Dividing cookies: what do students discern?

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This paper is on problem posing with 6-year-old students in Swedish preschool class. First, 77 students worked on a problem-solving task where they had to divide cookies equally amongst themselves. After that, the students were asked to pose a similar task to a friend. The focus of this paper is on similarities and differences between the initial problem-solving task and the tasks posed by the students. Almost all of the posed tasks were classified as mathematical tasks where the majority dealt with the same mathematical content (division) as in the initial problem-solving task. However, there were several tasks where other mathematics was needed to solve the task, as well as tasks where the context differed. One implication is that previous experiences and the circumstances in which problem posing is introduced seem to have an impact on the tasks posed by the students.

Introduction

The context of this paper is a longitudinal educational design research study in which problem solving and problem posing are used as a starting point for teaching mathematics with six-years-olds in Sweden. In Sweden, as well as in many other countries, problem solving is emphasized in the syllabus, aiming to educate students to become competent problem solvers (National Agency for Education, 2019). According to Niss and Højgaard (2019), problem posing is also to be considered a part of the problem-solving competence, implying that to become competent problem solvers, students should encounter tasks where they both solve and pose problems. However, there is more research on problem solving than on problem posing (Cai & Hwang, 2020), which is why we know more about students' ability to solve problem-solving tasks than their ability to pose (problem-solving) tasks. This is especially true for younger students, which is why the rationale for this study is to deepen our knowledge on young students' problem-posing abilities.

The empirical material in this paper is from one design cycle within the longitudinal educational design research study, in which the students worked on both problem solving and problem posing. Seventy-seven students from four classes first worked on a problem-solving task with the mathematical content *division*. After that, the students were asked to pose a *similar* task to a friend. The focus of this paper is on how the young students interpret *similar,* thus on similarities and differences between the initial problem-solving task and the tasks posed by the students. More specifically, the following question will be elaborated on: When asked to pose a similar task to a friend, what aspects of the initial problem-solving task are visible in the tasks posed by the students?

Problem posing

By incorporating problem posing as part of problem solving, students' problem-posing as well as problem-solving skills can be developed (Ellerton, et al., 2015; Palmér & van Bommel, 2020). Posing tasks is cognitively demanding (Cai et al., 2020) and offers challenges through the 'low-floor but high ceiling opportunities' (Zhang & Cai, 2021, p. 962). Further, when posing tasks, students get an opportunity to operate on different mathematical content that may develop their understanding (Brown & Walter, 2004).

Problem posing can be seen as an instructional activity and differs from asking questions as part of regular classroom discourse as well as from asking clarifying questions (Cai et al., 2020). The posing activity can take place before, during or after problem solving, where the students either are asked to generate a new task without a model to follow or to reformulate a given task (Silver, 1994).

In this study, the problem posing takes part after problem solving and the students are asked to pose a *similar* task which can be understood as reformulation of a given task (Zhang & Cai, 2021). This approach, where students are asked to pose a task relating to a given stimulus or a given situation, is sometimes referred as problem posing in a semi-structured situation (Stoyonova & Ellerton, 1996). Posing a similar task involves reflections on the original task and therefore reformulation can inform about the students' interpretation of what the original problem-solving task was really about (Carrillo & Cruz, 2016; Palmér & van Bommel, 2020). When reformulating, students can be asked to pose a task with a structure or a content similar to that of the original problem-solving task, or to pose a task that can be solved by using a similar method (Carrillo & Cruz, 2016). The students in the study presented here were not given any such explicit directions on how *similar* was to be interpreted.

Young students and division

Understanding division means understanding the relationship between the number of parts and the size of each part, which pre-schoolers often practise successfully in everyday situations where objects are to be divided equally. For instance, when sharing something between themselves, they are usually very careful that everyone gets the same amount or the same number. Such equal sharing is then the basis for both division and fraction, as both of these are about relationships between parts and the whole. Most often, 3- and 4-year-olds perceive cardinality as wholes but have more difficulty reasoning about subsets (Dehaene, 1997). A basis for understanding division is understanding that all subsets must be equal or consist of the same number of objects. When working on sharing in everyday situations, at early ages, one-to-one distribution is often used for distribution, supported by knowledge in addition and/or subtraction (Parmar, 2003).

A division like 12/4 can be contextualized in different ways, resulting in different ways to obtain the answer, in different kinds of division. The task ‘divide 12 cakes evenly over 4 bags; how many will there be in each bag?’ describes a partitive division, where the 12 cakes (dividend) are divided over a given number of bags (divisor) to obtain the number of cakes in each bag (the quotient). However, the task can be posed as a quotative problem: ‘12 cakes have to be put in bags of 4, how many bags do you need?’ The numbers of the dividend, divisor and the quotient are the same, the task is still 12/4, but now the numbers represent other parts in the problem (Table 1).

