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Christer Bergsten, Eva Jablonka, Manya Raman

Evaluation and Comparison of Mathematical Achievement: Dimensions and Perspectives

Proceedings of MADIF 8

The Eighth Swedish Mathematics Education Research Seminar
Umeå, January 24-25, 2012

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Preface

This volume contains the proceedings of *MADIF 8*, the Eighth Swedish Mathematics Education Research Seminar, held in Umeå, January 24-25, 2012. These seminars, organised by the Swedish Society for Research in Mathematics Education (SMDF), aim at enhancing the opportunities for discussion of research and exchange of perspectives, amongst junior researchers and between junior and senior researchers in the field. The first seminar took place in January 1999 at Lärarhögskolan in Stockholm and included the constitution of the SMDF. The second meeting was held in Göteborg in January 2000, the third in Norrköping in January 2002, the fourth and fifth in Malmö in January 2004 and 2006, respectively, and the sixth and seventh in Stockholm in January 2008 and 2010, respectively. Printed proceedings of the seminars are available for all but the very first meeting.

The members of the 2010 programme committee were Christer Bergsten (Linköping University, chair), Johan Häggström (University of Gothenburg), Eva Jablonka (Luleå University of Technology), Kristina Juter (Kristanstad University), Manya Raman (Umeå University) and Andreas Ryve (Mälardalen University). The local organiser was Tomas Bergqvist (Umeå University).

The programme of *MADIF 8* included two invited plenary lectures (Peter Nyström and Yoshinori Shimizu), one plenary panel (M. Kathleen Heid, William McCullum, Tamsin Meaney, and Eva Jablonka who also acted as moderator), 21 paper presentations, and 15 short presentations. As the research seminars have sustained the idea of offering formats for presentation that enhance feedback and exchange, the paper presentations are organised as discussion sessions based on points raised by an invited reactor. The organising committee would like to express its thanks to the following colleagues for their commitment to the task of being reactors:

Mike Askew, Maria Bjerneby Häll, Lisa Björklund Boistrup, Anneli Dyrvold, Laura Fainsilber, Peter Frejd, Gunnar Gjone, Ola Helenius, Ingemar Holgersson, Kristina Juter, Per Nilsson, Hanna Palmér, Per-Eskil Persson, Manya

Raman, Eva Riesbeck, Frode Rønning, Håkan Sollervall, Ravi Subramaniam, Hans Thunberg, and Jorryt van Bommel.

In this volume the two plenary addresses, 16 research reports (papers), and 12 short presentations are included. The plenary addresses are published by invitation. In a rigorous two-step review process for presentation and publication, all papers have been peer-reviewed by at least three researchers and all short presentations by members of the programme committee. Since 2010, the MADIF Proceedings have been designated scientific level 1 in the Norwegian list of authorised publication channels available at http://dbh.nsd.uib.no/kanaler/.

The editors are grateful to the following colleagues for providing reviews: Mette Andresen, Mike Askew, Anette Bagger, Ewa Bergqvist, Tomas Bergqvist, Ole Björkqvist, Lisa Björklund Boistrup, Gerd Brandell, Martin Carlsen, Anneli Dyrvold, Andreas Ebbelind, Sharada Gade, Gunnar Gjone, Simon Goodchild, Stefan Halverscheid, Örjan Hansson, Ola Helenius, Kirsti Hemmi, Mikael Holmquist, Ilana Horn, Johan Häggström, Paola Iannone, Uffe Thomas Jankvist, Maria Johansson, Monica Johansson, Kristina Juter, Gulden Karakok, Angelika Kullberg, Troels Lange, Håkan Lennerstad, Stephen Lerman, Thomas Lingefjärd, Tamsin Meaney, Morten Misfeldt, Lars Mouwitz, Per Nilsson, Mikaela Nyroos, Hanna Palmér, Per-Eskil Persson, Jöran Petersson, Kerstin Pettersson, Chris Rasmussen, Mikaela Rohdin, Helena Roos, Andreas Ryve, Frode Rønning, Joakim Samuelsson, Håkan Sollervall, Erika Stadler, Ravi Subramanian, Attila Szabo, Roger Säljö, Eva Taflin, Anna Teledahl, Kjersti Waege, David Wagner, Keith Weber, Michelle Zandieh, and Magnus Österholm

The organising committee and the editors would like to express their gratitude to the organisers of *Matematikbiennalen 2012* for financially supporting the seminar. Finally we would like to thank all participants of *MADIF 8* for sustaining their engagement in an intense scholarly activity during the seminar with its tight timetable, and for contributing to an open, positive and friendly atmosphere.

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Evaluation and Comparison of Mathematical Achievement

Peter Nyström
Umeå university

This paper presents a few images of mathematics achievement in Sweden, but also some reflections on how achievement can be conceptualised and assessed. Overall the images paint a rather negative picture of mathematics achievement, including a negative trend. One question raised is if students acquire other mathematical competences instead? The conclusion is that evidence of this is hard to find and that achievement interpreted as learning outcome needs more serious attention and less excuses.

Introduction

This presentation will primarily focus achievement in mathematics and how measures of achievement can be interpreted and used. In order to understand and take a critical look at the evaluation of achievement there are many general assessment issues that can be considered, and some of them will be summarized briefly in this paper. Educational assessment is an area receiving a lot of attention, though it seems as this attention has grown over the last decade, at least in Sweden. One aspect of educational assessment that has received increasing attention is the day by day and moment by moment assessment taking place as part of instruction, often known as formative assessment. An increasing body of research points to the potential of developing this practice in the classroom (Wiliam, 2011). Another aspect, and the one focussed in this paper, is the summative characterisation of learning outcomes. In Sweden we have seen a political demand for grades earlier in school, more national tests, and an overall increasing use and interest of achievement data for evaluation and inspection of schools. There is no doubt that achievement measures play a role in the political discourse about school and this seems to be particularly relevant for mathematics in Sweden. The reason is probably the strong focus on mathematics in international comparative studies as well as national evaluation, which in turn build on the central position given to mathematics not only because of its necessity in the defence of democratic values (democracy requires people to be able to have an informed opinion in matters were numbers and mathematics relations do play an important role) but even more because of the assumed relationship between mathematics and economic growth. Furthermore, achievement in mathematics has become a subject of public debate in Sweden due to the poor achievement results found in most attempts to measure achievement in mathematics, both from an international perspective (i.e. in comparison with achievement levels in other countries) and from a national perspective (i.e. in relation to national goals). Poor results and negative trends seem to attract the attention of media and policy makers much stronger than anything else.

Debates about achievement are stained by simplified uses and prejudiced interpretations and it is vital that mathematics educators, assessment specialists, and other experts are prepared to criticize and publicly deconstruct the misuse of assessment results. However, such a critical stance does not necessarily lead to the conclusion that measuring achievement and performing large-scale assessment is all bad. On the contrary, I claim that well designed and well executed measures of achievement can contribute to an evidence base for decisions on education in mathematics. There is however a need for sound interpretations, critical perspectives and reminders of both affordances and constraints in the methods used to assess achievement.

We must also acknowledge that policy- and decision-makers need some evidence of the effects of reform and school-policy. It is up to the mathematics education community to inform these decision-makers of the limitations that are inherent in every research design, including large-scale assessment of mathematical achievement. Achievement results are important tools for policy- and decision-makers in school and the assessment of achievement is therefore important for anyone in the mathematics education community who wants to make a difference.

In this presentation, aspects of achievement will be addressed in four different sections. The first section deals with definitions of achievement and the second section raises some important issues about assessment in general. The third section presents some images of mathematical achievement in Sweden and discusses them briefly, and the fourth and final section raises the question of possible alternatives to conventional views on achievement: What are students learning if they are not learning what we define as achievement?

What is achievement?

Achievement is a word that is used in a technical sense when we are talking about educational outcomes, but it is also a word that is used in everyday language. As all common words there is a risk that everybody finds the meaning of it obvious, making definitions superfluous. Everyone believes that the meaning is clear and shared, but this might not be the case. In addition, the word "achievement" does not have an evident translation to all languages (e.g. to Swedish), which makes it even harder to use in international settings. One obvious starting point in the quest for an understanding of this concept is to look for a lexical meaning of the word.

Achievement as accomplishment

The definition found in Wikipedia in January 2012 is that an achievement is similar to an accomplishment. This implies that when you have achieved something you have done something extraordinary, compared to what others have done or compared to your own ability. According to Oxford dictionaries (http://oxforddictionaries.com/), achievement is "a thing done successfully with effort, skill, or courage: to reach this stage is a great achievement". Achievement can also according to the same source represent "the process or fact of achieving something: the achievement of professional recognition; assessing ability in terms of academic achievement; a sense of achievement". It seems as though achievement in general refers to an accomplishment such as climbing a mountain or swimming ten miles in open sea. In the context of school mathematics, "achievement" would then mean to actually do something impressive. To fail mathematics would then of course not be an "achievement" in this meaning, and a lot of the learning outcomes found in studies of mathematical learning in Sweden and other countries are actually no achievement at all.

Achievement as learning

In a recent report, the Student Learning, Student Achievement Task Force commissioned by the National Board for Professional Teaching Standards in the US argues that student learning and student achievement are closely related concepts (Linn, et al., 2011). However, the task force also argues that "while the two terms are often used interchangeably, they convey profoundly different ideas, particularly as they relate to teaching" (p. 28)

> In brief, student achievement is the status of subject-matter knowledge, understandings, and skills at one point in time. The most commonly used measure of student achievement is a standardized test. Such standardized assessments measure specific areas of achievement--for example, the extent to which a 3rd grader has mastered the English/language arts standards in his or her state or district--and are best understood as one measure of a subset of a body of skills or knowledge. (p. 28)

Learning is defined differently.

> Student learning is the growth in subject-matter knowledge, understanding, and skills over time. In essence, it is an increase in achievement that constitutes learning. Central to this notion of learning as growth is change over time. Knowing whether student learning has occurred, then, requires tracking the growth in what students know and can do. It is only by comparing student mastery at successive points in time that the nature and extent of learning can be gauged. Student learning is also reflected in a broad array of outcome

> measures, including attendance, participation, engagement, and motivation. (p. 29)

It is interesting to see that two differences between achievement and learning are highlighted: (1) *Achievement* reflects one point in time, while *learning* has to do with growth and change over time, and (2) a*chievement* focuses a subset of a body of skills or knowledge, while *learning* covers a broad array of outcome measures, including attendance, participation, engagement, and motivation. This could mean that if the measurement of achievement was performed repeatedly and if it would focus a wider range of outcome measures, including motivational aspects (like the questionnaires used in PISA and TIMSS), the difference between "achievement" and "learning" would more or less disappear. The first is a matter of quantity, not character, and the second is a matter of quality in the assessment of learning and (even more importantly) achievement. It seems like the argument going against what is here called "standardized testing" and equalled to achievement is comparing poor large-scale testing with ideal classroom assessment.

It is also noteworthy that achievement is said to be "measured" while learning is said to be "gauged". This use of words could be interpreted as a rhetorical strategy aiming at discrediting what you are opposing and promoting your own values. The dichotomy between these two concepts is not as clear as the authors seem to be claiming since "gauging learning" is described very similar to what in other contexts would be called "repeated measurement". What is described is poor measures of achievement. The problem focussed is the consequences of assessment rather than the quality of inferences (cf. Messick's (1989) concept of validity presented below).

Achievement as learning outcome

An additional specification of the relation between achievement and learning is to view achievement as learning outcome. This definition implies first of all that achievement is not so much about potential learning as it is of actual evidence of learning. This might seem obvious, but it is not uncommon that learning is taken for granted and that learning is assumed because of time spent on what seems to be activities promoting learning. Consequently, one aspect of achievement is that it is about acquired learning. Secondly, viewing achievement as equivalent to learning outcome implies that achievement is not just a score on a specific test. Learning outcome is instead the concept or latent variable that the test is aiming at. A simple illustration of this is the teacher who wanted to know if his students mastered subtraction of two-digit numbers. The teacher prepared a test with five items, all about subtraction of two-digit numbers. One student answered all five items correctly. What does that result mean? If achievement is seen as test result, the student is categorised as proficient and all is well. If achievement is seen as learning outcome, a more problematic and not so obvious image emerges of what this student knows and can do in this domain. All we know is actually that the student answered the teacher's five questions correctly, at this particular time and place. This opens for a discussion about error and interpretation and a discussion of how to construct tests that actually reveal what we are interested in and reveal something that can be used for inferences about learning outcome.

Achievement as test results focuses on the visible and achievement as learning outcome focuses on the invisible, the complex learning goals that the curriculum and instruction are aiming for. I hypothesise that there are two fundamental beliefs about achievement.

Assessment of the visible	**Assessment of the invisible**
• Focus on the observed • Behaviours • Single tasks contain a lot of information • Secure conclusions • Manifest variables	• Focus on something that cannot be observed • Competences • Single tasks have little information value • Unsecure conclusion • Latent variables

The view to the left is dominating in assessment in higher education, and possibly to a large extent in examinations in mathematics, since the test defines whether you pass the course or not. The view to the right is for example represented in TIMSS and PISA, has theoretical advantages and actually models the situation better.

Borsboom (2008, p. 30) captures the idea of latent variables:

> When we treat a variable as observed, we mean nothing more than that we assume that the location of a person on that variable can be inferred with certainty from the data. When we treat a variable as latent, we mean that the inference in question cannot be made with certainty. It is important to see that this formulates the distinction between latent and observed variables as a purely epistemological distinction. (Borsboom, 2008, sid. 30)

Even though the application of latent variable theory in psychology often builds on a realistic view of entities and theory, there are approaches using latent variables that are fully compatible with a constructivist view (Borsboom, Mellenbergh, & van Heerden, 2003).

Conclusion about 'achievement', operational definition?

Achievement is thus best viewed as representing learning outcome and achievement can be assessed as a latent variable. This implies that assessing achievement means attempting to construct a picture of the invisible through written tests and other modes of assessment, that assessment results are always somewhat vague, preliminary, possible to question, etc., and that the quality of the picture can be affected, it can be better or worse, the information can be more or less credible. The aspect of quality in the evaluation of achievement (and educational assessment in general) is briefly covered in the next section.

How can achievement be measured?

The measurement (or assessment, I treat these terms as synonyms) of achievement can of course take many forms and use many different modes of assessment. There is no fundamental rule saying that multiple-choice items must be used or that the number of students participating has to be large. It is of course not within the scope of this paper to discuss the variety of approaches used in probing achievement, but there are a few fundamental aspects that need to be considered in every form of assessment. I claim that in order to answer the question of how to assess, and in order to evaluate any assessment, four aspects need to be considered. This simple framework states that in order to construct an assessment or in order to critically examine and interpret an assessment we need to answer four questions (Nyström, 2004):

1. What is the purpose of the assessment?
2. Criteria for good assessment?
3. What is being assessed?
4. How are students expected to express what they know and can do?

Purpose (1) is of course fundamental, and this is not unique to assessment. If the purpose is unclear we can neither now how to optimize the design of the assessment, nor evaluate the credibility of the assessment results. Depending on the intended use of assessment results, different modes of noticing and communicating results can be used. For summarising achievement a short summative result will be useful, but for formative purposes a much richer and analytic statement about the performance is needed.

Criteria for good assessment (2) are best captured in the concept of validity. There is a huge literature on validity and the concept has developed over the years, and become more complex. One major contribution to the development of the concept was made by Samuel Messick who presented a unified concept of validity incorporating consequences of assessment. According to Messick (1989, p. 13):

> Validity is an integrated evaluative judgment of the degree to which empirical evidence and theoretical rationales support the *adequacy* and *appropriateness* of *inferences* and *actions* based on test scores or other modes of assessment.

He clarifies that the terms "test" and "test score" should be understood generically, as including "any means of observing or documenting consistent behaviours or attributes" (p. 13).

Messick sets the scene for a view of quality including both the quality of the inferences from assessment results, i.e. what are the results telling us beyond a test score and beyond simple conclusions based on the observed results, and also including the consequences of assessment. This means that the assessment must be valued not only based on what evidence of learning it can substantiate, but also on the wanted and unwanted consequences when it comes to e.g. use of test results. In addition, Messicks theory stresses the crucial role of the purpose of assessment in order to claim validity. Furthermore, the theory is open for the trade off that can be found between different aspects of validity. For example, optimizing reliability in an assessment, which is equal to minimising random error, might make it necessary to lower the alignment between assessment tasks and curriculum. To represent complex competences in the curriculum, and at the same time reach high levels of reliability, can be almost impossible.

I find it important to acknowledge that important theories and ideas about assessment come from at least two different origins. One origin is assessment in educational settings and another is the attempts to measure psychological features. During the last 100 years earlier domination by psychometrics (psychological measurement) has been increasingly balanced by a focus on educational assessment (learning perspective), and hopefully we can make good use of both. There has been a development from psychological measurement to educational assessment (Shepard, 2000), from statistical theory towards a theory of assessment in mathematics (Webb, 1992). Ideally this development will incorporate useful ideas from different perspectives rather than discard concepts because they are believed to be exclusive to a certain perspective. One example is the concept of reliability, which is strongly connected to a psychometric approach and therefore often not considered in the educational context. However, the negative view of this concept can to some extent be contributed to a confusion between the concept of reliability and measures of reliability. If achievement is learning outcome and we acknowledge that observed results are not necessarily stable and "true", lack of reliability is a threat to proper conclusions and use of assessment results in every assessment. Furthermore, understanding different perspectives can enable a deeper understanding of results and designs. An example from mathematics is the kinds of problems were students can choose different ways towards a solution. Different students make different choices, depending on their personal preferences and on their level of understanding. For the "educationalist" this is a rich problem giving many opportunities for students to show what they know and can do. For the "psychometrician", this is a biased task, and should be changed in order to measure the same thing for everybody.

In addition to the purpose of the assessment and theories and frameworks underpinning the general issue of quality in assessment, the question of what to assess and the question of the character of what students have learned , need to be addressed. Both of these aspects are highly subject specific, i.e. in the development of assessment in mathematics, the specific character, culture, content and norms of mathematics need to be taken into account.

What is being assessed (3) is a matter of learning objectives. One common critique of achievement tests is that they are causing the negative behaviour in teachers and students known as “teaching to the test”. Teaching to the test is however not a problem if the test is aligned with curriculum. Therefore, curriculum alignment (Webb, 1997) is central and an important question in all assessment is how well the questions asked represent what students have been given the opportunity to learn. This means that learning objectives relevant to assessment is not only about what is formulated in written documents but even more about the emergent (Graybill, 1998) or implemented (see e.g. Mullis, Martin, Ruddock, O'Sullivan, & Preuschoff, 2009) curriculum.

How students are expected to express what they know and can do (4) is closely related to the question of what is being assessed, but the importance of this epistemological aspect motivates a category of its own. One example of this is that immediately after a learning sequence, students can be expected to have detailed knowledge of a domain. But what is reasonable to expect in an achievement test taking place further away from the actual learning sequence? Are long-term traces of learning something else than the short-term results? Perhaps they must be? Another example is the way students are expected to show what they have learned. Is it enough to be able to actually solve a problem in mathematics or do you always have to able to explain what you did and why?

We must bear in mind that the four aspects presented above, which are fundamental to assessment in all its applications, are related of each other. Putting a lot of effort into representing content and epistemology might conflict with the possibility of making reliable categorisations of students because of time restrictions or because of difficulties in making the judgement of student work stable in the sense that it is possible to repeat.

These aspects of assessment quality are equally important for all assessments, whether we call them “formative”, “summative”, “diagnostic”, or anything else. With these aspects in mind we now turn to some images of mathematics achievement in Sweden.

Images of mathematics achievement in Sweden

Presenting images of achievement in Sweden means by necessity to choose a few out of all possible candidates. The sample presented here is chosen based on requirements to be of a fairly large scale and to have something to say about trends. Each contributing source, and the results from that source, is described very briefly. The images of achievement come from grades in mathematics, national tests, tests at the entrance of tertiary education, and international comparative studies (TIMSS). These sources represent a variation in purpose and character as well as "stakes", i.e. the importance of results to the individual being assessed.

Grades

Grades are given at different points in the Swedish school system and the purpose is varying and also subjected to public debate. The purpose can be seen as information about achievement in relation to national curricular goals to students and parents, evaluation of schools on an aggregated level, and a basis for selection to higher studies in the final grade of comprehensive school and in upper secondary school. Stakes are high, and this is particularly the case in upper secondary school where students need good grades in order to compete for a place in prestigious programmes at the university.

The character of Swedish grades is that they are criterion-referenced, with grades being summative statements about how far each individual student has reached in relation to national assessment criteria. One example from the recently implemented new grading criteria is that for the grade C in upper secondary school mathematics (Mathematics 1a, the initial course for students in natural science and technical programmes):

> Students can formulate, analyse and solve mathematical problems. These problems involve **several** concepts and require **advanced** interpretations. In their work students re-express and transform realistic problem situations into mathematical formulations by **choosing and** applying mathematical models. Students can in a **simple** assessment evaluate the plausibility of their results, and also that of selected models, strategies, methods **and their alternatives**. (Skolverket, 2012a, p. 14; emphasis in original)

By the end of a course, this is part of what is expected of students in order to get the grade C, and since receiving a grade at least theoretically is a measure of learning outcome, it can be viewed as a measure of achievement. From 2011, grades are given in a scale from A-F. National assessment criteria are formulated for grades A, C and E, and students not reaching the level defined by E are considered not passing the course (grade F). When grades are used as a selection instrument for admission to higher studies, and when they are used for evaluation of school results, the grades A-F are quantified by a simple measurement model. The highest grade (A) is given a score of 20, the second highest (B) a score of 17.5 etc., down to the lowest passing grade (E) which is given a score of 10. The grade representing not passing (F) is given a score of zero.

A mathematics grade average for a cohort of students is calculated using these numbers, and the national mathematics grade averages for the period 1998-2011 are presented in Figure 1 (upper curve).

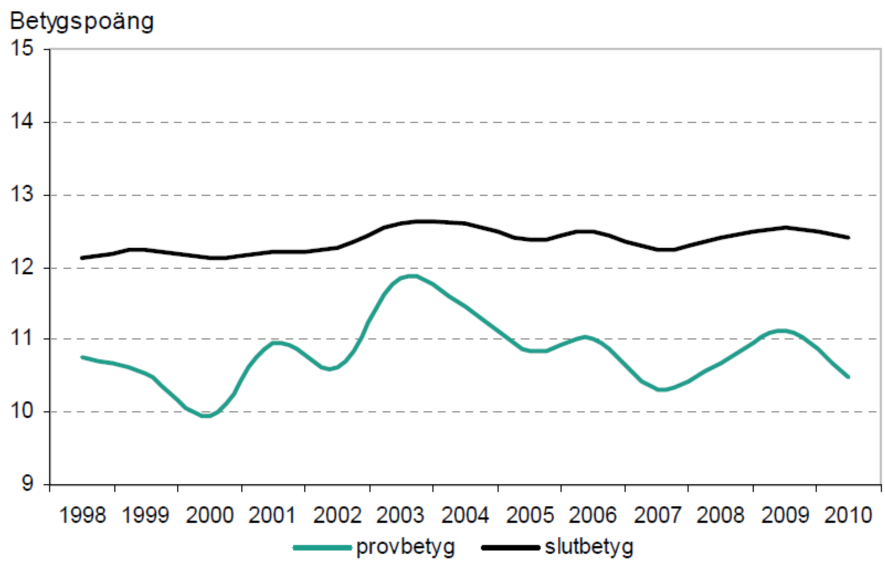


Figure 1. Mathematics grade average for the test-grade of the national test in mathematics (lower curve) and course grade in mathematics in the final year of comprehensive school. Source: (Skolverket, 2011)

Two features of Figure 1 are worth commenting. The first is that the level of achievement, measured as mathematics grade average, is fairly low. The average of about 12.5 is modest in a scale from 10 to 20. Furthermore, the mathematics grade average is low in comparison to other subjects in the Swedish comprehensive school. This result could indicate tougher grading criteria in mathematics compared to other subjects, poorer performance in mathematics, or a combination of the two. The second observation is that course grades are higher than test grades, meaning that on average students get a higher course grade than the grade implied by the national test. This discrepancy can be explained by the fact that teachers are supposed to use all information they have, not just results from national tests, when grading their students. There is however an imminent risk that what students know and can do at the end of a course, which can be seen as what the achievement grades are supposed to reflect, are confused by what students have done in relation to specific sub-domains during the course. This confusion makes grades rather difficult to interpret.

A present Swedish debate questions the reliability and trustworthiness of grades as measures of learning outcome. As a part of that discussion, research has pointed to grade inflation in Sweden, even though the degree of such inflation is disputed. From Figure 1 we can conclude that the grade point average has not changed substantially from 1998 to 2010 in mathematics in the final year of comprehensive school. However, inflation means that you get less value for your money and translated to grades this could mean that the same grade actually represents lower achievement. The question is whether the mechanisms of grading can calibrate the scale so that grades can be used for evaluation of achievement over time?

National tests

In the presentation of grades as measures of achievement in mathematics we have already touched upon national tests in mathematics. Sweden has a long tradition of giving teachers responsibility for grading students. There are no formal examinations in comprehensive school and upper secondary school but teachers are supported in their grading by compulsory national tests. Until recently, the purpose of these tests has been just as much to influence schools and teachers in the direction of the curriculum as to supply support for grading students. Currently, the official purpose of national tests in Sweden is however almost entirely focussed on supporting fair grading and on supplying data for evaluation on local and national level. Future consequences of this change are yet to be seen.

Stakes are low by definition, since individual students' grades are not intended to be decided by their results on the national test. However, due to the character of end-of-course-tests and due to the value that teachers and students, but also policy-makers and school-inspection, are giving these tests, they have in practice fairly high stakes for many students.

The character of the national tests in mathematics is that they are dominated by a written test but supplemented by an oral part. The tasks in the written test are partly of a multiple-choice or short-answer type and partly requiring extended answers. These tests have proven to be balanced in relation to the national curriculum (Boesen, 2006), but are not specifically designed to measure trend.

Figure 2 shows the percentage of students not passing the national test in mathematics for the final year of comprehensive school (year 9). According to these results, over the period 2004-2011 at least 12 percent of the students have not reached the lowest level defined. Even though trends are not generally observed in national tests in Sweden, a tendency towards increasing percentages not passing the test can be observed for this particular test.

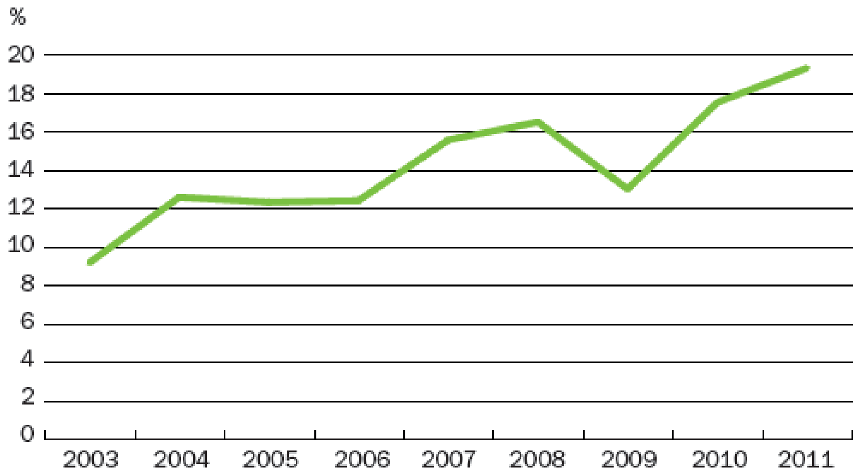


Figure 2. Percentage of students not passing the Swedish national test in mathematics, grade 9. Source: (Skolverket, 2012b)

To make the national tests more useful for longitudinal research would require changes in the tests. For example, assessing fewer content areas would make it justified to make inferences on how well these areas are mastered. Furthermore, equating tests with the help of anchor items would make it possible to study changes in attainment over time for individuals and groups.

Changes like these would increase the usefulness of national tests for longitudinal research but they would most likely have a negative effect on the validity of these tests for other purposes. In particular the most important purpose, which is to support teachers in their grading of students, would suffer because fewer content areas would make the tests less representative of the course and possibly lower the credibility of the tests in the eyes of the teachers, and repeated anchor items can cause secrecy problems and will occupy a substantial part of the limited testing time.

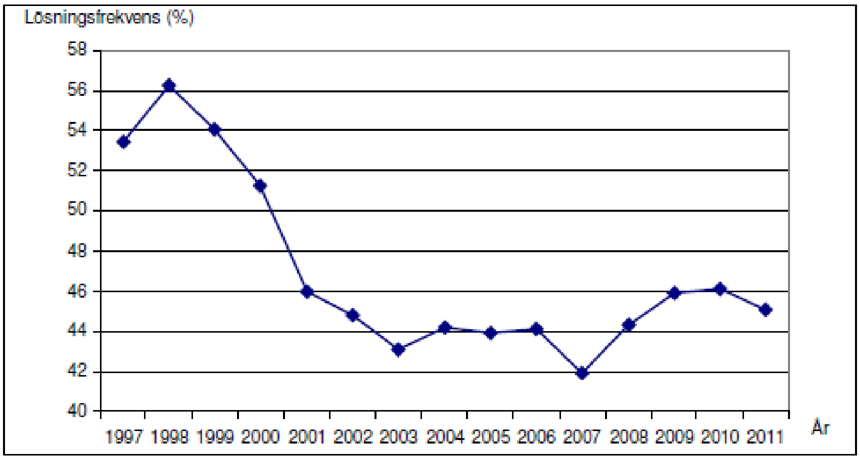
Tests at the entrance to tertiary education

Two examples of tests given at the entrance to higher education will be discussed briefly. The first test is a diagnostic test in mathematics given to students who have just started their first course in mathematics at a technical university. This test has a long history and can give some insight into changes over time. The other test is used for selection to higher studies and has recently changed to include more mathematics.

Diagnostic tests for students in mathematics

Several mathematics departments in Sweden have a long tradition of using diagnostic tests for students beginning their studies in mathematics. These tests have an interesting element of comparability over time since they have been virtually unchanged for a number of years. One example is the diagnostic test in mathematics used at the Royal Institute of Technology (KTH) in Stockholm.

The purpose of this particular test is not stated explicitly in the report from 2011 (Brandell, 2011) but it has definitely highlighted the level of basic knowledge among beginners at university and also the big variation, which can be seen as an important input for teachers of mathematics in this particular institution. Stakes can be considered low since students answer anonymously, but to what extent this low impact of results on individuals affect their motivation to do their best is not known. The test consists of 14 tasks representing basic mathematics and has been unchanged since 1997.



Figur 3. Average percentage correct for the diagnostic test for new-beginners in mathematics at the Royal Institute of Technology. Source: (Brandell, 2011)

The results, shown in Figure 4, indicate a steep decline in mathematical achievement 1997-2000, but fairly stable levels since then.

Swe-SAT

The Swedish Scholastic Assessment Test (Swe-SAT) intends to measure aptitude for higher education studies and offers an alternative way into higher education for students who cannot compete with grades from upper secondary school. Stakes are high, since good results on this test can open the way to virtually all programmes at university, including the most prestigious ones. The test consists of several parts, but in general these parts can be categorised as either focussing a verbal or a numerical component. The composition of the test has recently been changed, and now includes more typical mathematical tasks. Two examples of items used in the test given in November 2011 are shown below.

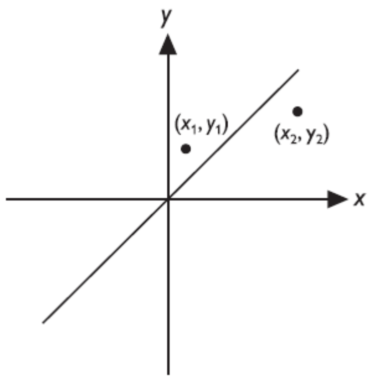
In the first example the correct answer (D) was given by 35 % of the participating test-takers, and among students from the natural science program in upper secondary school 54 % got it right.

Calculate $\frac{9}{0.01-0.001}$

A 90
B 100
C 900
D 1000

The other example from Swe-SAT comes from a part of the test were test-takers are prompted to evaluate which of two quantities is the larger.

The line $y = x$ and the points $(x_1; y_1)$ and $(x_2; y_2)$ are shown in the figure below.



Quantity I: $\frac{y_1}{x_1}$

Quantity II: $\frac{y_2}{x_2}$

A I is larger than II
B II is larger than I
C I is equal to II
D The information is insufficient

The correct answer (A) was given by 39 % of the test-takers, and 56 % among students in the natural science programme.

The tasks shown above are interesting examples of attempts to probe conceptual understanding in large-scale assessment. Both build on fundamental mathematical concepts but puts this basic knowledge in a new setting requiring test-takers to show something else than procedural understanding. The results indicating that these tasks are solved correctly by slightly more than half of the students from the natural science programme in upper secondary school, the students who have studied most mathematics at this level, can be seen as unsatisfactory.

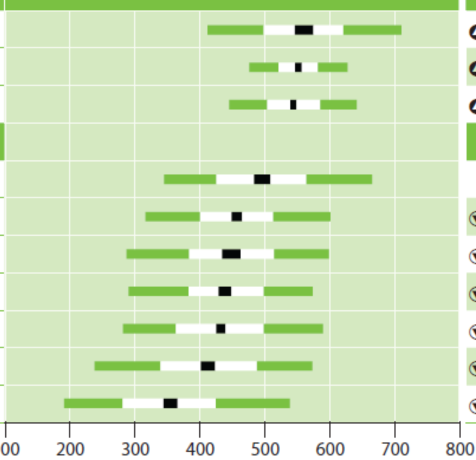
International comparative studies

The purpose of international comparative studies such as PISA and TIMSS are well described in international publications and frameworks from each study (see e.g. Mullis, Martin, Robitaille and Foy (2009) and (Niss, Emanuelsson, & Nyström, 2013). In Sweden at least, these studies are part of the national evaluation and play an important role in public debate about achievement. Stakes are extremely low in TIMSS and other international comparative studies since the results have no significance at all to the individual student. This is not causing so much problems for younger students since they seem to attempt their best anyway, but it is definitely problematic for older students (Eklöf, 2010). The character of these tests is that they are dominated by multiple-choice items and items requiring students to give a short answer. Items are strictly chosen in order to represent the assessment framework defining both content- and cognitive domains.

One example of results from international comparative studies is given in Figure 4. These are the overall results for mathematics in TIMSS Advanced 2008.

Exhibit 2.1 **TIMSS Advanced 2008 Distribution of Achievement in Advanced Mathematics**

TIMSSAdvanced2008 Advanced Mathematics



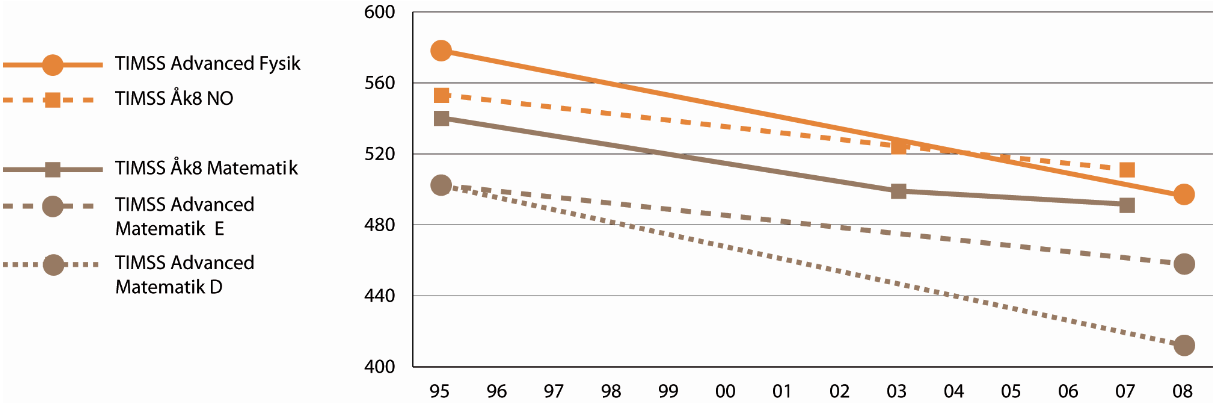
Country	Average Scale Score	Advanced Math-ematics Coverage Index	Years of Formal School-ing*	Average Age at Time of Testing	Human Develop-ment Index**
Russian Federation	▲ 561 (7.2)	1.4%	10/11	17.0	0.813
† Netherlands	▲ 552 (2.6)	3.5%	12	18.0	0.953
Lebanon	▲ 545 (2.3)	5.9%	12	17.9	0.772
TIMSS Adv. Scale Avg.	500				
Iran, Islamic Rep. of	497 (6.4)	6.5%	12	18.1	0.759
Slovenia	▼ 457 (4.2)	40.5%	12	18.8	0.917
Italy	▼ 449 (7.2)	19.7%	13	19.0	0.941
Norway	▼ 439 (4.9)	10.9%	12	18.8	0.968
Armenia	▼ 433 (3.6)	4.3%	10	17.7	0.775
Sweden	▼ 412 (5.5)	12.8%	12	18.8	0.956
Philippines	▼ 355 (5.5)	0.7%	10	16.4	0.771

SOURCE: IEA TIMSS Advanced 2008 ©

Figure 4. TIMSS Advanced 2008 Distribution of achievement in advanced mathematics. Source: (Mullis, Martin, Robitaille, et al., 2009)

This particular study focussed students in the final year of upper secondary school (year 12) who had studied advanced mathematics. Sweden's results have dropped dramatically since 1995 and this drop seems to be consistent with the decline observed in the diagnostic test from KTH described above. Another important aspect of these results is the Advanced Mathematics Coverage Index presented in Figure 4. This number describes the percentage of 19-year-olds that has been taught the advanced mathematics covered by the test. It is noticeable that for the top-scoring country, Russia, this level of education is rare for students in upper secondary school. Even though achievement is much more modest in Sweden, the percentage of students educated to that level is much larger (12.8 %). The most challenging results are actually from Slovenia where 40 percent of the students are taking advanced mathematics, reaching an achievement level way above Sweden.

Achievement trends in mathematics in Sweden are also illustrated in Figure 5, were results from comprehensive school are also revealed.



Figur 5. Trends in average achievement results for Swedish students in TIMSS. Source: (Skolverket, 2009a)

The decline in achievement from 1995 to 2007 is fairly consistent for mathematics and science, both for comprehensive school and upper secondary school students studying advanced mathematics and physics. Looking at different performance levels, TIMSS achievement has decreased for both low- and high-performing students. A recent Swedish PISA-report shows that in comparison to the OECD mean, results have decreased the most for low-performing students from 2000 to 2007. This is a remarkable result since Sweden has focussed low-performing students in particular during this period.

As an example of what the changes in overall score can mean concerning what students can do in mathematics, an item used in both TIMSS Advanced 2008 and TIMSS 1995 is presented in Figure 6, along with results from both years. For this particular item, the percentage correct has decreased from 54 to 41. At the same time, the percentage of students answering with alternative C has increased substantially. These changes are not easily understood or explained.

M3_04

	Tot. 2008	Tot. 1995
A	8,1%	5,9%
B*	41,5%	54,4%
C	24,6%	15,1%
D	5,8%	4,0%
E	11,1%	16,1%
Ej svar	8,9%	4,5%

* markerar rätt svarsalternativ

24 $\int_1^2 \left(x - \frac{1}{x^2}\right) dx =$

(A) $-3\frac{1}{8}$

(B) 1

(C) $2\frac{5}{8}$

(D) 4

(E) $4\frac{1}{2}$

MA13024

Figure 6. An advanced mathematics item given in both TIMSS 1995 and TIMSS Advanced 2008. Source: (Skolverket, 2009b)

Overall image

The images of mathematical achievement described above definitely represent a variation in several aspects, including purpose, stakes and character. Such differences are of course important in trying to understand the overall picture they are painting.

Student grades in mathematics are fairly poor and show no visible change for mathematics over the last ten years or so. The credibility of grades in Sweden is however challenged today and the mechanisms for securing quality in grading are weak.

National tests in Sweden also indicate fairly low achievement, and even lower than grades. These tests are not specifically designed to measure trend, but looking over time the results have fluctuated somewhat but not generally indicated any trends. One exception was presented above, showing a tendency towards lower achievement for students in the lower range of achievement.

Two mathematics tests used at the entrance to higher education were described, with one indicating a negative trend in mathematical achievement up until 2000 and fairly stable levels since.

International comparative studies were represented here by TIMSS, which has a strong connection to curriculum and supplies the most reliable measures of trends. According to TIMSS, mathematical achievement had a peak in Sweden in 1995 and has since seen a substantial decline. Swedish students perform poorly in relation to students in many other countries.

How can these differences be understood? Overall, the measures of achievement presented here indicate a fairly low achievement level for Swedish students in mathematics, both compared to national goals and in comparison with other countries. There also seems to be some consistency in evidence of a decline in mathematics achievement during the last 15 years, a decline that seems to have been steepest in the beginning of that period. How can the trends be understood? Have students learned something else, something that is not reflected in these traditional measures of achievement?

Alternative views of achievement (what are students learning if they are not learning the traditional stuff?)

The low levels of achievement in mathematics indicated above are to some extent unexpected in a country like Sweden where a lot of money is spent on education. From other perspectives we find images of Swedish schools as successful, for example when it comes to making students enjoy being in school. The unexpectedly low achievements are often discarded as artefacts from flawed measurement and it is claimed that students learn other things in school.

In Sweden as well as in other countries, voices are raised claiming that "traditional" ways of defining achievement fail to capture the outcome of theoretical and practical teaching interventions. An example of a reaction to unexpected developments in conventional measures of achievement was given in a recent article in New York Times analysing a school district which has invested heavily in ICT (Richtel, 2011). In the article it is described how "hope and enthusiasm are soaring here. But not test scores". In 2005 the district invested $46.3 million for laptops, classroom projectors, networking gear etc. An argument for this investment was "If we know something works, why wait?", and the pitch was based not on the ideas that test score would rise, but that technology represented the future. In the article it is described that some backers of this idea say "standardized tests, the most widely used measure of student performance, don't capture the breadth of skills that computers can help develop". In other words, students are learning something else which is not very well represented in standardized tests. What is this "other" learning outcome?

The elusive character of the "other thing" students are learning in mathematics instead of definable mathematical competencies is definitely a problem because the formulation and negotiation of learning goals and standards has such a prominent place in the educational discourse. Can we contemplate the possibility that students are actually *not* learning something else that can be described as mathematical instead of the aspects covered by the measures of achievement described in this paper?

Final remark

Even though I find it difficult to explain the decline in achievement in mathematics in Sweden with evidence of students developing other aspects of their mathematical understanding, there are definitely reasons for constructive criticism of existing measures of achievement.

It is obvious that important learning outcomes are not measured, and even though the overall conclusions about mathematical achievement might not be affected by this, the consequences for what students and teachers value as important might be substantial. There are also other threats to validity that need to be taken seriously, for example the risk of making reading ability too influential on achievement results in the effort of making mathematics relevant to life outside the mathematics classroom. Furthermore, existing programmes for evaluating achievement almost exclusively focus cognitive aspects of mathematics, even though the teaching and learning of mathematics also aims for affective goals. Finally, large-scale generalisable results based on quantitative data need to be complemented by other perspectives.

It is quite obvious that standardized testing (in particular TIMSS and PISA) assesses a restricted curriculum partly due to necessities imposed by the large-scale format, and partly due to beliefs about what mathematics should look like (and frameworks that actually intend to create possibilities of comparison). If possible, a widening spectrum of assessment formats would be welcomed. However, the necessity of this is to some extent an empirical question. If what is measured correlates strongly with what is not measured, then this is not a necessarily a problem.

Finally, returning to the initial reflections on the different meanings of achievement, I want to conclude that the assessment of achievement, as well as the interpretation of images of achievement, are highly dependent on how achievement is conceptualised. Viewing achievement as learning outcome makes it harder to assess, but definitely more relevant for policy-makers and others. I find the rather problematic image of mathematics in Sweden difficult to dispute on the basis of arguments that students learn something else in mathematics today that they did not learn earlier. However, I am happy to continue the quest for this elusive learning outcome. Meanwhile, we need less excuses and more serious discussions about how mathematics learning can be improved. Furthermore, we need to take the question of measurement of learning outcomes seriously and refrain from making false dichotomies like achievement vs. learning or summative assessment vs. formative. It is important to have a critical but informed discussion about quality in the measurement of achievement based on considerations of validity in relation to the constraints imposed on assessment.

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Large-scale External Assessment and Improvement of Teaching in Mathematics Classrooms: A Japanese Perspective

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University of Tsukuba

The current national assessment of academic ability in Japan was introduced in 2007 for the first time in 43 years. Aligned with the goals and contents specified in the national curriculum standards, the new assessment in mathematics focuses on students' ability of functional use of mathematics as well as on the basic knowledge and skills. Issues were raised by the introduction of the new assessment. In particular, there is a tension between large-scale assessment and classroom assessment in their differences of purpose, method, emphasis, and audience. Although the alignment of assessment with curriculum standards is a key for improving classroom teaching and learning, assessment should also be aligned with and central to teaching mathematics in classrooms.

Introduction

Mathematics education in Japan is currently in its transition period and faced with some issues in implementing the new educational policy. The new National Course of Study was released in 2008 and has been implemented at elementary schools in 2011 and at junior and senior high schools in 2012. The new national assessment of academic ability was introduced in 2007 for the first time in 43 years, prior to the release of the new curriculum standards. Aligned with the goals and contents specified by the national curriculum standards, the new assessment in mathematics focuses on students' ability of functional use of mathematics as well as on the basic knowledge and skills.

I will here discuss the issues raised by the introduction of the new national assessment and prospects for the use of assessment results for improving classroom teaching and learning. The framework for the mathematics assessment and several sample items are reviewed to describe key findings from the new assessment and how assessment items and their results can be used for the improvement of classroom teaching and learning. There is a tension between large-scale assessment and classroom assessment in their differences of purpose, method, emphasis, and audience. Although the alignment of assessment with curriculum standards is a key for improving classroom teaching and learning, assessment should also be aligned with and central to teaching mathematics in classrooms.

Background: The Japanese context

The Japanese education system is comprised of 6 years of elementary school, 3 years of lower secondary school, 3 years of upper secondary school, and 2-4 years of postsecondary school (e.g., 2 years of junior college or 4 years of university). Recently, secondary schools of 6 years are also available. The first nine years of schooling belongs to the compulsory education.

The basic guidelines for school curricula at elementary and secondary education to be used nationwide are prescribed in the National Course of Study, which is issued by the Ministry of Education, Culture, Sports, Science, and Technology (MEXT) and revised for every about ten years. The document includes the objectives and contents of all the school subjects. Each school sets up and implements its own curricula in accordance with the guidelines, taking into account the conditions of the local community and the school, the stages of growth and the characteristics of students, as well as other conditions for students' learning.

The new National Course of Study has been released in 2008 and implemented at elementary school in 2011, and at junior and senior secondary school in 2012. The new national curriculum standards emphasizes the importance of fostering students' abilities to think, represent, and make decisions as well as of facilitating "language activities" in classrooms in each school subject. In mathematics, teaching mathematics through mathematical activities is valued with the emphases of process aspects of mathematics such as problem solving and reasoning, mathematical modelling, representation, and communication.

The MEXT has also introduced a new national assessment at the final grades of elementary and lower secondary schools to monitor student's academic ability in Japanese and mathematics and environments of students' learning for the first time in 43 years. In the current Japanese education system, students' learning is evaluated by criterion-oriented evaluation from four different viewpoints. For mathematics, evaluation of students' learning has covered the following four categories.

- Interests in, volition for, and attitudes toward mathematics
- Mathematical ways of thinking
- Ability to represent and process mathematical objects
- Mathematical knowledge and understanding

The new national assessment discussed in the current paper aims to assess students' learning with a focus on the last three categories by using paper and pencil tests, as well as on the first category by the questionnaires.

In sum, mathematics education in Japan is currently in its transition period for implementing the new national policy. These movements in education have required key changes in various aspects in education in general and raised some issues in mathematics education in particular. Key changes can be found in the implementation of the new curriculum in classrooms and evaluation of students' learning in the new framework of assessment. Thus, examining the impacts of the newly introduced national assessment on improving teaching and learning in classrooms gives an opportunity to see prospects and to identify the challenges ahead in school mathematics in Japan.

Recent national assessments in Japan

Assessment has been a long-standing problem in Japanese mathematics education, as it has been in other countries (e.g. Pettersson & Boistrup, 2010). Japanese mathematics educators have struggled for decades with many of the same assessment issues that plague educators in many other countries, asking questions such as:

- What influences, both positive and negative, are exerted by external assessment on classroom-based assessment?
- How can the results of external assessments be used to design activities to move students' thinking forward, in addition to providing evidence of their present levels of knowledge and skills?
- How can teachers become more familiar, through assessment, with the abilities, skills, and thinking of their students, and thereby more appropriately able to plan and modify their classroom instruction?

In April 2007 MEXT conducted its national assessment of academic ability in the school subjects Japanese and Mathematics. In 2012 the assessment of academic ability in Science has also been conducted. The nationwide assessment aims to monitor students' academic ability and backgrounds of their learning, to examine and improve educational policies, and to provide key information to local boards of education and schools so that they can improve classroom practices. The new external assessment, started with the entire cohorts in grade 6 and 9, had strong impacts on classroom practices in those subjects.

In any school subject, in general, and in mathematics, in particular, a tension between large-scale external assessment and classroom assessment exists in their differences of purpose, method, emphasis, and audience. One of the key, but sometimes not noted, issues for classroom teachers with large-scale external assessments is to think about how to utilize the released results for improving classroom practices.

In the following sections, I will discuss how large-scale external assessment and classroom assessment in mathematics can be linked to enhance students' learning with a particular reference to the case of the new national assessment of academic ability in Japan. After the Japanese contexts of the introduction of the national assessment are described briefly, the framework for the new mathematics assessment and several sample items are provided to describe how the test items and the results can be used for the improvement of classroom practices. It is then argued that large-scale assessment needs not be seen as completely different from classroom assessment and that external assessment like the one discussed in this chapter can be used in certain ways to enhance students' learning.

Large-scale assessment in mathematics

Besides the large-scale international assessments such as IEA's TIMSS (Trends in International Mathematics and Science Study, e.g. Mullis et al., 2008) and OECD's PISA (Programme for International Students Assessment, OECD, 2010), several types of large-scale assessments, including paper-and-pencil tests with questionnaires administered to students, teachers, and schools, have been conducted in Japan. In particular, since the 1980s, three different types of large-scale assessments have been implemented; the National Assessment of Academic Ability and Learning Environments, the Assessment of Implementation of National Curriculum, and the Assessment of Specific Issues in Students' Learning. Each of these assessments has different aims and objectives for different school subjects with different student groups as shown in Table 1.

