsburg (2013). Sustainability and shift in reformed ownership are not discussed.

## Scaling as a theoretical reflection on experience

In their theoretical paper, based on experiences from different PD programs and existing literature on the subject, Cobb and Jackson (2015) address all dimensions of Coburn's scaling. The authors suggest that the depth of teachers' learning involves:

- (a) clarifying the goals for teachers' learning with respect to the products of the classroom design study, and (b) documenting teachers' current instructional practices and relevant forms of knowledge and conceptions about teaching and learning. (p. 1029)

For the sustainability of innovation, the same group of teachers must get opportunities to continue to collaborate. For spread, an organizational level for extending dissemination designs is important. An extending dissemination design on the organizational level is also necessary for an adequate plan for a shift in ownership.

The authors conclude that PD programs, where teachers leave their classrooms to learn how to change their practice, are not enough for a successful implementation. "We argue that high-quality pull-out professional development is essential but not sufficient, and go on to consider teacher collaboration and one-on-one coaching in the classroom as additional supports." (Cobb & Jackson, 2015, p. 1027)

## Discussion

We asked the question: What designs are used to support teachers to adopt new ideas in their practice and what dimensions of scaling are considered in the studies? We found that the facilitating design for supporting teachers to enact new ideas within their practice spread over Kennedy's (2016) four categories. The papers were distributed as follows: prescription (2 papers); strategies (2 papers); insight (4 papers); and, a body of knowledge (2 papers). We can of course only hypothesize based on this small sample, but from the dimension of scaling that was addressed in these papers, scaling seems to align with the choice of design for change in teaching practice.

Depth as in shift in teachers' beliefs, norms for communication, and pedagogical practices is crucial for studies using prescription as facilitating design. Prescription calls for fidelity to program theory in teachers' implementation. Yet, while teachers who both believe in and know the program theory will be likely to implement the program with fidelity to its core ideas, teachers that disbelieve or lack knowledge will focus on surface manifestations and circumvent the pedagogical ideas (Gregoire, 2003). Conversely, striving to understand the intentions behind a program and implement it with fidelity seem to enhance a shift in teachers' beliefs, norms for communication, and pedagogical practices (Guskey, 1986). It is uncertain what precedes what, but it may be the case that depth can arise both due to the teachers already having beliefs in line with the program ideas, or because the teachers carry out the program according to the prescriptions, i.e., in line with the model by Clarke and Hollingsworth (2002).

PD programs facilitating by applying the designs strategies and insight seem to have a broader approach to the dimensions of scale. All four dimensions of scale are discussed in the PD program Mastering Math (Prediger et al., 2019). In the other studies, facilitating by applying the design strategies or insights, one gets implicit information of considerations of scaling in all four dimensions. It is of course reasonable to assume that the authors have chosen to foreground certain parts of the program, leaving others out.

For PD programs using a body of knowledge as a design for teacher change, the scaling opportunities seem to narrow down. Sustainability and shift of reform ownership are not discussed. Perhaps this is due to the implementers losing control of future development of the innovation, when a university course is completed.

Results from the students to the participating teachers are not measured and reported in six of the studies. One reason may be that while it is doable to observe a change in teaching practice or teacher knowledge and beliefs, it is more complicated to measure what effect the changed practice has on students. Or it is the focus of another paper. For example, both studies using prescription (Clements et al., 2011; Corcoran, 2018) report on students' outcomes. Also, the studies on PD programs relying on a body of knowledge reports on students (Buchholtz & Kaiser, 2013; Gainsburg, 2013).

The study from Clements et al. (2011) shows positive results on student-level together with acceptable fidelity to the program theory in teachers' implementation. The study from Corcoran (2018), on the other hand, reports weak to non-existing differences between the experimental groups and the control group. A lack of fidelity to the program theory may, according to the author, explain these results.

Buchholtz and Kaiser (2013) measure teacher students' mathematical content knowledge and mathematical pedagogical content knowledge after taking innovative courses for prospective teachers. Gainsburg (2013) observes whether or not recent graduates implement program-emphasized teaching

practices in their classrooms. Both studies working with the design body of knowledge report on weak results for university courses to equip teacher students for reformed teaching.

There are no clear cuts between the four categorizations used in our review: Instructional sequences on mathematical concepts, and/or competencies, (19); curriculum materials (21); professional development (PD) projects (25); and curriculum reform (30). Since most PD programs include more than one category, we needed to decide on the most prominent one in the paper. The same reasoning can be applied to Kennedy's designs and Coburn's categorizations. Large programs often cover several designs, which is why our categorization is grounded in what is written in the papers. Further information in the PD programs than what is provided in the papers might of course change the picture.

Of the 25 papers categorized as PD programs, only 10 empirical papers and 1 theoretical paper discussed scaling of the innovation. It is reasonable to assume that many of the other PD programs reported on in the literature had dimensions of scale as well, yet without discussing it explicitly. In the next cycle of our review, we shall leave the systematic approach and move on to a heuristic approach. The insights from the systematic review shall guide us to search for theoretical papers together with all the 95 papers from the systematic review. With this fuller picture as a basis, we hope to be able to address the question of how to potentially create an empirically founded theoretical framework for IR in MER concerning large-scale development programs.

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### Note

- 1 We are aware that there is a plethora of papers discussing PD-programs. However, since it is "implementation" that is in focus here we do not aim to review all papers on PD-programs from our 20-journal sample.



# Programming to learn mathematics – exploring student teachers’ instrumental genesis

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Programming is now a prescribed part of the curriculum in mathematics in both primary and secondary school in Sweden, as in many other countries. Teacher training must thus prepare students for the challenges of teaching mathematics with programming. We explore how student teachers see their own training in programming in relation to mathematics and what opportunities to learn mathematics they believe that programming can offer pupils in school. An instrumental approach is used to analyse observations and a questionnaire on secondary school student teachers’ experience of a programming lab, where they investigate Riemann sums with programming. We find that students feel challenged by both the programming and the mathematical content, and that they see the challenges as useful, both for themselves and for their future pupils.

Technological and digital competence have become extremely important in society and have an increasing impact on people’s lives. As a consequence, programming has been introduced in the school curriculum in many countries. In Sweden, programming content was added to the national mathematics curriculum for compulsory school and upper secondary school in 2018 and reformulated in 2021. In the syllabus for the advanced mathematics courses at upper secondary school, programming is included as *a tool in problem solving, data processing or application of numerical methods* (Skolverket, 2011). An important factor for the successful integration of programming into mathematics education is the teachers’ subject knowledge. However, today, there is a gap between the need for programming in school and what many teachers feel their knowledge is in programming (Misfeldt et al., 2019). To bridge this gap and meet the new curricula, a programming strand is included in the secondary school mathematics teacher program at the University of Gothenburg. There is no course in programming per se. Instead, programming assignments are integrated in all the mathematics courses.

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The present study is a part of a larger project aimed at exploring the potential for learning mathematics through programming (Pejlare, 2021). In particular, we study how programming can be integrated in mathematics education, both at upper secondary school and in the teacher education program. In this study we focus on students preparing to teach mathematics at the secondary school level. The aim is to investigate the student teachers' understanding of their own learning of mathematics through programming, as well as their understanding of the prospects for programming as a tool in mathematics education in secondary school. To meet this aim, we use observations of secondary school student teachers during a programming lab, where they investigate Riemann sums with programs in Python, and responses to a follow-up questionnaire on their experience during the lab.

## Background

The connection between programming and learning in mathematics has been discussed since the late 1960s. During the 1970s, Seymour Papert developed the programming language LOGO with the aim of teaching children mathematics (Papert, 1980). The idea behind LOGO was that the visual language would make it easier to enable children to explore mathematical concepts. One advantage Papert could see in using programming in mathematics education was that it gave students an increased understanding of the problem-solving process.

There is a strong connection between programming and mathematics, partly because programming is based on logic, procedures, and functions (Misfeldt & Ejsing-Duun, 2015). Programming can help students gain insight into different mathematical concepts and develop mathematical abilities. For example, 12- to 13-year-old pupils developed understanding of the concept of variable using Logo-based tasks (Ursini-Legovich, 1994) and 6th grade pupils working with coding activities using Scratch showed significant improvement in mathematical thinking (Calao et al., 2015). Moreover, programming can be a good starting point for working with real world mathematical problems and connecting mathematics to applications in a new way, and thus has the potential to influence pupils' motivation and attitudes toward mathematics (Forsström & Kaufmann, 2018). In a study by Ke (2014), it was found that middle school pupils participating in Scratch-based mathematical game designing activities developed a significantly more positive attitude toward mathematics.

Nevertheless, it is not enough to know basic programming to be able to use programming in a mathematical context. In one study it was found that upper secondary school pupils with basic knowledge in Java programming experienced difficulties when using programming as a technical artifact to solve mathematical problems (Borg, 2021). These pupils had difficulties in translating mathematical ideas into programming code. In another study, in an activity where 5th grade pupils were expected to learn mathematical concepts through

programming in Hopscotch, it turned out that the programming exercises themselves were not enough for the pupils to develop mathematical competences, but that with some help from the teacher, the pupils could understand how programming and mathematics were connected (Misfeldt & Ejsing-Duun, 2015). For teachers, the challenge of using programming to teach mathematics has multiple layers. It requires thinking through the programming aspects, the mathematical aspects, and the coordination between the two (Kilhamn et al., 2021).

## An instrumental approach

Since the mathematics curricula for Swedish upper secondary school focus on programming as a tool, we use an *instrumental approach* (Trouche, 2004) to theoretically frame the study. This theory provides views concerning learning processes in complex technological environments. The key idea is the distinction between an artifact and an instrument: An artifact refers to a technological object that can be used as a tool, while an instrument is made up of the artifact together with a psychological component (Verillon & Rabardel, 1995). The artifact, such as for example a programming environment, becomes an instrument when the subject has integrated it into her activity. This process is called *instrumental genesis* and evolves in two directions: The *instrumentalization process* is directed towards the artifact itself, for example when a student learns how to handle the artifact, and the *instrumentation process* is directed towards the subject, for example when the student uses the artifact to represent a mathematical concept, changing the student's own understanding of the concept (Trouche, 2004).

An instrument involves the *techniques* and *mental schemes* that are developed and applied by the subject while using the artifact. The notion of a scheme was introduced by Piaget (1936) and redefined by Vergnaud as the "invariant organization of behavior for a given class of situations" (Vergnaud, 1996). Rabardel (1995) described a scheme organizing an activity with an artifact associated with the realization of a task as a *utilization scheme*. He distinguished between two types of utilization schemes: *usage schemes* that are oriented toward the management of the programming environment, and *instrumented action schemes* that are oriented to the carrying out of the specific mathematical task (Rabardel, 1995). To develop schemes and to build an instrument is a social process – it happens in a classroom environment with a collective process of instrumental genesis (Drijvers et al., 2010). Since the schemes cannot be observed directly, we can instead focus on the observable techniques, that is the interactions between the subject and the artifact.

## Methodology

The participating students are secondary school mathematics student teachers taking their first semester of mathematics courses. The data collection was

conducted about three quarters into the semester. The students were either in the end of their first or third year of the teacher education program, depending on whether mathematics is their primary or secondary subject. They had been introduced to programming in mathematics within their first mathematics courses on algebra and geometry. There they had built a block program in Scratch to draw regular polygons, using loops and variables. Thereafter, in the Calculus course, they are offered an optional workshop with an introduction to programming in Python, followed by a programming lab where they investigate how to approximate the area under a curve using narrower and narrower rectangles (Riemann sums) with Python. Participation in the lab is mandatory. In the workshop, the students were introduced to standard concepts of programming, such as arithmetic operations, variables, and data types, before they were guided through some basic programming exercises involving for-loops, nested loops, if-else conditional expressions, the definition of functions, and some basic 2D graphic.

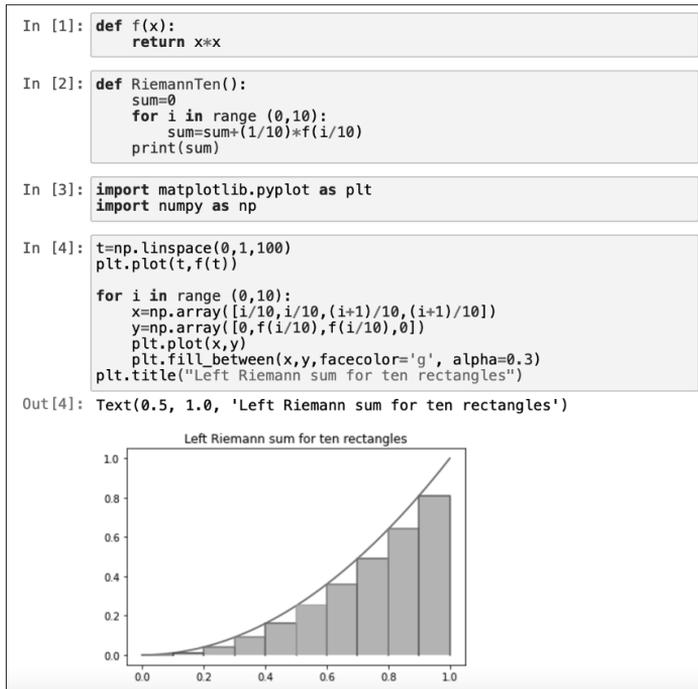


Figure 1. The code that the students received at the Python programming lab.

In the lab the students were reminded of the idea of a Riemann sum, which had been mentioned in a lecture, and were given some code: the definition of the function  $f(x)=x^2$ , the loop to compute the Riemann sum of this function from

0 to 1 with 10 intervals, and the code to visualize the graph of the function and the ten rectangles (see figure 1). The students were asked to make changes in it to investigate what happened if they changed the number of intervals when the Riemann sum was computed. Thereafter they were asked to investigate the difference between the left Riemann sum, the right Riemann sum, and the mid-point Riemann sum, and to see which one would converge the fastest. To do this they had to change the parameter in the definition of the Riemann function (see Ln [2] in figure 1) as well as the coordinates, i.e., the parameters in the two lists, in the loop that produces the graphic representation of the rectangles (see Ln [4] in figure 1). Moreover, the students were asked to investigate Riemann sums for other functions, such as  $f(x)=\sin(x)$  and  $f(x)=e^x$ , and other intervals  $[0, B]$ ,  $B \in \mathbb{R}^+$ . Finally, they were asked to compute, with the help of Riemann sums, the  $x$ -values of the quartiles of the function representing the standard normal distribution. At the lab the students followed the use-modify-create progression (Lee et al., 2011); they first had to copy and run the code, then to modify it and finally they – in the last exercise – had to write their own program. The students worked either alone or in pairs. At the end of the lab, they had to present their code and explain how it worked to one of the teachers in the course.

After the lab the students were asked to answer a questionnaire consisting of 12 questions: five multiple-choice and seven open-ended questions. The questions investigated their earlier experience of programming, their experience of the workshop and the two labs, and their understanding of programming in mathematics education in school and in teacher education. 38 students passed the lab and 26 of them answered the questionnaire. The data was collected in the spring of 2021, and due to the pandemic, the workshop and the lab were done on-line, using zoom as a conference environment. The questionnaire was also administered on-line. One multiple-choice question and two open-ended questions of the questionnaire focused on the earlier Scratch lab as well as students' expectations on the teacher education program and are not addressed in this paper. The remaining multiple-choice answers were analyzed statistically and the answers to open-ended questions were analyzed using qualitative content analysis. We looked for traces of the instrumentalization and instrumentation processes and focused on how students expressed the relationship between learning to program and learning mathematics, in general and in the context of this lab.

