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Editors: Christer Bergsten, Eva Jablonka, Tine Wedege

Mathematics and Mathematics Education: Cultural and Social Dimensions

Proceedings of *MADIF 7*

The Seventh Mathematics
Education Research Seminar
Stockholm, January 26-27, 2010

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Preface

This volume contains the proceedings of *MADIF 7*, the Seventh Swedish Mathematics Education Research Seminar, held in Stockholm, January 26-27, 2010. The seminars, organised by the Swedish Society for Research in Mathematics Education (SMDF), aim at enhancing the opportunities for discussion of research and exchange of perspectives, amongst junior researchers and between junior and senior researchers in the field. The first seminar took place in January 1999 at Lärarhögskolan in Stockholm and included the constitution of the SMDF. The second meeting was held in Göteborg in January 2000, the third in Norrköping in January 2002, the fourth in Malmö in January 2004, the fifth in Malmö in January 2006 and the sixth in Stockholm in January 2008. Printed proceedings of the seminars are available for all but the very first meeting.

The members of the 2010 programme committee were Christer Bergsten (Linköping University), Eva Jablonka (Luleå University of Technology), Katarina Kjellström (Stockholm University), Thomas Lingefjärd (University of Gothenburg), and Tine Wedege (Malmö University). The local organiser was Katarina Kjellström.

The programme of *MADIF 7* included two invited plenary lectures (Paul Dowling and Tine Wedege), one plenary panel (Jo Boaler, Paul Dowling, Stephen Lerman, with Christer Bergsten as moderator), 20 paper presentations, and 18 short presentations. As the research seminars have sustained the idea of offering formats for presentation that enhance feedback and exchange, the paper presentations are organised as discussion sessions based on points raised by an invited reactor. The organising committee would like to express their thanks to the following colleagues for their commitment to the task of being reactors:

Katalin Földesi, Gunnar Gjone, Simon Goodchild, Ola Helenius, Johan Häggström, Barbara Jaworski, Maria Johansson, Troels Lange, Steve Lerman, Tamsin Meaney, Peter Nyström, Elisabeth Persson, Per-Eskil Persson, Kerstin Pettersson, Mikaela Rohdin, Frode Rönning, Erika Stadler, Allan Tarp, Paola Valero.

In this volume the two plenary addresses, 19 papers, and 17 short presentations are included. All contributions were peer-reviewed by two or three researchers before presentation, and by members of the programme committee before publication. The editors are grateful to the following colleagues for providing reviews:

Paul Andrews, Iiris Attorps, Jonas Bergman Ärlebäck, Ewa Bergqvist, Tomas Bergqvist, Christer Bergsten, Maria Bjerneby Häll, Lisa Björklund Boistrup, Ole Björkqvist, Morten Blomhøj, Jesper Boesen, Johann Engelbrecht, Peter Frejd, Sharada Gade, Peter Galbraight, Gunnar Gjone, Simon Goodchild, Brian Greer,

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The organising committee and the editors would like express their gratitude to the organisers of *Matematikbiennalen 2010* for financially supporting the seminar. Finally we would like to thank all participants of *MADIF 7* for sustaining their engagement in an intense scholarly activity during the seminar with its tight time table, and for contributing to an open, positive and friendly atmosphere.

Christer Bergsten, Eva Jablonka, Tine Wedege

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Abandoning Mathematics and Hard Labour in Schools
A *New Sociology of Knowledge* and Curriculum Reform

Paul Dowling
Institute of Education, University of London, UK

This paper introduces and develops aspects of social activity method (SAM) in transaction with mathematics education. The outcome of this transaction is the identification of three key issues: disciplinarity, the tendency of practices—especially school mathematics—to mark themselves out from rather than constituting themselves in functional relationship to other practices; the new massification of schooling, the widespread distribution of erstwhile elite performances; and conceptualisation, the insistence on the acquisition of mythologised (in this case, mathematical) objects that is a consequence of the 'forensic science' of assessment practices that are themselves facilitated by 'teaching' as the particular mode of pedagogic practice that is prevalent in schooling. In addressing the third issue, the paper will also give some attention to the esoteric domain of school mathematics, which has been under-developed in earlier work.

The expression 'new sociology of knowledge' is emphasised in my title because it could, in terms of what I want to say, stand as the whole title. The main title is a proposal that derives from my sociology; that which follows 'new sociology of knowledge' is perhaps something of a wish, a wish that, looking at the curriculum from this sociological perspective might lead to a change; as a sociologist, I am not optimistic. As for 'knowledge', this, ironically, is not to be thought of as the object of my sociology, so much as that which is imagined by the practices that I want to consider; it's a myth.

For a general audience (and perhaps for a less general audience as well), I should explain what I mean by 'sociology'. This is how I described it in 1998:

> By the use of this term I mean that the theoretical space in which I am interested is concerned with patterns of relationships between individuals and groups and the production and reproduction of these relationships in cultural practices and in action. (Dowling, 1998, p. 1)

More recently (e.g., see Dowling, 2009) I have taken to borrowing from cybernetics and describe the theoretical space—the sociocultural—as that which is defined by strategic, autopoietic action directed at the formation, maintenance and destabilising of alliances and oppositions, the visibility of which is emergent upon the totality of such action, rendering them available as resources for recruitment in further action—same thing, really. This is my central principle or 'guiding thread', but it is also important to make explicit what I take to be the status of the work that I produce: essentially, what I am aiming to do is generate principled interpretations of the empirical, the principles being constituted as a developing theoretical framework, or method—*social activity method*, SAM. The approach makes no claims to exclusivity, either theoretically or empirically; its deployment is interrogative, rather than prescriptive; and its use-value is to be assessed pragmatically. SAM is not science, but then, as has been widely documented, science is not science, in terms of the stereotypical ways in which we frequently think of it, either [1].

What I propose to do in this paper is to present an interpretation of school mathematics and, in doing so, also to introduce some of the theoretical structure that has arisen out of the transaction of my central principle with school mathematics as an empirical setting and that, only subsequently, motivates the interpretation. My general argument in this paper is intended to lead to the following conclusions. Firstly, school mathematics is better thought of as marking itself out from other practices rather than as functional to them; I shall refer to this as the prevalence of disciplinarity. This situation would seem to call into question the role of mathematics in holding a compulsory and core place on the school curriculum. Secondly, trends that privilege the use of metrics in public discourse on education may be leading to a change in the distribution of erstwhile elite performances in mathematics and in schooling more generally that I shall refer to as the *new massification* of schooling. To the extent that this is a valid observation, schooling becomes less of a mechanism for selection and differentiation—perhaps other than at genuinely elite levels of performance—and more of an industry in service of governmental propaganda. Thirdly, I shall argue that it is the particular form of pedagogic relations in schooling that opens up a space for the development of the forensic science of assessment and the consequent mythologising of competence in the form of *conceptualisation*. In deconstructing this category in school mathematics I will argue that we need to move away from the dominance of *push* strategies that privilege the artificial subjects of school knowledge and towards a kind of practitioner-research-based curriculum that might more appropriately serve as an introduction to the diversity that is the collection of legitimate human activities.

A pedagogic device, recontextualisation and disciplinarity

A central point of departure for SAM is the work of Basil Bernstein (1971, 1977, 1990, 1996, 2000). I shall not reprise my engagement with Bernstein's theory here [2] but will illustrate it by reference to and departure from his category, the pedagogic device. The device is analogous, in some respects, to Chomsky's language acquisition device (to which Bernstein (1990) makes explicit reference), except that it is a social rather than a psychological mechanism and, according to Bernstein, constitutes a site of struggle for those having objective interests in the form taken by schooling. The device comprises three sets of rules, rules of: recontextualising, distribution, and evaluation. Very briefly, recontextualising rules delocate discourses from their fields of production—the university, say—to establish pedagogic discourse within the field of reproduction—the school. Pedagogic discourse is distributed to school students, differentiated on the basis not only of age, but also of socioeconomic class, gender, and other objective categories of social difference. Finally, evaluation rules determine what counts as successful performances. This simple structure [3] is persuasive in the organising of descriptions of major curriculum developments, such as the modern mathematics movement of the 1960s, as I have illustrated in Dowling (2008a). The question, however, is what does this achieve? Bernstein's project is also one of interpretation, but his interpretive framework is too distant from the empirical. As I have argued in Dowling (2009), Bernstein's primitive categories—classification and framing—are too easily operationalisable through such oppositions as between/within, space/time, what/how and recognition/realisation so that they put no pressure on the empirical and, partly in consequent of this, fail to learn from it. The theory, like so much social theory, stands apart from the empirical.

Bernstein's theory is predicated upon generative social structures—such as the pedagogic device, but also more general characteristics, such as 'the division of labour in society'. It seems to me that the postulation of generative structures is radically inconsistent with an interpretive approach. Rather than inspiring interpretation in front of the empirical, so to speak, Bernstein uses data to illustrate or access causal entities that lie behind it; his approach is an example of what I refer to as forensics. In what follows, I shall attempt to illustrate the move from forensics to what I call constructive description. As I have indicated, my starting point will be the pedagogic device and its three sets of rules. A further concern that I have with the pedagogic device is the apparent arbitrariness of its threefold construction. This is of crucial interest in respect of a theoretical construct; my approach strives to achieve motivated theoretical completeness. However, my departing from the pedagogic device will involve a shift into the empirical field, so that 'recontextualisation', 'distribution' and 'evaluation' will be taken to index three empirical sites. Completeness in the context of the empirical is addressed in terms of sampling strategies. However, I am not here aiming, here, at empirical completeness in terms of either representation or criticality (Dowling & Brown, 2010, see also Brown & Dowling, 1998), but intend, rather, to foreground conceptual interpretations.

The first component of the pedagogic device is its recontextualisation rules. For Bernstein, recontextualisation is achieved via the apparent action of the device—a social organ. My concept involves the transformation of a text or practice implicated in one activity by agents of another so that the text or practice is brought into alignment, in some ways, with the recontextualising activity. Clearly, we need to take a look at both activities in order to reveal the effects of recontextualisation. In order to do this, I shall consider a mathematics lesson described by Eric Gutstein (2002). He was concerned to get across the idea of expected values. His resources included graphing calculators and data on police traffic stops in Illinois and on the ethnic profile of the state. Gutstein explains:

> In mathematics, expected value is based on theoretical probability. If 30 percent of drivers are Latino, we would expect that 30 percent of random stops would be of Latinos—but only in the long run. This does not mean that if police made ten stops and five were of Latinos that something is necessarily out of line, but it does mean that if they made 10,000 stops and 5,000 were of Latinos, that something is definitely wrong. (Gutstein, 2002, no page nos)

In evaluating the lesson, Gutstein reports that:

> Students learned important mathematical ideas about probability through considering actual data about “random” traffic stops and compared these to the theoretical probability (what we should ‘expect.’) Graphing calculators can easily simulate large numbers of random ‘traffic stops’ (since they have a built-in ‘random’ number generator). (ibid.)

What was learned is revealed in this ‘fairly typical response’ (ibid):

> I learned that police are probably really being racial because there should be Latino people between a range of 1-5 percent, and no, their range is 21 percent Latino people and also I learned that mathematics is useful for many things in life, math is not just something you do, it's something you should use in life. (ibid.)

Emancipatory potential—albeit rather slender—was also apparent:

> What did emerge was students’ sense of justice (‘Why do they make random stops? … just because of their race and their color?’) and sense of agency, as well as perhaps a sense of naïveté (‘And Latinos shouldn’t let them [police], they should go to a police department and tell how that person was harassed just because of a racial color’). (ibid.)

The curriculum object—expected value—is explicit in Gutstein’s text. Of particular interest, however, is the appearance of the term ‘random’, with and without quotes. The extracts seem to suggest that the police only pretend at randomness, whilst the graphing calculator is able to reveal what real randomness would look like using imagined 'traffic stops'. A mathematical and political success, it would seem.

But here's the thing: random traffic stops are illegal in the US, being a breach of Fourth Amendment rights; police have to be able to demonstrate probable cause for their interpretation that an offence has been committed [4]. In fact, one might suppose that police are often not able to estimate the ethnicity of a driver until after they have made the stop. This would seem to suggest that, if there is a correlation between ethnicity and the probability of being stopped, then we might look for the presence of intervening variables for an explanation; a correlation between ethnicity and relative poverty and the association of the latter with the use of elderly and poorly maintained vehicles having visible defects, for example.

Statistics can be used in all sorts of way, of course. One Illinois Department—the Wilmette Police—used their data on traffic stops to demonstrate that stops for different ethnic groups and genders were, in fact, in proportion to their representation in the community, thus demonstrating that 'Wilmette police officers are engaging in bias free traffic enforcement' (Carpenter, 2004, p. 66). One possible interpretation might be that, if the stops are non-random (as the law requires), then behaviour that might lead to a stop being made is evenly distributed in terms of ethnicity. Another might be that there has been some quota stopping going on.

My very brief discussion of this issue is intended to illustrate that, whilst statistical methods might usefully be deployed in the investigation and interrogation of the activities of traffic police, both the mathematics lesson and, in this case, the annual reporting of police activities by a police department, have privileged a particular object from probability theory—expected value—and, in doing so, have recontextualised police actions to the point of rendering them illegal! Rather more comprehensive reports are produced annually for the Illinois Department of Transportation (for example, Northwest University Center for Public Safety, 2007). Again, though, the presumption that the expected value of stops for each category of driver is presented as the ideal state and any deviation is *prima facie* evidence of bias. We can describe what has happened here using the schema in Figure 1.

	Content (signifieds)	
Expression (signifiers)	I^+	I^-
I^+	*esoteric domain*	*descriptive domain*
I^-	*expressive domain*	*public domain*

$I^{+/-}$ represents strong/weak institutionalisation.

Figure 1: Domains of action.

I am conceiving the activity of mathematics education as a loose kind of alliance between mathematics educators that is characterised by a practice—school mathematics— that varies in terms of its strength of institutionalisation of modes of expression and of content (that which expressions signify). Those regions of the practice for which expression and content are most strongly institutionalised (I^+) form what we might regard as the nonnegotiable part of school mathematics. I refer to this as the *esoteric domain* of the practice. Practitioners of school mathematics have been apprenticed into this domain in the sense and to the extent that it regulates what constitutes legitimate mathematical utterances and actions on their part.

But school mathematics should also be seen as a hybrid activity that articulates the strictly mathematical with what we might loosely describe as pedagogic theory (see Dowling, 2008b). The latter will require the active subject of school mathematics—for example, the teacher or textbook or test author—to cast a *gaze* beyond mathematics *per se* as has happened in Gutstein's mathematics lesson involving traffic stops data. The result is a recontextualisation of a police activity that brings it into alignment with the esoteric domain of school mathematics as mathematics. In fact, in this case, the recontextualisation has occurred in two stages: the first stage has involved the collection of statistical summaries of policing events; the second stage is Gutstein's recontextualising of these as a pedagogic resource. The first stage constitutes an illegal (ie random) ideal traffic stop and the second stage fixes this by its emphasis on its pedagogic objective, the expected value. Now, by and large, the language of the responses to Gutstein's lesson (as reported in his paper and illustrated above) was not couched in esoteric domain language: neither expression nor content are I^+ mathematical language, but look far more like everyday language, albeit rather politically charged. Here, expression and content are weakly institutionalised (I^-); this is *public domain* language. [5]

The two other domains presented in Figure 1 are hybrids. The *descriptive domain* employs mathematical language to refer to non-mathematical content. This is the language of mathematical modelling. The expressive domain deploys non-mathematical language to refer to mathematical content; this is the domain of pedagogic metaphors, a fraction is a piece of cake, an equation is a balance, and so forth (see Dowling, 1998, 2007, 2009).

Figure 1 allows us to talk in a consistent way about how one practice—here, school mathematics—talks about another. In the case of Gutstein's lesson, the public domain seems to be operating in a janusian kind of way. On the one hand, it is presented as a portal into the esoteric domain: 'Students learned important mathematical ideas about probability …'. On the other hand, students also got the political message—'I learned that police are probably really being racial …'—but looking outwards from mathematics. Whilst the esoteric domain objective is mathematically legitimate, the public domain message is suspect, to say the least; policing has been recontextualised to make both a mathematical and a tendentious, political point. You might learn mathematics like this, but you're going to get a naïve view of the nonmathematical world that it recontextualises as its public domain.

If it is the case—and I maintain that it is—that school mathematics always constitutes its public domain as a collection of distorted or mythologised practices, then this would seem to undermine the use-value of school mathematics as providing the basis for competences that can be transferred to other activities. This is not entirely my contention. Rather, school mathematics fails to provide transferrable competences in *push* mode, which is to say, the mode that characterises Gutstein's lesson in which mathematics—the pushing activity—is privileged at the expense of the object of the gaze—traffic enforcement. It is an empirical question as to whether mathematical competences may be productively useable in *fetch* mode, that is, from within another activity that is, so to speak, recruiting resources from elsewhere and that, of necessity, recontextualises them in this recruiting by privileging the fetching activity. I am inclined to the view that even in fetch mode, school mathematics is far less used than is often supposed by mathematics educators. Indeed, to the extent that all activities are dependent on the particularities of the contexts—performances being accountable within their respective alliances—that define them, then the whole idea of transferrable knowledge and skills is problematised. Understanding practices as alliance/activity-specific entails that migratory competence is an imaginary category, imagined by the pushing activity or the fetching activity or by a meta-activity. In the case of the latter, the development of migratory competence contributes to a discursive and/or nondiscursive unifying totality that is either projected behind the plane of human activity (naïve realism) or constructed in front of it.

I want to refer to the alliance-specific nature of all activity as *disciplinarity* and to claim that the abstracted practices that constitute the disciplinarity of school mathematics are, for the most part, substantially detached from other disciplinarities, both in the school and beyond. This does not entail that we cannot constitute continuities between school mathematics, university mathematics, school and university physics, domestic practices and so on, but that such continuities as we can identify are likely to be rendered differently in each, as I have illustrated and argued above. Thus, we do not find repetitions of elements of the mathematics syllabus on the physics syllabus or vice versa—as a perusal of the Swedish school syllabuses will illustrate (Swedish National Agency for Education, 2008)—even though they may be thought to be closely related subjects. This is not surprising; the training and appointment of teachers—at least, in high school—the school timetable and national assessments and international tests (for example, Pisa and TIMSS) are generally organised on the basis of what Bernstein (1977, 2000) refers to as a collection code, the component fields of which must deploy disciplinarity strategies to constitute public singularities [6]. General curriculum principles and policy must be constituted at a higher level of analysis by a meta-activity or activities as I have suggested in relation to transferrable knowledge and skills.

Thus the curriculum is constituted as a collection code via disciplinarity strategies; meta-activity must, insofar as it casts a principling gaze on multiple disciplinarities, recontextualise the collection code as something resembling Bernstein's integrated code. The two levels collapse in the form of a substantive integrated code only in the absence of strategies of disciplinarity (which seems empirically unlikely). [7]

The effect of disciplinarity strategies in constituting the school curriculum as a collection of self-referential esoteric domains and—at least in mathematics—a collection of mythologised, public domain practices is curricular solipsism. This being the case, it is unclear how the continued status of mathematics and other members of the collection as compulsory curriculum subjects can be justified. Because of the importance of the public domain as the apparent guarantor of utility in mathematics, this subject, in particular, is potentially dangerous as the brief discussion of the mathematising of traffic stops demonstrates. Elsewhere (Dowling, 1998, 2009) I have referred to this danger as the myth of participation: that mathematics provides a necessary supplement to the practice in non-mathematical activities. In Dowling (2007, also 2009) I have made a similar argument in respect of the construction of what I referred to as mathematicoscience in TIMSS test items as privileging rational argument (mathematics) and objectivity (science) as, in effect, both necessary and sufficient conditions for the enactment of public decision-making and problem-solving. Of course, real decisions are not generally made nor real problems solved in public, but in private, where the discourse is, we might think, more likely to be irrational and prejudiced.

Distribution and the new massification

The second dimension of Bernstein's pedagogic device is its distribution rules. Now, in my earlier analysis of school mathematics textbooks (Dowling, 1991a, 1991b, 1995, 1996, 1998) I argued that school mathematics—as constituted in these books—served, in effect, as a translation device for converting socioeconomic class into mathematical 'ability' as 'objective' properties of students. It does this by realising socioeconomic class characteristics (such as occupational difference) in curricular tracks that are differentiated in terms of 'ability'. Further, by apprenticing high socioeconomic class/'ability' students to esoteric domain mathematics and low socioeconomic class/'ability' students to the mythical collection of practices that constitutes the public domain, school mathematics provides a career path for the former, but not for the latter. Essentially, the curriculum for high 'ability' students is about mathematics, whilst the curriculum for low 'ability' students presents mythologised versions of the students' own lives.

The introduction of the concept of disciplinarity, however, suggests that I should revise my description. I have now described the esoteric domain—and I shall have more to say about this domain in the next section—as constituting a self-referential region of practice, an element of a collection of such self-referential regions of practice that comprise the school curriculum. Students—the high socioeconomic class/'ability' ones—apprenticed to this domain are also being admitted to a mythologised practice; yes, they have a career within school mathematics, but this is a dead end job! It will be pointed out, of course, that entry into the esoteric domain of school disciplines—in contrast to restriction to the public domain collection—gives potential access to symbolic capital in the form of qualifications and that these can be 'exchanged' for further symbolic capital (university entrance) or direct economic capital through higher paid employment. However, what I want to term *new massification* strategies in education entail that erstwhile elite educational performances are now becoming much more widely distributed. Such performances are evidenced in the UK, for example, in terms of the increasing proportions of 16-year-olds obtaining grade A or A* at GCSE—now 20% according to *The Guardian* (27th August 2009), the increasing proportion of A grades being awarded at A-level year-on-year for the past 27 years reaching 26.7% in 2009, with an overall pass-rate of 97.5% (*The Guardian*, 20th August 2009), the participation rate in Higher Education for males, in 2007-8 was 38% and for females, 49% (DIUS). [8] Ultimately, the new massification strategies tend to foreground one-dimensional abstractions, numerical performances in respect of certification, university registration, national assessments, international tests and so forth. These simple statistics are recruited as metrics for institutional and governmental performance (see, for example, Smithers, 2008). Whether or not these developments have led to a lowering of standards, the new massification of elite performances is certainly reducing their value as symbolic capital as schooling performances are increasingly based upon criterion rather than norm referencing and schooling assessments increasingly resemble driving tests: all might reasonably be expected to pass (eventually) or, as Melanie Phillips (1998) put it, *All Shall Have Prizes*. Governments can use metrics as evidence of their achievements towards this goal, setting measurable targets in the same way as the British government has set targets for the number of medals to be won at the 2012 Olympic Games; but then, the Olympic Games are exclusively for elite performers. The prevalence of strategies of disciplinarity presents a valid case for abandoning mathematics as a compulsory school subject—at least in secondary schooling: the prevalence of metrics in new massification strategies sentences children to twelve years at hard labour in service, not of themselves, but 'democracy'.

Evaluation and conceptualisation

The third dimension of the pedagogic device is its rules of evaluation. We don't need the introduction of an arcane social organ to tell us that evaluation, in one form or another, is implicated in most if not all of what we do. The crucial issue here, however, concerns what is being evaluated, how, and to what effect. Pertinent to this is an episode reported by Mike Cooley from his research conducted in the aerospace industry. [9]

> At one aircraft company they engaged a team of four mathematicians, all of PhD level, to attempt to define in a programme a method of drawing the afterburner of a large jet engine. This was an extremely complex shape, which they attempted to define by using Coon's Patch Surface Definitions. They spent some two years dealing with this problem and could not find a satisfactory solution. When, however, they went to the experimental workshop of the aircraft factory, they found that a skilled sheet metal worker, together with a draughtsman had actually succeeded in drawing and making one of these. One of the mathematicians observed: 'They may have succeeded in making it but they didn't understand how they did it.' (Cooley, 1985, p. 171)

Which team was the more highly appreciated by the management, I wonder.

Before discussing Cooley's story, I want to introduce a schema relating to the contexts of pedagogic transmission [10]. Firstly, transmission may be institutionalised within the context of the production and/or elaboration of the practice. This mode characterises the 'legitimate peripheral participation' of Lave and Wenger (1991) (though not necessarily all of their examples) and also what might be described as traditional apprenticeships (see Coy, 1989a). The craft apprentice—the apprentice Tugen blacksmith, for example (Coy, 1989b), learns his (*sic*) craft in the forge, alongside the master. Alternatively, transmission strategies may be elaborated by relayers of the practice, who mediate between the mythologised practice (the 'knowledge' of the expert practitioner, experienced member, etc). In this mode, pedagogic theory will tend to take the foreground and the practice to be transmitted will be constituted as a curriculum. This is clearly the mode that is prevalent in schooling, where the emphasis is on the transmission of the mathematical expertise, but not the teaching expertise, of the teacher. On the other hand, if schooling itself is the practice to be reproduced, then it may be more appropriate to think of transmission strategies that are directed at the apprenticing of the newcomer into the community of school students, or school teachers, and so forth, in unmediated mode.