	Dividend	Divisor	Quotient
12/4=3	12	4	3
Divide 12 cakes evenly over 4 bags	12 cakes	4 bags	3 cakes in each bag
12 cakes have to be put in bags of 4	12 cakes	4 cakes in each bag	3 bags are needed

Table 1. Dividend, Divisor and Quotient in 12/4 = 3

Seventy years ago, research on division for young students (2nd grade elementary school) was conducted (Gunderson, 1955) describing differences in how young students worked on partitive and quotative division. Gunderson found that the students managed quotative division to a higher degree then partitive division. However, other research has suggested the opposite: that partitive division is easier for students as it follows the everyday experience of sharing and dividing things evenly and equally (Ching & Wu, 2021; Frydman & Bryant, 1988). In a study by Palmér (2008), 3-year-olds worked on partitive division. They had no problems when working on tasks like 9/3 but when faced with tasks like 9/4 they struggled with how to divide equally, as an equal amount or equal number.

Method

Educational design research encompasses the design and implementation of teaching as part of the research, with the aim of developing theories as well as new forms of instruction (Anderson & Shattuck, 2012). Thus, educational design research intends to be of value for both research and practice (Bakker, 2018). To develop theories that inform and guide the practice of teaching and learning, the studies are conducted in an iterative cyclic process of designing and testing interventions situated within an educational context. Each design cycle includes preparing for teaching, implementing the teaching, and finally, conducting a retrospective analysis of the teaching and learning (Cobb & Gravemeijer, 2008).

As mentioned, the empirical material in this paper is from one design cycle within the longitudinal educational design research study in which four preschool classes (6-year-old students) were involved. Based on several years of collaboration, the teachers in these classes were familiar with problem solving and problem posing, as well as the concept of educational design research and the aim of the study (Palmér & van Bommel, 2021). The teachers are educated as preschool teachers, which implies that they have completed a three-year university course in preschool teacher education. In line with the ethical guidelines provided by the Swedish Research Council (2017), the guardians of the students were given information about the study and consented to the participation of their child. All students participated in the lessons given by their regular teacher. After the lesson, documentation was gathered if consent was obtained.

The problem-solving and the problem-posing lessons

Two lessons were conducted within the design cycle. The two lessons were planned in collaboration between the researchers and the teachers on Zoom (because of the covid situation) and then documented in writing by the researchers and sent to the teachers. In the first lesson the focus was on problem solving. The students were divided into groups of three students in each group. Each group received 15 biscuits in clay, where the first task was to divide the biscuits equally between them (15/3, partitive division). One group at a time worked on the task. Then the teacher said that she also wanted to take part, whereby the students were asked to divide the 15 biscuits equally between four (15/4, partitive division).

In the second lesson, the focus was on problem posing. The students were reminded of the task they had worked on in the previous lesson and were asked to pose a *similar* task to a friend. This time they worked individually. The tasks posed by the students were documented on paper and the teacher helped with the writing if needed. The teachers also took notes. There were 77 students posing tasks but as one student posed two tasks, 78 tasks were analysed.

Analysis

The analysis focused on the paper-and-pen work from the students when asked to pose a similar task to a friend. When analysing reformulated tasks posed by students, context, structure and content are of interest (Carrillo & Cruz, 2016). Classification of posed tasks based on such aspects enables a focus on the connection between the posed tasks and the mathematical topic in the initial task, the structure of the tasks (alike or different questions) and the context of the tasks (same or different context). Based on a classification scheme developed by Carrillo and Cruz (2016), the scheme used in this study was inductive, developed to cover the different tasks posed by the students (Figure 1). First, the scheme focuses on the *content* and the tasks are divided into three groups: (1) tasks based on the same mathematical content (division) as the initial task, (2) tasks based on a different mathematical content to the initial task and (3) tasks without mathematical content. Then, the *structure* of the question in the division tasks is analysed: same as in the initial task (partitive division), other division questions, different from the initial task. Finally, an additional classification is made identifying the *context* in the posed tasks: whether the context is identical to the context in the initial task (cookies), or whether a different context (or no context at all) is offered in the task. This last classification is made for all the posed tasks.