	National Assessment of Academic Ability and Learning Environments	Assessment of Implementation of National Curriculum	Assessment of Specific Issues in Students' Learning
Major Aims	To monitor students' academic ability and background of learning nationwide to check and improve educational policy. To establish the PDCA (Plan-Do-Check-Action) cycle in educational policy. To improve classroom practices in each school.	To monitor the implementation of new national course of study. To improve classroom practices in each school.	To investigate specific issues in teaching and learning which are not explored by the Assessment of Implementation of National Curriculum.
Targeted Grades	Grade 6 and 9	Grades 5 through 9	Depending on the subject (Grades 4 through 9 for Mathematics)
Survey Style	Complete (2007-2009) Sampling (2010-)	Sampling	Sampling
School Subjects	Japanese and Mathematics (2007-) Science (2012)	Japanese, Mathematics, Social Studies, Science, and English (only for junior high schools)	All the school subjects

Table 1. Types of recent large-scale national assessments in Japan.

The new national assessment

The new nationwide test has been implemented to assess the academic achievement of sixth-graders in elementary schools and third-graders in junior high schools. Their scores in the test can be considered to give a good indication of how much progress they have made at those stages of their education.

From 1956 to 1966, there were national achievement tests covering random sample (5-10 %) of all the students, and another test for all students in Grade 8 and 9. However, these tests were suspended as they seemed to accelerate competitions among schools. The new nationwide test in 2007 was a response to public concerns over the deterioration in academic skills that became evident since 2002, when the school week was reduced from 6 days to 5, the content of textbook was reduced by roughly 30 % in relation to the revision of National Course of Study, and the Japanese ranking "went down" from PISA 2000 where Japan was on the top of the list of countries and regions in terms of students' achievement in mathematics to PISA 2003 in the sixth place.

The framework for mathematics assessment

The new national assessment consists of two bundles, A and B, for both Japanese and Mathematics. Each of the two bundles covers "Knowledge" and "Functional Use" respectively in each subject as described in the following.

- Bundle A, Items for assessing "Knowledge": Knowledge and skills needed for further learning in schools and for applying them in the real life situations
- Bundle B, Items for assessing "Functional Use": Competencies for applying knowledge and skills to the situations in the real life, and for planning, implementing, reflecting, and improving the plan to solve problems

Students at grade 9 worked on each bundle in the targeted subjects for 45 minutes, followed by another 45 minutes for the questionnaire. Bundle A includes multiple choice and short answer tasks, while bundle B includes open construction tasks as well.

This is a summative assessment in nature based on the current curriculum. The results give the ministry vital information on students' academic performance nationwide, which is later provided to schools and students. From 2007 to 2009, about 1.2 million sixth-graders at 22,000 elementary schools and 1.2 million third-year students at 10,500 junior high schools took the test. From 2010, it was decided that only a random sample of the targeted students took the test. In year 2011, however, the assessment was suspended because of the earthquakes and Tsunami in March and assessment tasks were sent to those schools that decided to use them in their schools.

Every year a couple of months after its implementation, the MEXT releases the results of assessment to the local governments, boards of education, and schools that participated in the assessment. Also, the students who participated in the test obtain feedback on their papers and other information, including charts showing statistical information on the test. Finally, classroom teachers are provided with documents that describe detailed information of the intention of items and results from related items in the previous assessment, as well as recommended lesson plans so that they can use the assessment tasks in their classrooms.

The framework for mathematics: Bundle B

While each item in bundle A is intended to assess students' basic knowledge and skills, the items in the bundle B are to assess students' functional use of mathematics in various contexts such as daily lives, learning in other school subjects like science and social studies, and learning within mathematics. Key phrases that describe the abilities the assessment tasks require of the students in bundle B for grade 9 are as follows (National Institute for Educational Policy Research, 2008).

- Observing events around us by focusing on numbers, quantities, and figures to grasp their key features
- Classifying the given information to select an appropriate one
- Thinking logically to draw a conclusion and looking back on one's own thinking
- Interpreting events in the real world or in the mathematical world, and expressing ideas mathematically

There are three dimensions to the mathematics assessment tasks in bundle B:

- Mathematics content specified in the National Course of Study
- Situations and contexts
- Mathematical processes

There is a similarity between these three categories and those in the assessment framework of OECD/PISA mathematics (Table 2). First, each item is aligned with the National Course of Study that specifies goals and content of school mathematics. Second, there are three categories for situations and contexts with which students are faced in the test: mathematics, other school subjects, and the real world. Third, there are three strands α, β, and γ in mathematical processes:

- α: Competencies for applying knowledge and skills to the situations in the real life
- β: Competencies for planning, implementing, evaluating, and improving the plan to solve problems
- γ: Related to both α and β

Category	The mathematical processes
Competencies for applying knowledge and skills to the situations in real life	α1: Mathematizing phenomena in everyday life. α1(1) Observing things by focusing on numbers, quantities, and shapes α1(2) Grasping key features of things around us α1(3) Idealizing and simplifying α2: Functional use of information α2(1) Classifying and organizing the given information α2(2) Select needed information appropriately to make decisions α3: Interpreting and expressing phenomena mathematically α3(1) Interpreting phenomena α3(2) Expressing own idea mathematically
Competencies for planning, implementing, evaluating, and improving the plan to solve problems	β1: Drawing up a plan to solve problems β1(1) Thinking in a logical way β1(2) Making a plan β1(3) Implementing the plan β2: Evaluating the result and improving the entire process β2(1) Looking back at the result β2(2) Improving the result β2(3) Extending the result
Related to both α and β	γ1: Making connections from one phenomenon to the other γ2: Integrating different things γ3: Considering things from multiple perspectives

Table 2. Types of recent large-scale national assessments in Japan (National Institute for Educational Policy Research, 2010, p.11)

Table 2 shows the detailed descriptions of mathematical processes to be assessed. Each item in bundle B is developed within the framework and described based on three dimensions. The key feature of items in bundle B, as embedded in the real world contexts with an emphasis on mathematical processes, is new to the teachers.

Emphasis on explaining mathematically

The new National Course of Study emphases the importance of “language activities” in classroom to be facilitated in each subject to enhance students’ learning. In accordance with the emphasis in the revised curriculum guidelines, each of the tasks in bundle B includes open-construction tasks that requires the students to explain things in one of the following forms of explanation:

- Explaining the observed facts and properties in a situation
- Explaining approaches and methods for solving a problem
- Justifying the reasons for the facts and properties

The first category corresponds to tasks that ask students to describe a mathematical fact or property, mostly in the form of a proposition. Tasks correspond to the second category ask students to describe the approach to a problem by specifying both “what is used” and “how it is used” as described in the example of "Mt. Fuji" shown below. It should be noted here that an "answer" is not required and the focus is on the method to be used. The third category includes tasks that ask students to explain the reason for the facts and properties. Construction of a proof falls into this category.

General results

For the past four years, results of average percentage of correct responses in each bundle and each item were released with specific suggestions for the improvement of classroom teaching. Table 3 shows the result of each bundle in mathematics at grade 6 and 9.

		2007	2008	2009	2010
Grade 6	Bundle A	82.1	72.3	78.8	74.4
	Bundle B	63.6	51.8	55.0	49.6
Grade 9	Bundle A	72.8	63.9	63.4	66.1
	Bundle B	61.2	50.5	57.6	45.2

Table 3. Result of mathematics assessment: Percentages of correct responses in average, 2007-2010.

As Table 3 shows, in general, while the percentages of correct responses were larger for the knowledge and skill items, there were several items that show the students' difficulties in mathematics. Also, the percentages of correct response to items in bundle B were relatively small for each year. While other statistics such as median and standard deviation are available, the results of each content area and of the common items with other assessments such as past TIMSS released items, as embedded in the item set, are also available.

Students' difficulties in the use of their knowledge and skills in contexts have been identified (National Institute for Educational Policy Research, 2007; 2008; 2009; 2010). Further, students' difficulties have appeared with particular contents such as number and quantity sense, understanding the meaning of operations, use of literal symbols, understanding the significance of proof, and concepts related to function.

Sample items and their results

The results of assessment were released a couple of months after its implementation, on July 31st in the case of year 2010. Some sample items are shown below.

(1) Common and decimal fractions (Grade 6, A3 (2), 2007)

Choose the largest number among 0.5, 7/10 and 4/5, and then show the locations of the numbers on the number line.

The percentage of students who answered the correct choice was 55.9. About 17% of the students chose "7/10" as the largest number. The result reveals that students have difficulties in understanding and expressing the size and meaning of common and decimal fractions.

(2) The relationship between divisor and quotient (Grade 6, A3, 2008)

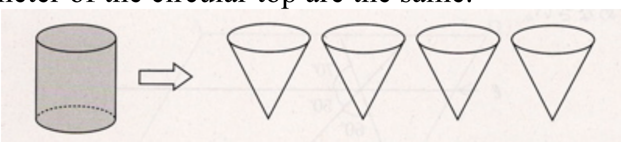
In the following expressions, "@" denotes the same number which is not 0. Choose all the expressions so that the result of computation is larger than the original number @.

a. @ × 1.2   b. @ × 0.7   c. @ ÷ 1.3   d. @ ÷ 0.8

The percentage of students who answered the correct choices (a and d) was 45.3. Those who chose "a" and "c" (12.0%) thought that the number in the expression, being larger than 1, will make the result of computation larger than "@". The small group of students (4.4%) who chose "a" and "b" might think that multiplication makes the result of computation larger than the original number.

(3) Volumes of cylinder and cone (Grade 9, A5 (4), 2007)

The following figures represent containers with the shapes of cylinder and cone. Their heights and the diameter of the circular top are the same.



When we move the water in the cylinder to the cone, how many cones are filled by water? Choose the correct figure from the following five choices.

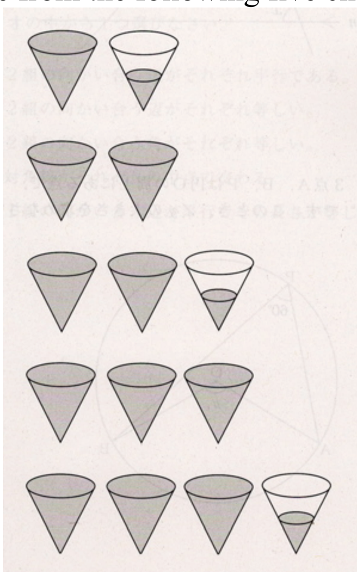
a

b

c

d

e



Only 38.1% of the students chose the correct answer (d). Roughly the same percentage of students (36.7%) chose the wrong choice "b". The result clearly shows that students' understanding of the relationship between the ratios of

volumes of cylinder and cone is very weak and that this may be caused by the lack of students' experience to have the activity in mathematics classrooms.

(4) Two-digit numbers (Grade 9, A2 (4), 2010)

When we express a two-digit number by using x for the tenth digit and y for the unit, which of the following expressions is correct?

a. xy b. $x + y$ c. $10\,xy$ d. $10x + y$

The result of the item is shown in Table 4. Many students had difficulties in representing a two-digit number using letters; they could not correctly make connections between the ten base numeral system and literal symbols.

Category	Choice	Response rate (%)
1	xy	11.5
2	$x + y$	11.0
3	$10xy$	8.9
4*	$10x + y$	67.7
9	Others	0.2
0	No Answer	0.8

* correct answer to the item

Table 4. Results of the Item A2 (4), Grade 9, 2010.

(5) "Mt. Fuji": An item from bundle B (Grade 9, B5, 2008)

The item (see Appendix) is intended to assess students' ability to apply a linear function to real data to solve a problem in context. It involves using data in a given table and a graph of a linear function to interpret phenomenon in the real world. Students need to use mathematics and explain a method for estimating the temperature at a certain location.

In this situation, air temperature (y) is treated as a linear function of altitude (x), and students are asked to explain a method for finding the air temperature at the altitude of 2500 m by describing both "what is used" and "how it is used". Here the category of "what is used" includes graphs, expressions, tables, numerical values, and so on. The category of "how it is used", on the other hand, includes drawing a straight line to identify the value of y when x = 2500, finding the expression of a linear function from the data given, and examining the rate of change from the table, and so on.

The correct response rates were, 54.7%, 25.0%, and 13.3% for Question 1, 2, and 3, respectively. The rates were quite low for Questions 2 and 3. Most students had difficulties in finding and applying a linear function in the given context. Also explaining their approach to the problem by specifying both "what is used" and "how it is used" was difficult for them, when the "answer" was not requested. The no response rate for Question 3 was 58.5%, the highest in all the items in the year.

Impacts of the new national assessment

The introduction of the new national assessment had strong impacts on various aspects of education in general, and mathematics education in particular. There are big debates among politicians and lay people, as well as educational researchers, on the participation by each local government, the costs, and effects of the introduction of such large-scale assessment. Whereas all the students in each cohort were targeted by the assessment in the first three years, the assessment in 2010 was conducted with the sample students (roughly 30 % of the students in each cohort) with the cut of budget. For those schools that decided to use the item sets for their own marking, the assessment tasks were sent to them for their own use. In year 2011, because in some prefectures the implementation was faced with difficulties by the earthquake and Tsunami in March, the assessment tasks were again sent to those schools that decided to use them. Thus, the design and implementation of the new national assessment have been influenced by many factors beyond the mathematics education community.

From the beginning of introducing the new national assessment in 2007, there was a discussion on the roles and functions of such a large-scaled assessment. The assessment aims to monitor student's academic ability and backgrounds of their learning and to examine and improve educational policies, and to provide key information to local boards of education and schools so that they can improve classroom practices. Thus, on the one hand, the assessment should be designed and implemented to be scientific from an educational measurement perspective. On the other hand, the results need to be useful enough for the improvement in classroom teaching and learning from a teacher's perspective. As there is a tension in the design and implementation in this regard, the design of the entire framework needs to be re-examined, in order for monitoring the "trends" of students academic ability, for instance.

When we look into the new textbooks series, we can find more emphases on the use of mathematics in real world contexts. At the transition period of changing emphasis in the national educational policy documents, the presentations of real world contexts in the selected mathematics textbooks series have shown certain shifts in the role of sources and the tasks related to real world contexts. Shimizu (2011) reported the results of an analysis of the inclusion of real world contexts in the new Japanese mathematics textbooks at the elementary school level. The results reveal that textbooks are aligned with the emphasis in educational policy documents on fostering students' abilities for functional use of mathematics in daily life as well as in mathematics.

Another observed major impact of the new national assessment includes the changes in the discourse of classroom teachers when they talk about teaching with their colleagues. In lesson study meetings, for example, teachers have started to use the test items to describe new emphases in mathematics education. Namely, the items in the assessment in previous years are used as illuminating examples of new emphases in mathematics education. Also, the assessment items and the results are introduced and used in various professional development courses.

Further, the introduction of the new national assessment had certain impacts on research in mathematics education. The accumulation of the results in each year provides mathematics education researchers with opportunities for exploring research possibilities in relation to learners and learning. For example, inclusion of open-construction items provides ample opportunity for the researchers to analyse students' abilities in expressing their ideas mathematically. While we need to examine the results from each item carefully, we also need to synthesize the results from different perspectives as a coherent body of description of the reality of the learners (Shimizu, 2005).

Challenges ahead

Linking large-scale external assessment to classroom practice

As mentioned earlier, there is a tension between large-scale assessment and classroom assessment. Classroom assessment is designed or used by classroom teachers for making instructional decisions, monitoring students' progress, or evaluating students' achievement (Wilson & Kenney, 2003). On the other hand, large-scale assessments are summative in nature and external to the course of regular classroom activities. Nevertheless, even in the case of large-scale external assessment, "assessment should enhance mathematics learning" (NCET, 1995, p. 13). Given the fact that taking a test, even an external one, is a key learning opportunity for students, assessment tasks in the external test need to be considered as the platform for enhancing students' learning. Also, it should be noted that the strong and close relationship between assessment and instruction has great potentials for improving classroom practice (Heuvel-Panhuizen & Becker, 2003). How can we make a link between large-scale external assessment and classroom instruction in mathematics?

Using external assessment task to improve classroom practices

In the context of the current educational reform in Japan, the relationship between the revised national curriculum framework and the national assessment is a key to the implementation of a new curriculum. If the emphasis in the new curriculum is reflected in the new national assessment, the test items play a key role in sending messages on the emphasis in the new curriculum standards to classroom mathematics teachers, just as "what is tested is what gets taught". In the case of the new national assessment in Japan, inclusion of open-constructed items with an emphasis on "explaining" is aligned with the emphasis on "language activity" in the new National Course of Study.

While the new national assessment focuses on fundamentals in school mathematics, the test items, those in bundle B, in particular, are new to most teachers, as they are embedded in the real world or intra-mathematics contexts with an emphasis on mathematical processes that are key features of the new National Course of Study. Teachers can become more familiar, through the new types of assessment tasks, with abilities, skills, and thinking of their students that need to be fostered, and thereby more appropriately able to plan and modify their instruction.

The MEXT has started to disseminate leaflets that include both the results on a few items from the assessment and recommended lesson plans related to those items. The leaflets are sent to all the schools to promote new visions among teachers on classroom teaching with the new emphasis. In this sense, external assessments can be used to design lessons to move students' thinking forward, in addition to providing evidence of their present levels of knowledge and skills.

Using the results of external assessment to help students

The assessment result of each item can be used to inform classroom teaching with more attention paid to those students with difficulties related to the item. This is the case when an assessment task is aligned with the mathematics content specified in the National Course of Study. In other words, test items can be connected to students' learning, even though they are external to the course of regular classroom activities, in terms of mathematics content in the curriculum.

As for the example of how teachers can help their students by using the assessment results, look at Table 3. Choice a, "xy", corresponds to those students who just replaced the numbers in each digit with letters x and y. Choice b, "x + y", corresponds to those students who did not understand the magnitude of the number in the tenth digit. Finally, choice c, "10xy", corresponds to those students who understood the magnitude of the number in the tenth digit but could not represent by using literal symbols. Assessment results suggest how the teacher can help the students. If you have the students who tend to choose "b" in your classroom, for example, you may show to the students a two digit number, say, 24, and make sure that the sum of x and y equals 6, not 24.

The MEXT has provided classroom teachers with suggestions and implication on what they can do with the students with difficulties as specified with the coding for analysis for all the items (e.g. National Institute for Educational Policy Research, 2010). As the "Learning" standard from the NCTM Assessment Standards (NCTM, 1995) suggests, the assessment task provides "learning opportunities as well as opportunities for students to demonstrate what they know and can do" (p. 13). From the teachers' perspective, the results of each item in large-scale assessments provide information about expected students' difficulties.

Concluding remarks

Classroom teachers often do not have direct influence on external assessment programs but such programs do have significant influence on what happens in classrooms. Although there is a tension between large-scale assessment and classroom assessment, as mentioned earlier, large-scale assessment needs not be seen as completely different from classroom assessment. Rather, external assessment, such as the one discussed in this chapter, fits with classroom assessment and results can be used for anticipating and considering students' thinking in each content domain.

Assessment should be aligned with and central to teaching mathematics. Students nowadays are living in the era of external assessment. Given the fact that taking a test is also a key learning opportunity for students, assessment tasks in an external assessment can be a platform for enhancing students' learning and for improving classroom teaching.

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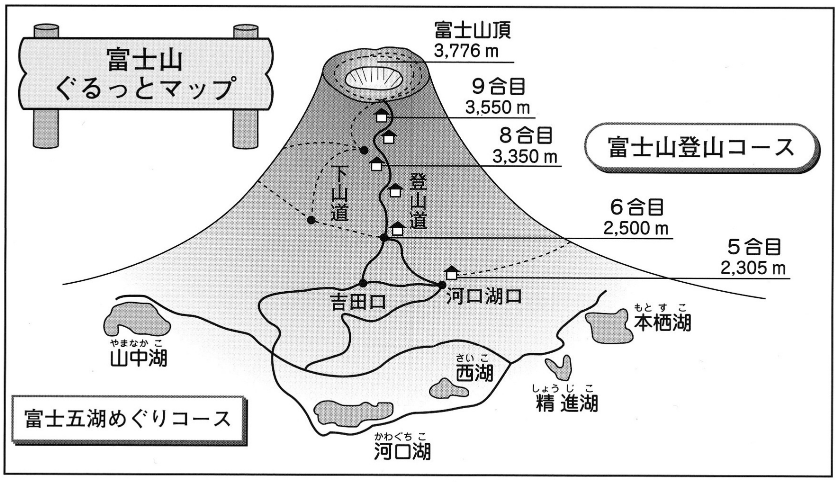
APPENDIX

Rina and her friends are planning to visit the Five Lakes of Mt. Fuji and then climb up to the sixth stage of the mountain this August.

A Map of Mt. Fuji Climbing and the Five Lakes of Mt. Fuji



Question 1.
You will take photos at two lakes among the five. How many different choices of two lakes do you have, if we ignore the order of the visits?

Question 2.
Rina and Ken-ichi are talking about the temperature of the sixth stage of Mt. Fuji.

> Rina: I have tried to investigate the temperature of the sixth stage, but I couldn’t find it because there is no observatory on the stage.
> Ken-ichi: It is known that the temperature falls at a constant rate as one climbs higher until an altitude of 10,000 meters.
> Rina: We may use the fact to find the temperature of the sixth stage.

If we hypothesize that the temperature falls at a constant rate as one climbs higher until an altitude of 10,000 meters, what is the relationship that holds anytime between altitude x meters and temperature y °C? Choose the correct one from the following:

y is proportional to x.
y is an inverse proportion to x.
y is a linear function of x.
Sum of x and y is a constant.
Difference of x and y is a constant.

Question 3.

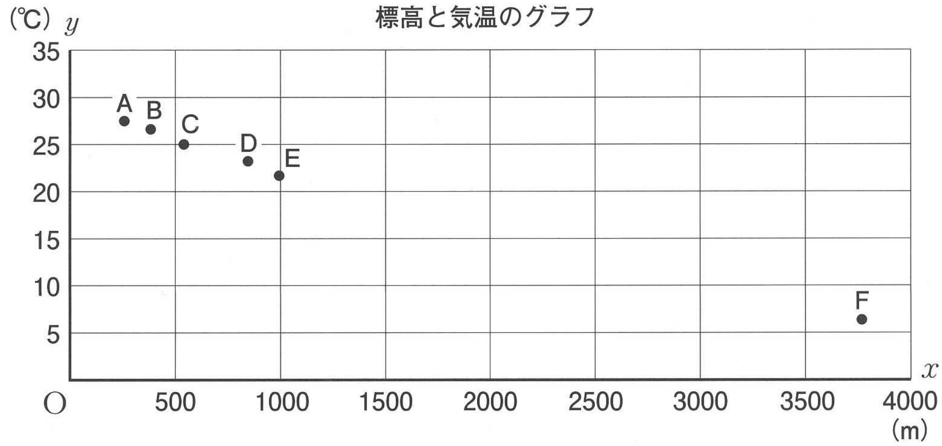
Rina investigated the mean temperature in August on the top of the mountain and around Mt. Fuji. She completed table below and drew a graph, measuring altitude as x meters and temperature y °C.

Altitude and Mean Temperature in August at Observation Points

Observation Points	Altitude (m)	Mean Temp (°C)	Observation Points	Altitude (m)	Mean Temp (°C)
A (Kofu)	273	27.7	D (Kawaguchiko)	860	23.3
B (Katsunuma)	394	26.7	E (Yamanaka)	992	21.7
C (Furuseki)	552	24.9	F (Fujisan)	3775	6.4

(Data Source: Meteorological Agency)

Graph of the relation between Altitude and Temperature



Rina understood that the temperature falls at a constant rate as one climbs higher. Then, she tried to estimate the temperature of the sixth stage of Mt. Fuji using data at the points D and F in the given table and the graph. Explain your method of estimating the temperature at the sixth stage (2,500m). You do not need to actually find the temperature.

Mathematics in the Upper Secondary Electricity Programme in Sweden: A Study of Teacher Knowledge

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Mathematics teachers' and electricity teachers' explanations of a mathematical task in the electricity context have in this study been compared and characterized. An overview of what teacher knowledge in mathematics and in electricity the teachers drew upon in explaining this task shows both similarities and differences in kinds of teacher knowledge. Detailed descriptions of specific explanations of the same topic by two teachers have been studied and this will further contribute to a suggestion of how the teachers' arguments and reasoning can be characterized in specific and general knowledge.

Introduction

Mathematics is taught in many vocational courses in the Swedish upper secondary education and mathematical knowledge is often a prerequisite to learn a profession. At least one mathematics course is mandatory for students enrolled in the Swedish gymnasium and according to the Swedish national curriculum documents, the teaching of the mathematics course should be adapted to the specific programmes (Skolverket, 2000a). The practical use of mathematics would be visible to the students, by using material and experiences from the vocational programme (Lindberg & Grevholm, 2011). It is not clear, however, that linking the mathematics course to the specific programme that the students are studying at really does enrich the vocational courses and the mathematics course.

For example at the Electricity Programme, the national curriculum documents state that being able to do correct calculations is a prerequisite in order to exercise the profession and the education should therefore develop the students mathematical knowledge (Skolverket, 2000b). Students at the Electricity Programme are taught mathematics in two different courses: their mathematics course and their electricity course. The students are taught by two different teachers, the mathematics teacher and the electricity teacher, and those teachers have different education and background. Some mathematics is needed to do electricity work, but what does that mathematics look like at the secondary school level? Students are actually getting two very different kinds of treatments of mathematics by their two different kinds of teachers (Straesser, 2007). But what, exactly, is this difference? That is what this article aims to start exploring.

In this study mathematics and electricity teachers are interviewed about what explanations they would give to students to a mathematical electricity task, a task commonly used in the first electricity course and that could be used in the mathematical course at the Electricity Programme. The task presented in this article involves calculating the total resistance in a parallel electrical circuit with two given parallel resistances.

The aim of this study is to explore the similarities and differences in mathematics and electricity teachers' explanations of mathematical electricity tasks to start understand how mathematics in the electricity context looks like when presented to students at the Electricity Programme. The following specific questions will guide the study: What teacher knowledge in mathematics and electricity do mathematics and electricity teachers at the Electricity Programme use in explaining a mathematical electricity task?; Are there characteristic similarities and differences in the teacher knowledge that the teachers draw upon to explain these mathematical electricity tasks?

Teacher knowledge

Teacher knowledge is a widely researched area. Shulman (1986) introduced the notion of pedagogical content knowledge (PCK) as an important part of teacher knowledge, besides the subject matter knowledge (SMK) and general pedagogical knowledge domains. To be able to compare the teachers in this study, the teachers' PCK and SMK in both mathematics and electricity are studied, using Shulman's definitions. SMK refers to the amount and organization of the subject in the teachers' minds and goes beyond knowledge of facts and concepts in order to understand the subject's structures. PCK refers to explanations and representations of the subject that could help learners understand it, knowledge of what makes the learning of the subject easy or difficult, what preconceptions learners are likely to have, and knowledge of strategies that are likely to help those learners. In this study Subject matter knowledge (SMK) is defined as knowledge of the content, both conceptual and procedural knowledge and Pedagogical content knowledge (PCK) is defined as knowledge of useful representations, examples and illustrations to make the content accessible to students. Also included in PCK is knowledge of common students' conceptions, misconceptions and difficulties with the content

The notion of teacher knowledge has been used in a wide range of studies in mathematics education (Hill, Rowan, & Ball, 2005; Loewenberg Ball, Thames, & Phelps, 2008; Rowland & Ruthven, 2011) and in studies comparing different mathematics teachers (Baumert et al., 2010; Krauss et al., 2008; Ma, 1999). Teacher knowledge has also been used in science education (Loughran, Mulhall, & Berry, 2004; Magnusson, Krajcik, & Borko, 1999) where electricity is included as a topic.

Method

In this study I have interviewed three mathematics and five electricity teachers, who are teaching in the first year at the Electricity Programme at four different schools in northern Sweden. They have been given a mathematical electricity task and been asked how they would explain and help students on the Electricity Programme with this task and also what alternative explanations they have. This method is used to understand the nature and extent of teacher knowledge (Hill, Sleep, Lewis, & Loewenberg Ball, 2007) and has been used in studies comparing different mathematics teacher groups (Ma, 1999). The tasks used are examples commonly used in the first electricity course at the Electricity Programme (Niss & Højgaard Jensen, 2002). The interviews were audio recorded and the teachers' written solutions were videotaped. The interviews, that lasted approximately one hour each, were transcribed.

Analysis of data

What the teachers say and do (calculate, point to, draw pictures and so on) is regarded as indications of knowledge that they have and draw upon to explain the interview task. The teachers' statements and actions are used as indications of their teacher knowledge, where "something will count as knowledge, and be modeled as knowledge, if it appears to be used as such by the individual being modeled" (Schoenfeld, 2011).

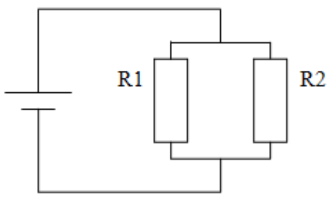
A summary of the teachers' statements and actions during each interview task was made. This summary consists of identified portions of explanations, expressing one idea each (Marks, 1990). One teacher's explanation of a task consists of a list of several different portions of explanations. The analysis started with the teachers' summaries. Each portion of explanations in the summaries was coded as indications of mathematical knowledge or electricity knowledge, depending on the main idea of the statement. For example a teacher's explanation of the formula for the total resistance in a parallel electrical circuit, with a simple number example, and calculations of this example with both decimal numbers and fractions, was categorized as indications of mathematical knowledge. A teacher's explanation of the same parallel circuit task, talking about electrons moving in the circuit, was categorized as indications of electricity knowledge. After the first categorization in mathematics or electricity knowledge, all the indications of knowledge in the mathematical category were divided into SMK or PCK and the same was done with the electricity knowledge category. Teachers' statements and actions of facts, concepts and structures of the subject are categorized as SMK. Explanations and representations of the subject and also knowledge of students' preconceptions are categorized as PCK. For example teachers' calculation of the total resistance in a parallel circuit was categorized as SMK in mathematics and a teacher's explanation with a simpler number example to help a student understand the formula is categorized as PCK in mathematics.

These categories of SMK and PCK in mathematics and electricity were studied to explore the similarities and differences between the two teacher groups.

Results

This section gives an overview of all the teachers' explanations of the interview task (see Figure 1). Then a presentation of detailed analyses of one specific explanation given by one mathematical teacher and one electricity teacher are given.

Your students have started to work with electrical circuits with parallel resistance. Many students have difficulties in using the formula to calculate the total resistance in a parallel circuit.



R1 is 220 ohm and
R2 is 120 ohm

Task:
Calculate the total resistance in the circuit.

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

How would you explain how to solve this task to a student?

Figure 1. Interview task given to the teachers during the interview.

An overview of the teachers' explanations of this task is presented in a picture showing the teachers' explanations in four rows representing SMK and PCK in mathematics and SMK and PCK in electricity, see Figure 2.

Overview analysis

The teachers were in the interviews asked to present all possible ways of explaining the task to a student (Figure 1). The overview picture (Figure 2) shows all explanations presented by all teachers, and in this overview it appears as many of the teachers' explanations are similar, but also that there are several explanations given by one of the teacher groups and not the other teacher group. This overview shows that in SMK in mathematics there is one explanation exclusively used by mathematics teachers and the same for SMK in electricity, where the electricity teachers have explanations exclusively used by electricity teachers. The electricity teachers have some indications of knowledge in the category PCK in mathematics and in PCK in electricity that the mathematics teachers did not mention, both concerning their experience with the subject and the students.

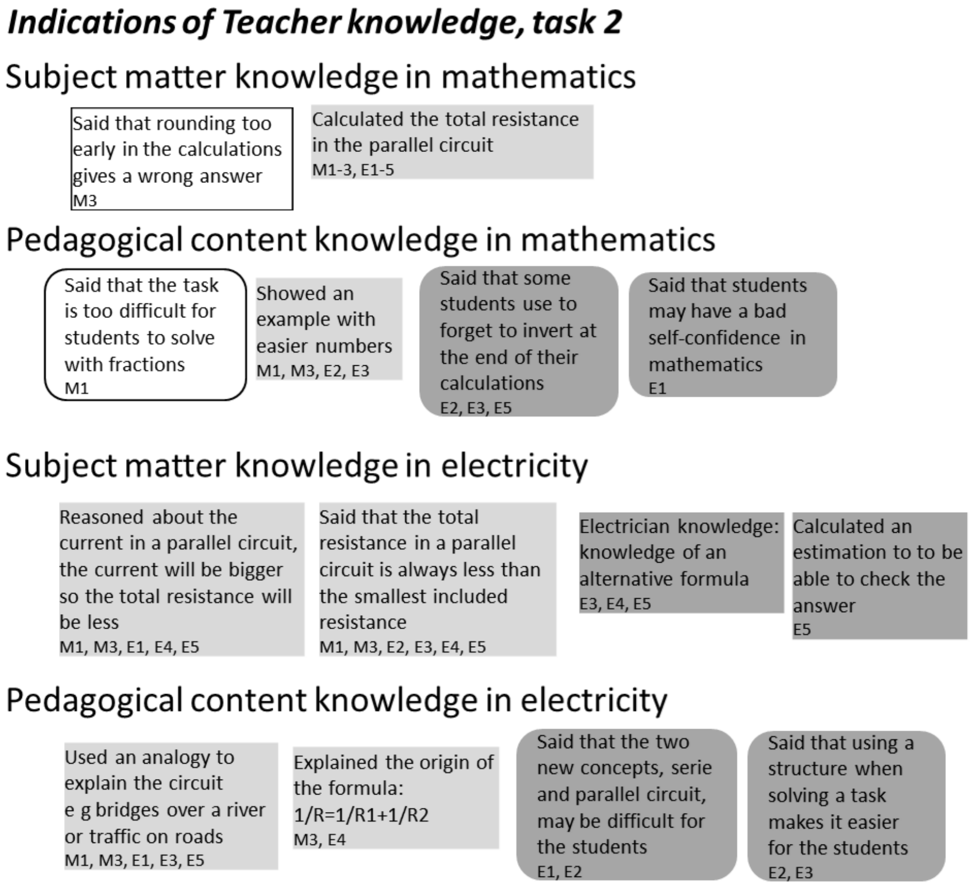


Figure 2. White boxes show explanations given only by mathematics teachers; dark grey boxes show explanations given only by electricity teachers and light grey boxes show explanations given by both mathematics and electricity teachers. Boxes in the PCK category with rounded corners represent explanations based on the teachers' experience with students' pre-knowledge or common mistakes. M1-M3, represents mathematics teachers, E1-E5 represents electricity teachers.

One mathematics teacher, M2, educated in mathematics and biology, appears only once in this overview picture. He calculated the task correctly but had no experience in helping students with tasks like this. Experience in teaching this task seems to be one part that is important for the teacher knowledge of the task. Shulman points out the teachers' capacities to transform content knowledge into explanations that are pedagogically powerful as a key to distinguish the knowledge base of teaching (Shulman, 1987) and the teachers' capacities to transform knowledge may develop with experience of teaching.

Detailed analysis

The overview (Figure 2) leaves out differences in the details of the teachers' explanations. These differences do not show up in this first analysis, so a new analysis section began, where explanations were selected that in this overview were categorized in the same category, but at a closer look had differences. In this paper, a detailed analysis of one mathematics teacher's and one electricity teachers' specific explanations of the same topic has been done, and a suggestion for how to characterize the differences between the teachers' explanations is made. The explanation that is analysed is the second grey box in the bottom row in the overview, Figure 2, and it shows one mathematics teacher's and one electricity teacher's explanation of the origin of the formula for the total resistance in a parallel circuit.

The detailed study indicates that the teachers draw upon different kinds of teacher knowledge - an explanation was supported by different knowledge of different types. The teachers' explanation could be supported by both mathematical knowledge and by electricity knowledge. Both the mathematical and the electricity knowledge could be of either specific or general type. In this analysis, an explanation based on general mathematics consists of a general algebraic solution, and an explanation based on specific mathematical knowledge consists of a solution with specific numbers for the specific task. An explanation based on general electricity knowledge consists of theoretical electricity arguments and reasoning. An explanation based on specific electricity consists of a real world illustration or a concrete example to illustrate this task, see Table 1. One teacher's explanation could consist of several arguments and reasoning based on both mathematical and electricity knowledge of both general and specific types. In the next section, one mathematics teacher's explanation and one electricity teacher's specific explanations of the same topic, will illustrate how these constructs of general and specific mathematical knowledge, and general and specific electricity knowledge could be used in an analysis to highlight the differences in teachers' explanations.

Knowledge/Type	**Specific**	**General**
Mathematical knowledge	Concrete numbers	Algebra
Electricity knowledge	Real world illustration/ concrete example	Theoretical/Physics

Table 1. Definitions for the detailed analysis.

Detailed analysis of data

In this example of a detailed analysis, two teachers explain the origin of the formula for the total resistance in a parallel circuit. Both teachers started with Kirchhoff's current law ("Kirchhoff's laws", 2009) stating that the sum of the part currents in the parallel branches equals the head current. After that the mathematics teacher used an algebraic way to derive the origin of the formula. The electricity teacher on the other hand, verified the origin of the formula by reasoning and calculating in the electricity circuit.

The mathematics teacher

Teacher wrote:	*Teacher did:*
$I = I_1 + I_2$ $U = I \cdot R$ $I = \frac{U}{R}$	The teacher started with writing Kirchhoff's law, and explained it while pointing to the part currents and head current in the circuit diagram. He said that the voltage is the same over the two resistances and that the currents could be rewritten by Ohm's law.
$\frac{U}{R_{ers}} = \frac{U}{R_1} + \frac{U}{R_2}$	The teacher substituted the currents in Kirchhoff's law with expressions for the currents derived from Ohm's law.
$U \cdot \frac{1}{R_{ers}} = U \cdot \left(\frac{1}{R_1} + \frac{1}{R_2}\right)$	The teacher then factorized the expressions on both sides of the equal sign,
$U \cdot \frac{1}{R_{ers}} / U = U \cdot \left(\frac{1}{R_1} + \frac{1}{R_2}\right) / U$	and he divided by U on both sides of the equal sign.
$\frac{1}{R_{ers}} = \frac{1}{R_1} + \frac{1}{R_2}$	He cancelled U/U and got the formula for the total resistance in a parallel circuit.

In this way the teacher derived the origin of the formula for the total resistance in a parallel circuit, by using algebra. The teacher used electricity knowledge in the beginning when he used Kirchhoff's law and when he explained that the voltage is the same over the two resistances and the same as the voltage source. This is categorized as general electricity knowledge, since it is broad knowledge about electrical circuits. Thereafter he used mathematical knowledge characterized as general mathematical knowledge, merging two formulas and simplifying the merged formula, which is not specific to only this task. In summary:

Knowledge base	**Type**	**The teacher's explanation**
Mathematical	General	Uses algebra to rewrite Ohm's law for the part currents
Mathematical	General	Merge Kirchoff's law with Ohm's law for the part and the head current
Mathematical	General	Breaks out U from the expressions on both sides of the equal sign in the Kirchoff's law equation and cancels those U
Electrical	General	Starts with Kirchoff's law, $I=I_1+I_2$
Electrical	General	Reasoning that it is the same U over R1, R2 and the voltage source

The electricity teacher

The electricity teacher said that in class he normally shows the origin of the formula by calculating the total resistance in the circuit with the aid of different chosen voltages. The electricity teacher's explanation is described below:

Teacher wrote:	*Teacher did:*
$I = I_1 + I_2$	The teacher started with Kirchhoff's law, and said that the sum of the part currents in the circuit diagram equals the head current.
$U = 9V$	He showed in the circuit diagram that it is the same voltage over the both resistances. He made an experiment choosing the voltage 9 V in the circuit
$I_1 = \frac{9}{220} A = 0.041A$	and calculated the part current I_1 using the voltage 9V,
$I_2 = \frac{9}{120} A = 0.075A$	and the part current I_2 also using the voltage 9V.
$I = I_1 + I_2 = 0.041 + 0.075 = 0.116A$	The teacher calculated the head current as the sum of the part currents.
$R_{ers} = \frac{U}{I} \Omega = \frac{9}{0.116} \Omega = 77\Omega$	The total resistance was calculated with Ohm's law and the voltage and the head current in the circuit.
$\frac{120}{R_{ers}} = \frac{120}{R_1} + \frac{120}{R_2}$	After that the teacher said that he uses to try other voltages, and he gets the same total resistance independent of what voltage he chooses, for example 12V or 120V.
$\frac{1}{R_{ers}} = \frac{1}{R_1} + \frac{1}{R_2}$	The teacher said that in the given formula the voltage is 1 V. He said that as long as it is the same voltage as numerator in the formula the total resistance will be the same. He also said that the advantage with having the voltage 1V is that you have the button on the calculator (1/x), which makes it easy to calculate this.

The mathematical knowledge this teacher used is categorized as specific mathematical knowledge, as he is calculating the electrical magnitudes in this particular circuit. Also the electricity knowledge the teacher used is categorized as specific electricity knowledge, when he chose different voltages to use in his calculations. The teacher concluded that the total resistance in the circuit is independent of what voltage there is based on his calculations and this was categorized as specific mathematical knowledge, as this conclusion is built of specific number examples. The teacher also used general electricity knowledge when he introduced Kirchhoff's law and explained that the voltage is the same over the two resistances and the same as the voltage source. In summary:

Knowledge base	**Type**	**The teacher's explanation**
Mathematical	Specific	Calculating the part and head currents in the circuit for different chosen voltages
Electrical	Specific	Chooses a voltage to be able to calculate currents and calculates part and head currents and total resistance in the loop
Mathematical	Specific	Makes the conclusion that the total resistance in a parallel circuit is independent of what voltage there is, based on the concrete examples that have been calculated
Electrical	General	Starts with Kirchoff's law, $I=I_1+I_2$
Electrical	General	Reasoning that it is the same U over R1, R2 and the voltage source

The teachers used different argumentation and reasoning in their explanations. The mathematics teacher used general mathematical and electricity knowledge and the electricity teacher used specific mathematical and electricity knowledge.

Conclusions and discussion

This interview study of mathematics and electricity teachers' explanations of a mathematical electricity task shows both similarities and differences in the knowledge the teachers draw upon, but as only eight teachers were interviewed this gives only a glint of how this could look like at schools. At the overview level, with the teachers' explanations categorized in subject matter knowledge (SMK) and pedagogical content knowledge (PCK) in both mathematics and in electricity, it appears that the two teacher groups' explanations are mostly similar. But detailed analysis of a specific explanation of a single topic suggests that a teacher's argumentations and reasoning could be characterized in general and specific mathematics and electricity knowledge, and further studies are needed to refine this. The detailed analysis showed that the mathematics teacher used general mathematical knowledge, whereas the electricity teacher did not. Other studies have shown that general mathematics is not part of workplace mathematics which may also be a reason in this study. Noss, Hoyles and Pozzi (2000, p. 32) wrote: "From a mathematical point of view, efficiency is usually associated with a general method that can then be flexibly applied to a wide variety of problems […] The crucial point is that orientations such as generalizability and abstraction away from the workplace are not part of the mathematics with which practitioners work". The electricity teacher in this study used specific mathematical and specific electricity knowledge, sufficient for his explanation and also discussed in workplace mathematics research: "For the practitioners in our studies, the computational and estimation methods of routine activity – in all their many forms – were more than adequate for their purposes." (Noss et al., 2000, p. 32).

This study shows that explanations from mathematics and electricity teachers could differ in their use of specific and general teacher knowledge and this raises questions regarding students' abilities to reconcile the different approaches.

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The Emergence of Mathematical Meaning and Disciplined Improvisation

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This paper argues, theoretically, for a pedagogy of disciplined improvisation, a pedagogy promoting the emergence of mathematical meaning. Problem solving provides a context for disciplined improvisation with the balance of structure and openness needed for meaning to emerge. This is illustrated by an example from a lesson in a primary (elementary) school demonstrating the mathematics that can emerge when conditions are set up for disciplined improvisation. Explicitly introducing teachers to the discipline behind disciplined improvisation is thus argued to provide pedagogic strategies that teachers can use to structure lessons for emergence of meaning.

Introduction

Matusov (1998) distinguishes between two views of development – through internalisation or through participation – arguing that different world-views underlie a preference for one or other of these positions. Daniels (2001) develops this by distinguishing between 'skills and functions' in the 'internalisation thesis' and 'meaning' in the 'participation thesis'. The effects of these world-views can be detected in the tensions and contradictions within which mathematics teachers find themselves positioned. On the one hand there is increased pressure from centrally mandated curricula and regimes of testing ostensibly to ensure that students achieve particular, pre-specified, learning outcomes. Advice to teachers on ensuring such outcomes can be situated in the internalisation view of development and a skills and functions view of learning mathematics whereby effective teaching techniques can be identified, specified and implemented. Thus teaching (and learning) is considered to function as a *complicated* system (Davis & Sumara, 2006) and, like other complicated systems, such as analogue clocks, can be 'engineered' to bring about pre-determined outcomes.

On the other hand there are calls to teachers to encourage learners' creativity and to help them develop 'productive dispositions' that go beyond learning mathematics as a collection of techniques. Such 'soft' learning outcomes cannot be engineered into being so advice to teachers is couched in terms of 'facilitation' and 'affordances'. Teaching for creativity and productive dispositions is rooted more in the participation thesis, viewing classrooms and schools as *complex* systems. In contrast to the high predictability of complicated systems, in complex systems the precise outcomes of events cannot be predicted. If the analogue clock is the archetypal metaphor for a complicated system, then gardens are typical of complex systems. Cutting down, say, a garden shrub will bring about change, but whether that change will result in the system flourishing or deteriorating cannot be known. Changes in a complex system may not be predictable, but nor are they random – theoretically at least, changes are explainable after the event. In contrast to the language of mechanics and engineering that characterises complicated systems, complex systems are described in terms of organics and emergence (e.g. Johnson, 2001; Senge, 2006).

So while theorists (and some teachers) might argue for recognising teaching and learning as a participatory complex system, policy directives, in the main, continue to take a internalisation position, positioning teachers as 'deliverers' of mathematical knowledge that, given appropriate instruction, should not be problematic for learners to acquire. How then are teachers to reconcile these two positions? Sawyer's (2004) construct of 'disciplined improvisation' provides a metaphor for reconciling such oppositions, as will be argued for in this paper.

Theoretical background

Collaborative emergence

Sawyer (2001) traces the origins of the concept of emergence to 1875 and the philosopher George Henry Lewes' distinction between two types of effects: resultants and emergents. In emergent effects, Sawyer argues, outcomes cannot be fully understood or predicted by studying the constituent parts, as illustrated by Lewes' example of the effect of water emerging from the combination of oxygen and hydrogen. The properties of water cannot fully be understood by reduction to studying the properties of oxygen and hydrogen (although quantum physics now overturns this claim). Emergent phenomena are non-reductionist and multiplicative rather then additive in their nature (Davis & Simmt, 2003).

Although the concept of emergence has been developed since Lewes' time, particularly in the physical sciences, it began to have impact on educational research with the development of artificial intelligence systems displaying intelligent behaviour based on simple, local rules of interaction and without the need for a central processor; models of insect colonies creating complex structures are canonical examples of emergent living systems (Clark, 1997). Group behaviour can be theorised as emergent when there is no structured plan to follow, and where there is no leader directing the group (Sawyer, 1999).

Classrooms and students are, however, fundamentally different from robots and ants in the range of actions and agency available to the participants. Sawyer's phrase of collaborative emergence to encompass phenomena "that result from the collective activity of social groups" (1999, p. 449) helps to delineate systems where there is interaction and agency, in the sense that individuals within the system can intentionally change what is emerging.

Many social activities might be considered to be examples of collaborative emergence, for example, football matches or jazz performances. The analogy of theatrical improvisation ('improv') is helpful when applied to teaching and learning, and best described as 'disciplined improvisation' (Sawyer, 2004).

Disciplined improvisation

Everyday use of the word 'improvise' suggests a lack of structure and spontaneity. Talk of a lesson being improvised smacks of a teacher being not prepared or 'winging it'. Theatrical improvisation is, however, highly disciplined. To the audience it may appear that an improvised scene is simply the result of actors 'freewheeling', but from the performers' perspectives what emerges is grounded in rules and principles. 'Disciplined improvisation' draws attention to improvisation as an aspect of teaching that brings its own discipline.

Improvisation in classrooms must also be disciplined in the sense of knowing when enough is enough – when to accept and build on learners' offerings and when to direct the lesson back in a particular direction. A challenge is to plan lessons that are sufficiently open to allow for the possibility of learners making contributions that can be built upon but at the same time sufficiently structured to allow for the mathematics that emerges to be worthwhile. Problem solving provides one approach to planning for a balance of structure and openness, for disciplined improvisation. Teachers 'over-structuring' problem solving lessons can close down chances for meaning to emerge (e.g. Henningsen & Stein, 1997). Explicitly introducing the discipline behind improvisation may help teachers to appreciate the potential within problem solving and provide pedagogic strategies that can bring structure to lessons without closing them down. An example from a lesson in a primary (elementary) school illustrates how improvised solutions to a problem can form the basis of emergent meaning.

The school context

This example comes from a two-year teaching experience in a primary school. At that time I was looking to go back to do some school teaching to explore the difficulties of enacting pedagogical reform. Several years of research had revealed scant evidence of problem-solving based pedagogies that are written about and I wondered if teachers were right in sometimes claiming that, given the constraints of schooling, problem-solving based teaching was not possible. Research had also shown that calls for pedagogic reform could lead to a change in teachers' discourse but not change in their practices (Askew, 1996; Handal & Herrington, 2003). Thus I wanted to study change in real-time – in approaching the local authority for a school to work in, Bow Bells was suggested.

Bow Bells school is located in a working-class area that more recently attracts an immigrant population. At the time of this teaching, pupil performance on national tests showed around 45% of eleven-year-olds were attaining the expected level, compared to government targets of 80%. Alongside this, inspection reports painted a picture of a school in difficulty. The publicity surrounding test results and reports did not attract teachers to apply to work there and the school was thus in a downward spiral. Thus the children we worked with were not ‘privileged’ or had had the kinds of experiences that might pre-dispose them to problem-based pedagogies.

Initial meetings with the teachers revealed two cultural beliefs about learners at the school. First, teachers commented on the limited language facility of the children (even for those children for whom English was their first language) and expressed the view that there was little point in asking the children to talk about mathematics. Second, and linked to the first, there was a general sense that the children had little to contribute to mathematics lessons: children had to learn the ‘basics’ before they would be able to engage in any form of problem solving or reasoning. This attention to the ‘basics’ permeated the school in all grades with a predominant teaching style across all the years of teachers showing a method on the board and the children subsequently completing practice worksheets.

A colleague, Penny Latham, and I argued with the staff that rather than accept these perceived limitations as givens our focus in improving learning outcomes would be on supporting the children to talk about mathematics and to develop their mathematical understanding through problem solving.

The example that follows is typical of the work we did. It comes from a class of eight- and nine-year olds, towards the middle of a year of work with them and their teachers. The lesson was co-taught by the regular class teacher and myself.

The children had been taught fractions previously, largely through worksheets with a focus on ‘recognising’ fractions as a part of a unit whole, not as the result of the operation of division with non whole number answers. The problem we chose to work on was a division problem that would result in fractions as the answer, similar to one described by West and Staub (2003, Chapter 7). Two aspects distinguish the approach we took from that described by West and Staub. First, the teacher and I did not try to anticipate likely solution methods: we wanted to see what would emerge and not be inclined to steer the pupils in particular directions (although in the event the solutions produced are very similar to those described by West and Staub). Second in setting up the problem we did not present any visual images to the class – we were interested in the representations the learners created and how they would work with these.