The second dimension of transmission strategy can be introduced by reflecting on two different examples of craft apprenticeship. The first is the apprenticeship of Japanese mingei folk potters, described by Singleton (1989). This looks very much like legitimate peripheral participation. The initial part of the apprenticeship involves minimum risk labouring work and observation around the factory. When the apprentice is permitted to work at the wheel, they are told that they must first make ten thousand sake cups. For the most part, the apprentice's products are thrown, unfired, into the clay bin for recycling until, eventually, the cups are rated as satisfactory and are sold in the shop—without the potter's mark—as seconds. Here, the apprentice as acquirer is relatively untheorised; their competence will (or may not) develop in time. Rather, the emphasis in this mode is the production of adequate products, which is to say, on performance, rather than on competence.

The apprenticeship of the mediaeval scribe seems to operate differently. In a 'school for scribes' described by Aliza Cohen-Mushlin (2008), the master (*sic*) scribe would pen a few lines as an exemplar and then the pupil would take over. When the pupil's performance was inadequate, the master would produce another exemplar. If the pupil progressed, they would be permitted to advance to more challenging tasks, such as rubrication and eventually take on the role of master. Here, there is clearly a sense of what is an adequate performance. However, a work completed in this mode will contain instances of both adequate and inadequate performances as the pupils' work would not be scrapped; parchment would have been too costly for this and, presumably, such a procedure would have introduced too great a delay into book production. This leaves the emphasis of the apprenticeship far more on the competence of the apprentice than on the quality of the final product, which is always going to be imperfect. The distinction between this mode and that of the pottery apprentice is, as is generally the case, one of emphasis, almost nuance, perhaps, but nevertheless discernible. Pedagogic theory—or what we can know of it—is light, in the scriptorium, seemingly confined to the provision of exemplars, imitation, and correction, nevertheless it is there.

	Transmitter Focus	
Mediation	Competence	Performance
Unmediated	*delegating*	*apprenticing*
Mediated	*teaching*	*instructing*

Figure 2: Transmission strategies.

The cartesian product of the two dimensions of transmission strategy gives rise to the relational space shown in Figure 2. Two of these strategy modes, teaching and apprenticing, are quite familiar and commonly opposed as, indeed, they are here, though in what I think is an original way. The commodity outputs of schooling might be said to be various forms of credentials that attest to competence. We might say, then, that the tools of the school are the curriculum and assessment protocols and its raw materials are its students. The performances produced by the students are generally of little importance once they have been assessed. The commodity outputs of factories are the performances of their staff, so the situation is the reverse of that of the school and it is unsurprising that we find novices confined to low-risk (and probably low paid), peripheral activities until their performances are judged to be satisfactory.

The leading diagonal of Figure 2 opposes delegating and instructing. The mode exhibited in Cohen-Mushlin's scriptorium has been labelled, delegation, which is here being understood as a strategy of transmission rather than a strategy of management (though one might presume that the latter generally entails the former). Here, unlike the situation in teaching, master and pupil performances are the principal products of the activity, yet the emphasis is on the development of a community of competent practitioners, rather than or, at least, as well as, on the quality of any of these products. I have no other empirical instances of this mode, though one might look to amateur, hobbyist activities. I have also encountered the sharing of repertoires of skills within informal (ie based in the public house) communities of jobbing builders and delegation might be an appropriate description of transmission strategy here. Consultancy work (the consultant being in the position of the transmitter) might be explored for evidence of this mode as might activities around succession planning in institutions.

Instruction is also frequently opposed to teaching and it constitutes the form of mediated transmission strategy that does not involve any developed pedagogic theory. I suppose sets of instructions accompanying consumer goods would frequently be described in this way. Not all instruction books are exhaustively described like this, however; the users manuals accompanying the professional grade cameras that I use tend to attempt to cater for incompetent users by including some teaching on the basic principles of photography and, in this respect the manuals differ from some of the reviews on websites concerned with photography.

To revisit the Cooley's afterburner episode, it would appear that the mathematician is privileging the evaluation of competence over performance. However, competence, in the form of 'understanding', is being measured in terms of the presumed need to devise an explicit, mathematical formulation of the problem. We might suppose—Cooley provides no evidence here—that the draughtsman and toolmaker 'understood' the problem in relation to its formulation according to different technologies that prioritised manual—what I refer to as low discursive saturation (DS^-), rather than intellectual or high discursive saturation (DS^+) practices and that they would have evaluated their activity—at least on this occasion—in terms of performance rather than competence. The social class implications are clear. Here, however, it is also interesting to note that, whilst (if my supposition is correct) the manual workers would have evaluated their performance on the basis of technologies and an apparently successful prototype product that are directly and routinely implicated in the activity of production, the intellectual worker appears to have been introducing an evaluative apparatus—mathematics—that might be more readily associated with mediated transmission and/or with a different field of production—in the university: it seems to have been relevant that all four mathematicians were 'of PhD level'.

The aerospace example is not, of course, an incidence of transmission, but it does involve evaluation and, in this respect, the referential objects supposedly used by the manual workers seem more at home in the context of production than do those of the mathematicians. I want to propose that both transmission in the context of production—delegation or apprenticeship—and transmission focusing on performance—apprenticeship or instruction—will tend to privilege the objects of production as the principal (which is not necessarily to say exclusive) referential objects for evaluation. This is because all three strategies must emphasise production as the main task in hand, closing down on opportunities for non-productive tasks. Uniquely, mediated, competence-oriented transmission—teaching—opens up a space for dedicated pedagogic action, because performances *per se* are arbitrary and ephemeral. Furthermore, because competence is postulated rather than directly visible, pedagogic action directed at its evaluation may be described as *forensic*, which is to say, directed at revealing things as they are (Dowling, 2009), in this case, 'things' referring to competence as a putatively objective (though potentially changeable) property of the evaluee. An assessment industry has developed in the field of educational studies. Forensic evaluation is theorised and standardised tests are constructed and deployed nationally and internationally. As I have noted in the previous section, performances on such tests are recruited as metrics in demonstrating or challenging the success of government education policy and so forth so that performance as such is important. However, these are not production competences; in and of themselves, they do not matter other than insofar as they are interpreted as forensic indicators of underlying competences.

In mathematics education—and in other subjects in the curriculum collection—competence is often described as 'understanding', as in the Cooley episode. Here is Jeff Vass—a sociologist and former researcher in mathematics education who now works in social theory:

> On leaving ed research I had got to the point where I thought it doesn't matter what is taught in maths. maths ed seemed to me to be full of spurious, hybrid psychological speculation justifying this or that teaching method. things that i thought might be useful, or have been useful to me, (like knowing by rote ones multiplication tables) were regarded with horror by people i met in education. They said children need to 'understand number' - i said this had never occured to me when learning tables by rote, and while contemplating 'what is number' might be something i could see myself doing at some point i could get along

> without understanding anything at all when converting fahrenheit to centigrade. didn't go down well. (Personal email)

I want to refer to this emphasis on competence or understanding as *conceptualisation* and I want to claim—and I suspect I will find few challenges in the field of mathematics education here—that this is a central strategy in the teaching mode of transmission. Another realisation of the conceptualisation strategy is the contention that learning mathematics is primarily concerned with the cognitive acquisition of mathematical objects, which, Raymond Duval (2006) argues, are rather different from the objects relating to the 'other domains of scientific knowledge' (p. 107):

> From an epistemological point of view there is a basic difference between mathematics and the other domains of scientific knowledge. Mathematical objects, in contrast to phenomena of astronomy, physics, chemistry, biology, etc., are never accessible by perception or by instruments (microscopes, telescopes, measurement apparatus). The only way to have access to them and deal with them is using signs and semiotic representations. That means that we have here only a single access to the knowledge objects and not a double access, mainly non-semiotic and secondarily semiotic, as is the case in the other areas. (Duval, 2006, p. 107)

And further,

> Mathematics is the domain within which we find the largest range of semiotic representation systems, both those common to any kind of thinking such as natural language and those specific to mathematics such as algebraic and formal notations. And that emphasizes the crucial problem of mathematics comprehension for learners. If for any mathematical object we can use quite different kinds of semiotic representation, how can learners recognize the same represented object through semiotic representations that are produced within different representation systems?
>
> […]
>
> This functional difference between the various semiotic representation systems used in mathematics is essential because it is intrinsically connected with the way mathematical processes run: within a monofunctional semiotic system most processes take the form of algorithms, while within a multifunctional semiotic system the processes can never be converted into algorithms. For example, in elementary geometry, there is no algorithm for using figures in an heuristic way […] and the way a mathematical proof runs in natural language cannot be formalized but by using symbolic systems. Proofs using natural language cannot be understood by most students […]. (Duval, 2006, pp. 108-109)

So:

1. There is no perceptual contact with mathematical objects, but there is in other activities;
2. Mathematics consists of a complexity of semiotic systems for the representation of its objects;
3. There is, in general, no unambiguous transduction between representations in different semiotic systems;
4. This generates difficulties for students of mathematics.

Duval also marks a distinction between proof and argumentation in mathematics and outside of mathematics, respectively:

> We can observe a big gap between a valid deductive reasoning using theorems and the common use of arguments. The two are quite opposite treatments, even though at a surface level the linguistic formulations seem very similar. A valid deductive reasoning runs like a verbal computation of propositions while the use of arguments in order to convince other people runs like the progressive description of a set of beliefs, facts and contradictions. Students can only understand what is a proof when they begin to differentiate these two kinds of reasoning in natural language. (Duval, 2006, p. 120)

Thus:

5. Even within a single semiotic system, there are (what I would call) fundamental discursive differences between mathematics and other activities.

Duval's realist methodology is problematic, for me, in two respects. Firstly, I do not want to ontologise the objects of mathematics nor, indeed, those of astronomy, physics, chemistry, biology, etc. Let me put it this way. The natural sciences—say, endocrinology— might be interpreted as coordinating theory and method, where the latter is, in part, constituted by the instrumentation that Latour and Woolgar (1979) referred to as inscription devices and by the principles of deployment of this instrumentation. The objects of endocrinological activity, then, are, like those of any other activity, including mathematics, to be taken to be constructed within the activity rather than noumenal objects sending messages to the endocrinologist. This, incidentally, is not an anti-realist claim, in a naïve sense, merely a form of a-realism that places its interest in the specific constructions of human activity rather than on faith in the metaphysical.

Secondly, for my purposes I do not find it helpful to essentialise semiotic systems— language, visuals, …—or even registers—algebraic and other forms of discursive representation—nor to consider the phases or aspects of signification; given a mathematical context, I cannot hear the word 'circle' or see a visual representation of a circle without the one calling up the other (although the specific visual may depend upon the context). Rather, I want to focus my attention on strategies that structure the esoteric domain, here of school mathematics.

So, instead of taking mathematics to comprise a range of semiotic systems, I suggest that it is more appropriate to say that it is characterised by a mixture of strategies that includes: i) discursive definitions, principles, theorems and so forth; ii) visual exemplars, most obviously in the area of geometry; iii) formal nomenclatures (the decimal representation of number, for example) and heuristics; and iv) instrumentation (calculators, computers, geometric instruments, and so forth). Now this empirically based list can be reconceptualised as a complex apparatus that exhibits variation in semiotic mode—discursive (available within language)/non-discursive (not available within language)—and action—interpretive/procedural. This gives rise to the schema in Figure 3.

	Semiotic Mode	
Mode of action	Discursive	Non-discursive
Interpretive	*theorem*	*template*
Procedural	*procedure*	*operational matrix*

Figure 3: Modality of esoteric domain strategy.

Now, quite clearly, the categories constituted in Figure 3 refer to general/ generalisable aspects of the esoteric domain; a template would be of little use if it constituted a unique instance. These general modes are then repeated as local instances, giving rise to a three-dimensional schema, represented in Figure 4.

I'll take an example that I've modified from Duval (2006) to illustrate the schema; this is the mathematical problem, what is the relationship between the perimeter of triangle ABC and the lengths, AE and AF in Figure 5. The verbal statement of the problem is a localising of theorem, that is, an enunciation; the grammatical issue that it is in the form of a question rather than a statement is not relevant here. Figure 5 itself is a graph, so a relevant procedure would be, identify a suitable template. This has been done in Figure 6.

	Semiotic mode	
Mode of action	Discursive	Non-discursive
Interpretive	*theorem*	*template*
Procedural	*procedure*	*operational matrix*
Mode of local action		
Interpretive	*enunciation*	*graph*
Procedural	*protocol*	*operation*

Figure 4: Modality of general and local esoteric domain apparatus

The template serves (in my reading) to articulate theorem and operational matrix as follows (I'll restrict myself to plane geometry):

Theorem

1. A circle is the set of points that are equidistant from a fixed point that is its centre.
2. There are two tangents to a circle from any point outside the circle.
3. These tangents are of equal length.
4. The tangents are perpendicular to the radius of the circle at the point at which they touch the circle.
5. The two tangents to a circle from a point outside of the circle, the line joining this point with the centre of the circle, and the radii of the circle at the points at which the tangents touch the circle form two congruent, right-angled triangles.

Operational matrix

6. Circles and line segments may be constructed by straight edge and compasses or using draw software on a computer.

A simple solution to the problem lies in recognising the template, Figure 6, in Figure 5; it can be seen to occur three times, as shown in Figure 7.

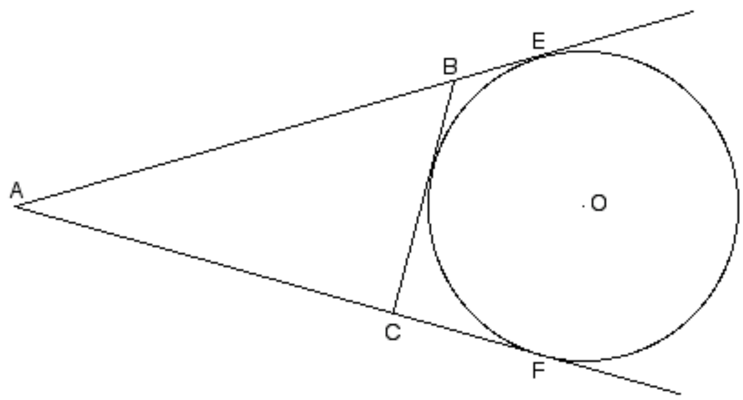

Figure 5: A Geometrical Graph.

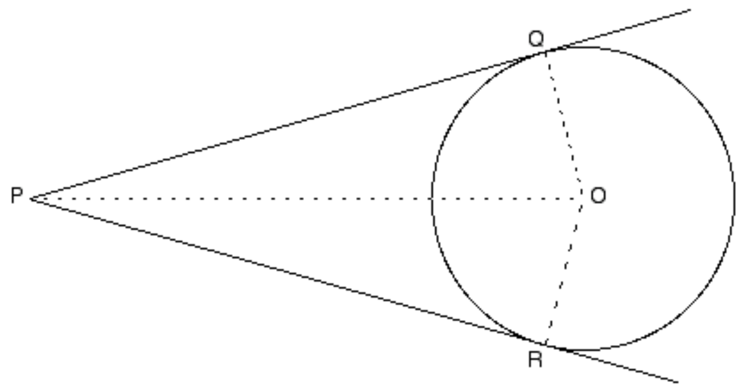

Figure 6: A Visual Template.

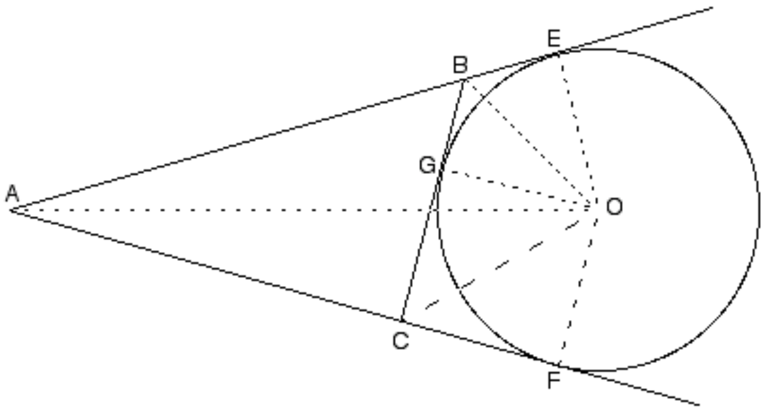

Figure 7: Realisations of the Template in the Original Graph.

Figure 7 reveals equivalents to the kite shape, PQOR (Figure 6) as AEOF, BEOG and CGOF, so we know that BE = BG and CG = CF and so we can conclude with the solution: the perimeter ABC = AE + AF. This has arisen out of the selective articulation of the original enunciation and graph with the procedure and template and its associated theorem.

Presumably, the problem might have been produced entirely as an enunciation. I will not attempt this here, but I suspect that the result would be quite tortuous; the visual texts enable us to recollect and organise the esoteric domain apparatus rather more efficiently, at least, than would be the case without them. They work very effectively here, and in school mathematics more generally, because the template, Figure 6, is strongly institutionalised within and by the practice as are other geometrical templates (images of standard geometrical forms and relationships; a pair of parallel lines traversed by a third line being another example).

Now I've referred to the discursive elements of problem and solution as enunciation, which is to say, local rather than general discourse (theorem). However, the lack of specific measurements (for the lengths of line segments or the magnitude of angles) does tend to generalise the figure, so there is a sense in which the solution is constituted by both enunciation and theorem strategies. Alternative solutions to that offered by the argument presented here might involve operational matrix/operation strategies. For example, it is clearly possible to measure the lengths directly from the graph and draw inferences from the results. This may be deterred in practice by another enunciation to the effect that the graph is not drawn to scale, but the operational solution would clearly eliminate theorem from its strategy.

In the consideration of number, we might consider that it is more helpful to think of, say, 351 as a text rather than as a sign [11]. As a spoken text, we would say 'three-hundred-and-fifty-one', marking out the place value system by the addition of words ('hundred') and inflections (five becomes fifty). This simple enunciation is associated with procedures that have to do with arithmetic. However, 351 is also a graph in the sense that the signification of each digit (digits are signs) is given by its spatial position relative to the others. A relevant operational matrix might be a spike abacus (there is no other obvious (and certainly no other obviously simple) way to move from the local instance, 351, to a more general text). Setting up the abacus as 351 is an operation. The abacus provides a simple technology for addition and subtraction within the positive integers, but is less successful (which is to say, rather complicated) for multiplication and division. We are left, then, without a non-discursive strategy for these latter operations; all we have are procedures. Clearly, we can produce alternative graphs for multiplication, rectangular arrays of dots, for example, transducing one graph, 352 x 792, into another—a rectangular array of 792, rows each containing 352 dots. Alternatively, we can produce a procedure for multiplication in columns or we might attempt to use a number line for all four operations. As Duval points out, there is clearly scope for considerable confusion.

Duval also implies that there are, in general, no unambiguous transductions between semiotic registers. This translates into my schema as the principle that, whilst there may be legitimate and illegitimate specific articulations between modes, the possible legitimate articulations may outnumber those that are necessary for the particular purpose at hand. In the geometry case above, for example, we do not need to know that the angles AEO and AFO are right angles, nor how the diagram was constructed. This entails that there is a problem of selection from the possible articulations, but the localisation that the template achieves clearly reduces this. For a discursive example, we might note that $2x + 3 = 0$, cannot be unambiguously transformed, but its recognition as a particular category of mathematical problem (procedure) is likely to narrow things down. We would, for example, expect most students making this recognition to compute $2x = -3$ and then $x = -1.5$ and this is the key: becoming a successful school mathematician entails the acquisition of a complex apparatus of interpretation and procedure. The solution of a problem is then a matter of selecting one or more suitable interpretation or interpretations and one or more procedure or procedures. The mathematical apparatus also constitutes the basis for the mathematical gaze that constructs the descriptive, public and expressive domains of action. To the extent that typical problem texts are available in these domains, one might suppose that facility in problem solving outside of the esoteric domain may also be acquired in a similar fashion to esoteric domain problem solving facilities. In school mathematics we certainly find expressive domain pedagogic graphs, for example, drawings of fractions as shaded parts of geometrical figures, equations as balances, and so forth.

As I have suggested, it may be possible to reduce the whole of school mathematics to theorem and enunciation strategies. It may further be the case that the resulting discourse would be characterised as DS^+ and highly coherent, which is to say that it would construct its objects in a consistent way. But it would not be teachable; we can hardly introduce the Peano axioms before the number lines that adorn the walls of elementary school classrooms. To render it teachable, we, in effect, pedagogise it. In order to achieve the discourse, the practice has to diversify its apparatus as in Figure 4 and weaken the institutionalisation of its significations in extending beyond the esoteric domain. The latter also serves as a marketing device in enabling school mathematics to be presented as something other than a closed mystery.

Now my interpretation of Duval's argument is that the work of signification in school mathematics is rendered complex because the objects of mathematics are signified in diverse semiotic registers between which there are no unambiguous transductions, so an essential aspect of the work of mathematics education lies in the coordination of these registers. This entails, of course, coordination around the objects that they signify. However, it may be that the basis of the problem lies not in a lack of coordination, but in the perceived necessity of coordination that itself derives from the ontologising of mathematical objects, conceptualisation. Such ontologising is facilitated in the sciences, according to Duval, by the fact that we have non-semiotic, perceptual access to the objects of the sciences albeit often mediated by technology. But, of course, we do not have access to anything at all that is not semiotically mediated, though the successfully marketed articulation of discursive and nondiscursive apparatus in the natural sciences may have misled us into believing that we do.

School science does often seem to be very 'thingy'. We arrange 'experiments' to find out or confirm or sharpen up our knowledge about 'things'. We can separate the components of a mixture of sand, wood, iron, and salt by taking advantage of what we know about them (sand sinks in water, wood floats, salt dissolves and iron is attracted to a magnet); we can make graphs to predict the extension of a spring under different loads (and this is how a spring balance works); we can cut up a mouse to see where the various bits are; and so forth. The objects of science are, of course, not, generally, these things themselves, but the objects constructed in science discourse: specific gravity; solubility; magnetisability; elasticity; anatomy (and its components); and so forth. This language is available to describe these things and other things like them. We do not—at least not below high school—very often go much beyond describing. We often do the same kind of exercise in mathematics, but the contexts are often rather different. Firstly, the things in the mathematics class are often presented, at least initially, in public domain terms; they are presented as if they are the way they are in our daily lives (let's think about shopping, for the time being). This, of course, is part of the point of mathematics as it is sold in the mathematics classroom: to enable us to engage more effectively in everyday situations (the myth of participation).

However, because we are to constitute the things in mathematical discourse and not within the everyday practices from which they are taken, what students see is a contortion of their own lives. Science, on the other hand, is not generally claiming to assist in managing one's daily lives. The things in science are, right from the beginning, placed in an unusual context and often in an unusual room—the laboratory. 'Suppose you (we) have a mixture of sand, wood shavings, salt and iron filings (not bits of a bicycle or car engine) and we want to separate out the various components; what can we do?' As with mathematics, there is no unambiguous transduction from the enunciation and its associate graph (the beaker containing the mixture) and no necessary separation between scientific discourse and everyday practice: we're not interested in the molecular structure of the salt nor its taste. School maths seems to be telling you lies about your own life; school science seems to be telling you new stuff about things that you're kind of familiar with, but perhaps have not really thought about.

School mathematics also has its 'laboratories', of course (generally not specially designed rooms, though). Here, we construct geometrical figures, play with abacuses and blocks of wood and hoops and chalk circles and so on. This must seem an arcane game. It's not immediately obvious why we should be interested in such things, but, even if we are, mathematics is not really about them at all; they seem to be pointing at something else. Whilst the entry to school science seems to be via the descriptive domain, the entry to school mathematics is often via the expressive. In formal terms, both school mathematics and schools science technologies construct their objects; school science defers entry into its discourse, perhaps, whilst school mathematics will not. This is not a philosophical or perhaps even a psychological problem, it's a pedagogic problem.

