

Exploring the use of Fermi Problems and the FPAT-framework with pre-service primary teachers to bring real-life contexts into classrooms

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In this paper we analyse 11 pairs of pre-service primary teachers' solutions to two Fermi problems. The solutions were expressed using the descriptive and analytic FPAT framework (Fermi problem Activity Template), and our analysis focus on characterizing the FPAT representations produced by the pre-service teachers. The results show that almost half of the produced FPATs in principle solved the problems but that only three solutions provided enough detail to be practically implementable. Multiple key constructs were used to structure the solutions in the first problem, but not so in the second problem. The variation of different activities and ways of working suggested in solving the sub-problems in the two problems also differ in a significant way.

Keywords: Mathematical modelling, real context problems, Fermi problems, teacher training

Introduction, motivation, aim and research questions

Mathematical modelling, generically understood as using mathematics to solve real-world problems, is now a common curricula goal in many countries, to, among other things, show the potential of mathematics to describe complex phenomena and to promote critical thinking (Niss & Blum, 2020). However, modelling has not yet been established as a regular classroom activity (Borromeo Ferri, 2021). One reason for this is the openness of mathematical modelling activities, which puts new demands on the teacher to cope with and handling the students' diverse thinking and decisions about how to approach the problems being modelled. In this context specific training for teachers and available suitable resources are essential to enable teachers to implement mathematical modelling in the classroom (Garfunkel et al., 2021). One way to make modelling and mathematics relevant to students, is for teachers to connect their everyday mathematics teaching to interesting real-life contexts as well as current affairs and topics in the news in ways that normally transcends what is offered in textbooks and traditional curricula materials. Peter-Koop (2009) has showed how modelling activities in terms of so-called Fermi problems can facilitate bringing in everyday situations and contexts into the primary school mathematics classroom in meaningful ways. However, to first identify instances in out-of-school-contexts where mathematics comes to the fore or plays an important role, and then to make the connections to school mathematics, is non-trivial

and requires solid and broad mathematical knowledge (Geiger et al., 2021). From a teacher education perspective, the challenge is how to prepare pre-service teachers for this task, in particularly in the case of the training of primary teachers, since normally only a limited number of mathematics courses are part of their teacher training programmes.

Building on Peter-Koop's (2009) work, and to support pre-service primary teachers (PSPTs) in developing ways of connecting and bringing real-life contexts into their mathematics classroom in a productive way, we have designed a teaching sequence where PSPTs are confronted with a set of *Fermi problems* which they both solve and analyse using a special framework called *Fermi problem Activity Templates*. Our aim in this paper is to develop a characterization of the FPATs generated by the PSPTs in our first implementation of the teaching sequence to gain a basic understanding of the structure and content that came to the fore in the PSPTs' FPATs. The research question we address in this paper is: What characterize the FPATs produced by the PSPTs in terms of (a) to what extent their proposed solutions reasonably solve the problems under consideration; (b) the key concepts and procedures that structure their solutions; and (c) the mathematical activities proposed in their solutions?

Fermi problems and mathematical modelling

The physicist Enrico Fermi introduced, used and popularized so called Fermi problems (FPs) as time-saving and effective tools to illustrate the power of deductive thinking as well as making preparations before engaging in experimental work. FPs, which at first glance might seem impossible to solve, are often sparsely worded and provide little or none of the explicit information needed to solve the problem (Efthimiou & Llewellyn, 2007). FPs have been used in problem-solving activities but have gained the attention of the Mathematics Education community as activities in which mathematical modelling is worked on, as they are firmly rooted in a real-world context generally (cf. Lesh & Zawojewski, 2007). FPs are foremost characterized by the way in which their solutions are achieved. This method is known in the literature as the Fermi (estimates) method and entails the decomposition of the problem into a number of stringed together sub-problems, and then using common sense assumptions, estimates and educated guesses to engage in simple calculations to solve the original problem (Carlson, 1997). Students can generate many different ways of solving a FP following this method, depending on the variables of the problem on which the solver focuses. In this sense, FPs allow addressing one of the needs for learning problem solving identified in recent years, that of generating and discussing different ways of solving a problem (Tjoe, 2019). Although educational research on FPs generally emphasises estimation, it has been suggested that the activity of estimation can be replaced by other (classroom) activities to find the numerical information needed to solve the problem (Sriraman & Knott, 2009).