Although the question was cast in an everyday-like context, the learners knew that a school-mathematics-like answer was expected (involving school mathematics objects like fractions). While an answer like “They would not get a full pizza but quite a bit each" might do in the "real" context, we talked through with the children that we were looking for mathematically accurate solutions. The point of the problem context is for the children to bring their knowledge of 'fair shares' and pizzas to the emergent mathematics. The mathematics emerges from the mathematisation of the context (Freudenthal, 1991).

Example: Sharing pizzas

The lesson began with a whole class discussion setting up the context for the problem – going out with friends or family for a pizza. We did not rush to introduce the actual problem, taking time to set up an atmosphere in the classroom that we hoped conveyed the spirit that the teachers were interested in what the children would bring to the problem. Finally we posed the problem, orally:

> 12 friends went out for a pizza. It was towards the end of the month, so they only had enough money to order 8 pizzas. They ordered the 8 and shared them equally. How much pizza did each person get?

The children cooperated in small groups, recording workings and solutions on one large piece of paper. While they worked on the problem the teacher and I encouraged groups to show and explain their methods. As answers emerged we selected two groups to share their solutions with the class and told these children to be prepared to come up to do this. The first method (Figure 1) was typical of what many groups had done: everyone got half a pizza each and the remaining two pizzas could be sliced into sixths (with evocations of the old Egyptian procedure of working with unit fractions). So the friends each got 1/2 + 1/6 pizza.

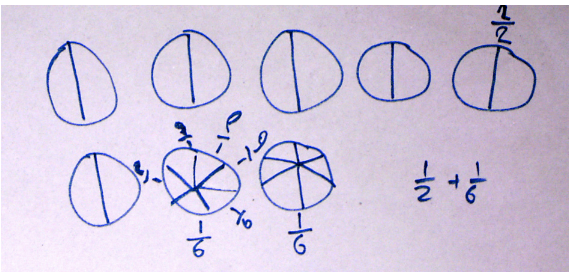

Figure 1. Everyone gets 1/2 + 1/6

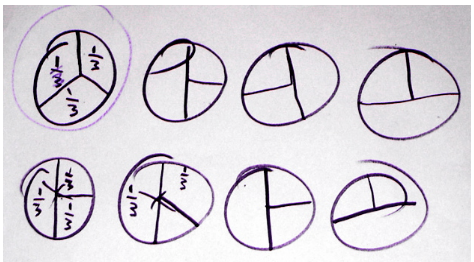

Figure 2. Everyone gets 2/3

The other solution was based on sharing four pizzas so that everyone would get 1/3 pizza (Figure 2). The second four pizzas would provide another 1/3 each. So in this group each person would get 2/3 of a pizza. (Figure 2 shows that some work needed to be done later on in helping the children represent thirds as equal sized pieces, but the principle of their argument was correct.)

After both solutions had been presented and discussed, the class was in agreement that each was correct. But what was happening here? Going out with one group could mean getting 1/2 + 1/6 of a pizza to eat; going with the other group could give you 2/3 of a pizza. Were these the same? If you really liked pizza which group would you want to go out with?

Now the interplay between the improvised solutions and the formal mathematics comes into being. The children knew from their everyday knowledge that, given the sharing was fair in each case, the portions would have to be equal. The challenge was to sort out why, mathematically, these two solutions appeared to be different. We challenged the groups to come up with representations that would show whether the two amounts were equal or different. Figure 3 is typical of the diagrams they produced to show equality.

The plenary discussion about what they had learnt indicated that many children were developing an understanding of equivalence. However, as Brousseau (1997) points out, it is easy to read into answers a knowledge that learners do not possess. To investigate the depth of their emergent understanding we asked the children to write, individually, an account of why they were certain that the two solutions led to equal amounts of pizza. This transcription of a letter is typical of what the children produced (note the evidence for the beginnings of the use of fractions as quantities – a half – and as operators – half of a third).

> I know that two thirds is the same as four sixths because three sixths are a half and two thirds and half of a third is the same as three sixths. So the other half of a third is the same as the other sixth on the diagram.

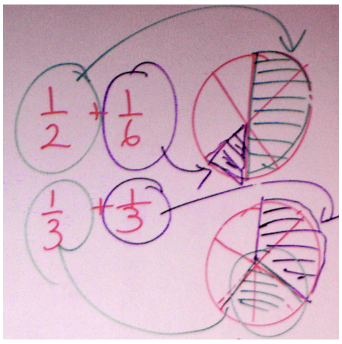

Figure 3. Everyone gets the same

A pedagogy of disciplined improvisation

It is easy to suggest that this example is merely a manifestation of a move from a teacher-centred to a learner-centred pedagogy, but I want to argue that such labelling diminishes both the work of the teacher and the children. I do this by linking Davis and Simmt's theoretical conditions for emergence to the practices of disciplined improvisation and the pizza problem as an example of disciplined improvisation in the classroom. Drawing on complexity theory (e.g., Johnson, 2001) Davis and Simmt (2003) identify five conditions that they consider necessary (but not necessarily sufficient) for the emergence of collective meaning: 1

internal diversity; 2 internal redundancy; 3 decentralised control; 4 enabling constraints; 5 neighbour interactions.

Internal diversity is the result of variations among the group and is necessary for a range of possibilities to emerge. Internal diversity is a given in improv – it drives the improvisation forward. As players pick up each other's offerings these get changed through the diversity. Take, for instance, the miming of offering a gift. Although in the mind of the player making the offering it might be clear that the gift is a necklace, the player accepting it could announce anything from an anklet to a zebra skin or any of an infinite number of possibilities.

In the pizza example the class was of heterogeneous abilities organised in mixed attainment groups. The discussion in these groups and the posters the learners created of their different solutions were outcomes of this diversity. From this perspective of disciplined improvisation, diversity in classrooms is desirable and an asset rather than something that teachers need to 'manage' (through differentiated activities) or 'reduce' (through ability grouping practices).

Internal redundancy is a complement to internal diversity. Redundancy in Davis and Simmt's sense means a degree of sameness in the form of "commonalities of experience, expectation, and purpose" (p. 310). Internal diversity promotes variety, while internal redundancy both facilitates the participants' interactions and allows for gaps in any one participant's contributions to be compensated for by the others. Internal redundancy is built into improvisation through the offers that are not taken up, and also in the way that, as a scene emerges, the range of possibilities is reduced and players work within the parameters of what the discourse that has emerged and the possibilities for how this might develop within the parameters of narrative structures. In the pizza problem the choice of context built in internal redundancy. The children had enough common knowledge of pizzas to be able to build on each other's offering. For example no one produced a representation of a pizza that could not be made sense of by everyone in the group and everybody had a good understanding of 'fair shares'.

Decentralised control is based on the view of complex systems as self-organising. Although the behaviour of a collective may look as though there is some central coordination, this sense of wholeness actually emerges from locate interactions (Varela, Thompson, & Rosch, 1993). The dominance of the 'if … then' thinking arising from complicated systems may be a cultural origin of assuming that events are centrally determined through cause-and-effect rather then emerging through complexity (Johnson, 2001). Decentralised control is a core tenet of improv (unlike scripted drama with the control of the director). As soon as one player tries to 'hog' a scene or another 'wimps out' the piece falls apart. Good scenes emerge from sharing the control. In the pizza problem control over the methods of solution was decentralised with the teachers learning from the students and using this to shape the plenary discussion rather than delivering a scripted explanation of equivalent fractions. I agree with Davis and Simmt's observation that it is not helpful to think of such activity in terms of being teacher-centred or learner-centred, mainly because 'the phenomenon at the center' was not an individual, but a collective phenomenon of a shared insight (p. 311).

I suggest that a key practice for establishing decentralised control in classrooms is spending time getting the learners to 'buy into' the problems. In improv, the opening minutes of a scene are spent establishing it; from the interplay of offers and take-ups players establish who the characters are, their relationship to each other and where the scene is taking place. Establishing this shared context is similar to what Becker (2000) in his analysis of jazz improvisation calls "a real shared interest in getting the job done" (p. 175). Like other researches leading to rich pupil solutions (e.g. Fosnot & Dolk, 2001) the time spent at the beginning of the lesson setting the context for the problem was not simply window-dressing or a device to make some unpalatable calculations acceptable. There was a general 'suspension of disbelief' created by spending time setting up the scenario, in getting buy-in from the children. This is in contrast to some views that artificial problems do more harm than good. While I would agree that the quick word problem about shopping, followed by another about cooking does not encourage engagement, more use could be made of more extended narrative scenarios to hook children in, as, for example, explored by Zazkis and Liljedahl (2008).

Enabling constraints permit decentralised control and enable the emergence of phenomena: "Complex systems are rule-bound, but those rules determine only the boundaries of activity, not the limits of possibility" (Davis & Sumara, 2006, p. 311). In improv enabling constraints set boundaries that, somewhat paradoxically, enable to the emergence of scenes that are more than the sum of the offerings of individual players. Key rules here are 'yes and …' — accepting and building upon the offers of others and not 'pre-scripting' the scene — mentally writing how you want it to go in advance of its emergence. In the pizza problem the children were organised into groups of three or four and the problem to work on carefully structured, but within that organisation there was variation in the methods and approaches to finding solutions, variation that could then be drawn upon to deepen the pupils understanding (Marton, Runesson, & Tsui, 2004).

Neighbour interactions, for Davis and Sumara, have to be considered as operating beyond the literal interpretation of learners working together. "Rather, the neighbours that must 'bump' against one another are ideas, hunches, queries, and other manners of representation" (p. 312). All improv scenes are built upon neighbour interactions and players words and actions – the offerings – 'bump' against each other. In the classroom neighbour interactions were occasioned at two levels: the interactions at the group level and the subsequent interactions between the groups. One key feature arising from such bumping together of ideas is that the meaning attributed to an offering may only become apparent after the event, of 'reverse causality' (Matusov, 1998, p. 330). Experienced improvisers testify to reverse causation. At the beginning of a scene, improvisers have a whole range of options open to them (one of the disciplines of improv is to keep these options open for as long as possible), once the form and content of the scene starts to emerge, actors will talk afterwards of the scene 'writing itself'.

For example, the group that solved the problem by sharing the pizzas out in thirds did not set out with that intention in mind: the size of paper and their choice of how big to draw the pizzas resulted in them producing a representation of two rows of four. This representation occasioned their use of dividing each pizza into thirds, but this decision emerged in the course of creating the representation (as opposed to deciding to divide the pizzas into thirds first and drawing four pizzas to facilitate this division). The meaning emerging from the representation of two rows of four was thus "distributed across time, space, and participants, interpreted and renegotiated" (ibid., p. 330). Meaning thus emerged from participation in the improvisation of a solution rather than this being a manifestation of some pre-existing understanding that the learners had already acquired and were deliberately bringing into being in solving the problem.

Conclusion

Disciplined improvisation provides a structure for looking at and working with Davis and Simmt's suggestion that we need to move attention away from what must happen in lessons to being open to possibilities, from a perspective of prescription to proscription. Proscription, Davis and Simmt argue, is more about setting out what is forbidden in contrast to a prescriptive stance of what is allowed (and by implication everything else is forbidden). In the case of the problem worked on here, the children were not allowed to work alone, and the expectation had been set up that they needed to show on paper how they were setting about solving the problem. In contrast to previous problem solving lessons in the school, which followed the teaching of particular topics and so cued learners into the mathematics that they were expected to use, the learners were encouraged to solve the problem in whatever way they chose (although, as pointed out above, there was a common expectation that this would be a mathematically acceptable solution). Thus a range of possibilities was opened up for meaning to emerge. Explicitly working with teachers on improv is one way of introducing them to the possibilities arising from proscription rather than prescription.

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Relating Vocabulary in Mathematical Tasks to Aspects of Reading and Solving

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This paper focuses on relationships between vocabulary in mathematical tasks and aspects of reading and solving these tasks. The paper contains a framework that highlights a number of different aspects of word difficulty as well as many issues to consider when planning and implementing empirical studies concerning vocabulary in tasks, where the aspect of common/uncommon words is one important part. The paper also presents an empirical study where corpora are used to investigate issues of vocabulary in mathematical tasks. The results from the empirical study show that there are connections between different types of vocabulary and task difficulty, but the connections could be mainly an effect of the total number of words in a task.

Introduction

When using written tests to assess students' mathematical ability, one aspect of validity is whether tests measure mathematical competence and nothing else. However, there is always a possibility that the test also measures the students' reading ability. Some researchers describe language as separated from mathematics (e.g. Greenlees, 2010), and issues of language are then not supposed to add difficulty to mathematical tasks. However, it might also be reasonable to include a language component in the concept of mathematical knowledge, which is supported by the aspect of communication presented as a part of mathematical proficiency in several frameworks (e.g. NCTM, 2000). The complex relationship between language and mathematics described here is the starting point for several studies that we are currently carrying out or planning (see also Österholm & Bergqvist, 2012a, 2012b). The overarching goal of our research is to better understand and be able to describe the connection between reading and solving mathematical tasks. In this particular paper we focus specifically on connections between vocabulary in mathematical tasks and aspects of reading and solving such tasks. In particular, we discuss and examine different types of words: everyday words and mathematical words.

Background

Many studies highlight the importance of vocabulary, both in relation to aspects of reading comprehension and in relation to the solving of mathematical tasks, and several different methods are used to study aspects of vocabulary. *Counting letters and/or syllables* to measure word length is one. Using syllables to measure sentence length or word length is an important linguistic measure of readability, and the number of letters as a measure of word length is also part of some readability formulas (DuBay, 2004). When *judging word difficulty and/or familiarity,* experts sometimes judge the words (e.g. Shaftel, Belton-Kocher, Glasnapp, & Poggio, 2006). A third type of method is using different *lists of words* to compare with. For example, Helwig, Rozek-Tedesco, Tindal, Heath, and Almond (1999) define *word familiarity* using a list of words that contains information about grade level and percentage of students that correctly could identify the meaning of the word.

An alternative method is to *calculate frequencies in particular corpora* in order to characterize words as being familiar or not. This method is based on the assumption (supported by empirical results, see e.g. Breland, 1996) that more common words are also more familiar. In modern linguistics, a corpus can be defined as "a collection of pieces of language text in electronic form, selected according to external criteria to represent, as far as possible, a language or language variety as a source of data for linguistic research" (Sinclair, 2005, p. 23). Different corpora are composed to represent different types of language and therefore they consist of words from different sources of text, for example newspapers or scientific articles.

Many studies, including several of the ones mentioned above, use statistical methods to investigate the relation between linguistic properties of a tasks and the students' performance on the tasks. However, these methods have serious limitations. The information gained from statistical computations usually concerns to what extent different linguistic aspects of a task (e.g. the amount of long words) correlate with the difficulty of the task, but it does not inform us on *why*. For example, longer words can be more difficult to decode phonologically, an issue that relates specifically to aspects of reading ability, but perhaps mathematically complex concepts are usually represented by longer words, an issue that relates specifically to aspects of mathematical ability. Therefore, if we only rely on statistical correlations between variables describing a task and students' performance on the task, we cannot conclude that a language aspect is simply related to reading ability. Other methods have been utilized in order to overcome this issue, often by using data of students' results not only on a mathematics test but also on a reading test of some kind. However, methods that use correlations and regressions in different ways have problems with aspects of validity or reliability (Österholm & Bergqvist, 2012a). Based on our previous methodological analysis, we suggest an approach using principal component analysis to measure a task's demand of reading ability. This method is described in detail in a previous publication (Österholm & Bergqvist, 2012a) and is summarised in the section *Empirical study* below.

Purpose

This paper presents the first part of a study where the purpose is to increase the understanding of the connection between different aspects of the vocabulary of a mathematical task and other aspects of the task, in particular difficulty and demand of reading ability. First, we propose a framework for the study of *difficult vocabulary* in mathematical tasks, that is, we focus on properties that can be seen as potentially causing difficulty (of any kind) for students when reading and solving the task. The framework gathers different perspectives on vocabulary from prior research and also includes discussions of empirical methodology, issues we see as missing in prior research. Second, we present a pilot study that introduces an empirical method using corpora to investigate issues of vocabulary in mathematical tasks. The method explores benefits of combining information from different corpora and to not only use information about whether words are "universally" common or uncommon, as is usually done in previous research. Our research questions are:

- What factors are there to consider when issues of difficult vocabulary in mathematical tasks are to be studied?
- Is there a connection between the difficulty or the demand of reading ability for a mathematical task and whether the words in the task are common or uncommon in mathematical language and/or everyday language?

A framework for the study of difficult vocabulary

Here we describe a first version of a conceptual framework regarding the notion of *difficult vocabulary*. In this framework we include what can be seen as different aspects of difficulty regarding the words in mathematical tasks, together with perspectives on how to analyse these aspects in empirical studies. However, for the present paper we focus our attention on the one aspect of common/uncommon words.

The framework has been created based on issues highlighted in previous research (in particular, see the background in this paper) together with our suggestions of alternatives to the perspectives described in existing research literature. Such alternatives have been noticed as relevant and important while planning the empirical study described below. Presented here is a first version of the framework, which at this point primarily consists of a structured description of issues highlighted in previous research. Note that some parts of the framework are here described more briefly, due to space restrictions.

Analysing the difficulty of a word

Here we focus on the analysis of singular words. We include in our framework the following five aspects of word difficulty, of which we only elaborate on the fourth aspect in the present paper: (1) word length, (2) word form (e.g. verbs in a passive voice or nominalised verbs/adjectives), (3) word type (e.g. pronouns or modal verbs), (4) common/uncommon words (word familiarity), and (5) word meaning (e.g. complexity of a concept or a word's potential ambiguity).

We here include four issues to take into consideration regarding the notion of common/uncommon words, and thereafter we discuss the process of analysing these issues in empirical research.

1. When, where, how and who? When a word is labelled as common or uncommon, this needs to be in relation to a certain population or discourse community. For example, oral everyday language can be seen as specifying the language used in a certain type of situation (where) and in a certain form/modality (how). This issue also includes considerations of whose vocabulary is referred to (e.g. regarding age or ethnicity) and the question of when, since language changes continuously.

2. Discourse-specific vocabulary. This issue refers to a relationship between different discourses (or populations), regarding information about whether a word is specific to a certain discourse.

3. Derivations and lexemes. This issue highlights the question of whether to focus on a specific word as being common/uncommon or to focus on its components or *lexeme*. Lexeme refers to the set of different forms a specific word can have. For example, "stand", "stood", and "standing" are elements of the same lexeme. An argument for focusing on lexemes instead of words is that even if a specific word is relatively uncommon, a reader can perhaps directly see the word as a form of a more common word from the same lexeme. Thus, the word is not as difficult as could be believed from how uncommon it is (e.g. see Dempster & Reddy, 2007). However, a keyword in the above argument is "perhaps", that is, we cannot know if or when students make this connection between a word and its lexeme, in particular for more complex types of forms of words (also discussed by Dempster & Reddy, 2007).

4. The context. Since the same word can have different meanings in different contexts, it could be necessary to distinguish between different meanings of words when analysing how common/uncommon they are.

With these four general issues as a basis, we now turn our attention to more practical issues, regarding the planning and implementation of empirical studies about common/uncommon words in mathematical tasks. We focus on the use of some type of explicit reference material in the analysis, in particular word lists and corpora. We describe three steps in the analysis of how common/uncommon a word is using a reference material.

A. Choosing/creating reference material. A few questions can be asked regarding any type of reference material. First, we have the question of *what material to include*, which refers primarily to the first general issue; for the material to be representative of some specified population, situation, and time. For example, to have a corpus for "school language" we need texts from all subjects. Second, we have the question of *what type of meta-information to include*, if any. For corpora, different types of linguistic meta-information can be included. Word lists usually include only the most common words, possibly together with some type of meta-information, for example about frequency of words or about the fraction of students at different school levels who know the meaning of each word (Helwig et al., 1999).

B. Searching for words in reference material. When searching for a word in a reference material, you need to decide what to count as a word and what to count as the same word. Primarily this decision can be about issues number 3 and 4 above. For example, when a corpus includes lexical meta-information, it is possible to search for lexemes and not only specific words. Also, you might need to take into consideration some issues at a more practical level, such as hyphens and spaced words (e.g. if "lifestyle", "life-style", and "life style" are seen as the same word).

C. Characterizing words. The frequencies of words can be used in different ways to measure how common words are. In corpora, a relative frequency can be used as a direct measure, comparable between corpora, of how common a word is. It is also possible to use frequencies, absolute or relative, as a basis for ranking words, and the ranking would then be possible to compare between corpora. The words can also be labelled as common or not in different ways, not directly based on relative frequencies, but on whether a word is included in a given list of (the most) common words (e.g. Dempster & Reddy, 2007) or similarly, whether a word is included among the 1000 (or any chosen number) most common words in a certain corpus (e.g. Österholm & Bergqvist, 2012b).

Analysing mathematical tasks regarding difficult vocabulary

After analysing each word in a mathematical task, the next question is how to use the information about each word to characterize the task. Here we discuss two issues that were necessary to handle when planning the empirical part of this paper, and that need to be handled when planning this type of empirical analysis in general.

Characterizing the "amount of difficulty" in a task. This issue can be measured in several different ways, for example by focusing on: the mere existence of difficult words; the number of difficult words; the proportion of difficult words; the mean of some quantified word property (e.g. to calculate the mean of the frequency of all words); and the spread of some quantified word property (e.g. to calculate the standard deviation of the frequency of all words).

Different parts of the task text. Included here are decisions about whether and how to include certain parts of the task text in the analysis, for example regarding: sub tasks and the leading text (i.e. the part of the task text that is common for all sub tasks); background information and the prompt (i.e. the question or description of what to do); tables, diagrams, figures, and symbols; and the repeating of a word.

The questions concerning both these issues can be seen as empirical questions, regarding which method most truly captures a potential difficulty in a task. For example, the creation of separate difficulty variables for different parts of task texts could make it possible to examine whether a potential difficulty in the leading text of the actual question is of most importance for a certain set of tasks.

Empirical study

This *pilot study* has the purpose to examine whether and how the presence of common or uncommon mathematics or everyday words is connected to the demand of reading ability or to the difficulty of a mathematical task. At this point, details of the analysis and choices made should be seen as preliminary and the results as tentative.

Corpus analysis has been used in mathematics education research by for example Monaghan (1999) who argues for further corpus analyses in order to get a more thorough understanding of the mathematics register. The present study explores how corpora could be used in an analysis of the vocabulary in mathematical tasks. More specifically, our focus is on common/uncommon words, which is relevant since more common words are also more familiar (see Breland, 1996). An earlier study (Österholm & Bergqvist, 2012b) did not show a significant correlation between demand of reading ability and frequency in test tasks of common words in any of two corpora. We therefore simultaneously use information from the two different corpora, in order to examine the existence of different types of words, in particular technical vocabulary, that is, words that are common in mathematical texts (especially textbooks) but uncommon in everyday language.

Method in the empirical study

We utilize results from PISA in order to have access to data from many students and many tasks, both mathematical tasks and reading tasks, which is crucial for our analysis of the demand of reading ability. Since the same mathematical tasks were used in 2003 and 2006, we combine results from these years in our analyses.

Our analysis consists of three steps. First, values for the variables *demand of reading ability* and *difficulty* are calculated for each mathematical task. Second, two different corpora are used to determine how common the words in the mathematical tasks are in two contexts (mathematics and everyday contexts). Third, correlations between the information from the first and second steps are analysed, and based on the results we discuss what the presence of words common/uncommon in different contexts means for the difficulty or demand of reading ability of the tasks.

In order to measure a mathematical task's *demand of reading ability*, a principal component analysis (PCA) is used. This method is presented and discussed in more detail in previous papers (Österholm & Bergqvist, 2012a, 2012b) and only briefly described here. In this study, all Swedish students' scores on all PISA mathematics and reading tasks from 2003 and 2006 are entered into the PCA, from which the first two components are extracted, which are expected to correspond to mathematical ability and reading ability. For each mathematical task, the loading value on the reading component is taken as a measure of the demand of reading ability.

As a measure of *task difficulty* we use the percentage of credited responses for the task (the p-value), which means that if the sum of all credits that the students get are 75 % of all possible credits, the p-value for that task is 0.75.

In order to determine how *common* or *uncommon* particular words in the mathematical tasks are in different contexts, we use two different corpora. The corpus we use to represent everyday language (of society in general) is composed of 58 novels (about 4.7 million words) and newspapers from the years when the PISA tests were distributed (i.e. 2003 and 2006; about 42 million words) [1]. To represent mathematical language, we use a corpus consisting of two mathematics textbooks intended for year 8 students (the same age group as the students that take the PISA tests; about 70,000 words), which are part of the OrdiL project (Lindberg & Johansson Kokkinakis, 2007). Since a purpose with this pilot study is to explore the use of corpora, we have at this point chosen the corpora partly based on easy access, and not created any new corpora specifically for the present study.

We analyse the words in the PISA mathematical tasks by searching for them in both our corpora, and retrieving the *frequency of each word in each corpus*. The search is made on the specific form of each word, and not on lexemes (see the framework), mainly due to non-existing meta-information in the corpora. When words contain a hyphen, the hyphen is included as part of the word during the search. Due to technical limitations in the search procedure, words with "strange" mixtures of upper and lower case letters (e.g. "woRd") are treated as separate words, but "Word", "word", and "WORD" are treated as the same word. All words in the mathematical tasks, also from tables, diagrams and figures, are included in our analyses, but "words" that consist of or include symbols, numbers, or punctuation marks (except hyphens) are excluded, for example labels such as "P5". Words consisting of only one letter are also excluded since they sometimes denote variables.

Based on the information on word frequency in each corpus, we sort the words into *four categories*, by labelling each word as common or uncommon in each of the two corpora. Separately for each corpus, based on the frequencies, we divide the group of unique words into two groups of equal size, so that half of the words are labelled common and the other half of the words are labelled uncommon. The words with zero frequency are excluded in the creation of the two groups since these words are seen as representing a flooring effect in the data, but they are thereafter included in the group of uncommon words for the continued analysis.

For the mathematical tasks, we define *four different variables*, as the number of words in the task in each of the four categories of words. Sometimes a group of tasks are preceded by a common introductory text. In those cases, we include the words in the introductory text in each of the tasks, since at this point no analysis of the need of the leading text for the understanding of each task is done.

To determine whether the types of words used in the tasks are related to the demand of reading ability and/or the difficulty, two-tailed non-parametric correlations are used (Spearman R coefficient), with a significance level of 0.05. Besides the four different types of words described above, the total number of words in tasks is also added as a variable in the correlation analysis. This variable is included in order to examine whether any significant correlations to the number of certain types of words in tasks could be an effect of the total number of words in tasks.

Empirical results

Vocabulary property	**Demand of reading ability**	**Difficulty**
Uncommon both	r = -0.071 (p = 0.578)	**r = 0.366 (p = 0.003)**
Uncommon math, common everyday	r = -0.230 (p = 0.070)	**r = 0.451 (p = 0.000)**
Common math uncommon everyday	r = -0.142 (p = 0.267)	r = 0.059 (p = 0.648)
Common both	**r = -0.275 (p = 0.029)**	**r = 0.497 (p = 0.000)**
Total number of words	r = -0.232 (p = 0.068)	**r = 0.442 (p = 0.000)**

Table 1. Correlations between the number of words of different types in tasks and the tasks' demand of reading ability and difficulty (N=63).

Table 1 shows that the total number of words in the tasks correlates in a significant way with difficulty and almost significantly with demand of reading ability. These results make it difficult to draw conclusions about any relationships between the number of *different types* of words in the tasks and difficulty or demand of reading ability. All correlations to difficulty are positive, while the correlations to demand of reading ability are all negative. That is, tasks with more words tend to be more difficult but also tend to have lower demand of reading ability, and the opposite is true for tasks with fewer words. However, there is also a significant negative correlation between the difficulty of a task and its demand of reading ability ($r = -0.589$, $p = 0.000$), making it even more difficult to interpret the correlations in Table 1.

Conclusions and discussion

Our empirical analyses show that there are clear quantitative connections between aspects of vocabulary and task difficulty, but these connections could be mainly an effect of the total number of words in a task and not different *types* of words. Connections between vocabulary and demand of reading ability are generally weak, but existing tendencies may also be an effect of the number of words in general, rather than of any specific types of words. More studies are needed in order to handle the uncertainties in these conclusions.

Our result showing that the effect of the total number of words might be primary, questions the results from other studies where the number of "difficult" words has been included as an important factor without considering the effect of the total number of words. However, Shaftel et al. (2006) use regression analyses to examine both the total number of words and several different aspects of difficult vocabulary, with task difficulty as independent variable. Their results show a non-significant regression coefficient for the total number of words, but significant coefficients for several other vocabulary properties, including the feature they label as "Math vocabulary". However, in their study they use expert judgments for which the description of "Math vocabulary" is somewhat unclear: "unusual or difficult but specific mathematics vocabulary words" (p. 126). More in-depth analysis is needed in order to explain the discrepancies between their study and our results.

Through the presented framework we have created a structured description of factors to consider when analysing issues of difficult vocabulary in mathematical tasks. In future development of our framework we will include relationships to theories of reading comprehension in order not to limit the framework to *describing* practical aspects of empirical research but also to include an explanatory dimension, for example to include *why* and *how* different properties of vocabulary can be seen as causing difficulty for students when reading and solving a mathematical task.

Note

1. From the Swedish Language Bank (http://spraakbanken.gu.se/korp/): the part with novels is labelled *SUC-romaner* and the parts with newspapers are labelled *GP 2003* and *GP 2006*.

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Researching Classroom Assessment in Mathematics: Theoretical Considerations

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The main theme in this paper is what two theoretical approaches, multimodal social semiotics and discursive/institutional theories, can convey when researching assessment displayed through feedback in mathematics classrooms. The theories were operationalized in a project finalised in 2010, and the findings of this project comprise the secondary theme of this paper. These findings consist of four discourses of assessment in mathematics classrooms, in which also roles of semiotic resources (e.g. graphs, gestures, speech) are taken into account. The discourses and how these are present in institutional traces (for example, decisions on a municipality level) are means for addressing the mathematics classroom as part of and affected by a broader institutional context.

Classroom assessment in mathematics

The basis for this paper is a research project finalised at the end of 2010 (Björklund Boistrup, 2010). Here, I focus on major theoretical considerations, while simultaneously presenting the main findings. Consequently, other parts of the research project, such as methods, are described briefly.

In the project, teacher-student communication in mathematics classrooms were examined with an interest in assessment displayed through feedback in day-to-day communication. One research question foregrounded in this paper is "What discourses of classroom assessment in mathematics can be construed and what affordances can be connected to students' learning and active agency?". I also connect to a question about what institutional traces can be identified in relation to the construed discourses. The research discussed in this paper is one answer to a call since Black and Wiliam (1998; see also Hattie & Timperley, 2007) regarding studies in depth on classroom assessment. Moreover, it is aligned with the argument that studies on assessment based on a social perspective are needed in mathematics education (Morgan, 2000) and, hence, the mathematics classroom is in this paper viewed as part of a broader institutional context. I will address the implications of this view in the last section.

During data collection, two students in each of five grade 4 classes were randomly chosen (for ethical reasons) with data collected via video recordings and written material. Subsequent analyses focused on the interaction between the two student participants and the teacher. Assessment is taking place explicitly through feedback provided when students receive mathematics test results but also implicitly during teacher-student interaction (see Figure 1).

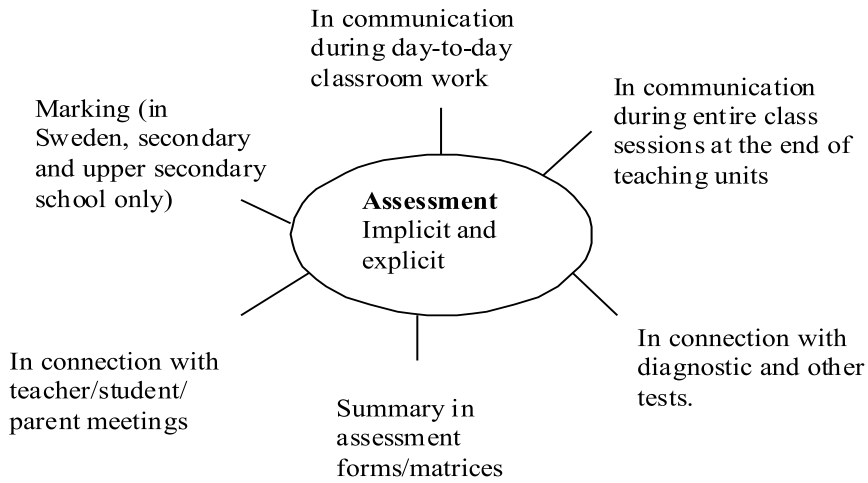

Figure 1. Assessment: A concept with broad boundaries (adapted from Björklund Boistrup & Lindberg, 2007, poster).

Social semiotics and discourses

In social semiotics, the interest is directed towards communication, with special attention given to a broad range of semiotic resources (for example graphs, speech, gestures) and their relation to each other and the social practice (Kress et al., 2001; Van Leeuwen, 2005). Consequently, many kinds of semiotic resources need to be taken into consideration, both in assessment in mathematics and in research on assessment. Assessment of learning is from a social semiotic perspective about acting on signs of learning, as shown by semiotic resources. In adopting a social semiotic perspective, a central notion is that what a semiotic resource represents and communicates is dependent upon the interest of the person using that semiotic resource, the existing situation, and the broader institutional context. In O'Halloran (2000) there is an interest in three semiotic resources: mathematical symbolism, visual display, and language, and the author addresses the impact that the multisemiotic nature of mathematics has on classroom discourse. In this paper, the range of possible semiotic resources is broader, including gestures and gazes and other non-verbal components. Here, learning is understood as meaning making towards an increased readiness to engage in the world with an increased use of semiotic resources in a discipline such as mathematics (Selander & Rostvall, 2008).

Inspired by Halliday (2004), social semioticians usually discuss three communicative meta-functions: the ideational, the interpersonal and the textual. In Morgan (2006), these functions are used with a focus on the construction of the nature of school mathematics activity. In this paper the meta-functions contribute to the construed discourses. The interpersonal meta-function outlines how language (used in a broad sense in this paper) enacts “our personal and social relationships with the other people around us” (Halliday 2004, p. 29). Here it concerns what kind of assessment particularly in the form of feedback is occurring in the interaction between teacher and student. The ideational meta-function is related to human experience and representations of the world (Halliday 2004). In this paper it concerns what aspects relative to the mathematics classroom are represented and communicated in the assessments. The textual meta-function is related to the construction of a “text”, and this refers to the formation of whole entities which are communicatively meaningful (Halliday 2004). Here the interest lies in what roles different semiotic resources play in assessment.

Drawing mainly on Foucault (1993; 2002), the second of the two main perspectives in this paper is an institutional/discursive perspective. Assessment in mathematics education is taking place in school with institutional aspects present, aspects which have both direct and indirect effects. Decisions may be made at different “levels” in the school system, and have a direct impact on the classroom work. There are also indirect aspects, such as classificatory systems, norms and dominant discourses (traditions) developed over time.

According to Foucault (1993; 2002) discourse is conceptualised as a broad notion that incorporates not only all statements but also the rules that affect the formation of possible statements. Consequently, the discourse is more than the entirety of what is communicated and the way it is communicated. It can be construed from what is not communicated, or what is communicated through gestures, attitudes, presentations, patterns of actions, and the rooms and furniture. For the people who are part of a discursive practice, the rules of the discourses affect what potential actions are allowable. For example, there are certain things that are acceptable to be communicated, and particular formats in which to communicate them. Discourse according to Foucault (2002) is to be conceptualised in line with a dynamic view (Björklund Boistrup & Selander, 2009). This dynamic view holds that the participants are not to be seen as imprisoned in a discourse. They can engage in resistance towards the discourse and then be part of a long-term change of the discourse or “leave” it and take active agency in another discourse. This dynamic view involves a strong position for the individual, and agency is another concept operationalised in the analysis (see also Mellin Olsen, 1993). Agency is understood here as a capacity for people, in this study mainly students, to make choices and to impose those choices on the world. This can be seen in terms of a person being active or passive.

Operationalising theories

The transcription was performed multimodal according to the social semiotic perspective as this enabled the capture of various feedback as well as aspects of the focus of the feedback. This is shown in a later section where some of the main findings are exemplified. In the study, the notion of discourse is used as an analytical concept. One way to describe assessment practice in a mathematics classroom is through the discourses that may be construed from the classroom communication. A starting point was a dichotomy of "traditional" and "active participant" discourses (Björklund Boistrup & Selander, 2009). In the analysis, these two discourses were identified in the data, but variations on them began to appear. Several tentative discourses emerged during the analysis, and I considered the ones that appeared to be clearly different to each other but not too confined. The basis for the construal of the discourses was the findings of the three initial analyses, which are connected to the three social semiotic meta-functions: (1) What kinds of assessment acts are present (drawing on Hattie & Timperley, 2007); (2) What are the focuses of the assessment acts in the mathematics classroom (drawing on Hattie & Timperley, 2007, and Skovsmose, 2005); and (3) What roles do semiotic resources play in the assessment acts. Two aspects, affordances for learning and students' active agency in the mathematics classroom, were essential when construing the discourses.

To summarise, the process of construing the discourses (drawing on Foucault, 2002) included these steps: (a) using the dichotomous discourses in an early attempt to interpret the material, (b) broadening the first two discourses by capturing deviations from, and contradictions to them, (c) choosing the clearest but still quite general among the proposed discourses, (d) using the social semiotic meta-functions as a basis for the construal and, (e) rechecking the discourses against the material, including discussions with colleagues.

Four assessment discourses in mathematics classrooms

The first of the four construed discourses is called "Do it quick and do it right", and has similarities to the traditional discourse mentioned in the previous section. The second, "Anything goes", is in opposition to a traditional discourse and the students' performances that can be regarded as mathematically incorrect are left unchallenged. The third, "Openness to mathematics", has similarities with the active participant discourse mentioned in the previous section. Finally, the fourth, "Reasoning takes time", goes one step further with a slower pace and an emphasis on mathematics processes such as reasoning and problem-solving.

Do it quick and do it right: In this discourse, the *feed back* is communicated mostly from teacher to student. Questions posed by the teacher are not often open and there are rarely follow-up questions. The *feed forward* concerns non-reflective action not connected to mathematics processes, and the teacher rarely challenges students' reasoning. *Feed up* (feed back and forward related to goals) is not present in this discourse. The focus is not on mathematics processes but on whether an answer is right or wrong, or the number of accomplished items. The *semiotic resources* used are not considered in terms of the learning of mathematics. Both teacher and student communicate in short utterances, and there are rarely longer silences. As a consequence, the lack of focus on mathematics processes produces low affordances for students' learning of mathematics. The main agent in this discourse is the teacher, and the affordances for students' active agency are not high.

Anything goes: There is limited articulated *feed back* in the discourse "Anything goes", apart from occasional approval. Here too, the *feed back* is mainly from teacher to student. Open questions also occur. The student is rarely challenged with respect to mathematics. There is a lack of constructive discussions about students' solutions, and answers possible to consider as mathematically incorrect can be left unchallenged. Different *semiotic resources*, including artefacts, are welcomed, and semiotic resources are rarely restricted. The teacher and students use short sentences, and there is not much silence. Often in this discourse, the teacher is the most active agent. Sometimes the teacher takes a more passive role in the discourse. S/he then does not interfere with students' reasoning even though something potentially mathematically incorrect is demonstrated. In the analysis I considered the affordances for students' learning and active agency in this discourse as low.

Openness to mathematics: There are several instances of *feed back* and *feed-forward* in this discourse, both from teacher to student and vice versa. Quite often the questions posed are open. Additionally the teacher and student show interest in mathematics and there is also an awareness of students' alternative interpretations of tasks. Sometimes the student is challenged with respect to her/his continued learning. Often the *focus* is on *processes* like knowing facts, practicing and routine. "Wrong" answers are also starting points for a talk, but, in the end, it is always clear what can be considered mathematically correct. Different *semiotic resources* are acknowledged and at times the teacher promotes, whilst at other times restricts, the use of semiotic resources dependent upon the meaning making and learning process demonstrated by the student(s). The lengths of teacher-student interactions are quite short. In this discourse, there are affordances for students' learning of mathematics and active agency.

Reasoning takes time: In this discourse, *feed back, feed forward* and *feed up* are present and in both directions between teacher and student. Recognition of students' demonstrated knowing, sometimes in relation to stated criteria, is frequent. The students are often challenged towards new learning. The *focus* is on *processes*, with emphasis on *processes* like inquiring/problem-solving, reasoning/arguing, defining/describing and occasionally constructing/creating. Different *semiotic resources* are acknowledged, and the use of *semiotic resources* can also be promoted or restricted when serving a certain process. Here, silence is common and the possibility (for both teacher and student) to be silent seems to serve the mathematics focus. In this discourse, the affordances for students' learning of mathematics are considered to be high, including a wide range of mathematics processes. Similarly, the affordances for students to take active agency are high.

Two of the four discourses construed from classroom communication

In addressing the first discourse, "Do it quick and do it right", we encounter a lesson where the students are working on their own in the textbook. Cecilia (Teacher) arrives at Catrin's (Student) desk to check a completed diagnostic test, and they both look at her work.

Time	Speech	Gestures	Body and Gaze
15:29	Cecilia (T): *One.* (*silence 2 s*) *"Which angles are straight?" A and?*	Cecilia (T) has a red pencil in her hand, ready to write. Catrin (S) holds a pencil.	Cecilia (T) is standing behind Catrin (S) and leaning over her.
15:35	Catrin (S): *B.*		Catrin (S) looks at the angles in the textbook.
15:36	Cecilia (T): *Yes, good.*		
15:37		Cecilia (T) writes an R in Catrin's (S) notebook.	

Excerpt 1. Multimodal transcript from video material.

In Excerpt 1, a pattern is clear, which continues for two more questions; Cecilia (T) reads a question from the diagnostic test (after 15 minutes and 29 seconds of the lesson) and Catrin (S) answers the same thing she has written in her notebook (at 15:35). Cecilia (T) marks R with her red pencil. In the following this pattern changes when Cecilia (T) comments on the writing of digits for the items in Catrin's (S) notebook. "What big digits you've made!" Cecilia (T) writes the digits in ordinary size in the margin of the page and tells Catrin (S) to do the same in the future.

The reasons for considering this to be an example of the discourse "Do it quick and do it right" are: (a) The only feed back and/or feed forward is in the direction from teacher to student; (b) There is a focus on the correct answers of the tasks (which is communicated at the very beginning of the sequence by the red pencil in Cecilia's (T) hand), and there are no follow-up questions. Later on, the focus is not on mathematics, but on the correct way to write and draw in the notebook (a task focus); (c) No considerations are made concerning semiotic resources, and there are few silences and short utterances; (d) The lack of focus on mathematics processes provides low affordances for the student's learning of mathematics and there are few affordances for the student to take active agency.

In the following sequence, from which "Reasoning takes time" was construed, Eddie (S), Enzo (S) and Ed (S), are working on an assignment. They are presented with five different solutions to the same task (376 – 149 =), shown in Excerpt 2. The students are told that the objectives for this assignment are cooperation and subtraction. They should find the suitable solution as well as determine what can be regarded as mathematically incorrect with the other four.

1. 370-150=220 220+6-1=225	2. 380-150=230 230-4+1=227	3. 300-100=200 200-30-3=167
4. 300-100=200 70-40=30 6-9=3 200+30+3=233	5. 376-100=276-40=236-9=227	

Excerpt 2. Transcript from written material. Assignment presented to students.

After Erika's (T) instructions at the beginning of the lesson, the groups start working. Erika (T) stands for several minutes in front of the class observing the students' work. Eddie (S), Enzo (S) and Ed (S) discuss the solutions. Enzo (S) raises his hand and calls for attention. Erika (T) arrives and Enzo (S) poses a question about there being two solutions with the same, and mathematically appropriate, answer: solutions 2 and 5. Erika (S) leans over their desks, looking at their work and posing questions to the three students about the purpose of the task (that only one solution is correct). She also asks how they have reasoned so far. Part of the communication is shown in Excerpt 3.

Time	Speech	Gestures	Body and Gaze
15:05	Erika (T): *What is your thinking then?*		Erika (T) looks at the worksheets.
15:07	Enzo (S): *Look.*		Ed (S) looks at 376 – 149.
15:08	Ed (S): *Well, that's two hundred twenty seven.* [Enzo (S): *And that is*] *That one can't be right.*	Ed (S) points at 376 – 149. Enzo (S) points at solution 4. Ed (S) points at solution 2.	Enzo (S) looks at solution 4. Ed (S) looks at solution 2. Enzo (S) looks at solution 2.
15:11	(silence 2 s) Ed (S): You take plus four when it should be minus four.		
15:16	(silence 3 s) Enzo (S): *No, minus four, that's six plus one, that's also the same.* (*silence 3 s*)	Enzo (S) points at solution 2. Enzo (S) stops pointing.	Students look at worksheet. Enzo (S) looks at Ed (S). Enzo (S) looks down.

Excerpt 3. Multimodal transcript from video material. Speech in brackets, [ ], signals simultaneous speech.

As shown in Excerpt 3, there are substantial pauses in the communication. Sometimes these silences are followed by reasoning from one of the students. After a while, the students' reasoning becomes more intense with a sustained focus on the mathematics involved in the task. Here, the students communicate their ideas for several seconds each. In one instance, Erika (T) points at solution 5 and asks whether they have done a calculation in that way before in class. The students answer no, and then there is a short discussion about solution 2. Before leaving, Erika (T) tells them that they get a few minutes more to think and also advises them to write down what is wrong with the ones that they know are definitely wrong.

The reasons why this is considered to be an example of the discourse "Reasoning takes time" are: (a) There are several instances of feed back and feed forward. Erika (T) communicates feed back and feed forward to the students about their work. The students ask for feed forward on the task and their demonstrated knowing is used as feed forward by Erika (T) for her future acts; (b) The focus of the feed back and feed forward is mainly on mathematics processes. The processes that are present, even after Erika (T) has left, are mainly reasoning/arguing and inquiring/problem-solving. Before leaving, Erika (T) also initiates the process of defining/describing since she tells them to write down their reasoning so far; (c) The feed back and feed forward from Erika (T) are realised several times by questions to the students. There are many instances of silence followed by utterances from the students as well as from Erika (T). There is also a silence when Erika (T), just prior to this sequence, stands in front of the class observing the students' work. She introduces semiotic resources and then promotes the process of describing when she tells them to write down their work so far; (d) They communicate a great deal about mathematics by way of speech, gestures, symbols on the paper etc. also in longer utterances. There are affordances for students' learning of several mathematics processes here. Moreover, there are considered to be affordances for students' active agency here, and the students take active agency in the sequence.

What the theories made possible

The engagement in social semiotic theory conveyed a possibility to view assessment as one aspect of classroom communication. In the analysis, the multimodal transcriptions revealed assessment communicated through semiotic resources such as gaze and gestures. Additionally, it gave a clear picture of the focus of the assessments through pictures, gestures, writings and so on. In the analysis, the semiotic resources were shown to play essential roles in the assessments. One example is whether there were any considerations made with regard to which semiotic resources the students could use when displaying mathematics knowing. It also provided a structure, through the meta-functions, around which I could organise the first three analyses as well as means for the construal of the discourses.

When it comes to the institutional/discursive perspective, I want to stress that I do not claim that discourses are anything more than an analytical concept. Moreover, in each classroom it was possible to construe at least two of the four discourses and often there could be changes during one lesson. One advantage of this theory is that it provides means to discuss and understand the assessment in mathematics classrooms as well as viewing the classroom as part of a broader context. The discourses in this study were construed in the institution of school, where acts in one assessment discourse are taken for granted. There are acts that are unlikely to appear in other assessment discourses in the mathematics classroom. One example where "Do it quick and do it right" is present is when a teacher communicating with a student about the student's performance on a diagnostic test focuses the feedback mainly on how to keep the student's notebook in order. She states that mathematics is a "clear-cut subject". In an alternative assessment discourse, for example "Openness to mathematics", there could, of course, be feedback on the preference for mathematics notes being kept in reasonable order. But in this discourse, this would be related to the importance of mathematics processes not getting lost in the student's notes. Here, the acts could be described as following the "rule" "mathematics processes are the primary focus in the mathematics classroom". Institutional traces like discourses are more indirect than decisions made by authorities, decisions that teachers have to follow; nevertheless, they can be perceived to be as strong.

Direct institutional traces, such as a decision on municipality level regarding the use of a certain assessment material, are considered to "carry" (introduce and/or maintain) all four discourses in the study. This occurs when a situation also includes a direct institutional trace and may have had an impact on which discourse that could have been construed from the situation. This illustrates that why things are the way they are with respect to assessment in the mathematics classroom is far from simply being a question of the individual teacher. For politicians, decisions are sometimes made on a national or municipal level that counteract what is stated in steering documents.

Hence, what assessment discourses may be construed in mathematics classrooms is a matter of a complex interplay between steering documents, decisions made on different levels within and outside the educational institutions, and dominant traditional discourses, as well as alternative discourses and agents in discursive practices. A positive change in affordances for students' learning and active agency of mathematics with respect to classroom assessment is a question of looking at every part of this interplay as a whole (Pettersson, 2010). One issue is that decisions about education must be coherent with regard to these affordances and thus not counteract one other. The discourses presented here can then be a basis for discussions and decisions concerning assessment in mathematics classrooms on different levels within the institution of school.

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Modelling Assessment of Mathematical Modelling – a Literature Review

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This paper presents a critical review of literature investigating assessment of mathematical modelling. Written tests as described in the reviewed papers draw on an atomistic view on modelling competencies while studies adopting more holistic approaches are rare. In order to assess the quality of students' work with mathematical models, an elaborated view on the meaning of quality of mathematical models seems to be needed.

Introduction

The role of mathematical modelling in the official Swedish upper secondary curriculum guidelines has been strengthened during the last 50 years (Ärlebäck, 2009). In the present curriculum mathematical modelling is one of seven mathematical "abilities" to aim for, which is to "interpret a realistic situation and design a mathematical model and to use and validate a model's properties and limitations" (Skolverket, 2010, p. 91, my translation). Modelling competence is assessed in the national course tests (NCT), which are developed in line with the curriculum. However, there are indications that modelling is difficult to assess. Frejd (2011) analysed test items in the NCT and concluded that there existed an uneven emphasis on different aspects of mathematical modelling. Frequently occurring aspects, such as to use an already existing model to calculate a result, were put in favour over other aspects that occurred sparsely or were left out, such as to critical asses conditions and validate results. Not a single item was found in that assessed all aspects of modelling (holistic view).

Looking at the result from Frejd (2011) and following the principle by Blomhøj and Hoff Kjeldsen (2006) that "the pedagogical idea behind identifying mathematical modelling competency as a specific competency is exactly to highlight the holistic aspect of modelling" (p. 166), raises some questions related to national assessment in Sweden regarding mathematical modelling. Is it possible to assess all aspects of modelling in the NCT? If yes, how? If it is not possible to assess modelling in written tests in a holistic way, what other assessment modes are being used or suggested? The aim of this paper is to review a selection of literature focusing on mathematical modelling in mathematics education in order to analyse approaches used or suggested to assess students' modelling competencies.