In addressing the pedagogic problem, one is led to ask, if scientific discourse is dispensable, at least in the earlier phases of school, might this not also be the case for mathematical discourse. Indeed, whether or not a fully principled, DS^+ version of school mathematics is possible, the integrating discourse would have to be constructed by meta-activity. At any point within school mathematics—which incorporates templates, procedures and operational matrices as well as discourse—the impossibility of general, unambiguous translation or transduction between school mathematical texts constitutes aporias and I want to suggest that it is the resolution of these aporias that motivates conceptualisation. I want to speculate that this is because mathematical integrating discourse cannot abide a vacuum. This is why Gödel's inconsistency theorem is so profoundly disturbing; this is why Foucault is justified in describing mathematics as 'the only discursive practice to have crossed at one and the same time the thresholds of positivity, epistemologization, scientificity, and formalization' (Foucault, 1972, p. 188). Duval's distinction between mathematics and 'other domains of scientific knowledge' has some validity after all. However, the difference is to be sought, not in ontology, but in the opposing strategies relating to the empirical that are prevalent within the respective activities: science must confront an empirical universe and must, therefore, be perpetually incomplete; mathematics must quickly discard the potentially corrosive empirical world even though its temporary introduction may be necessary for pedagogic reasons.

Conceptualisation, then, is a strategy that produces mythical mathematical objects and directs pedagogic action to their transmission. An alternative might be to consider rolling back on the conceptual approach to mathematics education that has held sway for the past sixty years or so and focus, instead, on the study of 'things'. I am suggesting that we refrain from erecting an arcane and only putatively totalising structure and mythical mathematical objects as mathematical discourse and concentrate, instead, on the fostering of, shall we say, *petits reçits*: procedure, templates and operational matrix strategies that are recruited within the study of 'things'. What's the best way to learn the functional use of a foreign language: to attempt to acquire generative grammatical discourse; or to acquire useable chunks? To the extent that grammar is never generative, but only at best interrogative, then the answer would seem to be the second alternative; grammar can, if we absolutely must have it, come later.

A new sociology of knowledge and curriculum reform

The new sociology of knowledge, part of which I have presented here, is called *Social Activity Method* (SAM). It is explored in greater depth and breadth in Dowling (2009), though the schemas in Figures 2, 3 and 4 have been developed since the earlier work went to press. I have emphasised three aspects of SAM that mark it out from other sociologies of knowledge and, in particular, from the sociology of Basil Bernstein, which has been particularly influential in the sociology of education. Firstly, the central proposition that guides analysis is that the sociocultural space is animated by strategic, autopoietic action directed at the formation, maintenance and destabilising of alliances and oppositions, the visibility of which is emergent upon the totality of such action and this renders them available for recruitment in subsequent action (and all action is, of course, 'subsequent' to action already completed). The schemas that are presented in this paper (see Figures 1, 2, 3 and 4) and others [12] emerge from the transaction of the central proposition with the empirical world, which is to say, with the world not already 'consumed' by SAM. Secondly, if we think of the empirical sociocultural world as being available as an unmotivated collection of texts and settings, then its transactions with SAM is constituted as a meta-activity that integrates small parts of this collection. The integrating metadiscourse is being conceived as what I refer to as *constructive description*; it is an artefact of the analysis that does not claim to be accessing motives that lie behind the empirical collection. Rather, the metadiscourse presented here may be thought of as standing in front of its object texts. This is not forensics; its evaluation is not to be assessed in relation to truth, but rather in terms of its pragmatic value in organising a disordered collection. The coherence of the metadiscourse is not irrelevant, here. However, I (Dowling, 2009; Dowling & Brown, 2010) maintain that an undue level of coherence—closure—is ultimately unhelpful in that a fully closed discourse can see only itself and cannot learn; it becomes sclerotic and necrotises the empirical. Thirdly, the method pushes analysis to the point of binary categories of strategy and, most commonly, considers the spaces opened up by taking the Cartesian product of two such categories. A relational space produced in this way is clearly logically complete. However, texts and settings are not totalised by these spaces. Firstly, whilst a particular strategy may dominate in a given text or setting, the expectation is that most texts/settings will be describable in terms of more than one, so that a particular space will provide the basis for mapping regions and trajectories within the text/strategy and the results may be aggregated to produce descriptions at higher levels of analysis. Secondly, a given space provides an analytic schema in terms of, generally, only two variables; further spaces may be deployed or generated to produce more complex pictures of the text/setting.

I have chosen, in this paper, to depart from Bernstein's construct, the pedagogic device and its three sets of rules. These rules and my departures are summarised in Figure 8. I have also used Bernstein's distinction between collection and integrated curriculum codes as a point of departure. In Bernstein's scheme, these are alternative forms of curriculum organisation. I have argued here that integrating principles must be generated within a metadiscourse [13]. I should, perhaps, reformulate this as: push strategies recontextualise their objects as integrated codes; fetch strategies recontextualise their objects as collection codes; a metadiscourse is, of course, a push strategy. We can go further and suggest that the integrated code constituted by push strategies might be constituted as the hegemonic imposition of integration on an inevitably unintegrated activity. This is precisely the offence of the mathematician in Mike Cooley's anecdote. This is precisely the offence of governmental new massification strategies in constructing a single, privileged career trajectory for all school students. This is precisely the offence of Eric Gutstein in rendering illegal traffic stops made by Illinois police. This is precisely the problem with formal schooling.

Pedagogic device rules	Departure	Implications
Recontextualising	*disciplinarity*	i) school curriculum as collection of selfreferential disciplines; ii) domains of action schema (Figure 1); iii) school mathematics recontextualises practices originating in other disciplines (esoteric domain provides integrating schemes for public domain collection); iv) push and fetch recontextualising between members of collection; v) integrating meta-activities; vi) myth of participation.
Distribution	*new massification*	i) school mathematics distributes public domain to low class/ability students and esoteric domain to high class/ability students (both mythical practices); ii) symbolic capital decreasing in value as access to previously elite performances expands.
Evaluation	*conceptualisation*	i) teaching transmission strategy opens space for forensic assessment; ii) multimodal esoteric domain apparatus exhibits aporias; iii) metadiscourse mythologises mathematical objects and 'understanding'.

Figure 8: Departures from pedagogic device.

The solution is to redesign the curriculum around fetch strategies by freeing it from the stranglehold of disciplinarity, new massification and conceptualisation strategies. This is not the place to develop a complete proposal for a new curriculum, which, in any event, would run the risk of simply replacing one set of integrating strategies with another. However, if we start from the proposition that schooling—and this certainly extends to include university undergraduate programmes—might be re-directed to serve as an introduction to the diversity of legitimate activities in society (ultimately understood globally), then this has to be achieved other than through the oversimplifications and distortions of integrating disciplinarities. Academic research is one collection of such activities, of course, and these have traditionally been the main referents of a number of school subjects [14]. Taking science, as an example, here is the introduction to science studies, summarising its 'aims':

> The natural sciences have developed as a result of Man's [*sic*] need to find answers to those issues concerning his existence, life and forms of life, our role in nature and the universe. The natural sciences are thus a central part of the Western cultural tradition. The natural sciences can both stimulate Man's fascination and curiosity in nature and make it understandable. Natural science studies satisfy the desire to explore nature and provide scope for the joy of discovery. The aim of science studies is to make the results and working methods of science accessible. The education contributes to society's efforts to create sustainable development and develop concern for nature and Man. At the same time the education aims at an approach to the development of knowledge and views which resonate with the common ideals of the natural sciences and democracy on openness, respect for systematic investigation and well-founded arguments. (Swedish National Agency for Education, 2008, p. 40)

This looks rather more like a celebration of mythical science (and of the (also mythical) Western cultural tradition' and 'democracy' and, indeed, of 'Man') than an enjoining to engage in an exploration of what scientific activities actually entail. Real science is messy, unreliable, politicized—in terms of both its enactments and recruitings—and generally a very long way from the idealised scientific method that is generally expounded in schools and even on undergraduate programmes. This is not to deny, of course, that published scientific work—generally journal articles, not books—tends to recontextualise laboratory messiness as pristine structures of pure logic and objective observation, but then it's a general feature of the human condition that people do not do what they say they do [15].

Much of the collection of human activities owes little or nothing to the disciplinarities of schooling. This range of the collection probably includes many of the domestic activities that are the targets of push strategies from school mathematics. To claim this is not to suggest that performances in these activities might not be enhanced by the recruitment of technologies from other activities, including mathematics, but that the principles of evaluation of these performances properly takes place in the context of the enactment of the activities themselves and not that of integrating activities such as school mathematics.

Essentially, the kind of curriculum that I am proposing comes close to the idea of practitioner research or, more appropriately, proto-practitioner research, or even intern research. Clearly, this kind of curriculum would take much of the control away from the teacher and, indeed, away from the state and place rather more responsibility on school and non-school sites of activity and on teacher-student negotiation on matters of organisation and management. It will be pointed out that the activities to be explored are themselves positivities constituted by strategies of disciplinarity. This is correct and, indeed consistent with my description of sociology as being concerned with strategic, autopoietic action directed at the formation, maintenance and destabilising of alliances and oppositions, the visibility of which is emergent upon the totality of such action. Furthermore, if it is believed that activities resembling current school subjects incorporate useful resources that might productively become the targets of fetch strategies, then they will need to be transmitted and this will entail the deployment of fetch strategies in establishing public domain portals. However, this does not entail that all school students need to acquire and embody twelve years worth of the school mathematics collection, far less that this be attempted under the watchful eye of forensic assessment and integrating strategies of conceptualisation. Of course, moving in this direction would involve major social upheavals, including the abandoning by governments of new massification strategies and the abandoning by teachers and educationalists of strategies such as the organizing of conferences based upon school disciplinarities. As I say, as a sociologist, I am not optimistic.

Notes

1. See, for example, Collins & Pinch (1998, 2005); Feyerabend (1975); Fleck (1981); Knorr Cetina (1999); Kuhn (1970); Latour & Woolgar (1979); Turnbull (2000).

2. But see Dowling (2009), cc. 4 & 8 and also Dowling (1999).

3. Bernstein's own description of the device is, as I have illustrated in Dowling (2009) often very confusing, with terms being used apparently inconsistently. Precision in its definition is not, however, crucial here.

4. Decker et al. (2004) do argue that US courts have been very liberal in respect of what might count as probable cause. However, the principle that there must be a reason for a traffic stop does undermine the assumption in the mathematics lesson that the stops are intended to be random; they are not.

5. In fact, public domain practice does not necessarily imply everyday language, merely language that is not I^+ in terms of mathematics.

6. Bernstein refers to these as boundary maintaining strategies. My approach, however, is relational, so the concept of 'boundary' is unhelpful.

7. There is an important issue here that I shall not pursue further in the main text. There is a consistent hierarchy that runs through the whole of Bernstein's work that privileges the former in each of the following pairs: elaborated/restricted speech code; integrated/collection curriculum code; vertical/horizontal discourse; hierarchical/horizontal knowledge structure. In each case, the privileged category is constituted only via the objectification of the subordinate category, which is to say, through the actions of a meta-activity. The same is clearly true of Luria's theoretical and participative thinking that was the basis for Bernstein's oppositions. Bernstein's apparent failure to recognise this is consistent with his more general lack of attention to the movement between levels of analysis in switching between classification and framing (see Dowling, 2009; Dowling & Chung, 2009, and particularly Chung, 2009).

8. www.dcsf.gov.uk/rsgateway/DB/SFR/s000839.

9. I have used this story many times before (see, e.g, Dowling, 2004), principally for its pedagogic value, though not always in the way that I am using it here.

10. I make no apologies for the use of the term, transmission, here. Social and cultural transmission or reproduction certainly does not exhaust the intentions or consequences of pedagogy, but it is a central aspect of formal schooling and crucial to the concerns of this paper.

11. See Dowling (2009) for a discussion of the distinction between text and sign.

12. There are over 200 specialist terms relating to SAM in the glossary of Dowling (2009) and thirty or forty more have been generated since this work went to press.

13. If we accept that there can be no absolute primitive principles—all principles must be interpreted in terms of prior principles in an infinite regress—then integrating curricular principles can be achieved only in a metadiscourse, because a curriculum is, at best, a sequence with an arbitrary starting point, which, itself, must introduce aporias so as to constitute the curriculum as a collection. In practice, the school mathematics curriculum is constructed as a collection of topics and activities that do not form a single sequence.

14. Science, history, geography, and perhaps social studies, for example, though probably not mathematics, which—apart from the new mathematics blip—is substantially detached from research mathematics.

15. Even if only for the fact that saying what one does involves textualising it.

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Ethnomathematics and Mathematical Literacy: People Knowing Mathematics in Society

Tine Wedege
Malmö University, Sweden

Ethnomathematics and mathematical literacy are two central notions about knowing mathematics in the world. While ethnomathematics stresses people's competence developed in different cultural groups in their everyday life, the idea of mathematical literacy mainly focuses on the mathematical and societal requirements to people's competencies. Starting with a critical and constructive view on ethnomathematics and on mathematical literacy, I suggest sociomathematics as an analytical concept for a subject field (people's cognitive, affective and social relationships with mathematics in society) and a research field in mathematics education encompassing the study of the two competences.

What does it mean to know mathematics? The mathematics teacher may find answers to this fundamental question in the curriculum and at the same time her/his personal beliefs about mathematics involve a response to this problem and to others like, what does it mean to learn mathematics and why teach mathematics. The researcher in mathematics education also needs answers to these fundamental questions and a series of scholars have replied by defining mathematical knowledge (capability, capacity, competence, proficiency) (e.g. Ernest, 2004; Kilpatrick, 2001; Niss, 2003; Skovsmose, 1990). Starting from a broad socio-cultural perspective on mathematics education, it is obvious that any definition is value based and related to a specific cultural and societal context. This is evident when it comes to two of the central notions about people's mathematical competences in and for a culture respectively a society: ethnomathematics and mathematical literacy. In this paper my focus is on *knowing mathematics in the world*, which is a broad and slippery expression that I use to cover a wide spectre of notions and ideas about what it means to know, to develop and to use mathematics in different cultural and societal contexts and situations. Through a critical analysis of a series of these notions, I aim at a terminological clarification illustrating some decisive differences between conceptual constructions of ethnomathematics and of mathematical literacy. Against this background, the analytical concept of sociomathematics which combine complementary aspects of the two concepts is presented and briefly discussed.

First observation: Functionality and contextuality

Under the heading *Mathematical literacy*, Jablonka (2003) includes a long series of notions and concepts about knowing mathematics in the world – among which are ethnomathematics and mathematical literacy – in her chapter of the Second International Handbook of Mathematics Education. Her aim is to investigate different perspectives on mathematical literacy that vary with the values and rationales of the stakeholders (e.g. politicians or researchers in different cultural and societal contexts). Jablonka argues that every conception of mathematical literacy promotes a particular social practice – implicitly or explicitly. In the paragraph introducing the section "Defining mathematical literacies", she points out some problems following from the very idea of knowing mathematics in the world:

> Any attempt at defining 'mathematical literacy' faces the problem that it cannot be conceptualised exclusively in terms of mathematical knowledge, because it is about an individual's capacity to *use* and *apply* this knowledge. Thus it has to be conceived of **in functional terms as applicable to the situations** in which this knowledge is to be used. (Jablonka, 2003, p. 78 [my emphasis])

My first observation is that knowing mathematics in the world is about *functional* mathematical knowledge in different situations in domains like education, economy, culture, science, democracy, etc. This is not a trivial assertion as Skovsmose (1990) bases the distinction between mathematical knowledge as such and technological mathematical knowledge, which is knowledge about how to build and how to use mathematical models, on a thesis stating that by learning mathematics (understood as a specialised academic discipline) you do not automatically learn how to use it. Or, in other words, functional mathematical knowledge cannot be reduced to "pure" mathematical knowledge. Neither does one learn to evaluate other people's use of mathematics in mathematical models. This is the reason for Skovsmose's distinction of a third type of mathematical knowledege: reflective knowledge. In Table 1, I have collected a series of terms used in the literature to describe different notions and conceptions of functional mathematical knowledge.

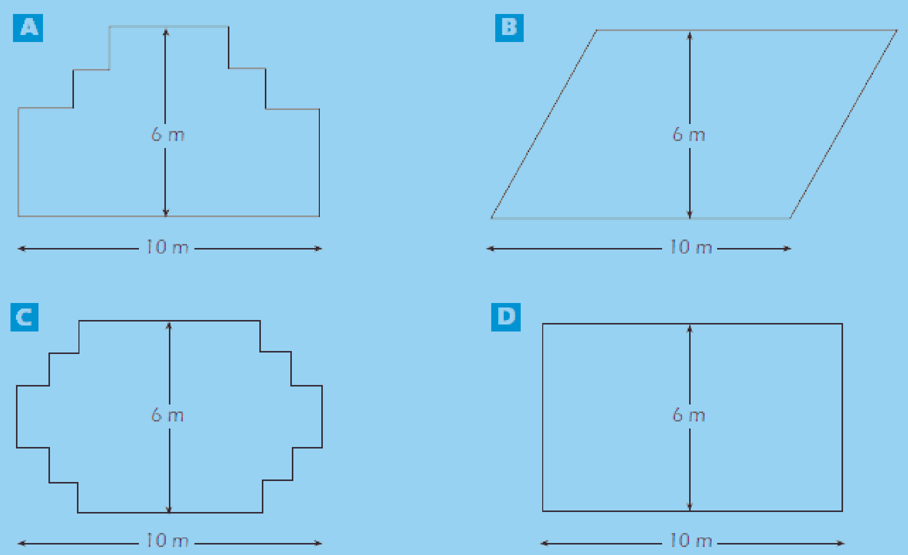
Table 1: Terms about knowing mathematics in the world.

folk mathematics		ethnomathematics
numeracy	quantitative literacy	
		techno-mathematical literacy
worker's mathematics-containing competences	street mathematics	
		mathematical proficiency
	mathematical literacy	
mathemacy		mathematical competencies
	adult numeracy	

When mathematical knowledge is claimed to be functional it is necessary to determine where (in school or everyday life) and for whom (society or individuals) (Johansen, 2004). In PISA (OECD, 2003), mathematical literacy is defined in functional terms and it is claimed that young people's readiness to meet the challenges of the future (mathematical literacy) is measured by means of mathematical tasks with so-called "real world" contexts. According to PISA, the example shown in Figure 1 is using content of the overarching mathematical idea "quantity" and is set in an occupational situation of a carpenter:

QUESTION 10.1

A carpenter has 32 metres of timber and wants to make a border around a garden bed. He is considering the following designs for the garden bed.



Circle either "Yes" or "No" for each design to indicate whether the garden bed can be made with 32 metres of timber.

Garden bed design	Using this design, can the garden bed be made with 32 metres of timber?
Design A	Yes / No
Design B	Yes / No
Design C	Yes / No
Design D	Yes / No

Figure 1: Sample question from PISA (OECD, 2009, p. 111).

However, the situation described does not have any similarity with the working situation of a carpenter. In order to give the correct answer to this task (A: Yes, B: No, C: Yes, C: Yes) one has to forget everything about timber, which has a physical extent in three dimensions, and think only of the mathematical object: the straight line and about transformations of areas that leave the circumference unchanged. The situation-context where the young people solve this task is in school within a special situation of participating in an international comparative test. The task-context is school mathematics (some basic geometry) though it is claimed to be a "real world" context.

Both functionality and contextuality of "knowing mathematics in the world" is stressed in the following definition of numeracy, which is the term often applied when it comes to adults:

> *Numeracy* consists of functional mathematical skills and understanding that in principle all people need to have. Numeracy changes in time and space along with social change and technological development (Lindenskov & Wedege, 2001, p. 5)

Functionality and contextuality are opening the discussion to one of the key issues in mathematics education and in research: the relationship between school mathematics and out-of-school mathematics when people are learning and knowing (see for example Harris, 1991; Hoyles, Noss, & Pozzi, 2001; Jablonka, 2009; Lave, Murtaugh, & de la Rocha, 1984; Nunes, Schliemann, & Carraher, 1993; Wedege, 2002).

Second observation: Knowledge developed or wanted in everyday life

Functionality is a common feature of the notions about knowing mathematics in the world represented by the terms in table 1. In this and the following section, I will discuss some distinctive traits starting from the same paragraph on some key problems related to defining "mathematical literacy" as quoted above from Jablonka:

> Any attempt at defining 'mathematical literacy' faces the problem that it cannot be conceptualised exclusively in terms of mathematical knowledge, because it is about an individual's capacity to *use* and *apply* this knowledge. Thus it has to be conceived of in functional terms as applicable to **the situations in which this knowledge is [to be] used.** (Jablonka, 2003, p. 78 [my emphasis and parentheses])

My second observation is that any notion about knowing mathematics in the world is based – implicitly or explicitly – on one of two meanings of the term *everyday knowledge* [1]:

- Knowledge **developed** in everyday life, i.e. knowledge that the individual has acquired in her/his everyday practice.
- Knowledge **wanted** in everyday life, i.e. knowledge that is supposed to be necessary/useful in people's everyday practice.

In the emphasized part of the paragraph from Jablonka ("situations in which this knowledge is to be used") it seems that she only includes the second meaning of "mathematical literacy" as knowledge wanted in everyday life. I have put parentheses around "to be" to open for the first meaning. Moreover, I suggest that another verb, "developed", is inserted: situations in which this knowledge is used and developed.

In mathematics education research and in international surveys, we find different concept constructions of knowing mathematics in the world. We have seen that they vary with the approaches, values and rationales of the researchers and the stakeholders (Jablonka, 2003). Furthermore, implicitly or explicity, they are based on different notions of mathematics and of human knowledge and learning (Wedege, 2003). As a first step, I have tried to distinguish between a series of concepts based on the meaning of everyday knowledge developed (type 1) and another series based on the meaning of everyday knowledge wanted (type 2) (see Table 2).

Table 2: Concepts about knowing mathematics in the world.

Type 1 (developed in everyday life)	**Type 2** (wanted in everyday life)
Ethnomathematics (D'Ambrosio; Bishop) *Folk mathematics* (Mellin-Olsen) *Street mathematics* (Nunes et al.) *Critical ethnomathematics* (Knijnik) *Worker's mathematics-containing competences* (Wedege) *Adult numeracy* (Evans)	*Mathematical literacy* (PISA, Hoyles et al.) *Numeracy* (Steen, ALL) *Quantitative literacy* (IALS) *Techno-mathematical literacy* (Kent et al.) *Mathemacy* (Skovsmose) *Mathematical proficiency* (Kilpatrick) *Mathematical competencies* (Niss)

Behind type 1 constructions, the main concern – and perspective – is the differences between school mathematics and out-of-school mathematics and the acknowledgment of people's informal knowledge. The focus is what people actually know and do, and their practices of using, adapting and producing mathematical competence are studied. In type 2, some of the constructions are mainly concerned with measuring, others with the relevance of mathematical knowledge and others with power relations. They are usually normative and conceived as intended outcomes in education. For the focus of this paper, I have chosen ethnomathematics (D'Ambrosio) and mathematical literacy (PISA) as the prototypes of knowledge developed in everyday life (type 1) respectively of knowledge seen as being wanted or required in everyday life (type 2). However, I have to stress that the tentative division of concepts in the two columns of table 2 is not a dichotomy. Some of the concepts put are actually dealing with the dialectics between the two types of knowledge as we shall see in the last section.

Jablonka (2003) has argued that any conception of knowing mathematics in the world (mathematical literacy) – implicitly or explicitly – promotes a particular social practice. She identifies five perspectives: Mathematical literacy for

- Developing human capital (OECD)
- Cultural identity (D'Ambrosio)
- Social change (Frankenstein)
- Environmental awareness (UNESCO)
- Evaluating mathematics (Skovsmose)

This analysis is based on the observation that different "conceptions of mathematical literacy are related to how the relationship between mathematics, the surrounding culture, and the curriculum is conceived" (p. 80). The names put in parentheses are examples of researchers and organisations found in her discussion of the five perspectives.

The perspective of mathematical literacy for *developing human capital* is based on a conception of mathematics as a powerful and neutral instrument for solving individual and social problems. Hence, mathematical literacy is defined as "a bundle of knowledge, skills and values that transcend the difficulties arising from cultural differences and economic inequalities" (Jablonka, 2003, p. 81). The perspective of mathematical literacy for *cultural identity* starts with a conception of mathematics being developed in all cultures where the mathematical practices differ in "the kinds of mathematics that are employed, in the purposes for employing that kind of mathematics, as well as in the associated beliefs about the nature of mathematics and in the values about the (mathematical) problem solution" (ibid, p. 82). Here we find yet another criterion for distinguishing the two constructions discussed in this paper. In ethnomathematics which is a construction for cultural identity, mathematics is value-laden and cultural-dependent, and in mathematical literacy which is a construction for developing human capital, mathematics is seen as neutral and universal.