## Results

We begin by presenting results concerning the students' experience of the lab and to what extent they found that it helped them in their understanding of Riemann sums. Thereafter we present their thoughts on programming in mathematics education.

## Does the Python lab help the students in their understanding?

The students have varying experience of programming when they start the teacher education program. Of the 26 students answering the questionnaire, a majority (14) claimed that they did not have any experience of programming, about one third (9) claimed to have some experience, and only three of the students claimed to have a lot of experience. The experienced students had knowledge of various programming languages such as C, C++, Java, JavaScript, Matlab and Visual Basic. Only four of them mentioned Python as one of the programming languages they had experience of.

All but two students choose to participate in the workshop. At the workshop the students were introduced to the programming environment and to the fundamental concepts needed for the lab. All students appreciated the possibility to participate in the workshop, but due to lack of earlier experience some wanted more time, more basic examples, and easier exercises. At the lab we observed that some of the students were still in the beginning of their instrumental genesis; they were struggling with handling the concepts, structuring the program, using formal syntax, and debugging. They were frustrated when their program did not work as expected as they did not have a strategy for how to solve problems. It became more obvious that some students had difficulties understanding important programming concepts when they in the questionnaire were given a short program including a loop where the variable sum was computed through the successive adding of areas of rectangles (see ln [2] in figure 1) and asked to explain the meaning of the variable. Even though many of the students could give a satisfying explanation, showing that they could use such a program as an instrument, six of them were not able to do this. One of these students wrote: "I just accepted that it should look like this. I understand that the sum changes depending on the interval, but it looks very unclear to me". This student had after the lab possibly not yet developed sufficient usage schemes related to basic programming concepts; she could accept that the variable changed values in the loop but could not explain how and why it changed.

When the students were asked how difficult they experienced the lab to be, a majority (15) answered that it was difficult. Four of the students answered that it was easy, and the rest (7) answered that either the mathematics or the programming was difficult. In all, more than 80% of the students thought that the programming part of the lab was difficult. It is not surprising that all students with no experience of programming thought that the programming was difficult, and that all students with a lot of experience thought that the programming was easy (see table 1).

One reason to include this programming lab in the analysis course was to support the students in using the experience of programming to get a deeper understanding of the concept of Riemann sums and the relation between integrals and the area under a curve, i.e., an instance of the instrumentation process.

Table 1. *The students' answers regarding their experience of programming and their experience of the difficulty of the programming lab*

How did you experience the difficulty of the lab?	Do you have experience of programming?		
	No experience of programming	Some experience of programming	A lot of experience of programming
It was easy	0	2	2
The programming was easy, but the mathematics was difficult	0	0	1
The mathematics was easy, but the programming was difficult	3	3	0
It was difficult	11	4	0

Asked whether the lab helped them to understand Riemann sums, half of the students (13) answered yes, about a third (8) answered no and the remaining 5 were not sure. The students who answered "yes" were asked to explain in what way the lab helped them. Their answers could be categorized into three types: *understanding the formula* (six students); *experimenting with parameters* (four students); *different representations* (three students).

The largest group answered that the lab helped them in *understanding the formula*. To write a correct program and make the intended changes, the students had to check each line of the program, in particular the parameters in the loop computing the sum. One student wrote: "I had to decompose the formula into smaller parts and that helped me understand how it was structured." A few students answered that they already understood Riemann sums, but that the lab clarified the formula for them.

Some students answered that *experimenting with parameters* helped them in their understanding. By changing the different parameters, they got an understanding for the role of each parameter in the program. One student expressed this in the following way: "I liked experimenting and changing the values. When I changed the number of rectangles, I could see how this affected the sum."

Finally, some students answered that using *different representations* helped them in their understanding. One of the students expressed that it is always easier to learn when the content is represented in different ways, and that she, after the lab, directly visualized the rectangles in the Riemann sum when she worked with integrals, demonstrating instrumentation. Another student wrote: "I could see the curve and how the rectangle changed depending on if it was left, right or midpoint Riemann sums."

Of the eight students who answered that the lab did not help them in their understanding of Riemann sums, all but one had no or little earlier experience of programming and thought that the programming part of the lab was difficult. It seems that these students were insecure mainly due to the programming since they had not developed sufficient usage schemes related to the management of

the programming language; they were struggling, felt confused, and did not think that they were well enough prepared. The last student had a lot of experience with several different programming languages and thought that both the lab and the mathematics involved were easy. This student apparently already felt a sufficient understanding of the mathematics involved, and therefore didn't see the lab as leading to new knowledge. For this student, the instrumented action schemes related to the exercises in the lab were already developed.

### Can programming offer pupils in school opportunities to learn?

To find out about the student teachers' understanding of programming in mathematics education, they were asked if they believed that programming can offer pupils in school opportunities to learn mathematics.

Out of the 26 students, 3 students answered "no". One student motivated this by writing that it takes too much time from the mathematics lessons. A second student motivated this through arguing that to be able to construct a program you already need to know the mathematics involved and therefore the programming will not offer pupils the opportunity to learn mathematics. Thus, this student did not acknowledge the instrumentation process, i.e., she did not believe that programming can help students gain a deeper understanding of mathematical concepts. The third student had struggled a lot with both the mathematics and the programming part during the lab. Common to these three students was that neither of them experienced that the lab helped them in their own understanding of Riemann sums.

The remaining 23 students answered that they believed that programming can offer pupils in school opportunities to learn mathematics. They motivated this in several different ways:

- Programming can support a *deeper understanding* of mathematics, concepts, formulas, algorithms and so on (eleven students).
- *Visual representations* with the help of programming will help pupils in the learning of mathematics (four students).
- *Explorative work*, and changing the values of parameters in a program, will help pupils in the learning of mathematics (three students).
- Programming can support pupils in their *problem solving* (two students).
- Programming can help pupils to develop *logical thinking* (two students).
- Pupils are interested in gaming and robots, and they know that programming is important in future professions and therefore programming will be a *motivating factor* in their mathematics education (five students).

Out of the 23 students who were positive to programming in mathematics, nine also mentioned that more programming knowledge was needed. Teachers must learn programming, and how to teach programming in mathematics. Finally, some of the students mentioned that pupils need to work with programming from the early grades and throughout their mathematics courses. When the pupils have fundamental knowledge in programming and coding it will be easier to work with programming in a rewarding way and to offer pupils opportunities to learn mathematics.

## Discussion

This study involved just one class-sized group of students and the specific observations may not represent teaching of programming in general. Rather, it provides a close-up view of how a session can attend to both programming and mathematical development. The students taking part in the study were generally positive regarding programming in mathematics, even though most of them had no, or only little, previous programming experience. It can be expected that they before the lab had not developed sufficient usage schemes related to fundamental computational concepts such as variables, parameters, and loops. Thus, most students had not come far in their instrumental genesis (Trouche, 2004) before the lab. The lab setup, applying a use-modify-create progression (Lee et al., 2011) where students were encouraged to help each other and solve difficulties together, offered opportunities for students to develop utilization schemes. In the instrumentalization process, they discovered the programming environment and explored the basic computational concepts. The instrumentation process includes developing schemes for knowing when to perform an action, plan the action, and understand what the action does. During the lab, some students had difficulties reflecting on their work, but were pressed to do so in order to explain their code to each other or to one of the teachers.

Many students struggled with the mathematical content of the exercises during the lab. With a clearer guidance from the teachers, it may have been easier for the students to understand how the constructed program and the mathematical content were connected (Misfeldt & Ejlsing-Duun, 2015). Moreover, many of the students did not have enough programming experience to be able to solve the exercises without support from other students or from the teachers. It can be difficult to construct exercises where programming is used to explore mathematics, or to solve mathematical problems, in such a way that neither the instrumental genesis, the programming nor the mathematics constitutes an overwhelming challenge for the students. However, this study shows that it is possible to attend to instrumental genesis and to mathematical learning in parallel.

A majority of the students in this study, even though they were only in the beginning of their instrumental genesis, recognized the potential of programming in mathematics teaching. Nevertheless, these future mathematics teachers face a complex challenge. They will be expected to establish conditions for pupils to acquire fundamental programming knowledge and to develop schemes related to the mathematical content at the same time. Here, the teacher education program has an additional responsibility: besides teaching programming and mathematics, we must also give student teachers opportunities to develop their competence in teaching programming in mathematics so that they, in turn, will be able to orchestrate their pupils' instrumental genesis.

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# Exploring the use of Fermi problems and the FPAT-framework with pre-service primary teachers to bring real-life contexts into classrooms

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In this paper we analyse 11 pairs of pre-service primary teachers' solutions to two Fermi problems. The solutions were expressed using the descriptive and analytic FPAT framework (Fermi problem activity template), and our analysis focuses on characterizing the FPAT representations produced by the pre-service teachers. The results show that almost half of the produced FPATs in principle solved the problems but that only three solutions provided enough detail to be practically implementable. Multiple key constructs were used to structure the solutions in the first problem, but not so in the second problem. The variation of different activities and ways of working suggested in solving the sub-problems in the two problems also differ in a significant way.

Mathematical modelling, generically understood as using mathematics to solve real-world problems, is now a common curricula goal in many countries, to, among other things, show the potential of mathematics to describe complex phenomena and to promote critical thinking (Niss & Blum, 2020). However, modelling has not yet been established as a regular classroom activity (Borromeo Ferri, 2021). One reason for this is the openness of mathematical modelling activities, which puts new demands on the teacher to cope with and handling the students' diverse thinking and decisions about how to approach the problems being modelled. In this context specific training for teachers and available suitable resources are essential to enable teachers to implement mathematical modelling in the classroom (Garfunkel et al., 2021). One way to make modelling and mathematics relevant to students, is for teachers to connect their everyday mathematics teaching to interesting real-life contexts as well as current affairs and topics in the news in ways that normally transcends what is offered in textbooks and traditional curricula materials. Peter-Koop (2009) has showed how modelling activities in terms of so-called Fermi problems can facilitate bringing in everyday situations and contexts into the primary school mathematics classroom in meaningful ways. However, to first identify instances in out-of-school-contexts where mathematics comes to the fore or plays

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an important role, and then to make the connections to school mathematics, is non-trivial and requires solid and broad mathematical knowledge (Geiger et al., 2021). From a teacher education perspective, the challenge is how to prepare pre-service teachers for this task, in particularly in the case of the training of primary teachers, since normally only a limited number of mathematics courses are part of their teacher training programmes.

Building on Peter-Koop's (2009) work, and to support pre-service primary teachers (PSPTs) in developing ways of connecting and bringing real-life contexts into their mathematics classroom in a productive way, we have designed a teaching sequence where PSPTs are confronted with a set of *Fermi problems* which they first solve, and then analyse using a special framework called *Fermi problem activity templates* (FPATs). Our aim in this paper is to develop a characterization of the FPATs generated by the PSPTs in our first implementation of the teaching sequence to gain a basic understanding of the structure and content that came to the fore in the PSPTs' FPATs. The research question we address in this paper is: What characterize the FPATs produced by the PSPTs in terms of (a) to what extent their proposed solutions reasonably solve the problems under consideration; (b) the key concepts and procedures that structure their solutions; and (c) the mathematical activities proposed in their solutions?

## Fermi problems and mathematical modelling

The physicist Enrico Fermi introduced, used and popularized so-called Fermi problems (FPs) as time-saving and effective tools to illustrate the power of deductive thinking as well as making preparations before engaging in experimental work. FPs, which at first glance might seem impossible to solve, are often sparsely worded and provide little or none of the explicit information needed to solve the problem (Efthimiou & Llewellyn, 2007). FPs have been used in problem-solving activities but have gained the attention of the mathematics education community as activities in which mathematical modelling is worked on, as they generally are firmly rooted in a real-world context (cf. Lesh & Zawojewski, 2007). FPs are foremost characterized by the way in which their solutions are achieved. This method is known in the literature as the Fermi (estimates) method and entails the decomposition of the problem into a number of stringed together sub-problems, which are solved using common sense assumptions, estimates and educated guesses, and simple calculations to solve the original problem (Carlson, 1997). Students can generate many different ways of solving a FP following this method, depending on the variables of the problem on which the solver focuses. In this sense, FPs allow addressing one of the needs for learning problem solving identified in recent years, that of generating and discussing different ways of solving a problem (Tjoe, 2019). Although educational research on FPs generally emphasises estimation, it has

been suggested that the activity of estimation can be replaced by other (classroom) activities to find the numerical information needed to solve the problem (Sriraman & Knott, 2009).

In addition, FPs have been portrayed as miniature-modelling problems capturing the essence of full modelling problems (Robinson 2008), having the advantage of being more well-defined and delimited real-world problems, and hence less complex and more manageable for both teachers and students. At the primary level, Peter-Koop (2009) found that students working with FPs successfully can generate their own models and use the Fermi method to estimate the number of cars in a motorway traffic jam. The students used a variety of strategies and developed new mathematical knowledge to arrive at their solutions. These results are consistent with those of Albarracín and Gorgorió (2019), who used a number of different FPs and also showed how the students could adapt their solutions strategies to new contexts. In addition, working with FPs has been shown to have a positive effect on the development of primary students' modelling skills, making them aware of the phases of modelling and develop modelling sub-competences such as simplification, mathematization, interpretation and explanation of real phenomena (Haberzettl et al., 2018).

## FPATs – Fermi problems activity templates

In previous research we have discussed FPs both as integrators between the STEM disciplines and as facilitators for learning in the STEM disciplines (Årlebäck & Albarracín, 2019). Based on our review of the research on FPs in the STEM disciplines, we identified the four types of mathematical activities that most commonly are used in determining the unknown but needed numerical values of important quantities to solve a given FP: *Guesstimation*, *Experimentation*, *Looking for data* and *Polling or statistical data collection*. Below we briefly describe and provide examples of these four activities, and we discuss how these align with the problem-solving process of FPs.

*Guesstimation* is the activity of answering a (sub-)problem based on solely simple calculations involving educated guesses and estimates of the unknown quantities involved. The type of rough answer to a problem guesstimation results in, can be adequate and productive when working with ill-defined problems or when detailed solutions are not required (Shakerin, 2006).

*Experimentation*, with the original intent of Fermi to support the development of laboratory skills and experiment planning, is the activity to conduct an (physical) experiment and take measurements to determine adequate values for the relevant quantities needed to solve the given problem.

*Looking for data* is the activity to seek the relevant quantitative data needed using external records and sources, such as national statistical institutes resources, Wikipedia, or more topic specific resources as illustrated by

Phillips and Milo (2009) in the project B10NUMB3R5 (reliable and validated experimentally derived values of quantities relevant for research in biology, [bionumbers.hms.harvard.edu](http://bionumbers.hms.harvard.edu)). With respect to existing records and resources, FPs can be tools for critically evaluating such sources and data.