Mathematical modelling, modelling competence, and assessment

There is not one single unambiguous definition of the notion of mathematical modelling shared in mathematics education, but rather the definitions used and/or descriptions of modelling given depend on the theoretical perspective adopted (Frejd, 2011). In this paper I have chosen the approach taken by Blomhøj and Højgaard Jensen (2003) to describe mathematical modelling. Their description consists of the following six sub-process: *formulating a task in the domain of inquiry*; *selecting the relevant objects, relations and idealising*; *translation into a mathematical representation*; *using mathematics to solve the corresponding mathematical problem*; *making an interpretation of the results in the initial domain of inquiry*; and *evaluating the validity of the model* (p. 125). In addition, Blomhøj and Højgaard Jensen write that "[b]y mathematical modelling competence we mean being able to autonomously and insightfully carry through all aspects of a mathematical modelling process in a certain context" (p. 126), a definition I have adopted for this paper. The motivation for using this framework is that a similar view of modelling forms the basis of the construction of the guiding questions used in Frejd (2011) to examine NCT items.

While the term assessment is frequently used unproblematically in research in mathematics education, Niss (1993) points to the complexity of assessment endeavours in mathematics education. Instead of declaring a specific definition of assessment, the intention in this paper is to be open to investigate everything that is called assessment and relates to mathematical modelling.

Methodolodgy

According to Bryman (2004), central features of any qualitative literature review are first that relevant and adequate research literature related to the aim is identified, and secondly that the analysis of the literature serves the purpose(s) to relate, organise, and connect the literature (constructing intertextual coherence) as well as to point at research literature that seems to be incomplete (problematizing the situation). Therefore, the method used includes three phases: 1. Identifying adequate papers, 2. Categorising the papers, and 3. Extended analysis.

Phase 1, to identify relevant literature addressing modelling assessment from the vast amount of papers, articles, and books about mathematical modelling in mathematics education, is difficult. An overview or a state of the art in this research domain may be found in the *ICMI14 Study: Modelling and applications in mathematics education* (Blum et al., 2007) [1]. Other cited works relate to the proceedings from *the international conference on the teaching and learning of mathematical modelling and applications* (ICTMA), as well as the proceedings from the thematic working groups explicitly focusing on issues related to mathematical modelling in the *Congresses of the European Society for Research in Mathematics Education (CERME)*. In this survey the ICMI14 Study, all 15

ICTMA proceedings, and all proceedings from the 'modelling working group' at CERME have been examined. In addition, the special issues of ZDM 2006 focusing on mathematical modelling (issues 38(2) and 38(3), respectively) and one other 'older' paper (Berry & O'Shea, 1982) have been analysed. The method used to identify the relevant articles in the chosen literature are based on key words in the titles (assessment, assessing, evaluating, etc), an examination of all abstracts in book sections relating to assessment, and a search for the word assessment in the index.

There are several alternatives to synthesise different works (phase 2) and to analyse the identified papers (phase 3). One alternative is to use grounded theory (Strauss & Corbin, 1998). A second alternative could be to design questions and argue for their relevance. A third alternative could be to use already designed questions used for a similar purpose.

Niss (1993) argues that some questions in relation to assessment of mathematical modelling are "highly relevant" (p. 50) to answer. Those concern issues about *why* assess modelling, *what* should be assessed using *which* kinds of *tasks*, *who* (individuals, groups, ...) should be assessed *when* and *how* (design, mode, outcomes, reports) and by *whom* (Niss, 1993, p. 49).

However, these questions, which seem to be derived from well articulated common sense arguments, are quite general and fundamental in all types of assessment. The choice for phase 2 (synthesis) and 3 (further analysis) in this survey is therefore a combination of the alternatives discussed above. Phase 2 is based on a coding strategy inspired from grounded theory, in order to be open for a wide set of alternatives of categories. However, it is not possible to describe the actual coding process here, due to limited space, which is a limitation of this paper. Similarly, the presentation of phase 3 is also constrained by the limited space. The further analysis of the articles focuses on different *modes,* mainly in written tests, as a consequent of the initial concern as stated in the introduction. *Mode* in this paper refers to different types of assessment methods and relates to Niss' (1993) question *how*, but as *how* depends on *why* and *what,* these are also implicitly analysed.

Results and discussion

Underpinnings of assessment frameworks

To analyse approaches for assessing modelling, it is important to know about types of research studies to understand whether the criteria used in a framework or mode of assessment are derived from a theoretical analysis, based on literature, ad-hoc constructions, experience from assessment situations or empirical studies of students' work. 75 articles relating to modelling assessment were identified (out of total 707 papers from the ICTMA-, ICMI study14-, the CERME 'modelling group'-proceedings) and categorised as *empirical* (case studies), *philosophical* (based on argumentations), *theoretical* (building frameworks), *overviews* (literature reviews, introductions), *description of practice* (curriculum descriptions, course descriptions) and *competitions* (see figure 1).

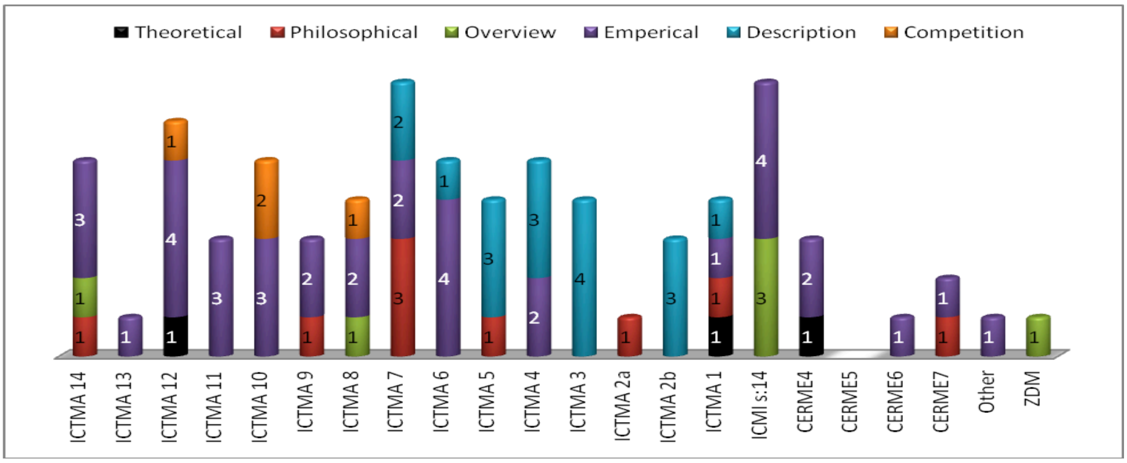

Figure 1. The frequency of the identified categories

The most frequent category in Figure 1 is *empirical studies* (36 out of 75) and it includes a large variety of topics studied, such as: developing and evaluating research tools assessing modelling (e.g. Zöttl, Ufer,& Reiss, 2011; Izard, 2007); analysis of test items in written tests (Frejd, 2011b; Turner, 2007; Stillman, 1998; Naylor, 1991); investigations of differences between teachers' marking of students work (Berry & O'Shea, 1982); poster assessments (Houston & Breedon, 1998; Houston, 1997; Wake, 2010); investigating how middle school students solve linear pictorial patterns (Amit & Neria, 2010); comparative study of modelling outcomes between two groups, where one group worked with the modelling process and the other group worked with just examining models (Legé, 2007). There are six literature *overviews* identified: an introduction to assessment chapters (Blomhøj, 2011; Galbraith, 2007); a review of assessment methods used (Houston, 2007); an overview of the expert and novice issue related to modelling (Haines & Crouch, 2007); overviews more in general about modelling, with parts related to assessment (e.g. Galbraith et al., 1998). Nine papers refer to more *philosophical* issues, such as why, whom, when, etc. (Burton, 1997; Niss 1993; Oke & Bajpai, 1986); descriptions and arguments on 'how to assess' based on examples and experience (e.g. Henn, 2011; Brown, 2001; Izard, 1997); and assessing mathematical models (Jablonka, 1997). There are three *theoretical* papers attempting to develop some type of framework for assessment and marking (Højgaard Jensen, 2007; Hall, 1984; Henning & Keune, 2006). Finally, there are four papers relating to modelling *competitions* like the China Undergraduate Mathematical Contest in Modelling (Jiang, Xie, & Ye, 2007) and 17 papers related to descriptions of practice, 'this is the way we work with assessment at our school' (e.g. Batteye & Challis, 1997; Swam 1991); or 'description of the curriculum situation in for example northern Ireland/Australia (e.g. Coxhead, 2007; Money & Stephens, 1993).

To summarize, of the investigated proceedings every tenth paper relates to assessment, and out of these identified papers most were categorised as *empirical studies* and the fewest as *theoretical* studies. This is true for most research domains in mathematics education, where there is a need for a large number of empirical studies to explore, compare and evaluate complex issues as well as to underpin the development of theoretical frameworks. However, in this study, there seem to be 'very' few papers focusing on theoretical aspects on assessment (i.e. trying to create frameworks) and the development of these frameworks is not grounded in case studies. Notable is also the large proportion of identified papers (almost 30% of the investigated papers) relating to descriptions of practice and descriptions of competitions.

In the third phase of the analysis, there were several *modes* identified in the sample. These are *written tests* (36 papers), *written project reports* (25 papers), *written project reports including oral presentations* (6 papers), *contests* (4 papers), *poster sessions* (4 papers), and *students' portfolio* (2 papers). Eleven of the papers above involve more than one mode. The written tests analysed or used refer, inter alia, to multiple-choice questions, shorter tasks, extended tasks, unseen tests (traditional tests) and seen tests, final exams, and shorter classroom tests.

Written tests

Almost every third paper (11 out of 36 papers) relating to a written test is dealing with multiple-choice questions that stem from Haines, Crouch and Davis (2000). This multiple-choice test originally consisted of 12 questions, each with five alternatives, grouped in pairs to assess six aspects of the modelling process (i.e. to be used in pre- and post settings). The number of test items has been extended to a total of 22 items testing 8 aspects. These aspects relate to a modelling cycle and address phases such as making simplifying assumptions, formulating the problem, assigning variables, parameters, and constants, etc. The test, together with a partial credit assessment model (the scores 0, 1, 2), makes it "possible to obtain a snapshot of student' [modelling] skills at key developmental stages without the student carrying out a complete modelling exercise" according to Haines et al. (2000, p. 10). This test instrument has been used in a variety of settings with different aims such as to investigate the levels of students' modelling competencies (Frejd & Ärlebäck, 2011). However, even if this test instrument is widely used some critique has been raised. Frejd and Ärlebäck (2011) found in their study that only two of the pairs of items were comparable in the respective aspect of the modelling process. They also argue about the lack of ICT and collaborative work, which are other important aspects of modelling, though the main point refers to the atomistic view. Haines and Crouch (2007) notice that “these items do not address the full range of modeling skills, for they do not, as yet, cover *solving mathematics*, *refining a model* and *reporting*” (p. 420).

Another written test to assess modelling competency is developed by Zöttl, Ufer and Reiss (2011). Their test consists of a sample of 12 items (out of 36) divided into four categories, where three categories relate to different aspects of the modelling process, and the fourth category to “short, but complete modelling tasks” (p. 432). They argue that “adequate modelling tasks should always require the performance of a complete modelling process” (p. 428). An example of a ‘complete modelling task’ is to ”[e]stimate the total area of Spain by using the [given] map” (p. 432) where Portugal’s total area is stated. To theoretically underpin the test Zöttl et al. (2011) use Højgaard Jensen’s (2007) three dimensions of modelling competency: degree of coverage, radius of action and a technical level. Degree of coverage relates to which parts of the modelling process are to be used and to what extent the students perform autonomously and use reflections. The radius of action concerns the range of contexts in which a student may perform his/her modelling ability, and the technical level refers to how advanced the mathematics is which the student uses. Højgaard Jensen (2007) illustrates these levels by a geometrical box (i.e. as three independent vectors). One may question this illustration, because in case of modelling the three levels of modelling competency do not work independently, but are (as the author sees it) intimately interwoven (especially since one aspect of modelling is about ‘pure’ mathematics). Thus, if a student possesses a large toolbox (good technical level) it will imply more mathematical options to solve a problem and create more opportunities for reflections (a better degree of coverage) than for a student who possesses a low technical level. Zöttl et al. (2011) only measure the degree of coverage and the technical level, since they let the radius of action be specified concerning geometry. To assess the items, they used a dichotomous scoring, because they find partial credits too complex. How they scored the ‘complete modelling task’ stated above is not described, but it appears to be related to the scoring of PISA items, because the task is similar to a PISA item discussed by Turner (2007). However, no references are given, and it can be questioned to what extent it is a complete modelling task: A clue is given in the text, “you can draw onto the map if it helps you” (p. 432), and a solution to the task seems straightforward (e.g. to estimate the area of Spain by covering the map of Spain with copies of the map of Portugal).

The PISA framework (OECD, 2009) is developed for large-scale international students’ assessment. The role of modelling in this framework is however problematic, according to Jablonka and Bergsten (2010), who call it “a circular construction” (p. 30): On the one hand mathematising, which is the primary building block of mathematical literacy, is described as a modelling process, while on the other hand, this primary building block consists of eight competencies, where one of these competencies is modelling competence and equal to mathematising. There are some papers about PISA items in the identified sample, and according to Turner (2007) the items do promote an interest in modelling but have a low level of complexity of modelling activity. Henning and Keune (2006) have reformulated PISA items to assess three levels of modelling competence. The three hierarchic levels, which Henning and Keune (2006) have developed by adopting "the competence levels of mathematical literacy" (p. 1667) are: 1. Recognize and understand modelling (to describe the modelling process); 2. Independent modelling (to solve a modelling problem and interpret the result); and 3. Meta-reflection on modelling (to critically analyze and reflect upon the modelling process). Henning & Keune seem to think that meta-reflection is an "extra" activity, which is not needed when someone makes a model. They state "[a]t this third level of competence, the overall concept of modelling is well understood" (p. 1669), but at the same time "[a]t this level, it is not absolutely necessary to have previously solved problems by means of modelling techniques" (p. 1669). This statement is contradictory to other research like Blomhøj and Hoff Kjeldsen (2006) who claim that "[m]odelling competency is developed through the practice of modelling" (p. 166)

Vos (2007) used the alternative practical assessment tasks (hands-on tasks) developed by TIMSS in a case study. The test was composed by several tasks where the students instead of solving realistic problems described in words were given concrete equipment such as rubber bands to work within a laboratory-like situation. It was found that both students and teachers had a positive attitude towards the test, but the coding of the open-ended questions was problematic.

Projects

It is often argued that using written tests is not the best way to test modelling. According to Berry and Le Masurier (1984, p. 59) "*project* is the ideal method", and Niss (1993) claims that the use of traditional modes (i.e. written test) is difficult or even impossible, but that "[o]ne particular appropriate type [of assessment] is *projects*" (p. 47). There are also some evidence, in relation to the assumptions above, presented by Antonius (2007) who argues that ”the different competences seem to be more visible in project examination" (p. 414) than in traditional examination because it is more extensive (includes both written reports and oral examinations).

Descriptions of projects as part of assessment are quite frequently appearing among the identified papers. The projects are extended 'realistic' problems that students try to solve during a longer time period (like 40 hours in the case of Berry and O'Shea, 1982). The students explain their solutions in written reports and in some cases (in 6 papers) defend them in an oral presentation. The main question regarding projects is how they can be assessed. Approaches to assess projects found in this sample are impression marking (Gillespie, Binns &

Burkhardt, 1989), formal marking schemes (e.g. Berry & O'Shea 1982), grounded theory (Maaß, 2007), and observations (Dunne & Galbraith, 2003; Herring, 1991). The most frequent approach is the use of more or less formal marking schemes (i.e. a set of defined criteria to be followed). However, none of the marking schemes used in the papers are justified in the sense of describing why just these criteria are being used and no others. The fact that most criteria are quite generally written in order to cope with many different projects makes it difficult to apply them and accounts for considerable variation between markers (e.g. Berry & O'Shea 1982; Haines, 1991). Hall (1984) argues for double blind marking to increase reliability together with his framework to calculate the final outcome (i.e. geometric mean value), but according to Haines' (1991) case study Hall's 'geometric model' does not differ much from traditional marking.

A project may also include a poster session, which is discussed in three papers that also relate to peer-assessment. Wake (2010) does a case study focusing on formative assessment and poster presentation. He argues that peer-assessment provides feedback in a language that the students understand and that learning is most effective if students are aware of the objectives to be learned. A conclusion drawn is that the use of a modelling approach on teaching and learning with formative assessment in day to day practice changes the teachers' roles, so that "both [teacher and student] are now active with learners struggling to solve a task and make reflective judgments about their ability to do so using new rules of assessment that focus on process as opposed to outcomes" (p. 2093).

Concluding remarks

What is found in this study clearly relates to the complexity of any assessment endeavor (Niss, 1993a), illustrated by Izard's (1997) statement that "[n]o single assessment method is capable of providing evidence about the full range of achievement" (p. 109). The written tests as described in the reviewed papers draw on an atomistic view of assessment focusing more on the product than on the process. A confirmation of Frejd's (2011) findings of a lack of a holistic approach (in the NCT) was also found in other countries (Stillman, 1998; Naylor, 1991). The question still remains if it is possible to construct justifiable holistic approaches to assess modelling in national tests.

According to Jablonka (1997), the most crucial aspect to assess in students' work with modelling is to judge the quality of a mathematical model. The framework developed in Jablonka (1996; summarised on pp. 209-212) for analysing mathematical models is also intended to be used for assessing students' work with mathematical models. She lists a number of critical questions to ask to the modelling work, organised under the two main headings evaluation of efficiency ("To what extent does the model fulfill its main goal?") and assessment of usefulness ("What is the contribution to the solution of the main problem and how can the goals and consequences be evaluated?”). While the efficiency questions can be used to form the basis for developing a set of assessment criteria for students' models, assessment of students' work with mathematical modelling also needs to take into account the extent to which the students have considered the usefulness questions while studying or developing mathematical models.

When it comes to assess within a holistic approach the use of projects is suggested (e.g. Niss, 1993). Here assessment criteria could be guided by the critical questions suggested by Jablonka (1996). Recent developments draw on projects (Antonius, 2007), formative assessment (Wake, 2010), and alternative assessment (Vos, 2007). However, while in “pure” mathematics one usually has some common mathematical ground forming the base for classifying and assessing students’ work, without shared views on how to judge the quality of a mathematical model one cannot expect shared views in a debate about assessing students’ mathematical modelling.

Note

1. Due to the large number of references, papers published in ICTMA-proceedings and the ICMI-14 Study proceedings are listed in the appendix.

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Appendix

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Antonius, S.	**(2007)**	Modelling based project examination.
Battey, A., & Challis, M.	**(1997)**	Deriving learning outcomes for mathematical modeling units within an undergraduate program.
Berry, J., & O'Shea. T.	**(1982)**	Assessing mathematical modelling.
Berry, J., & Le Masurier, D.	**(1984)**	O.U. Students do it by themselves.
Blomhøj, M.	**(2011)**	Modelling competency: teaching, learning and assessing competencies-Overview.
Brown, R.	**(2001)**	Formulating and assessing a mathematical modelling problem in a technological environment
Burton, L.	**(1997)**	The assessment factor - by whom for whom, when and why.
Coxhead, C.	**(1997)**	Curriculum development and assessment in north Ireland.
Dunne, T., & Galbraith, P.	**(2003)**	Mathematical modelling as pedagogy-Impact of an Immersion program.
Frejd, P., & Ärlebäck, J. B.	**(2011)**	First results from a study investigating Swedish upper secondary students' mathematical modelling competencies.
Galbraith, P.	**(2007)**	Assessments and evaluation-Overview.
Galbraith, P., Haines, C., & Izard, J.	**(1998)**	How do students' attitudes to mathematics influence the modelling activity?
Gillespie, J., Binns, B., & Burkhardt, H.	**(1989)**	Assessment of mathematical modeling.
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Haines, C., & Crouch, R.	**(2007)**	Mathematical and applications: Ability and competence frameworks.
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Izard, J.	**(2007)**	Assessing progress in mathematical modelling.
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Jiang, Q., Xie, J., & Ye, Q.	**(2007)**	An introduction to CUMCM.
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Stillman, G	**(1998)**	The emperor's new clothes? Teaching and assessment of mathematical applications at the senior secondary level.
Swam, M.	**(1991)**	Mathematical modeling for all abilities,
Turner, R.	**(2007)**	Modelling and applications in PISA
Vos, P.	**(2007)**	Assessments of applied mathematics and modelling: using a laboratory like environment.
Zöttl, L. Ufer, S. & Reiss, K.	**(2011)**	Assessing modelling competencies using a multidimensional IRT approach.
The ref. to all ICTMA proceedings and the ICMI- study can be found at http://www.ictma.net/literature.html		

Empowerment and Control in Primary Mathematics Reform – the Swedish Case

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We explore the recent national school reform in Sweden from the perspective of primary teachers, using Bernstein's concepts of classification and framing and focusing especially on the teachers' experiences of policies of control and empowerment. Almost all the teachers participating in our study (n=41) relate positively to the reforms in general. Several empowering experiences are connected to the national examination in particular, while many teachers still experience the steering document as not especially helpful in guiding their instruction. There is a tension between the positive experiences of the control policy and worries about violating children's right to develop at their own rate based on their prerequisites.

Introduction

The assessment of students' achievements in mathematics has been in focus at international and national levels for several decades. The declining results of Swedish students' mathematical achievements in the assessments organized by the Programme of International Student Assessment (PISA) and the Trends in International Mathematics and Science Study (TIMSS) have been used by the Swedish Government as argument for the recent reforms concerning mathematics. After a period of weak classification and framing (see Bernstein, 2000, and Harling, Hansen & Lindblad, 2008), during which schools and teachers had a great deal of space to decide both the mathematics content to be dealt with and the rate of instruction in the classrooms, the Swedish mathematics curriculum has now been reformed in order to offer teachers clearer guidelines regarding both content and knowledge requirements, already in the first three grades (Skolverket, 2011a). The reforms have already met some criticism, due to the possible negative effects of quantitative assessment and the risk that the teachers' professsional freedom will be restricted (e.g., Liedman, 2011).

Research suggests that the way curricular guidelines are implemented in the classroom is greatly influenced by teachers' experiences and views (e.g., Corey & Gamoran, 2006; Remillard, 2005). Hence, to understand how reforms are implemented and, on a larger scale, what influence the reforms have on teaching and learning, it is important to understand how teachers experience and relate to them. This is true not least in Sweden, where the national curriculum previously has left substantial space for local interpretation and application.

In this paper we investigate the curriculum development and the implementation of the recent reforms from the perspective of primary teachers using Porter, Archbald and Tyree's (1990) definitions of two different curriculum policies, *control* and *empowerment*. Control refers to guidelines, tests and examinations that the schools and teachers are obligated to use, while empowerment refers to various strategies for supporting teacher professionalism. These two policies can coexist and interact, as will be shown in this paper. We found also Bernstein's theories on classification and framing (Bernstein, 1990; 2000) useful in our analysis. We begin the first section with a brief description of Bernstein's concepts and their relation to the two policy strategies defined by Porter, Archbald and Tyree (1990) as well as to the teachers' experiences of them. Then we analyse the recent Swedish developments using these concepts.

Curriculum development and the policy of implementation

Classification refers to the relations *between* categories, and classification (both strong and weak) always carries power relations (Bernstein, 1990; 2000). *Internal* classification concerns the boundaries between different mathematical contexts within the subject, while *external* classification concerns the relations between mathematics and other subjects as well as everyday practices. In our analysis, the focus is on external classification. Framing refers to the control of communication in local interactional pedagogic relations. This control can be over the selection of the communication along with its sequencing and pacing, the criteria or the assessment (Bernstein, 2000). The *control policy strategies* (prescriptions of the goals, the contents, national examinations and school inspections) can be more or less prescriptive and lead to weak or strong external and internal classification with respect to the boundaries between various mathematical contents and between mathematics and other subjects, and framing concerning sequencing and pacing of the content as well as the criteria of the assessment. Teachers' *experience of empowerment* has to do with inclusion and participation in constructing and transforming the social order. Their experience of control refers in our study to expressions of restrictions of their professional knowledge. The teachers' voices are not only seen as individual constructions but also structurally determined.

The pedagogic practice in Swedish mathematics classrooms (grades 1-5) has obviously been relatively weakly classified and framed during the past two decades (cf. Harling et al., 2008). Before the recent reform, only very general goals for the fifth and ninth grades in compulsory school were presented. The fulfilment of these goals was monitored through national examinations, the first taking place during the fifth grade and testing a minimum level of pupils' achievement. Hence, there was no instruction regarding the content to be covered in order to reach the teaching goals. Also, the boundaries between mathematics as a discipline on the one hand and pupils' everyday life on the other were weakened during this period due to the mainstream pedagogical trends.

Invisible pedagogy is characterized by both weak classification and framing. The integration of different school subjects as well as the emphasis on students' free choice regarding their learning activities and their rate of learning have been connected to invisible pedagogies (e.g., Fowler & Poetter, 2004). The strong emphasis on students' motivational aspects and the child's right to develop at his/her own rate in the Swedish context also indicates invisible pedagogies. The criteria for assessing children's performance might have been implicit and unclear to students and parents, and sometimes, even to the teacher. Research indicates that students whose parents have non-academic backgrounds are even more disadvantaged by invisible than by visible pedagogies (e.g., Bernstein, 1990; Morais & Neves, 2001; Lubienski, 2004). This development was also recently observed in the Swedish context, and has been used by the government as one of the arguments for the recent reforms (Wester, 2011).

The first steps (implemented in 2008-2009) of the current reforms were to define specific goals for the third grade and introduce national examinations assessing the achievement of these goals. The second step (implemented in 2011) was to reform the whole compulsory and upper secondary school curriculum. The new Swedish national curriculum can still be considered only a framework as it does not suggest textbook materials or teaching methods, lesson plans or tests. Instead, teachers can freely choose the textbooks they use, or choose not to use any textbooks at all (cf. Fowler & Poetter, 2004). In the new curriculum, there is also a strong emphasis on students' everyday experiences, which should guide the content dealt with in the mathematics classrooms (the word *everyday* is mentioned 23 times in the ten-page document).

Bernstein uses the concepts *horizontal* (e.g. everyday common knowledge) and *vertical* (e.g. subject knowledge) when explaining why attempting to weaken classification and framing in progressive forms of education, for example by introducing informal discourses, does not make the instruction more effective but rather the opposite. According to Bernstein, the vertical discourse of schooling is inevitably strongly classified but the essential characteristics can be "'masked' from all or some participants, or made clear and explicit to all, so that all involved can understand it" (Bourne, 2004, p. 63). The blurring of the boundary between everyday knowledge and school knowledge has also been connected to invisible pedagogies (cf. Morais & Neves, 2001). Yet, finding alternative pedagogies is not unproblematic, as Bourne (2004) and Jablonka & Gellert (2010), for example, show.

Concerning the *policy of empowerment,* teachers had the possibility to comment on the suggestions for the current national curriculum while it was being written, if they were informed of this possibility and allowed time to engage in the process by school leaders. Other empowering strategies were also applied in the Swedish policy, like attempts to strengthen the teacher preparation programs, and professional development activities organized by various actors, in order to strengthen the effects of the recent reforms (see Johansson, 2010). In addition to national examinations, the fulfilment of the curricular objectives is also monitored by school inspections. Further, the municipalities (as school authorities) as well as the local school leaders can conduct other kinds of monitoring in their areas. Hence, both control and empowerment strategies are used, in order to create and implement the new curriculum (cf. Porter, Archbald & Tyree, 1991).

According to the National Agency for Education, the purpose of the national examination is to support an equal and fair judgement and marking, to offer a basis for analysing the fulfilment of the knowledge requirements at school, school authority, and national levels, and to contribute to concretizing the syllabuses and raising the goal fulfilment of students (Skolverket, 2011b). The national examinations also have a diagnostic purpose (PRIM-gruppen, 2011) and, hence, should enhance the *formative* use of the results. The notion of formative assessment has been used for several decades in many countries, and its benefits have also been in focus in the Swedish context during the past decade (see, for example, the National Agency for School Development, 2007). Formative assessment refers to the continuous cycle of observing, testing and improving instruction according to the observations and results of the tests. Swedish teachers can choose their own diagnostic materials, or can choose not to use written tests or diagnoses at all. The National Agency for Education offers material for the diagnostic and formative assessment of students, but there is no obligation for the teachers to use it. Consequently, no national control is exercised here so the extent and character of formative assessment can vary between schools and classrooms depending not only on the policy of the respective municipality and school but, above all, how important teachers consider the value of formative assessment with respect to both their own teaching and students' learning.

Data gathering and analysis

The explorative study presented in this paper is part of a larger project investigating the implementation of the primary level reforms in one Swedish municipality with a special focus on certain schools as cases. The municipality was chosen because of the on-going cooperation project between the municipality and our university.

In the present paper, we investigate the 'voices' of primary teachers and analyse the relevant parts of the data gathered during 2010-2011 through questionnaires and interviews. The first questionnaire, with both open questions and closed statements, was delivered (via mathematics developers [1] and head-masters) to 32 primary schools (grades 1-3) in the municipality in focus in our study. Yet, we cannot be sure that all the teachers received the questionnaire, and even less that they were given time to respond to it. A total of 13 teachers (11 females and 2 males) responded to the first questionnaire in 2010. A shorter version of the questionnaire, with only open questions, was given to 28 third-grade teachers when they attended an information meeting organized by the municipal mathematics developer in 2011. None of them had responded to the first questionnaire and all of them (25 females and 3 males) responded to the questions in the short questionnaire. Further, semi-structured interviews were conducted with four teachers. The aim of the interviews was to explore some issues from the questionnaires more in-depth. The four teachers were chosen because they work in the schools that are in focus in our larger study. All the teachers (41) who participated in our study had the qualifications needed for primary level teaching, but not all of them had the qualifications for teaching mathematics. Further, most of the teachers had more than five years teaching experience, but there were also some who had no experience of teaching but were to start to work this autumn (2011). Although about one-fourth of the primary teachers responded to our questions, we cannot claim that our sample is representative of all the primary teachers in the municipality. Still, the data we have obtained are rich and the analysis reveals several interesting issues to delve more deeply into in further studies.

For this paper, we have particularly analysed the questions dealing with how the teachers relate to and experience the introduction of the goals and national examination as well as formative assessment: *Do you think it was positive to introduce goals and national examinations for the third grade? Motivate your answer*; *Has this reform somehow influenced your teaching, and if so how? How do you judge the mathematics knowledge of individual students? Have you been informed about your school's results in the earlier national examinations in mathematics? Do you think that the new steering documents support you in planning and carrying out your teaching?* (The last question was posed only in the questionnaire directed at the third-grade teachers.)

Besides these questions, the teachers had the possibility to talk about issues relevant to the study. The interviews were transcribed and the responses to the questionnaires were compiled before the analysis. We followed a qualitative data analysis but made some quantification where appropriate. The responses were coded with respect to categories connected to feelings of empowerment and restriction. In the next section, we offer several authentic expressions in order to make our analysis more transparent.

Experiences of empowerment and control

We present the results by first describing and exemplifying the teachers' voices concerning the *goals and content descriptions* in the curriculum documents. Then, we focus on the experiences of *the national examinations* as well as formative assessment in general.

Almost all the teachers in our study declare that it was positive to introduce the new steering documents already in the third grade, and several statements show experience of empowerment rather than control. A number of teachers consider the new curriculum clearer and more concrete than the previous one, but at the same time feel it is still quite open to interpretation, concerning both the knowledge requirements and the content. They state that it would be helpful to have examples of tasks in order to concretize the frame. One teacher criticizes the goals for defining only the minimal level, and fears that they do not encourage teachers and students to "strive ahead".

Yet, the majority of the teachers state that the steering documents in mathematics for the third grade support them in their planning and actual teaching. "It's good to have concrete national guidelines that create consciousness about what is important." They also experience that the new steering document has already influenced their teaching in certain ways; for example, they now feel more conscious of the goals and experience that the goals now steer their teaching. Several teachers state that they have started to work with LPP (local pedagogical plans) since the first reform was introduced. Other changes mentioned are "more mathematical discussions", "more teaching", "more work with elaborations with concrete materials and problem-solving", and "more practical work with weights, volumes, etc.". Hence, it seems that the policy of control also has empowering effects as it seems to enhance teacher professionalism.

Concerning the national examinations, the teachers state that they make it easier for them to work with their own assessment as it is now clearer what the students should know and achieve. Hence, the tasks in the national examinations help the teachers concretize the curricular goals: "We get a response about the level of teaching, whether we're occupying ourselves with the right things." One teacher even writes that there are good tasks and ideas for mathematics lessons in the national examinations. Concretizing the goals is also one of the purposes the National Agency has declared for the examinations. This can be connected to *stronger framing*, which makes the requirements of the pedagogic practice more visible to the teachers. Many teachers also state that there is a greater chance that pupils will be assessed equally (also one of the national purposes), although one teacher experiences that it is difficult to correct the examinations objectively.

Some also state that they now think more about preparing the students to pass the examination, something that has often been pointed out as a limiting effect of testing.

Concerning the formative use of the national examinations, several teachers experience it as positive to conduct an earlier check as this lets them take action for more and relevant student exercise, and support students who do not meet the knowledge requirements: “We need an earlier check so that there won’t be unpleasant surprises later”; “It’s a good time to ‘measure’ ”; “It’s too late in the fifth and six grades”; “It’s good to have an earlier check, so we can get an idea about the child’s development”; “You get information about what students should train at more and what kind of support is needed.” Hence, the teachers experience a great deal of positive effects through this monitoring instrument.

Yet, the results do witness not only experiences of empowerment but also those of negative monitoring and restrictions on the teachers’ professional freedom. According to some teachers, the examination supports them in judging students’ mathematical knowledge on the one hand, but at the same time they feel that the examination becomes “a very big thing” to students and parents or that they have difficulty communicating the results to students and parents. “What I think is tough is saying to a cute little child who’s only 8-9 years old that he/she does not really fill the bill, you’re not good enough! Of course we don’t say it like that, but maybe between the lines and masked in a number of nice formulations.” They state that a teacher should “play down the test” so students and parents can understand that it is merely a diagnosis for the teacher and helps him/her obtain information about what should be practiced more in the mathematics classroom. Again, the teachers talk about the examinations in a formative manner. Some, although positive towards the examination, experience the tests as very time-consuming: “On the other hand, the teacher already knows before the examination who’s struggling. One might dedicate more time to these children instead of putting an enormous deal of time into preparing, administering and correcting all these tests.” Hence, the teachers feel that the tests control their use of time and that they as professionals could do something better with the time they now spend on the tests.

There is also a tension between the advantages of the tests, mentioned above, and how some teachers feel about the student’s right to develop at his/her own rate: “We say on the one hand that the children should have the possibility to develop at their own rate according to their own ability… all (children) are different. And yet there’s the idea that all children should have achieved at least this minimum level in mathematics by the end of the third grade. That feels tough to me!”; “If the child is to develop at his/her own rate, it’s difficult to be ‘mature’ and pass the national examination in the third grade”. Only two teachers do not experience the reform as positive at all, and refer to the problem described in the previous quotes: “No! It prevents the children’s individual knowledge process. To believe that all students should reach equally far at the same time is the consequence of a wrong thinking process”; “It’s forced me to adapt the students to the level of the national examinations, instead of adapting the teaching to students’ individual knowledge processes”. Hence, these teachers express a worry that the national examinations force them to violate the natural maturation of the child. There is also a worry about the limited time in the scope of the mathematics lessons to allow all students to be able to achieve the goals.

The results reported above indicate that the teachers tend to think about the national examinations formatively and that they expect school leaders to also take action with students who fail the examinations: “The headmasters become more willing to find resources in order to support students who fail to achieve the goals, because the result of the school is made visible”. Whether the school leaders use the results formatively at a school level (e.g., learn about the weaknesses and strengths of the pedagogies applied in the school and take action) seems to vary, according to the teachers. The majority of the teachers declare that they have not been informed of the school’s results in previous examinations. Concerning formative assessment in general, most of the teachers state that they use tests like ALP (developed by Gudrun Malmer) and Diamant (from the National Agency of Education), textbook tests and discussions. Several teachers only mention observations during the lesson work, and some state that they do not yet know how they will assess the students because their class is new for them. This shows that formative assessment is not a priority for all teachers in planning their teaching before the term starts.

The majority of the teachers do not have any experience of empowerment policies in the form of in-service education concerning the reforms, or extra time for studying and discussing the new steering documents before the meeting attended by 28 of the total 41 teachers in our study.

Conclusions and discussion

The results of our study indicate that the primary teachers consider the introduction of the goals, content and national examinations for the third grade in general as something positive, something that strengthens their teacher professionalism rather than restricts it, assuming they are used in a correct manner (“not knocking out the students”). Hence, the strategies of control can be experienced by the teachers themselves as empowering the teacher’s professionalism. A great deal of positive expressions can be connected to stronger classification and framing, and making the requirements visible to teachers, school leaders, students and parents. At the same time, the expressions of negative monitoring are connected to worries concerning children’s right to feel that they are good enough and their right to develop at their own rate based on their prerequisites, a characteristic of pedagogic discourse that some researchers have connected to invisible pedagogies (see, for example, Fowler & Poetter, 2004). The national examination also entails some control over teachers' time, which some teachers experience as restricting their possibilities to make professional decisions concerning where to concentrate their effort. On the other hand, the new curriculum still leaves a great deal of space for teachers' interpretations and freedom to apply their teacher professionalism in their decisions, which many teachers also experience as problematic as they would prefer even more concrete guidelines about the content of their instruction.

Concerning the preparation of students to pass the examination mentioned by some teachers, there may be a danger of only working with similar tasks as those found in the examinations. The examinations only measure the minimal achievements and hence the goals of the instruction should be more extensive, as one of the teachers points out. The continuous formative assessment, which is not regulated at all, plays a substantial role in how the instruction is planned beyond the national examinations. The national examinations are experienced in a formative manner by the teachers. However, we do not know how this is realized at school level. There is no monitoring of the use of other kinds of formative assessment in the municipality, and our results indicate that there is a substantial variation between how teachers relate to it.

Neither the status of primary teachers nor that of primary teacher education is especially high in Sweden compared with certain other countries, and primary teachers seldom take part in the school debate at a national level. It is possible that one reason why the teachers experience the reforms as generally positive, besides the clearer guidelines, is that the importance of primary teachers' work has become more visible in society at a national level. We found the theoretical underpinnings drawn on Bernstein's theories of pedagogical discourses fruitful for our analysis. Yet, more work is needed to illuminate the primary teachers' role and positioning in the time of the recent reforms.

Note

1. The municipal mathematics developers are a part of the national developmental project initiated by the Swedish Government in 2006. They have participated in the regional conferences organized by the National Centre for Mathematics Education (NCM).

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Recognising Knowledge Criteria in Undergraduate Mathematics Education

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As part of a larger study on the transition between upper secondary and tertiary mathematics education, this paper reports findings from an investigation of the relation between the students' understanding of the criteria for legitimate mathematical knowledge and their achievement during the first year in their undergraduate mathematics studies. As a methodology, we used interviews in which engineering students were given excerpts from different, more or less formal, mathematics textbooks and asked whether they could rank these texts as being "more or less mathematical" and explain why. The results of our case study indicate differences in the students' views that are related to their achievement.

Introduction

The analysis reported in this article is part of a larger research project (funded by the Swedish Science Foundation) about what has become called the "transition problem" from secondary to university undergraduate mathematics education. The goal of the project is to develop an integrated view of mathematical, didactical and social aspects of the transition problem. In this report we draw on one of several interviews conducted throughout the first year of the students´ university enrolment. In this interview we attempted to get insight into their awareness of the type of mathematics in the beginning undergraduate mathematics courses, in lectures and in exams (as compared to upper secondary school mathematics).

Related studies

Students enrolled in different programs, such as engineering and civil engineering, physics, other natural sciences and in teacher education have to pass mathematics courses. The outcomes of national surveys and of international research studies point to a range of issues that describe discrepancies, problems and difficulties arising from the transition from school to university mathematics learning these students face. A couple of studies suggest that discrepancies are not only a problem of specific topics to be mastered, but of a change in the type of mathematics. The notion of "a fundamental conceptual divide between school and university mathematics" (Hoyles, Newman, & Noss , 2001, p. 832) has been used in this context. Mathematics at university is also presented in a comparatively advanced technical language, which students perceive as more cumbersome (Hemmi, 2008). Raman (2002) infers from an analysis of textbooks that in the transition from pre-calculus to calculus students lack opportunities for co-ordinating informal and formal aspects of mathematical meaning, a problem addressed also by Bergsten and Jablonka (2010). Österholm's study (2008) shows the difficulties students face when confronted with a mathematical text containing symbols in comparison with a less technical version of the same text.

In a study at universities in France, Spain and Canada, the tasks to be dealt with are perceived by the students as more "abstract" (Guzmán et al., 1998, p. 749). The authors point out (pp. 752-753) that tertiary mathematics includes "unifying and generalising concepts", which set up new demands, often described as a switch from intuitive to formal mathematical thinking. In the lectures this is often related to a "Euclidean style" of presentation in the form of definition, theorem, and proof (Weber, 2004).

In a Swedish report (Högskoleverket, 2005, p. 32), 75% of the students find the mathematics courses difficult, and 85% of the students say that the university sets up new demands. On an open question, the most common answers were "higher demands of understanding" (c. 330 in the sample of 2379, comprising 59% of the target population) and "higher level/higher demands/more content" (c. 190). The meaning of "higher demands on understanding" and "higher level" is not specified in the students' answers.

Theoretical background and research question

The studies cited above indicate changes in knowledge criteria (e.g. "higher" or more "abstract" mathematics and new forms of presentation). Our investigation attempts to clarify the extent to which the undergraduate students are aware of such changes and how this awareness is linked to their achievement. We draw on discursive approaches, in particular on Bernstein's theory of pedagogic discourse. For the purpose of the analysis reported here, the concepts of *classification* and *recognition rules* are of particular relevance (e.g. Bernstein, 1981). We start with the assumption that students, throughout their mathematics education, move through a range of different mathematical discourses. Differences in mathematical discourses can, for example, be described in terms of qualitative differences, e.g. foregrounding empirical or metaphorical abstract references, informal (often called 'intuitive') or formalised arguments, inductive or deductive reasoning, or in terms of the extent to which the principles of a range of mathematical activities are made explicit. When moving from high school mathematics through undergraduate mathematics to "higher levels", the knowledge becomes more strongly *classified*. The concept of *classification* describes the strength of the boundaries between discourses and groups of actors (e.g. Bernstein, 1981). Strong classification means that strong boundaries between subjects are maintained and that informal and formal knowledge are more strictly separated. The knowledge classification of undergraduate university mathematics creates specific subject-related recognition rules that differ from school mathematics and from more applied or more advanced mathematics.

In order to be successful in university mathematics activities, students need to understand the principles for distinguishing between the university context, and the context of doing high school mathematics: They have to recognise the speciality of the discourse, in which they engage; they must be in possession of the *recognition rules* (ibid.). According to Bernstein, this is a necessary condition for their capacity of producing what counts as a legitimate mathematical contribution in this new context.

We investigated whether and on which grounds the undergraduate students were able to recognise weaker or stronger principles of knowledge classification, and whether this recognition is correlated to differences in their achievement.

Methodology

In order to grasp students' possession of the recognition rules, they were, in individual interviews, confronted with four different mathematical texts and asked which of those appear "more mathematical" to them. For the selection and description of the different texts, as well as for the analysis of the students' responses, we employed analytical tools developed in the context of systemic-functional linguistics (cf. Halliday & Hasan, 1989), where language is modelled as interacting with the social context of its use. Each "text" (oral and written productions) is an instance of the process and a product of the social meanings in a particular context. Different aspects of meanings embedded in a text allow "predictions" of corresponding features of the context. Understanding the speciality of the context of doing undergraduate mathematics (*recognition rules*) implies recognising the speciality of the corresponding texts and vice versa.

Halliday and Hasan (1989, p. 44-45) state that a learner, while listening or reading, has to (1) "understand the processes being referred to, the participants in these processes, and the circumstances [...] associated with them [EXPERIENTIAL]," as well as the "relationship between one process and another or one participant and another, that share the same position in the text [LOGICAL]". The learner also needs to (2) "recognise the speech function, type of offer, command, statement, or question, attitudes and judgements embodied in it, and the rhetorical features that constitute it as a symbolic act [INTERPERSONAL]". He or she also has to (3) "grasp the news value and topicality of the message, and the coherence between one part of the text and every other part [TEXTUAL]."

The capitalised terms in brackets refer to different aspects of meaning, which in turn correspond to different aspects of the context, that is to (1) the field, (2) the tenor, and (3) the mode of a discourse. The field refers to the activity and topic with which the participants are engaged in which the language figures as an essential component, the tenor to the (socially significant) relationships, status and roles of the participants, and the mode to what the language is expected to achieve in the context. However, there is no straightforward relationship between the features of the text and features of the context – all meanings in a text are functions of the context and vice versa (Halliday & Hasan, 1989, p. 55). The employment of the framework as a methodology allows a differentiated description of knowledge classification, both in the texts and the students' recognition rules.

The students

For this study, 20 first year civil engineering students from a university in Sweden were selected from five different study programmes (mechanical engineering, computer technology, physics and electric engineering, industrial economy and a programme with a focus on energy and environment, here denoted M, C, P, I, and E, respectively), so that within each programme there were students with all different combinations of low and high achievement on the diagnostic mathematics test at the beginning of their studies and the mathematics exams during the first year of study, respectively.

The interviews and the texts

The interviews on which we draw in our analysis were conducted after about half a year of the students' enrolment at university after their examinations in the introductory linear algebra and/or calculus courses. Here we only focus on one part of these individual interviews that dealt with the students' recognition of different types of mathematics. In the interviews, the students were shown four texts (1-2 pages), all excerpts from Swedish language mathematics textbooks at undergraduate level. The texts were selected so as to correspond to different strengths in the classification of the field and variation of mode and tenor. The students were invited to compare the texts and asked whether and how they perceive them as "more or less mathematical" and to rank the texts along this dimension, if possible. The interviews were audio-recorded, transcribed and relevant parts coded according to the students' focus on field, mode and tenor of the discourse. In the following, the four texts are described in terms of how features of the texts refer to these categories.

Text A (Tengstrand, 1994, pp. 52-53). Field: The text gives an example of organic growth, specified as the growth of a bacteria colony, which is described as a recursive function in two steps, and then generalized to $t = n$. By definition, $1 + p = a$, is introduced as growth factor. Substitutions into the formula are made for non-integer time periods ($t = 1/2$, $t = 1/3$). Exponential growth for the bacteria colony is then declared after deriving $N(t) = N_0a^t$. Then an example is calculated (deriving N(t) from 2 values). Tenor: An anonymous knowledgeable author speaks to an unknowing student (employing a general "we", and a "reader-we", e.g. in "as we have seen earlier"). The worked example is introduced with an imperative. Mode: The overall mode of the text is expository, and the steps are logically connected (*then*, *so*, *now*, *as*, etc.). The semantic choices reflect a narrative structure (bacteria growing in time), even though the tense is present tense. The coherence of the topic is achieved through repetitive use of technical terms or respective mathematical symbols, and through reference to a statement earlier in the text. Equations are printed aligned to the centre. The example at the end is framed as a procedure in symbolic notation.

Text B (Lennerstad, 2002, pp. 238, 240-241). Field: The text deals with power functions. It introduces symbolic notations for positive exponents and basis, and then for a root function as inverse power function, justified by its suitability with "computational laws for exponents". A section on negative exponents then culminates in two theorems about growth properties of monotonous functions, and of power functions in particular. There are no proofs of these, but they appear as consequences of the exposition. The text contains a footnote about mathematical meanings of "root", which is presented as a dialogue between Hjalmar and Inge. For this "parallel text", the corresponding field is the verbalized learning activity of two students. Tenor: The text employs a "we" including both reader and author (*we summarise*), as well as a general "we" commonly used in mathematical writing. The relationship between author and reader is constructed as one between a friendly teacher and students who try to understand. The dialogue between students in the footnote offers identification with a group of readers who try to understand the same text. Mode: Generally, the mode is expository, with an introduction about what is to come in the section, suggesting a didactic mode. Heading, sub-headings and the two theorems are numbered, which foreground technicality. The text also contains four graphs of functions. The description of the functions in the graphs employs non-technical terms (*approaching*, *raising*, *falling*, *coming closer*, etc.), while the rest of the text (except for the dialogue) shows a high degree of technicality. Two equations appear centred ($\sqrt[k]{x} = x^{\frac{1}{k}}$, $x^{\alpha} = \frac{1}{x^{-\alpha}}$), but also within the running text there are equations and inequalities in symbolic notation. The text is composed as a series of successive generalisations and contains grammatical metaphor.

Text C (Hyltén-Cavallius, & Sandgren, 1956, p. 184). Field: The text starts with a statement and proof (also called "proof") of one form of the intermediate value theorem, which appears under the name "theorem 10" with reference to a graph of a function with several local maxima and minima. Reference to another theorem in the same book is made, and the proof in one part employs the technique of indirect proof and explicitly states assumptions. The text continues with a note that invites to conduct a "thinking experiment" in relation to whether the theorem would also be true for a function defined only for rational numbers. Tenor: Throughout the text speaks an anonymous knowledgeable author to an unknowing reader in the form of an exposition. Mode: There is a reference to the definition of continuity in the same book. The text is expository and it foregrounds technicality as well as grammatical metaphor in all its parts except for the note. It frequently uses a general imperative (*consider*, *assume*, *let*) and a general "we" typically used in mathematics texts. Coherence is achieved through logical relation and substitution of symbols.

Text D (Hellström, Morander, & Tengstrand, 1991, pp. 382-382). Field: The text has the heading "work with varying force". It starts with a general description of a process (a moving body) and poses the question that is going to be answered: *How does one calculate the work if the force K(x) changes with the distance x from the starting point?* A generalised "formula" for the problem (work as an integral) is developed through heuristic reasoning and applied to a problem of a falling body. For doing so, the formula according to which two bodies attract each other with a force proportional to their mass is stated. Throughout the text, uncommon sense interpretations of "force" and "distance", "interval" etc. are suggested by immediate use of a symbol after the words. Except for these word-symbol groups, the text employs non-specialised language, including estimation modifiers such as in "equals nearly" or "about". Statements only including symbolic notations are printed aligned to the centre. Some mathematical symbols are used in a non-technical way ($\sum$, Δx, $\rightarrow$). Tenor: Throughout the text, an anonymous knowledgeable author/teacher speaks to an unknowing student (frequently employing a general "we"). Mode: The overall mode is expository and in parts didactical (e.g. questions as introduction to an exposition). The text appears logically coherent, even though new themes are introduced quickly.

According to a common characterisation of mathematics texts as focusing on technicality and grammatical metaphor, dealing with proof rather than with calculations or applied examples, as well as the expository mode and impersonal style, the ranking of the texts (from strongly to weakly classified) is CBAD.