Ethnomathematics

By the end of the 1970's and at the beginning of the 1980's there was a growing attention to cultural and societal aspects of mathematics and of mathematics education (Gerdes, 1996). The Brazilian mathematician and researcher in mathematics education Ubiratan D'Ambrosio lanced his "ethnomathematical programme" and presented it at the Fourth International Congress on Mathematics Education (ICME4) in 1984. He put *academic mathematics*, i.e. mathematics taught and learned in schools, towards *ethnomathematics* which

> is the mathematics practiced by cultural groups, such as urban and rural communities, groups of workers, professional classes, children in a given age group, indigenous societies, and so many other groups that are identified by the objectives and traditions common to these groups. (D'Ambrosio, 2006, p. 1)

This definition, which is taken from his recent book "Ethnomathematics: Link between traditions and modernity" (D'Ambrosio, 2006), does not differ in meaning from the origin definition (D'Ambrosio, 1985, p. 45). However, what were then called "identifiable cultural groups" are now explained as "groups identified by the objectives and traditions common to these groups". The political background for the ethnomathematical movement was the cultural imperialism of the transplanted, imported mathematics "curriculum" which is emphasised to be alien to the cultural traditions of Africa, Asia and South America (Gerdes, 1996). According to D'Ambrosio, the students' mathematical capacities for using numbers and measure, and for handling geometrical forms and concepts are replaced by other forms of practice, which have gained status as mathematics:

> the mathematical competencies, which are lost in the first years of schooling, are essential at this stage for everyday life and labour opportunities. But they have indeed been lost. The former, let us say spontaneous, abilities have been downgraded, repressed and forgotten while the learned ones have not been assimilated either as a consequence of a learning blockage, or of an early drop-out, or even as a consequence of failure or many other reasons. (D'Ambrosio (1985) Sociocultural Bases for Mathematics Education, UNICAMP, Campinas, quoted after Gerdes, 1996, pp. 912-913)

The mathematical everyday capacities of children and workers are downgraded, repressed and forgotten. In the same period in another part of the world (Sweden) and in another research field (adult education), Alexandersson (1985) did an empirical study of the relation between adults' knowledge acquired in school and in everyday practice. His study showed that people's mathematical everyday competence was algorithmetised through schooling and their problem solving ability decreased. From Norway, Mellin-Olsen (1987, p. xiii) revealed the question which had driven his research through twenty years: "why so many intelligent pupils do not learn mathematics whereas, at the same time, it is easy to discover mathematics in their out-of-school activities". His definition of folk mathematics as knowledge biased by culture or social class is of type 1 like ethnomathematics. Mellin-Olsen finds that the definition of folk mathematics as mathematics and the very recognition of folk mathematics "as important knowledge is a political question and thus a question of power" (p. 15).

In a study building on international ethnological and ethnomathematical investigations, Bishop (1988) has taken the power of definition and identified six types of mathematical everyday activity, which he claims are cross-cultural:

- *Counting* (the use of a systematic way to compare and order discrete phenomena),
- *Localising* (exploring one's spatial environment and conceptualising and symbolising that environment, with models, diagrams, drawings, words or other means),
- *Measuring* (quantifying qualities for the purposes of comparison and ordering, using objects or tokens as measuring devices with associated units or 'measure-words'),
- *Designing* (creating a shape or design for an object or for any part of one's spatial environment),
- *Playing* (devising and engaging in games and pastimes playing by rules with more or less formalised rules that all players must abide by),
- *Explaining* (finding ways to account for the existence of phenomena, be they religious, animistic or scientific).

Mathematical literacy

In the late 1990's, OECD lanced the International Programme for Student Assessment (PISA) which intends to estimate and compare between countries the store of "human capital" defined as: "The knowledge, skills, competencies and other attributes embodied in individuals that are relevant to personal, social and economic well-being" (OECD, 1999, p. 11). According to PISA *mathematical literacy* is

> the capacity to identify, to understand, and to engage in mathematics and make well-founded judgements about the role that mathematics plays, as needed for an individual's current and future private life, occupational life, social life with peers and relatives, and life as a constructive, concerned, and reflective citizen. (OECD, 2000, p. 50)

Mathematical literacy is established as an answer to the question of what mathematics is needed in an individual's life. At the societal level, Niss (2003) states that a well educated population is needed "to actively contribute to the shaping of society, and a broadly qualified work force, all of whom are able to activate mathematical knowledge, insights, and skills in a variety of situations and contexts" (p. 115). Niss, who is also a member of the international expert group in PISA, presents the conceptual framework of eight mathematical competencies as an answer to the question "What does it mean to master mathematics?" pure and simple. The competencies are almost identical with the eight competencies defined in the PISA framework (OECD, 1999), but here they are grouped in two categories:

(1) Ask and answer questions in and with mathematic
1. Thinking mathematically
2. Posing and solving mathematical problems
3. Modeling mathematically
4. Reasoning mathematically

(2) Deal with and manage mathematical language and tools
5. Representing mathematical entities
6. Handling mathematical symbols and formalisms
7. Communicating in, with, and about mathematics
8. Making use of aids and tools (Niss, 2003).

In his article "Understanding mathematical literacy", Kilpatrick (2001) gives an answer to the question "What does successful mathematics learning mean?" and presents what he is calling an elaborated view of mathematical literacy. However terms like "mathematical literacy" and "mathematical competence" are rejected as non suitable and *mathematical proficiency* is defined in terms of five interwoven strands: (a) conceptual understanding, (b) procedural fluency, (c) strategic competence, (d) adaptive reasoning, and (e) productive disposition, which includes the student's appreciation of mathematics. Kilpatrick states that these strands are to be developed in concert and claims that "it was clear from the existing research that problem solving offered a context in which all the strands of mathematical proficiency could be developed together" (p. 107).

While it is possible to argue that "mathematical modelling" is the key competency in Niss' framework, it is obvious that "problem solving" is the crucial activity in a mathematics curriculum based on Kilpatrick's construction of mathematical knowledge.

In the late 1960s, according to Rubenson (2001) UNESCO introduced lifelong learning as a utopian-humanistic guiding principle for restructuring education. The concept disappeared from the educational policy debate but, in the late 1980s, it reappeared in a different context and in a different form. The debate was now driven by an interest based on an economistic worldview, emphasising the importance of highly developed human capital, science and technology.

It is possible to see the two constructions of ethnomathematics and of mathematical literacy as illustrations of the two generations of lifelong learning respectively with recognition of people's informal knowledge developed in everyday practice and arguments for formal education. From the late 1990's, according to Rubenson (2001), it seems that the restricted, economistic view on education of the second generation, which was severely criticised, has been succeeded by a third generation (*economistic – social cohesion)* with active citizenship and employability as two equally important aims for lifelong learning – at least on the rhetoric level because of the conflicting agendas between the two ideologies behind.

Third observation: Capacity or performance

Functionality is a common feature of all conceptions of ethnomathematics and mathematical literacy, and – as mentioned – all definitions refer to situations where mathematical knowledge is (to be) used or developed. However, any mathematical activity is carried out by someone – directly or indirectly; human beings are involved. Hence I have chosen the grammatical structure "knowing mathematics in the world" instead of "mathematical knowledge in the world" to denote my focus in this paper. For the last time I return to Jablonka's text about difficulties in defining mathematical literacy:

> Any attempt at defining 'mathematical literacy' faces the problem that it cannot be conceptualised exclusively in terms of mathematical knowledge, because it is about **an individual's capacity to *use* and *apply* this knowledge**. Thus it has to be conceived of in functional terms as applicable to the situations in which this knowledge is to be used. (Jablonka, 2003, p. 78, my emphasis)

In the late 1990's, the international education discourse changed from "qualification" to "competence", and today the term "competence" is almost hegemonic in educational discourses, with "mathematical literacy" and "numeracy" as prominent examples of constructs in mathematics education, where they are often divided into partial competencies (e.g. competency to interprete quantity & numbers and competency to indentify dimension & shape). Qualification can be defined a priori in terms of skills and knowledge and has to do with formal education and certification, while competence deals with people's capacity – based on knowledge and authority – to handle a specific type of situations (Wedege, 2003). In the OECD context of defining general concepts of qualifications and competences in the mid 1990's, a voice was heard introducing the individual in the discussion. Fragnière (1996) stated that the competencies are composed by the subjective ability to use one's qualifications, know-how and knowledge to accomplish something: "In fact, there are no "objective" competencies capable of being defined independently of the individuals in which they are embodied. There are no competencies in and of themselves; there are only competent people" (p. 47).

In constructions of ethnomathematics based on the definition "mathematics practiced by cultural groups" which was introduced by D'Ambrosio (1985), human activities such as counting, measuring, locating etc. and people's capacities to handle situations through these kind of activities are acknowledged as important. In the PISA definition of mathematical literacy from 2000 presented above, there was not any individual to use and engage with mathematics. But in the 2003 survey, where mathematics is the major domain, the individual is introduced in the definition [2] and, at the same time, the criteria of employability is removed and by that the citizenship perspective is accentuated:

> *Mathematical literacy* is **an individual's** capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen. (OECD, 2003, p. 37, my emphasis)

However, the way of modulating and describing individuals' mathematical literacies in a framework of eight "objective" competencies defined independently of the individuals is still the same. So is the way of testing young people with standard mathematical tasks across countries and cultures as the example in Figure 1.

On the cross road between the second and third generation of lifelong learning, educational discourse and terminology have changed from qualification to competence but, in the political and educational context, some of the qualities of a scientific competence concept have been lost. For example the dissolution of the classical dichotomy of knowledge versus skills which allows the recognition of tacit knowledge and of knowing and learning in practice/action. In its basic sense of human capacity, competence unites the complex mix of knowing and doing. In the article entitled "Constructions of competence concepts", I have argued that in many educational documents, like policy reports and curricula, the discourse is not guided by a *logic of competence*, where the dualism between the individual and the situation is constitutive, but rather by a *competency logic* like the one we find in the international surveys. The former refers to inherent properties of the concept, while the later is meant to apply to a given context, once a concept of competency has been specified (Wedege, 2003). According to Bernstein (2000), in a *recontextualising process*, discourses are relocated from their scientific fields of production (philosophy, anthropology, psychology etc.) to establish pedagogic discourse within the field of reproduction (education) (see also Dowling's paper in this book). In Bernstein's analysis of the recontextualisation of "competence", he contrasted two pedagogic models: Competence and performance (see table 4).

In the competence models, emphasis is upon the realisation of competences that "acquirers" already have, or are thought to have. The pedagogic text reveals the acquirer's competence development cognitively, affectively and socially. In the performance models, emphasis is upon a specific output of the acquirer, upon specialised skills. The pedagogic text is the acquirer's performance objectified by grades. The competence models, like educational constructions of ethnomathematics, focus on the individuals' mathematical capacities in different contexts, while the performance models, like educational constructions of mathematical literacy, focus on the required qualifications in mathematics predefined in terms of competencies. However, as pointed out by Jablonka and Gellert (2010) when ethnomathematics is imported into the classroom discourse via a curriculum, there is a risk that the purpose of the recontextualisation is misuse in terms of traditional school mathematical topics.

Table 4: Recontextualised knowledge (Bernstein, 2000 p. 45).

	Competence models	*Performance models*
1. *Categories:*		
space time discourse	weakly classified	strongly classified
2. Evaluation or ientation	presences	absences
3. Control	implicit	explicit
4. Pedagogic text	acquirer	performance
5. Autonomy	high	low/high
6. Economy	high cost	low cost

Returning to lifelong learning as a guiding principle for education, one might conclude that the two competence models co-existing within mathematics education as ethnomathematics and mathematical literacy illustrate the third generation (economistic – social cohesion).

Sociomathematics

I introduced the expression “knowing mathematics in the world” to cover a wide spectre of notions and ideas about what it means to know and to use mathematics in different cultural and societal contexts and situations. After observing that functionality is a common trait of these notions and that contextuality is either involved or ignored in the different constructions, and after observing that any concept is based on one of two meanings of everyday knowledge as developed or wanted in everyday, I choose ethnomathematics (D’Ambrosio) and the human capital version of mathematical literacy (OECD) as prototypes. Basic conceptions of mathematics as respectively value-laden & cultural dependent and neutral & universal are other distinguishing traits. Finally, I contrasted two conceptions of mathematical competence in the discourse ruled by logic of competence with the individual’s capacity as the core respectively by logic of competency with performance in focus.

In any empirical study of people knowing mathematics in the world it is possible to take a *subjective approach* starting with people’s subjective competences and needs, and an *objective approach* starting either with societal and labour market demands or with the academic discipline (transformed into "school mathematics"). The following two workplace studies in mathematics education illustrate the conflict between the general and the subjective approaches. In a study on proportional reasoning in expert nurses' calculation of drug dosages, Hoyles et al. (2001) compared formal activities involving ratio and proportion (general mathematical approach) with nurses' strategies tied to individual drugs, specific quantities and volumes of drugs, the way drugs are packaged, and the organization of clinical work (subjective approach). In their large project involving 22 case studies, Hoyles et al. (2002) research questions were about employers' demands for mathematical qualifications, competencies and skills (general societal approach) and about what skills and competencies the employees felt were needed for the job, and what they currently possessed (subjective approach). However, to understand the cognitive, affective and social conditions for people knowing mathematics in the world, one has to take both dimensions into account (FitzSimons, 2002; Wedege & Evans, 2006). In PISA, the approach leading to the theoretical framework is general (OECD, 1999). PISA claims that the starting point is societal and labour market demands. However, the framework is based on a conceptual construct of academic mathematical knowledge in terms of competencies and not on empirical research on people's needs of mathematics in society.

I have introduced the concept of *sociomathematics* for a subject field where people, mathematics and society are combined, and for the research field where the societal context of knowing, learning and teaching mathematics is taken seriously into account (Wedege, 2010). The subject field encompasses constructions of knowing mathematics in society e.g. mathematical literacy and critical ethnomathematics. As a research field, sociomathematics combines general and subjective approaches when studying people's relationships with mathematics in society. Any sociomathematical study is based on the idea of dialectic interplay between the following two dimensions of everyday knowledge:

- Knowledge developed in everyday life, i.e. knowledge that the individual has acquired in her/his everyday and societal practice.
- Knowledge required in society, i.e. knowledge that is relevant/ useful in people's everyday and societal practice.

In a sociomathematical construction, people's everyday mathematics is recognized as mathematics and, at the same time, the powerful position of academic mathematics is acknowledged. Critical ethnomathematics, in Table 2, type 1, as defined by Knijnik (1999) is an example. Her study of landless peasants' mathematics is not just about people's competences in a well-defined cultural context but about a larger political context, where power relations are made visible. Mathemacy (type 2) defined by Skovsmose (2006) is also an example of a concept based on a dialectic interplay between individual needs and dispositions and societal needs and requirements.

Notes

1. This distinction is made with inspiration from the research project “Everyday knowledge and school mathematics” (Wistedt, 1990).

2. In the first PISA framework published in 1999, the definition of mathematical literacy also started with “an individidual’s capacity” (OECD, 1999, p. 41), but a different definition was used in the 2001 report.

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A Theoretical Model of the Connection Between the Process of Reading and the Process of Solving Mathematical Tasks

Ewa Bergqvist and Magnus Österholm
Umeå University, Sweden

In this paper we suggest a theoretical model of the connection between the process of reading and the process of solving mathematical tasks. The model takes into consideration different types of previous research about the relationship between reading and solving mathematical tasks, including research about traits of mathematical tasks (a linguistic perspective), about the reading process (a psychological perspective), and about behavior and reasoning when solving tasks (a mathematics education perspective). In contrast to other models, our model is not linear but cyclic, and considers behavior such as re-reading the task.

Introduction

The relation between reading ability and mathematical ability is important to examine, especially since written tests are the predominant form of assessment in mathematics teaching. However, this relationship is also complex. On the one hand, tests which are intended to measure achievement in mathematics should not measure reading ability, which the framework for PISA highlights by noting that: "The wording of items should be as simple and direct as possible" (PISA, 2006, p. 108). On the other hand, a student has to be able to read and write in order to pass a paper and pencil test. Also, communication is one of the competences brought forward within frameworks describing school mathematics worldwide; see for example NCTM (2000) and PISA (2006). Reading (and writing) mathematics can then be seen as an important part of knowing mathematics. Thus, we cannot completely separate reading ability from mathematical ability.

If you, the typical reader of this article, read a mathematical task in order to solve it, when does the solving begin? It can be difficult to separate the reading process from the solving process (e.g. see the analysis by Österholm, 2007). In some cases, the process of reading seems to be clearly separated from the process of solving, and in other cases the two processes seem to be highly integrated. Sometimes you start to think of the solution before you are done reading. This might happen when you read the following task: "Seven girls had a backpack each, and in each backpack there were seven cats. Each cat had seven kittens, etc." Sometimes you have solved the task at the same time as you are finished reading. For example, this might happen when you read the following task: "What equals 2+3?" Other times you have to read the task several times in order to solve it, but when you finally have figured out what the task's wording actually meant, you already know the answer. This might happen when you read the following task: "If Anne was half Mary's age when Mary was 14, then how old will Mary be when Anne is 50?" In fact, once you have decoded the text, the solution is right at hand. It seems unclear whether this process should be seen as an instance of utilizing a reading ability, a mathematical ability, or a mixture thereof. Through these examples we highlight, again, the complex relationship between reading ability and mathematical ability, and also the complex relationship between reading and solving a mathematical task. In addition, these simple examples highlight the need for a theoretical model of reading and solving mathematical tasks that takes these relationships into consideration.

The purpose of this paper is to present a theoretical model that describes aspects of both reading and solving mathematical tasks and that takes into account research from three perspectives; linguistics, psychology, and mathematics education. By examining existing research about relations between reading and solving mathematical tasks in each of these perspectives, we argue for the necessity of a model that combines them. More long-term goals, not to be fulfilled in this paper, are that the model should be possible to use to explain empirical data and as a guide for planning and designing empirical studies about the relationship between reading and solving mathematical tasks. The model is of a theoretical nature since it is not based on empirical data, and the purpose is not that it should act as a theory for analyzing all aspects of task solving or of reading comprehension. The purpose is to capture the dynamic relationship between reading and solving tasks, in order to be able to study this relationship in more detail.

Three perspectives on reading and solving

In the existing literature about reading and solving mathematical tasks, we find that articles tend to focus on properties of the task text (a linguistic perspective), on the reading process (a psychological perspective), *or* on behavior and reasoning when solving a task (a mathematics education perspective). The literature survey by Österholm (2007) shows that not much research exists that directly focuses on the relationship between reading and solving mathematical tasks, and in line with this result we have found that only in a few cases does a single study consider a combination of at least two of the mentioned perspectives.

In the following three sections we characterize each of the three perspectives, in particular regarding if and how they view connections between reading and solving mathematical tasks. We do not discuss many (empirical) studies in relation to reading and solving within these three large research areas, but the purpose is now to localize and discuss frameworks and models that seem most useful for our purpose of creating a model that characterizes the connection between the process of reading and the process of solving mathematical tasks.

The linguistic perspective – the wording and grammar of tasks

Within linguistic research the object of analysis is often the wording or the grammatical properties of the text. There are some features of mathematical texts and tasks that might decrease the readability, according to linguistic research. Examples of such properties are technical vocabulary, multiple semiotic systems, and grammatical patterns (Schleppegrell, 2007). Linguistic research also considers in what ways readability formulas can predict the readers' difficulties understanding a text or a test item (Homan, Hewitt, & Linder, 1994). The underlying assumption is that texts with higher readability indices are more difficult to read than texts with lower indices. These studies often use predictors, for example the number of difficult words or word length, as basis for the calculation of an index (Homan et al., 1994). The implicit model of the reading comprehension of a mathematical task within linguistic research can therefore be seen as a simple function of two variables, the reader's prior knowledge (including different kinds of abilities) and the complexity of the text; that reading comprehension increases with better prior knowledge and with lower text complexity.

The psychological perspective – the process of reading

There exists plenty of research from a psychological perspective focusing on reading comprehension and the process of reading. In this kind of research, reading seems most often discussed through the characterization of mental processes and mental representations. Such characterizations are done in different ways in different theoretical frameworks, for example by relating to Bloom's taxonomy regarding mental processes (Graesser, León, & Otero, 2002) or by describing multiple levels of representation (van Oostendorp & Goldman, 1998).

There is no room here to cover the complexities of different types of characterizations of mental processes and mental representations in order to discuss different possibilities for describing the reading and solving of mathematical tasks as seen from a psychological perspective. However, there are some frameworks about reading, as seen from a psychological perspective, which have also been applied or related to the solving of mathematical tasks, which is an important aspect in our choice of framework. Kintsch's (1998) theory of *comprehension* will be used as the main framework describing aspects of reading in our model, a choice based on three reasons: (1) the work of Kintsch has had a great general influence on research about reading comprehension (Weaver, Mannes, & Fletcher, 1995), (2) the theory includes detailed models about both the mental process and the mental representation in reading comprehension, and

(3) the theory has been applied to the situation when students solve mathematics word problems.

Kintsch (1998) describes the mental representation of texts by distinguishing between three different levels, or components, of the mental representation; the surface component, the textbase, and the situation model. The surface component refers to when the words and phrases themselves, and not their meaning, are encoded in the mental representation. The textbase represents the meaning of the text, that is, the semantic structure of the text. The situation model is a construction that integrates the textbase and aspects of the reader's prior knowledge.

Besides the components of mental representation, Kintsch's theory also includes a model, the construction integration model, which describes how a mental representation is created in the comprehension process, in particular how the utilization of prior knowledge occurs through associative activation. This model describes a fundamental cognitive functioning when interpreting something 'external', which for example can refer to a given text and also a text you create yourself, such as when starting a calculation in the solving of a task.

When Kintsch (1998, chapter 10) applies his theory of comprehension to word problems, another component is added to the description of mental representations; the so called problem model, which is a mathematization of what is described in the text. The problem model consists of a schema from the reader's long-term memory that is activated based on some properties of the situation model. This reliance on schemas has been criticized by other researchers, for example since there are empirical results about word problems that do not seem compatible with a schema theory (Thevenot, Devidal, Barrouillet, & Fayol, 2007). Kintsch (1998, p. 354) also acknowledges that there are limitations to this part of the theory, since it assumes the existence of "full-fledged schemas […] that need only to be applied correctly", while other studies he refers to show that this is usually not the case, and that acquiring such schemas "is a major facet of learning". Instead, the solving of word problems usually utilizes knowledge that is "less orderly, less abstract, and more situated" (p. 357).

It seems like a model of the reading and solving of mathematical tasks from a psychological perspective could be divided into two cases. First, we have a model that is based on the utilization of schemas. This model seems to be limited to tasks that are of a very familiar type for the reader, since it assumes the existence of "full-fledged schemas", which have been created through abstraction from plenty of experiences of similar types of tasks. Second, we have a model that turns "away from the abstract schema concept and toward the notion of situated cognition" (Kintsch, 1998, p. 355). In this model it seems unnecessary to introduce the notion of problem model, since this notion seems closely connected to abstract aspects and not situational aspects.

The mathematics education perspective – the process of solving

Much mathematics education research related to reading and solving mathematical tasks consider *word problems*. Although there is plenty of such research, a literature survey among journal articles showed that "not many studies exist that in a direct manner examines the relation between reading and problem solving among the 199 references about word problems" (Österholm, 2007, p. 141). Instead, most studies about word problems seem to focus on aspects of modeling and relationships between the task and the "real world" (e.g. see Verschaffel, Greer, Van Dooren, & Mukhopadhyay, 2009), where the *process of reading* is most often not problemized. In addition, studies about word problems that do focus on aspects of language and reading seem to see a separation between the process of reading and the process of solving, which is not congruent with our own view. For example, Roth (2009, p. 63) characterizes the problem an informant has as "related to understanding the text, not one of providing the sought-for problem" and that students could need assistance in "understanding first the text and then the task". Therefore, instead of using perspectives on reading and solving from studies about one genre of mathematical tasks (word problems), which tend to focus on aspects not directly relevant for other types of tasks, we choose to direct our attention to frameworks that more generally describe mathematical task solving.

Within mathematics education research there are many theoretical frameworks that can be used to describe the *solving* of mathematical tasks, see for example Vinner (1997) and Lithner (2008). These authors both present two fundamentally different concepts that are relevant when describing how students solve a mathematical task; pseudo-analytic and analytic thinking (Vinner, 1997), and imitative and creative reasoning (Lithner, 2008). The first concept in each framework is connected to the student *knowing how* to solve a task without necessarily knowing why the method works, and the other to the student *knowing both how and why*. Lithner (2008) sees the wording of a task as a surface property that the solver sometimes can use in order to identify the type of solution strategy (cf. schema) appropriate to solve the task. Vinner (1997) similarly discusses how the solver can determine the appropriate solution procedure to a given question by using mental schemes to determine the *similarities* between the question and so called *typical questions*. He does not explicitly say so, but it is reasonable to assume that these similarities could for example be based on the wording or grammatical structures of the tasks. Schoenfeld (1985) considers the activity of reading as an important part of problem solving but as a behavior rather than as a mental process. Pòlya (1945) includes issues connected to reading in what he calls the first phase of problem solving; understanding the problem. He argues that the wording of the problem needs to be understandable for the student, but he does not further examine what the connection is between reading and solving.