*Polling or statistical data collection* means to engage in data collection and statistical analysis to get the values of the relevant quantities needed to solve a FP. Besides providing the values, engaging in data collection and statistical analysis has the potential to increase awareness and provoke a critical stance toward various problems in society and the environment. Such examples, as suggested by Sriraman and Knott (2009), are for instance the wastage of food or the fresh-water consumption.

Solving a FP using the Fermi method entails identifying several sub-problems to address and solve. Then, it is the subsequent solutions and coordination of these sub-problems that results in the solution to the original FP. Anderson and Sherman (2010) put forward a simple geometrical diagram representing the structure of such a solving process for the FP *How many hotdogs are consumed at the Major league baseball (MLB) games each season in the US?* (see figure 1).

Inspired by Anderson and Sherman (2010), we expanded their representation by incorporating the four identified different types of activities and the structure of the solution process of a FP to a framework for design and analysis called Fermi problem activity templates, FPATs (Albarracín & Ärleback 2019).

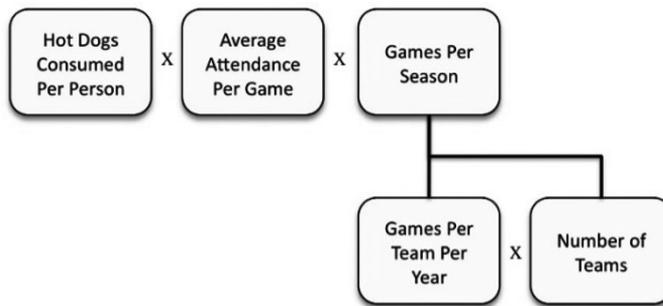


Figure 1. Structure provided by Anderson and Sherman (2010)

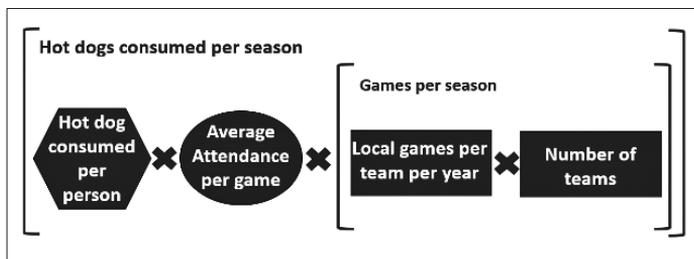


Figure 2. A FPAT based on the figure 1 structure

In a FPAT, the intended activity (if used as a design tool) or performed activity (if used as an analytical tool) to solve a given sub-problem is denoted by a specific geometrical shape: Guesstimation (Ellipse); Experimentation (Trapezoid); Looking for data (Rectangle); Polling or statistical data collection (Hexagon). One possible FPAT for the solution in figure 1a (by Anderson & Sherman, 2010), with specific activities for each sub-problem, is given in figure 2.

## Setting, the teaching sequence and methods

The data we analysed in this paper comes from a collaborative research project involving a Catalan and a Swedish setting aiming at designing, implementing, and evaluating teaching sequences that introduce PSPTs to FPs as well as FPATs as didactical tools. The teaching sequences differed somewhat in their design in the two settings due to different circumstances and boundary conditions, and in this paper we only report on the Catalan setting and data. The Catalan teaching sequence consisted of three 3-hour working sessions mixing brief lectures/instructions, individual problem solving as well as problem solving in pairs/groups, and whole class discussions. During these sessions 22 PSPTs worked with a series of four FPs and the following two problems:

- A How many toilet paper rolls are needed at school in a year? How much space do they occupy?
- B How many ambulances are needed if we want to attend any emergency in less than 8 minutes in any place in Catalonia?

Problem A is set in a known and tangible situation for the PSPTs. In contrast, problem B presents a complex situation about which the PSPTs have less experience. In a similar problem to problem B (optimally positioning rescue helicopters in a mountain area), Kaiser (2005) identified different approaches to the problem, based on the specific definitions of "optimal" and solving strategies. The PSPTs worked on these tasks in parallel in five stages: (i) individually wrote and explained their proposed solution plans for the two problems; (ii) discussed their proposals in pairs and agreed on joint solution strategies; (iii) in pairs, identified and connected the curricular mathematical content and procedures needed to solve each of the problems; (iv) after receiving a 20 minute introduction to the FPAT framework, in pairs, created and wrote down their FPATs representing their solutions of the problems; and (v) reflected on their work so far and revisited stages (ii) and (iii). The PSPTs studied their fourth and last year of the primary teacher training degree and had previously studied 3 courses of mathematics and mathematics education, but not modelling or real-life problem solving. We collected various documents generated by the PSPTs during the teaching sequence, but in this paper, we only focus on the 22 FPATs developed for problems A and B in stage (iv).

## Analysis

The analysis is based on the mathematical structure of the PSPTs' solutions and the quantities (Thompson, 1994) they chose to mathematize and include in their solutions. To qualitatively characterize the FPATs the PSPTs produced our analysis focused on (a) to what extent the solutions mediated by the FPATs reasonably solve the problem in question; (b) what the key concepts and procedures the PSPTs used to structure their solutions around were; and (c) the number of sub-problems and what types of FPAT-activities the PSPTs proposed to use in solving these.

To characterize to what extent the produced FPATs in terms of the suggested division into sub-problems relate to the posed problems in a way that potentially lead to a viable solution, we looked at both logical consistency and accuracy. These two aspects (captured under the rubrics "Solve?" and "Close?" respectively in tables 1 and 2 in the result) are binary descriptors (yes/no) of the PSPTs' FPATs. With respect to logical consistency, we focused on whether the sub-problem structure provided actually lead to valid solution to the problem or not. For instance, the PSPTs might have misunderstood the task or context rendering it impossible for them to reach a reasonable answer, or that key and crucial variables are not considered in the solution. With respect to accuracy, we looked if the FPATs potentially could result in a sufficiently good rough estimate or not, since although all important aspects and variables might be incorporated in the solution, there in addition needs to be enough detail specified at an adequate level about how to actually go about in producing the answer (FPATs with activities considered to be specific enough in this respect are indicated by an asterisk (\*) in the # Sub-problems column in tables 1 and 2).

It seems natural to assume that the number of quantities and sub-problems involved in solving a FP is a key element in determining its complexity (Greefrath & Frenken, 2021). Hence, to characterize the complexity of the FPATs, we identified and counted the number of sub-problems and the number of quantities (cf. Thompson, 1994) considered in the proposed solutions. By identifying which quantities the PSPTs focused on, we particularly noted what key concepts they used to structure and organize their solution and FPATs around. In addition, we also considered the nature of the activities proposed for each sub-problem and determined whether these really were suitable and viable for solving the respective sub-problem. There are instances in the data where PSPTs proposed activities that cannot be carried out in a realistic or meaningful way to provide the solution to the sub-problem they are supposed to solve. For example, the suggestion to search for the exact information about the distance an ambulance can travel in 8 minutes is not viable, but rather needs to be broken down into further sub-problems or found based on conducting an experiment.

Figure 3 shows an example of a PSPT produced FPAT for problem B. Here, the whole area of Catalonia (looked up from some data source) is to be divided



Figure 3. Pair 2's FPAT for problem B

by the area that can be covered by an ambulance (without making clear what type of activity to use in arriving at this quantity: experimentation or guesstimation), and to multiply by the number of ambulances per area (a guesstimation). In our analysis we have characterized this FPAT as solving the problem, and that it considers the essential aspects that allow to obtain a reasonable rough estimate. The key construct used to structure the solution is the area covered by an ambulance in 8 minutes, and the solution involves three sub-problems and two main quantities (area and number of ambulances). Note however, that we consider that it is possible to refine this solution by introducing more sub-problems and to be clearer about the type of activities proposed to achieve more reliable results.

## Results

The results, summarized in tables 1 and 2, show that the PSPTs produced FPATs that in principle adequately solved the problems in eight of the cases for problem A and in one case for problem B. Two of the eight FPATs for problem A were explicit and precise enough to provide unambiguous solutions, whereas this was not the case for the one FPAT for problem B. The number of quantities considered in the PSPTs' solutions are greater in problem A (average = 3.18) compared to in problem B (average = 2.09). Turning to the identified key construct used by the PSPTs, a great variation can be seen in what was central to their solution structure in problem A, whereas in problem B, all FPATs revolved around the single construct of how far an ambulance reaches in 8 minutes.

Focusing on the number of sub-problems, we see that although the number of sub-problems the PSPTs used in their FPATs on average are equal on problem A and B (4.45 and 4.36), and that in both problems the number of sub-problems is between two and eight, the distribution of sub-problems is quite different. The distribution is more centred around the mean for problem A (Std Dev 1.63) compared for problem B (Std Dev 2.49).

If we look at the proposed types of activities in the FPATs, the pattern is similar for problem A and problem B in the sense that Looking for data is the most frequently used, followed by Guesstimation, then Experimentation and lastly (and not at all in the case of problem B), Polling (which is suggested by

Table 1. *Characteristics of the FPATs the PSPTs produced – problem A*

Pair	Solves?/ Close?	# Sub- problems	# Quant	Guess	Expr	Stat	Look f. data	Key construct
1	Yes/No	3	3	1	2	0	0	Consumption per toilet
2	Yes/No	5*	4	3	0	0	2	Times we go to toilet
3	No/No	8*	4	2	3	0	3	Length of paper used
4	Yes/No	4*	3	1	1	0	2	Rolls used per week
5	Yes/Yes	5*	4	3	0	0	2	Length of paper used
6	Yes/No	4	3	0	0	0	4	Rolls per week
7	No/No	5*	2	1	1	2	1	Time roll lasts per class
8	No/No	4	3	1	1	1	1	Time roll lasts per class
9	Yes/No	3*	3	1	1	0	1	Rolls per toilet
10	Yes/Yes	6*	4	2	1	0	3	Length of paper used
11	Yes/No	2*	2	1	0	0	1	Rolls per week
Mean	–	4.45	3.18	1.45	0.91	0.27	1.81	–

Table 2. *Characteristics of the FPATs the PSPTs produced – problem B*

Pair	Solves?/ Close?	# Sub- problems	# Quant	Guess	Expr	Stat	Look f. data	Key construct
1	No/No	4*	2	2	0	0	2	Ambulance reach (AR)
2	Yes/No	3	2	1	1	0	1	AR
3	No/No	7*	3	3	0	0	4	AR
4	No/No	3*	2	1	0	0	2	AR
5	No/No	8*	4	4	0	0	4	AR + density
6	No/No	4	3	1	0	0	3	AR
7	No/No	6*	2	4	0	0	2	AR
8	No/No	7	2	0	4	0	3	AR
9	No/No	2*	1	1	0	0	1	AR
10	No/No	2*	1	0	1	0	1	AR
11	No/No	2*	1	0	0	0	2	AR
Mean	–	4.36	2.09	1.54	0.54	0	2.27	–

two pairs in problem A). However, it is notable that the relative proportions of the suggested activities within the two problems are quite different: in problem A these are much closer to one another compared to the corresponding proportions in problem B. When it comes to the activities the PSPTs proposed for each sub-problem and if these could be considered suitable for solving these, we found that this was the case in 16 of the FPATs (see \*-markings in table 1 and 2 respectively). When we look at the FPATs containing activities not corresponding to the sub-problem in an adequate way, four of these come from pairs 6 and 8.

## Discussion and conclusions

Although we in this paper report an exploratory study involving only a small sample of work from 11 groups, the analysis shows the complexities inherent in solving FPs set in real and everyday contexts. The characterizations of the FPATs of the two problems A and B show the importance of choosing the FPs the PSPTs work on in the teaching sequence with great care – especially regarding the context of the FPs. Multiple factors revealed by the analysis indicate that many of the PSPTs' difficulties in solving problem B are related to their lack of previous experience in the real world relevant to the problem. In contrast, the context of problem A connects to a more tangible situation to which they easily can relate. The disparity in contexts is reflected in the number of successfully proposed FPATs as well as the proportions of the different types of activities suggested in solving the sub-problems. This result in fact illustrates that the FPATs the PSPTs produced can help to identify the nature of the difficulties they have in solving a problem; since the variety in key constructs seems to be an indicator of the previous knowledge they have about the context of the problem. In addition, looking at the PSPTs' produced FPAT using the characterization, can also be used as a tool that allows identifying the nature and level of knowledge that the PSPTs bring to a given problem. In other words, we suggest that this characterization can be useful in teacher training to (a) design teaching sequences with FPs; and (b) function as a didactical tool for PSPTs and teachers in connecting and bringing out-of-school contexts into the classroom, as well as (c) using the FPATs to make different ways of solving a problem explicit (Tjoe, 2019).

Regarding the activities suggested by the PSPTs to solve the sub-problems, we acknowledge that working with and applying the FPAT-analysis is a new way of thinking about this type of problem-solving for the PSPTs, and that they need more experience working with both FPs and FPATs. The fact that three of the pairs only proposed Guesstimation and Looking for data as activities solving their suggested sub-problems, make us think that it is necessary for the PSPTs to have more exposure and training in the design of experiments and data collection approaches. A greater variety of activity types open for more opportunities to connect to out-of-school contexts. It should be stressed that by the nature of a FP all sub-problems can be solved by Guesstimation, but whether this is a viable approach or not strongly depend on whether the solvers have the necessary extra-mathematical knowledge. Since we in this study unfortunately do not have access to this type of data, we cannot further analyze this aspect. However, this is an interesting venue for future research. We also notice that in future studies it would be interesting to ask PSPTs for even more information about the specific ways in which they intend to carry out their suggested experimental activities or data collection activities.

Even if the PSPTs were presented with FPs using real contexts, the PSPTs struggled to solve the problems, meaning that just situating the problems in everyday contexts were not enough to develop proper solutions. Although FPs are small-format modelling activities, they still require solvers to make a connection between the real context and the mathematical content. In this sense, FPATs make this connection explicit revealing themselves as promising tools, but it is still necessary for PSPTs to develop the skillset needed to identify mathematizable aspects of a real phenomenon. However, the results presented in this paper suggest to us that FPATs are promising tools for working on problems with real contexts with future teachers also at the primary level. FPATs reveal the structure of the model developed and the mathematical procedures to be implemented, establishing a specification that allows learning opportunities for PSPTs. More research is needed about how preservice teachers understand and use FPATs, as well as how teaching sequences need to include, exemplify, and connect to all four different activities in the FPAT, to further facilitate bringing out-of-school context into the classroom. However, we understand that a proposal based on FPATs has the potential to support teaching interventions with PSPTs about mathematical modelling.

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# Problem-solving in Swedish curricula in a time of change

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It is long known that students' learning in mathematics is facilitated by problem-solving activities. Therefore school authorities all over the world have incorporated problem-solving in their curricula. However, problem-solving does not have a clear definition, and its meaning risks being watered down in the process of implementation. In this study, we examine how problem-solving is described and used in Swedish syllabi, commentary materials and national tests for school year 6–10, before and after the 2021 and 2022 revisions. Our results show that "problem-solving" is increasingly conceptualised as a goal rather than a means for learning and that, as a goal, problem-solving competency is reduced. As a guidance for teachers the policy documents are often vague and even contradictory. Implications for teaching practice and Swedish students are discussed.