Findings

Our general question whether recognition of the differences between the texts would be necessary for success, something "predicted" by the theory, can be answered positively. From the 20 students, only four students have ranked the texts in the order CBAD (students C6, C9, M8, P6). While one of these has moved from high (diagnostic test) to low achieving (course examination), the three others achieved high scores on both occasions. Four other high achieving students (E4, I9, M7, P3) chose CBDA, while still another (E3) offered two alternative rankings, CBAD and ABCD, and expanded on the meaning of "more mathematical". These outcomes reflect that recognition of the knowledge classification is necessary but not sufficient for success, another implication of the theory. Indeed, the three students with low scores in both diagnostic test and examination (I1, I12, M1), did not choose the ranking CBAD, but instead CBDA, DABC, CA (BD unclear), respectively. Of the remaining students, three high achieving (M5, P2, P7) ranked BC and three students who increased their achievement (E6, E8, I6) ranked CB as most mathematical, while two students with lower achievement (C7, M9) ranked A as the most mathematical text.

In the following, we present some of the arguments provided by the students. In general, they referred mostly to features reflecting the field (experiential and logical meanings) and to some extent to the mode of the discourse (textual meaning), but not so much to features that reflect the tenor (interpersonal meaning). Arguments for ranking text C as "most mathematical" and others as "less mathematical" by pointing to terminology (as reflecting the field) included:

> …because they take up more mathematical things (C6) …almost exclusively mathematical terms (C9) …easy to count the number of words that have nothing to do with math (E4) …variables, they have a curve here where the variables are declared and shown and Greek letters (I1) …strange words and only *f* of *a* not equal to *f* of *b* and all that (I9) …very arbitrary numbers *a b* and such…much palaver about bigger than zero and such stuff (M7) …one says let *y* be an arbitrary number (P3)

References to the field in favour of text C occasionally also referred to its theoretical nature and generality, such as:

> …proof for continuous functions…one defines mathematics (M7) …this gives no examples from reality (P3) …powerful mathematical proof (P6) …this is proof…with lots of intervals and continuous (E4) …these here now [C and B] deal more with the mathematics itself…describe things within the mathematics…this is then within pure mathematics…inner-mathematical (E3)

References to the mode included statements about coherence and inaccessibility:

> …theorems refer to theorems (P3) …first they say something and then they prove it…with the help of certain assumptions (M8) …strict (P6, C6) …more sectioned…with theorem and proof (E8) …a normal layman does not understand then what one talks about there (I1) …even worse (M7) …if one missed a lecture and would try to read further into it so it should somehow be such one (M8) …this is about how our teachers or lecturers go about things (M7)

A couple of students described texts C and B, and A and D, respectively, as similar and chose rankings with the first of these groups as "more mathematical", while the order within the groups varied. Many referred to the somewhat more didactical mode of the text B and to the structure, such as:

> …more explaining (C6) …explanation…as they show in the math book (C9) …also mathematical but more understandable and so because this is more text and less only expressions and symbols…more words more text more explaining [than C] (I9) …but something does that it feels less mathematical…can't

> actually point my finger at what this is (M7) …explains a little shows and explains …than just lining it up (M8) …not really a proof but they just explain what something means (P6) …written more in words actually (E4) …well more like a just flowing text (E8)

No student saw making assumptions as a feature of text B, in contrast to text C.

The texts A and D were often distinguished from the others by reference to the field. The ones who ranked D as the "least mathematical" mentioned:

> …very much examples…physics formulas (C6) …physics or what one should call that…this is kind of no well type gravitation *g* (C9) …also more a little physics about it…a relation kind of between a body and a length (M9) …not so much general…this physics…text (P7) …useful in physics (P6) …I think somehow more physics (E6) …it more applies mathematics (E3)

Also reference to the mode was made by those who ranked text D last, such as:

> …example of a 'if we do like this so it should become like this' (C6) …some values down there…one discusses oneself forward to things that seem reasonable (P7) …not so rigorous (P6) …several different formulas…that one should transform them or do some more math than just put in a numeral (E3)

Two low achieving students ranked text A as the most mathematical, however for different reasons. One described texts B and C as equally containing "more advanced mathematics". As a general reason for the ranking ADBC, the student focussed on the mode in terms of the reader's access, for example:

> …I see as most mathematical actually to be able to get someone to understand compared with just to write formulas straight forward (C7)

The other one conveyed two conflicting views in the interview and based his final ranking on his own preferences in terms of what mathematics means and that it should be "easy to understand" (student M9):

> …yet mathematics is for me numbers and tasks…math is it is to calculate tasks to be able to apply it…it is easier to understand the mathematics when it is written with with numbers than with text…somehow more only text [in C]

However, the distinction between pure and more applied mathematics (field) was also made in the same interview:

> …they explain pure mathematics [B and C]…this one feels kind of physics [D]

As this student was one of those who improved their achievement in the examination compared to the initial test, the ambivalence could express a "transition" between different recognition rules. There was, however, a group of low achieving students, who explicitly stated that the experiential and logical meaning of the texts, that is, the field of the discourse, remained hidden to them:

> …one does not get to grasp what about what that is what one reads [C] one takes up maybe this and this [A, D] more than those two…this one I

understand actually a little more [B] this one now one can really understand [D] (C7) …this was kind of better [B] (I6) …that the brain registers more easily if it is written each in one's line [A]...that one I do not like [C]...that one I like…structured and so [D] (I12) …that one is the best [A]…simply harder to understand…here they assume things all the time…very very much theory [C] (M1) …math one should also be able to understand kind of so that there is nothing more mathematical just because it becomes more complicated (P2)

Discussion

The outcomes of our study show the potential of a discursive approach as well as of the theoretical framework based on Bernstein's theory of pedagogic codes. In sharing the assumption of functional linguistics, in which language is modelled as interacting with the social context of its use, we assume, while talking about features of the text, what the students say reflects their experiences in the context. We observed a relation between the students' understanding of the principles for knowledge classification (*recognition rules)* and their achievement, as detailed above. While the higher achieving students were more specific when talking about the field of the discourse, the low achieving students seemed to find it challenging to describe what the texts actually are about. These latter students do not seem to recognise that the principles of knowledge classification have changed in comparison with high school mathematics.

Some students took the access to the field of knowledge provided in the text as a criterion for a "more mathematical" text. The higher achieving amongst them took the esoteric nature of the texts as a criterion for ranking them as the "most mathematical". This suggests that they attribute a specific mode, namely inaccessibility, to mathematical discourse. Two low achieving students also mentioned inaccessibility, but they took it rather as the rhetorical function of the texts, that is, they referred to the tenor of the discourse. Without access to the field of the discourse, these students could naturally only focus on their status as readers. They demanded, however, that a mathematics text should be more accessible and one student suggested that the field of the discourse should not be generalised hypothetical statements, but rather numbers and tasks. This suggestion reflects the knowledge criteria from school mathematics.

Only very few students pointed to the typical "Euclidean style" of presentation in the form of definition, theorem, and proof, but pointed to making assumptions as a characteristic feature of mathematics. The ones who described the texts that deal with examples as less mathematical, also indirectly referred to the generality of mathematics. The differences in knowledge classification between applied and pure mathematics seemed to be more obvious to most of the students, as well as the distinction between procedures and principles.

Almost all students referred to the anonymous author in plural as an active subject in their formulations. “They”, perhaps the community of mathematicians or the collective of mathematics teachers, have written the texts. The students seem to assume a common tenor of the discourse. After all, despite subtle differences in tenor, all the texts speak from a position of an expert who teaches a group of similar students. The students experience themselves as participants in a community of knowers who distribute their message to the ones who do not know yet. This experience does not differentiate between high and low achievers.

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Candy or Equation? Why do Students Get Different Explanations on the Same Problem?

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The individualisation of school mathematics teaching might be necessary in the face of the inhomogeneity of classrooms. However, the choices that the students and the teacher face, might actually reinforce differences in achievement. In the following article I present and analyse data from a study about the emergence of disparity in achievement in mathematics classrooms. In the example we can see some of the effects of individualisation. The investigation shows differences in the teacher's explanations on the same problem in the interaction with different students.

Introduction

A critical issue in mathematics education is why some students are more successful in school mathematics than others. One approach to answer this question is in terms of psychological traits, that some students are more "talented" in mathematics than others. The study reported from in this paper instead sets its focus on the actions in the classroom and not on theories about natural ability. In the course of an overall international project investigating the emergence of disparity among students in mathematics classrooms, two Swedish classrooms have been video-taped for about three weeks (see Jablonka, Johansson & Rohdin, 2010). In this contribution, conversations from a first year Swedish upper secondary class, occurred in the third lesson at the beginning of the studies, are analysed. When students and teachers meet each other for the first time, it does not take long before both the students and the teacher seem to have realised or got an idea about who apparently is good and who is not so good at mathematics. One question we ask in the research project is what kind of actions creates these images, and whether they are stable and how these images affect the teaching of mathematics. In a mathematics classroom we find different domains of mathematical activity, in which the students engage. One question we ask is whether, within the same classroom, different students might be initiated into different domains.

The first observation we can do is that in mathematics classrooms an "intermediary domain" is constructed that functions as mediation between the everyday discourse of the students and the formal mathematical discourse with its specific grammatical features and technical terms:

> As Anna Sfard shows us, in discussing the limits of mathematical discourse, the differences in the 'meta-discursive' rules between everyday discourse and mathematical discourse require us to develop a well-defined intermediary between the two. (Umland & Hersh, 2006, p. 9)

Typically, this "intermediary domain" in mathematics classrooms consists of different types of word-problems. But there are other traces of the everyday domain in the form of images, didactic material, and metaphors. However, the intermediary domain is not "well-defined". It becomes a shared field of activity through a process of institutionalisation in school mathematics as its "public domain" (Dowling, 2009). In our approach, we are in general interested in the potential of using intermediary domains as a base for developing mathematical concepts and methods, especially through interaction with students. This includes also possibly limiting effects of such interaction if a reproduction of everyday discourse is the outcome. Our interest focuses on the type of knowledge to which different groups of students might or might not gain access. There might be a general problem in a classroom practice that offers tasks for students without specifying the knowledge domains that form the basis for their solution (Gellert & Jablonka, 2009; Jablonka & Gellert, 2011). In the conversations analysed below, one sees that students and teachers' actually operate in different knowledge domains when they work out solutions to contextualised tasks.

Theoretical background

School mathematics, in its different versions, is related to different curriculum conceptions. These can be seen as an outcome of a process of dual recontextualisation (Jablonka & Gellert, 2010). Recontextualisation consists in subordinating one practice to the principles of another (Bernstein, 2000). In school mathematics, features of the practice of developing mathematical knowledge in an academic practice (such as investigation, systematisation, proof and refutation) are subordinated to pedagogic principles. School mathematics, though, also recontextualises domestic everyday practices, and in some curricula, also vocational practices. Consequently, word-problems are a result of a dual recontextualisation. Thus they constitute hybrids between domestic and mathematical knowledge (the "intermediary domain"). When students have to solve tasks that do not contain a specification of the knowledge domain on which a solution is to be based, different strategies allow the development of more or less generalised mathematical knowledge or more or less localised knowledge. In order to account for the emergence of such differences, the pedagogic strategies described by Dowling (2009) can be used as a methodological tool for interpreting interaction on the micro-level in terms of these strategies. The following strategies are reinterpreted to be used as a methodological tool:

Specialising consists in distinguishing different cases of a method or concept and developing specialised means for dealing with them, including specialised terminology. The range of application of a practice is thus reduced by specialising.

Generalising includes describing different cases in relation to a common principle and so expands the range of application of a practice.

Specialising and generalising offer a route to developing a mathematical practice of which the principles are made discursively explicit and can be shared, that is, institutionalised.

Localising is the strategy of constructing one particular local example and thus offers no route to a principled mathematical discourse.

Articulating consists in pointing out different examples of a method, without making the underlying principles explicit.

The strategies of localising and articulating refer to (school) mathematical activities, for which the criteria are not, or cannot be made explicit. These include, for example, developing a mathematical structure that describes empirical processes and structures ("modelling"), finding a deductive proof for inductively developed principles, solving a word-problem, making an appropriate drawing to illustrate an argument, or applying a solution method to a range of different examples.

In the conversation analysed in this article, the students are engaged in solving a word problem. For this activity there are no explicit criteria available to the students and there are no hints in which knowledge domain they have to solve it.

The research setting

The data is from a classroom with students aged fifteen to sixteen, attending the first year of Swedish "gymnasium" (i.e. upper secondary school). This is the third lesson after their very beginning at the new school. This means that the students and the teacher do not know each other from before. The students are working mostly individually with the textbook tasks and they are sitting in small groups with four students in each group. The task is from the first chapter of the textbook and deals with understanding and analysing tables and charts. The episodes have been chosen by identifying those tasks that most students ask about (not necessarily in the same lesson). From these conversations about the same task, two have been chosen that differ most in the use of strategies. The students are working with the task individually or occasionally in small groups. All excerpts are from the same lesson where these two students, here called Sara and Marcus, are asking the teacher for assistance with the task. The two students are not sitting close to each other and ask the teacher independently.

The task

The task in Figure 1 was used in this study. It is a word problem and, at a first glance, could be placed in the public domain. The task appears to be authentic (in the public domain) but it is actually placed in the intermediate domain because you need to use just the right amount of everyday knowledge and mathematics. You need to know that the table provided is not the bill, it is information about the subscription and that is everyday knowledge, or as Dowling (2009) would say, public domain knowledge. On the other hand you need to realise that in this specific task, it does not matter what kind of calls you make because the cost is the same whether it is a call to another mobile phone or to a regular phone, and that is in the intermediate domain. In the everyday the kind of calls you make, makes a difference in what costs you get. Furthermore the question about how long he has been talking is not clearly connected to the phone bill.

One month Fredrik got a phone bill of 226 SEK. He had dialled 37 calls and sent 14 sms. How long has he been talking?

Mobile for all	Fees
Single payment	250 SEK
Fee/month	59 SEK
Opening fee	0,59 SEK
To mobile net fee /minute	0,59 SEK
To regular phone fee /minute	0,59 SEK
SMS	0,75 SEK
MMS	2,25 SEK

Figure 1. Textbook task used in the study (Szabó, (2007); my translation)

The data

The following transcript is translated from Swedish. It is translated in a way that also shows the way of using the language in these conversations. Due to interruptions and other disturbances it might be hard to follow every step of the conversation but this shows clearer the use of the different strategies and the divergence in the conversation. The analysis is presented to the right of the transcript. The first student is Sara. She is raising her hand and the teacher walks up to her.

1	Teacher:	hi is everything ok	
2	Sara:	Yes	
3	Teacher:	mm okay	
4	Sara:	on this	
5	Teacher:	he gets a phone bill of two hundred twenty six crowns	Localising
6	Sara:	mm	
7	Teacher:	dials thirty seven calls and sends fourteen sms and then there is a fee for	Localising

		how long he talks	
8	Sara:	mm	
9	Teacher:	what kind of strategy do you intend to use here	Question
10	Sara:	I was thinking/	
11	Teacher:	what do you want to know	Localising
12	Sara:	how long he's been talking	Localising
13	Teacher:	yes	
14	Sara:	isn't it the average of every call you calculate	Specialising
15	Teacher:	yahh	
16	Sara:	then you can check how much one sms costs	Localising
17	Teacher:	yes	
18	Sara:	that times fourteen	Specialising
19	Teacher:	yes perfect	Specialising
20	Sara:	take away that from that	Specialising
21	Teacher:	yes and then he has dialled thirty seven calls then he has	Localising
22	Sara:	yes that's the thing I don't know then it's that times thirty seven	Specialising
23	Teacher:	yes that is to and only to be allowed to start talking	Localising
24	Sara:	[inaudible]	
25	Teacher:	yes and that is what is left	
26	Sara:	that is how much he has been talking	Localising
27	Teacher:	you know right that is what he has called for	Localising
28	Sara:	then you take that and divide with	Specialising
29	Teacher:	and then you check what every call costs eh… and then you have an amount of money and you know how much every call costs then you can calculate how many calls that was	Localising
30	Sara:	[inaudible]	
31	Teacher:	mm start with sorting out and check how much money he has got left	Localising
32	Sara:	ok	
33	Teacher:	so you figure out how to calculate how	Localising

		many	
34	Sara:	ok	
35	Teacher:	if it does not work I'll come back	

The following conversation occurs 8 minutes later after the teacher has been talking to other students. Sara is raising her hand again and the teacher walks up to her.

36	Sara:	I can't get it to work	
37	Teacher:	was it still wrong	
38	Sara:	yes	
39	Teacher:	ok now you have	
40	Sara:	it should be this much but I don't know how to do that	Specialising
41	Teacher:	eh is this that	
42	Sara:	no this was it	
43	Teacher:	yes there	
44	Sara:	that has nothing to do with	
45	Teacher:	no ok … is that what he has called for	Localising
46	Sara:	yes that is what is left when I took away	Localising
47	Teacher:	ok /sms/ if that had been two crowns per minute mm and you have had this much money mm how would you have calculated that	Localising
48	Sara:	no idea	
49	Teacher:	if you had ten crowns and you by candy that costs two crowns each mm how many candies have you bought	Localising
50	Sara:	five	
51	Teacher:	how did you do	
52	Sara:	I took how many times two becomes [inaudible] ten	Specialising
53	Teacher:	yes perfect	
54	Sara:	it's ten	

In line 5 we can see that the teacher is localising by repeating the task and continues to repeat the task in line 7 and hence is still localising. In line 9 the teacher is posing an open question when asking about the strategy and in lines 10-11 the teacher interrupts Sara before she answers the question and localises again by returning to the specific question in the task. In line 12 Sara responds to the teacher's question and is localising, but then in line 14 she is asking a question on how to calculate; even though she refers to the task she is using specialised language. Then Sara continues to localise in line 16 by referring to the task, in particular to the sms. In line 18 Sara is using a mathematical specialised language and is not referring to the task, in line 19 the teacher confirms Sara's specialising. Sara continues without referring to the task, by means of specialised mathematics language in line 20. However, in line 21, we can see the teacher localising again by returning to the task. In the following lines we can see the same pattern: Sara is asking for the method or answering with specialised language (except in line 26 but there it is in response to the teacher).

In line 33 the teacher is asking or stating "so you figure out how to calculate". In Swedish the teacher uses "klurar ut". The English "figure out" does perhaps not sufficiently capture that it is something difficult and not straight forward. Thereafter Sara is left alone for about ten minutes, while the teacher is talking to other students. In the conversation that follows a similar pattern occurs: Sara is asking for ways how to calculate and the teacher is localising. In line 47, the teacher is asking for a way of calculating with specific data. Sara's answer, ”no idea", amounts to an even more localised question where the teacher (line 49) introduces prices for candy and asks how many she could buy with a given amount. In the context of the whole conversation with Sara, this is an articulating strategy through pointing to another example for the same method.

Altogether, in this conversation Sara is localising and specialising by means of reference to the task, but is also asking for the principle of how to calculate. The teacher is localising within the context of the task and also in his articulation of the example with the candy.

In the following, Marcus and the teacher talk about the same task. This conversation occurs 10 minutes after the second conversation with Sara in the same lesson.

55	Teacher:	is there a problem	
56	Marcus:	nnn that one…take away…and one point zero eighty crowns	Localising
57	Teacher.	what are the costs in the two hundred twenty six crowns	Localising
58	Marcus:	the sms the calls sms	Localising
59	Teacher:	sms and so the thirty seven calls and that one … right … opening fee	Localising
60	Marcus:	then those two	Localising
61	Teacher:	yes there is one more thing	Localising
62	Marcus:	fee per month	Localising
63	Teacher:	right mm now we find out how much has he spent of those two hundred twenty	Localising

		six with the fee opening fee per call and those sms	
64	Teacher:	and those hundred fifty six and fifty	Localising
65	Marcus:	Mm	
66	Teacher:	that should be enough for the calls	Localising
67	Marcus:	mm ok and take away fifty nine times thirty seven too… since I know it's thirty seven calls	Localising
68	Teacher:	yes but that's the thirty seven calls he made the question is how long has he been talking together during these thirty seven calls	Localising
69	Marcus:	how do you know	
70	Teacher:	have you noticed it is the same price yes per minute to mobile as to regular phone so it does not matter what kinds of calls	Localising
71	Marcus:	I know one more cost that is thirty seven times zero point fifty for the opening fee because that has to be per call	Localising
72	Teacher:	but that you took	Localising
73	Marcus:	no that was the sms	Localising
74	Teacher:	no you didn't take that it was only the sms yes perfect then	Localising
75	Marcus:	fifty nine times	
76	Teacher:	mm ... ok ten fifty mm… if you have that left to call for how many calls does he have or how many minutes has he been talking if it costs fifty nine pennys ya	Localising
77	Marcus:	two hundred twenty eight point	
78	Teacher:	well the point we don't care about only whole minutes	Localising
79	Teacher:	mm have you reasoned that you take each cost separately and then what's left and then you took the next cost and then what was left	Specialising
80	Marcus:	Mm	
81	Teacher:	another way	Generalising
82	Marcus:	Division	Specialising
83	Teacher:	well yes that is also something you could do x calls and then set up everything and	Specialising

		put it equal to the two hundred twenty six that's something one could do	
84	Teacher:	and if you should be able to calculate the total cost first before without those calls fee calculate the fifty nine plus thirty seven times zero fifty nine plus fourteen times sms zero seventy five and so get a total sum and then subtract and see what is left	Specialising
85	Marcus:	Yes	
86	Teacher:	it is possible to solve in a number of ways	Generalising
87	Marcus:	mm I did the equation first but I could not solve it because I forgot that fee	Specialising
88	Teacher:	you forgot the fifty nine well I did the same over there	
89	Marcus:	Ya	

In lines 55-67 Marcus and the teacher are both discussing the task and they hold on to a localising strategy as they refer to the context of the task in each utterance. Then the question about what kind of calls is posed in lines 68-70. Here Marcus is using everyday knowledge by realising that there usually is a difference between the cost to a mobile phone compared to a regular phone and hence he is localising even more. In line 79 the teacher is specialising by describing Marcus' ways of calculating, then, in line 81, the teacher is generalising by asking for another method to solve the task. In line 83 the teacher is specialising by describing a solution to the task in mathematical language including variables. In line 84 the teacher introduces another method for how to solve the task and hence he is specialising. What we can see, generally, is that Marcus is localising and specialising whilst the teacher is localising at first and then specialising and also generalising. In fact we can see clearly in line 83 that the teacher is giving Marcus an insight into how this task could be solved in a *mathematical* way. But from the discussion above we cannot see anything from Marcus' side that would have initiated this kind of discussion.

Discussion

There are obviously different strategies in operation in the conversions with Sara and Marcus. We can see the teacher using different strategies although Marcus and Sara are using the same. As discussed above, only generalising and specialising strategies offer access to a principled mathematical discourse. Thus, the different strategies used by the teacher can be seen as contributing to the disparity in achievement in this classroom. It seems that Marcus gets more chances to solve the problem as a *mathematical* problem. One question to ask is on what the difference could be based. Do the teacher's contributions and questions emerge from within the particular conversations, forced by the interactional dynamics? This would mean, the teacher tries in each case to follow up the strategies used by the students. However, a look at the students' turns only reveals that both, Sara and Marcus, use specific data from the task and do not attempt to generalise. In fact it is Sara who requests a general strategy and Marcus is using even more localising strategies.

In light of the outcomes of the analysis of the conversations presented above, the question to be followed up in the future course of this research project is to trace back all conversations these two students had previously with the teacher. As it is only the third lesson, it is important to find out how and when the differential discourse of the teacher started. Also, relating the outcomes to the students' social and economic background and to their mathematics achievement at the end of the course, might inform on patterns of stratification in the mathematics classroom.

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The Validity of Students' Conceptions of Differentiability and Continuity

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University students' conceptions of differentiability, continuity and relations between these concepts were studied to reveal their choices of representations and their strategies to justify their relational claims. Questionnaires and interviews were used to collect data. The results were analysed and categorized through a framework based on understanding as connections between concepts, and a theory of three modes of developing mathematical knowledge. The students showed ambiguous representations opposing their own statements in some cases. The most common feature of the students' descriptions of a continuous function was incorrectness, implying a need to develop the students' concept images in that area.

Background

Mathematical proficiency entails ability to justify claims through adaptive reasoning using well-selected strategies productively and to understand the concepts involved in the processes (National Research Council, 2001). Students at university level taking their first calculus course deal with many new, or already known concepts, in a short period of time. They have some pre-knowledge of concepts to consider when learning new aspects of them and they also have to adjust to university studies where the pace is higher and the mathematics is taught in more formal detail than they are used to from upper secondary school. This may have effects on the learning outcome in terms of details in knowledge representations and tendency to adjust existing representations of concepts. Older insufficient or invalid (in the sense that they do not cohere with commonly accepted mathematical theory) conceptions may linger in parallel to new and correct ones causing confusion. This was the case in a previous study about limits of functions where some students thought that all limits are unattainable from misinterpreting the limit definition at the same time as they stated that a limit was attained in an example (Juter, 2006).

The concepts of differentiability and continuity are key concepts in calculus and are closely linked through their definitions. The relation is not symmetrical in terms of implication, i.e. continuity does not imply differentiability but differentiability implies continuity. The aim with this paper is to understand how students explain the concepts and the relations between the concepts, i.e. what kind of strategies and representations they use in their explanations, and how they justify their claims, by answering the following research questions:

1. How do students explain derivatives and continuity?
2. How do students perceive and explain the relations between these concepts?

Theoretical frame

Students' perceptions of mathematical concepts are reflected in their solutions, reasoning and other actions as traces of their *concept images* (as defined by Tall & Vinner, 1981). Their strategies to justify their mathematical hypotheses have developed in the community of the classroom within its frames of rules and traditions. Understanding a concept and being able to solve tasks involving the concept may be regarded as synonyms for some students, particularly if being able to solve tasks is enough to pass courses. However, in the literature a distinction has been made between these two ways of dealing with mathematics, according to their core features. Hiebert and Lefevre (1986) defined *conceptual knowledge* (p. 3) as a web of pieces of information well linked together with meaningful connections. Relations between concepts are abundant and significant. They defined *procedural knowledge* (p. 6) as knowledge requiring an input which the learner recognises and is able to perform in a step by step procedure, like an assembly description of a piece of furniture, to obtain an outcome. No relational understanding is required for the process to be carried out. Strong and valid connections between concepts, i.e. conceptual knowledge, help learners to understand more as new information is embedded in, and supported by, their existing knowledge (Hiebert & Carpenter, 1992). Rich connections between concepts also lessen the burden of remembering pieces of knowledge and make transfer easier. Either way to understand a new concept requires mathematical development of existing representations. Tall (2004) introduced a model describing development in three different modes, *the conceptual-embodied world* with an emphasis on exploring activities, *the proceptual-symbolic world* focusing concepts' dual features as objects and processes expressed in symbols or procepts (Gray & Tall, 1994), and *the formal world* where mathematical properties are deduced from the formal language of mathematics in definitions and theorems. Students' concept images develop through the worlds with different emphasis on the three modes allowing them to understand concepts differently.

Based on the above-mentioned definitions of knowledge and understanding (Hiebert & Lefevre, 1986; Hiebert & Carpenter, 1992) together with Tall's theory of three worlds, a set of categories to classify students' links between concepts was created and used in a prior study, as presented in Table 1 (see Juter, 2009, 2011 for further details). Examples of classifications of students' links from the earlier study are provided in the table to clarify the categories.

Type of link	Definition
Valid link, procedural	True relevant link with focus on calculations or applications, ex: *Derivative of velocity gives acceleration*
Valid link, naturally conceptual	True relevant link revealing a core feature of the concept, not formal, ex: *Derivative is the slope of the tangent in a point*
Valid link, formally conceptual	True relevant link formally revealing a core feature of the concept, ex: *If the limit* $\lim_{x \to a} f(x) = f(a)$ *exists in every point then f(x) is continuous*
Irrelevant link, no reason	No actual motivation for the link is provided, ex: *Limits have something to do with derivatives*
Irrelevant link, no substance	Peripheral true link without substance relevant for the concept, ex: *You can add derivatives*
Invalid link, misconception	Untrue link due to a misconception of the concept, ex: *Continuous means the same change everywhere*
Invalid link, counter perception	Untrue statement contradicting prior statements ex: *sinx is continuous and continuous means linear*

Table 1. Definitions of links between concepts. Examples in italics.

The last four types of links are not desirable for the students, who often are unaware of the quality of the links, particularly if irrelevant or invalid links are mixed with valid ones (Juter, 2011). Links are formed in different situations, e.g. at lectures, with peers or in solitude. Textbooks, lecturers' selections and general interests of the group of students frame the learning environment and therefore affect the representations students are using. Representations used when learning a certain topic may become vague if they are not endurable enough, e.g. not sturdily linked to other concepts (Hiebert & Lefevre, 1986). If a person learns a new mathematical topic in the embodied world and her abilities then develop to symbolic treatment she has changed the way of thinking to a proceptual-symbolic mode (Tall, 2008). If the learning phase in the conceptual-embodied world has been too short or otherwise inadequate, there may develop disjoint or vague parts of the concept image rendering the person unable to explain core features of the concept.

Viholainen (2008) conducted a study on students' incoherent conclusions about differentiability and concluded that erroneous conclusions sometimes came from linking correct parts of knowledge in an incorrect manner. He also concluded that erroneous conceptions may come from the individuals' earlier knowledge structures. The student in Viholainen's study worked with four piecewise defined functions and was asked to determine whether they are differentiable and continuous. The students' first standpoint was that differentiability requires continuity, but after thinking about one of the given functions (which is neither differentiable nor continuous) he changed his mind and said that differentiability does not require continuity since he thought the function was differentiable but not continuous. The changed standpoint came from his memory of an invalid method of checking if a piecewise defined function is differentiable. The student did not possess deep enough understanding of the concepts to be able to see what they really mean. Students who recognize the efforts required to make necessary adjustments of their concept images to understand a concept may choose to only learn the concept shallowly, in a procedural manner, to be able to manage routine tasks to pass the exam (as expressed by a student in a study about students' development of learning limits of functions (Juter, 2006)). All influences on students' mathematical behaviour interact and possibly cause spin off effects.

Methods and sample

In this paper I have chosen to study students' representation choices and proving strategies from openly stated questions. Two groups of students, a total of 43, enrolled in their first calculus course at university level were studied. The first group of students, Group A, consisted of 13 males (M) and 9 females (F) (a total of 22 students). The second group, Group B, took the same course the following semester and consisted of 16 males and 5 females (a total of 21 students). The duration of the course was 10 weeks and included basic calculus with limits, continuity, derivatives, integrals, differential equations and Taylor's formula. There were a written exam and an oral exam, both individual, to assess the students after the course. The students in both groups were taught by the same lecturer.

The students answered a questionnaire when they had covered continuity and derivatives in the courses. The questions are (Question 1 is about upper secondary education and omitted here):

2. Explain what a derivative is as if the one you explain to has never heard of the concept.
3. What are derivatives used for?
4. What features do continuous functions have?
5. Are all continuous functions differentiable? Justify your answer.
6. Are all differentiable functions continuous? Justify your answer.

The aim was to see how they justified their claims. The questions were posed using only the concepts without examples to allow the students to select whatever way they wanted to explain, with the purpose to make them reveal their perceptions. This strategy is different than that used by Viholainen (2008) who started with examples and then asked about general aspects of the properties. A drawback with a general approach is that some students may answer shortly, but the openness of the method allows some diversity in the descriptions, which was the aim. The students' responses to the questions were categorized according to validity and type of arguments used through the categories in Table 1. The category *irrelevant link, no reason* is not used in this analysis since it is not applicable on this set of data (the students are not asked to explicitly link concepts together: they use their links in descriptions of concepts in a more implicit manner).

After the course, a selection of students from Group A (11 of them) were interviewed. The students were asked about the questions from the questionnaire and the students' answers, proving, examination forms and attitudes to mathematics. Some of the interviewed students took the follow up multivariable analysis course and four of these students were interviewed again. The focus was the same as before with the experiences of the first course and implications on the second in terms of studying strategies and examination forms.

The students from Group B were studied at their oral exams to reveal their strategies for proving theorems and reasoning. The studied oral exams were conducted by their lecturer and by me. In this paper, the results from the questionnaire will be presented.

Results

The results are presented and discussed in the order of the tasks as given in the questionnaire. The open questions resulted in a large number of different answers from the students. Some of these answers were placed in more than one category since they featured more than one way to explain. The number of answers in a category may hence exceed the number of students represented in that category. The students are represented as two groups, Group A and B. The number of answers in each category shows the preferred views of the students in the groups. The most common answers to Questions 2 to 6 are presented with the number of students in each of the Groups A and B in brackets after the answers. Examples of students' arguments are provided after Questions 5 and 6.

> **Question 2**. Explain what a derivative is as if the one you explain to has never heard of the concept:
>
> **Most common answers:**
>
> Slope of a curve or tangent, (14 from A, 16 from B of which 3 from A and 3 from B wrote slope of tangent)
>
> Rate of change or a measure of change, (15 from A, 12 from B)

Two from Group A and four from Group B described the derivative as a function in itself, describing slope or rate of change of another function. A vast majority of the students explained derivatives without many formal details and used words as change and slope which implies that they have a natural intuitive sense of the concept, *valid link, naturally conceptual* in Table 1. Only two students explained derivative with a stronger focus on the processes of calculations or definition. One of them stated that "The derivative is one degree less than the original function". This student probably refers to procedural knowledge (Hiebert & Lefevre, 1986) of polynomial functions' derivatives and her statement does not show any of the core features of the concept of derivative in general, *irrelevant link, no substance* in Table 1. The other student described derivative as "The difference in y divided by the difference in x" which is a core feature of the definition of derivative and hence categorized as *valid link, formally conceptual* in Table 1. One student claimed to have "no idea" how to explain what a derivative is. He nevertheless answered the follow up question about the use of derivatives as a means for dealing with graphs, so he had an idea of the use of the concept.

Question 3. What are derivatives used for?
Most common answers
Studies of change or rate of change, study curves, (23 from A, 11 from B)
Maximum and minimum, (5 from A, 11 from B)

Most answers were similar to the answers of Question 2, but six students used links to other mathematical concepts to describe what derivatives are used for, e.g. limits, integral calculations, inequalities, asymptotes and differential equations (2 students), categorized as *valid link, procedural* in Table 1.

Question 4. What features do continuous functions have?
Most common answers
They are differentiable, (8 from A, 8 from B)
They are integrable, (5 from A, 6 from B)

16 students thought that a continuous function is also differentiable in general, revealing that their concept images of continuity and differentiability need further development, *invalid link, misconception* in Table 1. Seven students from Group A and three from Group B stated that there are no leaps in the graphs, *valid link, naturally conceptual* in Table 1. Seven from Group A and two from Group B stated that the intermediate value theorem is valid and three from Group A and four from B used the definition of continuity showing that the limit in each point is equal to the function value, *valid link, naturally conceptual* and *valid link, formally conceptual* respectively in Table 1.

Question 5. Are all continuous functions differentiable?
Most common answers
No, (16 from A, 11 from B)
Yes, (8 from A, 5 from B)

The most common way to justify the answer "No" was to give a counter example. Ten students from Group A and four from B presented $|x|$ as a counter example and one student from Group A chose e^{x^2} as a counter example. The students' "Yes" answers had more varied justifications. For example, four students (three from Group A and one from B) claimed that there is a certain slope of the curve and it is hence differentiable. Two students from Group A and two from B argued that left and right limits will be the same. There are 24 answers from the 22 students in Group A. The reason is that two students answered the question both ways. One wrote: "Yes, if we do not count functions with absolute values".

If the results from Question 4 and 5 are compared it shows that eight of the students who claimed that a continuous function is differentiable in Question 4 also claimed that continuity implies differentiability in Question 5. They were non-ambiguous in their conceptions even though they were incorrect. On the other hand, five of the students who incorrectly claimed that all continuous functions also are differentiable in Question 4 correctly stated that continuity does not imply differentiability in the following question, so their first answers would be categorized *invalid link, counter perception* in Table 1. One student simply answerer "No", another justified his answer by saying that the function needs to be harmonic, one stated that differentiability implies continuity, and two had similar reasoning, one about pointy graphs where there is no derivative at the peak and the other stating that there can be different derivatives in the same point. The last two examples show traces of conceptual knowledge (Hiebert & Lefevre, 1986) where characteristic features of the concepts are being used.

Question 6. Are all differentiable functions continuous?
Most common answers
Yes, (16 from A, 11 from B)
No, (5 from A, 6 from B)

Seven students from Group B just answered "Yes" without any explanation. None from Group A did. One from each group just answered "No". Six students from Group A correctly used the definitions for derivative and continuity to explain their answers ("Yes"). None of the students from Group B used such explanations. Four students (three from A and one from B) used reasoning about connectedness to justify their "Yes". Five students thought that a function can be differentiable but not connected and hence not necessarily continuous (two from Group A and three from B) revealing a very poor concept image of these concepts

Discussion

The students' explanations of derivative show a rather uniform view in terms of slope and rate of change, which was an expected result. According to the students, derivatives are used for studies of change and rate of change and for studying curves or for determining extreme values for functions. The former type of answer was given by a majority of the students in Group A and only a few chose the latter type. The students' answers in Group B, on the other hand, were evenly distributed in the two categories (11 in each category). This result implies a stronger emphasis on solving problems in Group B than in Group A. The first category comprises more generally expressed descriptions of what derivatives are used for. Answers to Question 6 also reveal a difference between the groups where seven from Group B answered correctly without an explanation and none of the students in Group A did. Six students from Group A correctly used definitions of derivative and continuity to justify their claims while none of the students in Group B did. These results imply a conceptual approach in Group A and a more procedural one in B, in terms of Heibert and Lefevre (1986). The students in Group A also more readily used formal representations than the students in Group B did. The students in Group B seemed to be more in the proceptual-symbolic world of mathematics (Tall, 2004), focusing on the processes, whereas the students in Group A in many cases worked within the formal world. The students had the same teacher, but the groups showed different characteristics in terms of generality and ability to use theory. This difference might be due to the fact that the groups took the course in different semesters with different circumstances such as other courses requiring mathematics or peer influence in the groups.

The students' explanations of continuity also reveal a trend, but in an undesirable direction as the most common answer was that all continuous functions are differentiable (16 students thought so, eight from each group). This result was more surprising since they had worked with these properties explicitly in the course. The students had, despite this, not been able to create valid links to help them see the specifics of the concepts. At this point, there were no signs of development trends of the two groups through Tall's three worlds of mathematics, i.e. none of the groups showed evidence of working within one of the worlds more than the other group did.

Justifications used by the students were often not sufficient to be called proofs, mostly due to a lack of arguments. In the cases where proofs were actually correctly conducted (by seven students in Question 4 and six students in Question 6) almost all proofs were according to the textbook. The answers to Questions 4 and 5 show an uncertainty about the concepts and their relation to each other. Five students gave opposing answers to what they had answered before. Two of them used valid arguments, pointing at core features of differentiability, for their claim that continuity does not imply differentiability. The students knew that there can only be one derivative at each point of the function for it to be differentiable. This property was not evoked when they answered to Question 4 about properties of continuous functions. The students did think of derivatives when prompted to think about continuity, but not of the crucial aspects of the concepts. This shows how variations in prompts may evoke different parts of the concept image. Questions 4 and 5 are both about continuity, but the first is open and the second demands a certain investigation. The five students in this case made opposing statements about the same thing from the questions in the timespan of a couple of minutes.

Eight of the 16 students claiming that continuity implies differentiability in Question 4 kept that standpoint in Question 5. That the function has a certain slope at all points and hence is differentiable, was one argument used by four students in Question 5, and two students reasoned about left and right limits being the same. These students investigated thought examples to be able to answer the question. The examples were, as it seems, chosen without taking specific properties of differentiability and continuity into consideration since they all were differentiable to begin with. A strong mental representational web supporting their conceptions of the concepts (Hiebert & Carpenter, 1992) could help the students select examples with relevant features to determine if continuity implies differentiability. A large number of students used the absolute value function as a counterexample to show that continuous functions do not need to be differentiable, ten from Group A and four from Group B. The example is used as a generic example of a non-differentiable continuous function that helps the students remember the property and understand why there cannot be an unvarying derivative at the peak. There was, however, an example of a student using the example as an exception to what he thought was the rule that continuity implies differentiability. Either he did not know what properties of the function made it an exception or he knew the properties but not how to use that information in a general context to see that the implication is not true.

Vague memory or misunderstood pieces of knowledge can sometimes be a reason for the students' uncertainty about concepts' properties. The student in Viholainen's (2008) study exemplifies such uncertainty, as well as some of the students in the present study. Different parts of a concept image dominate in different situations influencing the students' actions, as we can see in the ambiguity of the answers to Questions 4 and 5. Memory and other cognitive factors have a fundamental impact on students' learning abilities. An established memory of a concept may be hard to change and even harder to maintain changed. In the formerly mentioned study about limits of functions (Juter, 2006), a couple of students were certain that limits are always unattainable by the function approaching the limit. During a calculus course, this was sorted out and in an interview the students showed that they understood why some limits are attainable and some not. A year after the course was over, the students were interviewed again and their former standpoint was back, i.e. limits of functions are never attainable. The memory of the first view (often linked to their interpretation of the formal definition of limits) dominated over the second view of examples where limits are attainable and a correct interpretation of the definition. It can take a long time to establish new views of already established notions as it requires the creation of new connections to produce a valid rich web of representations (Hiebert & Carpenter, 1992)

This is a first analysis of the data collected. Awareness of the validity of one's own mathematical representations is a first step to improvement of them.

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The Tail Wagging the Dog? The Effect of National Testing on Teachers' Agency

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Australia introduced national testing in literacy and numeracy in 2008 in order to ensure that all children reached basic benchmarks. In January 2010, school results were published online, making these tests high-stakes, especially for schools likely to have poor results. In 2009, a research project was conducted on what supported and hindered teachers in a school in a low socio-economic area to take up mathematics professional development opportunities. This paper explores the impact of national testing on their perceptions of their agency, particularly the constraints it imposed on taking up these opportunities.

National Assessment Program – Literacy and Numeracy

In 2008, as part of a push to gain greater control over the schooling sector, the Australian Federal Government replaced tests done by individual Australian states with the National Assessment Program – Literacy and Numeracy (NAPLAN). NAPLAN was designed to determine whether Australian students had reached minimum standards at Years 3, 5, 7 and 9 (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2011) and thus contributed to the rhetoric about 'raising standards', particular for students from disadvantaged backgrounds (Donnelly, 2009).

In January 2010, the Federal Government opened a website, *My School*, which compared NAPLAN results for individual schools against schools with students from similar socio-economic backgrounds (Jensen, 2010). Consequently, these tests became high stakes (Lingard, 2010), with teachers in schools in low socio-economic areas concerned about the impact on their reputations as educators (Lange & Meaney, 2012). In the previous State-based tests, comparisons of school performance had not been possible. *My School* marked a major change in the influence of accountability practices on mathematics teaching in Australia, and was indicative of the increased trend towards the abstraction and quantification of education (Hardy & Boyle, 2011).

However, the use of multiple choice and short answer questions and a focus on achieving minimum standards means that NAPLAN assesses only a limited type of mathematics: recalling factual knowledge, such as the names of 2-dimensional shapes, and completing computations (for examples, see web page http://www.naplan.edu.au/tests/naplan_2010_tests__page.html). This means that there are no opportunities for “problem-solving, investigation, mathematical modelling, and the communication of mathematical ideas” (Barnes, Clarke, & Stephens, 2000, p. 624) that are valued in Australian curriculum documents.

It is accepted in educational research that assessment, especially high stakes assessment, becomes the de facto curriculum for teachers (Resnick & Resnick, 1992)–a feature known as “the assessment tail wagging the curriculum dog” (Barnes et al., 2000, p. 624). In the US, Ellis (2008) reported how the rhetoric around direct-instruction methods improving basic computational test results was connected to beliefs that computation was what children needed. The solving of rich tasks was not seen as important. Yet, there is little research on how the relationship between high stakes assessment and teaching operates in the detail of daily school reality. In this paper, we explore how NAPLAN affected perceptions of teachers’ agency in regard to the teaching of mathematics.

Agency

Agency has become a frequently used term in mathematics education research (see Andersson & Norén, 2012). Yet, there seems no consensus on what it is or how it operates. Researchers such as Boaler and Greeno (2000) and Brown and Redmond (2008) used Pickering’s (1995) ‘dance of agency’, which focuses on the tensions between an individual’s own agency and the agency of the discipline of the subject, such as mathematics. However, when tensions in mathematics teaching and learning are explored, then resistance as a form of agency can become over emphasised (see Wagner, 2007). For Klein (1999) agency is an ability to act in certain ways that conform or resist societal expectations, expressed through discourses that describe, for example, what a typical teacher or learner is:

> A form of agency may be realised when teachers recognise the constitutive power of discourse and how teaching interactions position learners in ways that can authorise and empower, or alienate and prevent them from acting in powerful ways. (p. 89)

Common ground between the definitions includes a recognition that agency involves the meaning making and actions of an individual that occurs within a socio-historical context. Nevertheless, these definitions can be reduced to a tension between agency and structure (Biesta & Tedder, 2006). Rather than seeing agency in opposition to structure, we have chosen to use Emirbayer and Mische’s (1998) definition of agency as being embedded within structure:

> the temporally constructed engagement by actors of different structural environments—the temporal relational contexts of action—which, through the interplay of habit, imagination, and judgment, both reproduces and transforms

> those structures in interactive response to the problems posed by changing historical situations (p. 970; italics original).

The three interrelated elements, habit, imagination and judgment, which emphasise the historical and social nature of agency, metaphorically form a "*chordal triad of agency* within which all three dimensions resonate as separate but not always harmonious tones" (p. 972; italics original):

- habit, the iterational element, is "*the selective reactivation by actors of past patterns of thought and action, routinely incorporated in practical activity, thereby giving stability and order to social universes and helping to sustain identities, interactions, and institutions over time*" (p. 971; italics original)
- imagination, the projective element, is "*the imaginative generation by actors of possible future trajectories of action, in which received structures of thought and action may be creatively reconfigured in relation to actors' hopes, fears, and desires for the future*" (p. 971; italics original)
- judgment, the practical-evaluative element, is "*the capacity of actors to make practical and normative judgments among alternative possible trajectories of action, in response to the emerging demands, dilemmas, and ambiguities of presently evolving situations*" (p. 971; italics original*).*

Emirbayer and Mische (1998) suggested that all three dimensions are interrelated, but at different moments "it is possible to speak of action that is more (or less) engaged with the past, more (or less) directed toward the future, and more (or less) responsive to the present." (p. 972). Although acknowledging that habitual actions were also agentic, they saw problematic situations as more likely to make actors reflective, thus leading to the possibility for alternative actions and for situations to be restructured. In our data, the high-stakes nature of NAPLAN became problematic for the teachers in regards to thinking about their own teaching. Consequently, we considered Emirbayer and Mische's definition of agency to be valuable in better understanding the data.

The participants, the school, the project and the research study

The school was located in a regional centre in New South Wales (NSW), Australia and serviced a low socioeconomic area. It taught children from 5 to 12 years old and had a high Indigenous population as well as children from defence service families resulting in a 60 percent turnover during the year. Poor results in the 2008 NAPLAN test meant that the school received funding for a range of professional development (PD) activities. At the end of 2008, we were invited by the principal to set up small projects that would suggest possibilities for longer term projects in 2010 (see Meaney & Lange, 2010). None of the projects related directly to the sort of mathematics tested in NAPLAN. One of them was a PD project on increasing writing in mathematics to support students' reflective thinking. This project allowed us to investigate enablers and constraints on teachers' take-up of PD opportunities.

As part of the research, we initially interviewed: 4 teachers, although one later withdrew; some students from each of their classes; one parent; two deputy principals and the principal. Over the four weeks of the professional development, we filmed one lesson from each teacher weekly, audio-recorded the commentary of the teacher and a researcher when looking at each filmed lesson, and audio-recorded the meeting that was held each week and attended by the teachers and researchers. At the end of the project, we interviewed the teachers and the students again. Although we never specifically asked about NAPLAN, it was mentioned by the teachers, the school executive, staff and the parent, especially in the initial interviews but also in the shared meetings. Our project occurred as the results from the second year of NAPLAN were being sent to parents and the *My School* website was being discussed in the media. Although it is not surprising that NAPLAN came into the discussions, it does show that these participants saw it as important because they voluntarily discussed it.

NAPLAN and teachers' perceptions of their agency

The comments in the data about NAPLAN showed that all three dimensions of agency were drawn upon. However, as suggested by Emirbayer & Mische (1998), their interaction was not always harmonious. Although participants could project possible alternative courses of action, only those which involved utilising NAPLAN results seemed to have any likelihood of becoming a reality. In the transcript extracts, all names are pseudonyms.

From the initial interviews, it was clear that participants considered that NAPLAN would affect teaching at the school. The two deputy principals, interviewed together in August, 2009, suggested that the schools' NAPLAN results would result in mathematics becoming the focus in the following years.

> Kylie: And I suppose next year for infants [five to seven year olds] it will become more oral focused, because that's going to be our focus next year, that and maths. We think we've done literacy and language well now. That's been our focus for a while, but after our NAPLAN, we need to focus on maths now.
>
> Harriet: I think our NAPLAN result will show what we need to do, but we need to be doing, we need to have a whole school direction.

Using their knowledge of the impact of a literacy focus on teachers developing children's understanding and skills, the deputy principals *imagined* that a focus on mathematics would have similar results because they could *predict* the teachers' actions (Emirbayer & Mische, 1998). Poor NAPLAN results were the catalyst for the shift, as they showed "what we need to do." It provided the situation that needed to be changed and *their past experiences* offered *a potential* *solution to transform the situation*. However, it is the deputy principals' agency that is in focus. For the teachers, the schools' response to poor NAPLAN results was another feature of the situation in which they negotiated their agency.

The principal's view about the sort of mathematics that the children at his school needed and the relationship with NAPLAN was more complex but suggested a similar potential resolution.

Tamsin: How do you think maths should be taught?

Paul: I think as educators and I guess particularly as higher education, we need to have people looking at where our society and where our numeracy needs are heading, not to be marching up and down in the one spot, doing the same thing that we did 20 years ago. The important part of numeracy is giving kids the ability to be able to use numeracy to benefit themselves, and then what we need to look at is how best to do that. … Is there a different sort of logic that we can use as more related to what young people in High School and then subsequent employment might want to use? They're the sort of bigger questions that – I think if we can get higher education looking at alongside the schools situation, that will give us that idea of what we should be teaching in numeracy and some more effective ways. …

Tamsin: So, what have you done at this school to help teachers teach in that way?

Paul: I mean what we are trying to do, and again in our context, it is not about straight teaching, … it is as simple as making sure some children have had something to eat, … I mean the things I dealt with this morning – a good example is food, clothing, a death in the close family, and kids need to be able to talk about those things. Now not all those things are in every family, so what we try to do is understand the children's welfare needs. We have put in place [a number of professional development projects in pedagogy, literacy and mathematics]. The other thing that we have been doing under the School Learning Support Coordinator Programme is to go back and do some significant data analysis of test items. Now we're doing both the NAPLAN and the PAT-Test and both of those have the facility to go back and identify student response and then do error tracking, all that sort of stuff.