In addition, FPs have been portrayed as miniature-modelling problems capturing the essence of full modelling problems (Robinson 2008), having the advantage of being more well-defined and delimited real-world problems and hence less complex and more

manageable for both teachers and students. At the primary level, Peter-Koop (2009) found that students working with FPs successfully can generate their own models and use the Fermi method to estimate the number of cars in a motorway traffic jam. The students used a variety of strategies and developed new mathematical knowledge to arrive at their solutions. These results are consistent with those of Albarracín and Gorgorió (2019), who used a number of different FPs and also showed how the students could adapt their solutions strategies to new contexts. In addition, working with FPs has been shown to have a positive effect on the development of primary students' modelling skills, making them aware of the phases of modelling and develop modelling sub-competences such as simplification, mathematization, interpretation and explanation of real phenomena (Haberzettl et al., 2018).

FPATs – Fermi problems Activity Templates

In previous research we have discussed FPs both as integrators between the STEM disciplines and as facilitators for learning in the STEM disciplines (Ärlebäck and Albarracín, 2019). Based on our review of the research on FPs in the STEM disciplines, we identified the four types of mathematical activities that most commonly are used in determining the unknown but needed numerical values of important quantities to solve a given FP: *Guesstimation*, *Experimentation*, *Looking for data* and *Polling or Statistical data collection*. Below we briefly describe and provide examples of these four activities, and we discuss how these align with the problem-solving process of FPs.

Guesstimation is the activity of answering a (sub-)problem based on solely simple calculations involving educated guesses and estimates of the unknown quantities involved. The type of rough answer to a problem guesstimation results in, can be adequate and productive when working with ill-defined problems or when detailed solutions are not required (Shakerin, 2006).

Experimentation, with the original intent of Fermi to support the development of laboratory skills and experiment planning, is the activity to conduct an (physical) experiment and take measurements to determine adequate values for the relevant quantities needed to solve the given problem.

Looking for data is the activity to seek the relevant quantitative data needed using external records and sources, such as national statistical institutes resources, Wikipedia, or more topic specific resources as illustrated by Phillips and Milo (2009) in the project www.bionumbers.org (reliable and validated experimentally derived values of quantities relevant for research in biology). With respect to existing records and resources, FPs can be tools for critically evaluating such sources and data.

Polling or Statistical data collection means to engage in data collection and statistical analysis to get the values of the relevant quantities needed to solve an FP. Besides providing the values, engaging in data collection and statistical analysis has the potential to increase awareness and provoke a critical stance toward various problems in society and the environment. Such examples, as suggested by Sriraman and Knott (2009), are for instance the wastage of food or the fresh-water consumption.

Solving an FP using the Fermi method entails identifying several sub-problems to address and solve. Then, it is the subsequent solutions and coordination of these sub-problems that results in the solution to the original FP. Anderson and Sherman (2010) put forward a simple geometrical diagram representing the structure of such a solving process for the FP *How many hotdogs are consumed at the Major League Baseball (MLB) games each season in the US?* (see Figure 1).

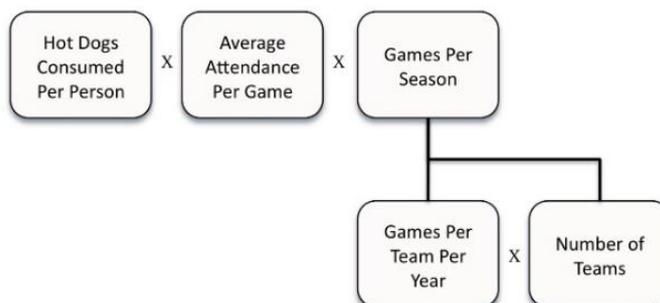


Figure 1. Structure provided by Anderson and Sherman (2010).

Inspired by Anderson and Sherman (2010), we expanded their representation by incorporating the four identified different types of activities and the structure of the solution process of an FP to a framework for design and analysis called Fermi Problem Activity Templates, FPATs (Albarracín and Ärlebäck 2019). In an FPAT, the intended activity (if used as a design tool) or performed activity (if used as an analytical tool) to solve a given sub-problem is denoted by a specific geometrical shape: Guesstimation (Ellipse); Experimentation (Trapezoid); Looking for data (Rectangle); Polling or Statistical data collection (Hexagon). One possible FPAT for the solution in Figure 1a (by Anderson & Sherman, 2010), with specific activities for each sub-problem, is given in Figure 2.