Mathematics education research agrees that teaching through problem-solving has benefits (Munter & Correnti, 2017). Teaching mathematics is often regulated by governing texts (for a longer discussion, see e.g. Boesen et al., 2014). Sweden has foregrounded problem-solving in mathematics in curricula since 1994 (Utbildningsdepartementet, 1994). However, the term "problem" is given different meanings by different people – not all associated with learning benefits – and the meaning and role of problem-solving can be diluted during curricular reforms (Gravemeijer et al., 2016). Lately, Swedish mathematics syllabi have been revised every year, with the most recent changes being implemented in 2021 for upper secondary school (years 10–12) and in 2022 for compulsory school (years 1–9). In this study, we examine the meanings of problem-solving expressed in Swedish school curricula, by analysing available syllabi, commentary materials and national tests before and after the latest revisions.

## Problem-solving in mathematics education research

In this study, we define "problems" as tasks where the solution method is not known in advance, in contrast to routine tasks that only requires the use of a

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known procedure (NCTM, 2000; Schoenfeld, 1985). In this sense of the word, solving problems is a far more complex process than solving routine tasks. This complexity is reflected in the various models for problem-solving processes developed by researchers, describing different phases, such as interpretation, exploration and suggestion of potentially fruitful approaches (Rott et al., 2021). The importance of evaluation, including describing and verifying the solution, and identifying alternative approaches, is also often highlighted as key for learning (Koichu et al., 2021).

Some researchers focus on prerequisites of problem-solving or foreground problem-solving as a goal of mathematics education (e.g., Schoenfeld, 1985). While it is acknowledged that problem-solving could be seen as a competency and assessed as a learning outcome, it is still an open question how this should be done (Niss & Højgaard, 2019). In fact, mathematics tests mostly consist of short, structured items that do not assess problem-solving competency (Jones & Inglis, 2015). Mathematics education research more commonly frames problem-solving as a learning activity and it has been shown that problem-solving give richer opportunities to develop mathematical competence than applying given methods on routine tasks (Boaler, 2014; Downton & Sullivan, 2017; Jonsson et al., 2014; Lester & Cai, 2016). In sum, the research field supports the use of problem-solving as a means for teaching and learning, while effects of assessing problem-solving competency is seldom studied or advocated.

## Problem-solving in Swedish curricular reforms

Problem-solving was introduced in Swedish curricula at the same time as the school management model changed from a rule-based to a goal-based system (SOU 1992:94; SOU 2007:28; Utbildningsdepartementet, 1994). The intent of this reform was to refrain from directly regulating teachers' practice and instead govern schools through testing students' performance in relation to given goals. This was initially done by two means: the curriculum, including learning goals within the syllabi for each subject, and the national tests. The first syllabi specified "goals to strive for" that were meant to form a basis for the planning and design of teaching, and "goals to attain" that were meant to be assessed (Skolverket, 2004), with problem-solving competency included in the former. However, evaluations showed that the goals were hard to interpret for teachers and that the teaching was based on the goals to attain rather than the goals to strive for (Boesen et al., 2014; Skolverket, 2004; SOU 2007:28). This led to calls for commentary materials further explaining the goals and teachers turning to the national tests for help in interpreting the syllabi (SOU 2007:28). In 2011, the two sets of goals were reformulated as mandatory content and knowledge requirements in terms of competencies, of which problem-solving is one. The goal-oriented system remains, and the guidance for teachers now consists of the

syllabus, the commentary materials and national tests for grade 3, 6, 9 and each course in upper secondary school. In contrast to research, the Swedish system thus foregrounds problem-solving as a competency to be assessed, while the choice and development of means for teaching and learning is left to teachers.

## Aim and research question

The Swedish curriculum and the national tests play a key role in shaping teachers' view of teaching and learning in mathematics in general, and problems and problem-solving in particular. This view may in turn impact the teaching offered to students. Therefore, we aim to investigate the concept "problem-solving" in Swedish national policy documents, by focusing the following research question: How is "problem-solving" described and used in syllabi, commentary materials and national tests for school year 6–10 before and after the 2021 and 2022 revisions?

## Methods

To answer our question, we conducted a concept analysis according to Nuopponen's (2010) four steps: compilation of the material, elaboration of a preliminary concept system, systematic analysis of each source separately, and synthesis and comparison of the results for each source. The analysis was both descriptive: describing the state and use of concepts before and after the revisions respectively, and contrastive: exploring and clarifying similarities and differences between the previous and revised policy documents (Nuopponen, 2010). The four steps were conducted as follows below.

### Compilation of the material

As we aim to study the concept problem-solving in national policy documents, our analysis was source-restricted (Nuopponen, 2010) to the three policy documents governing Swedish mathematics instruction: the mathematics syllabi, the commentary materials for those syllabi, and the national tests. We further restricted our analysis to school year 6–10. We excluded year 11–12 since a large part of students do not study mathematics in those years, and year 3 as the national test for year 3 does not specify what tasks assess problem-solving. For year 10 we chose the syllabus and test for the intermediate track course Mathematics 1b. There are no national tests available based on the new syllabi, and such tests could therefore not be included in the data collection.

Our data thus consisted of the mathematics syllabi for compulsory and upper secondary school from before and after the revision (Skolverket, 2019; 2021a; n.d.a), the commentary materials for these syllabi (Skolverket, 2017; 2021b; 2021c; n.d.b), and the most recent publicly available national mathematics tests

and scoring instructions for school years 6, 9 and the course Mathematics 1b. Each syllabus consists of a short introduction and an aim, followed by a list of mandatory content and knowledge requirements specified for grade levels E, C and A. From the tests only the tasks assessing problem-solving competency according to the scoring instructions were included, a total of 55 tasks.

### Elaboration of a preliminary concept system

An early step in concept analysis is to form preliminary concept definitions and relations to use as a framework for the analysis. Since the commentary material and the national tests are intended to interpret the syllabi, we used the syllabi to form our preliminary concept system.

As stated, problems are tasks where the solution method is not known in advance. Problem-solving is the activity students engage in when trying to solve problems and a means for learning. Problem-solving competency is the set of abilities needed to engage in problem-solving and a goal for learning. Three aspects of problem-solving competency are: solving problems, describing and evaluating solutions, and suggesting alternative approaches. These three aspects and their intended progression over years as well as different levels of them are further explicated in the left side of figure 1.

### Systematic analysis of each source separately

As the national tests is a different kind of source than the syllabi and commentary materials, we devised separate methods of analysis for these two types of sources.

#### *Analysis of syllabi and commentary materials*

First, all instances of the term "problem", including compounds, were marked. In the commentary material, mere quotes from the syllabi were excluded. Next, each sentence was coded as regarding either problems, problem-solving or problem-solving competency. The syllabi sentences coded as problem-solving competency were the knowledge requirements and were not further analysed as they had already been used to create the preliminary concept system (left side of figure 1). The sentences in the commentary material concerning problem-solving competency were coded as related to either progression or grade level, or both. Each progression and/or level sentence was then analysed in relation to the three aspects of problem-solving competency, identifying both further explanation of and contradictions to the knowledge requirements.

#### *Analysis of National tests*

Of the 55 tasks 25 were rewarded points on grade level E, 30 on grade level C, and 18 on grade level A. In order to understand the tasks' requirements, we first formulated the solution steps students most likely would take to get the point specified in the teachers' scoring instruction (Jäder et al., 2020). Then, we used

the preliminary concept system to analyse the tasks, scoring instructions and plausible solutions, considering: 1) Whether the task was a problem, by identifying what (if anything) the student needed to do besides methods included in the mandatory content. 2) What aspects of problem-solving competency the task assessed, by identifying: a) the context of the task. b) whether the student had to solve the whole task to receive the point, and c) whether the student had to describe or evaluate any part of the solution was considered, and d) whether the student had to suggest alternative approaches. The analysis first considered each task, then summarised the result for the set of tasks giving points at each grade level and school year, and finally identified similarities and differences between grade levels and school years.

### Synthesis and comparison of results regarding each source

In the final step of the analysis, the results from the separate analyses of the syllabi, the commentary materials and the tests were synthesised and compared. This was done by identifying further explanations, additional examples and contradictions between different sources concerning the same syllabus and what had changed or stayed the same in the revisions.

## Results

For all grades and both before and after the revisions, the syllabi describe problem-solving as both a means and goal for learning. However, the commentary materials have shifted in focus. Before the revisions, more sentences focused on problem-solving as a means (48 for year 6–9, 39 for year 10) than on problem-solving as a goal (17, 0). After the revisions the focus has changed (means: 31, 16; goal: 34, 22). While formulations regarding problem-solving in the aim and mandatory content of the syllabi are largely the same before and after the revisions, the knowledge requirements have changed remarkably. Below, we present in more detail how the knowledge requirements are stated in the syllabi (figure 1) and interpreted by the commentary materials and national tests first before and then after the revisions, by attending to each of the main aspects of problem-solving competency – “solving problems”, “describing and evaluating”, and “alternative approaches”.

### Problem-solving competency before the revisions

The knowledge requirements specifying the progression and different levels of problem-solving competency grade 6–10 are shown on the left side of figure 1. For year 6–9, the commentary material gives some further explanations of the progression, as well as the differences between grades. However, for year 10, no sentences concerning progression or grades were found, and hence no references to the commentary material for year 10 are made below.

	BEFORE THE REVISIONS			AFTER THE REVISIONS		
	School year 6	School year 9	School year 10	School year 6	School year 9	School year 10
SOLVING PROBLEMS	The student can solve simple problems in situations close to the student	The student can solve different problems in familiar situations	The student can formulate, analyse and solve mathematical problems	The student solves		
	E in a mainly functioning way		E of simple character	E simple problems		
	C in a relatively well-functioning way		C	C relatively complex problems		
	A in a well-functioning way		A of complex character	A complex problems		
	by selecting and using strategies and methods with		These problems include			
DESCRIBING AND EVALUATING	E some adaptation to the character of the problem.	E a few concepts and requires simple interpretations.	E	The student evaluates the plausibility of the results and		
	C relatively good adaptation to the character of the problem.	C several concepts and requires	C			
	A good adaptation to the character of the problem.	A advanced interpretations	A			
	The student describes approaches					
	E in a mainly functioning way and reasons	The student reasons	The student can, with			
ALT. APPR.	E in a simple and somewhat substantiated way	E simple judgements	E	evaluates strategies and the plausibility of the results and assesses the plausibility of the results.		
	C in a developed and relatively well-substantiated way	C simple judgements	C			
	A in a well-developed and well-substantiated way	A nuanced judgements	A			
	E in a mainly functioning way and reasons	about approaches and	evaluate the plausibility of the results as well as selected models, strategies and methods			
	E contribute to some suggestion for an alternative approach.	E	E contributes to some suggestion			
ALT. APPR.	C give some suggestion for an alternative approach.	C	C	and		
	A give suggestions for alternative approaches.	A	A			
	E in a simple way		E in a simple way			
	C in a developed way		C in a developed way			
	A in a well-developed way		A in a well-developed way			
ALT. APPR.	E contribute to some suggestion for an alternative approach.	E	E	and		
	C give some suggestion for an alternative approach.	C	C			
	A give suggestions for alternative approaches.	A	A			
	E in a simple way		E in a simple way			
	C in a developed way		C in a developed way			
ALT. APPR.	E contribute to some suggestion for an alternative approach.	E	E	and		
	C give some suggestion for an alternative approach.	C	C			
	A give suggestions for alternative approaches.	A	A			
	E in a simple way		E in a simple way			
	C in a developed way		C in a developed way			

Figure 1. An overview of the syllabi knowledge requirements in school year 6, 9 and 10 before and after the revisions (authors' translation)

### Solving problems

From year 6 to 10, the main progression lies within the character of the problems students solve. In year 6, students solve simple problems in situations “close to the student”. The commentary material describes such situations as familiar to the student and as everyday situations; it thus seems like the problems in year 6 should be connected to situations that students have personal experience of. The analysis shows that half of the tasks in the national test have an everyday context, but it is unlikely that the students have had a personal experience of many of the task contexts. The syllabus states that students in year 9 should solve different problems in familiar situations. The commentary material describes this as a broadening of problem-solving, in the sense that students can encounter more complex everyday situations that the students have not necessarily experienced personally, for example making a budget. However, half of the tasks in the test for year 9 are purely mathematical. In year 10, students solve mathematical problems, which implies that problems do not have to have a real-world context. Still, a majority of the problems in the year 10 test have an everyday context. In fact, the proportion of tasks that are purely mathematical, i.e., lacked an everyday context, slightly decrease over school years, contradicting the commentary materials’ interpretation of progression.

The grade levels are characterised in different ways in compulsory and upper secondary school. In year 10, the difference between grades lies in students being able to solve problems of different complexity. This implies that students at different grade levels solve different problems, which is also reflected in the test. In year 6 and 9, the difference lies in the quality of students’ solutions, implying that students at all grade levels are to work on the same problems but

solve them with various success and sophistication. The commentary material and the year 6 and 9 tests give a different view, more in line with the year 10 requirements. The commentary material states that:

A student that have gotten far in their knowledge development can view a task as a routine task if he or she knows a solution method. Another student encountering the same task can, however, need to explore and try different approaches to reach a solution. (Skolverket, 2017, p. 7, authors' translation)

The quote reflects the idea that tasks can be problems for students at lower grade levels but not for students at higher grade levels, also in year 6 and 9. This view is supported by the year 6 and 9 tests, in that they mostly reward points for different grades for different tasks. While about a third of the tasks in year 6 and half of the tasks in year 9 are rewarded problem-solving points for a partial solution, the complete solution will in those cases most often not give higher level problem-solving points. Only in 16 tasks do students have the opportunity to show problem-solving competency at two or three levels in the same task. In fact, an increase in complexity of tasks from E to A level can be identified in the tests for all school years. At C- and A-level, students need to, to a greater degree, interpret, and structure the information in the task and make calculations so that the right conclusions can be drawn to find a solution.

The analysis also showed that students can, not seldom, receive E-level problem-solving points by presenting a full solution that merely applies methods included in the mandatory content. This signals that students at E-level are not expected to learn standard methods to a level where they can recognise when they are applicable and use them confidently.

### *Describing and evaluating*

Only a few sentences in the year 6 and 9 commentary material concerns the progression or level of describing and evaluating. One sentence states that the “depth of students’ reasoning regarding approaches and plausibility of results” should increase over school years and grades, and another that students should develop an awareness of the efficiency of their approaches with respect to the problem, through being “given knowledge of evaluation of choices”, but this knowledge is not further described. Of the 55 tasks analysed, 36 prompted students to write down her solution, but no examples were found that explicitly asked for evaluation of solutions. The apparent idea of a progression starting in description and successively adding more evaluation and reasoning regarding one’s approaches is not further elaborated in neither commentary material nor national tests.

### *Alternative approaches*

Alternative approaches are only mentioned once in the commentary material for year 6 and 9, and then as a mere statement that higher grades require

higher degrees of contribution from the student. No tasks on the national tests explicitly required students to formulate alternative approaches.

### Problem-solving competency after the revisions

In the revisions, the knowledge requirements have been substantially shortened. The progression over school years and the differences between grades is now easier to overview, while the three aspects are still represented (right side of figure 1). A larger proportion of the sentences about problems and problem-solving in the commentary materials now concern progression and differences between grades for both compulsory and upper secondary school.