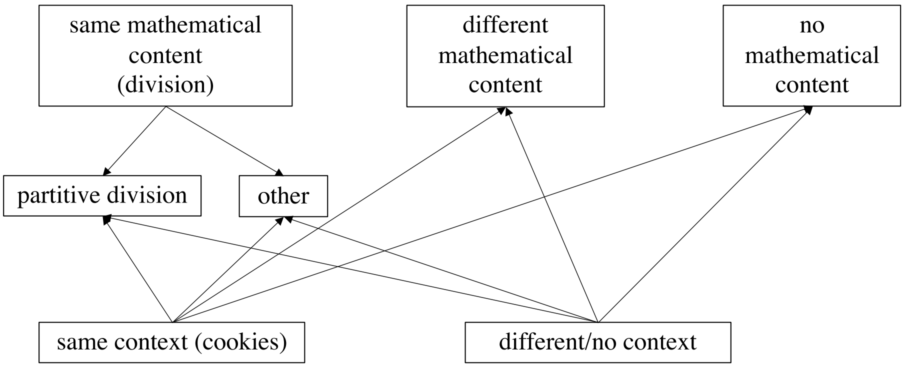

Figure 1. Classification scheme used to analyse the students' tasks

Results

In this section, we will first describe the results regarding the content of the tasks posed by the students, after which the results regarding the context will be dealt with. Table 2 shows a summary of the classification of the 78 tasks posed by the students. This classification is based on the steps described in Figure 1. Table 2 will be used for the upcoming analyses.

	Same mathematical content (division)		Different mathematical content	No mathematical content	Total
	Partitive division	Other			
Same context (cookies)	18	8	8	2	36
Different/no context	19	2	16	5	42
Total	37	10	24	7	78
	71				

Table 2. Result of classification of the students' tasks

Focusing on the content of the posed tasks

Of the 78 posed tasks, 71 were classified as mathematical tasks and 7 as non-mathematical tasks. Some of the non-mathematical tasks did have some mathematical elements; for example, the question posed to the picture on the left in Figure 2 was *Colour the shapes*, where typical mathematical shapes were used, in this case a circle, a triangle and a square. However, no mathematics is needed to solve the task. Another example of a non-mathematical task is the picture on the right in Figure 2, with the posed question *What letter is it?*

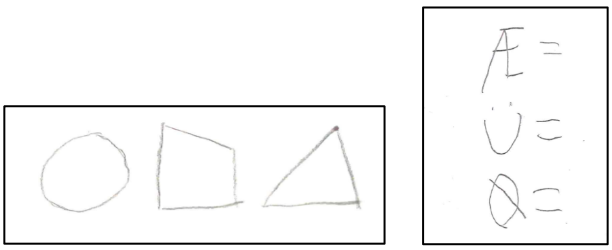

Figure 2. Example of two non-mathematical tasks

Of the 71 tasks where mathematics was needed to solve the task, 47 tasks dealt with division, as in the initial problem-solving task. In the remaining 24 tasks, other mathematics was needed to solve the task. One such content was addition; for example, the question posed to the picture in Figure 3 was *How many cookies will there be if you put all cookies together?* Another example of a task on addition was *You have nine rings and get two more, how many are there in total?*

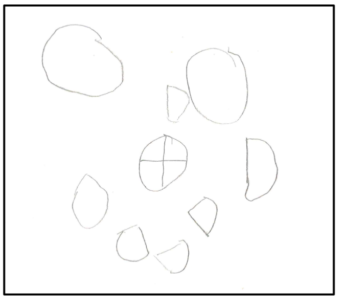

Figure 3. Example of a posed task with the mathematical content 'addition'

Multiplication, *How many cookies do you need if all three children will get two each*, counting, *Count the number of dots on the cookies*, patterns, *How does the pattern continue?* (Figure 4), and subtraction, *200-109, what is the answer?* are some examples of other mathematical content used.

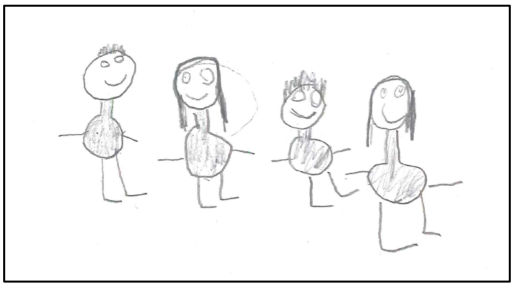

Figure 4. Example of a posed task with the mathematical content 'patterns'

Of the 47 tasks on division, most tasks were on partitive division as in the initial problem-solving task. Here the quotient could be a natural number, such as *Two children will divide 28 liquorice* or *12 pieces of dough to be divided by four persons*, but also a rational number, such as *Four chocolate bars have to be divided by three persons.* In most posed tasks it was inferred that the division should result in equally 'much' and some students stated this explicitly in their posed questions. Some of the students, however, focused on equally 'many' as in the task in Figure 5, *Divide all the cookies so each girl gets equally many.*

Figure 5. Example of a posed task focusing on equally many

In some of the posed tasks, the dividend was a whole number but represented as a rational number. One example is Figure 6 below, where the posed task is to divide four whole cookies and four half cookies between two persons.