Figure 2. An FPAT based on the Figure 1 structure.

Setting, the teaching sequence and methods

The data we analysed in this paper comes from a collaborative research project involving a Catalan and a Swedish setting aiming at designing, implementing, and evaluating teaching sequences that introduce PSPTs to FPs as well as FPATs as didactical tools. The teaching sequences differed somewhat in their design in the two settings due to different circumstances and boundary conditions, and in this paper we only report on the Catalan setting and data. The Catalan teaching sequence consisted of three 3-hour

working sessions mixing brief lectures/instructions, individual problem solving as well as problem solving in pairs/groups, and whole class discussions. During these sessions 22 PSPTs worked with a series of four FPs and the following two problems:

- A. *How many toilet paper rolls are needed at school in a year? How much space do they occupy?*
- B. *How many ambulances are needed if we want to attend any emergency in less than 8 minutes in any place in Catalonia?*

Problem A is set in a known and tangible situation for the PSPTs. In contrast, problem B presents a complex situation about which the PSPTs have less experience. In a similar problem to problem B (optimally positioning rescue helicopters in a mountain area), Kaiser (2005) identified different approaches to the problem, based on the specific definitions of ‘optimal’ and solving strategies. The PSPTs worked on these tasks in parallel in five stages: (i) individually wrote and explained their proposed solution plans for the two problems; (ii) discussed their proposals in pairs and agreed on joint solution strategies; (iii) in pairs, identified and connected the curricular mathematical content and procedures needed to solve each of the problems; (iv) after receiving a 20 minute introduction to the FPAT framework, in pairs, created and wrote down their FPATs representing their solutions of the problems; and (v) reflected on their work so far and revisited stages (ii) and (iii). The PSPTs studied their fourth and last year of the primary teacher training degree and had previously studied 3 courses of mathematics and mathematics education, but not modelling or real-life problem solving. We collected various documents generated by the PSPTs during the teaching sequence, but in this paper, we only focus on the 22 FPATs developed for problems A and B in stage (iv).

Analysis

The analysis is based on the mathematical structure of the PSPTs’ solutions and the quantities (Thompson, 1994) they chose to mathematize and include in their solutions. To qualitatively characterize the FPATs the PSPTs produced our analysis focused on (a) to what extent the solutions mediated by the FPATs reasonably solve the problem in question; (b) what the key concepts and procedures the PSPTs used to structure their solutions around were; and (c) the number of sub-problems and what types of FPAT-activities the PSPTs proposed to use in solving these.

To characterize to what extent the produced FPATs in terms of the suggested division into sub-problems relate to the posed problems in a way that potentially lead to a viable solution, we looked at both logical consistency and accuracy. These two aspects (captured under the rubrics ‘Solve?’ and ‘Close?’ respectively in Tables 1 and 2 in the result) are binary descriptors (yes/no) of the PSPTs’ FPATs. With respect to logical consistency, we focused on whether the sub-problem structure provided actually lead to valid solution to the problem or not. For instance, the PSPTs might have misunderstood the task or context rendering it impossible for them to reach a reasonable answer, or that key and crucial variables are not considered in the solution. With respect to accuracy, we looked if the FPATs potentially could result in a sufficiently good rough estimate or

not, since although all important aspects and variables might be incorporated in the solution, there in addition needs to be enough detail specified at an adequate level about how to actually go about in producing the answer (FPATs with activities considered to be specific enough in this respect are indicated by an asterisk (*) in the # Sub-problems column in Tables 1 and 2).

It seems natural to assume that the number of quantities and sub-problems involved in solving an FP is a key element in determining its complexity (Greefrath & Frenken, 2021). Hence, to characterize the complexity of the FPATs, we identified and counted the number of sub-problems and the number of quantities (cf. Thompson, 1994) considered in the proposed solutions. By identifying which quantities the PSPTs focused on, we particularly noted what key concepts they used to structure and organize their solution and FPATs around. In addition, we also considered the nature of the activities proposed for each sub-problem and determined whether these really were suitable and viable for solving the respective sub-problem. There are instances in the data where PSPTs proposed activities that cannot be carried out in a realistic or meaningful way to provide the solution to the sub-problem they are supposed to solve. For example, the suggestion to search for the exact information about the distance an ambulance can travel in 8 minutes is not viable, but rather needs to be broken down into further sub-problems or found based on conducting an experiment.