These and other similar frameworks do not focus on the reading *process*, but put more emphasis on reading as a type of activity or behavior when describing the solving of mathematical tasks. However, they do sometimes mention or consider the wording in the mathematical tasks when discussing the students' solution strategies.

Lithner's (2008) research framework will be used as the main framework describing the solving process in our model, since we see it as the most detailed framework, including more well-defined components, compared to similar frameworks, and since it is in many ways representative of the type of frameworks discussed above, for example Vinner (1997). Also, this framework neither focuses on nor excludes word problems. Lithner characterizes different types of reasoning and in this paper we apply the framework to the solving of mathematical tasks in particular. Therefore we see "reasoning" as any method that can be used to solve tasks (a standpoint that basically coincides with the definition of reasoning within the framework) whether the solution is correct or not. Lithner divides all types of reasoning into two major categories, imitative and creative reasoning. The basic types of imitative reasoning are memorized reasoning (i.e. remembering a whole answer) and algorithmic reasoning (i.e. remembering an algorithm and calculating an answer) and both are fundamentally different from creative reasoning.

The basic steps when performing imitative reasoning are *choosing a strategy* and *implementing the strategy* (Lithner, 2008). The choice is made based on surface properties of the task, for example the wording. The implementation consists of writing the answer down (memorized reasoning) or following the steps of the algorithm (algorithmic reasoning).

Creative reasoning is based on the intrinsic mathematical components of the task, and the reasoning is new (to the solver) and flexible. In order to solve a problem (a task that is basically new to the solver) it is necessary to use creative reasoning, since it is not possible to identify the task and choose a memorized method based on its surface properties.

A suggested model of reading and solving mathematical tasks

Criticism of existing models

Empirical studies related to Kintsch's (1998) theory of different components of mental representation in reading comprehension confirm the need to distinguish between these components. These studies also show an intricate relationship between text properties (in particular text coherence), the reader's prior knowledge, and the reader's reading comprehension. In particular, results show that "readers who know little about the domain of the text benefit from a coherent text, whereas high-knowledge readers benefit from a minimally coherent text" (McNamara, Kintsch, Songer, & Kintsch, 1996, p. 1). These results from the psychological perspective highlight some shortcomings in the linear model of reading comprehension that is used, or implied, in linguistic research. We are however not dismissing linguistic research as a relevant influence on the process of creating our model. The plethora of research that statistically connect different linguistic properties of tasks to either reading comprehension or success rate in solving the tasks need to be considered in our model, which is not done in this paper, but planned for in future studies.

Even if psychological theories relate to, and make predictions of, students' solving of mathematical tasks, they are in principle limited to the description of mental representations. However, the behavioral aspects of solving mathematical tasks cannot be reduced to something directly and unambiguously determined by mental representations of texts. As Kintsch (1998, p. 333) mentions: "What the student remembers and what the student does are related in informative ways and *mutually constrain each other*" (emphasis added). This relationship, in particular the mutual constraint, is not described within the psychological models of reading and solving mathematical tasks, since these models focus on describing mental representations (i.e. content and structure of memory). However, this relationship is not included in theories from mathematics education research either, since such theories try to explain phenomena through aspects of students' explicit strategies and reasoning, thus focusing on behavior rather than mental representations. This type of research ignores the process of reading the task and the good predictions that can be made from psychological theories that focus on details of the reading process. Thus, an important starting point for our model is to analyze connections between these two perspectives; one that focuses on mental representations (the psychology perspective) and one that focuses on aspects of behavior (the mathematics education perspective).

A study of Hegarty, Mayer, and Monk (1995) focuses both on aspects of mental representation and on aspects of behavior when students are solving mathematical tasks, since it relates data on students' reading behavior to the creation of different types of mental representations. However, there are some inconsistencies within this study, in particular regarding the relationship between the two steps of problem solving the authors refer to (p. 19); constructing a problem representation (how a student understands a problem) and solving a problem (including computational procedures and problem solving strategies). The eye fixation method used by the authors is described as a method "to gain insights into the nature of the comprehension processes" (p. 19), but the empirical results make it plausible that "the eye-fixation protocols cover both the comprehension and planning stages" (p. 25). Instead of forcing the descriptions of method and data into a model that presupposes a linear separation between processes of comprehension and of solving (i.e. between aspects of mental representation and aspects of behavior), we argue for a model that can more easily describe the eye-fixation data, and that also includes central theoretical components from the perspectives of psychology and mathematics education.

The structure of our model

What is presented here is the first version of our model, which includes a basic structure of the relationship between aspects of reading and solving a mathematical task. This structure is a contrast to the common linear types of models, which are problematic since they, as described earlier, presuppose a separation between reading and solving. In addition, we use a structure that takes into account the mutual constraint between aspects of mental representation and aspects of behavior. Therefore, our model includes a cyclic component (see Figure 1) that allows for the behavioral component to affect the mental representation, and not only the other way around.

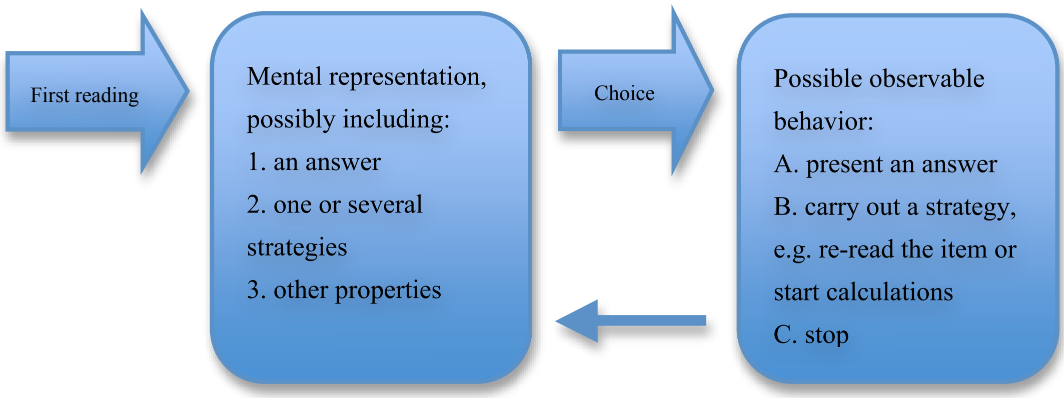

Figure 1: A visualization of the suggested model of reading and solving mathematical tasks.

The first step in the model is the *first reading* of the text, a reading that does not have to include the whole text from beginning to end. This reading creates a mental representation (MR) of the text through a process of comprehension (Kintsch, 1998). Through activation of prior knowledge (i.e. a creation of a situation model), this MR can include the explicit answer (1 in Figure 1) to the question asked in the task (e.g. when '5' is activated in the reading of the task 'What equals 2+3?') or a strategy (2 in Figure 1) that can be used to get (closer) to an answer (e.g. when 'derivation' is activated when reading a question about maximum or minimum values). The word 'strategy' is here used in a wide sense; it could mean a mathematical strategy (e.g. an algorithm, cf. Lithner, 2008) or a more heuristic strategy, for example including the step 're-read the item' or 'ponder the question'.

These properties of the MR then guide the behavior, the choice, for example so that the solver presents the answer (A in Figure 1) if it is a part of the MR or so that he or she can carry out a strategy (B in Figure 1) that has been activated. However, it can also be other properties (3 in Figure 1) of the MR that guide the behavior (other than a direct activation of an answer or a strategy), for example that the MR is very fragmented, which could make the reader choose to re-read the text. Note that this strategy of re-reading could also have been activated in the MR, but it seems impossible to separate these possibilities in practice. Furthermore, even if the MR could activate a behavior that can be directly "applied", a type of choice is always present, since the MR does not directly determine behavior. This is in line with Lithner (2008, p. 257) who sees the concept of *strategy choice* in this setting in a wide sense; "choose, recall, construct, discover, guess, etc." The third option, to stop (C in Figure 1), includes giving up but also leaving the task for various other reasons (e.g. lack of time or becoming bored).

If you merely either present an answer (A) or stop (C), your behavior has little potential to affect the MR. However, every kind of *strategy implementation* (B) will affect the MR since what you do is directly related to the task and it thus becomes part of the comprehension of the task. For example, some types of activities might be seen as mostly adding something to the MR (e.g. carrying out calculations) while other types might be seen as mostly changing the existing MR (e.g. re-reading). These changes to the MR can, in the same way as it is done in the first reading of the text, activate parts of prior knowledge and can cause the new version of the MR to include new answers or strategies, thus starting a new turn in this cyclic process.

Other studies and theories in relation to our model

Let us now return to the study by Hegarty et al. (1995). Their data consist of patterns in students' eye fixations as they read and re-read (parts of) the task text. Within our model, we have the first reading of the task as a first step, whereas the re-reading of the task is a behavioral aspect that has been performed based on some property of the MR created in the reading process. As mentioned, this property can be of different types, for example that the MR includes a specific strategy for creating the solution (e.g. to add two specific numbers given in the text) where the re-reading is performed in order to recall some specific information given in the text (e.g. the specific numbers to add), or it could be that the MR includes a more general strategy for how to handle mathematical tasks (perhaps of a specific kind). Since Hegarty et al. (1995) used relatively simple and familiar types of tasks it can be assumed that the students had knowledge of specific methods for solving these tasks. That is, the type of reasoning active for these students was of an imitative type (as labeled by Lithner, 2008). Thus, in our model, this type of reasoning can be described as the activation in the MR of a direct answer (corresponding to Lithner's memorized reasoning) or of a strategy (corresponding to Lithner's algorithmic reasoning). In the present version of our model it is these types of reasoning that can be described and modeled more specifically.

However, also aspects of creative reasoning could be located within our model. In particular, the constant “flow of information” in both directions between behavior and MR, which is present through the cyclic property of our model, can open up for a more dynamic and flexible pattern of thinking and behavior, which are typical criteria for creative reasoning (e.g. in Lithner, 2008). In addition, the process of comprehension (as defined by Kintsch, 1998) is active in many parts of the model, and this process includes the automatic associative activation of prior knowledge, which can include aspects of creative reasoning. For example, if you have never thought about X and Y at the same time before, but a task presents both these aspects, the activation in the process of comprehension can cause new connections to be made between X and Y in particular but also between associations to X and Y. This process can be seen as creative in a novelty aspect (a criterion given by Lithner, 2008).

Conclusions and further research

Our model is based on the assumption that we cannot separate the reading and solving of mathematical tasks. This assumption is argued for by reflecting on our own reading and solving (see the introduction) and by highlighting difficulties in the use of models that assume the opposite (see the description of our model). In addition, our model can be combined with more elaborate models related to the two main components of our model; Kintsch’s (1998) theory in relation to mental representations and Lithner’s (2008) theory in relation to behavioral aspects of task solving.

In the continued refinement of and argumentation around our model we will relate to more types of empirical results, in particular results from the linguistic perspective, and we will also refine the structure described in Figure 1, in particular regarding the meaning of the arrows used to denote some kind of transition and/or influence. Besides this type of continued research, we will also use this model as a basis for empirical studies in order to test the usefulness of the model and to make direct verifications or rejections of (parts of) the model.

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A Remark on Didactic Transposition Theory

Christer Bergsten[1], Eva Jablonka[2] and Anna Klisinska[2]
[1]Linköping University, [2]Luleå University of Technology, Sweden

With reference to a historical study on the relation between the production and distribution of mathematical knowledge, using calculus as an example, some assumptions in didactic transposition theory, as introduced by Yves Chevallard, are discussed. Given the prominent status of this theory, the paper intends to initiate a debate that could help lifting it out from its relative isolation within mathematics education as a research domain.

Introduction

The relation between mathematical practices in educational institutions and other mathematical practices has been a key concern for mathematics education. The practice of academic mathematics at universities is still meant to be a model for conceptualising school mathematics in post compulsory mathematics, even though there are other conceptions of curriculum. Elements of this practice are selected and 'transformed' for the purpose of teaching.

A theory that has been concerned with this process and has gained considerable attention, also in the Nordic community of mathematics education, is the "didactic transposition theory" (Chevallard, 1985; 1991). This theory has been employed also in subject areas other than mathematics, has initiated an extended research programme (Bosch & Gascon, 2006), but has also met critique (see e.g. Beitone, Decugis, Dollo, & Rodrigues, 2004, pp. 62-69; Freudenthal, 1986). The aim of this paper is to discuss and challenge some basic premises of didactic transposition theory by way of an example, based on a historical study of the didactic transposition of a specific body of mathematical knowledge, the calculus (Klisinska, 2009). The discussion points to the role of the pedagogic discourse for the social construction of mathematical knowledge. By this we attempt to draw attention to possible productive interactions of the didactic transposition theory with other theorising employed in mathematics education.

The didactic transposition

The first comprehensive outline of the theory of didactic transposition is found in Chevallard (1985), with an extended edition in Chevallard (1991) and it was presented at the first *Ecole d'été de didactique des mathématiques* in 1980 (Chevallard, 1991, p. 7). The term 'transposition didactique' was used earlier (Chevallard, 1978). It aims at producing a scientific analysis of 'didactic systems' and is based on the assumption that the knowledge set up as a teaching object ('savoir enseigné'), normally has a pre-existence as *scholarly knowledge* ('savoir savant'), that is "a *body* of knowledge, not knowledge in itself", ranging from "genuinely scholarly bodies of knowledge to scholarly-like or even pseudo-scholarly ones" (Chevallard, 1992a, p. 228). "For many bodies of knowledge taught at school the integrated whole required existed outside school. School mathematics, for example, has essentially evolved from mathematicians' mathematics" (Chevallard, 1989, p. 57). Thus, the mathematics taught at school has been generated outside school and moved ('transposed') to school by a series of adaptations before being accepted for teaching.

According to Chevallard (1991, p. 43) there is often an immense difference between these objects of knowledge. He defines the didactic transposition as the work done during the transformation from *scholarly knowledge* via *knowledge to be taught* and the actual *knowledge taught* to *learnt knowledge* (see e.g. Bosch & Gascon, 2006). The first step in this sequence of transformations of knowledge is taking place in the *noosphere*, a non-structured set of experts, educators, politicians, curriculum developers, recommendations to teachers, textbooks etc. Analysing the 'knowledge to be taught' through the agents and materials from the noosphere reveals the conditions and constraints under which it is constituted.

The analysis of the didactician aims at making visible the difference between the transposed (taught) object and the scholarly object, a difference not spontaneously perceived by the teacher. In addition, while ruled by norms and values attached to the educational institution, the teacher does not always take responsibility of the epistemological consequences of this difference. There is an *illusion of transparency*, a feeling that the knowledge to be taught is not to be questioned, which may lead to an 'epistemological rupture' of the knowledge objects (Chevallard, 1991, p. 42-43). As to the notion of transparency, Chevallard refers to Bourdieu, Chamboredon, and Passeron (1973). In their critique of naïve sociology, they propose that the transparent unmediated, common sense knowledge about the facts of social life are hard to overcome because of common metaphors lurking in language. Chevallard links the illusion of transparency to the teachers' conception of the supposedly mathematical facts thought at school. The difference between the teacher and didactician resembles, then, the difference between the holder of naïve sociological knowledge and the sociologist.

The didactic transposition implies a *textualisation* of knowledge, as well as a *depersonalisation*, thus producing an objectification possible to be made public and to form a basis for social control of the learners by developing systems for testing (Chevallard, 1991, pp. 61-62). The 'text of knowledge' produced thus serves as a norm for knowledge and for what it means to know, as well as for the progression of knowledge, authorising didactical choices.

Knowledge is "living" in institutions. Chevallard (1989, 1992b) distinguishes "knowledge in use" or "practical knowledge" from teachable knowledge. Based on Bourdieu's notion of "practical logic", he makes the distinction between "institutional systems of acquaintance (connaissance)" and knowledge (savoir) and suggests that in order to become teachable the former have to be transformed into knowledge. Such "body of knowledge", must be produced before it can be taught or "utilised" and not only "put to use" (1992b, p. 162).

A question that can be analysed from the perspective of didactic transposition is what different institutions define as legitimate knowledge. An essential difference between research mathematics and school mathematics is seen in the principles that govern knowledge growth. In the scholarly field, a problem is the driving force of knowledge construction, while in teaching the progression is run by the contrast between the old and the new objects (Chevallard, 1991). "Bodies of knowledge are, with a few exceptions, not designed to be taught, but to be used. To teach a body of knowledge is thus a highly artificial enterprise" (Chevallard, 1989, p. 56). Hence, research-type mathematical behaviours and attitudes are difficult, if not impossible, to obtain in the mathematical practice of classrooms. This becomes clear when, for example, proving theorems is at issue.

Intellectual roots of the theory

The idea of a didactic transposition of scholarly knowledge was adapted and elaborated by Chevallard from the sociologist Michel Verret (1975), who emphasised that knowledge can not be taught in the way it was produced in the scientific community: the 'transmission didactique' induces a selection as it privileges the success, continuity, and synthesis of knowledge, not typical characteristics of the production of knowledge (pp. 140-141). Due to the separation of subjects in teaching institutions, and the need for evaluation, a didactic transposition process is defined by decomposition, depersonalisation and development of a detailed teaching sequence of knowledge (pp. 146-147). These three notions are used by Chevallard. The process of transposition presupposes that knowledge to be taught is clearly defined and open to social control.

Other intellectual roots are mostly mentioned as sources of inspiration rather than in the form of specific references and often remain implicit. Ideas concerning the need of a 'transmission didactique' to make teaching possible were expressed already by Auguste Comte (1852) in his *Catéchisme positiviste.* When discussing the teaching of religion, he provides the following argumentation for his choice of the dialogue as the format of his didactic text (pp. 10-11):

> A discourse, then, which is in the full sense didactic, ought to differ essentially from a simple logical discourse, in which the thinker freely follows his own course, paying no attention to the natural conditions of all communication. [1]

To avoid the tedious logical elaborations of a lecture, Comte argues for dialogue:

> One should use a dialogic format, appropriate in any true communication, for explaining such notions that are at the same time important and mature enough […] Far from indicating an excusable negligence just towards secondary matters, this format, being well institutionalised, instead constitutes the only mode of exposition which is truly didactic: it suits equally well all levels of intelligence. [2]

He concludes that (our emphasis)

> this transformation for the purposes of teaching (orig. *transformation didactique*) is only practicable where the doctrines are sufficiently worked out for us to be able to distinctly compare the different methods of expanding them as a whole and to easily foresee the objections which they will naturally elicit. [3]

This statement highlights that it is only for the teaching of an elaborated body of knowledge that Comte sees a need of a 'transformation didactique', which changes the principles of the discourse, to make it conceivable for the intended audience. In the dialogue outlined in his book, which can be seen as a textbook in religion, the scholarly knowledge (to use the term of Chevallard) is presented by a sanctioned representative of the field in focus, a priest. In his presentation of the didactic transposition, Chevallard does not refer to Comte, though he mentions Comte when discussing other issues (Chevallard, 1991, pp. 215, 218).

Verret was concerned with teaching at university level, while Chevallard uses examples mainly from secondary school mathematics in France. This is emphasised by Beitone et al. (2004, p. 57) who claim that didactic transposition theory is even more useful today than in the mid 70's for analysing university teaching. Both Verret and Chevallard also discuss different types of knowledge and which types can be seen as scholarly knowledge, and note that certain kinds of practical and tacit knowledge are not, and cannot, be taught outside their own fields of practice.

Critique of the theory

Soon after the first publication of Chevallard's (1985) exposition of his theory, Hans Freudenthal (1991) presented a critical review of the book, written in French. While appreciating the eloquence of the language, though wondering what is hidden between the serious and the ironic, Freudenthal questions the whole idea of a didactic transposition mainly because of the vagueness of the term 'savoir' (knowledge), and in particular the term 'savoir savant' (translated to 'scholarly knowledge' by Chevallard). What is the scholarly knowledge that is transposed, for example, into the teaching of arithmetic or algebra at elementary school levels? From the example given by Chevallard in the text (further elaborated with M-A. Johsua in the 1991 edition), Freudenthal concludes, ironically, that scholarly knowledge must refer to the 'good mathematics' produced by some great mathematicians from history, now to be transposed to the level of understanding of the youth (Freudenthal, 1986, p. 325). He points out that school mathematics and the students of today are concerned with technological aspects of knowing how to do rather than learning transposed versions of more or less ancient scholarly mathematical knowledge, which has only a marginal influence in the modern technological culture.

After this and other critique of his theory, Chevallard notes that “Experience shows that the theory of didactic transposition is an easy prey to misunderstanding” (1989, p. 51). According to Beitone et al. (2004), the theory has been challenged by three main points of critique. One argument states that scholarly knowledge is the source of knowledge for teaching and cannot have different value or character depending on the institution that handles it. As the transposition process in the theory is conceptualised as ‘degrading’ knowledge, it should be avoided by the participants (such as teachers) by not taking part in it. According to the second argument, knowledge as treated in school is not a simple derivation of scholarly knowledge with a logic imposed by academic mathematics. The view of knowledge as described by the didactic transposition theory has to be seen as an expression of elitism not in line with modern pedagogy. In particular, school subjects not similar to mathematics and science in terms of knowledge principles remain out of the scope of the theory, as for example the teaching of language. Finally, reference knowledge for teaching in school also has other origins than in scholarly fields, such as different kinds of tacit knowledge involved in social practices. This is, for example, the case in school curriculum conceptions underpinned by ethnomathematics, or in university mathematics courses for some traditionally non-academic vocations (such as nurses). However, the issue has been recognized by Chevallard (1989).

As to the possibility of identifying a clearly delineated body of scholarly knowledge as a blueprint for judging its didactic transposition as legitimate, Chevallard recognizes a difficulty:

> In most cases […] a given body of knowledge will appear only in fragments. […] The first step in establishing some body of knowledge as teachable knowledge therefore consists in making it into a *body* of knowledge, i.e., into an organized and more or less integrated whole (ibid., p. 57).

This quote points to a possible influence of the distribution of knowledge on the production of knowledge, an issue that questions the premises of the didactic transposition theory. Further, it suggests the pre-existence of the “body of knowledge” that has been produced outside the teaching institution. These issues were highlighted in a study with a focus on the didactic transposition of proof, which used the Fundamental Theorem of Calculus as an example (Klisinska, 2009).

The didactic transposition of calculus

In the development of the mathematical sub-area today known as calculus, the Fundamental Theorem of Calculus (FTC) played a key role in linking integration and differentiation. Through a historical study Klisinska (ibid.) investigated the dynamic between the adaption of the FTC and the calculus for teaching and its establishment as part of the scholarly knowledge. In an interview study, she also investigated how mathematicians at universities (producers of mathematical knowledge) interpret the FTC and how they see this area when they act as ‘transposers’ of this body of knowledge for teaching. Our exposition below will refer to the outcomes of the historical study only. [4]

The development of the statements connected with the FTC, which initiated the institutionalisation of a new body of knowledge, was studied with reference to original works of researchers and classical works about the history of calculus. By institutionalisation we refer to a process of crystallisation of a specific discourse. We take the regularity by which significations are recognised as belonging to the distinctive discourse of a practice, and the extent of the stability by which these significations are employed, as an indication of their institutionalisation. In Chevallard’s (1992, p. 144) terms, this is the process by which an object (of knowledge) comes into existence for an institution, and he states that this presupposes a recognised denomination. “Institution” here does not necessarily refer to a formal organisation, but there must at least be some alliance amongst a group. Our conception of institutionalisation is in line with that of Chevallard. For an expansion of the notion of institution, which in his theorising is a “primitive term”, see Hardy (2009).

As indicators of institutionalisation of the FTC, reference to a sub-area or to a proposition with a common name and textbook or handbook appearances were considered. Textbooks were separated from research publications by their intention to address an audience with less specialized knowledge in the area to which the sub-area belongs. Only some examples will be discussed here (for further details on the sources and the methodology, see Klisinska, 2009).