Although he recognised the issues that children at his school dealt with, it did not stop the principal from *imagining* what their numeracy needs may be as adults. At the same time, students' responses to the NAPLAN test are seen as useful data to identify what mathematical learning problems they may have. Juxtaposing these two ideas shows the tension between having to deal with the situation as it is, whilst at the same time *envisaging a different kind of future*. Although the principal could imagine alternative actions, his role required him to deal with the requirements of the present situation. In doing so, he drew on *established habits* of instituting professional development projects in order to direct teachers' attention to what they needed to do. Without ongoing opportunities to reflect on alternatives, then *habitual* processes would be reinstated, with NAPLAN results driving the direction that teachers would be asked to address. Ultimately, this may lead to a restriction of his and ultimately the teachers' agency to a mere *manoeuvring between repertoires of habitual actions* when making the possibilities suit the specific situation (Emirbayer & Mische, 1998). Barnes et al. (2000) described how the tradition of NSW teachers was to focus on the high stakes assessments and be less focussed on curriculum outcomes.

The teachers discussed how they felt they were pressured by NAPLAN. Although the youngest year level that NAPLAN tested was Year 3 students, the Year 2 teacher, Sarah, felt that she had internalised the pressure and at times this was in contradiction to what she felt was beneficial to her students:

> Sarah: I like kids to get in there, and you know – explore but I guess, I just do it. And then, I'm still very conscious of where that fits with the syllabus and with testing and stuff like that, because you know, I'm just not a fan of NAPLAN. It is our yardstick so it's not much, you know, I can't see that changing, …
>
> I don't have – "and show me your program" [from the school executive or parents] – I don't have that pressure, as such. But I certainly feel the pressure of the gaps in my kids, and where the syllabus benchmarks are, and my obligation to the year 3 teacher and my pick up off the kinder[garten]-[Year] 1 skills they didn't get, and so, I guess I'm a little bit – but it's probably me that applies that pressure, not – you know – it doesn't seem to be coming external. … So, yeah, I guess most of the pressure is syllabus based and preparation for NAPLAN – well it's never a forethought. It's more of a back thought, you know. I get on with what I'm doing.

As part of the practical evaluation element, Sarah's agency is expressed through her *decision* to provide her students with opportunities to explore. However, there is an obvious tension if she must also ensure the students have the minimum standards assessed by NAPLAN. Her vision of herself as a teacher is caught in a conflict that can only be resolved when the two aims – allowing children to explore and achieving minimum standards – are achieved together. With limited possibilities to achieve this balance, then it becomes difficult "to act rightly and effectively within particular concrete life circumstances" (Emirbayer & Mische, 1998, p. 999), and agency becomes restricted or almost paralysed.

This tension was seen in much of the discussion between the teachers. The following extract came from the second week of the project, at the end of the hour-long meeting. The teachers discussed how the emphasis on literacy and numeracy affected their teaching, and how assessments such as NAPLAN and Best Start (New South Wales Department of Education and Training, 2011), an assessment of children's literacy and numeracy during the first three years of school, affected their perceptions of young children and their learning:

Kathy: And the other thing I've been thinking about a lot too, lately is that there's such a big push for literacy and maths now, and it's been made like a focal point, I guess, is that what else are we missing that could allow any of our children to be achievers – like are we missing the art side of it? Are we missing the music? Are we missing the performance? And are we missing the drama? And all of the things in our curriculum, which allows children to become more confident in their skills, which then, relates to maths, or which then relates to literacy. And so, they're coming at literacy and maths, and everything else just sort of getting pushed aside. Maybe, I don't know, I just sort of think it's gone a bit all arse up, would be the word I'd use, because you've got to have these other things to give them that. …

Geoff: And that's the pressure of NAPLAN. …

Sarah: And we're getting that pressure younger, too. Like, look at us now, we've got kinder[garten] Best Start things that are assessing little people before they've even had time to do anything and –

Kathy: It used to be known as "Kindergarten, learn to play" didn't it?

Sarah: you know, like where's all the playing with blocks?

The opportunity provided by the meetings allowed the teachers to *problematise* their current situation through characterising the past in the work of their joint reflection. As Emirbayer and Mische (1998) stated "the problematization of experience in response to emergent situations thus calls for increasingly reflective and interpretive work on the part of social actors" (p. 994). Although in this discussion they do not offer possible alternative actions, the teachers focus their attention on the impact on their teaching of the system's emphasis of literacy and numeracy, through assessments such as NAPLAN and Best Start. For them, activities such as music, drama, play are squeezed from their teaching and replaced by literacy and numeracy. The situational constraints on their agency to make choices about their actions are clear. Conforming to these constraints requires them to some degree to give up these visions of "good teaching." Similar comments have been made in other high stakes testing situations. In the United States, teachers stated that they had reduced the amount of social studies and science that they taught in order to focus on literacy and mathematics (Taylor, Shephard, Kinner, & Rosenthat, 2003).

The teachers were able to envisage alternatives but only those where NAPLAN and other mandated testing were removed from the situation. The following extract comes from a meeting in which Kathy had described how she withdrew small groups and then ascertained what they knew and designed a program to move them forward:

Sarah: I used to argue here when we had all the audits and stuff that, maybe we should actually have the curriculum as a guideline, and just say, for this 3 years, we're not actually going to worry about benchmarks. We're going to go back where we are and just, you know, do all your data, so you can prove what you've done, have starting points, have end points, but actually throw away any preconceived notion of where we should get to, and see what happens over 3 years. … I don't know, I still think that we would probably find that we'd actually be faster, because we'd go back and check and then we'd just move because that pressure's kind of gone. There's no NAPLAN test, there's no, just see what happens.

Kathy: It's just pulling out what they don't know along the way and just fit all that in and then keep going.

Emirbayer and Mische (1998) suggested that "in proposing new social ends as well as different means for arriving at them, actors draw upon—and sometimes extend, rearrange, and transform—the master frames extant in the broader political culture" (p. 993). These teachers could *envisage* a line of action where their concerns about 'good teaching' could be enacted. However, the hierarchical nature of schools embedded within a wider schooling system did not provide teachers with openings for transforming the frames in which they operated. As much as Sarah could *envisage* a reality without NAPLAN, NAPLAN was not going to disappear. As teachers, they had no alternative, but to comply with how the system insisted the curriculum/syllabus should be implemented. Their vision was unlikely to be *judged* as an appropriate alternative and sanctioned by the schooling system. Consequently, their agency was restricted by the structure in which high stakes tests reified the benchmarks that they had to teach towards, regardless of whether their students reached them.

Conclusion

Although the teachers had not yet been completely coerced into teaching to the test, by October 2009, their ability to enact their agency was curtailed. The teachers perceived that the problem that they faced was how to provide 'good teaching' which would support their students to learn within their current situation. Sarah and Kathy were vocal about how it was necessary to start their teaching from where the students were at mathematically, and not where the syllabus indicated that the children should be. Sarah felt that her students needed opportunities to explore, whilst Kathy wanted an opportunity to use drama and art to develop students' confidence, so they would be better able to tackle literacy and numeracy learning. Although they both saw that there were some opportunities within their current situation to implement these ways of teaching, they also identified them as being in conflict with the testing regime which emphasised literacy and numeracy outcomes. In contrast, the school's poor NAPLAN results were seen as a problem by the executive staff. Their solution was to use those results to focus the school on mathematics and to provide professional development on problematic areas as determined by NAPLAN. The responses that were envisaged at the school level were habitual in that they drew on what had worked in regard to improving literacy results.

Rather than feeling that they should change their teaching so that their students would do better on the NAPLAN tests, the teachers imagined ignoring NAPLAN. However, given its strong institutional support, this was unlikely to happen. Without being able to envision alternative courses of action, their agency was restricted and it was likely that teaching to the test would become a stronger feature in the following years.

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Swedish Preschools, Play and the Learning of Mathematics

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Recently, attention has been focussed on the mathematics learnt in preschools and how this contributes to children's subsequent learning in schools. This paper explores the dilemma of trying to increase preschool children's learning of mathematics, whilst allowing their play to guide that learning. In Sweden, the revised curriculum for preschools specifies more mathematics to be covered. Yet, like other countries, Sweden traditionally has seen preschools as places where learning arises from children's play. We suggest two avenues for further research that would contribute information to increasing the likelihood of play supporting mathematics learning.

Mathematics and preschools in Swedish society

An analysis by Greg Duncan and colleagues of six longitudinal studies suggested that early mathematics knowledge is the most powerful predictor of later learning including the learning of reading (Duncan et al., 2007). Combined with concerns about preschools inhibiting children learning deep mathematics (Clements & Sarama, 2007), this has led to preschool mathematics education becoming a focus in recent years (Barber, 2009; Perry, Young-Loveridge, Dockett, & Doig, 2008). For Sweden, there is a dilemma of wanting both to ensure that children begin school with stronger mathematical understandings and to adhere to the philosophy that preschool children should learn through play. This is a dilemma that some see as irreconcilable (Carr & May, 1996; Lee & Ginsburg, 2009). In this paper, we explore this dilemma and suggest two possibilities for further research that could contribute to it being resolved.

Although it is not compulsory for young children in Sweden, by 2008, more than 90 percent of children aged 2 to 5 years attend preschools (Broman, 2010). Unlike other Western countries which also expanded the number of preschools to meet parental demands, the systematic intervention of the government in providing not just physical spaces but also highly educated staff is considered unique to Sweden (Broman, 2010).

Originally in Sweden, play was considered the foundation for preschool children's learning experiences. In the revised version of the preschool curriculum, play still retains a central role as the medium through which children are expected to learn.

> Play is important for the child's development and learning. Conscious use of play to promote the development and learning of each individual child should always be present in preschool activities. Play and enjoyment in learning in all its various forms stimulate the imagination, insight, communication and the ability to think symbolically, as well as the ability to co-operate and solve problems. (Skolverket, 2011, p. 6)

In Swedish, to play in a situation without rules is "lek" and this is the form of play mentioned in the curriculum. Play is acknowledged as being difficult to define (Samuelsson & Carlsson, 2008). Docket and Perry's (2010) definition combines many of the features identified by Samuelsson and Carlsson (2008):

> The process of play is characterised by a non-literal 'what if' approach to thinking, where multiple end points or outcomes are possible. In other words, play generates situations where there is no one 'right' answer. … Essential characteristics of play then, include the exercise of choice, non-literal approaches, multiple possible outcomes and acknowledgement of the competence of players. These characteristics apply to the processes of play, regardless of the content. (Dockett & Perry, 2010, p. 175)

Although play retains a place of importance in the curriculum, there has been a shift in government documents suggesting that preschools should prepare children for school, through a focus on literacy, numeracy and other subjects (Broman, 2010). For example, the revised version of the preschool curriculum, (Skolverket, 2010), implemented in July, 2011 increased the attention on mathematics. In the 1998 version, one objective stated that children "utvecklar sin förmåga att upptäcka och använda matematik i meningsfulla sammanhang, utvecklar sin förståelse för grundläggande egenskaper i begreppen tal, mätning och form samt sin förmåga att orientera sig i tid och rum" (develop their ability to discover and use mathematics in meaningful contexts, develop their understanding of the basic properties of the concepts number, measurement and shape and their ability to orient themselves in space and time) (Skolverket, 1998, p. 9; our translation). In the revised curriculum, this was expanded to "develop their understanding of space, shapes, location and direction, and the basic properties of sets, quantity, order and number concepts, also for measurement, time and change, develop their ability to use mathematics to investigate, reflect over and test different solutions to problems raised by themselves and others, develop their ability to distinguish, express, examine and use mathematical concepts and their interrelationships, develop their mathematical skill in putting forward and following reasoning," (Skolverket, 2011, p. 10).

The emphases in the revised curriculum suggest that children can learn through play and so there is no dilemma. Nevertheless, as described in the next section, some researchers indicate that this combination may not be achievable.

Mathematics, play and direct teaching

Play, as defined above, involves children exploring their world through fantasy, and physical manipulations, using 'what if' thinking. They make choices and so control what occurs. In preschools, there are predominantly two kinds of play, free play, in which children uses the resources around them without adult intervention, and guided play where a teacher sets up a situation but allows children's own interests to guide the play and the learning. As well, direct teaching can occur in preschools. The teacher not only sets up the situation but prescribes what actions the children are allowed to carry out. Children may still enjoy this learning but they can make limited, if any, choices about what they do.

Concerns have been raised about whether children are able to explicitly explore mathematical ideas during free play:

> Children do indeed learn some mathematics on their own from free play. However, it does not afford the extensive and explicit examination of mathematical ideas that can be provided only with adult guidance. … Early mathematics is broad in scope and there is no guarantee that much of it will emerge in free play. In addition, free play does not usually help children to mathematise; to interpret their experiences in explicitly mathematical forms and understand the relations between the two. (Lee & Ginsburg, 2009, p. 6)

On the other hand, an adult watching or participating in child-initiated play can develop children's mathematical ideas by stimulating their curiosity and language use (Doverborg, 2006). Björklund (2008) showed that adults were important in setting the parameters for children's opportunities to engage with mathematical ideas. Nordahl (2011, p. 13; our translation) provided an example of this guided play from her research where she monitored the play of children aged between 1 and 3 years:

> Nancy, Minnie (2.5 years) and Jonna (3 years) build with wooden blocks. Minnie builds towers of as many blocks as she can, and when it collapses she laughs delightedly and then simply starts again. Jonna first builds a base and then continues on top of this.

Eva (förskollärare): Vad bygger du Jonna?	Eva (preschool teacher): What do you build Jonna?
Jonna: Jag bygger vårt hus, det har fyra våningar. Där bor jag (pekar) på trean.	Jonna: I build our house, it has four floors. I live there (points) on the third.
Eva: Oh, jag bor på ettan, mitt hus har bara en våning.	Eva: Oh, I live on the first, my house has only one floor.
Eva *vänder sig till Mimmi som balanserar upp ännu en kloss på sitt torn:* Du bygger riktigt högt.	Eva *turns to Minnie who balances yet a block on her tower:* You build really high.
Mimmis torn rasar och hon skrattar förtjust och utbrister: Inte mer!	Mimmi's tower collapses, and she laughs delightedly and exclaims: No more!

Eva:	Nä det har du rätt i nu är det inte högt längre (skrattande).	Eva:	No you're right now, it is not high anymore (laughing).
Nancy	bygger bara ett lager och med ”hålrum” emellan – nästan som en ritning.	Nancy	builds only one layer and with "cavities" between - almost like a drawing.
Eva:	Det är ett stort hus du bygger, Nancy.	Eva:	It's a big house you build, Nancy.
Nancy:	Nej inte stort. Långt.	Nancy:	No, not big. Long.
Eva:	Ja jättelångt. Lika långt som du nästan.	Eva:	Yes very long. As long as you almost.
Nancy	blir förtjust och lägger sig ned bredvid och konstaterar samtidigt att hon behöver fylla på med klossar.	Nancy	is delighted and lies down next to and acknowledges that she needs to fill up with blocks.

In this example, the adult encourages Jonna’s use of ordinal terms and provides opportunities for Mimmi and Nancy to use comparative terms to do with height and length. The role of the teacher is crucial in reinforcing the use of these mathematical terms. Nevertheless, the children control the play.

On the other hand, if a teacher does not recognise the potential mathematics within a situation, then they are unlikely to extend children’s curiosity or language use. In Sweden, concern was expressed about preschool teachers’ use of resources designed by Fröbel, the German pedagogue, instrumental in setting up preschools in the nineteenth century. Doverborg (2006) cited a study by Leeb-Lundberg (1972) which found that deep mathematical understanding was required for teachers to support children’s use of some of Fröbel’s equipment. When this was not provided in their teacher education, it was impossible for teachers to develop children’s mathematical understanding.

In 2003, 100 preschool teachers in Sweden were surveyed about their teaching of mathematics. Only 3 teachers explicitly addressed the curriculum goals in their planning (Doverborg, 2006). Many felt that learning occurred naturally as part of children’s everyday lives and so did not have to be planned for. Nordahl (2011) reported similar anecdotal experiences:

> My colleagues … often perceive mathematical development to only occur in the form of "learning to count." This has meant that they have not noticed when the children's mathematical development took place. Instead, they may even have impeded it by interrupting or trivialising the mathematical discoveries of pre-school children, such as size perception. (p. 11, our translation)

Doverborg (2006) also felt that preschool teachers needed to see mathematics as more than “sifferskrivning och ramsrakning (writing numerals and reciting counting rhymes)” (p. 7; our translation).

A consequence of these concerns, especially in English-speaking countries, has been the implementation of a number of direct teaching programs in preschools. An American project, *Big Math for Little Kids*, was founded on the view that children needed to be presented with activities in a cohesive manner, but that these activities should be joyful and contribute to developing children's curiosity about mathematics (Greenes, Ginsburg, & Balfanz, 2004). Repetition of the activities provided opportunities for the mathematical ideas to be extended. For Greenes et al. (2004), the development of mathematical language was a key to helping children reflect on their learning.

Preschool mathematics programs of this type are generally sequenced with an expectation that children move along development progressions. For example, in another American project, *Building Blocks*, a set of activities were provided, based on learning trajectories for children (Sarama & Clements, 2004). Teachers who understood the learning trajectories were better able to provide "informal, incidental mathematics at an appropriate and deep level" (p. 188). Papic, Mulligan and Mitchelmore (2011) implemented an intervention program on repeating and spatial patterning in one preschool over a six month period. Children were grouped according to how they performed on an initial diagnostic interview and then provided with tasks for their level. A combination of individual and group time was provided. Children progressed to the next level if they showed competency in their current level. Papic et al. (2011) found that, after one year at school, the children performed better on a general numeracy assessment than children from a control group.

Yet, many feel that direct teaching in early childhood settings could lead to "learned helplessness and a feeling of failure" (Farquhar, 2003, p. 21). Many preschool and early school programs, such as those described by Papic et al. (2011) and Clarke, Clarke and Cheeseman (2006), include assessing children before, or as, they enter school on their mathematical knowledge. Such assessments risk children being labelled as "behind" or "at-risk" at a much earlier age. Although designed to support teachers to target their teaching to the children's levels, this early assessment has the potential to lower teachers' expectations about children's capabilities. It may also affect children's perception of themselves as learners of mathematics. Learning through play, where the children themselves have control and can adjust it to the competencies of participants is less likely to result in these sorts of consequences.

It is also unclear whether direct teaching in preschool has a lasting impact on children's academic performances. In a study of children from 3 preschools with different pedagogical approaches, children varied in academic performance at different ages (Marcon, 2002). At the last stage of the study when children moved into their sixth year of school, children who had attended a preschool that was academically focused showed the least progress. "Grades of children from academically directed preschool classrooms declined in all but one subject area (handwriting) following the Year 6 transition" (Marcon, 2002, p. 20).

Historically preschool programs often were established for children perceived as being "at-risk" of academic failure (Samuelsson & Carlsson, 2008). In recent years, the role of preschools in overcoming academic disparities has become prominent again (Clements & Sarama, 2007, p. 462). Yet, Clements and Sarama's suggestion that poor Black Americans came to school with pre-mathematical understandings and were unable to generalise, whilst other children started school with mathematical understanding has been heavily criticised by Martin (2010).

A program, designed for children thought to begin school "at-risk" of academic failure was *Building Blocks* (Sarama & Clements, 2004). Yet achievement gains of children do not seem to be long-lasting. Clements et al. (2011) found that after the first year at school, the gains from participating in *Building Blocks* at preschool were reduced and after the second year there was no substantial gain at all. They detailed other studies which showed similar results.

The correlation between mathematics knowledge on entering school and later learning has resulted in many calls for direct teaching of mathematics in preschools (Clements & Sarama, 2007). Yet, the circumstances of children's lives contribute to the knowledge that they show at all ages. In reporting on a longitudinal project, in New Zealand, that followed about 500 children till they were 10 years old Wylie (2001) found that:

> children who started school with low literacy and mathematics scores were much more likely to improve their scores if their parents were highly educated, or if their family had a high income. Good quality early childhood education and experiences at home, or later out-of-school activities using language, symbols, and mathematics, also made improvement more likely. (p. 11)

The circumstances that meant that young children did not have "good quality childcare" may be the same circumstances that did not provide them with rich out-of-school activities. As Marcon (2002) warned there are many variables that affect children's later school achievement, not just their preschool programs.

Consequently, there is a need to be very cautious in making suggestions about how young children should engage in mathematics in preschools. It is not simple to increase children's mathematical understanding through play but there are a number of concerns about instituting a direct teaching approach. In order to better understand how teachers can develop children's mathematical understanding through play, there is a need to document what mathematics children currently engage with in preschools and how teachers support children's mathematical learning.

Research possibilities for resolving the dilemma

The dilemma of increasing children's mathematical knowledge prior to school, through using their play as the basis for learning is not simple to navigate. We suggest two possibilities for research that could provide information to help navigate through this dilemma. The first is to document what mathematical ideas arise during both free and guided play. As well, although the importance of the adult in preschool children's play is well documented, there is a need to know more about how they develop children's mathematical curiosity and language. We discuss these ideas in the following sections.

What mathematical knowledge do young children use in play?

With the perception that mathematics in preschool should be about preparing children for school, most research has focussed on the development of number knowledge (see Clarke, Clarke, & Cheeseman, 2006). Yet, the abstract nature of number terms means they are more difficult to learn than relational terms such as heavy, empty, etc (Hore & Meaney, 2008). In an example of free play based around their eating of cornflakes, Björklund (2008) explored how children in a Finnish preschool discussed amounts in different ways, including using their body:

> Elisa (2:7) and Adam (3:1) are sitting at the table eating conflakes [sic]. Elisa says 'you have little, I have a lot' pointing first at Adam's plate and then at her own plate. Adam shows his index finger and thumb, measuring a couple of centimeters, saying 'this little', then widening the space between his fingers and says 'I have this much'. Elisa says 'look, I have much, much, much' and circles her finger over the plate. Adam continues 'later I want much, I want this much' showing his five fingers on one hand. When Adam gets more cornflakes he says 'I got much, Elisa!' Elisa answers 'I will also have much, much more, this much' and shows both her hands with all ten fingers shown. (p. 88)

Children have knowledge about a range of mathematical topics (Clements & Sarama, 2007). Documenting what is discussed and used in preschools and how it matches what is suggested in the curriculum may support teachers to broaden their conceptions of mathematics. This is likely to have a flow-on effect to their work with children. Presently, there is little research that systematically documents the mathematical activities that children engage in during guided and free play in preschools in Sweden (see Doverborg & Samuelsson, 2011). One project is that of Nordahl (2011) who used Bishop's (1988) 6 types of mathematical activities to classify the activities that 1-3 year olds engaged in at her preschool. She observed the children using numbers, different measurement ideas and shapes in their everyday play. However more research of this kind, especially with 4-5 year old pre-school children is needed.

How do preschool teachers develop children's mathematical curiosity?

Research with Swedish preschool teachers suggest that preservice and in-service teachers would benefit from professional learning which illustrates how mathematical ideas can be discussed with children while they play (Nordahl, 2011; Doverborg, 2006). Currently, Delacour (2012) is investigating 4 preschool teachers' use of the revised curriculum (Skolverket, 2010) in planning and implementing lessons. Nevertheless, more research is needed into how teachers recognise and then utilise the possibility for mathematical discussions.

To be able to recognise opportunities for mathematical discussions may require preschool teachers to make use of the mathematics understanding that children bring from home. In Australia, Clarke and Robbins (2004) worked with families in low socio-economic areas to document home practices that required numeracy understandings. They found that there were many activities and these were highly valued by families. For Clarke and Robbins (2004), "the challenge for preschool and early years teachers is to connect and build upon this rich base of mathematical experiences in ways that acknowledge and support the family's role" (p. 181). Nordahl (2011) also reiterated the need for Swedish preschool teachers to make use of children's everyday knowledge of mathematics.

Although the revised Swedish curriculum for preschools (Skolverket, 2011) implies that children's play can be the basis for learning mathematics, research both in Sweden and elsewhere suggests that this may not be simple to implement. This can lead to a dilemma where mathematics learning and children's play are constituted as two different possibilities, thus resulting in a dilemma. In this paper, we suggest two avenues for research that would provide valuable information resolving the dilemma. To develop and support children's mathematical learning through their play in Swedish preschools requires the documenting of the complexity of teachers' work and children's play in which mathematical ideas arise.

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The Presence of Test Anxiety and its Relation to Mathematical Achievement in Grade 3

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In this study the presence of test anxiety and its relations to performance in different mathematical areas is investigated. Overall, in or study of Swedish Grade 3 pupils, test anxiety explained 20% of the variance for the total mathematical score, with the subscale "thoughts" as the significant predictor. The model of test anxiety also explained Number understanding, Mass and Time, Patterns, and Mathematical problems; however Mental arithmetic and Written arithmetic algorithms were not significantly explained by the model. Test anxiety seems not to be a major problem in this sample; still, significant negative correlations were found, which likely might influence the pupils in some aspects.

Introduction

In 2006 the Swedish Government decided as a first step to introduce national educational goals in mathematics and Swedish for grade 3 and as a second step to introduce national tests. The first mandatory national tests were implemented in spring 2010. National tests in these subjects will consist of several short subtests designed to assess some parts of the mathematics and Swedish syllabus goals for this age group. These tests are to be administered within a ten-week period, on dates decided by the school. With the aim of lessening the impact of the test situation, the mathematics and Swedish tests are connected by a story of two children on an adventure (Skolverket, 2009).

It is well recognized that exams can trigger intense emotions (Pekrun et al., 2004). There seems to be no difference between positive and negative emotions, both are reported by pupils to the same extent (Spangler *et al.*, 2002) and also appear to be necessary for optimal performance in achievement situations (Hopko et al., 2001). For example to motivate the child to put adequate effort into performance on a test and reach his/her full potential, a certain degree of anxiety is seen as beneficial (McDonald, 2001). Wolf and Smith (1995) found, however that in testing situations high motivation in combination with high anxiety in pupils resulted in almost the same levels of performance as low motivation in combination with low anxiety. Thus, although high motivation on a test may be necessary, the anxiety accompanying increased motivation levels could have unfavourable consequences. For some pupils the stress might be too big and seriously impact on their performance in the wrong direction. At present there is convincingly research pointing to these debilitating effects of test anxiety on academic performance (Eum & Rice, 2010).

The preamble paragraph in Swedish education is equity across curricula, *i.e.* equitable opportunity to learn in school. Thus, one important purpose with national tests in Sweden is to support the individual learning, and identify weaknesses and strengths for each pupil (Skolverket, 2010). Test anxiety could represent a bias factor that impedes pupils from reaching their true potential. In these cases, test results are misleading and not educationally justified (Zeidner & Matthews, 2005). To meet equity in testing fairly and justly practice in testing must be attained. Although the impact of test anxiety on exam performance might be modest, its potential influence on children leading to underperforming on an exam is considerable for the individual and it is necessary to attend to this in education (McDonald, 2001). Accordingly, the present study focuses on the potential negative feeling that might accompany pupils when taking a test, and the relation of this to actual performance in mathematics. To gauge the Swedish sample it is also being compared to the American reference sample for the test anxiety instrument used here. Many studies have been conducted in the field of test anxiety. However, these have mainly focused on older pupils; in younger pupils and in a Swedish context research is less complete in test anxiety, and thus, the present study is explorative in nature.

Test anxiety

Test anxiety is seen to be learnt, typically evoked in educational settings, and developed during early school years (Pekrun, 2000). The occurrence of test anxiety differs considerably; 10% to 40% in pupils (e.g. McDonald, 2001; Bodas, Ollendick, & Sovani, 2008). In children test anxiety is viewed to be a situation-specific trait being manifested during formal evaluative situations and argued to include cognitions, somatic symptoms and test-irrelevant behaviours that may weaken academic performance (Wren & Benson, 2004). The construct of test anxiety can be theorised as multi-dimensional. Among older pupils two empirically distinct but interrelated components of test anxiety are found to be worry and emotionality (Zeidner, 2007). Worry represents the cognitive aspect of anxiety and concerns the person's own evaluation of his/her performance. Compared to emotionality worry is experienced for a longer period of time, some days before the exam, and throughout the test. Emotionality is the person's subjective awareness and understanding of physiological reactions in different test situations. This affective component is present immediately before a test and then diminishes. Worry is believed to have a stronger negative influence on test performance than emotionality (Eum & Rice, 2010). Worry can be categorised as a trait or a predisposition, and emotionality as a state, temporary and depending on the context of a given test and testing environment (Zeidner, 2007). Children's test anxiety is likely first to be dominated by emotional-affective responses, and later by cognitive concern or worry (Wigfield & Eccles, 1989).

Test anxiety seems to be present in all age groups (Connor, 2003), although older compared to younger pupils report more test anxiety (Zeidner, 1998). Test anxiety is thought to peak around grade 4 (Araki, 1992) up to junior high school, where gender differences also become more marked (Lowe & Lee, 2008). Still, few studies report on test anxiety as a widespread experience in the lower ages (Zeidner, 1998). Commonly, girls compared to boys report more test anxiety. However, there might be other reasons than actual higher levels of test anxiety behind; girls are believed to be socialised to show their feelings publicly and boys to hold back or hide them (Bodas et al., 2008). Aspects related to schooling are also believed to be more feminine and hence are not valued by males (Skelton, 2001). Among school subjects, mathematics in particular seems to cause more stress in test situations (Putwain, 2008) and failure in it attributed to the learners' own competence and not to the task in question (Boekaerts, Otten, & Voeten, 2003).

Method

Participants and setting

Seventy-four Swedish pupils (34 girls, 40 boys) aged 9-10 years from six grade 3 classes in five schools in different demographic settings participated in the study. During a ten weeks period the pupils took the national tests and filled in the questionnaire. Data collection complied with the ethical guidelines regarding information, consent to participate, scientific use and confidentiality. Consent was obtained from the parents, who were informed by letter and at an open meeting. All of the children gave their assent to participate. Schools were selected on the basis of representativeness of national average on demography.

Instruments

The *Children's Test Anxiety Scale* (CTAS; Wren & Benson, 2004) contains 30 items tapping: (a) "thoughts", 13 items, (e.g. "While I am taking tests I worry about failing"); (b) "autonomic reactions", 9 items, (e.g. "While I am taking tests my belly feels funny"); and (c) "off-task behaviours", 8 items, (e.g. "While I am taking tests I check the time"). The CTAS is a refined and modernised self-reported pen-and-paper questionnaire. Participants rate their level of agreement with each statement on the CTAS based on a Likert scale from 1 (almost never) through 4 (almost always). The CTAS has satisfactory reliability coefficient (0.92) and high practicality in naturalistic field settings (Zeidner, 2007). Results from a recently conducted study provide evidence for the reliability and validity of the CTAS with Scandinavian younger pupils (Nyroos et al., 2012).

Academic performance was assessed by the *National test*[1], which included seven subtests: Patterns, Mass and Time, Number understanding, Mental arithmetic, Mathematical problems, Written arithmetical algorithms, and Communication and Concepts. The latter is excluded as it was a group assignment. The tasks varied in form (e.g. simple numbers, complex tasks) and required different methods of expression (e.g. drawing, writing). The national test was administrated by the class teachers, and the CTAS by the researchers.

Data analyses

Data analyses were made in SPSS Version 19.0 (IBM). The raw score counts for each subtest (descriptive statistics in Table 2) are viewed as reasonable approximations of continuous, interval scale measures (Wright & Linacre, 1989), and are being used as an indication of a possible measure of the latent trait. The pattern of results was similar for boys and girls (Table 1), thus, the total sample is presented in the text. Possible scale ratings on the scale levels between boys and girls and mean ratings between the Swedish sample and the reference sample (Wren & Benson, 2004) were analysed through *t*-test (Table 1). Degree of relationship was measured by a Pearson correlation analysis of the results for the mathematical areas and the CTAS measurements (Table 3). Finally, a series of regression analyses, using each mathematical area as the dependent variable and the three CTAS measurements as independent variables are presented to examine how the CTAS predicts the outcome of the different mathematical ability assessments (Table 4). In addition, the overall Math total scores were examined within the regression analysis. Data had normal distribution (histogram and probability plot), missing data are excluded, model assumptions are fulfilled and no influential cases were detected when checked with residuals properties and statistics. One class differed notable in mean from the other five; however, this did not affect the mean, mode and median greatly. Where significant differences were flagged, Cohen's *d* effect sizes were calculated in order to estimate the magnitude of the difference between mean values. The subtests and CTAS items had relatively high internal consistency, Cronbach's alpha .76 and .91, respectively.

Result

The Swedish sample on average reported significantly lower levels of CTAS measurements than the reference sample (Wren & Benson, 2004) did (Table 1). Cohen's effect size values suggested a moderate practical significance for "thoughts" (d = -0.77) and "off-task behaviours'" (d = -0.63), and a high practical significance for "total" (d = -1.05) and "autonomic reactions" (d = -1.25). Since the reference sample starts school one year earlier than the present sample, *t*-tests were also conducted for grade 4 of the reference sample. For the significant differences Cohen's effect size values suggested a moderate practical significance for "thoughts" (d = -0.59) and "off-task behaviours" (d = -0.71), and a high practical significance for "total" (*d* = -0.86) and "autonomic reactions" (*d* = -0.94.)

	Mean (Sd)					Swedish sample: Gender		
CTAS	*Swe grade 3* (n = 74)	*Reference sample grade 3* (n = 46)	*t*-test	*Reference sample grade 4* (n = 55)	*t*-test	*Girls* (n = 34)	*Boys* (n = 40)	*t*-test
thoughts	23.16 (8.11)	29.54 (8.51)	-6.705**	28.07 (8.65)	-5.077**	22.13 (7.65)	24.10 (8.50)	1.082 *ns*
autonomic reactions	11.95 (3.77)	17.15 (4.53)	-10.138**	16.25 (5.25)	-7.234**	11.81 (4.20)	12.07 (3.41)	.304 *ns*
off task behaviours	13.27 (4.51)	16.07 (4.36)	-5.672**	16.65 (5.06)	-5.899**	12.97 (4.57)	13.54 (4.50)	.550 *ns*
total	48.00 (13.15)	62,76 (14.81)	-8.573**	60.98 (16.61)	-6.722**	46.79 (14.00)	49.03 (12.46)	.725 *ns*

*p < .05, **p < .01

Table 1. Means and standard deviations (in parentheses) of scores in CTAS measurements and *t*-scores.

Patterns	*Mass and Time*	*Number Under-standing*	*Mental Arithmetic*	*Mathematical Problems*	*Written arithmetical Algorithms*	*Mathmatical Total*
8.12 (1.06)	8.78 (1.83)	11.65 (1.81)	24.99 (4.12)	8.63 (1.93)	16.30 (3.35)	78.60 (10.32)
max 9	*max 11*	*max 13*	*max 28*	*max 10*	*max 20*	*max 91*

Table 2. Means and standard deviations (in parentheses) of scores in Indicated Subtests in Mathematics. Maximum score in italics.

Correlation analysis was conducted for the different mathematical areas, total math scores, the different CTAS measurements and total (Table 3).

Variables	1	2	3	4	5	6	7	8	9	10	11
1. CTAS-thoughts		.611**	.320**	.897**	**-.307****	**-.443****	**-.406****	**-.379****	**-.380****	**-.322****	**-.485****
2. CTAS-autonomic reactions			.348**	.786**	**-.390****	**-.335****	-.155	**-.280***	-.222	-.130	**-.317****
3. CTAS-off task behaviours				.648**	-.020	-.018	-.148	-.067	-.062	-.001	-.053
4. *CTAS-total*					-.282	**-.350****	**-.312****	**-.281***	**-.297***	-.196	**-.382****
5. Patterns						.376*	.575**	.375*	.349*	.316*	.723**
6. Mass and Time							.521**	.553**	.512**	.283	.758**
7. Number understanding								.522**	.551**	.385*	.810**
8. Mental Arithmetic									.480**	.249	.721**
9. Mathematical Problems										.140	.752**
10. Written Arithmetical Algorithms											.502**
11. *Mathematics total*											

*p < .05, **p < .01. Note: Significant correlations are highlighted in bold. Two-tailed.

Table 3. Correlations for the CTAS measurements and subtests.

As can be seen in Table 3, "thoughts" significantly correlated with all mathematical areas and Math total. "Autonomic reactions" significantly correlated with Patterns, Mass and Time, Mental arithmetic, and Math total. "CTAS total" significantly correlated with Mass and Time, Number understanding, Mental arithmetic, Mathematical problems and Math total.

Dependent variables	Independent variables	R^2	ß	F
Patterns	*CTAS total*	.16**		4.30**
	thoughts		-.10	
	autonomic reactions		-.36*	
	off task behaviours		.14	
Mass and Time	*CTAS total*	.20**		5.99**
	thoughts		-.32*	
	autonomic reactions		-.22	
	off task behaviours		.17	
Number Understanding	*CTAS total*	.13*		3.33*
	thoughts		-.38**	
	autonomic reactions		.09	
	off task behaviours		-.07	
Mental Arithmetic	*CTAS total*	.09		2.28 *ns*
	thoughts		-.24	
	autonomic reactions		-.08	
	off task behaviours		.00	
Mathematical Problems	*CTAS total*	.11*		2.94*
	thoughts		-.35*	
	autonomic reactions		.04	
	off task behaviours		-.03	
Written Arithmetical Algorithms	*CTAS total*	.08		1.94 *ns*
	thoughts		-.33	
	autonomic reactions		.07	
	off task behaviours		.06	
Mathematics total	*CTAS total*	.20**		5.68**
	thoughts		-.41**	
	autonomic reactions		-.09	
	off task behaviours		.07	

*$p < .05$, **$p < .01$

Table 4. Regression analyses with subtests as dependent variables and CTAS measurements as independent variables.

CTAS contributed different proportions of the explained variance for the subtests (Table 4). For the collapsed Math Total Score, 20% of the variance was significantly explained by the overall CTAS with significant predictor “thoughts”. For Pattern 16 % of the variance was significantly explained by CTAS (significant predictor “autonomic reactions”). The significant contribution of CTAS to the variance observed for Mass and Time was 20%, for Number understanding 13%, and for Mathematical problems 11% significant predictor in all cases being “thoughts”.

Discussion and conclusion

Test anxiety is not a harmless experience but related to several severe conditions and constraint career advancement (Stöber & Pekrun, 2004). It is however, possible to learn how to cope with it early in life, and therefore it is important to investigate the existence of test anxiety in younger pupils. In general, research on test anxiety in children is a neglected area, and in Sweden few studies on test anxiety have been conducted. Even if test anxiety is a well recognised syndrome and has been found to be equally present in diverse geographical settings, there are reports on culture-specific aspects influencing (Bodas *et al*., 2008). Thus, the purpose of this study was to explore the existence of test anxiety in a sample of Swedish grade 3 pupils, and to investigate the relations between test anxiety and mathematical performance.

The present sample reports on test anxiety but compared to the reference sample (Wren & Benson, 2004) to a significantly less extent; the effect sizes indicate medium to big differences. Despite that the test conditions in US differ from the Swedish conditions it offers a benchmark to gauge against. For example pupils in US have had high-stake standardised testing since 2001 (Lowe et al., 2008). In general high-stake tests are perceived as being stressful, resulting in anxiety (O'Neil & Abedi, 1992). Test anxiety is also believed to be a learned condition (Pekrun, 2000) and therefore the present sample might not have incurporated such experience. Further, one important aspect stressed by the National Board of Education was to play down the test situation, thus deemphasising possible negative consequences of large scale testing. The pupils being studied thus might not have experienced any high levels of test anxiety. Notwithstanding, the more test-anxious pupils in the sample performed more poorly in mathematics; and in size and direction this is consistent with other studies of test anxiety and academic performance (Putwain, Connors, & Symes, 2010). No difference was observed between boys and girls in reported test anxiety, which also is in line with previous research on this age group (Lowe & Lee, 2008).

Test anxiety is a learnt condition with many aspects influencing. Order and delivery of teaching, individual dispositions and context are some. Mathematics is also not a one-dimensional subject but requires many competencies. Several significant correlations were observed between mathematical performance and CTAS measurements. The regression analysis also revealed that the CTAS contributed different to performance in subtests. Overall, the CTAS explained 20% of the variance for the total mathematical score, significant predictor being "thoughts". Thus, on a general level it seems like pupil were a bit worried for the national test. In Patterns 16% of the variance was significantly explained by the model of CTAS with significant predictor "autonomic reactions". One reason for this outcome might be the assignments in hand; the pupils were asked in several tasks, to draw own detailed pictures or patterns, which was time consuming and perhaps stressful at that time. "Autonomic reactions" is a state of test anxiety being experienced when having the test, and not a feeling brought with the pupil to the test like "thoughts" (Zeidner, 2007). The total model of CTAS significantly explained 20% of the variance for Mass and Time; 13% for Number understanding; and 11% for Mathematical problems; significant predictor in all subtests being “thoughts”. Mass and Time is a mathematical area commonly connected with difficulties. Children and adults find those tasks hard to solve (NCTM, 2000). Pupils categorised as having major difficulties with mathematics in general have problem with number sense. These pupils lack a basic innate understanding of numbers (Jordan *et al.*, 2006). Problem solving involves a complexity of conceptual knowledge to understand the situations described in those problems. Thus, this is a demanding task to manage, requiring a good working memory capacity (Andersson, 2007). High demands on the working memory as in those competencies mentioned, together with anxious thoughts steal additional capacity of the working memory resulting in poor solving (Hadwin, Brogan, & Stevenson, 2005). The subscale “thoughts” as the significant predictor to the variance for Mass and Time, Number understanding and Problem solving can here be interpreted as a predisposition loading on the working memory (Zeidner, 2007). Regarding Mental arithmetic and Written arithmetic algorithms, which had no significant relations with CTAS this could be due to being typical routine mathematics text book tasks, which the pupils are used to work with. They felt familiar with the design and contents of the tasks (Araki, 1992).

The present study is done with a rather small sample, limiting the conclusions that can be made based on these results. Thus any far reaching conclusions about the national situation drawn from this small sample of 74 pupils may not be convincing. Even if the variances could be significantly explained by CTAS, the variances in the subtests are also small and no other variables are included in the model like parents background, teacher’s role *etcetera*; however, test anxiety does have some influence since the correlations are negative and significant. The authors therefore suggest two educational implications; primary teachers should start to actively intervene to counteract test anxiety and its negative effects. Different mathematical competencies also need to be dealt with separately in regards to test anxiety. Nevertheless, the impact test anxiety will have on Swedish pupils is for the future to tell. Swedish primary pupils till now have not had national tests, but with increased experience in test and evaluative situations, the question is if they might become accustomed to testing, developing a resistance to anxiety, or acquire the learning of test anxiety.

Note

1. For information, see http://www.prim.su.se/matematik/amnesprov_3.html

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Assessment as a Tool in the Professional Identity Development of Novice Mathematics Teachers

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The empirical material and the results presented in this paper are taken from a study investigating novice primary school mathematics teachers' professional identity development. This paper focuses on how novice primary school mathematics teachers use assessments as feedback in their professional identity development. The respondents equate pupils' results in assessments with understanding and learning, and they use the assessments primarily as confirmation in identity development as mathematics teachers and not as material for planning lessons. In this paper, confirmation through assessment is illustrated by the case of Helena, one of the respondents in the study.

Introduction

The empirical material and the results presented in this paper are taken from a study investigating novice primary school mathematics teachers' professional identity development as experienced from the perspective of the teachers themselves. According to McNally, Blake, Corbin and Gray (2008) the transfer from teacher education to teaching is to be seen as a shift in identity, where becoming accepted as a teacher by colleagues but also by oneself is central. In the study of novice primary school mathematics teachers' professional identity development, confirmation has turned out to be one of several elements in this professional identity development. The aim of this paper is not to analyse novice teachers' professional identity development as a whole but to present how the novice teachers in the study use confirmation through assessment to develop a sense of themselves as *a kind of primary school mathematics teacher.*

In their development of a professional teacher identity the novice teachers in the study try to obtain confirmation of themselves being a teacher who is doing a good job and whose students (therefore) are learning. This confirmation is obtained in many ways, for example, confirmation through praise from pupils and parents and confirmation through approval from colleagues. Another way of obtaining it is *confirmation through assessment* which is the focus of this paper. This confirmation regards how the novice teachers use the results from assessments in mathematics as confirmation of their own teaching. The assessments they focus on are written assessments, for example, assessments offered in textbooks and by the Swedish National Agency for Education. The novice teachers' need for feedback on their own mathematics teaching results in them using these mathematical assessments primarily as an instrument to verify themselves as [good] mathematics teachers rather than as a teaching instrument.

First in this paper, the study, its theoretical framing and its setting regarding assessment will be presented. After that, feedback in professional identity development as a mathematics teacher by confirmation through assessment will be illustrated using the case of Helena, one of the novice teachers in the study. The paper ends with a final discussion.

Theoretical framing

There are many studies regarding novice mathematics teachers and how they teach, or more often how they do not teach, as intended based on their teacher training. For example, several of the studies that Cooney (2001), Phillip (2007) and Sowder (2007) refer to in their research reviews show that teacher education has little effect on student teachers, and that what students learn in teacher education tends to transform when they start working as teachers. These studies often provide an external perspective where the researcher observes and evaluates teaching. Based on these studies, the perspective of the novice teachers themselves became important in the present study.

According to Gee (2000-2001), identity is to be recognised (by oneself or others) as a *kind of person* in a given context, which would imply that professional identity as a teacher of mathematics is being recognised (by oneself and/or others) as a teacher of mathematics in a given context. As such identity has both individual and social elements. Similarly Morgan (2010) writes that establishing a (positive) professional identity as a mathematics teacher involves positioning oneself "within discourses of education in general and mathematics teaching in particular (p. 109)" in ways that allow one to be seen by others and oneself as a (good) teacher of mathematics.

Peressini, Borko, Romagnano, Knuth and Willis (2004) argue for using a situative perspective in studies of mathematics teachers' teaching. The term situative refers to a set of theoretical perspectives and lines of research which conceptualise learning as changes in participation in socially organised activities and individuals' use of knowledge as an aspect of their participation in social practices. In the study, two situative theoretical perspectives, communities of practice (Wenger, 1998) and patterns of participation (Skott, 2010; Skott, Moeskær Larsen, & Østergaard, 2011), are coordinated in a conceptual framework aiming to capture both the individual and the social part of identity development (Palmer, 2010), involved in the over-described recognition as a kind of person. The individual's patterns of participation in different communities of practice affect how they are recognised, by oneself and others, as a *kind of primary school mathematics teacher*. To be seen by others and oneself as a primary school mathematics teacher and, through that, develop a sense of self as a kind of primary school mathematics teacher, the individual needs to develop a pattern of participation in communities of practice that enables such recognition.

Design and analysis of the empirical material

This study of primary school mathematics teachers' professional identity development is a case study with an ethnographic approach, where seven novice teachers have been followed from their graduation and two years forward. The novice teachers in the study were selected because they chose to write their final teacher education Bachelor theses on mathematics education and therefore also hopefully would be interested in teaching mathematics after graduation.

The ethnographic approach has been used to make visible the process of professional identity development and to sustain the perspective of the participants. Ethnography is a way of looking at, listening to and thinking about social phenomena where the main interest is to understand the meaning activities have for individuals and how individuals understand themselves and others (Arvatson & Ehn 2009; Aspers, 2007; Hammersley & Atkinson, 2007). According to Aspers (2007), gaining such an understanding requires interaction which implies that the researcher participates with, observes and interviews respondents in the field of study. Similarly, Charmaz (2006) emphasises the importance of researchers producing rich data that goes beneath the surface to reveal the respondents' views, feelings, intentions and actions.

The empirical material in the study has been collected through self-recordings made by the respondents, observations and interviews. To accomplish a balance between an inside and outside perspective in line with the ethnographic approach (Aspers, 2007), the observations have been both participating and non-participating. For the same purpose the interviews have been both spontaneous conversations during observations and formal interviews (individual and in groups) based on thematic interview guides. These varying empirical materials have different characteristics but are treated as complete-empiricism (Aspers (2007) implying that all the material constitutes a whole that the analysis is based on in order to shed light on the professional identity development of the novice mathematics teachers. When doing their self-recordings the respondents knew the purpose of the study based on interviews and observations but they were told to record whatever and whenever they wanted and that it was up to them to decide what was important for the researcher to know about starting to work as primary teacher. Neither in formal interviews nor in observations were questions asked about the respondents' use of assessment.

The results presented in this paper have been developed gradually based on an interplay between fieldwork and analysis of observations, interviews and self-recordings. The starting point of the analysis is the meaning the respondents themselves infer on the situations studied (Aspers, 2007; Hammersley & Atkinson, 2007). The analysis in the study has been done using grounded theory methods which implies building and connecting categories grounded in the empirical material by using codes (Charmaz, 2006). Coding the empirical material does not imply using pre-constructed codes, but labelling the empirical material, line-by-line, with as many codes as possible (Kelle, 2007).

When coding the empirical material a pattern was discovered regarding expressions (words and/or actions) where the respondents made connections between their mathematics teaching (past and present) and/or utterances from colleagues, students and parents, where the later were expressed as a positive confirmation of the former. These segments were coded as *confirmation* inferring on the respondent's establishment of the correctness [1] of their mathematics teaching. The segments within the code were then sorted based on the source of the confirmation where *confirmation through assessment* regarded segments where the source of the confirmation was assessment. Not all assessments were used as confirmation as an example will show from the empirical material.

When coding line-by-line, the researcher also writes memos about the codes and, by writing memos and developing and refining the codes, categories are developed. *Confirmation through assessment* was later incorporated into the category *feedback used by the individuals to recognise themselves as a kind of primary school mathematics teacher.* Other kinds of confirmation were also included in that category, however, in this paper, only the subset confirmation through assessment is focused on. The expression *kind of primary school mathematics teacher* refers to Gee's (2000-2001) identity as being recognised as a *kind of person* in a given context and that is also the significance when using the expression further on in the paper.

Assessment in mathematics

This paper is not about assessment *per se* but about novice teachers' patterns of participation regarding assessment in mathematics, about the meaning assessment has for them. In this section however, a brief illustration of research regarding teacher's use of assessment will be presented. Also a brief illustration of what is written about assessment in various steering documents, which the novice teachers in the present study refer to when talking about assessment, will be presented.

According to Lyon (2011) research on assessment at the classroom level usually takes an assessment-centered, teacher-centered, or student-centered approach. The most researched of these is the assessment-centered approach (Wiliam, 2007; Lyon, 2011). In this paper the teacher-centered approach is the one focused which according to Lyon (2011) and Shavelson et al. (2008) primarily is concerned with how teaching and students' learning can be improved by the use of formative assessment. Another direction in the teacher-centered research is teachers' beliefs about assessment. There are also studies regarding teachers' use of self-assessment or self-reflection as a technique for self-improvement (Elbaz, 1988, 1991; Ross & Bruce, 2007). In those studies self-assessment or self-reflections are used to change teacher practice. However, using the search words teacher, assessment, identity, development and evaluation, in various combinations, produced no results regarding studies about relationships between teachers' use of assessment in everyday teaching and their professional identity development.

In various steering documents in Sweden formative assessment nowadays are in the foreground. According to the primary school curriculum (Skolverket, 2011), teaching shall take its starting point in pupils' prior experience and pre-knowledge. Teaching is to be constantly examined and evaluated. Also, teachers shall evaluate and inform pupils, parents and principals about the knowledge development of the individual pupils.