### *Solving problems*

The requirements for different grades in year 10 has now replaced the requirements for all school years. This means that the difference between grades in year 6 and 9 is no longer the quality of students' solutions, but the complexity of the problems students are able to solve. This also means that there is no progression over years evident in the knowledge requirements. However, the commentary material for year 6 and 9 relates increased complexity to both progression and grades. For year 10, the increased complexity required for higher grades is exemplified in seven aspects. For year 6 and 9, three of those aspects are used to specify difference in grade level (familiarity, level of abstraction, combining content), while three of the aspects specify progression (more advanced interpretations, more advanced calculations, more advanced concepts) and one exemplifies both (less guidance).

### *Describing and evaluating*

There is only one sentence specifying the knowledge requirements related to "evaluation" in each of the commentary materials. For year 6 and 9, evaluation still concerns both results and strategies, but the commentary material only specifies evaluation of results: "the student needs to relate conclusions and results to the original mathematical questions". For year 10, evaluation is now restricted to results and is described as "the student as a rule, identifies absurdities, even if there can be exceptions".

### *Alternative approaches*

Being able to assess "alternative approaches" has been removed as a criteria from year 10, but contributions to alternative approaches is still required in year 6 and 9. Only one sentence in the commentary material comments on this but does not further explicate or exemplify different levels of contribution: "the student can see the problem and the methods and strategies used from different perspectives".

## Conclusion and discussion

Our results show that "problem-solving" is increasingly conceptualised as a goal rather than a means for learning in Swedish policy documents, and that as a goal, problem-solving competency is reduced in complexity. The commentary material only provides vague guidance for teachers, due to the documents' absence of rich descriptions of how to handle the different aspect of problem-solving. In addition, descriptions and examples given in the syllabi, commentary material and the national tests are often contradictory.

Before the revisions, the descriptions of problem-solving competency were rather complex, but reflected important aspects of problem-solving as a process and as a learning activity. They gave value to less efficient approaches, contributing to collective solution processes and reasoning about strategies and results. In this respect, they reflected what we know about problem-solving phases (Rott et al., 2021; Schoenfeld, 1985) and the practices that enhance mathematics learning (Munter & Correnti, 2017). However, neither the commentary materials nor the national tests fully reflected these aspects. Specifically, they gave very limited guidance regarding two of the three aspects: "describing and evaluating", and "alternative approaches", even though they are known to be both important and hard to engage students in (Koichu et al., 2021). Rather than providing the called for support for teachers' understanding of the complexity of problem-solving (SOU 2007:28), the revised curricula has simplified and narrowed the description of what problem-solving entails.

A significant change in the syllabus for compulsory school is that grades are now separated by the level of complexity of problems students solve. The previous syllabus implied that students at all grade levels should work with the same problems but might contribute at different levels of proficiency. Such an approach facilitates other activities that are beneficial for learning, such as communicating, comparing, contrasting and critiquing one's own and others' ideas (Munter & Correnti, 2017). If students instead solve different problems, such activities can become much more difficult to orchestrate. Furthermore, our results show that before the revisions, E-level tasks seldom require more than application of methods included in the mandatory content, which – if problems are to be tasks for which the solution method is unknown – implies either that students at E-level are not expected to know the mandatory methods taught or that problem-solving is not required at E-level at all. The new requirements risk increasing the difference between learning opportunities given to different students and as a result hamper students' development of mathematical competence, counteracting the aim of the subject as stated in the curricula (Skolverket, 2021a; n.d.a) and contradicting research showing that students at all ages and levels benefit from problem-solving activities (Boaler, 2014; Downton & Sullivan, 2017; Jonsson et al., 2014; Ridlon, 2009).

Another significant change is that the knowledge requirements no longer reflect any progression with respect to the types of problems students should be able to solve in different school years, aside from covering different mandatory content. Since it is long known that problem-solving competency goes beyond application of given methods (Schoenfeld, 1985), it is unclear whether this can be seen as a progression in problem-solving – especially for students who stay at E-level over years. In this sense, the policy documents provide questionable guidance for teachers' development of students' problem-solving competency.

In sum, this study raises doubts regarding the revised curricula's potential for supporting teachers in creating rich opportunities for learning mathematics, especially for struggling students. Based on the results of the study we suggest an enrichment of the syllabi concerning problem-solving, so that teachers can get a deeper understanding of what is to be taught and how, and thus support richer learning in mathematics.

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# Developing and testing a framework for analysing, comparing and visualizing content matter in and across mathematics textbooks

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In this methodological paper we discuss the development and testing of a framework for analysing and visualizing the overall structure of the content and tasks in mathematics textbooks. The aim of the framework is to provide a lens on mathematics textbooks at the secondary level that is mathematics topic- and content-area-unbiased to facilitate an as objective as possible comparisons between both (a) different contents, topics, and tasks within a textbook; and (b) contents, topics, and tasks across different series of textbooks or within a textbook series. The paper explores the results of the framework applied to four mathematical textbook chapters visualized in terms of so-called hierarchy charts. Inter-rater coding reliability with respect to a developed coding manual, and future development and applications of the framework is also discussed.

Textbooks are important and fill multiple purposes in the teaching and learning of mathematics (Fan et al., 2013; Schubring & Fan, 2018). Indeed, the influences of textbooks on the mathematics classroom are well documented both internationally and in the Swedish context (Valverde et al., 2002; Bergqvist et al., 2010a,b). In particular, much of the mathematics the students meet and work on in school is mediated through the mathematical tasks in terms of problems, exercises, and activities in the textbooks (Henningsen & Stein, 1997). Hence, textbooks, their content and use, have been an area of research in mathematics education going back at least to 300 years (Fan et al., 2013).

Fan et al. (2013) made a systematic literature review of 100 journal papers that all focused on analysing textbooks, and identified three main research foci roughly equally distributed: (1) *the role of textbooks* (non-empirical and more philosophically oriented papers about the role of the textbook in mathematics teaching and learning); (2) *textbook analysis and comparisons* (studies of textbook contents and topics as study-object in themselves as well as within and across different textbooks); and (3) *textbook use* (research on teachers' and/or students' use of textbooks). Fan et al. (2013) also used a fourth category, *other*, in their categorisation. Within the second research foci, textbook analysis

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and comparisons, which is the focus in this paper, Fan et al. (2013) further identified five sub-foci, namely research focusing on and analysing: (i) *mathematics contents and topics*; (ii) *cognition and pedagogy*; (iii) *gender, ethnicity, equity, culture and values*; (iv) *comparisons of different textbooks*; and (v) *conceptualization and methodological matters*.

From a Swedish perspective, and focusing on the secondary level, most research carried out to date is in line with foci (2) or (3). An example of research on textbook use (3) is Johansson (2006), who illustrated and discussed how three mathematics teachers use the textbook as an instrument in different ways to organize their mathematics lessons. However, research focusing on analysing and comparing various aspects of the actual textbook appears to be more common. There are multiple examples of Swedish studies that involve two or more of the sub-foci identified by Fan et al. (2013). Brändström (2005) for example investigated the cognitive demand of tasks in relation to differentiated tasks (ii) in the topic area of fractions (i) and compared textbooks from three different textbook series (iv). On the other hand, Jakobsson-Åhl (2006, 2008) studied how algebra (i) and algebra-oriented word problems (i) had been treated and changed in textbooks over a 40-year period (iv). As a last example, we have Brehmer et al. (2016) who looked at and characterized various aspects of problem-solving tasks (i) in three upper secondary mathematics textbook series (iv): type of reasoning required (ii), placement (i), level of difficulty (ii) and task context (i).

From the literature on research on textbook analysis one can note at least two things. First, regarding the analytical tools and developed framework in general, and in particular the one used in the three latter examples of Swedish research, many tools contain at least one analytical dimension which require some type of value judgement (level of cognitive demand; phenomenographic- and hermeneutic interpretations, and; reasoning type and difficult level respectively). Secondly, the results of the textbook analyses are often presented in tabulated forms listing proportions or/and percentages, cross-tables and statistical comparisons of how the textbook were coded. Not seldom is it hard to get an overview of the coding and results.

In this paper, we want to develop a flexible framework that provides a general structural overview of the textbook and a rough categorisation of the tasks in the textbook that will facilitate

- a comparisons between different contents, topics and tasks *within* a textbook;
- b comparisons between contents, topics and tasks *across* different series of textbooks or within a textbook series;
- c comparisons in such a way that only more or less objective aspects of the textbooks (i.e. not including categories or classifications open to subjective interpretation such as cognitive demand, level of complexity etc.); and

- d a direct, intuitive and accessible way to provide an overview and visualization of the result of applying the framework, that point out major trends and patterns in the data as well as suggest what aspects of the analysis that might benefit from a closer scrutiny.

## Motivation, aim, and rationale for the framework

As mentioned above, the aim of the framework presented in this paper is to provide a general description and overview of the structure of mathematics textbooks and the tasks found in them. In addition, we want this general description and overview to be possible to visualize in an intuitive and direct way. Our long-term motivation for developing the framework is to investigate to what extent different contents and content-specific tasks are presented and treated in similar ways. We are especially interested in characterizing how the content matters of *statistics* and *probability* are presented and treated compared to other mathematical contents at the secondary level. Hence, we are in particular interested in investigating the textbooks used for both the lower- (grades 7–9) and upper level (grades 10–12: "gymnasium"). However, in this paper we only discuss the framework itself and not the comparative study in which the framework is being used. In other word, using Fan et al. (2013) categorisation of textbook research foci, this paper qualifies in the category (v) *conceptualization and methodological matters*.

The methodological goal for the framework is three-fold: (1) to code *all content* of the pages of the textbook (i.e. *all area* of the pages should be coded); (2) to identify and code *all tasks* in the book; and (3) to explore so-called *Hierarchy charts* to see if this type of representation and visualization of the coding provides a good overview and entry point to the analysis based on the framework. Hence, the framework has two types of codes with two different foci: One set of *structural codes* that categorise what types of content the pages of the textbook contain; and one set of *task codes* that categorise the tasks in the textbook in three rudimentary types of tasks depending on how they are presented in the textbook. The codes emerged collaboratively in an iterative processes inspired by previous frameworks used in research on textbooks (e.g. Brändström, 2006; Fan et al., 2013; Glasnovic Gracin, 2018; Valverde et al., 2002) and by going through multiple cycles of testing, evaluating and revising the codes as the framework was applied to empirical material in terms of secondary mathematical textbooks.

## Codes to characterise the overall structure of the textbooks

The codes that characterise the overall structure of the textbooks are intended to cover the total area of each page and can broadly be divided into two main types. The first type of code aims at capturing the overall structure of the textbook in terms of *passive content*. With passive content we here refer to content that not per se ask the students to actively perform any calculations or other

mathematical work. There are eight such passive codes in the framework, such as for example *Central content/goal* (often a bullet list entailing the official leaning goals as specified in the national curricula documents) and *Instructional narrative* (a written presentation of the concepts, methods, and theory treaded in the *chapter*<sup>1</sup>).

The second type of code focuses on *active content* in the sense that they capture instances in the textbook where students are encouraged to actively engage in mathematical work. The framework has seven such active codes, such as for example *Exercises* (often coherent sections of the textbook with tasks where the students are offered opportunities to practice on the content and topic in the chapter) and *Activities* (suggested activities, either to be done in groups or individually, often involving some investigating, experimenting or other practical activity, aimed at motivating and getting the students interested in the content and topic at hand). For the definition of *task* adapted in the framework see below. Table 1 lists and briefly explains all the codes. Two codes, *History & social* and *Other*, contain both active and passive content.

### Codes to characterise the tasks in the textbooks

For the framework to have a mathematics topic- and content-area-influenced-free conceptualization of the notion of mathematical tasks, we define a *task* broadly to be a clearly designated self-contained set of instructions that asks the students to engage in some mathematical work and/or activity. This definition stands in contrast to many other conceptualisation of mathematical tasks used in research, such as the one used by Stein and Smith (1998) who define that a task is "[...] a segment of classroom activity that is devoted to the development of a particular mathematical idea. A task can involve several related problems or extended work, up to an entire class period, on a single complex problem" (p.269). The framework differentiates between three types of mathematical tasks (collectively abbreviated to *XIE*):

- (X) *eXercises*: Often briefly formulated tasks that normally only requires one or a few calculations and/or manipulations based on a formula or given method (c.f. Glasnovic Gracin, 2018).
- (I) *Intra-mathematical tasks*: Word problems that are completely formulated in a strictly mathematical context with no coupling to contexts or situations outside mathematics and the real word (c.f. Niss & Blum, 2020).
- (E) *Extra-mathematical tasks*: Word problems that are formulated in an extra-mathematical context that is more or less authentic, connected to the real world or set in an imaginary scenario (c.f. Niss & Blum, 2020).

The *XIE* codes work as sub-codes in the framework, and all tasks in six of the seven active structural codes, as well as the tasks in the *History & social* code,

were assigned to exactly one of the above XIE codes. The XIE codes were not applied to the sections of the textbook coded as *Activities*. Note that the *Diagnostic*, *Filling* and *Summary*, have additional (not XIE) sub-codes (see table 1).

Although the framework tries to minimize the use of strongly subjective codes to increase inter-rater (inter-coding) reliability, some connections and parallels are possible to discern with previous used research frameworks. For example, typically tasks that Jakobsson-Åhl (2006, 2008) considered to be word-problems are exclusively found coded as (I) or (E), as is the problem-solving tasks as described by Brehmer et al. (2016).

## Applying and piloting the framework

To apply and pilot the framework we used the computer software NVivo12 (QSR, 2018) to code digitalized versions (high-resolution colour pdfs) of four mathematics textbook chapters: two from the Swedish lower secondary mathematics textbook *Prio matematik 7* (Cederqvist et al., 2012); and one each from the Swedish upper secondary mathematics textbooks *Matematik5000 1a* (Alfredsson et al., 2011a) and *Matematik5000 1c* (Alfredsson et al., 2011b). We iteratively developed a coding manual containing: (a) elaborated definitions of all the codes and illustrating examples taken from the iterative process that resulted in the framework; and (b) explicit instructions and guidelines about how to mark and select the sections in the textbooks that are to be assigned structural codes (using a region-/area-selection tool in NVivo) as well as how to mark and select the tasks to be assigned task codes (using a text-selection tool in NVivo). The first tentative coding manual was put together inspired from (a) non-value-laden codes from other frameworks (c.f. Brändström, 2006; Glasnovic Gracin, 2018; Jakobsson-Åhl, 2006; 2008); (b) literature on different types of mathematical tasks (c.f. Niss & Blum, 2020); and (c) by inspecting and discussing the content and tasks in assorted mathematics textbooks. The coding manual was then successively developed and refined iterative by four coders applying the manual to code randomly selected pages in the textbooks until the process resulted in a saturated framework (see table 1).