Classical outlines of the history of mathematics commonly trace the ideas of calculus back to ancient Greece. It is also common to refer to Leibniz and Newton as “inventors” of the modern calculus. However, Leibniz and Newton did not invent the same calculus, and did not set out calculus as a well-defined sub-area of mathematics as they differed in problems studied, approaches taken, and methods and notations used (Boyer, 1959; Baron, 1987). The early development of the limit concept [5] was crucial for the process of institutionalisation of the calculus. By using limits as the basis for definitions, Cauchy’s work established new criteria by integrating definitions and proofs with applicable methods. In the 19th century the formal ε-δ definition of limit by Weierstrass, the definition of the Riemann-integral and a set theoretic definition of function were added in accordance with the development of the criteria for legitimate knowledge.

Throughout the history of calculus, the institutionalisation of knowledge for the purpose of teaching was one driving force for the change of knowledge criteria. However, in the historical development, it is not easy to differentiate between criteria for the producers and distributers of knowledge. While the first developments in calculus were communicated entirely within the field of knowledge production through personal communication, textbooks for the wider distribution of calculus soon appeared. The first printed textbook in differential calculus appeared in 1696, written by de l'Hospital with the help of Johann Bernoulli. From the introduction (1716 edition) it becomes clear that the name "Calcul integral" was already in use. Thus, by having a specific name it had gained an 'official' status as a specific part of knowledge to which one could easily refer. However, what was signified by this name changed considerably.

Cauchy's *Cours d'analyse* from 1821 and *Résumé* from 1823, written for The *École Polytéchnique* in Paris, were the first textbooks in which calculus appeared as an integrated body of knowledge with clear borders towards other mathematical areas. It included a proof of a proposition (with no name) very similar to what now is called the FTC. The textbook can be seen as an attempt to provide access for a wider audience to a knowledge based on the same criteria as those promoted by the producers of mathematical knowledge.

Also in other early textbooks the propositions related to what is now called FTC are not named, but in the *Course d'analyse mathématiques* from 1902 by Goursat, translated into English already in 1904 and widely spread, the expression "fundamental theorem" is used for the fact that "every continuous function f(x) is the derivative of some other function". In the textbook *An introduction to the summation of differences of a function* by Groat, printed in 1902, the expression "the fundamental theorem of the integral calculus" appears, as well as the more short "fundamental theorem". In *The theory of functions of a real variable & the theory of Fourier series*, published in 1907 by Hobson, one chapter has the title "The fundamental theorem of the integral calculus for the Lebesgue integral". That the name of the theorem serves as a chapter title and is extended to a more general application indicates a strong level of institutionalisation. Wiener refers several times to "the fundamental theorem of the calculus" in *Fourier transforms in the complex domain* from 1934. That this name became institutionalised is evident from the classical book *What is mathematics?* from 1941, where Courant and Robbins write (p. 436):

> There is no separate differential calculus and integral calculus, but only one *calculus*. It was the great achievement of Leibniz and Newton to have first clearly recognized and exploited this *fundamental theorem of the calculus*.

The development of the calculus shows that the process of institutionalisation of a body of knowledge has to be seen in relation to the practices of publication and education.

Discussion

As the FTC and its proof, in the version that has become institutionalised, links two different fields of investigation, it can be attributed a systematising function. Cauchy's further systematisation by means of introducing a set of basic concepts for an outline of the theory was developed in the context of the teaching at the École Polytéchnique. This clearly shows that there is a dynamic relation between the production of knowledge and its transmission in the development of knowledge criteria. For example, reference to Cauchy's scholarly work is commonly done by drawing on his textbooks. That his textbooks became popular and that his exposition of the calculus became generally adopted (Boyer, 1959, pp. 282-283) can be explained by the combination of its influence on the producers of knowledge through applying criteria for knowledge that became internally socially shared, and the relative autonomy of teaching that accounts for its distribution. Action as 'producer' and at the same time as 'transposer' affects both the unmediated and the pedagogic discourse.

There are other prominent examples of a dynamic relationship between the development of 'knowledge for teaching' and the intention of re-organising and re-describing a set of related outcomes of research in different sub-areas for the purpose of presenting it in a coherent way. Felix Klein's *Elementarmathematik vom höheren Standpunkte aus* from 1908 is an example of a work that provided insights for both teachers of mathematics and researching mathematicians. The same year Godfrey Harold Hardy published his *A course of pure mathematics* (reprinted in many new editions), which "was intended to help reform mathematics teaching in the UK" and more specifically to prepare students to study mathematics at university [6]. It is not easy to locate such publications in relation to their role for a didactic transposition of scholarly knowledge. Transposition might include a 're-systematisation' of knowledge, for example, from a hierarchical structure of embedded specialised theories into an organisation by shared techniques within different specialised areas. This is often the case when the outcomes of basic research are to be used in applied research.

The necessity for a didactic transposition assumes a separation between the producers of knowledge and teachers. If this separation includes a division of labour, the producers of knowledge are not the ones responsible for a transposition of the outcomes of research into knowledge for teaching. Several examples from the history of calculus (Bernoulli-l'Hospital, Lagrange, Cauchy) show that such a division of labour did not always exist. Prominent researchers in the area worked as 'transposers' of the knowledge produced by themselves for the purpose of the very introduction into that area. In this case, the transposition is not achieved by another group of agents, who belong to the 'noosphere' in the theory of didactic transposition. In the situation during the time of Cauchy the social base for the noosphere was different than today as there was no independent textbook industry and no massification of higher education. In addition, the relatively low degree of specialisation of the discipline reduces the gap between levels of mathematical knowledge in terms of a hierarchical knowledge organisation. The time-span, after which a piece of scholarly knowledge becomes a potential object for teaching, depends on the level of specialisation of the knowledge to be taught in relation to the goals of the teaching institution. For example, Lebesgue integration soon entered advanced university courses, but still is not the standard approach in introductory calculus courses. The textbook by Hobson, published in 1907, includes the Lebesgue integral that was published in 1904.

The historical study also shows that it is hard to find a distinct point in time when the calculus as a delineated body of knowledge has become institutionalised. Consequently, the scholarly body of knowledge named 'calculus', which could be the starting point for a didactic transposition, cannot easily be identified. The investigation of the names for the FTC used in research publications and textbooks shows that different names were used for similar versions of *the* FTC, but also that the same name was used for different versions of it, or no specific name was used. There is no distinct transposition of a clearly identifiable piece of scholarly knowledge, but a series of re-descriptions. The institutionalisation seems to be happening within the 'knowledge for teaching' rather than within the scholarly knowledge. An example of this is Cauchy and his *Cours d'analyse* from 1821 and the follow up in the *Résumé* from 1823, which was to meet the new demands arising in higher education after the French revolution, even explicitly at "the urging of several of his colleagues" (Katz, 2004, p. 432). Our example shows that the original scholarly body of knowledge that is to be the starting point for a didactic transposition is not as fixed, organised and stable as suggested by the theory. It is indeed hard to trace back the original elements of different mathematical discourses from the field of knowledge production that are manifested in a curriculum. [7]

The relation between the mathematical knowledge in institutions that are not only established for the purpose of education, and its forms developed for apprenticeship into the discipline, is but one facet of the life of knowledge in society ("la vie des savoirs dans la société"; Chevallard, 1991, p. 210), which is the focus of concern for Chevallard and the reason for naming the further elaboration of his theory the 'anthropological theory of didactic' (ATD; see Bosch & Gascon, 2006). Even if this name does not suggest so, the theory introduces a programme that sets out to develop a sociology of (mathematical) knowledge by studying 'didactic systems'. However, scholars working within the ATD have not, to our knowledge, engaged in a critical discussion of its basic principles and relations to other frameworks such as Activity Theory or to Bernstein's work that is concerned with the production, reproduction and distribution of knowledge, and in particular with the process of knowledge recontextualisation. From the viewpoint of Bernstein's theory (e.g. 2000), pedagogic discourse is defined by the fact that it recontextualises a practice by moving it from its original site in order to use it for a different purpose. As the discourse moves from its original site (to become a pedagogic discourse), there is always space in which ideology can play. An ideologically transformed discourse is not the same discourse anymore. This is reminiscent of the description of 'didactic transposition' accomplished in the 'noosphere' in the ATD, though the ideological transformation of knowledge is not a major concern, neither is its distribution in relation to social structures and their (re-)production through education. Hence, we see a potential for productive interaction of the ATD with other languages of description used in mathematics education research by relating it to theories that share some of the (often implicit) intellectual roots of the didactic transposition theory.

Notes

1. "Le discours pleinement didactique devrait donc différer essentiellement du simple discours logique, où le penseur suit librement sa propre marche, sans aucun égard aux conditions naturelles d'une communication quelconque."

2. "On réserve la forme dialogique, propre à toute vraie communication, pour expliquer les conceptions qui sont à la fois assez importantes et assez mûries. [...] Loin d'indiquer une négligence excusable seulement envers les cas secondaires, cette forme, quand elle est bien instituée, constitue, au contraire, le seul mode d'exposition qui soit vraiment didactique: il convient également à toutes les intelligences."

3. "cette transformation didactique ne devient réalisable qu'envers des doctrines assez élaborées pour qu'on puisse nettement comparer les diverses manières d'exposer leur ensemble, et prévoir aisément les objections qu'elles devront susciter."

4. For the full report, see Klisinska (2009).

5. For example in the influential textbooks *Théorie des functions analytiques* from 1797 by Lagrange and *Traité élémentaire* from 1802 by Lacroix.

6. Wikipedia online: <en.wikipedia.org/wiki/A_Course_of_Pure_Mathematics>

7. When discussing transformations of academic mathematics for school mathematics, Ernest (2006) makes a similar general claim that "for any particular theory or area of mathematics, there is no fixed or unique mathematical theory formulation as a source. There are always multiple formulations by different mathematicians and groups of mathematicians constructed and published at different times" (p. 73). He also adds that for topics in school mathematics which today are no longer topics of research mathematics, at the time when they were, the academic texts written (with Stevin's *Decimal fractions* as one example) "were both advanced academic treatises for scholars, as well as teaching texts" (p. 73).

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Classroom Assessment in Mathematics – a Review of Articles in Two Journals

Lisa Björklund Boistrup
Stockholm University, Sweden

Assessment has the possibility to influence students' learning and can facilitate students' active agency in mathematics teaching and learning processes. This makes assessment a critical aspect to be addressed in research on classroom practice in mathematics education. This paper presents a study on to what extent there are articles with an articulated focus on classroom assessment in journals of mathematics education and how classroom assessment is conceptualised within the articles. In the discussion potentials for further studies are discussed.

Assessment in mathematics: A background

Today there is a lot of interest in assessment, including classroom assessment, in educational research. A search on the abstracts from the EARLI (European Association for Research on Learning and Instruction) conference 2009 [1] for the term "assessment", gave more than 300 hits [2] out of approximately 1700 abstracts (EARLI, 2009). This means that more than one sixth of the contributions to this general learning and education conference had a strong connection to assessment. One cause for this interest is that learning and assessment are strongly linked, that is, how assessment is done has impact on student's learning (e.g. Gipps, 2001; Pettersson, 2007; Shephard, 2000). Since at least Black and Wiliam (1998), there has been a call for in depth classroom studies in this area, and there still is (e.g. Hattie, 2009).

This paper takes its departure in a tentative literature review focused on definitions and boundaries in classroom assessment in general through a search on Libris [3] (Björklund Boistrup, 2009). In the present paper I direct the interest solely towards classroom assessment in *mathematics education.* The study investigates (1) to what extent there are articles with an articulated focus on classroom assessment in journals of mathematics education. The aim is also to (2) find how classroom assessment is conceptualised within the mathematics educations field, that is, within what boundaries the articles are positioned. In order to find some answers to these questions, I have performed a new literature review, this time focussing on two journals, Educational Studies in Mathematics (short: ESM) and The International Journal on Mathematics Education (called ZDM). Further on, similar reviews will be performed on other journals in the field of mathematics education and consequently, the selection of these two journals is based on practical considerations. When performing this review, I used the analytical structures, concerning boundaries of assessment, used in and derived from the earlier review (Björklund Boistrup, 2009). These include possible mathematics classroom situations, and students' and teachers' roles in mathematics classroom assessment. I also describe to what extent institutional aspects are included in the literature.

The background of this paper is a larger study on classroom assessment in mathematics (Björklund Boistrup, work in progress). In that study, I analyse the assessment that is present in the interaction between teacher and student(s) in the mathematics classroom. When performing such an analysis, a decision has to be made about what to include as signs of assessment in mathematics, and what to exclude. During this analysis, I came to question something that I thought was already clear to me, namely "What *is* assessment in mathematics*?*" A question close to this is: "What boundaries of classroom assessment in mathematics can be found in the research literature?" The review presented in this paper serves as a base for discussing these issues.

The foundations for the analytical structure used in this review are the research interests that drive the larger study, the findings from the earlier review and, most importantly, the theoretical perspectives. The latter will be described shortly in the next section.

Theoretical perspectives

The overarching theoretical framework for this paper, as well as for the larger study, is social semiotics (e.g. Hodge and Kress, 1988; van Leeuwen, 2005) combined with institutional/discursive theory (e.g. Foucault, 1975/2003). In social semiotic theory, according to Hodge and Kress (1988) and van Leeuwen (2005), all semiotic resources of communication are recognised. This means that all kinds of semiotic resources have to be taken into consideration (in assessment in mathematics and in research on assessment), such as gestures, and gazes, pictorial elements and moving images, sound and the like. Semiotic resources are seen as socially and culturally designed in different processes of meaning-making, so their meaning changes over time. The semiotic resources "chosen" in a specific situation reflect the interest of the sign maker, and they are therefore not arbitrary. Kress (2009) argues for the importance of understanding multimodal communication to be able to fully understand a phenomenon such as assessment. In this paper, this perspective brings about a focus on the aspects of interaction and semiotic resources.

Institutions can be taken for granted by a researcher who "knows" the situation. But without some idea of the communicative situation, it is very difficult to draw conclusions from, for example, a conversation (Selander & Rostvall, 2008). Here "the institution" in its historical context is also included (Foucault, 1975/

2003). The interactions between teacher and student are situated in a context characterized by dominant mathematics education discourses, the use of artefacts developed over time, framings in terms of specific resources for learning, division of time, structures within and between schools, classification of students into schools and learning groups, established routines, classroom structure and authoritative rules. Institutional framings have both direct and indirect effects. Decisions may be made on different "levels" in the school system, which have a direct impact on the classroom work. There are also indirect aspects, such as classificatory systems, norms and 'traditions' developed over time. In this paper, this perspective brings about a focus on to what extent institutional aspects are present in the articles and how these are considered.

The review

In the literature search, I searched in the journals ESM and ZDM on the search terms listed in the first column in Table 1 (articles from 2000 to 2009). For providing a reference measure, I also searched in ERIC and added "mathematics" and "education" in the search.

Table 1: Search results for different search terms in ESM, ZDM and ERIC (Final search performed April 2010).

	ESM	ZDM	ERIC
"classroom assessment"	4	4	25
"formative assessment"	4	6	30
assessment + feedback	49	47	114
"educational assessment"	9	3	266

My next step was to go through the abstracts of the found articles in ESM and ZDM and to choose the ones to be further analysed. The selection criteria were: "An explicit focus (can be one out of several) on one or several aspects of teacher and/or student initiated classroom assessment and a relation to compulsory school". There were ten articles, five articles in ESM and five in ZDM, that met these criteria and they are marked with * in the reference list.

A first outcome in relation to the first objective is hereby at hand. It is easy to conclude that classroom assessment is not to a high extent addressed within these two journals. On average one article with this focus explicitly stated, is published in the respective journal approximately once every second year.

The issue stated as the other objective of this review, that is, within what boundaries assessment is positioned in the articles, needs more elaboration. First I address the possible kinds of situations that are focused on in the articles.

Possible situations for assessment

When it comes to *when* classroom assessment can occur I have looked at three kinds of situations and to what extent the authors address them. The same categories were used in Björklund Boistrup (2009) and they are an outcome of the theoretical background of the larger study (Björklund Boistrup, work in progress).

1. Explicit classroom assessments in the form of individual tests/diagnoses.
2. Other explicit assessments in other forms in the teaching. It can be portfolios for assessment, planned teacher-student meetings etc.
3. Implicit assessments in the everyday teaching. It can e.g. be the teacher who gives feedback to a student who is working in the text book in mathematics.

All of these kinds of situations are represented in the articles. Watson (2000) respectively Watt (2005) address all three kinds of assessment. Watson (2000) presents findings where she has identified components of teachers' practices when acting as assessors in the normal course of classroom mathematics work. Among the components of practice of teachers as assessors are explicit assessment in the form of e.g. tests; other explicit assessments are students' self-assessment. Among the mentioned implicit assessments are exercises, observation of students, intuition etc. Watson concludes by addressing equity problems with these kinds of implicit assessments, since the same student's performance most likely would be differently assessed by different teachers. Watt (2005) holds another view and presents results of a survey study on secondary teachers' attitudes towards the use of alternative assessment. The results show that teachers were satisfied with traditional tests as valid measures of student ability. Explicit alternative methods asked about in the survey are oral tasks, practical tasks, student self-assessment and parental assessment, and (more) implicit methods are observation and student journals. Watt emphasises the importance of alternative assessment methods in mathematics classrooms, and she states that the teachers' reluctance seem to rely on an assumption that these methods are not reliable.

In some of the articles only one kind of situation is addressed. Walen and Wiliams (2002) have a focus on explicit assessment in the form of tests and they address emotional reactions to timed tests in mathematics education. Mercier, Sensevy, and Schubauer-Leoni (2000) also address only one of the kinds of situation, in this case teachers' implicit assessments of students' mathematical capabilities and their connections to social interactions within class. Similar to Mercier et al. (2000), Zsur (2007) has a focus solely on implicit classroom assessment in mathematics, but when it comes to teachers' and students' roles in the assessment, there are quite some differences between these two. These roles are discussed in the next section.

Teachers' and students' roles in the assessment

I have looked at what roles the main actors in the assessment, students and teachers, are given according to different authors. The following different processes, related to students and teachers, are found in the general assessment literature (Björklund Boistrup, 2009):

1. Teachers' assessment of each students' (shown) knowing/learning
2. Teachers giving feedback to students about students' learning
3. Students' assessment of own (and others') learning, possibly together with other students
4. Teachers' assessment of students' shown knowing and learning as a feedback for their teaching
5. Teachers' reflection on their own assessment, possibly together with peers

The first four of these processes are represented in the articles from ESM and ZDM. In some, the focus is on solely one of them, and in some there is a focus on several of the processes. The fifth process, teachers' reflection on their own assessment is not specifically addressed in any of the articles.

As mentioned in the previous section, both Mercier et al. (2000) and Tzur (2007) have a focus on implicit assessments in the mathematics classroom. When it comes to teachers' and students' roles, Tzur addresses the teachers' assessment of students' learning and in doing so he emphasises the soundness of a theoretical stage-distinction regarding concept formation. Mercier et al. also focus on teachers' assessment of students', but they have another interest when they discuss how teachers' assessment of students' actions are affected by respective student's social position in the group of students. Through excerpts they show examples of these processes and they also address how the teacher gives different feedback to students, and also this connected to social position. To some extent they address how the teacher uses the assessment as feedback for the teaching. Li (2000) addresses the same processes as Mercier et al., with a different method. She conducts, with a historical perspective, a review of the development of assessment practice in China. Li's review covers different kinds of assessments, where classroom assessment is in focus in two sections. When addressing classroom examination Li discusses teachers' assessment of each student's knowing. When addressing instructional assessment she discusses teachers giving feedback to students as well as teachers using students shown learning as feedback for the teaching. Moreover, she discusses students' self-assessment. This latter process is the main focus in Brookhart, Andolina, Zuza, and Furman (2004), and also in Kaur (2008). Brookhart et al. present results from an action research project. Their study suggests that students' self assessment, when students really are involved in the process, can add reflection and meta-cognition to rote memory lessons, such as when learning the multiplication tables. Kaur

(2008) addresses, through an interview study in Singapore, what really matters to teachers and students when it comes to teaching and learning mathematics. The findings of the teacher interview data showed that teachers attached importance to (among other activities in the classroom) students' self assessment.

In a few of the articles there is a multimodal stance when it comes to assessment and communication. Moskal and Magone (2000) argue for the acknowledgement of semiotic resources, as one feature, in a study on making sense of what students know about decimal numbers. One aspect is the importance of sensitivity to ways in which the semiotic resources used in tasks' presentations affect students' responses. They also argue for the benefits when students are free to choose the semiotic resources to use when answering the tasks since this can give insight into the students' reasoning.

Institutional aspects

When adopting an institutional perspective, an interest in student-teacher interaction in 'isolation' is not sufficient. These interactions are then seen as institutionally situated and by this they are affected by, as well as part of, a broader context. This interest in institutional aspects affects what I look for in this review. Is classroom assessment in mathematics limited to being situated merely in the classroom, or is there a broader scope addressed in the literature? When institutional aspects are focused, these different ways of bringing them in are found in the general assessment literature (Björklund Boistrup, 2009):

A. Present mainly through advise of how to encourage teacher development in conscious work, on municipality or school level, where teacher collaboration on assessment issues is promoted

B. Elaboration on how decisions on national and/or municipality "level" have impact on classroom assessment.

C. A focus on institutional aspects in the form of traditions

I have compared these categories with the ten articles analysed in this paper. In four of the articles from ESM and ZDM one of these aspects is focused. Fan and Zhu (2007) describe results from a study on problem solving where assessment is explicitly mentioned as one aspect of classroom practice. They describe results from a review, in which successful intervention programs, comprising alternative assessments, are included. This fits into the category A,

In one article, institutional aspects in the form of traditions (C) are part of the analysis, and that is in Walen and Williams (2002).

Summary

The following table (Table 2) is an attempt to capture the outcomes of this literature review in an integrated way. The columns represent the three kinds of situations of classroom assessment. The rows represent the five processes concerning teachers' and students' roles in the assessment. The letters in italic, *A, B* and *C*, after some of the author names indicate the three institutional aspects. In parenthesis there is *M* or *m* to indicate where multimodal aspects are more or less present. No letters indicate that neither institutional nor multimodal aspects are clearly addressed in the article.

Table 2: The outcomes of the results of the review. T=teacher, S=student, *A, B* and *C*=Institutional aspects present; *(M)* and *(m)*=multimodal aspects present.

	1. Explicit – tests	2. Explicit – other	3. Implicit
1. Teacher assesses	Fan & Zhu *A* Watt Walen & Williams *C* Moskal & Magone *(M)* Li *B* Watson *B (M)*	Fan & Zhu *A* Watt Li *B* Watson *B (M)*	Tzur Mercier et al. *(m)* Watt Watson *B (M)*
2. T gives feedback to S	Li *B* Lin & Li *B*	Li *B* Lin & Li *B*	Mercier et al. *(m)*
3. S self- ass.	Brookhart et al.	Fan & Zhu *A* Watt Kaur Li *B* Brookhart et al. Watson *B (M)*	
4. T views S's learning as feedback for own teaching	Moskal & Magone *(M)* Li *B*	Li *B*	Mercier et al. *(m)*
5. T self- ass.			

This overview shows that the wide boundaries for classroom assessment used as analytical framework for this paper are almost, but not totally, filled when looking at the articles as an integrated whole. Teacher assessment in the form of tests and other explicit assessments, such as students' self-assessment, are the ones most frequently addressed in the articles. Also other explicit teacher assessments as well as implicit teacher assessment are addressed in several articles. Implicit assessment in relation to self-assessment is not present in any of the articles. As already mentioned, teachers' reflection on their own assessment is not addressed, which, on the contrary, was the case in the previous review of general assessment literature (Björklund Boistrup, 2009). When it comes to institutional aspects, these are addressed in some of the articles but not to a high extent. This holds also for multimodal aspects.

An additional point in relation to reviews like this, is the fact that the literature is written in English and several articles are from the perspective of Anglo-Saxon countries. In this review some of the articles represent an 'Asian perspective'. The tradition and climate in Sweden concerning assessment is quite different compared to these countries. Cultural aspects are something to be conscious of when partly relying on literature originally written in English or reflecting practices and institutional traditions in other countries.

Conclusions

The objective for this paper was to (1) investigate to what extent there are articles with an articulated focus on classroom assessment in journals of mathematics education. The aim was also to (2) find how classroom assessment is conceptualised within the mathematics educations field, that is, between which boundaries assessment is conceptualised in the articles. In the previous review, which was on general assessment literature (Björklund Boistrup, 2009), the number of hits was extensive but this time my interest was to investigate a part of the literature within the field of mathematics education in relation to assessment in mathematics classrooms. The number of articles that fit the criteria was, as earlier shown, small and one conclusion is that the interest in these matters does not seem to be high in the field. This is the case despite the fact that comprehensive research shows that assessment has impact on students' learning (see the introductory section, above). In the small number of articles that was found, the wide boundaries for classroom assessment used as analytical framework for this paper were almost, but not totally, filled. There are some "blind spots". In the following I give an account of some of these blind spots, including parts of the frame-work that were addressed shortly (see above), and I also discuss what aspects of mathematics education could be addressed if these empty spaces became to be (more) "filled" in the research literature in the future.