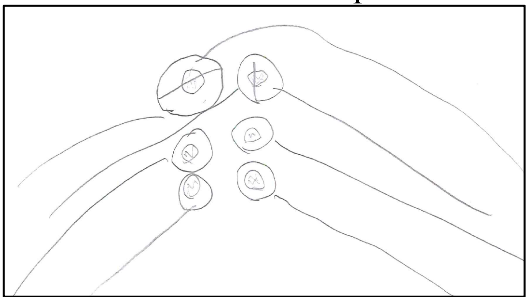

Figure 6. Example of 6/2. (4 + 4*(1/2)) /2

Ten of the posed tasks with the content division rephrased the question in such a way that it differed from the initial task. In one of those posed tasks, 15 cookies were to be divided between a dad, a mum and a child. The cookies were of different size and division came with a rule that dad should have the biggest cookies, mum should have the medium-sized cookies and the child was to get the smallest ones. Other tasks within this category used quotative division; for example, *How many quarters will you have if you divide three cookies?* or a combination of partitive and quotative division, *How many halves will each person get if they divide two whole cookies between the two of them?* Yet another example was *Two old men will divide 15 cookies, but they divide each cookie, so how many half cookies will they get each?*

Focusing on the context of the posed tasks

Cookies were present in 36 of the 78 tasks. This context was used in both posed mathematical (see for instance Figures 5 and 6 above) and non-mathematical tasks, for example *Find the divided cookie*.

The context used in the posed mathematical tasks (71) differed. In 34 of these tasks 'cookies' were used to operate on. Most other contexts were comparable to cookies, such as sweets or pie. One student explicitly stated to the teacher, *You know, you can actually divide other things as well and draw four ice-creams to divide amongst two persons.*

In some of the posed tasks other objects were used, resulting in divisions where division in fractions could be seen as problematic. One of the students seemed to recognize this and changed the context of his posed task. The task started with five 'Dracula-teeth' to divide between two persons, but then the student changed his picture, added a tooth and stated, 'so, now both can have three'. Another student drew a bathtub and stated that both children in the bathtub should have equally much water. This student then changed and posed another task.

Some of the tasks did not use any context at all, for example *200-109, what is the answer*. Also, in tasks in which students were to *draw equally many*, no specific context was used.

Discussion and implications

The aim with educational design research is to develop theories that inform and guide the practice of teaching and learning. As mentioned in the introduction, we know less about young students' ability to pose (problem-solving) tasks than their ability to solve problem-solving tasks (Cai & Hwang, 2020). In the design cycle presented in this paper, problem posing took part after problem solving and the students were asked to pose a *similar* task that can be understood as a reformulation of a given task (Zhang & Cai, 2021). The question focused on is what aspects of the initial problem-solving task are visible in the tasks posed by the students? The interest in this question is two-fold, partly relating to the ability of these young students to pose tasks and partly to their reflections on the original task. The tasks posed by the students inform us about the students' interpretation of what the original problem-solving task was really about (Carrillo & Cruz, 2016; Palmér & van Bommel, 2020).

As presented in the results, only 7 of the 78 posed tasks were non-mathematical tasks. Thus, the students seemed to interpret problem solving as a mathematical activity. This differs from a previous study by Di Martino (2019) where students in kindergarten and 1st grade (5–6-year-olds) most often did not refer to mathematical problems when being asked about problem solving. In Di Martino's study, the students referred to everyday life problems. An important difference between Di Martino's study and the study presented here is that problem posing in our study was closely connected to problem solving. However, in the study by DiMartino (2019) students in grade 3 and 5 were studied as well, and they had the same conditions as the students in kindergarten and 1st grade. Only grade 3 and 5 posed mathematical tasks. Thus, not age but previous experiences and the circumstances in which problem posing is introduced seem to have an impact on the tasks posed by the students.

Regardless of how the students interpret *similar*, their posed tasks show their experiences of mathematics. There were several posed tasks on mathematical content that is fairly uncommon in preschool class, for example multiplication. Another example is shown in Figure 6, where the posed task is to divide four whole cookies and four half cookies between two persons – thus, a more complex variation of 6/2: (4 + 4*(1/2)) /2.

In this study, 47 of the 71 mathematical tasks (where mathematics was needed to solve the task) dealt with division, as in the initial problem-solving task. Thus, the content of the original task was discerned by a majority of the students. For the remaining 24 tasks, other mathematics was needed to solve the task. This does not, however, imply that these students did not discern division as the content of the initial task, rather it may imply that *similar* is interpreted differently by different students. What aspect of the initial task is to be similar in the posed task? As the students were not told how to interpret *similar*, some students may have interpreted *similar* as a task about cookies, others as a task on division and yet others as a mathematical task in general. Thus, for example, if a teacher wants to know how the students experience division, this needs to be made explicit when introducing problem posing. Or, if a teacher wants to know what the students themselves discern in the problem-solving activity, the problem posing ought to be as open as in this design cycle. Another pedagogical recommendation when working with open problem solving, as in this design cycle, is to explore the diversity of the posed tasks with the students – what do they discern as similar and different between the tasks posed by their classmates and the initial task?

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