## Inter-rater reliability

To investigate the robustness of the framework two coders independently coded four whole mathematics textbooks chapters: from *Prio Matematik 7* the chapters on *Statistics* (30 pages) and *Fractions and percent* (48 pages), and from *Matematik 5000 1a* the chapter on *Probability and statistics* (48 pages) and from *Matematik 5000 1c* the chapter *Graphs and functions* (44 pages). NVivo12 provides an automatic analytic comparison of two coders' coding with respect to both region-/area-coding using a pixel measure for the marked and selected section coded, as well as the coding of marked and selected text segments. In Nvivo, the agreement of the coders is measured using two methods:

Table 1. *The codes and sub-codes of the developed framework*

Category (Sub-categories)	Brief explanation/example
Central content/ goal	(Bullet) list specifying the curricula content covered in the chapter and/or the students' learning goals.
List of content	A list presenting the mathematical content in the chapter.
Introductory text	Brief introductory and motivating text introducing a chapter – not an Instructional narrative (see below)
Activities	Suggested (group) activity (investigation, experiment or other practical) aimed at motivating and inspire the students.
List of concepts	List of concepts, potentially entailing brief explanations.
Introductory tasks (XIE)	Introductory and warm-up tasks ("starters").
History (H) & Social (S) (XIE)	A section presenting mathematics in a: (H) historical context, its development or historical significance; (S) social context and stressing the role and function of mathematics in society.
Instructional narrative	A written (introductory) text describing the content of the chapter discussing and explaining theory and concepts.
Solved tasks (XIE)	Fully solved tasks presented.
Exercises (XIE)	Tasks to practices the topics and contents of the chapter.
Extension-tasks (XIE)	Tasks that go beyond what could be considered to belong to the standard course covered in the chapter; extra tasks or material deepening or extending concepts , theory and/or methods.
Application of abilities (XIE)	Tasks explicitly stated to practice or test the mathematical abilities found in the mathematics curricula (such as communication, reasoning, and problem solving).
Summary (List of concepts, Instructional nar- rative, Mindmap)	Section summarising the chapter. List of concepts: list of concepts containing explanations and examples; Instructional narrative: a summary written in the style of an instructional narrative; Mindmap: a network representation summarising the important concepts, strategies and methods of the chapter.
Repetition tasks (XIE)	Tasks in the end of the chapter aimed at giving a repetition of the content covered in the chapter.
Diagnostics (concept XIE, Proficiency XIE)	Tasks that test the contents in focus of the chapter; either focusing on concepts or proficiency.
Filling (Figures/ decorative pattern, Header, Footer)	Figure/decorative pattern such as a photo or confined area with decorative pattern (potential related to or illustrating a task, but not placed in an integrative way with respect to the task). Header and Footer are simply the pages' header and footer.
Other	Content in the chapter to which the above categories are not applicable, such as references to web-based material, pre-tests, lists of prerequisite knowledge for example.

(a) *percentage agreement* showing the proportion of all coded units the coders agree on divided by the total number of unique units coded by the two coders; and (b) *Cohen's kappa coefficient*,  $\kappa$ , which provides a number between -1 and 1, where  $\kappa \leq 0$  indicate no agreement and  $\kappa = 1$  indicates perfect agreement. Cohen's kappa is a statistical measure that factor in the chance of two coders assigning the same code to a unit by pure chance, and according to Fleiss et al. (2003),  $0.75 \leq \kappa \leq 1$  is considered very good inter-rater reliability;  $0.41 \leq \kappa \leq 0.75$  fair to good; and  $\kappa \leq 0.4$  poor.

The analysis of the inter-rater reliability in the four coded chapters showed almost a 100% agreement for the coding of the tasks into the XIE-categories. Only in two cases did the coding differ, and in one of these cases one of the

coders admitted to having accidentally picked the wrong code in the NVivo12 environment. In the other case one of the coders expressed that the coding manual allowed for some ambiguity with respect to the definition of what constituted an extra-mathematical tasks, rendering the divergent coding. The definition was discussed and the coding manual slightly revised, which resolved the ambiguity and divergent coding. With respect to the (sub-)codes XIE, we hence conclude the framework to reliably capture the three different (general) types of tasks: *exercises*, *intra-* and *extra-mathematical tasks*.

With respect to the overall structural codes the inter-rater reliability measures NVivo generates is not as straight forward to interpret as in the case with the XIE codes. This is due to the fact that part of the structural coding is to decide what area (in terms of a rectangle drawn in using the area selection-tool in NVivo) of the page it is that should be coded. In other words, there are two aspects to consider here: the (size of the) area that is coded and what code the designated area is assigned. Looking across the four chapters, the selected areas by the two coders overall agree to 96%. This relatively high level of agreement is partly due to that many of the areas of the pages in the textbooks that are coded in the framework are scaffolded by design features of the textbook. Some features, such as *heading*, *footer*, and (sets of) *Exercises* are clearly delimited in the textbooks with breaking lines, and the coloration in the books also provide structure that facilitate that the selection of the areas to code. The calculated Cohen's  $\kappa$  factors in and compares the size of the areas selected and what codes the two coders have assigned to these. Looking across the four chapter and all coded areas,  $\kappa$  on average is  $\bar{\kappa} = .90$ , with the lowest  $\kappa_{min} = 0.497$ , which indicates an overall very good inter-rater reliability (Fleiss et al., 2003). It is notable that both coders coded the exact same general areas (though slightly different in size) and exact the same tasks, indicating that the structure and procedure for the coding laid out and specified in the coding manual were adequate. Based on the above evaluation of the inter-rater reliability, we conclude that the framework works in a satisfactory and reliable way. Next, we explore the output of the framework applied to mathematics textbooks in terms of so-called *hierarchy charts*.

## Results in terms of hierarchy charts

Figures 1 and 2 below show the *hierarchy charts* generated by NVivo for the coding of the two chapters from the lower secondary mathematics textbook Prio Matematik 7. The hierarchy charts depict the 17 structural codes in rectangles coloured in darker nuances, with the size of each rectangle of a given code (colour) proportional to the total area of the chapter assigned that code (see further remarks below). The lighter nuances coloured rectangles found within some of the darker coloured rectangles show the proportions of how the tasks, or other sub-codes, within the particular structural code were coded. For example,

looking at the code *Diagnostics* (Diagnos) in the two hierarchy charts, we can see that tasks are roughly proportionally distributed similarly with respect to the relative focus on *concepts* (begreppstest) and *proficiency* (färdighetstest).

With respect to *Exercises* (övningar), the Statistics chapter chart shows that there is an overwhelming majority of *extra-mathematical tasks* (utommatematiska övningar) compared to *intra-mathematical tasks* (inommatematiska övningar) and *exercises* (mängdträning). This is not surprising given that the content matter of statistics invites the use of real data, diagrams, and extra-mathematical situations to work on. Looking at *Exercises* in the chart for the Fraction and percent chapter however, shows a more balanced distribution of *extra-* and *intra mathematical tasks* and *exercises*. Here, *exercises* is the largest category of tasks, followed by first *extra-* and then *intra-mathematical tasks*. This pattern of types of tasks found in *Exercises* in the Fraction and percent chapter is mirrored in *Repetition tasks* for the same chapter, which is in stark contrast to the distribution of tasks in *Repetition tasks* for the Statistics chapter.

## Remarks, conclusion, discussion, and future research

The framework we have presented in this paper have shown promising result with respect to its potential in providing a mathematics topic- and content-area-unbiased overview of the overall structure and tasks in mathematics textbooks. As such, the framework allows for a slightly new entry point when analysing mathematics textbooks in that it can function as the basis for making different

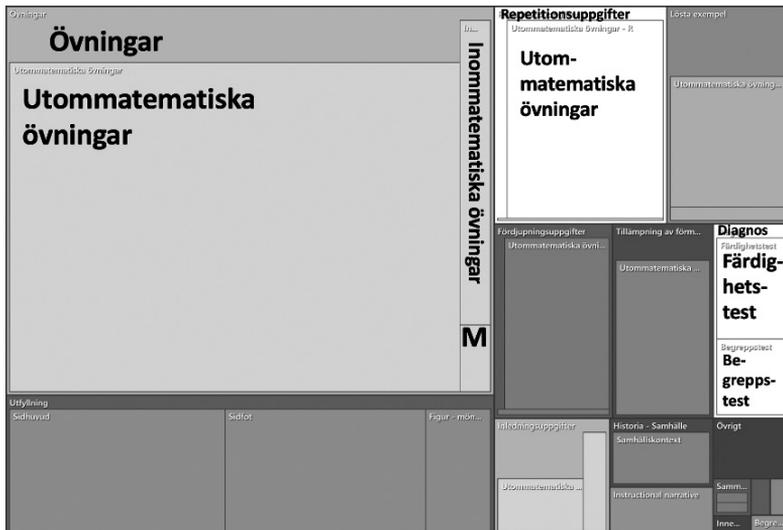


Figure 1. The hierarchy chart for the coding of the chapter Statistics (30 pages) in *Prio Matematik 7* (Cederqvist et al., 2012)<sup>2</sup>

types of comparisons across topics and contents within as well between textbooks. The analysis of our piloting of the framework showed very good inter-rater reliability, which is encouraging with respect to continue using and developing the framework.

The hierarchy charts proved to be useful to visualize the results of the coding and in identifying and recognizing both somewhat expected as well as more novel patterns, similarities, and differences in the chapters on *Statistics* and *Fractions and percent* in the Swedish lower secondary mathematics textbook Prio Matematik 7. Here, it should be noted that the hierarchy charts visualization produced in NVivo is dynamically scalable, meaning that printed in a different format, even the smallest regions will be more intelligible in terms of providing clearly readable codes for example. However, a much deeper and richer analysis would be possible if the hierarchy charts are complemented with the table explicitly showing the breakdown of the distribution of codes and sub-codes. This is one natural way to extend the analysis, in particular when it comes to gain more insight into the structural elements (codes) of the textbooks that are relatively sparsely present in the analysed textbook (and hence are small and hard to make sense of in the hierarchy chart).

Another way to develop the framework in the context of comparing different textbooks and textbook series, is to bring in a topic-based and content-centered framework to provide a more detailed content-specific analysis as a complement to the overall picture painted by the framework presented in this paper. With the teaching and learning of statistics as one of our more recent research interests,



Figure 2. The hierarchy chart for the coding of the chapter Fractions and percent (48 pages) in Prio Matematik 7 (Cederqvist et al., 2012)<sup>2</sup>

these two (add a break-down table of codes and a complementing topic-based and content-centered analysis) will be our next endeavours – to further and complement the analysis presented here.

## Notes

- 1 In defining the codes used in the framework presented in this paper, we use the notion of *chapter* to denote a selected continues text, such as a generic normal book chapter, section or sub-section of a textbook, that can be delimited by specifying a page-range (and if needed also on what line(s) on the first and/or last page the selection starts/ends).
- 2 The original labels provided by NVivo (in white font) are here reinforced in black for clarity.

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# Support in relation to problem solving – building a common knowledge base?

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We present three closely related projects concerned with supporting students' mathematical problem solving. The projects build on the assumption that problem solving activities are beneficial to students' learning but challenging for teachers to organise. Teachers must find ways to support students' progress in problem solving without removing necessary challenges. The projects deal with this support in different ways, something we intend to use to illustrate the risk that mathematics education research becomes fragmented, making it more difficult for teachers to access and use research results in their professional development. We welcome participants to discuss how closely related research projects like ours can collaborate and complement each other to contribute to a knowledge base that is accessible and useful to teachers.

## Introduction

Despite the huge potential to facilitate students' learning that research has attributed to mathematical problem solving (Jonsson et al., 2014), research has consistently found it uncommon to teach mathematics through problem solving (Boesen et al., 2014; Maass et al., 2019). If mathematics education research continues to advocate problem solving as a productive way to teach mathematics, research also needs to identify and propose solutions to the challenges associated with such an approach. Results from such research could offer opportunities for teachers to overcome the challenges and develop their teaching. One of the challenges that teachers face is supporting students during problem-solving, given that any form of support always risks reducing the challenge to the extent that learning opportunities are lost. Therefore, there is a need for tools and strategies that could help teachers to find the right balance between challenge and support. Attending to this need is the focus of three different Swedish research projects in mathematics education. These will briefly be presented below and during the symposium they will serve as an example of research projects that are closely related.

The first project, Ufer – using feedback to encourage students' creative reasoning is a longitudinal, on-going, design research project that involves cooperation with 4 teachers over the last 6 years. The project aims to empirically develop principles for teachers' actions in interaction with students' during

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problem solving (Teledahl & Olsson, 2019). Cooperation with teachers includes planning for problem solving with the intent to support students' creative mathematical reasoning (CMR), as described by Lithner (2008). The theoretical point of departure is that mathematics learning is enhanced by teaching that allows for CMR (Jonsson et al., 2014). During the problem-solving situations we document and analyse teacher actions that lead to students' continued CMR. Preliminary results suggest that students should be encouraged to (1) express independent reasoning, (2) develop their reasoning, (3) justify their reasoning and (4) find a way to test their results. Future research aims to describe teacher actions that achieve 1–4.

The second project comprises three interrelated, ongoing large-scale design research projects conducted in collaboration with 50 mathematics teachers at seven different schools, teaching school years 1–12 (Säfström et al., 2021). The aim is to develop and study design principles and tools for supporting teachers' interaction with students during problem-solving, in order to promote students' learning. It rests on previous research regarding the connection between qualities in students' reasoning and their learning (Lithner, 2017) and uses formative assessment (Black & Wiliam, 2009) as a structure for teachers' support. Over iterations of classroom experiments, analysis and development, teachers and researchers build knowledge on students' problem-solving processes, the difficulties students encounter in such processes, the opportunities for students' own construction of mathematical reasoning those difficulties entail, and how teachers can realise those opportunities in their interaction with students. The results include both theoretical insights and practical tools for teachers' interaction with students.

The third project, about how prompts to self-explain can nurture learning, is conducted in collaboration with a teacher in grade 4–6 (Dyrvold & Bergvall, 2019). Self-explanation prompts (SEPs) have previously been defined as questions or elicitations that aim to induce meaningful explanations for oneself to make sense of new information (e.g. Rittle-Johnson et al., 2017). In this project we seek understanding of the potential of SEPs in collaborative work, based on a social cultural perspective with an emphasis on language. An insight into activities that lead to progress can enhance teachers' possibilities to support students' development and accordingly, the aim of the project is to explore how meaning making takes place in students' joint discussions fostered by SEPs. Analyses of students' discussions during collaborative problem solving reveal five types of recurring meaning making activities in relation to SEPs. SEPs in combination with these activities can be used as tools to support meaning making for example when problem solving does not progress as expected or in a strive to understand a new concept.

At the symposium, we will present what aspects of support during problem solving are foregrounded in each project, how these perspectives contribute to

ideas or guidelines for support and how the results could be used by teachers. A natural next step for each project would be to promote and spread their own results, an approach we argue is not the most fruitful for either the research community or the teaching practice. Such an approach risks causing more fragmentation in the field of mathematics education and hinder teachers' access to and use of research results in their professional development. Therefore, the participants of the symposium are invited to a discussion on the following questions.

How can related but different projects collaborate to better serve mathematics teachers' professional development?

How can related but different theoretical perspectives and results complement each other and contribute to a common knowledge base in mathematics education research?