Teacher gives feedback to student

In research concerning teachers' feedback to students in mathematics education there are potentials of addressing to what extent the feedback is elaborated. These include 'feed-back' (What aspects of competence has the student shown?); 'feed-up' (How can the aspects shown, and future learning and teaching, be related to stated goals?) and 'feed-forward' (What aspects of mathematics competence might it be best to focus on in future teaching and learning?) (Hattie & Timperley, 2007; see also Björklund Boistrup, work in progress). Moreover the teachers' feedback could be addressed in a more focused way. This comprises the extent to which the teacher's feedback focuses on processes regarding learning mathematics or whether it is on the student as a person or on how to continue with the work in relation to the textbook. Neither of the latter two focuses have a positive impact on students' achievements (Hattie and Timperley, 2007; Björklund Boistrup, work in progress).

Students' self-assessment

Further research concerning students' self-assessments in mathematics education has a potential to address students' agency in mathematics teaching and learning. Mellin-Olsen (1993) addresses this when he poses the question about who the subject of assessment in mathematics is. Here are of course aspects concerning the focus on mathematics, as mentioned in previous paragraph, essential. Also implicit self-assessment in day-to-day classroom work plays a role here. Depending on teachers' feedback the students can be invited to communicate more or less self-assessment. Such communications offer potentials for students to take active agency in the continuing mathematics teaching and learning process.

Teachers assessment of students' shown knowing and learning as a feedback for their teaching

In the literature search it was possible to find articles with a focus on how teachers capture the students' shown knowing, but not labelling it as part of an assessment process (see for example Margolinas, Coulange and Bessot, 2005). The point made in relation to the objectives of the review presented in this paper are the opportunities, when addressing the teachers' particular responsibility in assessment processes, for viewing the students' shown knowing and learning as feedback for the teaching. That is, assessment is not just about valuing students' achievements but also to act on what is shown. This aspect also includes teachers' reflection on their own assessment.

Multimodal aspects

The assessments discussed in this paper are all communicated in multimodal ensembles where various semiotic resources play different roles. There is a potential for studies on these roles in relation to assessment in mathematics. One example is the restrictions that can be implied when students are only allowed to use a few semiotic resources when showing mathematics knowing during assessments. On the other hand, there might be certain mathematics processes promoted when some semiotic resources are excluded.

Institutional aspects

When addressing institutional aspects in relation to assessment in mathematics, there are means to go beyond mere descriptions of actions of assessment in mathematics classrooms. There are also possibilities to understand discourses of assessment in mathematics classrooms as part of a broader context including how institutional framings interact with the assessments in the classrooms.

The larger study

In the larger classroom study that was mentioned in the introduction, I use a broad scope for classroom assessment in mathematics, including institutional and multimodal aspects (Björklund Boistrup, work in progress). Hopefully the results from this larger study will be a contribution to the field of mathematics education that helps to fill some of the "blind spots" identified in the literature review.

Notes

1. www.earli2009.org
2. For searching on the conference' website there are fixed key word options. The options that mention assessment are: Assessment methods, Assessment of competence, and New modes of assessment. Overlaps of hits are excluded.
3. http://libris.kb.se/

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Revisiting Perspectives on Mathematical Models and Modelling

Peter Frejd
Linköping University, Sweden

This paper presents a literature review of the notions of mathematical models and modelling from three different theoretical perspectives in order to analyse students' descriptions in a future empirical study. Similarities and differences between the perspectives are discussed and suggestions regarding how to use the different perspectives and examples of possible research questions are given.

Introduction

The Swedish government's official curriculum documents for upper secondary school emphasises, in the section about the nature and structure of the subject mathematics, the cultural and social dimensions of mathematics. ”Over thousands of years of development, mathematics has contributed to our cultural heritage. Mathematics is a precondition for major developments in society and permeates the whole of society, often in ways that are invisible to untrained observers” (Skolverket, 2000, p.61). One of the aims of the school subject mathematics is to provide students with the ability “to analyze, critically assess and solve problems in order to be able to independently determine their views on issues important both for themselves and society” (Skolverket, 2000, p.60). According to Skovsmose (1994), a mathematical “critical competence” is of essential value in a democracy, when the forming of opinions and political decisions are based on mathematical models. This critical competence is one of *five arguments* stated by Blum and Niss (1991) for promoting modelling activities in education. The government has also increasingly explicitly pointed out the importance of mathematical models and modelling in the curriculum during the last 40 years (Ärlebäck, 2009), and in the present curriculum states that teaching of mathematics should aim to ensure that students “develop their ability to design, fine-tune and use mathematical models as well as critically assess the conditions, opportunities and limitations of different models” (Skolverket, 2000, p.61).

Even though mathematical models and modelling is prescribed in the curriculum as an essential point in the teaching and learning, it is not described how these notions are to be interpreted. Neither are everyday meanings of models nor modelling unambiguously defined [1]. Consequently, pertinent questions to ask refer to the meanings associated with models and modelling in the context of the teaching and learning of mathematics. In Frejd and Ärlebäck (this volume) it was concluded that upper secondary students in Sweden had no clear view about how to describe the notions of mathematical models and modelling and that further research about students' views of models and modelling are needed.

The purpose of this paper is to discuss different perspectives on models and modelling found in educational research to be able to better analyze upper secondary school students' descriptions of their own activities and thinking processes while engaging in some modelling activity and their views on the relevance and epistemological status of mathematical models in society (which I plan to investigate in a future empirical study in Sweden).

Different perspectives on the term 'mathematical model'

The 14th ICMI Study volume, *Modelling and Applications in Mathematics Education* (Blum, Galbraith, Henn, & Niss, 2007) presents the field of applications and modelling and attempts to clarify the basic notions and terms. The term model is discussed and related to the modelling process (described later in this paper) and some examples of standard models are given (linear, exponential or logistic growth, inverse proportionality, etc.), but it becomes apparent that there is no clear or shared definition of a "mathematical model". Common definitions stress the representational aspects of mathematical models: According to the Encyclopaedia Britannica a "mathematical model is either a physical representation of mathematical concepts or mathematical representation of reality". A physical model is for instance a three-dimensional surface made of wires to visualize some abstract mathematical concept. About a mathematical model of reality one reads that "anything in the physical or biological world, whether natural or involving technology and human intervention, is subject to analysis by mathematical models if it can be described in terms of mathematical expressions" Mathematical model (n.d. a). Wikipedia´s definition is: "a mathematical model uses mathematical language to describe a system" Mathematical model. (n.d. b). Representational aspects of models are also found in technology literature one can find definitions of a mathematical model as "a representation of essential aspects of an existing system (or a system to be constructed) which presents knowledge of that system in usable form" (Eykhoff, 1974, p. 1). In literature from mathematics education "*models* are conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation system, and that are used to construct, describe, or explain the behaviors of other system(s) – perhaps so that the other system can be manipulated or predicted intelligently. A mathematical model focuses on structural characteristics (rather then, for example, physical or musical characteristics) of the relevant systems" (Lesh & Doerr, 2003, p. 10), According to this description models are situated both in the mind of the learner and in representational media (equations, etc).

One can argue that Sfard's (2008) definition of a mathematical model would be: *A subsuming mathematical discourse* [2] (*with a mediating narrative*) [3]. She writes, for example, "discourse on function *subsumes* discourses on algebraic formulas, on curves and physical process" (p.175), and that functions can be models of real life situations (p. 122). One can also read, "in case of natural science the research-produced and practice-*mediating narratives* are likely to take the form of concise symbolic formulas, such as $E=mc^2$ or N2+3H2→2NH3, which are to present 'laws of nature'" (p. 35).

Jablonka (1996) outlines a critical discussion about mathematical models in mathematical education. She analyzes about 100 modelling examples from different teaching materials with a curriculum perspective (relevance/goals of mathematics education) to investigate what images of reality mathematical modelling constructs and what can be said about the epistemological status of the particular mathematical models. The models she found were from a variety of different domains and they were often simplified, taken from technical or scientific contexts where they had no educational purpose, and some were invented specifically for didactical use. One of the problematic issues she points to is the validation of mathematical models in the classroom situation. A critical question she raises is if it is useful, or even possible, for students to develop mathematical models for problems that are unknown to them if they do not know the techniques needed to solve them. She also questions the metaphor of a mathematical model as a reflection of reality.

To sum up, mathematical models are described in dictionaries, technology and mathematics education literature to be some kind of representation/ description/ explanation of "something" in terms of structural characteristics/ expressions from mathematics. These models may be situated in a variety of different places such as the mind of the learner, the discourse where the model is being used or in some form of representational media/ physical object. This "something" is described in terms of vague expressions, such as conceptual system, existing system, narrative, knowledge or reality. The process to create a model of "something" is called modelling.

Modelling through different perspectives

An important point to start out with is that mathematical modelling professionally and modelling in school are two different things (Blum, 1991; Borrowed, 2006). Modelling used in research is focused on special aspects concerning modelling and applications (such as new simulation techniques, particular mathematical properties, estimation of parameters), but in education the didactical approaches are distinguished by a normative orientation concerning the entire process of modelling. Modelling in school is a didactical activity that aims at different goals: Teaching mathematics, teaching modelling, teaching about the relevance of mathematics in society and in private life, teaching critical evaluation of models. These goals are stated explicitly or implicitly in the Swedish curriculum, but as the process of modelling is not explicitly described it opens up for the influence of different ideologies to come into play.

Kaiser, Blomhøj and Sriraman (2006) claim that they have “a global theory for teaching and learning mathematical modelling, in the sense of a system of connected viewpoints covering all didactical levels” (p. 82), but that this “theory of teaching and learning mathematical modelling is far from being complete” (p. 82). The theory or the “overall theoretical framework” (p. 82) they refer to, is “a commonly accepted idea about a general mathematical modelling process” (p. 83). The descriptions of the general modelling process may vary depending on different research aims (Borromeo, 2006) but the process is most often depicted as cyclic, starting in “real world”, moving into a “mathematical world”, modelling the former, and then validating by moving back to the “real world” (e.g. Blum & Leiss, 2007). According to Kaiser et al. (2006), all models of the general modelling process are described as a cyclic process with five up to seven sub processes and they are split into two parts, one called reality and one called mathematics. This modelling process is necessary to carry out “in order to create and use a mathematical model” (Kaiser, Blomhøj, & Sriraman, 2006, p. 82).

A short elaboration of such a cyclic process will be given here. The setting is called the *real situation* (the problem described in everyday knowledge). To understand the task is to *make a mental representation of the situation* (how the individuals are thinking about it). Then simplify/structure and idealize and filter the information to come up with a *real model* (external representations) and then continue to mathematize these criteria into mathematics and create a *mathematical model*. The last steps are to work mathematically with the model to get answers, *mathematical results*, interpret mathematical results to *real results* and to validate. If the validation shows that the result is more or less ‘wrong’ and other aspects have to be included, then the individuals have to go through another lap in the cycle.

In educational research literature one can find many supporters of this “cycle and split (reality & mathematical)” perspective of modelling (Blum & Niss, 1991; Maaß, 2006; Borromeo, 2006; Kaiser & Sriraman, 2006; Blum & Leiβ, 2007; Blum et al., 2007).

Critique of this description of a *general modelling process* has come from different authors. According to Jablonka and Gellert (2007), there is no straightforward translation from a *real model* to a *mathematical model*, because it is problematic to go from one context (discourse) to another, such as to quantify non-mathematical characteristics and relate them mathematically. In a classroom situation there is no validation according to their view, because the result is not put back into a "real" *real situation* and when students are in a mathematics class it is not possible to construct *real models* that are non-mathematical. In addition, Jablonka (1996) also found that in most of the modelling examples she investigated the validity issue was avoided or suggestions were made that objective criteria would exist for comparing the model with "reality". Some critical questions she discusses are: What are the validation processes for those models that are based on parameters that cannot be measured in reality? Why translate those "real world models" into mathematics and where do those models originate? Jablonka and Gellert (2007) continue:

> mathematisation within the circular process of mathematical modeling is – epistemologically regarded – a potentially misleading construct and it is – pedagogically – of debatable value. On the one hand, the circular model of mathematical modeling adequately acknowledges the contingencies of problem definition and formalization; on the other hand it tends to obscure the informative power of mathematics. Mathematics is not only the sphere where formalized problems find their solutions; mathematics is from the outset the vantage point from which problems are construed. (pp. 5-6)

Instead they discuss the work of researchers who concentrate on analysing mathematisation (an activity to organize, formulate, make judgment about reality) and demathematisation (rendering mathematics invisible or unnecessary, for example through development of materialised mathematics) as a social process, because "mathematics is a means for the generation of new realities not only by providing description of real world situations, but also by colonizing, permeating and transforming reality." (p. 6) Treffers [4] (1987) distinguishes between horizontal and vertical mathematisation. A horizontal mathematisation is the process of formulating a mathematical description between different spheres, such as a transfer from an everyday description to a more scientific description, and a vertical mathematisation is the process of digging deeper into the mathematical sphere by proving theorems and defining concepts. Demathematisation is connected to trivialization and devaluation (Keitel, Kotzmann, & Skovsmose, 1993), because new technology changes the preconditions for mathematics; some mathematics that was needed to carry out a procedure is taken over by machines and we start to rely on and trust the "black box". According to Keitel, Kotzmann and Skovsmose (1993), mathematisation may induce realised abstractions, which means that thinking abstractions become a part of reality and we just take them for granted and do not reflect on these abstractions any more, for example on economic systems. Demathematisation is not just discussed as a threat to democracy (Skovsmose, 1994), but also in school. One example is the debate on the introduction of CAS-calculators (computer algebra systems) into the classroom; teachers feel threatened that the use of methods/rituals about how to solve equations will disappear in favour of “pushing a button”.

As a comment to this last example, it was found by Persson (2009), in a review of research on calculators, that this realised abstraction (CAS) also offers new possibilities and “that calculators can be powerful cognitive tools for the enhancing of students’ skills and understanding of mathematics, and in particular algebra” (p. 70).

Ärlebäck (2009), in an empirical study about introducing modelling activities to upper secondary students, has not found any evidence of a cyclic process:

> Although the idealised view of mathematical modelling as described in terms of a modelling cycle has been much employed in mathematics education research, the discrepancy with what actually happens when students engage in modelling activities is palpable. My opinion is that this ‘inconsistency’ is something that researchers ought to take more seriously to refine current theories and methods to be able to better validate our research findings. (p. 356)

Mathematical modelling from a commognitive (cognitive combined with communicational) perspective (Sfard, 2008), stresses the importance of discourses and change in discourses. “Discourses permeate and shape all human activities, the change in discourse goes hand in hand with the change in all other human doings.” (p. 118). The process of discursive compression (modelling) is an act of saming or by conflating different discourses into a new subsumed discourse, for example,”[W]hat we used to call “modelling-real-life situations with functions” is the act of subsuming parts of the discourse about these real-life situations to the discourse on functions” (p. 122). In a possible interpretation of the modelling process it could be described as a *routine* [5] in the subsumed discourse. I interpret Sfard’s view of the modelling process according to the figure below:

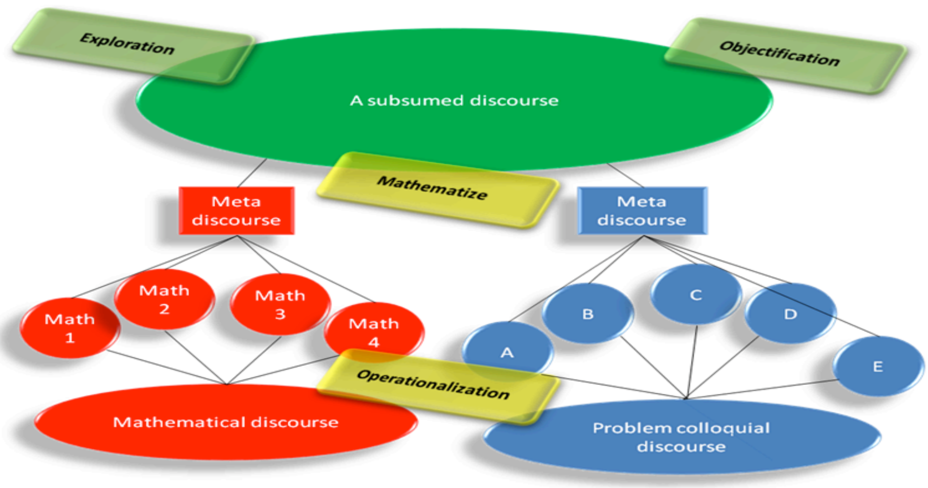

Figure 1: An interpretation of commognitive view on the modelling process.

This interpretation of the modelling process from a commognitive view will be described with the help of Figure 1 and with a problem for some soccer enthusiasts that discuss the best game ever seen (this example is inspired by an example in Jablonka and Gellert, 2007). Note that this routine (the modelling process) is taking place inside the *subsumed discourse*, by saming or conflating different discourses. The best game ever is the problem described in a *colloquial discourse* (everyday language) within the community of discourse (the enthusiastic soccer group). The first step is *operationalization*, that is, defining the criteria or the requirements that ought to be included in the best game ever. With help from the *meta discourse* the enthusiastic soccer group examines (cycles of modifications, *conjectures-test-evaluations*) and indentifies the criteria (for instance they may come up with (A) number of shots on goal, (B) importance of the game, (C) teams involved, (D) final score and (E) individuals' performance/achievement). "The next step is objectification, which begins with *mathematization of the metadiscourse*" (p. 124). To mathematize the criteria for an excellent soccer game, the meta mathematical discourse will help to define what mathematics is necessary to use in order to create a model and to validate the model with use of excellent and tedious soccer games (cycles of modifycations, *conjectures-test-evaluations*). The mathematics needed (or chosen) is taken from the *mathematical discourse*, that might be *math 1* functions, *math 2* statistics and so on. The subsumed discourse includes a certain vocabulary (mathematical and soccer), visual mediators (the model may be in a form of a symbolic artifact to simplify communication), routines (modelling, the pattern of *conjectures-test-evaluations*), and endorsed narratives (the final description of the solution to the soccer problem).

If the routine (modelling process) is described as "true", it is called *exploration*. "Exploration is a routine whose performance counts as completed when an endorsable narrative is produced or substantiated" (p. 224). If the endorsable narrative is substantiated then it is called objectified. *Objectification* is a "process in which a noun begins to be used as if it signified an extra discursive, self-substained entity (object), independent of human agency."(p. 300)

Kaiser and Sriraman (2006) tried to achieve a classification of current perspectives on modelling within education in the article *A global survey of international perspectives on modelling in mathematics education*, but they got criticism from the working group on modelling and applications at the CERME5 conference about the need to separate research perspectives and didactical approaches (Kaiser et al., 2007). The working group proposed a revised classifycation system with the following main categories: *realistic or applied modelling; contextual modelling; model eliciting approach; educational modelling; socio-critical and socio-cultural modelling; epistemological modelling; cognitive approaches; affective approaches; pragmatic teaching oriented approaches; and* *theoretical approaches* (ibid., pp. 2039-2040). They hope that this overview will identify both differences and commonalities between different perspectives on mathematical models and modelling to promote collaborations between researchers from different perspectives. The initiative for collaboration is valuable but according to this review the task seems well optimistic when there exist large gaps between different theoretical perspectives.

To sum up, the mathematical modelling process or modelling is described in all perspectives as a “process”. Depending on the theoretical perspective adopted, different words are being used to describe the modelling process, for instance: routines, cyclic processes and mathematisation. The goal of modelling and the way to accomplish the goal can be quite different in different perspectives. A solution of the real world problem is one goal, while other goals can be to reflect upon mathematics or to develop new mathematical theory (Kaiser et al., 2007). The process to accomplish the goal is everything from cyclic processes to creating new discourses. Viewing research literature of empirical studies of the cyclic process, I found no evidence that there would exist a cyclic process, and Borromeo Ferri (2007), for example, discusses *modelling routes* (see figure 5, p. 266). However, there might be some false interpretations of the cyclic process, if it is not expected to be linear, then some critics’ fall like the incompatible *modelling routes*. The suggestion that the result is *never* (instead of not often) put back in a real situation can also be disputed.

The debate of the modelling as a cyclic process will continue, and in my opinion “the cyclic process” should be revised or defined in a way so that no misunderstandings are allowed. It is clear from this review that there are numerous different theoretical perspectives on models and modelling used in educational research (other perspectives not discussed/mentioned here due to space limitations, are for instance ATD, constructivism etc.). Coming back to the purpose of this paper: What are the outcomes of this review in terms of analyzing upper secondary school students’ descriptions of their own activities and thinking processes while engaging in some modelling activity and their views on the relevance and epistemological status of mathematical models in society?

Conclusion

The three theoretical perspectives that have been discussed in this paper are “general modelling process”, ”mathematisation as a social process” and “the commognitive view”. What effect does a choice of perspective have on the research question (i.e. how may possible a research question be stated) and how can the chosen perspective actually be used? The “general modelling process” focuses on real world problems, on steps in a cycle and on the distinction between mathematics and reality. An initiating question could be *What parts of the modelling cycle do students emphasize while engaging in modelling activities* *and how do the students relate their mathematical models to the models in society?* The use of the framework could then be to connect and identify information to parts/steps in the cycle and to find discrepancies /relations between models the students know about and models in society. To use this framework one will need to clarify the meaning of real world problems, reality vs. mathematics and thinking process. These problems do not exist in the "commognitive view", which has a definition of thinking, and the distinction between mathematics and reality is resolved by conceptualising it as subsumed discourses. *What narratives do students endorse about (i) the role of the discourse (modelling) in learning mathematics and (ii) the connections between discourses of modelling and society?* This is a question that could be investigated from a commognitive perspective, because the adopted approach is to analyse the discourse and to "investigate transformations in discourses rather than "in people"" (Sfard, 2008, p. 276) which is a key stone of this perspective. The third perspective, "mathematisation as a social process", has many similarities with the commognitive perspective, such as that the start of the modelling process does not have to include real problems and that the structure of mathematics is *the (or a)* vantage point from which problems are construed. *How do students' views of the formative power of mathematics in a social context change (not change), when the students are enrolled in mathematisation activities?* This could be a possible question to be investigated from a perspective of mathematisation as a social process. One can also use "critical discussions" (Jablonka, 2003, p. 98) to find students' views of modelling and their view of the relevance and epistemological status of mathematical models in society, which is an important aspect of mathematical literacy (Jablonka, 2003) and also a goal in the curriculum (Skolverket, 2000).

Notes

1. See http://www.merriam-webster.com/dictionary/model

2. *Discourse* is a "special type of communication made distinct by its repertoire of admissible actions and the way these actions are paired with re-actions; every discourse defines its own community of discourse; discourses in language are distinguishable by their vocabularies, visual mediators, routines, and endorsed narratives" (Sfard, 2008, p. 296). *Mathematical discourse* is a "discourse with vocabulary that counts as mathematical" (Sfard, 2008, p. 299).

3. *Mediators* "are objects that are operated upon as part of communication" (Sfard, 2008, p. 133). *Narrative*: "a series of **utterances**, spoken or written, that is framed as a description of objects, of relations between objects, or processes with or by objects, and is subject to endorsement or rejection, that is, to being labeled as "true" or "false"" (Sfard, 2008, p. 300).

4. This paragraph summarizes an outline in Jablonka and Gellert (2007).

5. *Routines* are repetitive patterns characteristic of the given discourse (Sfard, 2008, p. 134)

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On Swedish Upper Secondary Students' Descriptions of the Notions of Mathematical Models and Modelling

Peter Frejd and Jonas Bergman Ärlebäck
Linköping University, Sweden

This paper analyses 263 Swedish upper secondary students' descriptions of the notions of mathematical models and modelling using a coding scheme inspired by grounded theory, resulting in 34 coding categories. These are reduced and brought together in seven broader categories capturing different dimensions of the students' expressed views of mathematical models and modelling. It is found that the students prominently connect mathematical models and modelling with aspects of problem solving and with applying mathematics in extra-mathematical situations. Mathematical models are also equated with mathematical formulae and modelling is considered to be a constructive and creative activity.