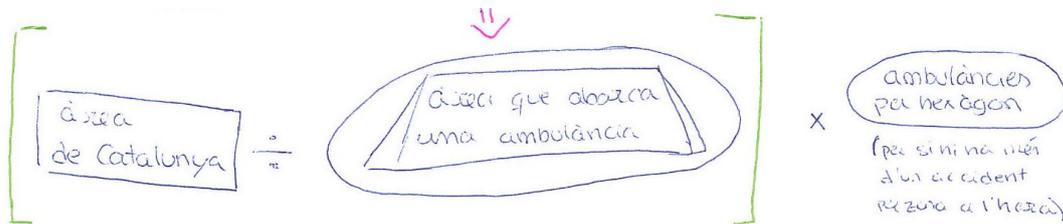


Figure 3. Pair 2's FPAT for problem B.

Figure 3 shows an example of a PSPT produced FPAT for problem B. Here, the whole area of Catalonia (looked up from some data source) is to be divided by the area that can be covered by an ambulance (without making clear what type of activity to use in arriving at this quantity: experimentation or guesstimation), and to multiply by the number of ambulances per area (a guesstimation). In our analysis we have characterized this FPAT as solving the problem, and that it considers the essential aspects that allow to obtain a reasonable rough estimate. The key construct used to structure the solution is the area covered by an ambulance in 8 minutes, and the solution involves three sub-problems and two main quantities (area and number of ambulances). Note however, that we consider that it is possible to refine this solution by introducing more sub-problems and to be clearer about the type of activities proposed to achieve more reliable results.

Results

The results, summarized in tables 1 and 2, show that the PSPTs produced FPATs that in principle adequately solved the problems in eight of the cases for problem A and in one

case for problem B. Two of the eight FPATs for problem A were explicit and precise enough to provide unambiguous solutions, whereas this was not the case for the one FPAT for problem B. The number of quantities considered in the PSPTs' solutions are greater in problem A (average=3.18) compared to in problem B (average=2.09). Turning to the identified key construct used by the PSPTs, a great variation can be seen in what was central to their solution structure in problem A, whereas in problem B, all FPATs revolved around the single construct of the how far an ambulance reaches in 8 minutes.

Focusing on the number of sub-problems, we see that although the number of sub-problems the PSPTs used in their FPATs on average are equal on problem A and B (4.45 and 4.36), and that in both problems the number of sub-problems is between two and eight, the distribution of sub-problems is quite different. The distribution is more centred around the mean for problem A (Std Dev 1.63) compared for problem B (Std Dev 2.49).

Pair	Solves?/ Close?	# Sub- problems	# Quant	Guess	Expr	Stat	Look f. data	Key construct
1	Yes/No	3	3	1	2	0	0	Consumption per toilet
2	Yes/No	5*	4	3	0	0	2	Times we go to toilet
3	No/No	8*	4	2	3	0	3	Length of paper used
4	Yes/No	4*	3	1	1	0	2	Rolls used per week
5	Yes/Yes	5*	4	3	0	0	2	Length of paper used
6	Yes/No	4	3	0	0	0	4	Rolls per week
7	No/No	5*	2	1	1	2	1	Time roll lasts per class
8	No/No	4	3	1	1	1	1	Time roll lasts per class
9	Yes/No	3*	3	1	1	0	1	Rolls per toilet
10	Yes/Yes	6*	4	2	1	0	3	Length of paper used
11	Yes/No	2*	2	1	0	0	1	Rolls per week
Mean	-	4.45	3.18	1.45	0.91	0.27	1.81	-

Table 1. Characteristics of the FPATs the PSPTs produced - Problem A.

If we look at the proposed types of activities in the FPATs, the pattern is similar for problem A and problem B in the sense that Looking for data is the most frequently used, followed by Guesstimation, then Experimentation and lastly (and not at all in the case of problem B), Polling (which is suggested by two pairs in problem A). However, it is notable that the relative proportions of the suggested activities within the two problems are quite different: in problem A these are much closer to one another compared to the corresponding proportions in problem B. When it comes to the activities the PSPTs proposed for each sub-problem and if these could be considered suitable for solving these, we found that this was the case in 16 of the FPATs (see *-markings in Table 1 and 2 respectively). When we look at the FPATs containing activities not corresponding to the sub-problem in an adequate way, four of these come from pairs 6 and 8.