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# Challenges when implementing the Elkonin-Davydov curriculum in mathematics

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Teachers interested in developing students' possibilities to take part in joint discussions in problem-solving often have to deal with challenges regarding norms in the mathematics classrooms. Dominating classroom norms is a factor in mathematics teaching, be it of social or sociomathematical type. Here we address challenges in relation to norms experienced when attempting to create possibilities for mathematics learning, starting at a general and algebraical point rather than a specific and arithmetical one. We draw on two projects that explored the Elkonin-Davydov curriculum in Swedish classrooms. The results indicate that norms, of different kinds, are impedimental in different ways, when performing a theoretical work in mathematics teaching.

The overarching aim of this symposium is to discuss experienced challenges concerning, what Yackel and Cobb (1996) describes as social and sociomathematical norms, among Swedish teachers who have tried the Elkonin-Davydov curriculum (ED-curriculum) (Davydov, 2008).

To create opportunities for students to develop their theoretical or algebraic thinking, the ED-curriculum is developed based on the theoretical foundation of learning activity (LA) (Davydov, 2008; Kaput, 2008; Schmittau, 2004). Mastering the general / theoretical that is built into a mathematical object or concept is an important starting point for a learning activity. In order for students to engage in LA, a content-rich problem situation must be provided. Such a situation should be complex enough to create some challenges for students in a way that their current knowledge is not fully sufficient. When planning a problem situation, possible learning models must also be constructed because it is with the help of the learning models that students in their inquiry work should be able to develop new knowledge. However, getting students to engage in LA is a very delicate process. The students' agency can easily be turned into something where they try to do what they think the teacher wants them to do, if so

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the students are no longer the active part of the problem-solving process. In part, balancing this delicate process is a matter of classroom norms, both social and sociomathematical (Yackel & Cobb, 1996).

The ED-curriculum for mathematics teaching has several similarities with inquiry mathematics teaching (e.g. Brousseau, 1997; Yackel & Cobb, 1996) which emphasizes sociomathematical norms as argumentative and explanatory, autonomous or agentive actions collectively directed at mathematical objects rather than procedural operations. Specific to the ED-curriculum is its overall goal: the development of the youngest students' algebraic thinking (Vygotskij, 2001). With algebraic thinking, students' understanding of a mathematical object and its conceptual and theoretical structures is in focus in the curriculum, and thus in the teaching situation. To use the concepts of social norms and sociomathematical norms, both inquiry teaching and ED-curriculum in some respects relies on similar norms. Enabling the students' agency presupposes that the teacher is able to establish social classroom norms of cooperation and collective problem-solving processes. Specific to LA is that a problem situation needs to be complex and demanding enough so that the students' current knowledge is always insufficient in some respect. As a common classroom norm, therefore, students have to presume that their current knowledge is insufficient and that new knowledge needs to be developed. Another social norm is that students use the experiences of others as a resource. In relation to the ED-curriculum, examples of a sociomathematical norm that needs to be established in the classroom concerns students' engagement in the problem identification process, and that solution-proposals are to be given in an algebraic form.

A curriculum developed in one cultural setting cannot just be implemented in another cultural setting without challenging the dominating norms (Stigler & Hiebert, 2009). The research projects we have been involved in indicate challenges that must be handled if the ED-curriculum is to be implemented in Swedish schools. Therefore, the purpose of this symposium is to, on the basis of two research projects in a Swedish context, exemplify and discuss some social and sociomathematical norms we have identified. The two projects are 1) Helena Eriksson's doctoral project and 2) a research project funded by the Swedish Institute for Educational Research.

### *Project 1: Challenging norms in joint work on learning models*

Project 1 is an ongoing research project that started in 2014 as a doctoral project (see Eriksson, 2021). In different extensions, twelve teachers, one researcher and about 150 students in preschool class – grade 5 participate. Most of the students were newly arrived in Sweden. This project aimed to explore the ED-curriculum as a tool for designing mathematics teaching in a multilingual primary school in Sweden. The data consists of about 90 research lessons and about 30 sessions when the teachers were planning the research lessons. Some results

from the project are presented concerning algebraic thinking. A challenge of norms when developing a learning activity that has been identified is the joint work on learning models in the research lessons. Traditionally, the students are used to following teachers' instructions about what model to use and how to solve a problem. But, in the research lessons the students were supposed to suggest, and argue about, which, why and how models should be constructed in order to discuss a specific concept aiming to solve the identified problem.

### *Project 2: Challenging norms in joint theoretical work*

Project 2 was implemented collaboratively between teachers and researchers in four schools during the years 2017–2019. Altogether, eight teachers, eight researchers and about 310 students participated in the project. The research lessons were completed in compulsory school grades 1, 5, 7 and in upper secondary school. A total of 17 video-recorded research lessons aiming to explore how teaching, including problem situations, can be designed in order to enable students to develop a capability to reason algebraically were conducted. Principles from the ED-curriculum guided the design and analysis of the research lessons. The results are similar across the school years and points towards challenges related to norms connected to, for example, students' voluntarily showing their suggestions on the board for everyone to see, or teachers choosing to listen to, and managing, students' taking turns when elaborating on each other's suggestions rather than asking for answers (Eriksson et al., 2021; see also Eriksson et al., 2019). Both factors are pivotal for creating and maintaining the dimension of joint theoretical work signifying a learning activity when working with the general structures and relationships in mathematical objects.

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# Exemplifying different methodological approaches of analysing textbooks in mathematics

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In this symposium, we will discuss different ways of analysing mathematics textbooks from a methodological point of view. The discussion will be based on examples from five separate ongoing analyses of Swedish textbooks divided into two methodological approaches; one where analysis is conducted within an established theoretical framework, and one where analytical tools are constructed through combining aspects of different theories. The symposium will be held at the conference MADIF13.

Textbooks play an important role as mediators between syllabi and teachers in the classrooms, or similarly, in translating policy to pedagogy (Valverde et al., 2002). By now, it is well established that mathematics textbooks have a great impact on teaching since they serve as an important resource for mathematics teachers (Lepik et al., 2015), form learning opportunities for students (Stein et al., 2007), and may even define mathematics as a school subject (Rezat & Strässer, 2014). Over the past two decades, textbooks have received increasing attention in the research field of mathematics education (Jablonka & Johansson, 2010; Rezat et al., 2019), especially considering the rapid development of digital teaching materials (Pepin et al., 2017). However, the field of mathematics textbook research is still developing and strives to establish its philosophical foundations, theoretical frameworks, and research methods (Fan, 2013).

In an attempt to classify literature on mathematics textbook analysis, Fan et al. (2013) came up with the following five categories: (1) content and topic; (2) cognition and pedagogy; (3) gender, ethnicity, equity, etc.; (4) comparison of textbooks; and (5) conceptualization and methodology. In the symposium, we will focus on methodology in relation to content analyses and comparisons of textbooks. Our aim is to increase the knowledge of how theory and method can be applied when analysing textbooks by discussing five ongoing Swedish

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textbook analyses. The studies exemplify two methodological approaches, where the first covers analyses where theoretical concepts and analytical tools are embedded in a broader theory, while the second covers analyses that combine and/or modify different theories in order to form an analytical tool.

### Approach I: analysing within one theoretical framework

We present two textbook analyses that are positioned within different theoretical frames; Variation theory and the Anthropological Theory of the Didactic (ATD).

Study 1 builds on variation theory and investigates mathematics textbooks for upper secondary vocational education, with the aim of exploring learning opportunities afforded to students with respect to both general mathematical knowledge, and its' use in vocational practice. In particular, characteristic properties of geometry tasks will be contrasted with the corresponding topics in textbooks for other programmes, as well as older vocational mathematics textbooks. According to variation theory, the object of learning and its corresponding critical aspects are in the focus of attention. To discern an aspect, the theory states that students must have the opportunity to experience a potential variation in that aspect (Marton, 2015). Therefore, patterns of variation are analysed in order to evaluate learning opportunities afforded to students by sets of textbook tasks (Kullberg & Skodras, 2018; Sun, 2011).

Study 2 is embedded in the ATD. By using praxeology as an analytical tool (Chevallard, 2006), algebra content in Swedish upper secondary textbooks for academic preparatory and vocational programmes is analysed and compared. Internationally, school algebra has already been investigated from an ATD perspective. For instance, it has been shown that secondary school algebra is often reduced to equation solving and manipulation of algebraic expressions (Bosch, 2015). It is therefore of interest to examine this from a Swedish perspective. The study forms part of a larger project exploring socioeconomic aspects of the didactic transposition (Chevallard, 2006) of algebra in Sweden. In the present study, the focus is on the didactic transposition of algebra from "knowledge to be taught" to "taught knowledge". The aim is to unpack differences in how algebraic content is offered in textbooks for academic preparatory and vocational programmes, and to discuss how these differences may affect students' opportunities for future education and participation in society.

### Approach II: analysing by modifying and/or combining theoretical frameworks

Within this approach, three studies are described. The first concerns digital teaching platforms and the following two programming content at different school levels.

In Study 3, the utilization of dynamic elements in digital teaching platforms is analysed. Building on a pre-existing model of 15 different elements present in digital textbooks (Pohl & Schacht, 2017), a typology for describing and analysing

dynamic functions in digital textbooks in mathematics was developed. A development into 23 elements was needed since the existing model did not capture all features deemed important. The typology was developed in relation to an analysis of the dynamic functions present in seven digital textbooks. The sections were divided into text elements, for example a paragraph or an image, and categorised according to their characteristics in terms of semiotic resource, modality, degree of dynamic function, interactivity, and degree of feedback. The 23 categories were then structured into a typology of five dynamic types (Bergvall & Dyrvold, 2021). These types define five steps of increasing dynamic function and increasing invitation to interact; from the first type defining static presence to the last type defining continuous dynamic feedback.

In Study 4, programming content in mathematics textbooks for grades 1–9 is analysed, with the aim of characterizing the programming content and discussing how it may affect students' opportunities to learn mathematics. In an initial study, programming tasks in textbooks for grades 1–6 were analysed (Bråting & Kilhamn, 2021) using an analytical tool that combines a theoretical framework of computational thinking (Brennan & Resnick, 2012) and one of actions (Benton et al., 2017). Initially, the programming tasks in the textbooks were analysed using the two frameworks separately, but neither of them could describe the content in a comprehensive way. The first framework did not highlight mathematical ideas, and the second did not sufficiently elaborate on computational concepts. An analytical tool was therefore constructed as a combination of concepts and actions, which, in addition, could be used to discuss the links between programming and mathematics. In an ongoing study, textbooks for grades 7–9 are analysed. Taking the previously constructed analytical tool as a point of departure, further developments are made to accommodate for text-based programming.

In Study 5, programming content in mathematics' textbooks for upper secondary school is analysed. As the theoretical frame an instrumental approach is used (Trouche, 2004). The key idea is the difference between an artifact and an instrument: Within the activity of a subject, an artifact becomes an instrument through an individual genesis, the so-called instrumental genesis (Verillon & Rabardel, 1995). Thus, the instrument does not exist in itself – it is made up of an artifact and a psychological component. In order to investigate in what ways the textbooks can offer support for students' instrumental genesis, programming tasks are investigated at three levels inspired by the already mentioned framework of Brennan and Resnick (2012). At the first level the artifact itself is in focus, when the computational concepts treated in the tasks are investigated. The second level is a psychological one, where the actions that students are requested to perform are investigated. At the third level, the categories emerge from the data, based on the different ways in which programming is used as a tool for simulating situations, constructing algorithms, and using different representations.

## Concluding remarks

In the symposium, we will compare the different approaches in order to discuss affordances and constraints that they bring on board. From a methodological perspective, we raise questions concerning the clear but at the same time restrictive frame created by the first approach and issues of validation and coherence in the second approach.

Moreover, although all five studies basically analyse what learning opportunities the textbooks offer, there is a variation in aims and foci. For instance, Studies 2 and 4 are primarily concerned with the mathematical content, while Studies 1, 3 and 5 look more at how the textbooks are, or potentially could be, used. Another difference can be noticed between the two analyses of programming content; while Study 4 aims to find out the role of programming in mathematics education, in Study 5 it is already assumed that the students are supposed to use programming as a tool in mathematics.

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# Short presentations

## En studie om elevers uppfattning om associativitet och hur det kan påverka algebraundervisningen

ROBERT GUNNARSSON

Jönköping University

Denna presentation beskriver en pågående studie om hur associativitet kan uppfattas och hanteras i matematikundervisning. Studien syftar till att beskriva elevers olika sätt att förstå associativitet och utifrån dessa designa lämpliga undervisningsinslag för att främja elevernas algebraiska förståelse. Vissa delresultat beskrivs.

## A tentative attempt for professional development: contingent moments in teaching mathematics with historical resources

MELIH TURGUT AND IVETA KOHANOVÁ

Norwegian University of Science and Technology

This presentation focuses on ongoing research related to the professional development of master students, regarding teaching mathematics with historical resources. The participants of the study are seven in-service and pre-service teachers enrolled in a master course on the historical and philosophical perspectives on school mathematics. We present a tentative analysis of challenges of master students' written responses to two (imaginary) contingent moments regarding student queries/ideas while teaching mathematics – particularly teaching algebra – with historical resources. Our tentative analysis implies two categories: surface content knowledge and neglecting student knowledge and student thinking.

## A comparison of two frameworks for the analysis of knowledge and skills for teaching statistics – MKT vs. RCM for PCK

PER BLOMBERG

Halmstad University and Karlstad University

This presentation is part of a Ph.D. project that aims to increase knowledge about how to support the development of teacher students' pedagogical content knowledge (PCK) for teaching statistical inference. Dealing with the existing diversity of theoretical approaches is a well-known challenge for the research community. The focus of this short presentation is to compare two reputable frameworks: *Mathematical knowledge for teaching* (MKT) and *Refined consensus model* (RCM) for pedagogical content knowledge. This comparison will highlight their contributions, merits, shortcomings, and possible connections to evaluate and guide an ongoing teaching and learning design in teacher education for primary school.

## Analysing argumentative processes during mathematical problem solving in small groups

HANNA FREDRIKSDOTTER

Uppsala University

This presentation reports from a study of young students' use of justifications when solving a mathematical problem in small groups. In the study, video recordings from two grade 6 classrooms were analysed, using Harel and Sowder's taxonomy of proof schemes as a base for an analytical tool. The analysis showed similarities between the young students' justifications and results of previous research on adult students' proofs, for instance regarding students' use of examples. The analysis also showed how students sometimes used calculations that contradicted empirical examinations, which could not be explained by the taxonomy of proof schemes. An ethnomethodological approach is suggested, to further the analysis of argumentative processes taking place during mathematical problem solving in groups.

## Response to intervention (RTI) in number sense – developing a method supporting students at risk in a Swedish context

LENA KARLSSON<sup>1</sup> AND HELENA ROOS<sup>2</sup>

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This presentation focuses a project on identifying and supporting students at risk of falling behind in their learning of number sense in early school years (from grade 1). Response To Intervention (RTI) is used as a model for monitoring and supporting students at risk in a staged series of research-based interventions focusing on number sense. In this project 113 students' number sense has been monitored from grade 1 to the end of grade 2. Students at risk have participated in tier 2 and tier 3. A control group of 37 students has been monitored. Tentative results indicate significant differences on group level where the intervention group show fewer students left behind.