The Swedish National Agency for Education provides both obligatory and optional tests. The purpose of these is, according to the Swedish National Agency for Education, to provide an equal judgement of pupils and to increase target achievement. The results of the tests can also be used by the teachers when planning lessons [2]. 'Diamond' is one example of an optional test offered by the Swedish National Agency for Education. Diamond consists of 55 diagnoses intended for use in primary schools. Diamond's aim is to map the pupils' knowledge development and to provide material for the planning of teaching with good "supposition of pupils reaching arrayed knowledge goals" (Löwing & Fredriksson, 2009).

In the different steering documents different terms such as 'evaluation' and 'test' are used, but in this paper 'assessment' will be used as an umbrella term.

The case of Helena

Analysis of the empirical material provided by all seven respondents showed that confirmation is a central part of their becoming primary school mathematics teachers. This confirmation is expressed in different ways where the confirmation through assessment presented here is one which relates to how novice teachers use assessment of pupils' knowledge as feedback in their own professional identity development. They use c*onfirmation through assessment* to recognise themselves as a kind of [good] primary school mathematics teacher. This use will be illustrated by the case of Helena.

Helena was chosen for the paper since directly after graduation she got a job as primary teacher [3] including teaching in mathematics and then continued this work during the whole time for the study. However, the patterns of participation regarding assessment that Helena illustrates refer to all respondents in the study.

After graduation, Helena starts working as a class teacher at an upper primary school. She teaches several subjects, amongst them mathematics. She works at the same school for two years after graduation but since her first and second classes at the school are sixth grade, she changes pupils three times. Her last class is a fourth grade class. However, independently of pupils or grade she uses and talks about assessment in the same way throughout the two years.

The section below is not to be seen as the wholeness of the code *confirmation through assessment* but as examples of empirical instances labelled within that code.

Mathematics lessons in Helena's class often start with a Diamond diagnosis, the optional test offered by the Swedish National Agency for Education presented earlier. In addition to the Diamond tests, Helena's mathematics lessons mostly consist of an introductory explanation by herself followed by the pupils working in their math books. The math book is structured with tests at the beginning, during and at the end of chapters. The math book and the Diamond test together frame the structure of Helena's mathematics teaching:

> A basic course is around ten pages and then the red or blue course is about six pages. After every second chapter there is a test. Preferably two to two and a half weeks for every chapter. It is about that. (interview)

> The plan is for them to reach the diagnosis this week so we can get further and start with fractions before Christmas. My plan is to do another Diamond test with them this week to check them a little. (self-recording)

Helena often talks, both with me and her pupils, about the curriculum and the importance of pupils reaching their goals. However, there are few signs of her using the results from different assessments, for example as the ones in the quotations above, when planning and performing her mathematics teaching. Instead the assessments are used by Helena to confirm her own teaching.

> Yesterday […] we had a little test in mathematics. I am testing them with tests from Diamond. And we have been working with sequences of numbers and simple shapes. And that result was really good which felt very good. […] I believe it feels quite nice. (self-recording)

Helena puts together the pupils' results in score tables which she often shows me but when using and talking about these assessments, in interviews, observations and self-recordings, she does not emphasise how she is going to perform future teaching based on the results. Instead, she talks about how the pupils' results make her feel and the students learning confirm her sense of self as *a kind of primary school mathematics teacher.* Assessment is primarily used by her to confirm her mathematics teaching, not to plan future teaching.

When performing assessments, Helena often equates the pupils' results with understanding, for example as below:

> Today the pupils started by doing "test yourself" without them first having worked with multiplication by ten, hundred or thousand in the math book. We wanted to test how much of it they understand now based on our explanations.

> It turned out to be much harder than we had thought. [...] But it was rather interesting that so many actually hadn't understood what we had been doing during quite a lot of lessons. (self-recording)
>
> After the lesson I actually felt that most of them understood. They worked and when I looked through their papers it felt good because the most of them have understood what it is about. (self-recording)

In the second quotation above, what is interesting is also what she does not say. She does not talk about the layout of the mathematics lesson (even though her saying that she has looked through "their papers" indicates that the pupils have written something connected to mathematics during the lesson) or about the mathematics content that the pupils had understood. What she singles out is that their understanding "felt good".

As mentioned not all actions and/or utterances in the empirical material regarding assessment were labelled within the code confirmation by assessment. In situations like in the above quote when pupils are not learning, Helena often emphasises different possible reasons for that.

> […] I cannot understand how they have been able to let him through fourth and fifth grade and half of grade six without reacting. […] In the action program it is actually written that him reaching the goals in every subject is right now not a vision. (interview)

One time, when three pupils failed a test in the math book, Helena explains to me that two of them have mathematics action programs and that the third was about to get one and that "he should have already had one in grade four." In a way, such explanations absolve Helena from the responsibility for the pupils' non-understanding and non-learning and enable her to focus on the confirmation of the pupils which indicates that they are learning mathematics. As such Helena uses assessment as feedback to get confirmation of pupils' learning mathematics to develop a sense of self as *a kind of primary school mathematics teacher.* Assessment indicating students not learning are on the contrary not connected to her mathematics teaching but to other circumstances, for instance the mathematics teaching in lower grades.

Final discussion

The way Helena uses and talks about assessment in mathematics can be understood as her pattern of participation regarding assessment in mathematics. The respondents use assessment to get feedback on students' learning. Through their use of assessment, their patterns of participation regarding assessment in mathematics, they get confirmation of students learning mathematics and recognise themselves as [good] primary school [mathematics] teachers. According to Gee (2000-2001) and Morgan (2010), professional teacher identity is to be recognised by others and oneself as a teacher of mathematics in a given context. The novice teachers in this study use *confirmation through assessment* to acquire this recognition. Through their use of assessment, their patterns of participation regarding assessment in mathematics, they recognise themselves as *a [good] primary school mathematics teacher.* This may also be the case for experienced teachers but that has not been a part of this present study.

Based on the different Swedish steering documents presented in this paper, assessment is intended to be used to evaluate and inform pupils, parents and principals about the knowledge development of the individual pupils, to provide an equal judgement of pupils, to increase target achievement, to map the knowledge development of pupils and to provide material when planning lessons. When talking about mathematics teaching in general, Helena sticks strictly to the curriculum but in her use of assessment; those aims are no longer the focus. She maps the pupils' results but during my observations she does not use these results, or talk about using them, as material for planning lessons. Instead she talks about the results as confirmation of her mathematics teaching. If being connected to the conceptual framework of this study, the content in steering documents can be understood as a shared repertoire in a document-based community of practice regarding assessments. An individual's pattern of participation is a merge based on different experiences in different communities of practices (Wenger, 1989). Through the use of ethnography in this study, the respondents' patterns of participation regarding assessments have been explored as a merge of the shared repertoire in that community of practice and the need for confirmation of student learning in the respondents' identity development as mathematics teacher. To be able to recognise themselves as *a [good] primary school mathematics teacher* they need confirmation of pupils learning.

Wiliam (2007) mentions evaluating the quality of educational programs as one purpose with assessment. In this present study the novice teachers use assessments to evaluate their own teaching. To get confirmation of student learning the novice teachers have developed a pattern of participation regarding assessment in mathematics that could bring about the wanted confirmation. Assessments (the tests in the math book and the Diamond diagnoses) designed for formative assessments (supporting learning) are used in a summative way (certifying the achievements or potential of individuals) as confirmation of pupils understanding and learning. According to the situative perspective used in this study there is not a one way relationship between teaching and learning (Peressini et al., 2004) but even so, the respondents, by their teaching and choice of tests, have a big influence on their own confirmation.

In a final group interview two years after their graduation, the respondents were united in wanting to show that they know how to teach mathematics. For example they said "I will show that I know this, and that I know how to teach" and "I feel one has a lot to prove". One way for the respondents to demonstrate that they know how to teach mathematics, both to themselves and others, is *confirmation through assessment*. In order to know that she is doing a good job, Helena keeps records (score tables) of the pupils' results. These make it possible for Helena to demonstrate that she has the qualities of a good mathematics teacher. As such, *confirmation through assessment* is a part of these respondents' becoming and being primary school mathematics teachers, a part of their professional identity development. However, *confirmation through assessment* is just a partial result in the present study and does not mould the professional mathematics teacher identity as a whole but it is part of a big picture.

Notes

1. http://www.ne.se/sve/bekräfta?type=DICT
2. http://www.skolverket.se/prov-och-bedomning
3. In Sweden, it is difficult to get a job as primary teacher as there are more qualified teachers than jobs. During the study several of the respondents had periods where they did not teach mathematics.

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The Threshold Concept of Function – A Case Study of a Student's Development of Her Understanding

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This paper describes a longitudinal case study of a prospective mathematics teacher and her development of understanding of the threshold concept of function during a semester of studying mathematics. Four interviews within a nine month period are analysed. The results show how the student at the beginning of the semester made linkages to everyday life. After being presented with an abstract definition her understanding changed. At the end of the semester she looked at functions as a platform to stand on and some months later her understanding allowed her to discuss functions as objects. The present study points out the time and work that is needed to transform the understanding of a threshold concept.

Threshold concepts

As an outcome from studies of learning in higher education Mayer and Land (2005) proposed the notion of threshold concepts. In a subject there are several concepts that have a potential to transform the understanding of the subject but also often are problematic for the students to learn. A threshold concept can be seen as a portal to a new and previously unreachable view of the subject area. It is a threshold to cross over, but when the threshold has been crossed the understanding has been transformed. There are several studies that have argued on the use of threshold concepts across a range of subject areas (see e.g. Meyer, Land, & Baillie, 2010). There have also been some discussions about how to conceptualise the transformation when crossing the threshold (e.g. Scheja & Pettersson, 2010). However, there are still many research questions regarding the nature of the transformation as students come to understand a threshold concept.

Meyer and Land (2005) characterised threshold concepts as initially troublesome, transformative, integrative and irreversible. However, they also emphasised that, because of individual differences in e.g. prior knowledge, these critical features of a threshold concept will be experienced in varying degrees by students. Threshold concepts also tend to serve as subject boundary markers, and may position students within a liminal space where their understanding is rendered unstable in the oscillation between old and new understanding. It is in the liminal space 'stuck places' may be experienced by students (Meyer & Land, 2006). Understanding a threshold concept will bring about a significant shift in students' perception of a subject or a part thereof. The transformation may be sudden, but it often occurs over a long period. Integrating prior understandings is part of the transformation and understanding the threshold concept will expose previously hidden relations between concepts in the subject area. The change in perspective is unlikely to be forgotten or will be unlearned only by considerable effort.

In the area of calculus e.g. the concepts of function, limit, derivative and integral can be seen as threshold concepts (Pettersson, 2008). Research on students' understandings of these concepts has emphasised learning problems that students experience (see e.g. Artigue, Batanero, & Kent, 2007). The focus for the present study is students' transformation of their understanding of the concept of function. The study is part of a larger study aiming to explore how university students' understandings of the threshold concept of function develop during a semester of mathematics studies. The case study presented here provides a deep analysis of how one student talks about her understanding. The research question is: How does the student's understanding of the threshold concept of function develop during one semester of mathematics courses?

Research on understanding the concept of function

In the early nineties Harel and Dubinsky (1992) edited a book on research about students' understandings of functions. Previous studies have shown e.g. the differences between the concept definition and students' concept images (Tall & Vinner, 1981) and that a common conception among students is that functions must be represented by an equation and this equation must include a variable (Ferrini-Mundy & Graham, 1994). Sfard (1991) discussed the duality of mathematical concepts. Several concepts, e.g. function, are introduced as processes. To make it possible to move on and use functions it is also important to understand functions as objects. More recent studies have also shown that students frequently view functions only as processes and that the reification to an object is difficult for the learner (Hansson, 2006; Viirman, Attorps, & Tossavainen, 2010). A transformed understanding of the threshold concept of function will include the ability to view functions both as processes and as objects. The reification will be a part of the transformation.

Method

The present case study is part of a longitudinal study involving 18 first-year prospective mathematics teachers taking introductory courses in calculus at a large Swedish university. In the first semester these students complete general teacher education courses, an introductory course in mathematics and a course in mathematics education. The semester in which data were collected was the students' second and included four mathematics courses: 'Vectors and functions', 'Calculus', 'History of mathematics', and 'Geometry and combinatorics'. Observations were made in the first part of the semester in the course 'Vectors and functions' and at the end of the semester in the 'Calculus' course. Individual interviews with students yielded data in four interviews over a nine month period. Data were also produced from three questionnaires administered over a period of four months. The first two questionnaires asked the students to explain what a function is and to rate their own understanding. The third one also asked if given graphs and formulas represent functions. All of the students that took part in the courses were informed about the research and voluntarily took part in the research. The present author, as the researcher, was not involved in the teaching or in the examination of the courses. The students were also informed that their answers in questionnaires and interviews would not be presented to the lecturers and examiners in a manner that would reveal individual identities.

To make it possible to present a deep analysis of the transformation of a student's understanding, the case study presented in this paper comprises the analysis of the interviews with one student teacher, Kim (fictitious name). Kim was chosen since she presented a rich meta-language in the interviews. She had thought about her conceptions before the interview and made her thoughts about her understanding visible through her language. In that way the interviews with Kim gave rich datasets. The first three interviews were conducted at the beginning, in the middle and at the end of the semester. The fourth interview was conducted at the end of the next semester. During that semester the student had taken courses in mathematics education, probability and statistics, and general teacher education courses. The interviews were semi-structured and took the student's responses in the questionnaires as a starting point. The duration of the interviews, conducted in a room in the math library, lasted about 30 minutes each. They were audio recorded and transcribed in full.

A qualitative analysis of interview transcripts was done with a specific focus on how the student's ways of talking about her experiences of understanding changed over time. The analysis applied a context-focused conceptualisation of the development of the student's understanding (Halldén, Scheja, & Haglund, 2008). Emphasis was placed on how Kim developed personal understanding of the learning material by putting it in a particular (cognitive) context or framework where it made sense for her in the perceived circumstances. The analysis focused on discerning varying ways of explaining the meaning of the concept of function, and on describing these in terms of contextualised conceptions supporting different understandings of the concept. The transcript was repeatedly read in parallel to listening to the audio file. Notes were made about the contexts the student used and a narrative about the student's development of her understanding was produced. The analysis was discussed with colleagues to improve the validity of the results.

The concept of function in the textbooks and the teaching

The literature for the course 'Vectors and functions' is comprised of two compendiums written by mathematicians at the university. In the first chapters in the compendium about functions, polynomial and exponential functions are presented. The graphs of these functions are studied by variable substitutions without the use of the derivative. In the last chapter there is a discussion about the concept of function and the concept inverse function is defined. Logarithmic functions are studied as examples of inverse functions. The definition of function that is given in this compendium is the following: "A function is a mapping that for each number x in a specified set maps the number to another number which is called the value of x for the function and is denoted $f(x)$" (personal translation). At the end of the course the students in a lecture worked with a text presented as an exercise to read mathematical texts. The text presented the following definition of function: "A function is a subset of the Cartesian product in which all elements x in the domain occur in exactly one pair (x, y)" (personal translation).

The literature in 'Calculus' included a text by Persson and Böiers (2010) as well as worksheets including some theory, exercises and reading instructions. The students used the worksheets during the lectures and said that they used the textbook just a little. Persson and Böiers provide in Chapter 1 the following definition: "A function is today understood as a rule or a process that in a well-defined and unique way remake (transform) some specified objects to new objects" (p. 7, personal translation). In the worksheets there is no definition or discussion about the concept of a function.

During the teaching there was only in one occasion a discussion about definitions of function. Related to the reading exercise, the lecturer discussed the definition of a function given in the text used. He used an example where the function was about marks assigned to some of the students in the class, also pointing out that no rule or arithmetic formula was used in that example.

Results

In this section the results from the four interviews will be presented, first each interview separately and then a summary will be given.

Interview 1 – Using linkages to everyday life

The student teacher Kim is going to be a teacher in mathematics and science in secondary school (grade 7-9). She is a university educated student who had worked as an economist before starting the teacher education programme. She passed the introductory course with good results but disclosed that she had to work a lot to reach the understanding she desired. In the first interview she talked about her impressions of mathematics and her learning during school and previous working life:

> For me mathematics has always been like a drawn curtain and behind that curtain there was a secret I didn't think I could get access to…or that it couldn't make itself available to me [---] and then and as time has passed it has sort of unveiled itself, so now I think I have a pretty, or I don't know, you know…you're confronted with things in life that are relationships, you meet with functions in real life [---] and then you understand and can calculate.

Her talking about mathematics as a secret behind a curtain and its unveiling while confronted with things in real life connected her current understanding mostly to everyday life. When she was asked about what a function was for her she answered:

> Well, I can feel a bit that it is this... you put in something and in there something happens, if you put in a value as in a box and something will happen there according to a recipe, so I think that it is the recipe that is the function for me.

This understanding of a function as a machine is not unusual when asking students about functions. To understand function as a process is a common starting point (cf. Sfard, 1991). However, interesting in Kim's utterance is also her understanding of the function as the recipe. This will change during the semester, as will be pointed out below. In the first interview Kim was also asked to give an example of a function:

> For me a function is a relationship between for instance speed and distance, so if I speed up I'll cover a longer distance within a particular time frame. Or a minute tariff on a mobile phone bill, like if I talk longer I use more minutes and so my bill will increase.

In this answer Kim again talked about and provided linkages to everyday life. Even though she talked during the interview about a function as a recipe she gave just one example including a recipe. At this time, in the beginning of the semester, her understanding of a function as a recipe seemed to be anchored in real life situations.

Interview 2 – Accessing abstraction

The second interview was conducted six days after the lecture when the reading exercise presented above was done. In that exercise Kim was presented with an abstract definition of a function and was also given an example by the lecturer. When asked about a definition of function Kim answered:

> Today I will say it is a relation between two... eh things, still it is [small laugh]... but what's new in my understanding of functions is that there doesn't have to be a rule that defines this relationship; sometimes there's just a relationship between two… chosen things and that the function is rather, that it's important that each element of the first given set gets its partner from the second set, and that for me is a new way of understanding functions.

In this utterance Kim displayed two different understandings. She had her understanding from before about function as a relationship, even though her choice of words was slightly different to that used in the first interview. However, she also had a new understanding; an abstract definition of function had in some way been added. She had recognised that it does not need to be a rule defining the relationship. In the previous interview she pointed out that the rule, or in her words the recipe, really is the function. It is also interesting to notice her hesitation when choosing her words "two... eh things". In the examples from everyday life given in the first interview the "things" put in were numbers. Now, after listening to the example given by the lecturer about giving marks (A-F) to students, she may have recognised that the "things" put in need not to be numbers.

The second interview indicated, both in direct utterances and in indirect ways from her utterances, that Kim had been influenced by the reading exercise and from the example given by the lecturer. But she also still had her original understanding, and she had not yet completely reconciled this understanding with her new, emerging, understanding. As she put it:

> I think I'm still like in between two understandings of this and I suppose I'm trying to find a way to put them together.

This utterance indicated that Kim, now in the middle of the semester, had entered a liminal space where her understanding of a function was unstable (Meyer & Land, 2006).

Interview 3 – Standing on a platform of functions

Before the third interview, conducted at the end of the semester when the calculus course was nearly completed, Kim and the other students had filled in a questionnaire asking if several graphs given in coordinate systems could be a representation for a function or not. Kim had answered correctly on all the graphs and was asked what principles she had used:

> Well, the principle, I look at how many y values I can get for each x value, I kind of look in that direction and then I check, is there only one alternative then it is a function, if there are several alternatives then it could not be a function.

This means that she had no need for checking if there is a rule. Nor was it a problem for her if the function was discontinuous or if the domain was discrete.

She handled functions at this moment, at the end of the semester, in a convenient way. Asked what she thought was the most problematic about functions she answered:

> You know all of a sudden I don't think that it's the concept of function that's so difficult any more, but more analysing and being able to juggle algebraically […] but the functions, I think they've suddenly become, well I don't know, a table, a platform, it's what we're doing at the moment and then the analysis of these functions suddenly has become the central topic.

Kim's utterance about functions as a platform indicated that she has passed through the liminal space. She presented a new and unified understanding. She was able to look up from the concept itself and could use functions as input in other processes.

Interview 4 – Establishing an understanding allowing reflection on process-object relations

Interview 4 was conducted at the end of the next semester, aiming at following up the understanding of the concept of function once the students had been able to get a perspective on the courses that discussed functions. Asking Kim again to explain the concept she answered:

> It is still the case that I all the time remember that I can have a value for x and then it has its correspondence, or it kind of reflects on just one value, and that's the way I live with it, and precisely that it's not needed to be continuous, and there don't need to be any patterns that you can follow... I think that was the big case that made me start to look at several other things as being functions, it just needs a correspondence and that could be totally arbitrary.

At this time Kim used the understanding following from an abstract definition, for every x in a set there is to be exactly one y, but she did not use the words from the definition given in the lecture referred to in interview 2. However, her understanding had now become unified; the two understandings she referred to in interview 2 seemed to have merged into one understanding allowing her to think about functions in a confident way. It could also be noted that Kim in this interview did not use linkages to everyday life. Asked about that, Kim said that she had put the linkages to everyday life away for a while, but also added that she would probably use such linkages again when teaching her students.

In interview 3 Kim talked about functions as a platform. This was followed up in interview 4 when she was asked if she understands functions as objects:

> I: In research, functions are sometimes talked about as objects. Could you feel that functions is kind of, a specific function like the sine function, that it could be like a "thing" in some way, something that you juggle with?

> K: No, I don’t feel that way at all. It’s more like a description of something… well, of course it’s an object…I mean it feels as I’m moving along something, sort of gliding along a scale in my… in this set I’m allowed to move…and that I can see what happens then, no not [an] object, it’s too an abstract…

After first strongly denying the understanding of functions as objects, Kim started to discuss, on her own, her understanding, in talking about gliding on a scale. So she was asked if she instead looks at functions as processes:

> I: Researchers talk also about functions as processes, that it is something you do; you have an input and get an output. If you talk about functions as objects, the process could be inside but you don’t need to look at it every time, you understand the function as a thing.
>
> K: In that case I think I’m leaning more towards an object, than a process. It’s more like, I can sort of move around within this entity but each value is there all the time, so nothing happens just because I choose a certain x; all those x are possible to pick all the time, so I think that’s the way I see it.

The notions of functions as processes and objects were new for Kim and it could be questioned to present these notions to her during the interview. As she pointed out, this was really abstract for her. However, her reasoning gave us interesting information about her understanding. She ended up talking about moving around within an entity, an understanding that could be interpreted as an objectification.

Development of Kim’s understanding of function

The four interviews revealed different stages in Kim’s transformation of her understanding of the threshold concept of function: using linkages to everyday life, accessing abstraction, standing on a platform of functions, and establishing an understanding allowing reflection on process-object relations. The change observed in her way of talking about a function can be conceptualised in terms of a process of contextualisation in which her repertoire of possible contextualisations (framings) of function was extended. This extended repertoire of contextualisations allowed Kim’s initial understanding of function, linked to concrete everyday life examples (“if I speed up I’ll cover a longer distance within a particular time frame”), to be gradually enriched to include a more abstract mathematical understanding (“there doesn’t have to be a rule that defines this relationship”) allowing functions to be seen as objects (“I can sort of move around within this entity”).

Discussion

The present case study illustrates and clarifies the complexity of the transformation involved in coming to understand a threshold concept. In Scheja and Pettersson (2010) it was argued that students’ shifting of contextualisation is an important part of the development of their conceptualisations of the threshold concepts and that the transformative aspect of threshold concepts could also be conceptualised in this way. These shifts of contextualisation also allow the student to become gradually more and more aware of the ways of thinking and practising in the subject. Through the present longitudinal study a possibility has been given to follow the transformation of understanding during a study year. Kim's development included shifting of contextualisations, from a concrete everyday context to a context allowing functions to be seen as abstract objects.

The findings also link to the notion of 'liminality' (Meyer & Land, 2006) describing a crucial stage in the process of coming to understand a concept or, indeed, a discipline. Kim seemed to enter this liminal space having two different understandings in parallel in the middle of the semester. She moved on and left this space with a transformed understanding.

Coming to understand functions as objects is not a neat and tidy process; it is highly dynamic and requires hard work (cf. Hansson, 2006; Sfard, 1991; Viirman, et al., 2010). For Kim this process took the whole semester and even several months later she was not comfortable when trying to analyse her understanding in terms of processes and objects. As has been pointed out, the understanding of a threshold concept often takes time to develop. Kim was interviewed over a period of nine months. This kind of longitudinal data is crucial if we want to explore the lengthy process of coming to understand threshold concepts.

It could be argued that Kim is not a typical student. Surely she is not. She has much more experiences of life including studying other subjects and working as an economist. She is also atypical in the way she expressed herself. Mostly it is hard for students to find words to explain their understandings and it is unusual for students to have thought about their conceptions. Kim had the words and could express her thoughts about the concept. That made it possible to elicit data about the process of transformation. Presented in this case study is Kim's way of talking about this process, a vignette of a personal journey. And, although it cannot be said that this is the process for every student, there are nevertheless important things to learn from Kim's description of her transformational journey towards the understanding of the threshold concept of function.

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Student-Initiated Communication with the Teacher: Field, Mode and Tenor

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This paper reports from one part of a study of upper-secondary mathematics classrooms in Sweden. In Sweden, individual work on textbook exercises constitutes a large part of upper-secondary mathematics education. During such work, the teacher is available to students as a resource to use in their work. In this part of the study, the focus is on how students in such textbook-based mathematics lessons make use of the available resources. In this paper, student-initiated communication with the teacher is discussed, in order to build on and extend a previous analysis of how the students make use of the teacher.

Introduction

The study within which this paper is written is part of an international research project, *The emergence of disparity in mathematics performance*, the aim of which is to analyse how student background is linked to success in school mathematics [1], [2]. This paper is part of an attempt to identify how differences in communication repertoire are linked to access to valued parts of the curriculum. Building on the analysis presented in Rohdin (2012), this paper focuses on student-initiated communication with the teacher during independent work.

Background

The overall research project draws on Bernstein's (2000) categories *visible and invisible pedagogies*, and the related concept pairs *classification and framing* and *recognition and realization rules*. In the Swedish context, the notion of framing turned out to be particularly interesting. The classrooms we have observed in the Swedish part of the study, including the one discussed in this paper, have had weak framing over pace, but stronger framing over sequencing. This is because the lessons are very much based on the textbooks, and follow the order of the books. In terms of pace, students are given a sheet of paper on which is written which chapters, pages or exercises they are meant to work with at a given time. They are, however, given the responsibility to choose which exercises to do and which to skip. The teacher in the class described in this paper explicitly tells the students not to spend time doing what they already know how to, and to move on to what they do not know. The teacher does not directly check whether the students have done (or skipped) the sections they are supposed to. Teacher-framing is (apparently) weak, and therefore it becomes very much up to the individual student how and even whether to learn. This applies in particular to initiating or not initiating communication with the teacher (see below).

Bourne (2003), in an analysis of language classrooms with weak and strong framing, respectively, claims that when much is left to the students' choices, "evaluation replaces instruction, and certain children are not given access to the academic discourses on which, Bernstein argues, the development of scientific concepts ultimately depends" (p. 498). This seems to be of particular relevance to the Swedish mathematics classrooms where framing is apparently weak and students are faced with many choices (cf. Jablonka, Johansson & Rohdin, 2010).

Student-initiated communication with the teacher

Context

In line with the aims of the overall project, the classes studied were beginning their first year at upper secondary school (Swedish *gymnasium*). This was because it was important to catch what happened during the first few weeks, when the students were still new to each other, the teacher and the school. The class discussed in this paper was a small one (less than 15 students), with a mix of social backgrounds and nationalities among the students, and there were more girls than boys. Most of the students were entirely new to each other when the school year started. Some had met before but did not know each other well. The teacher had not met the students before.

The first nine lessons of the school year were video-recorded. Two cameras were used when possible, following the teacher and the students, respectively. During these first nine lessons of the school year, the lesson structure is fairly typical of Swedish mathematics classrooms at the upper secondary level, as described by for example Skolinspektionen (2010), in that much of the time is spent on individual and independent textbook-work. It is important to remember, however, that what is described here is what happened during these first nine lessons of the school year. Thus, it is not necessarily representative of the rest of the year. For further details on the data, see Rohdin (2012).

Communication patterns and perceived ability—discrepancies

In Rohdin (2012), an analysis of the communication patterns in the classroom was presented, and the communication patterns were discussed in connection with the students' reasons for picking some students as the best at mathematics. An incongruity was pointed out, namely that students said they thought a student was good at mathematics because he or she seldom asked the teacher a question, while the communication-pattern analysis showed that the students most frequently picked out as good at mathematics did in fact ask the teacher questions, and often comparatively frequently. In order to look more closely at this discrepancy, the videos and transcripts of the instances of student-initiated communication between student and teacher were analysed.

Instances of student-initiated communication with the teacher

The concept scheme of *field*, *tenor* and *mode* (Halliday & Hasan, 1989) was used as a methodological tool for focusing the analysis of these communication instances. According to Halliday and Hasan, these concern *what a text is about*, *who the participants are* (including *their status*) and *what part the language is playing in the interaction*, respectively (1989, p. 24). Thus they cover many of the ways these communication instances could differ from each other. The field, tenor and mode together define particular *registers*, "the semantic configurations that are typically associated with particular social contexts" (Halliday & Hasan, 1989, p. 43), which, according to Halliday and Hasan reflect the nature of the activity in which language plays an essential part. The analysis in this paper is similar to that presented by Atweh, Bleicher & Cooper (1998), although their analysis focused on the whole-class communication of two different teachers working with two different classes.

The analysis presented in this paper uses data from lessons 2 and 9, focusing on communication instances in lesson 9. Lesson 2 is the first "real" lesson in the sequence (lesson 1 dealt with handing out books and calculators, and going through the goals and grading criteria of the course), and lesson 9 is the last lesson in the sequence. The language used in the classroom is English, which is not the native language of any of the participants involved (students and teacher). In the analysis, therefore, use has been made of the knowledge of the participants' native languages, and of how the participants use English during the lessons. This is particularly so in the case of the participants' use of the pronoun "you". In all three extracts presented (see below), the word "you" is used frequently. This should be understood as referring to the second person singular (Swedish or German "du"), rather than as an impersonal pronoun (Swedish or German "man", English "one"). This can be seen by looking more closely at the context in which the word occurs.

Table 1 shows the number of occasions and the number of minutes of student-initiated communication with the teacher of each student during lessons 2 and 9. The students picked most frequently as being good at mathematics were students A, D and E. Table 1 shows that they do not seem to initiate communication with the teacher any less than other students, and in the case of student D the opposite is the case. In lesson 2, three students (B, D and I) initiate communication with the teacher. In lesson 9, the number of instances has increased, as has the number of students involved: all students present except student H initiate communication with the teacher at least once during the lesson. There are notable differences in the amount of time the teacher spends with each student during such student-initiated communication, ranging from a total of a few seconds to over 17 minutes.

Student	Occasions in lesson 2	Minutes in lesson 2	Occasions in lesson 9	Minutes in lesson 9
A	0	0	3	3½
B	3	1½-2	absent	absent
C	0	0	absent	absent
D	3	1½-2	6	17½
E	0	0	2	8
F	0	0	1	<½
G	0	0	2	4
H	0	0	0	0
I	2	1	1	<½
J	absent	absent	1	1

Table 1. Number of occasions and number of minutes of student-initiated communication with the teacher during lessons 2 and 9.

By lesson 9, most students had started initiating communication with the teacher, as Table 1 shows. Three students (at different times during the lesson) ask for help with the same exercise. The transcript extracts from these three occasions are presented below, and the extracts are discussed separately in terms of field, tenor and mode, and then compared. The three extracts are presented in the order in which they occurred during the lesson. The extracts have been chosen because the topic, the time and the place all remain broadly the same across these three instances of student-initiated communication with the teacher. From the third extract, the wording of the exercise can be identified: “Find the range of each function when the domain is {-1, 0, 6}.” The function discussed in all three extracts is $t = 5 + r$.

The extracts are presented in their entirety, from the start, when the student catches the teacher’s attention, to the end, when the teacher leaves the student. The starting and ending times of the episodes in the lesson are given in brackets. The unit of analysis is the conversation, not the teacher’s utterances on their own or the student’s utterances on their own.

Student A (19:04-20:17)

Student A: [says teacher’s name]

Teacher: Mm.

Student A: [inaudible]

Teacher: Yes.

Student A: choose a number or?

Teacher:	Mm. The domain is the same as the x-values,
Student A:	[inaudible]
Teacher:	and in this case we have r, so this means that substitute the values for r and calculate for t. So the first value you get for your range that would be five minus one, that is four.
Student A:	Mm.
Teacher:	And then you take the second one and substitute that value in,
Student A:	Mm. [inaudible] so that's
Teacher:	so that will give you, yes, so you will have four, five
Student A:	[inaudible]
Teacher:	and [pause] the third one is six.
Student A:	So it should be
Teacher:	So that will give you five plus six [pause] eleven.
Student A:	Mm.
Teacher:	So your range would be the set of those three numbers.
Student A:	Okay. [pause] Okay. [inaudible]
Teacher:	Mm. But the domain that is always the set of the independent variables,
Student A:	Mm.
Teacher:	and the range the function values.

After getting the teacher's attention, student A starts by asking whether one way of doing the exercise is correct, or whether there is another way. The teacher does not answer explicitly, but instead starts explaining the task, which could be taken as meaning that the suggestion made by student A was incorrect.

The teacher does not mention the numbers in the domain, only what the word "domain" means. The teacher explicitly gives the definition of "domain" and "range". *En passant*, the teacher implicitly comments on the "non-standard" choice of variables (r and t instead of x and y)—"and in this case we have r". At the end there is a summing-up, "the domain is always the set of the independent variables, and the range [is always the set of] the function values", suggesting that this is the important point of the exercise. The teacher uses the words "domain", "range", "set", "substitute", "x-values", "calculate" and "function values". The tone of voice and the pace of the utterances give an impression of brisk, business-like interaction. There is no waiting-time for the student to "catch up". The student interjects, usually interrupting the teacher's utterance. The teacher also interrupts the student. There is an instance of "you" with an active verb, but mostly the "you" is a passive part of the process, not an agent. Things happen, and the "you" observes it happening. There is one case of "we", and it is a passive "we"—"in this case we have r". The image created is that of an objective mathematics, independent of what the student or the teacher does. There is an impression that the details are not so important and can be rushed over, while the overall structure is emphasised. There is some space for the student to do or say something, and the student occasionally tries to create more space by (mostly unsuccessfully) interrupting the teacher. The mode of the exchange appears to be a dialogue, but on closer inspection it turns out that the student's contributions do not in fact have any impact. Thus it is more like a monologue, although the student attempts to make it a dialogue. The main input from the student is the original question, although the teacher's response does not directly relate to it. Apart from this, the teacher's part of the conversation forms a coherent whole on its own.

Student J (20:21-21:23)

Student J:	[says teacher's name]
Teacher:	Mm.
Student J:	[points at an exercise in the textbook]
Teacher:	Yes. You know the domain, the values are minus one zero six, so that means the range would be the values for t so we take the first one minus one and substitute that one into the function. You will have five minus one that will be four.
Student J:	Mm.
Teacher:	So that's the first value and then take the next one that's zero. Five plus zero that's five. Six five plus six eleven, so this will be the range,
Student J:	Okay.
Teacher:	the corresponding function values.
Student J:	Okay.
Teacher:	You see it could have been written like y equals five plus x also, the choice of letter doesn't matter, you can choose whatever letter you want.
Student J:	Mm.

After getting the teacher's attention, student J says nothing at all, but points at the exercise in the book. The teacher says what the domain is by saying what the numbers are, not by giving a definition of "domain". There is an implicit definition of "range". There is no explicit "summing-up": once the answer has been reached the explanation is over. The teacher uses the words "domain", "range", "substitute", "function" and "corresponding function values". The teacher's utterances usually start with "you" and an active verb. The part corresponding to the explicit imperative with student G (see below) becomes a collaborative enterprise—"so we take the first one". This could also be a case of what is called "Krankenschwester-wir" ("nurse-we") in German (cf. Sachweh, 2006), that is a "we" that can mean "you" or "I", but not "we" [3], perhaps to create a caring, nurturing atmosphere. In this case "we" in practice means "I", since the teacher is the one doing the taking and the student does not visibly or audibly participate. This contrasts with the "we" in the first extract, where it can indeed be the case that "we [both] have r". The teacher at the end explicitly points out that "the choice of letter doesn't matter". The teacher explains this statement and expands, and once again there is a "you" taking an active part in the process—"you can choose whatever letter you want". There is a suggestion that this is the important part of the exercise. The image created is that of an operational mathematics, dependent on the student or the teacher doing something with it. On the other hand, there is no space for the student to actually do something, since the teacher does all the work. In this case, it is even clearer that the communication is a monologue, with a designated listener. The listener shows involvement by saying "mm" or "okay" at appropriate places, but no further contributions occur. The teacher's utterances form a coherent whole on their own, without the student's utterances.

Student G (32:38-34:29)

Student G:	[says the teacher's name] I need help.
Teacher:	Find the range of each function when the domain is minus one zero and six. So that would tell you this is the independent variables.
Student G:	Okay.
Teacher:	And your function is t equals five plus r so finding the range means that you should substitute each value and calculate what's the value of t because your t-values would then be the range.
Student G:	Okay. [inaudible]
Teacher:	So take the first one to start with and that's minus one.
Student G:	Mm. [pause] Is it t equals
Teacher:	Yes five [pause] minus one you will have the first value
Student G:	Is it plus minus one or just minus one?
Teacher:	You can just write minus because
Student G:	Yeah.
Teacher:	plus minus will be just minus.
Student G:	Mm.
Teacher:	So that will be four. [pause] And then you take your second value. [pause] And then the third.
Student G:	Yeah okay.
Teacher:	And then you can write the range with braces like this, so your values would be then?
Student G:	Four five and eleven.

Teacher:	Exactly. [pause] Neat. Nice.
Student G:	Okay.
Teacher:	And you notice also that the choice of letter doesn't matter.
Student G:	Yeah.

After getting the teacher's attention, student G says "I need help", but does not elaborate on what kind of help, or with what. The teacher first reads the text of the exercise aloud, and then "translates" the text for the student. "Domain" and "range", while not explicitly defined, are explained. The teacher explains what solving the exercise would amount to doing, and then proceeds to do that. The teacher uses the words "independent variables", "function", "range", "substitute", "calculate" and "t-values". There are quite a few instances of "you" with an active verb and a couple of instances where the "you" is a passive part of the process. There is one explicit imperative—"so take the first one to start with", and a few implicit ones—"and then you can write the range with braces". This is a reference to "the set" mentioned in the conversation with student A, but without the explicit use of the word "set". There is one explicit question—"so your values would be then?"—which becomes the "summing-up" part, checking that the student realises what the answer is. The teacher at the end again points out that "the choice of letter doesn't matter", but in contrast to the conversation with student J, the teacher does not comment further on this. The image of mathematics created is ambivalent between objective and operational. Here, there is more space for the student to do something, and the student's doing is recognised and praised at the end of the exchange—"Neat. Nice." It is partly a monologue, but with some of the parts supplied by the listener, in the sense that the student is filling in the details the teacher is expecting and leaving space for. There are exceptions, when the student makes a suggestion or asks a question that makes the teacher deviate from the intended direction for a while. This extract therefore has more dialogical features than the previous two. Without the student's utterances, the communication does not form a coherent whole.

Comparison

The three students initiate the communication in very different ways. The very first initiation is the same in all three instances—the students all use the teacher's name in order to get attention. After that, however, there are differences. Student A starts by asking a question about the exercise, student J quietly points at the exercise in the book and student G asks for (unspecified) help. After these initiations, most of the speaking is done by the teacher.

In the first extract, the teacher rephrases the exercise and then performs the steps to get to the answer, and finally rephrases the answer in the original terms. In the second extract, the teacher talks the student through how to get the required numbers. In the third extract, the teacher first tells the student how to go about answering the exercise, and then talks the student through the process, leaving space for the student to perform the calculations.

The students' initial behaviour is reflected throughout the conversation. Student A asks whether s/he has the right idea of how to go about doing the exercise, thus demonstrating a certain amount of competence and independence. During the conversation, the student tries to re-establish this competence, by occasionally interrupting the teacher with indications that s/he has understood before the teacher has finished explaining, and independence, by interrupting the teacher with contributions. Student J points at the book without saying anything at all, demonstrating helplessness. During the conversation, the student contributes indications that s/he is listening and accepting what the teacher says. This can be seen by the fact that the instances of "mm" and "okay" occur at places where the teacher leaves a space for them, expecting them. Student G asks for unspecific help, not offering any indication of having an idea of what to do, but indicating a certain amount of initiative. During the conversation, the student offers confirmation of listening/accepting at appropriate places, and also contributes a (partial) suggestion and a question.

There are also differences in the teacher's tone of voice and pace of speaking. These features are of course not possible to show in the transcripts, but add to the impression of the situation. With students J and G, the teacher's voice is softer and slower, and with student A the teacher's voice is sharper and quicker, and also darker. The conversation between the teacher and student A thus comes across as more formal, brief and "business-like", something which is reinforced by the more formal language used. The focus is on the general features of the exercise rather than on the specific numbers or letters used. The conversations between the teacher and student J and to some extent student G come across as more nurturing and encouraging. In the conversation between the teacher and student J the focus is on what is specific to the exercise. In the conversation between the teacher and student G, the focus is on getting the student to be able to arrive at the required answer.

Conclusion

The analysis in Rohdin (2012) indicated a need to examine more closely the occurrences of student-initiated communication with the teacher, in order to explain the discrepancy between students' reasons for considering particular students to be good at mathematics, and the findings of the communication-pattern analysis. An analysis of a selection of such occurrences shows that, although in this classroom all are engaged in the same social activity, there are variations of the teacher-student school mathematics register. There is more or less focus on technicality (field), the emergent role relationships differ slightly (tenor), and there are subtle differences in the mode, with some more didactical or expository and others less. In some of these cases the student is constituted as competent and independent, and in others as helpless and dependent. If such differences are induced by the students' initial openings of the conversations, they would depend not only on the teacher's different approaches, but also on the different students' different approaches. These interact with and reinforce each other during the conversations, and across time this could provide one mechanism through which students get access to different kinds of mathematics. The students' ways of communicating with the teacher, their different registers, therefore seem to influence their access to valued kinds of mathematics.

Notes

1. The Swedish part of the project is funded by Vetenskapsrådet.

2. For further details and a thematic literature review on the project, see web page at http://www.acadiau.ca/~cknippin/sd/index.html.

3. An example of the "Krankenschwester-wir" would be "How are we feeling today?"

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Novice Mathematics Students at University: Experiences, Orientations and Expectations

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In this paper, we report from an on-going study of novice university students in mathematics and the secondary-tertiary transition. A total of 146 students from three Swedish universities were given a questionnaire in the beginning of the semester. The aim was to characterize them as learners of mathematics. The results were summarized with descriptive statistics and Principal Component Analysis (PCA) was used to look for correlations. The results show that the teacher and the textbook play a crucial role in their learning of mathematics. Furthermore, the students can be characterized as either individual or interactive learners, which correlates with the choice of university.

Introduction

This on-going study concerns novice mathematics students and their transition to university studies in mathematics. In this study, the novice students are first-semester university students. The notion of transition should be understood as students' learning of mathematics in a new setting in light of their previous experiences of studying mathematics (Stadler, 2009). The comprehensive study is based on two questionnaires, one distributed at the beginning and the other at the end of the first semester. The aim of the first questionnaire is to give an account of their previous experiences of studying mathematics at the secondary level, their views of mathematics and learning mathematics, and their expectations regarding their forthcoming mathematics studies. The aim of the second questionnaire is to examine how these transition-related aspects are affected after the first semester of mathematics studies at the university. In this paper, we report on the results from the first questionnaire.

Background and methodological considerations

Mathematics students' encounter with the secondary-tertiary transition seems to be associated with a variety of problems (Gueudet, 2008). In different studies, attention has been paid to students' under-preparedness for mathematics studies at tertiary level (Brandell, Hemmi & Thunberg, 2008; Kajander & Lovric, 2005; Lawson, 2003; Thunberg & Filipsson, 2005), in particular students' lack of sufficient and suitable mathematical pre-knowledge and skills (Jourdan, Cretchley, & Passmore, 2007; Liston & O'Donoghue, 2009), their difficulties with mathematics as a scientific subject (Guzmán, Hodgson, Robert, & Villani, 1998) and the challenge of adapting to a new learning environment (de Abreu, Bishop, & Presmeg, 2002). These studies contribute to our understanding of some of the difficulties that can be associated with the transition, but without specific considerations of what is also a problem from a student perspective.

The transition is a multifaceted phenomenon that has been researched from different theoretical perspectives (Gueudet, 2008). However, choosing a theoretical perspective for a study of the transition in advance also entails deciding on the focus of research questions, methodological approaches and features of research results. The differences in definitions of the crucial aspects of the transition may result in incompatible research results and also complicate the accumulation of knowledge of the phenomenon over time.

For the current study, our main goal is to examine novice students' transition without simplifying the nature of this phenomenon. In order to accomplish that, we have based our quantitative study on a general conceptual model for describing students' learning of mathematics. This model consists of a categorization of the main features of learners of mathematics and was developed as part of a qualitative study on the secondary-tertiary transition from a student perspective without any pre-defined theoretical perspective (Stadler, 2009).

Choosing this model as a foundation for the questionnaire contributes to the on-going discussion about mixed methods and how a combination of qualitative and quantitative methods can result in a more comprehensive examination of a specific phenomenon (Winberg, 2006).

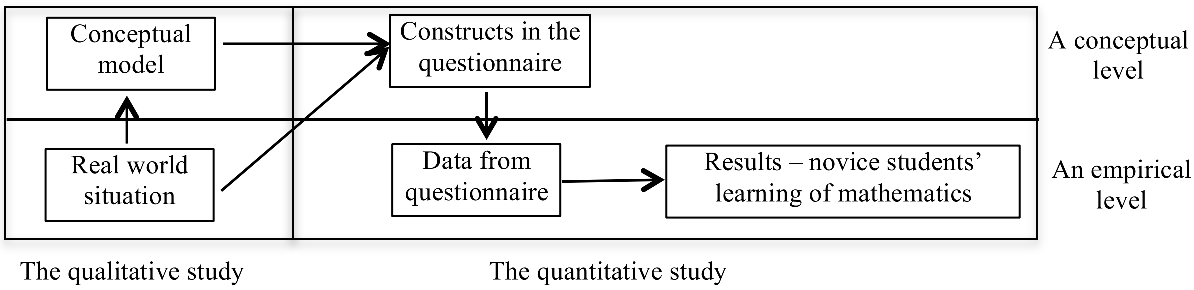

Figure 1. The methodological approach of the study.

The starting point for the qualitative study was the transition, as a real world situation (Figure 1). A conceptual model was generated as a theoretical description of the crucial aspects of students' learning of mathematics in a new setting in light of their previous experiences. This model was generated with an inductive qualitative approach, which did not involve any a priori assumption such as a pre-defined theoretical framework. To make use of these concepts for the quantitative study, they must be made operational (Bryman, 2002). However, in this case, the conceptual model is the result of a systematization of empirical data. Thus, to make the conceptual model operational, we have returned to the empirical data it was based on in the first place. These empirical data were used to formulate questions for the questionnaire. The questions were organized in themes that related to the conceptual model.

The conceptual model – three categories

The conceptual model consists of three relational categories that constitute significant aspects of learning and understanding mathematics from a student perspective (Stadler, 2009). These three categories are mathematical learning objects, mathematical resources and students' actions as learners.

The *mathematical learning objects* category refers to the students' view of the overall purpose of learning mathematics. It captures students' interpretation of what mathematics is and what learning mathematics is all about. For students in secondary school, an essential part of mathematics studies is working with textbook exercises. For these students, solving exercises can be a mathematical learning object in itself. Mathematical learning objects can consist of actions or knowledge. For example, students may focus on verbal explanations to other peers as a mathematical learning object, which not only involves the knowledge of how to explain something to someone else but also requires mathematical knowledge of what should be explained.

The *mathematical resources* category concerns those objects and phenomena that students need in order to learn mathematics. Textbooks, teachers, peers, mathematical pre-knowledge and logical thinking are some examples of potential mathematical resources that can constitute mathematical resources when the students use them as such. In discussions of students' learning of mathematics, mathematical resources can be labelled in accordance with how they can be used. As a mathematical resource, an explanation from the teacher can be regarded as dynamic since the teacher interacts with the students and can adjust or change the explanation based on how it is received by the students. On the other hand, the textbook can be regarded as a static mathematical resource because it does not change its content, whether the student understands it or not. The textbook is also a constantly available potential mathematical resource.

The *students' actions as learners* category captures students' actions, intentions and conceptions in relation to their learning of mathematics. It is a category that comprises both mathematical learning objects and mathematical resources in a mutual relationship. The students use mathematical resources they believe can be helpful with respect to a specific mathematical learning object. On the other hand, the availability of potential mathematical resources determines which mathematical learning objects students focus on.

The design of the questionnaire

As mentioned earlier, our study is based on two questionnaires. The first, which is reported on in this paper, aims at accounting for novice mathematics students' previous experiences of studying mathematics at upper secondary school, their orientations, i.e. beliefs, dispositions, values, and preferences about mathematics studies (Schoenfeld, 2011) and their expectations concerning their forthcoming mathematics studies. According to Hartas (2010), survey questions can be sorted into four main categories: knowledge, attitudes, behaviour and attributes. Students' preferences about mathematical resources can be captured by questions both about their attitudes and behaviour to determine how they have considered different mathematical resources and how they have been using them. To operationalize mathematical learning objects, i.e. what students regard as the main aim of studying mathematics, we have to focus on students' attitudes. Questions about what they regard as important aspects of mathematics and learning mathematics can capture their mathematical learning objects. On the other hand, the "students' actions as learners" category concerns their behaviour rather than their attitudes. Thus, what the students think they actually did when learning mathematics at secondary level will illustrate this category.

Based on these considerations, we designed a questionnaire with 15 themes:

1. Entry requirements (attributes)
2. Lesson activities in upper secondary school (behaviour)
3. Valuation of lesson activities in upper secondary school (attitudes)
4. Help-seeking behaviour during mathematics lessons in upper secondary school (behaviour)
5. Valuation of homework activities outside school (attitudes)
6. Help-seeking behaviour during homework in upper secondary school (behaviour)
7. Valuation of resources for the learning of mathematics (attitudes)
8. Valuation of the mathematics teacher's actions (attitudes)
9. Valuation of working with peers (attitudes)
10. Valuation of the textbook (attitudes)
11. Orientations towards mathematics and the learning of mathematics (attitudes)
12. Expectations concerning forthcoming studies of mathematics at university
13. Orientations about mathematics studies at university
14. Estimation of time required for self-studies
15. Other comments

Besides some initial questions in Theme 1 about the students' attributes and questions in Theme 2 about activities during an ordinary mathematics lesson, a majority of the questions concerned behaviour and attitudes, which corresponds to empirical instances of the three categories (Stadler, 2009). The 117 questions in Theme 3-13 were formulated as Likert scale questions with a five-step rating scale. For example, the initial questions about orientations towards mathematics and the learning of mathematics were formulated as follows:

11. Here are some questions about your views of mathematics and learning of mathematics.

	Strongly Disagree				Strongly Agree
a) It´s easy for me to learn mathematics.	1	2	3	4	5
b) I can solve most exercises by myself.	1	2	3	4	5
⋮					
w) I learn new concepts by solving exercises.	1	2	3	4	5

Data collection and analysis methods

The questionnaire was distributed during the first two weeks at the beginning of the first semester of the study program. It took approximately 15-25 minutes to answer. The participating novice students came from one university and two technical universities. The specific groups of students at each university were chosen according to availability. The majority of the university students (U) were studying in a 3-year programme but some were also studying in a 5-year programme. All the students at the technical universities (TU1 and TU2) were in a 5-year programme. All the students studied comparable introductory mathematics courses during the first year. In total, 146 students answered the questionnaire, 110 of whom were men and 35 were women. A total of 83% of the students were between 18 and 22 years old.