Pair	Solves?/ Close?	# Sub- problems	# Quant	Gues s	Expr	Stat	Look f. data	Key construct
1	No/No	4*	2	2	0	0	2	Ambulance reach (AR)
2	Yes/No	3	2	1	1	0	1	AR
3	No/No	7*	3	3	0	0	4	AR
4	No/No	3*	2	1	0	0	2	AR
5	No/No	8*	4	4	0	0	4	AR + density
6	No/No	4	3	1	0	0	3	AR
7	No/No	6*	2	4	0	0	2	AR
8	No/No	7	2	0	4	0	3	AR
9	No/No	2*	1	1	0	0	1	AR
10	No/No	2*	1	0	1	0	1	AR
11	No/No	2*	1	0	0	0	2	AR
Mean	-	4.36	2.09	1.54	0.54	0	2.27	-

Table 2. Characteristics of the FPATs the PSPTs produced - Problem B.

Discussion and conclusions

Although we in this paper report an exploratory study involving only a small sample of work from 11 groups, the analysis shows the complexities inherent in solving FPs set in real and everyday contexts. The characterizations of the FPATs of the two problems A and B show the importance of choosing the FPs the PSPTs work on in the teaching sequence with great care – especially regarding the context of the FPs. Multiple factors revealed by the analysis indicate that many of the PSPTs’ difficulties in solving problem B are related to their lack of previous experience in the real world relevant to the problem. In contrast, the context of problem A connects to a more tangible situation to which they easily can relate. The disparity in contexts is reflected in the number of successfully proposed FPATs as well as the proportions of the different types of activities suggested in solving the sub-problems. This result in fact illustrates that the FPATs the PSPTs produced can help to identify the nature of the difficulties they have in solving a problem; since the variety in key constructs seems to be an indicator of the previous knowledge they have about the context of the problem. In addition, looking at the PSPTs’ produced FPAT using the characterization, can also be used as a tool that allows identifying the nature and level of knowledge that the PSPTs bring to a given problem. In other words, we suggest that this characterization can be useful in teacher training both to (a) design teaching sequences with FPs; and (b) function as a didactical tool for PSPTs and teachers in connecting and bringing out-of-school contexts into the classroom as well as using the FPATs to make different ways of solving a problem explicit (Tjoe, 2019).

Regarding the activities suggested by the PSPTs to solve the sub-problems, we acknowledge that working with and applying the FPAT-analysis is a new way of

thinking about this type of problem-solving for the PSPTs, and that they need more experience working with both FPs and FPATs. The fact that three of the pairs only proposed Guesstimation and Looking for data as activities solving their suggested sub-problems, make us think that it is necessary for the PSPTs to have more exposure and training in the design of experiments and data collection approaches. A greater variety of activity types open for more opportunities to connect to out-of-school contexts. It should be stressed that by the nature of an FP all sub-problems can be solved by Guesstimation, but whether this is a viable approach or not strongly depend on whether the solvers have the necessary extra-mathematical knowledge. Since we in this study unfortunately do not have access to this type of data, we cannot further analyze this aspect. However, this is an interesting venue for future research. We also notice that in future studies it would be interesting to ask PSPTs for even more information about the specific ways in which they intend to carry out their suggested experimental activities or data collection activities.

Even if the PSPTs were presented with FPs using real contexts, the PSPTs struggled to solve the problems, meaning that just situating the problems in everyday contexts were not enough to develop proper solutions. Although FPs are small-format modelling activities, they still require solvers to make a connection between the real context and the mathematical content. In this sense, FPATs make this connection explicit revealing themselves as promising tools, but it is still necessary for PSPTs to develop the skillset needed to identify mathematizable aspects of a real phenomenon. However, the results presented in this paper suggest to us that FPATs are promising tools for working on problems with real contexts with future teachers also at the primary level. FPATs reveal the structure of the model developed and the mathematical procedures to be implemented, establishing a specification that allows learning opportunities for PSPTs. More research is needed about how preservice teachers understand and use FPATs, as well as how teaching sequences need to include, exemplify, and connect to all four different activities in the FPAT, to further facilitate bringing out-of-school context into the classroom. However, we understand that a proposal based on FPATs has the potential to support teaching interventions with PSPTs about mathematical modelling.

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