## Using heat maps from eye tracking in stimulated recall interviews

ANNELI DYRVOLD AND IDA BERGVALL

Uppsala University

This presentation discusses students' interpretations of heat maps from eye tracking. Heat maps are often referred to as 'just' eye candy because of their appealing nature and the somewhat 'hidden' data. Undoubtedly, there is valuable information in these visualisations and if attention is paid when conclusions are drawn, the data is a useful complement to quantitative measures. We explore pros and cons when using heat maps in stimulated recall interviews and contrast this method to stimulated recall using videos or the use of think aloud protocols. A conclusion is that the heat map can attract attention to what actually happened and thereby evoke valuable references to thought processes, but at the same time it may draw attention to actions instead of to reasoning and thoughts because the image represents the reader's activity ("I looked at...").

## Math teaching anxiety and teachers' pedagogic practice in Swedish preschools

LAURA GALEANO

Uppsala University

This presentation reports the results of a pilot study (N=50) testing the construct validity and reliability of a questionnaire in which teachers' ratings showed a negative, statistically significant ( $p=0.10$ ) correlation of moderate size between higher levels of math teaching anxiety and a lower frequency of math related language use in preschools ( $r=-0.360$ ). Math anxiety involves worry and feelings of tension manifested when manipulating numbers or solving math problems, in the classroom and in everyday life. As instruction quality of teachers with higher MA levels, and their own attitudes and beliefs about mathematics might impact students' MA and math performance, the results suggest that teachers' self-awareness of their language output in relation to their math teaching anxiety levels can contribute to mitigate the intergenerational effect of MA.

## Planning mathematics teaching in preschool

JOSEFIN ROSTEDT

Jonkoping University

This presentation takes its departure in the new curriculum for Swedish preschool where preschool teacher's responsibility for teaching subject matters such as mathematics, has been strengthened. The teaching is recommended to have a thematic approach, to be play-based, and intertwine care, development, and learning. Also, the curriculum states that teaching can be both planned and spontaneous. The task given by the government is complex and calls for a problematization of teaching planning and what role planning can have in preschool especially for mathematics teaching. The ongoing study aims to investigate teaching planning and how preschool teacher and child minders prepare and change their mathematics teaching during the time of planning.

## A discourse analysis on preschool class teachers' talk about assessment in mathematics

MARIA WALLA

Dalarna University

This presentation describes a discourse analysis on Swedish preschool class teachers' talk about assessment in mathematics. Since 2019, a mandatory assessment material (Find the Mathematics) has been available for assessing six-year-old students in Sweden. In this study, four focus group interviews were conducted with altogether 12 preschool class teachers. The results show four discourses with diverse meanings ascribed to equity: equity meaning students do the same thing; equity meaning students have different needs; equity at risk because of unsatisfactory conditions; and equity at risk because of limited resources.

## Sustaining students' participation in mathematics

MALIN GARDESTEN

Linnaeus University

Based on a classroom study conducted in a Swedish Grade 3, this presentation explores students' participation in mathematics. Video-recorded observations were analyzed based on two theoretical approaches connected to teachers' pedagogical and subject matter knowledge, and relational abilities. Instances, where the two theoretical approaches intersected, were analyzed focusing on students' participation in mathematics framed by a social practice theory. The tentative results show that when the teacher in interaction with the students, uses both her mathematical knowledge for teaching and her relational abilities, the students sustainably participates in solving an open number sentence.

## Designing a teacher-guide for de-ritualising teaching with GeoGebra

IDA FANTENBERG NIKLASSON<sup>1</sup>, NELLY WANNBERG<sup>1</sup>,  
CECILIA KOZMA<sup>1</sup> AND LISA ÖSTERLING<sup>2</sup>

<sup>1</sup>KTH Royal Institute of Technology, <sup>2</sup>Stockholm University

This presentation describes the development of a teacher-guide for using GeoGebra in secondary mathematics teaching. The methodology is an iterative development, and is part of a collaboration between student teachers, teachers, and researchers. Commognitive theory informs the development, and in particular the move from ritual to explorative routines. The results highlight the challenges of explorative learning, where teachers stated how learners were looking for short-cuts. Therefore, the teacher-guide supports a gradual move from ritualized short-cuts, towards explorative participation in mathematics with GeoGebra. In particular, the guide challenged teachers to reflect on their question-posing. We claim this supports both students and teachers to participate in an increasingly explorative mathematics discourse.

## Building a paradidactic infrastructure for teachers' professional scholarship in Sweden

YUKIKO ASAMI-JOHANSSON AND MIKAEL CRONHJORT

University of Gävle

This presentation describes analytical methods of our project that will investigate possibilities and conditions for a group of Swedish teachers who work to establish their own professional scholarship of teaching mathematics using two Japanese methods: structured problem solving and lesson study. An a priori analysis using tools from the anthropological theory of the didactic presents in what way the teachers transpose the Japanese model for structured problem solving and adapt its didactic techniques into the Swedish context. The analysis will also describe how or if lesson study can support construction of a paradidactic infrastructure that disseminates the formed knowledge, and the facilitators' role for the establishing process of the infrastructure.

## Student teachers' use of a general analytic rubric when scoring pupils' mathematical problem solving solutions

BIRGIT GUSTAFSSON

Karlstad University

This presentation is an on-going research study. The aim of the study is to deepen the understanding of how the scoring of mathematics in student solutions is affected when using a general analytic rubric on an algebraic problem solving task. Especially attention will be paid to how a general analytic rubric contributes to the scoring. The general analytic scoring rubrics are today a popular assessment tool among teachers. Even if there is a lot of research in the area of rubrics there is a lack of research about rubrics used in scoring of mathematics and on the arguments for the scoring. The data consists of recorded discussions, when six student teacher groups scored five pupil solutions. In the analyses of data the focus will be on what is considered in the scoring and what reasons they give for their scoring.

## Exploring new territories: a mathematics teacher's practice regarding programming with young learners

ØISTEIN GJØVIK, IVETA KOHANOVÁ AND MELIH TURGUT

Norwegian University of Science and Technology

In this presentation we report parts of ongoing research, where a mathematics teacher enters a new arena of teaching programming and computational thinking skills to young learners. In this project, the programming environment of Emil the robot is utilized for achieving the competency goals of the new Norwegian curriculum. Emil allows pupils to learn the fundamentals of programming and computing through plugged and unplugged activities. Emil does not give feedback or tell the pupils whether their solutions are right or wrong. It is intended that pupils should give feedback to each other while working in pairs. Our ongoing classroom observation replicates that the teacher's role is of crucial importance in setting classroom activities and orchestrating the pupils' learning and thinking.

## Cognitively activating mathematics lessons: a Nordic comparative study

JÓHANN ÖRN SIGURJÓNSSON

University of Iceland

This presentation reports preliminary results from a video observation study of two specifically selected lessons from each country, Iceland, Norway, Sweden, and Denmark, where each lesson had rich opportunities for cognitive activation. Cognitive activation is a dimension of teaching quality that involves how teachers facilitate students' cognitive activity through challenging tasks and mathematically rich practices. Classroom observations from Iceland have shown opportunities for improvement in this dimension. Preliminary results indicate differences in both lesson content and structure, but similarities in use of group work and discussions, explicit student roles in discussions, and ways of eliciting student thinking. Ongoing analyses of these lessons seek to address implications for the "Nordic model" of instruction.

## Multilingual mathematics teachers' professional identity in multilingual mathematics context

DANAI DAFNOPOULOU

Linnaeus University

This presentation discusses the relevance of investigating multilingual mathematics teachers' professional identity in multilingual mathematics contexts. Research on mathematics education still lacks cases of teachers with a diverse linguistic background more than the language of instruction. A participatory view of identity will be followed in order to provide empirical results on how is to be and become a multilingual mathematics teacher in multilingual contexts. The theoretical framework of Patterns of Participation will be adapted. An ethnographic longitudinal study of three teachers in lower secondary mathematics level in Sweden is proposed. The data will be generated from teachers observations in classroom and interviews about their educational and professional background, as well as instances of classroom interaction with students.

## Develop mathematical reasoning? – a literature review of tasks and their implementation

JIMMY KARLSSON

Karlstad University

Reasoning has been identified as a core component in mathematical knowledge. Less focus has been on synthesizing research including subprocesses of reasoning such as conjecturing, justification and generalization. Acknowledging the importance of tasks and their implementation, this paper aims at eliciting aspects that has the potential to foster students' mathematical reasoning by means of a systematic literature review. The current main findings suggest a focus on open-ended exploratory tasks and implementation that successively builds on students' mathematical work. Classroom culture and interaction are identified as important, and teachers balance between modelling and students' exploration is recognized.

## Student teachers' explanations of linear equations evaluated by comparative judgement

NICLAS LARSON<sup>1</sup> AND KERSTIN LARSSON<sup>2</sup>

<sup>1</sup>University of Agder, <sup>2</sup>Luleå University of Technology

This presentation reports from an ongoing study where Norwegian and Swedish student teachers evaluate scripts from an earlier data collection, where other groups of student teachers were invited to explain the steps of the solution to a linear equation. In the current study, the participants are invited to evaluate these previous scripts by repeated pairwise comparisons. This comparative judgement generates a ranking of the scripts. The purpose of this study is to explore the student teachers' ranking of the explanations of solutions to a linear equation, and what differences appear between judges from the two countries. Initial results indicate that Norwegian judges tend to rank Norwegian scripts higher. This suggests that differences between the two countries identified in the explanations collected earlier remain when another group evaluates these scripts.

## Students, mathematics textbooks, and agency

MALIN NORBERG

Mid Sweden University

This presentation reports on a study of 18 Swedish year 1 students' (7–8 years) work with mathematics textbooks analysed according to the concept of agency. The empirical data consisted of video material, students' representations, and mathematics textbooks. The result showed that some exercises enable agency, and some do not. Also, students' opportunities for agency are affected by the notion that, according to the students, mathematical symbols is the resource that should be used to be considered successful in mathematics. A conclusion from this is that the textbook needs to be used consciously, offering different learning situations based on both opportunities for agency and multimodal aspects to provide all students learning situations that benefit both learning and the opportunity to discover themselves as mathematical individuals.

## Mathematics, vocational education, and multilingualism: epistemic aspects

LISA BJÖRKLUND BOISTRUP, PETRA SVENSSON KÄLLBERG

AND ULRIKA RYAN

Malmö University

In this presentation, we discuss epistemic aspects with regards to mathematics in relation to vocational mathematics and language use. We draw on two theoretical frameworks which build on praxeology, and “Language as resource” in multilingual mathematics activities. In so doing, we explore and try to combine assumptions from them with specific attention to epistemic aspects, such as questioning hierarchies among disciplines, and how language use may open up opportunities as to how mathematics is conceptualised and understood.

## Mathematical modelling in social sciences

JÖRAN PETERSSON

Malmö University

This presentation outlines the mathematising of population pyramids in secondary education. The population pyramids are presented as reduced schemas, making them more accessible for use in both mathematics and social science classrooms. The outcome is that such schemes show useful for describing relations between the shape of a population pyramid and aspects of macro-economic status of country; both when comparing countries with different shapes of the population pyramids at the same time period and when comparing one country over historical periods with different shapes of its population pyramids. Hence, this model opens for several societal applications, which in turn form bases for students' discussion and interpretations of both simulation models and the principle of mathematical reductionism.

## Number sense in the app Vektor: mathematical progression and use of various modes

HELENA JOHANSSON<sup>1</sup>, MALIN NORBERG<sup>1</sup> AND

MAGNUS ÖSTERHOLM<sup>1,2</sup>

<sup>1</sup>Mid Sweden University, <sup>2</sup>Umeå University

This presentation describes an on-going study that takes a multisemiotic approach to analyse the app Vektor in order to understand how interaction between different modes in digital tools can support students' development of number sense. Preliminary results show that some interactions between modes highlight central aspects of number sense, while the purpose with other interactions are unclear.

## Spatial relations and other text features in the connections between mathematical symbols and written language

ULRIKA WIKSTRÖM HULTDIN, EWA BERGQVIST, TOMAS BERGQVIST, LOTTA VINGSLE AND MAGNUS ÖSTERHOLM

Umeå University

In this presentation, ways to communicate connections between mathematical symbols and natural language in text are described. In mathematical textbooks, the two semiotic resources are often used together to introduce and explain mathematical ideas, concepts, and methods. To make sense of the texts, readers need to combine language descriptions and symbolic expressions, and when reading order is not adamant, spatial relations become important. We present different categories of connections between written language and mathematical symbols that are based on spatial relationships together with other features, for example, visual links, primary syntax, and reading order.

## How natural language gives meaning to mathematical symbols in textbooks at different school years

EWA BERGQVIST, LOTTA VINGSLE, MAGNUS ÖSTERHOLM, TOMAS BERGQVIST AND ULRIKA WIKSTRÖM HULTDIN

Umeå University

This presentation focuses on an on-going study where we examine if and how there is any type of progression over school years in the use of natural language when addressing symbols in textbooks. The analyses produce a set of categories that describe different ways of using natural language for giving meaning to symbols, for example, how symbols are defined or explained in a very explicit language or how natural language can give information about a symbol in a bit more implicit or indirect manner.

## Connections between natural language and mathematical symbols in mathematics textbooks

TOMAS BERGQVIST, ULRIKA WIKSTRÖM HULTDIN, EWA BERGQVIST, LOTTA VINGSLE AND MAGNUS ÖSTERHOLM

Umeå University

This presentation will give an overview of a project focusing on mathematical symbols and natural language, which both can be found in mathematics textbooks. There often exists different types of connections between the two sign systems. In this study we examine these connections and changes over school years. Six types of connections were defined, and over 3000 connections were identified and coded in textbooks for school years 2, 5 and 8. The results show that connections are mostly based on spatial proximity in the early years (about 63%), and by symbols interwoven in sentences in later school years (about 80%). There is a development from separating natural language and symbols in year 2 to mixing the two sign systems in school year 8.

## Assessment discourse in mathematics curriculum: a hindrance for critical thinking and democracy?

CHRISTIAN H. ANDERSSON

Malmö University

This presentation discusses a find in a discourse analysis of the revised mathematics curriculum for upper secondary school in Sweden. The analysis is related to critical thinking, democracy, and the new digital technology of using mathematics on large quantities of data, such as our digital traces, that may challenge democracy, e.g. by generating discriminatory algorithms. Based on how the curriculum addresses society and how the teaching and learning goals are described, a discourse labelled society as an aim, but not to be assessed, was construed. The discourse portrays a tension between what is described as valuable knowledge on one hand, and what is assessed on the other. A question posed, how can this find be related to a wider discussion on assessment and any hindrance for educational aims such as critical thinking and democracy?

## Connecting teachers' use of curriculum resources in planning with mathematical knowledge for teaching

MARCUS GUSTAFSSON, JORRYT VAN BOMMEL AND

YVONNE LILJEKVIST

Karlstad University

This presentation reports on an ongoing study, which aims to create more knowledge on the relationship between different types of curriculum resources when identified in the practice of teachers planning collaboratively. These resources are described through the Design Capacity for Enactment framework, augmented with domains of the Mathematical Knowledge for Teaching framework. The aim is to identify and examine the connection, rather than to claim to explain the relation. Preliminary results show that there are many different types of resources used, both digital and analogue, and that teachers' Knowledge of Content and Students and Knowledge of Content and Teaching guide the reasons for what types of resources are used.