The quantitative data have been analysed using two methods. Firstly, we have used descriptive statistics to summarize data in order to describe the main features of the participating students. Secondly, Principal Component Analysis (PCA) was used to find the correlation pattern between categories as well as the relative importance of categories for discrimination between groups.

Results

Background attributes of participating students

The background of the participating students can be seen from Table 1. 80% of the students had conducted their secondary level studies within a national programme (natural science or technology) at the secondary school. For TU1 and TU2, the mathematics entrance requirement is Mathematics E from upper secondary school, while for U Mathematics D suffices.

	U	TU 1	TU 2	Total
Number of students	59	44	43	146
Access programme				
Upper secondary school	46	35	36	117
Adult education	10	2	7	19
Natural science/technology foundation year programme	3	6	0	9
Grades – Math D				
Pass	33	11	7	51
Pass with credit	20	12	8	40
Pass with special distinction	6	19	27	52

Table 1. Educational background of the participating students.

The students' previous experiences of mathematics studies

A clear majority of the students have had similar experiences of mathematics lessons at upper secondary school. A mathematics lesson begins with the teacher giving a lecture, which lasts 10-15 minutes of a 60-minute lesson. The rest of the lesson is spent on the students' work with textbook exercises. This work can be handled either individually or in interaction with others, depending on if they typically used to work by themselves with textbook exercises, or if the worked in cooperation with their peers. Teacher initiated discussions within the whole class, group work organized by the teacher or activities involving use of computers are all very rare.

The student's valuation of the importance of different lesson activities for their learning of mathematics in upper secondary school can be seen in Figure 2. The teacher's lectures at the board, individual work with textbook exercises and individual help from the teacher, which were considered to be the three most valuable activities, were also the most frequent during the lessons. Organized group work and discussions within the whole class, and computer activities is considered to be of less importance. This is most likely due to the fact that many students never experienced these activities in upper secondary school.

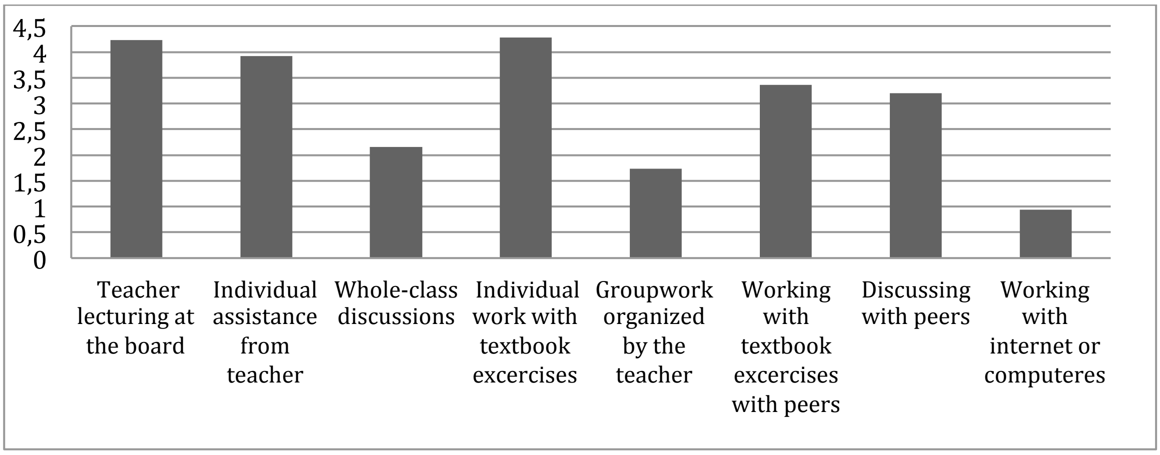

Figure 2. The students' evaluation of lesson activities.

Students were also asked to evaluate the importance of different potential mathematical resources for learning mathematics in upper secondary school. As illustrated in Figure 3, the two most important resources are the teacher and the textbook. Other important resources are their peers, the pocket calculator and the book of formulae. Working with tests and exams from previous years was of slightly less importance, and computer or Internet based resources received a very low score.

The students have a clear opinion that the most important contribution from the teacher is to provide whole class lectures. As can be seen in Figure 4, the students also regard the teacher's ability to motivate the students, and to provide help with textbook exercises on an individual basis, as very important.

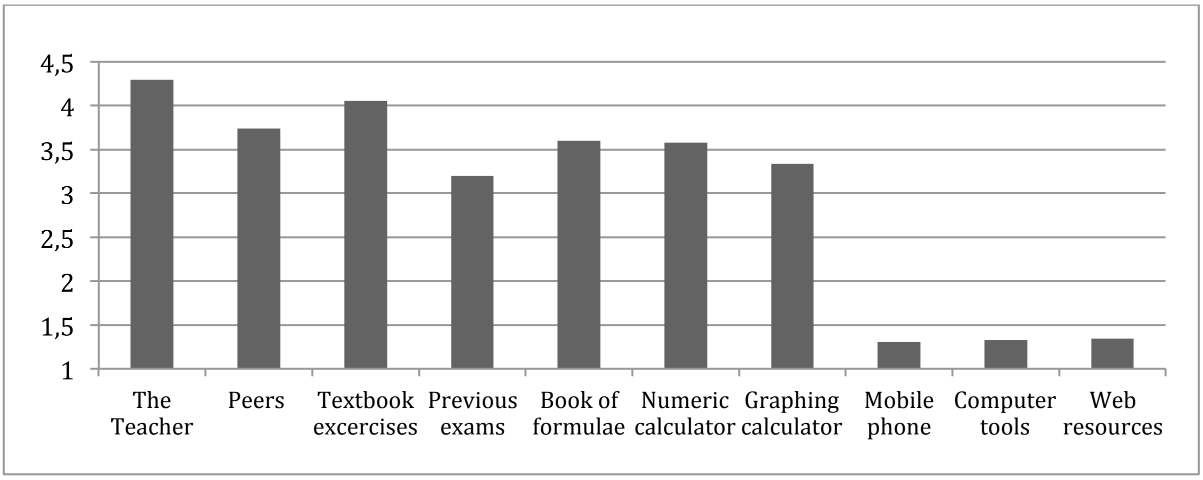

Figure 3. The importance of potential mathematical resources

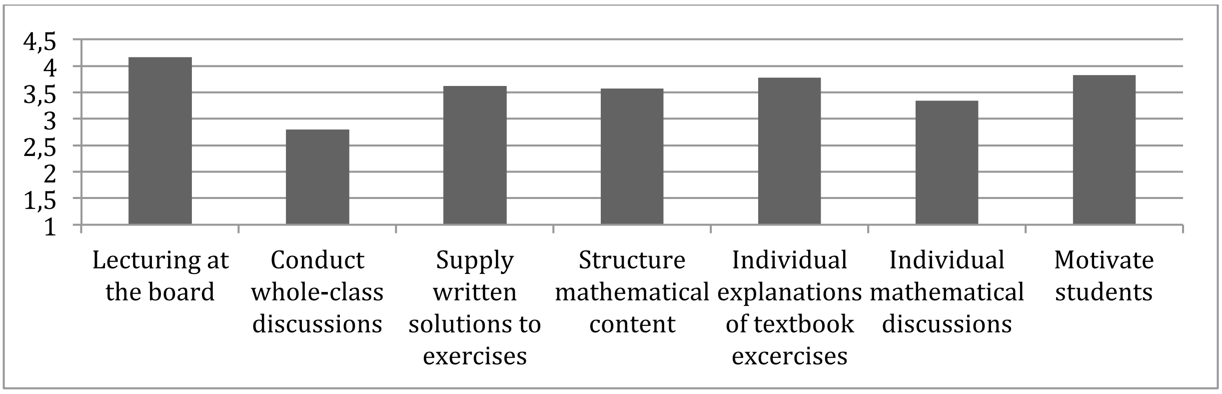

Figure 4. The importance of the teacher's contribution

Tables with formulas and short summaries, worked examples, exercises and answers to the exercises are the most highly valued features in the textbook, while text describing concepts and theory is considered somewhat less important. Peers are seen as an important resource primarily during lesson while working with textbook exercises. It is interesting to note that many students felt that providing explanations to other students was a situation of potential learning.

The students' expectations regarding their university studies in mathematics

In the questionnaire, there are also questions concerning the students' expectations on similarities and differences between upper secondary and tertiary level studies in mathematics. According to the novice students who participated in the study, the significant difference is that mathematics will be more difficult. To succeed with their studies, the novice students believe it is important to do what the teacher tells them to do, participate in all the teaching activities that are provided and spend a great deal of time on studying in addition to the scheduled teaching activities. In particular, over 90% of the students feel that they have to improve their time planning and take more responsibility for their studies at the university compared to what they are used to doing at upper secondary school.

Thus, spending a great deal of time studying independently is what the students regard as the most important difference between secondary and tertiary mathematics studies.

Results from the PCA analysis

In contrast to the descriptive statistics, Principal Component Analysis (PCA) can be used to show correlation patterns between the categories as well as the relative importance of categories between different themes. We performed a PCA on our data, using SIMCA P+ software (Umetrics, 2004), to examine the relations between different features of the novice mathematics students. Table 2 contains an overview of the fraction of total variation in student responses to items in each theme that can be described (R^2X) and predicted (Q^2) by the respective model. (A) is the number of components and (N) is the number of students who have responded to the items in each model. Q^2-values are reported for the single themes (i.e. how much of the variation in student responses on the items within the theme that can be predicted by the model) and the top model (i.e. variation in the student scores on the themes that can be predicted by the top model).

				Single theme	**Top model****
Themes	**A**	**N**	**R^2X**	**Q^2**	**Q^2**
Student competence profile	1	146	0,42	0,21	
Valuation of learning activities	2	135	0,54	0,05	0,61*/0,04*
Help seeking behavior during math lessons	2	145	0,43	-0,07	0,25*/0,12*
Valuation of math activity outside school	2	145	0,53	-0,14	0,07*/0,04*
Help seeking behavior during homework	1	144	0,30	0,08	0,32*
Valuation of resources for learning math	2	145	0,43	0,04	0,18*/0,00
Valuation of math teacher actions	1	145	0,46	0,24	0,47*/0,22*
Valuation of working with peers	1	144	0,62	0,45	0,22*
Valuation of the textbook	1	144	0,48	0,19	0,31*
Orientations towards math and math learning	3	146	0,37	0,06	0,70*/0,36*/0,0
Expectations of university math studies	2	144	0,36	0,01	0,13*/0,00
Orientations towards university math studies, including time requirements	1	142	0,36	0,21	0,33*
Top model	2	146	0,35		0,07*/0,14*
* Significant on the 95 % level ** For themes with more than one component, cumulative top model Q^2 is given for each component.					

Table 2. Overview of the fraction of total variation in student responses.

The descriptive statistics showed that the students tend to regard mathematics at secondary level as mainly individual or interactive. Therefore, we looked for a characterization of the students as learners in terms of individual or interactive in the PCA. This was obtained through some of the questions in Theme 11, focusing on the students' preferences to work alone or with others. Then, we divided Theme 11 into sub-groups according to different themes: the nature of mathematics, learning style, task-solving strategies and self-efficacy. The PCA showed that the students that preferred to work individually also had a higher score on self-efficacy and believed tasks could be solved if enough time was provided for thinking. To a lesser extent, these students asked teacher and peers for help or had discussions. The TU2 students, which also had higher grades, had a more individual learning approach than the others in the study. The students with a more interactive learning approach and task-solving strategies, i.e. preferred to work with their peers and ask the teacher for help and discussion, showed a lower level of self-efficacy and regarded it as more important to get acquainted with new learning materials in collaboration with others.

Discussion and concluding remarks

In this study, we have given an account of students' previous experiences of studying mathematics at secondary level, their views on mathematics and learning mathematics, and their expectations concerning their forthcoming mathematics studies. Our study confirms that the teacher and the textbook are the most important resources in upper secondary school, and that the students' experiences of mathematics studies at secondary level are dominated by a short introductory lecture by the teacher, followed by work with textbook exercises.

The results also indicate that the students' work with textbook exercises could be characterized as mainly individual or interactive, and the outcome of the PCA analysis indicates that the students can actually be characterized as having an individual or interactive learning approach. Concerning their use of mathematical resources, we conclude that the more individual students tend to use mathematical resources that are more static and available, and rely on their own capacity to think and learn for themselves. On the other hand, interactive students tend to focus on interactive, mutual and dynamic mathematical resources, for example their teacher and peers. A related result from the qualitative study (Stadler, 2009) showed that students could be characterized as independent or dependent as learners, whereby the latter group encountered more transition-related difficulties when the demands on independence in mathematics studies at university increased. This can also be related to the results showing that students with higher grades tend to be more individual. Thus, in the second questionnaire we also want to determine whether there is a correlation between individual and independent students and interactive and dependent students, and how these two groups of students experience the transition.

Even though our results indicate important insights about students in transition, we are aware of the danger of jumping into too far-reaching conclusions. Even though the sample of university students (U) contained students from different programs, the TU1 students came from one program and TU2 students came from another program. Thus, the sample is small and not representative for all mathematics students in transition. However, the results that have been presented in this paper give an indication of what can be of interest for further studies of novice mathematics students at the university.

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Applying Japanese Problem Solving Oriented Lesson Structure to Swedish Mathematics Classrooms

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Since the late 1990s, the ranking of Sweden in international surveys of education such as TIMSS, has dropped noticeably, especially in mathematics. The Swedish Schools Inspectorate (2010) points out two main reasons for the decline: 1. Students are not being equipped to develop different skills such as problem solving and the ability to make mathematical connections, nor to reason and express themselves mathematically; 2. The teaching is largely characterized by students working individually in their textbook. Problem solving centred teaching methods in Japan have been developed with an emphasis on finding ways to organise the classroom discourse to make the students active learners of mathematics, without losing the focus on the mathematical content. The "problem solving oriented" approach (PSO), which was developed by Kazuhiko Souma (1997), is one of the variations of Japanese teaching methods where teachers focus to enhance the students' attitude towards engaging in mathematical activities. The aim of this paper is to clarify how PSO, by extracting its *didactical organisation* with the help of the anthropological theory of didactics (ATD; e.g. Chevallard, 1999), could affect students' willingness to "reason and express themselves". ATD provides a framework for the analysis of how the didactic process relates to and transforms the mathematics taught, where the didactic process is described as an organised collective work aiming to construct a *mathematical praxeology (MO)*. A *praxeology* is described by structuring it into *tasks* and *techniques* (the *praxis*), together with its *technology* and *theory* (the *logos*). *A didactical organisation* (*DO*) is a praxeology developed by teachers to organise the work of establishing an appropriate *MO*.

Together with a teacher in a lower secondary school, I designed lesson plans according to PSO's basic lesson structure: 1. Show the problem; 2. Let students guess (a part of) the answer; 3. Give students opportunity to solve the problem and then discuss their solutions; 4. Summarise the lesson with references to the text book. In this presentation, I focus on an episode from a lesson with the topic "subtraction and multiplication with negative numbers". The students have been introduced to the positioning of positive and negative rational numbers on the number line and the interpretation of the absolute value as the distance from the origin. The task is to compare the value of following expressions: task A: $(+6)-(+2)$, $(+6)-(-2)$, $(-6)-(+2)$, $(-6)-(-2)$, and task B: $(+6)\cdot(+2)$, $(+6)\cdot(-2)$, $(-6)\cdot(+2)$, $(-6)\cdot(-2)$. The students' guesses for $(-6)-(-2)$ split into (-8) and (-4). They notice that the solution has something to do with the direction of the signs but it is difficult for them to explain clearly what. One student points at the two minus signs and says "Minus times minus is plus, that is why $-\,(-2)$ will be +2". Another student points out the position of (-6) on the number line and says, "Minus means going to the left, so $-(-2)$ may mean go to the left furthermore, so it is (-8)". Then the teacher asks the class, "What does the minus sign (in front of (-2)) actually express?" A third student says "It means to move from the current position to the opposite side – like in a mirror" and explains on the number line "It is supposed to go to the left side from (-6) by the subtraction, but because of the minus sign in (-2), conversely, it proceeds to the right". His description and method are accepted and used by many of his classmates when the class solves the next task in group B, $(-6)\cdot(-2)$.

The *techniques* are the simple arithmetic operations informed by visualisation on the number line, the *technology* consists of the interpretation of multiplication by (-1) as mirroring, the interpretation of addition by a number as translation left or right by the absolute value and the use of basic algebraic rules like the distributive law and associative law. The *theories* are those of basic arithmetic, the real or rational number system, and (largely implicit) the theory of one-dimensional vectors. The *didactical task* in the *DO* is to make the class start to absorb and for themselves construct this MO concerning negative numbers. The *didactical techniques* are: 1. Consideration of suitable ("rich") problems; 2. Encouraging initial guesses (in spite of the simple expressions, different guesses came out); 3. Techniques to steer and invigorate the whole class discussion; 4. Confirming and institutionalising by using the textbook. These techniques promote students' participation in the mathematical discourse. Without the *guessing* technique and other techniques of invigorating the class discussion, many students might just accept statements without actively participating in the construction of the MO. ATD is a macro theory, which views learning from an institutional perspective. The method of PSO is mostly motivated from a cognitive individual centred perspective, which emphasises students' motivation to participate in the discourse. But the MO provided in PSO fits the epistemological components of ATD.

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The Notion of Height – Through Variation Theory and van Hiele Levels of Thinking

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In this pilot study two different worksheets were tested and pupils' answers were analysed. The analysis focused on what information could be obtained on pupils' conceptions of the notion of height. The two worksheets were constructed using two different theories. The worksheet based on Variation theory captured pupils' understanding in relation to known misconceptions, whereas the van Hiele worksheet captured in what way pupils used formal notation.

Introduction

In Italy, Cannizzaro and Menghini (2006) constructed worksheets based upon the van Hiele theory of levels of thinking in geometry. For this study we choose to use one of these existing worksheets, and to construct a second worksheet with the same mathematical content, using Variation theory. The aim of the study is to see in what way the two worksheets assess pupils understanding of the notion of height. A subsequent question is then, if these worksheets assess differently, how can these differences be described?

Theoretical frameworks

Two theories are used in this study, the van Hiele levels of thinking and Variation theory**.** For this study the first van Hiele levels are of relevance, in which the pupil goes from a visual level of thinking, through a descriptive-analytic level to an abstract-relational level of thinking (van Hiele, 2004). Variation theory describes learning as based on the concepts discernment, simultaneity and variation. The focus of attention must be drawn to the phenomenon and its critical aspects, which can be done through different patterns of variation (Marton & Tsui, 2004).

Data

The pilot study was conducted in a lesson in two 7th grade classes (age 13) in Sweden with a total of 37 pupils. The data were collected during one lesson and resulted in each pupil answering one worksheet, distributed over 19 van Hiele and 18 Variation theory worksheets.

Analysis and results

Each of the questions on the existing van Hiele worksheet was analysed and adapted according to the theoretical ideas of Variation theory. Critical aspects were made possible to discern by making use of separation, fusion and contrast. Three critical aspects concerning height were taken into account when adjusting the worksheet. A *height* is a *straight line, perpendicular to the base* and *not depending on the position*. A new critical aspect appeared in the van Hiele worksheet: pupils marked CD as a height in a rectangle ABCD with base AB.

In our setting there was no dialogue between teacher and student. Pupils did not seem to be able to discern the different aspects of the notion of height themselves and generalization did not take place, in neither of the worksheets. The type of tasks in the van Hiele worksheet was more familiar to the pupils, which might have supported discernment but as the offered aspects on the notion of height were limited, generalization seemed more difficult.

Both worksheets addressed different aspects of the notion of height. The van Hiele worksheet showed in what way pupils were able to use correct notation and confirmed the pupils' existing conception of height. The Variation theory worksheet showed in what way the pupils could explain their answer, and purposely challenged misconceptions. In the Variation theory worksheet it became clear that pupils show a stable conception of the notion of height, although wrong or not complete. Most likely the pupils pupils working with the van Hiele worksheet have the same, wrong, notion but detection was not possible in that worksheet.

In the presentation we will show the choices made in the construction of the Variation theory worksheet. Furthermore the results will be exemplified.

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Teaching/Learning Geometry in Preschool: Children's Experiences and Discernment

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The increased interest among politicians and researchers in children's mathematical learning and achievement, imposes new requirements on preschool teachers' work with teaching. Many of the Swedish preschool teachers, in the study reported here, were educated before 1998 when the present Swedish preschool curriculum was introduced. This means that they did not have mathematics in their teacher education. In the municipality where the preschools in this study are located, mathematics is a priority and they are working with the implementation of the revised curriculum (2010, 2011). The aim of this short communication is to present praxis near research (as a part of my thesis) in three Swedish preschools.

The theoretical framework used in this study is variation theory (Björklund, 2007; Marton et.al., 2004; Runesson, 2006). Variation theory is a theory of learning that links teachers' actions with children's experiences. According to this theory, learning always has an object and in this study it is geometry; geometric figures and in particular circles. The object of learning is experienced and conceptualized by the children in varying ways. *Variation* is a primary factor and it supports children's learning. In order to understand what variations a preschool teacher can use to support learning, a critical point is to understand children's varying ways of experiencing something. That means that teachers have to find out what different experiences children have of the object of learning (geometric figures). The critical conditions are interacting parts of the entire learning process and to experience geometric figures the child must have opportunities to discern critical criteria related to the object of learning. Thus, according to the theory, *discernment* is also another factor that teachers have to be aware of. Some critical aspects of the object of learning need to be discerned *simultaneously*. How the child is experiencing geometric figures depends on if the child at the same time may reflect certain aspects of what is discerned. What aspects that occur simultaneously is another factor that preschool teacher need to have knowledge about.

The overall aim with this research is to find out if it is possible for preschool teachers to work goal-oriented (teach) and also come close to children's perspectives by focus on children's interests and experiences. The research seeks to answer to following questions:

1. How can preschool teachers design goal-oriented work with geometric figures in relation to variation theory?
2. How can preschool teachers work with the concept of circles (geometric figures) and variation in everyday activities?

The study is based on observations and interviews with 15 preschool teachers from three preschools. The teachers are working with the object of learning (geometrical figures) in fairy tales and other everyday activities. They design the learning situation together in the working team, decide which critical aspects they have to make visible for the children and how they will carry out the teaching. In this short presentation the selection of case includes different phases of the game *Cirlcehunt.* The purpose of the game is that children will have opportunities to discern circular shapes and learn circular criteria in a play situation. In this example one preschool teacher and six four year old children are playing together. The game starts with a *fairytale* and the goals are to find out what different experiences children have of the learning object and to give them opportunities to discern different shapes. After the fairytale the preschool teacher presents the object of learning and shows a circle's attribute (critical aspects), i.e. perfectly round with a curved line. Then the children get a problem to solve - sorting circles in different sizes and colours. They have to reflect on differences and similarities between circles. The preschool teacher listens, reflects and evaluates. She highlights some conclusions and after that she extends the game with new problems for the children to solve. Now they have to discern and pick circles from a sample of geometric figures like for example a hexagon and an ellipse.

The findings so far show that the preschool teacher's intentions and what they really do is not always what the children learn. The preschool teachers need knowledge of children's different experiences and they have to have the same focus as the children. They also need theoretical knowledge together with their practical knowledge in order to make the didactical situations become learning situations. The theoretical knowledge includes mathematical knowledge and knowledge about variation theory.

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Analysing the Discourse of Teacher Training

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Introduction

Traditional research on identity has focused on the individual (Phillips, 2007). However, in later years, there has been increased focus on the social setting (Skott, 2009). According to Morgan (2010), social perspectives on identity provide us with more and specifically different information about the development of an identity.

The present study adopts a qualitative approach, researching prospective teachers (primary school teachers, years 4-6) and follows them before, during and after courses, lectures, seminars, internships, study groups and examination work. The empirical data come mainly from interviews with voice/video recording, observation with field notes, written examinations in the subject of mathematics education. The intention is to study both the affordance and the prospective teachers' agency during the process of developing a teacher identity within teacher training. Affordance is here its potential use to a prospective teacher in a given teaching situation (Van Leeuwen, 2005). Agency is about the ability to make choices and act upon them within the teaching situation (Kress, 2010).

The aim of this presentation is to highlight the social semiotic perspective of System Functional Linguistics (SFL) as a theoretical perspective for studying and analysing prospective teachers' identity development. In this study SFL is a way of addressing and unfolding situated communication, to disentangle students' participation in past and present discourses. Morgan (2006) emphasises that this unfolding can serve as a crucial window when following processes, in this case to become a primary teacher.

Social semiotics, identity and SFL

According to Morgan (2006), there is a growing interest within the field of mathematics education in semiotics, which is about the systematic study of signs. A researcher can assume different perspectives on semiotics, such as the pure intention of the sign, the cultural interpretation of the sign and the social interpretation of the sign. The latter perspective highlights interaction and communication. Social semiotics is not a theory (Van Leeuwen, 2005) and needs to be applied to another field of interest, in this case, mathematics education.

When prospective teachers engage in communication about teaching and learning mathematics or engage in semiotic production provided by the mathematics educators, the outcome is a result of the enhancement of their capacity to act as a mathematics teacher. When learning, the capacity is at the same time a change in identity. In this way, identity and knowledge are linked together. Kress (2010) defines identity "as the outcome of constant transformative engagement by someone with 'the world', with a resultant enhancement of their capacities for acting in the world" (p. 174).

SFL draws upon the notion that a text is not something predefined. Rather, it is something that will be constructed in interaction with others. SFL regards a text as being handled in three different processes, so called meta-functions, simultaneously (Morgan, 2006). These functions will be presented through the following short excerpt from the first interview with prospective teacher Lisa: "I found it messy, stencils about this and that, very messy. In the early years, there was no structure. Everyone knows that you need a textbook in mathematics, to give it structure".

The ideational meta-function concerns statements that address the interpretation of one self in different discourses. An example of this is when Lisa expresses her view on teaching without textbooks: "I found it messy, stencils about this and that, very messy". When using language you negotiate or address social relations. Lisa continues: "In the early years, there was no structure". This is more active than the ideational function. It addresses persons or situations, in this case Lisa's early school years, and is called the interpersonal meta-function. The short excerpt is about something and addresses a situation and this is done within the textual meta-function which concerns the coherency of situated communication, in this case a meta-assumption: "Everyone knows that you need a textbook in mathematics, to give it structure".

SFL is used in this study to unfold situated communication, to be able to follow the process of becoming a primary teacher.

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On the Use of Emphasizing Brackets when Learning Precedence Rules

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Introduction

Brackets can be used with different intentions in mathematical expressions. Here we focus on two different intentions with the symbols – *brackets as part of* the precedence rules and *brackets to emphasize* precedence (what we here call emphasizing brackets). For instance the arithmetic expression $3+5\cdot 2$, according to the precedence rules (multiplication first, then addition), should yield a result of 13. If we would insert a bracket as in $(3+5)\cdot 2$, the expression should be calculated in order of brackets first and then multiplication, and give a result of 16. This follows the normal precedence rules that "brackets precede multiplication" and "multiplication precedes addition". If we instead would insert brackets as in $3+(5\cdot 2)$, the expression should be calculated to equal 13. In this latter expression the brackets have been used to *emphasize* the precedence rules in contrast to the expression above where the brackets were part of the precedence rules. The emphasizing brackets could be considered as mathematically useless. However, it has been suggested that emphasizing brackets should be inserted for didactical reasons into arithmetic expressions (Linchevski & Livneh, 1999). In addition, it has previously been demonstrated that emphasizing brackets can enhance the structure sense in algebraic expressions (Hoch & Dreyfus, 2004) as well as in basic arithmetic expressions (Marchini & Papadopoulos, 2011).

Aim of the study

The aim of this study is to explore if the didactical intention on using emphasizing brackets could be an obstacle for students when starting to learn the precedence rules. That is, the intention is to test in which way the introduction of, didactically motivated but otherwise useless, emphasizing brackets to students has a positive effect on calculating arithmetic expressions with mixed operations.

Methodology

The data was collected in a quasi-experimental study of young students at the age of 13-14 with a test group and a control group. Both groups were given the same pre-test (including 16 arithmetic tasks) and later post-test (including 16 other arithmetic tasks). In between, both groups were exposed to instructions on a simplified version of the precedence rules (first bracket, then multiplication, then plus and minus) including four examples of the type $(a+b)\cdot c$, and four examples of the type $a+b\cdot c$. In the test group the examples were articulated such that multiplication has higher priority and therefore can be interpreted as using extra (emphasizing) brackets around the product, though the word emphasizing was not used. In the control group brackets were not mentioned in the second type of examples. The pre- and post-tests did not contain any exercises with emphasizing brackets. In total, 169 students were investigated in this study.

Preliminary results

Computing from left to right was dominant in the students' pre-tests. The students who were exposed to the instructions using emphasizing brackets were less prone to abandon a left-to-right strategy for a precedence rule when computing $a\pm b\cdot c$ type arithmetic expressions without brackets. In detail, the increase (from pre-test to post-test) in the number of answers that can be associated to a precedence rule computation was about the same in both groups (increase by a factor of 2.2 and 2.1 in the test group and the control group, respectively). However, the number of left-to-right-related answers was decreased by a factor 2.4 in the test group compared to a factor 5.2 in the control group. Hence we cannot find support in our data for the didactical suggestion given by Linchevski and Livneh (1999) to insert brackets in order to emphasize the precedence rules in arithmetic expressions with mixed operations.

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Mathematical Reasoning Requirements to Solve Tasks in Physics Tests

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Introduction

Mathematics and Physics are closely intertwined and mathematical reasoning is assumed to be essential in the work of professional physicists, in addition when learning physics and thus when solving physics tasks. Some of the difficulties students encounter when learning physics likely relate to their use of mathematics and how they reason mathematically (Bing, 2008; Nguyen and Meltzer, 2003). Lithner (2008) discusses how learning difficulties in mathematics can be explained by what kind of mathematical reasoning is used by students. Another assumption is that National tests have an effect on both teaching and learning, not at least in stressing what is covered in the taught curriculum. This on-going study thus investigates the nature of mathematical reasoning needed from a student in Swedish Upper Secondary School, in order to solve physics tasks on Physics tests from the National Test Bank, in cases where mathematical reasoning is applied.

Theoretical framework

The study presented here uses the definition of mathematical reasoning and a framework developed by Lithner (2008). Depending on which mathematical foundation that is used, the framework distinguishes between Creative Mathematically Founded Reasoning (CMR) and Imitative Reasoning (IR). To be CMR there has to be some novelty in the solution and the argument supporting the strategy should be plausible and anchored in intrinsic mathematical properties. If it is enough just to recall an answer and writing it down, or if following an algorithm step by step will give the right answer without any demands of novelty the task is categorised as IR. The research question for the study is: *What is the distribution of tasks requiring either Creative mathematically founded reasoning or Imitative reasoning in the Physics tests from the Swedish National Test Bank?*

Method

The distinction described above is in this study used when analysing the kinds of mathematical reasoning required of upper secondary school students in order to solve tasks on ten tests from the Swedish National Test Bank in Physics. The object of study is the reasoning requirements of an average student and no students with their actual solutions are included. The method for the analysis is earlier used in e.g. Palm, Boesen and Lithner (2011). The tests in the National Test Bank are developed by the Swedish National Agency of Education as an assessment support and most of the material is classified as secret. Of the 36 tests developed so far, ten were randomly chosen and each test comprises approximately 21 tasks. In the analysis both textbooks in mathematics and physics are considered and also a physics handbook, which students are allowed to use during the tests. Physics tasks solvable without using mathematical reasoning, i.e. solutions only including physics facts or mathematical subject areas not covered in the textbooks in mathematics, are categorised as non-mathematical reasoning.

Result and Analysis

A preliminary result indicates that it is necessary to reason mathematically to solve three-fourth of the tasks. Approximately two-fifth of the tasks could be solved with IR and one-third required CMR. Considering the reduction of complexity, to equate the learning history with the textbooks, there could be a larger number of tasks for which it is sufficient with imitative reasoning. As mentioned above, previous studies have shown that imitative reasoning and rote learning can lead to learning difficulties in mathematics. It can then be reasonable to assume that using mathematical reasoning based on surface properties when solving physics tasks also can contribute to learning difficulties of physical concepts.

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Mathematical Knowledge Requirements for Learning Activity Design Supported by ICT

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Introduction

As part of an ongoing project in a lower secondary school related to the teaching and learning of mathematics supported by Information and Communication Technologies (ICT), I am currently engaged in the collaborative design of a specific mathematical learning activity with a focus on algebra. In this project, I work together with three mathematics teachers, following the methodology of co-design (Roschelle & Penuel, 2006). One reason for using this approach is the need for combining expertise in the areas of mathematics, pedagogy and especially technology.

Initial discussions with the teachers have resulted in the need to highlight the mathematical knowledge needed by teachers for their participation and contribution to this specific project. We specifically address the following research question: *What mathematical knowledge is required of a teacher in order to participate in the co-design of a mathematical learning activity supported by ICT?*

Theoretical background

As a theoretical framework I have used the MKT model (Figure 1), which provides a categorisation of Mathematical Knowledge for Teaching (Ball, Thames, & Phelps, 2008). This model does not explicitly address the issue of using ICT in mathematics teaching. Thus I have integrated elements from the framework for Technological Pedagogical Content Knowledge (TPCK: Koehler & Mishra, 2008). The MKT model and the TPCK framework are both based on Schulman's notion of PCK and address complementary issues of knowledge needed for teaching mathematics.

CCK: Common Content Knowledge
HCK: Horizon Content Knowledge
SCK: Specialized Content Knowledge
KCS: Knowledge of Content and Students
KCT: Knowledge of Content and Teaching
KCC: Knowledge of Content and Curriculum

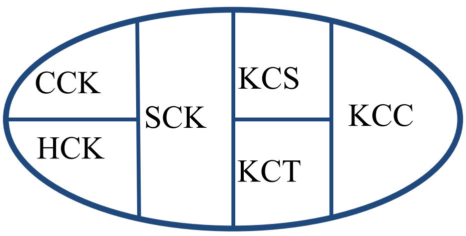

Figure 1. The MKT model

The TPCK framework focuses on knowledge about the affordances and constraints of technologies (Koehler & Mishra, 2008). Within the subject domain of mathematics, we specifically needed to consider affordances for representation and communication (Hegedus & Moreno-Armella, 2009). Technologies provide representational affordances for multiple and multi-modal representations, simulations, manipulation of data, and conversions of representations (Koehler & Mishra, 2008). They also provide affordances for communication, which may be regarded primarily as an issue within Specialized Content Knowledge (SCK) (Potari, Zachariades, Christou, & Pitta-Pantazi, 2008).

Preliminary results and analysis

Parts of a specific conversation with one of the teachers concerned the students' inability to make sense of the distributive law. The teacher was asked to explain how the equality could be justified. The only explanation the teacher could provide was an instruction, illustrated by arrows, how to manipulate and "move" the a in the expression $a\ (b + c)$ onto b and c and thus forming the expression $ab + ac$. Furthermore, the three teachers seemed to favour certain representations before others, and their choice of representations seemed to be based on individual taste rather than mathematical considerations.

The theoretical background emphasizes the role of representations. With this in mind, the preliminary results imply that teachers' mathematical knowledge requirement would be to consider and be able to judge and compare the didactical value of various representations in order to move forward and explore the affordances for representations provided by ICT. In this case I see a need for competence development of the SCK in terms of mathematical representations.

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Explanation as a Ground for Beauty?

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This work-in-progress report explores the question of whether mathematical explanation is, or should be, linked to the notion of mathematical beauty. If there is a connection between explanation and beauty, it is not a straightforward one – we will provide examples of explanatory proofs which are not beautiful, and beautiful proofs which are not explanatory. Still, the argument goes, there is some essential connection between beauty and explanation, and this connection can help render the question of what mathematical beauty consists in more tractable.

The main distinction we draw upon is between proofs that explain and proofs that demonstrate (Steiner, 1978; Hanna, 1990). Our claim is that proofs that explain are often found more aesthetically pleasing than proofs that merely demonstrate. To illustrate this point we will draw upon several examples from the literature and some pilot data of mathematicians making aesthetic judgements about different proofs. One such example is based on the following question:

> Suppose you decided to write down all whole numbers from 1 to 99999. How many times would you have to write the digit 7?

Consider the following two solutions (adapted from Dreyfus and Eisenberg, 1986), the first of which we refer to as a "Bookkeeping" solution, which systematically counts all of the 7's, but does not provide a sense of explanation:

> <u>Solution 1</u>: (Bookkeeping)
>
> Between 1-99: 20 (10 in 1's place; 10 in 10's place)
>
> Between 1-999: 300 (20 for each 100's, plus an extra 100 in the interval 700-799)
>
> Between 1-9 999: 4 000 (300 for each 1 000's, plus an extra 1000 in the interval 7000-7999)
>
> Between 1-99 999: 50 000 (4 000 for each 10 000, plus an extra 10 000 in the interval 70 000 – 79999)

The second solution provides more structure than the first, appealing to the symmetric character of the number 7 (it is not privileged over the other digits, we could have very well asked how many 3's there are). This solution is more explanatory than the first, in that it provides a sense of why we get this particular result, and allows for generalization (for instance it is easy to see from this solution how many times the number 7 appears between 1 and 100 000 000).

> Solution 2:
>
> Include 0 among the numbers of consideration (this won't change the answer since 7 is not a digit of 0). Now suppose all numbers from 0 to 99,999 are written down with five digits each, e.g. 306 is written as 00,306. In this set of all combinations every digit will take every position equally often, so every digit must occur the same number of times. There are 100,000 numbers that have 5 digits each, that is 500,000 total number of digits. Each of the 10 digits appears equally often, so each one appears 50,000 times. In particular, this is true for the number 7.

The explanatory proof was rated as more aesthetically pleasing to mathematicians in Dreyfus and Eisenberg's paper. This result was confirmed by the current author in a pilot study conducted with six mathematicians, and with a workshop of high school teachers. A similar pilot study was conducted using different proofs of the square root of 2 being irrational, and an explanatory proof was considered more aesthetically pleasing than several non-explanatory proofs [1].

The connection, if there is one, between beauty and explanation is not at all clear. But these pilot studies provide some empirical evidence that a connection exists, perhaps along the lines of what Rota (1997) refers to as "enlightenment". This would add a fourth category to what Natalie Sinclair (2001) has listed as three roles of the aesthetic: (1) motivating the choice of certain problems to solve; (2) guiding the mathematician to discovery; and (3) helping a mathematician decide on the significance of the result. The fourth category would be to engender understanding, something we know that explanation provides, while demonstration does not.

Note

1. For space considerations this example is left out here, but was given in the oral presentation.

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An Inclusive Perspective on a Pedagogy for Students in Special Needs in Mathematics

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This paper deals with a part of an on going project that investigates mathematics education for students in special needs in mathematics. The aim here is to initiate the development of an explanatory framework for understanding critical factors in the learning of mathematics for students in special needs in mathematics from an inclusive perspective. In order to investigate this we will take the perspective of pedagogues involved in the situation of students in special needs in mathematics.

A relational perspective on difficulties in mathematics stresses the need to consider in detail how the teaching and learning activities in question affect the students' learning of mathematics (Dalvang & Lunde, 2006). The present project adheres to the relational view in striving to reach an understanding of difficulties in mathematics from an inclusive perspective (Lindeskov, 2006).

In an inclusive perspective, students and their mathematical understanding are not considered isolated and de-contextualized units. Mathematical understanding is viewed as a cultural and social phenomenon. As a social phenomenon, we seek explanations to students' learning difficulties in the teaching of mathematics, where "we strive to identify and remove all barriers to learning for all children" (Ballard, 1999, p. 2). When inclusion is effective from a learning perspective, all students actively belong to and participate in the current practice and students' different abilities are seen as assets (Farrell, 2004). The base of inclusion is then to value diversity rather than assimilation (Ballard, 1999). Nilholm (2006) emphasizes this by stating that the inclusive school is based on the diversity of children. Consequently, with an inclusive perspective in mathematics education, all students' skills and abilities are taken into consideration promoting learning in the specific teaching situation. However, we claim that there is still much to learn regarding the meaning of inclusion and the identification of factors that appear critical in the students' learning and how different factors work and connect regarding inclusive teaching in mathematics.

This investigation of inclusion in mathematics education is grounded in a socio-constructivist perspective on learning. This means that we look carefully into how the learning of mathematics is integrated with how the learner perceives social and cultural demands, expectations and possibilities of the situation in which the learning takes place (Nilsson, 2009). The overall principle of this perspective is that learning is considered to be a process of belonging and participation. For the learner this means an engagement and contribution to the practice (Wenger, 1998). The engagement process involves both acting and knowing in the practice, which includes "both the explicit and the tacit" (ibid., p. 47). This means that the practice includes all visible representations and all implicit and underlying elements (Wenger, 1998).

We will investigate inclusion from the perspective of pedagogues. In terms of participation and contextualization this means that we look at how the teacher and the remedial teacher in mathematics allocate the problem of including students in special needs in mathematics to the mathematical practice of the class. To do this we need a more fine-grained framework, identifying ways to participate in the mathematical practice, to be included. Asp-Onsjö (2006) makes a distinction between *spatial, social* and *didactical* inclusion. Spatial inclusion basically refers to how much time a student is spending in the same room as his or her classmates. The social dimension of inclusions concerns the ways in which students are participating in the social interaction with the others. Didactical inclusion refers to the ways in which students' participation relates to a teacher's teaching approach and the way in which the students engage with the teaching material that the teachers may supply for supporting their learning of mathematics. These three analytical categories will serve as a base in developing a more fine-grained explanatory framework, aimed at increasing our understanding of how students in special needs in mathematics are participating, develop their way of participating or become restricted from participating in the school mathematics practice.

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Developing Student Interaction in Multilingual Upper Secondary Mathematics Classrooms

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Student communication in upper secondary mathematics classrooms

In this short communication, we review some of the literature on student communication in upper secondary, multilingual mathematics classrooms. We suggest that the amount of research in this area is limited. As a consequence, we suggest some research questions for a research project in multilingual schools in Malmö.

We see this topic as being valuable in understanding how to improve teaching of mathematics in upper secondary schools in Sweden, where Skolinspektionen (the Swedish School Inspection) has highlighted a number of issues. Regarding the first mathematics course, Skolinspektionen (2010) stated that students did not gain sufficient prerequisite skills, such as problem solving or being able to express themselves orally and in writing. Individual work dominated classroom work and discussions about mathematics were not given enough time. As well, students claimed that the mathematics lessons were boring and monotonous and that they only focused on getting correct answers and not on determining the reasonableness of their answers.

With a change in the mathematics syllabus in 2011 (Skolverket, 2011), communication in mathematics has been given more attention and, in the national mathematics tests, an oral component has been introduced. This is similar to approaches in other parts of the world, which also value communication in supporting students to think mathematically. However, little research appears to have been conducted in upper secondary classrooms on this issue (see Goos, Galbraith & Renshaw, 2002; Forster & Taylor, 2003).

Multilingual classrooms

The need for more communication seems particularly pertinent in multilingual classrooms. Van Eerde and Hajer (2009) claimed that "learning mathematics and second language appropriation cannot be separated" (p. 270). In Sweden, 18 percent of students in upper secondary schools have a *foreign background*, which means that they are born abroad or born in Sweden with both parents born abroad (Skolverket, 2010). In Malmö, 46 percent of the students have a foreign background.

Generally, research on communication in the multilingual mathematics classroom has been conducted in primary and lower secondary schools. This research suggests, for example, that it is “crucial that the pupils verbalize their ideas and thoughts and address dialogue partners” (Brandt & Schütte, 2010, p. 91). Also in multilingual classrooms, *code-switching*, which means that students switch between two or more languages, appears to be valuable (see Setati, 2002).

Establishing a research project on student communication

From the literature, we have found that research on student-to-student interaction in multilingual upper secondary classrooms is limited. We therefore suggest that there is a need for a research project that aims to answer the following research questions:

How do teachers and students perceive the impact on students' learning from increasing student-to-student interactions in multilingual secondary mathematics classrooms? What pedagogical practices do teachers and students find effective? What are the benefits for students who do not have Swedish as their dominant language in “talking more mathematics”?

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Mathematics Textbooks Related to Algebra Content

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Introduction

Mathematics textbooks as educational resources and artefacts are widely used in classroom teaching and learning. What is presented in a textbook is often taught by teachers in the classroom. Similarly, what is missing in the textbook may not be presented by the teacher. Textbook content reflects pedagogical intention. This study is based on an assumption that pedagogical content knowledge (PCK) (Shulman, 1986) is embedded in the subject content presented in textbooks. Textbooks contain both subject content knowledge (CK) and pedagogical content knowledge (PCK). The embedded PCK in textbooks varies depending on which teaching culture a textbook reflects. Quadratic equations as part of algebra content are taught at Swedish upper secondary school. This study is about analysing algebra content concerning different methods, including factorisation, for solving quadratic equations presented in Swedish mathematics textbooks, using the constructs PCK-CK (Shulman, 1986; Mishra & Koehler, 2006) as analytical tools. The study has been presented in the form of a licentiate thesis (Sönnerhed, 2011).

The primary aim of the study was to explore what pedagogical content knowledge (PCK) regarding solving quadratic equations is embedded in the mathematics textbooks. The secondary aim was to analyse the algebra content as subject content (CK) from the perspective of mathematics as a discipline related to the historical development of algebra.

Research methods

Content analysis with the PCK-CK as theoretical tools has been applied for the study. The criteria for analysing mathematics exercises in the textbooks were based on the previous research on textbook analysis. Four rounds of analyses were carried out on 12 Swedish upper-secondary mathematics textbooks. One of them was selected for a deep analysis. The results were generated accumulatively in every round of analysis.

Results

The results show that the selected textbooks all presented four methods for solving quadratic equations. There was an accumulative relationship among these methods with a final goal of presenting how to solve quadratic equations by the 'quadratic formula' (often called the pq-formula). It was found that one of the textbooks contained an overall embedded teaching trajectory with five sub-trajectories in the presentation of solving quadratic equations with the four solving methods (i.e. the square root method; using a factorization method to solve simple quadratic equations; completing the square method, and a direct solution with the quadratic formula). Instead of factorization, among the four methods the quadratic formula is emphasised as a final goal in the overall trajectory. The five sub-trajectories were organized and connected by four historically related geometrical models according to a part-whole relationship. These four geometrical models of areas for rectangles and squares represent basic algebra rules for building up the four different solving methods. That way, a complete teaching sequence on solving quadratic equations was offered in the textbook. The result of presenting the quadratic formula, the last among the four solving methods from the 12 textbooks, may imply that teaching in Swedish classrooms puts focus on solving quadratic equations by the quadratic formula (the pq-formula), which however will need further empirical evidence.

Teaching quadratic expressions has different focuses in different mathematics classroom cultures. In Singapore and China, for example, teaching the factorisation method (also called cross-multiplication method) is emphasised (Kemp, 2010; Leong et al., 2010). Consequently, this may lead students to solve quadratic equations with the factorisation method in focus. Learning to use the pq-formula may implement an instrumental understanding of quadratic equations while the factorisation method may provide students with opportunities for understanding quadratic structures and preparing them for the future study of factorising polynomials of higher degrees. However, the hypothesis requires further research. The continued study will compare the same algebra content between Chinese textbooks and the analysed Swedish textbooks.

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University Mathematics Teachers' Discourses of Functions – What is Made Possible to Learn?

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My thesis project concerns the teaching of functions in undergraduate mathematics, viewed as a discursive practice. In previous studies, early reports of which have been presented at CERME 7 (Viirman, 2011a) and PME 35 (Viirman, 2011b), I have used commognitive theory (Sfard, 2008) to describe the teaching practices of seven university mathematics teachers, from three different universities in Sweden, focusing on what characterizes the different discourses of the teachers regarding functions.

More precisely, I have used Sfards' characterization of discourses in terms of *word use*, *visual mediators*, *narratives* and *routines* (Sfard 2008, pp. 133-135). The analysis of this part of the project is not yet complete, but as examples of findings I can mention a classification of routines (discursive patterns), containing for instance *construction*, *substantiation* and *motivation* routines (Viirman, 2011a), as well as an analysis of the way word use impacts on the clarity and accessibility of teaching (Viirman, 2011b).

Having so far given a mainly descriptive account of the discursive practices of the teachers, in the last part of the project I wish to investigate the possibilities of learning afforded by these practices. To this end I intend to use variation theory (Marton, Runesson, & Tsui 2004), a theory focusing on conditions for learning, and also putting great emphasis on the role of language and discourse in learning, something which fits well with the discursive perspective I have used so far. I argue that the central theoretical constructs of variation theory can be made to fit into a discursive, participationist theoretical framework.

From the commognitive perspective, learning is changing one's discourse, and the *object of learning* central to variation theory then becomes changed in specific aspects of the discourse, for instance regarding certain discursive objects, like functions. From a discursive perspective, the *intended* object of learning is manifested in the teacher's discursive practices, while the *enacted* object of learning becomes the researcher's description of the discursive practices as they enfold in the classroom. This enacted object of learning, also called the *space of learning* (Marton, Runesson, & Tsui, 2004), is constituted by patterns of variation, making it possible to discern the critical aspects of the object of learning. These patterns of variation can be seen as aspects of the discursive activity: "the space of learning, which comprises different dimensions of variation, is constituted by linguistic means in the interaction between teacher and students." (Marton, Runesson, & Tsui, 2004, p. 24) The similarities are obscured by the objectified language used in variation theory, but once you realize that the space of learning can be viewed as a description of a discursive activity, the compatibility is more easily seen.

What I intend to do, then, is analysing the discourses of the seven teachers, looking for the patterns of variation, examining which aspects of the function concept are made possible to discern, and which remain hidden. As for determining these critical aspects, there is a lot of research done, both by myself (Viirman, Attorps, & Tossavainen, 2010) and a great many others (e.g. Harel & Dubinsky, 1992) concerning different aspects of the learning of the function concept, and this research will be tapped into for this purpose. Here I will only briefly give two examples. One concerns arbitrariness, a characteristic feature of the modern concept of function, which among other things means that the domain and range of a function can be any type of sets. If the term 'function' is used synonymously with 'real function of one real variable', as it is by some of the teachers in my study, without making this restriction clear, then this aspect is not made visible. A second example concerns linear transformations. One teacher in my study speaks of vectors $(x,y,0)$ in 3-space as two-dimensional vectors, thereby giving the impression that 2-space is always embedded in 3-space. This obscures the variation in dimension, since 3-space is not spoken of as embedded, making the transition to higher dimensions seem less natural.

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