Introduction

In Sweden, the government has emphasized the importance of mathematical models and modelling in the upper secondary mathematics curriculum. Indeed, using mathematical models is put forward as one of four important aspects of the subject mathematics that should permeate *all* mathematics teaching, and an explicit goal for the mathematics teaching is to develop the pupils' abilities to model mathematically (Skolverket, 2001). Nevertheless, the notions of mathematical models and modelling are only described in implicit terms, which opens up for various interpretations of the meaning of these concepts as well as of how they should be implemented in the classrooms (Ärlebäck, 2009). In a previous qualitative study of two Swedish upper secondary teachers' expressed views on mathematical models and modelling, it was concluded that the teachers could not give coherent and well formulated descriptions of the notions (Ärlebäck, 2010). This paper focuses on the upper secondary students and aims to give an overview and initial understanding of how the students describe the notions of mathematical models and modelling. The investigation will contribute to the understanding of the state and status of mathematical models and modelling in the Swedish school system, a field which generally is under-research. Furthermore, students' views and how they comprehend the goals of their learning are of importance for the teaching and learning of mathematics. The results presented here stem from the qualitative part of a study focusing on how upper secondary students describe the notions of mathematical models and modelling, and investigating their abilities to solve problems with various modelling characteristics. In this paper we will only report on results in relation to the students' descriptions. The research question studied is '*How do Swedish upper secondary students in the 12th grade describe the notions mathematical models and modelling?*'.

Mathematical modelling and modelling competency

There are many different views and perspectives one can take on the notion of mathematical modelling (Blum, Galbraith, Henn, & Niss, 2007; Haines, Galbraith, Blum, & Khan, 2007). We have chosen to follow the approach taken by Blomhøj and Højgaard Jensen (2003), who describe mathematical modelling as consisting of the following six sub-process: *formulating a task in the domain of inquiry*; *selecting the relevant objects, relations and idealising*; *translation into a mathematical representation*; *using mathematics to solve the corresponding mathematical problem*; *making an interpretation of the results in the initial domain of inquiry;* and *evaluating the validity of the model* (p. 125). Blomhøj and Højgaard Jensen define that "[b]y mathematical modelling competence we mean being able to autonomously and insightfully carry through all aspects of a mathematical modelling process in a certain context" (p. 126), which is the definition we have adapted. The motivation for using this framework is twofold; (1) it is more or less the same view of modelling that underlies the construction of the test items by Haines, Crouch and Davis (2000) which we use in our study [1]; and (2) it is similar to the interpretation of mathematical modelling done by Palm et al. (2004) of the governing upper secondary mathematics curricula documents, i.e. Skolverket (2001).

Method and methodology

For reasons elaborated in (Frejd & Ärlebäck, in press), a design of four different seven-item tests based on 14 test items originating from Haines et al. (2000) was used to investigate 400 Swedish upper secondary grade 12 students' mathematical modelling competency. Apart from solving the seven test items the students were also asked to state their gender; last taken mathematics course and the grade they received on this; if they previously had encountered the notions mathematical models and/or modelling in their upper secondary education; as well as an additional open question on what the notions of mathematical models and modelling meant to them. The reason for choosing an open question was to give priority to the students' choice of words in their descriptions. The test was constructed so that it started with the open question followed by the seven test items. The students spent approximately 20-25 minutes working on the test. To provide some context to the open question and to introduce the notions of mathematical models and modelling, three quotes from the curriculum [2] were given before the students were asked to state whether or not they had previously encountered the notion mathematical modelling during their schooling. The students were encouraged to '*describe in your [the student's] own words what meaning you ascribe to the notions of mathematical models and modelling*'. 263 of the students (66 %) answered this open question [3] using between one and 48 words (10 words in average), and each individual answer was taken as one *coding unit*. Inspired by the coding procedure in grounded theory, we approached this data consisting of the students' written responses using a "line-by-line analysis" (Strauss & Corbin, 1998, p. 57) in order to generate initial conceptual categories describing the data through *open coding*. During this first analysis we realised that the students seemed to interpret the exhortation quite heterogeneously in the sense that in most of the cases it was by no means clear *what question* the students actually were answering; if they referred to *mathematical models*, *mathematical modelling* or *both*. Some students even just commented the given quotes. As a consequence, we had to focus on getting some initial understanding of how Swedish upper secondary students describe both notions taken together.

Open coding

To get a first interpretation of the data and to form ideas about possible meanings of words, sentences and phrases in the open coding process, the 50 first coding units were analysed independently by the two authors, guided by the question '*How do the students describe mathematical models and modelling in their response to the open question?*'. According to Robson (2002), working in a group "will enhance the ideas pool about what the data are examples of, and it will assist in keeping individuals 'on task'" (p. 494). In total, 28 conceptual categories were identified and to each category a question was formulated to be asked to the coding units to facilitate the categorisation. Using these 28 categories, the remaining coding units were analysed individually by the two authors, and after checking against each other's progress after 100 and 200 coded coding units, the number of categories and corresponding questions increased in number to 33 and 34 respectively. The students' answers (*coding units*) were coded belonging to one or more categories, and in order to get "observer triangulation" (Robson, 2002, p. 174) a comparison was done of the independent coding using the Holsti's method (Holsti, 1969) for computing reliability:

Holsti's method for reliability (R)

$$R = \frac{2C_{12}}{C_1 + C_2}$$

C_{12} is the number of codes assigned and agreed upon by both coders. $C_1 + C_2$ is the total number of codes assigned by both coders.

According to Kaid and Wadsworth (1989) an *R*-value above 0.85 is satisfactory, but if a value below 0.80 is achieved the researchers should react and take suitable measures. Our first coding, at the time of 248 coding units, resulted in the reliability R=0.75 (C_{12}=450, C_1+C_2=1190), which we found unacceptable. We discussed and revisited our categories and questions with the objective to make the categories more distinct and the questions sharper, thereby facilitating to "systematically specify" (Strauss & Corbin, 1998, p. 102) the descriptions and the definition of the 34 categories [4]. Using these modified categories and questions, a second coding resulted in a reliability of R=0.96 (C_{12}=609, C_1+C_2=1266), which we found satisfactory. However, after the second coding there were 26 coding units which we disagreed upon. These were discussed and eventually our differences of opinion were settled and completing the opening coding process resulting in R=0.99 (C_{12}=648, C_1+C_2=1304). A set of 16 new coding units entered late and was added. This gave us a chance to test our categorisation schema resulting in R=0.90 (C_{12}=38, C_1+C_2=84) for the 16 new coding units and R=0.99 (C_{12}=686, C_1+C_2=1388) for the now in total 263 descriptions.

An example of a representative coding unit is: "A mathematical model to me is a formula or a way to calculate and to solve problems". Guided by the corresponding questions to the open categories this coding unit was coded into (C1) *Models are formulae* – Are models expressed in terms of formulae?; (C9) *Method* – Are models/modelling methods, solving strategies or algorithms?; (C10) *Calculating, unspecified* – Does modelling include calculations?; and (C13) *Problem solving* – Do models/modelling have to do with solving problems?

Axial coding

The process of linking the 34 categories together required many discussions on how to reassemble the data, relating new broader categories to the open coding categories and to identify the diversity of conditions associated with each category. Questions asked were '*how*' and '*why*' the open coding categories were related, and this process (described more in detail below) of *axial coding* resulted in that the numbers of categories decreased and transformed into seven new categories. See the table below and the appendix for the relation between the open and axial coding categories.

Table1: Resulting categories of the axial coding process.

Axial coding categories	
A. No clear view of modelling B. Using/applying mathematics C. Formulae and other connections D. Curriculum aspects	E. Problem solving aspects F. Focus on the aims of modelling G. Modelling is a creative activity

In the analysis of the open coding categories we realised that the category *Emotional and affective aspects* (C23) to a large extent (25 coding units out of 33) captures coding units in which the students explicitly emphasize that working with models and modelling is important [5]. Since our focus in this study was not on students' emotional inclinations toward models and modelling, this category was excluded in the axial coding process. In addition, also the *Metaphor* (C29) and the *Other* (C100) category were excluded since they captured the form of how the students wrote his/her answers and nonsense answers respectively. As a consequence, 39 coding units were excluded (33, 3, and 3). Our choice of grouping the open coding categories is of course only one possibility among others, and our rationale is provided below.

A large proportion (about 25 %) of the students expressed not to have a clear view of models and modelling and we linked and collected the open coding categories where students' descriptions explicitly indicated such an uncertainty into category A. Category B is related to how mathematics is used and applied in other contexts, especially in relation to the real world/reality. The category *Realistic examples* (C14), on how mathematics is or could be used in real world situations, was also taken to be one of the constituents of category B. '*How is a model represented mathematically?*' is the question used to discern category C and relations were found to the open coding categories of *Models are formulae* (C2), *Equations* (C12) and *Connections* (C17). We found that some of the open coding categories had aspects connected to curriculum and teaching methods, so we made category D an overall category describing these curriculum aspects. The largest open coding category was *Problem solving* (C13) and taken together with some of the other open coding categories also related to aspects of problem solving, category E was conceived. Turning to category F, *Focus on the aims of modelling,* it focuses and collects the open coding categories that could be seemed to address the question '*What is the aim and meaning of models and modelling?* The last category, G, *Modelling is a creative activity*, brings together the open coding categories *Creative activity* (C2) and *Description* (C28).

According to Strauss and Corbin (1998), the third step in a classical grounded theory approach is the *selective coding process* to refine and integrate a theory of students' conceptions (descriptions) of mathematical models and modelling and to discover the main theme of the research (*a central category*). However, in this case when the arguments and sentences are quite short and the overall aim is to explore students' descriptions of models and modelling, we decided not to carry through this third step and create a central category, but rather to try to theorize students' expressed views of mathematical models and modelling based on the axial coding categories.

Results

To facilitate the comprehension of the result, some definitions about codes and coding are briefly recapitulated: a *coding category* (open or axial) is one of the categories C0-C32, C90, C100 (open) or A-G (axial); a *coding unit* is a student description (a piece of data); and when a coding unit is considered to belong to a coding category the coding unit is assigned the *code* representing the specific coding category.

The frequency of the 34 open coding categories (C0-C32, C90, and C100) and the frequency of number of codes per coding unit are displayed in Figure 1.

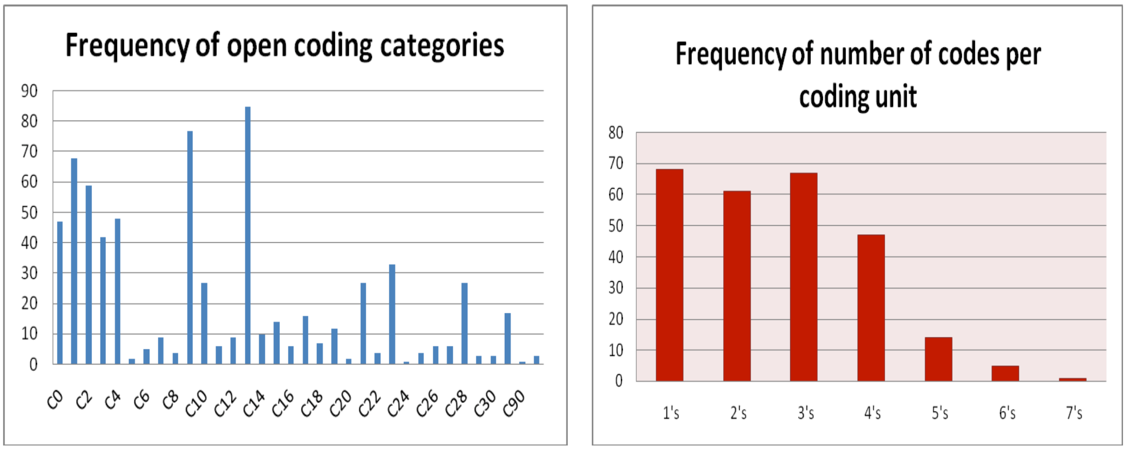

Figure 1: Frequency of the open coding categories and number of codes per coding unit.

The left diagram in Figure 1 shows that the codes *Problem solving* (C13), *Method* (C9), *Models are formulae* (C1), *Creative activity* (C2), and *To use/apply mathematics unspecified* (C4) are the most frequently appearing open codes in the data. *Do not know* (C0) also appears relatively frequently, and it is notable that 18 of the 34 open coding categories have got less than ten codes. In the right diagram in Figure 1 it is evident that the majority of the coding units have been coded with one to four open coding categories (2.6 in average) and the coding units assigned more than four open codes are together less than 10 % of the coding units. The total number of codes in the open coding procedure is 686.

The frequency of the seven axial coding categories (A–G) and the frequency of number of codes per coding unit are displayed in the diagrams below. In the left diagram in Figure 2 one can see that category E (*Problem solving aspects*), B (*Using/ applying mathematics*) and C (*Formulae and other expressions*) are the most frequently appearing codes in the data. About half of the coding units (51 %) are coded belonging to category E [6] (*Problem solving aspects*). Category B (*Using/applying mathematics*) and C (*Formulae and other connections*) are coded to every third coding unit (37 % respectively 33 %). The categories D (*curriculum aspects*) and F (*Focus on the aims of modelling*) were coded to about

10 % of the coding unit and the categories got less than 15 % together of the total number of codes. In the right diagram in Figure 2 it is obvious that the majority of the descriptions made by the students have been coded belonging to between one and three axial coding categories (2.1 in average). The coding units coded belonging to four or five (axial) codes are together less than 10 % of the total coding units. The total number of codes in the axial coding procedure became 512; 39 open codes were excluded (see the section on axial coding).

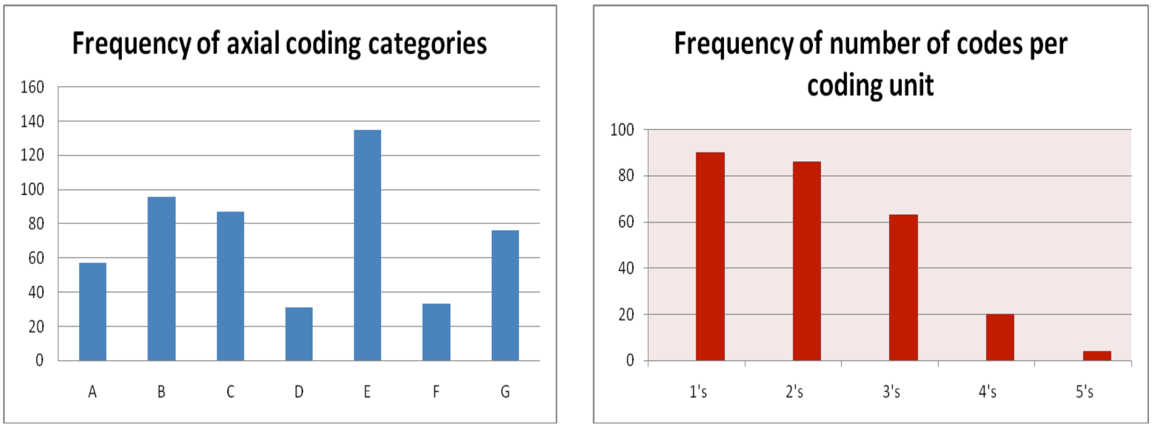

Figure 2: Frequency of the axial coding categories and number of codes per coding unit.

Conclusion and implications

66 % of the students responded to the open question and the study shows that out of them, one fourth expressed that they did not have a clear view on mathematical models and modelling. Generally, the descriptions made by the students were short in facts and in words (10 in average) and all together, since there were only few open codes per coding unit, this indicates a discrepancy between what is prescribed in the upper secondary mathematics curriculum and what the students expressed with respect to the notions of mathematical models and modelling. One reason could be that the students have not experienced these notions in the classroom, which is in line with results reported in Frejd and Ärlebäck (in press) where only 23 % stated that they had heard or used the notion of mathematical models and modelling before in their secondary education. Other reasons could be that the students have heard the notions but do not have a clear view about them or that they find it difficult to describe and express their views in writing.

The main conclusions we draw from interpreting the students' descriptions, as coded inspired by the grounded theory axial coding procedure, is that the students associate mathematical modelling with problem solving and with using/ applying mathematics as a tool in different situations, and mathematical models with formulas and equations. However, the fact that the given quotes from the curricula documents mentioned ‘problem solving’ twice might have drawn the students to give answers including problem solving.

A look at the more fine grained open coding categories revels that *Method* (C9), *Problem solving* (C13), *Models are formulae* (C1), and *Creative activity* (C2) are the four largest categories. In addition to the possible influence of the curricula quotes on the students' expressed emphasis on *Problem solving* (C13) above, the same comment may also apply to the expressed emphasis on mathematical modelling as a *Creative activity* (C2); the quotes use words like *design* and *fine-tune* (see [2] for details). One can note that the use and reference to *problem solving* in the upper secondary curriculum documents also lack explicit and clear definition or descriptions (Ärlebäck, 2009). As a consequence, the connection made between mathematical modelling and problem solving does not provide any helpful information or deeper understanding of either of the concepts per se. Regarding the categories *Method* (C9) and *Models are formulae* (C1) the connection to the given curricula quotes are not as easily inferred, but they rather seem to originate from the students themselves or other external influences. It is interesting to note that on the one hand the students express that mathematical modelling is methods, solving strategies and algorithms for problem solving, and on the other hand that mathematical modelling is a creative activity. To us this seemingly contradictory aspects of the nature of mathematical modelling expressed by the students need to be further researched. However, the result of the open coding process suggests that the students experience *Method* (C9), *Problem solving* (C13), *Models are formulae* (C1), and *Creative activity* (C2) as central aspects of what mathematical models and modelling are, but to investigate this closer further research is needed. One possibility to shed some light on these matters would be to ask separate questions on mathematical models and modelling respectively. Additionally it should be investigated what kind of modelling problems and what aspects of modelling processes the students have been tested on before and to what extent teachers believe modelling activity is a part of mathematics education and how these beliefs/views influence their teaching.

The indications found in Ärlebäck (2010) and in this paper concerning the difficulties to describe the notions of mathematical models and modelling seem to imply the need for a more explicit definition of these concepts in upper secondary written curriculum documents. Such a clarification might facilitate the teaching of mathematical models and modelling and to make the learning goals for the students more explicit and comprehensible.

Notes

1. The test items developed by Haines, Crouch and Davis (2000) have been used in a variety of contexts with different purposes; see Frejd and Ärlebäck (in press) for references.

2. Three quotes were given; "Problem solving, communication, using mathematical models, and the history of mathematical ideas, are four important aspects of the subject that permeate all teaching" (Skolverket, p. 61); "An important part of solving problems is designing and using mathematical models…" (Skolverket, p. 62); and " The school in its teaching of mathematics should aim to ensure that pupils:… develop their ability to design, fine-tune and use mathematical models, as well as critically assess the conditions, opportunities and limitations of different models" (pp. 60-61).

3. This is to be compared with the fact that 381 of the students (95 %) answered at least three of the seven test items on the tests. See (Frejd & Ärlebäck, in press) for details.

4. See the appendix for the 34 open coding categories as formulated after the last coding; C0-C32, C90, and C100.

5. Compare with the contextual quotation used in the beginning of the tests given in footnote 2: "An important part of solving problems is designing and using mathematical models…" (Skolverket, p. 62, italics added)

6. To clarify the calculation of 51%, take the 134 coding units of category E from figure 1 left diagram divided by total number of coding units 263.

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Appendix

Category No.	Name of category and its characteristic question	A	B	C	D	E	F	G
0	**Do not know** – *Is statements like 'Don't know' part of the coding unit?*	A						
1	**Models are formulae** – *Are models expressed in terms of formulae?*			C				
2	**Creative activity** – *Is modelling a creative activity?*							G
3	**Connection to reality** – Are *models/modelling to do with the extra-mathematical domain?*		B					
4	**To use/apply mathematics unspecified** –*Are models (unspecified) tools?*		B					
5	**To use/apply mathematics in future professions** – *Are models tools for use in future professions?*		B					
6	**To use/apply mathematics in everyday life** – *Are models tools used in everyday life?*		B					
7	**No experience** – *Does the coding unit relate explicit to no experience of models or modelling?*	A						
8	**Organize teaching** – *Are models/modelling a teaching method?*				D			
9	**Method** – *Are models/modelling methods, solving strategies or algorithms?*					E		
10	**Calculating, unspecified** – *Does modelling include calculations?*					E		
11	**Facilitate calculating** – *Does modelling facilitate calculations?*					E		
12	**Equations** – *Are models expressed in terms of equations?*			C				
13	**Problem solving** – *Have models/modelling to do with solving problems?*					E		
14	**Realistic example** – *Are models described as/with realistic example?*		B					
15	**Artificial example** – *Are models described as/with artificial example?*		B					
16	**Predict** – *Are models being used to make predictions?*						F	
17	**Connections** – *Are models expressed in terms of connections?*			C				
18	**Idealization** – *Is modelling about idealizing/simplifying a situation?*					E		
19	**Interpretation** – *Is modelling about interpreting mathematical representations?*					E		
20	**Proofs** – *Is modelling connected to proof?*				D			
21	**Understand** – *Do modelling facilitates understanding?*						F	
22	**Increased mental ability** – *Is modelling associated with the increasing the ability to 'think mathematically' and/or increase 'abstract thinking'?*						F	
~~23~~	~~**Emotional and affective aspects** – *Are there any emotions or affections expressed in the coding unit?*~~							
24	**One-to-one** – *Are there one-to-one correspondence between models and reality?*						F	
25	**Everything** – *Do models permeate everything in mathematics?*				D			
26	**Way of analyzing** – *Is modelling explicitly related to analyze a situation?*					E		
27	**Geometry** – *Is modelling related to geometry?*				D			
28	**Description** – *Is modelling relate to make a description of 'something'?*							G
~~29~~	~~**Metaphor** – *Are metaphors being used to describe models/modelling?*~~							
30	**Quotation referring** – *Are there any reference made to the given citations from the curricula in the coding unit?*				D			
31	**The structure of mathematic and mathematical knowledge** – *Are there any connections to the structure of mathematics or to any specific subject fields of mathematics?*				D			
90	**No comments** – *Does the coding unit relate explicit to no comments?*	A						
~~100~~	~~**Other**~~							

Narratives of Students Learning Mathematics: Plurality of Strategies and a Strategy for Practice?

Sharada Gade
Umeå University, Sweden

Reporting from a study conducted at a grade six classroom in Sweden, this paper takes a closer look at narratives from a Brunerian perspective. The features of what constitutes a narrative and what the conduct of narrative inquiry could offer are discussed. Research narratives of students are rendered providing insight into strategies beyond voices that students bring to situated learning. Robust recognition of narratives is argued for, as a means with which to strengthen teaching-learning practice and as implementable strategy with which to bridge research, theory and practice. In having potential to simultaneously be object, tool and result of the study of human development in practical activities, narratives seem also to satisfy a key premise of activity theory.

Advocating the active role of teachers in educational and professional research Lawrence Stenhouse is often quoted as "It is teachers who, in the end, will change the world of the school by understanding it". Yet the manner in which either teachers or research grasps the complexity of schooling remains wanting of appropriate means. In this paper I attempt to address this problematic area by utilising narrative inquiry and report from a study conducted during the spring of 2009 at a grade six mathematics classroom at a Swedish school. Obtained by way of observations and interactions with students and their teachers my approach recognises that as storylines, narratives both represent and resolve the situated nature of individual experiences. In sharing narratives, I portray not just the voices of students but also their strategies towards learning mathematics in their classroom. The question thus addressed is - what range of voices and strategies do students narrate and display within the classroom teaching-learning of mathematics. Towards these aims I first refer to literature that illuminates the need for recognising the narrative mode of meaning being made within schools, followed by its study in the form of narratives. I then sequence student narratives before bringing my discussion to conclusion.

Schooling and narrative

While reflecting on Bruner's legacy to educational theory, practice and research, Olson (2007) draws our attention to his exposition of intersubjectivity. Essential to the meeting between child and adult as also between mind and society, Olson points out that this notion of intersubjectivity is central to being able to understand our goals of education as well as the means with which to pursue them. Schooling in society for students, Bruner argues, is not only about the acquisition of valid knowledge and useful skills but also about being fulfilled and satisfied in experience - resulting in a deepening of understanding and an increasing level of the control. It is to arrive at such understanding that Olson (2007) reiterates the need for an anthropological study of schooling - to be in a position to appreciate its complexity and situatedness. This would also inform he says a pedagogy that encourages students as learners to formulate their own views and test them against those of others.

Bruner’s notion of intersubjectivity is accompanied by the attention he draws in particular to narratives. A principal means in which meaning is made and shared, a child’s narrative Bruner (1991) points out, is a vicarious means of accessing his or her meaning making and construction of reality. Bruner explains narrative (1985, 1990) to be a normative manner in which an individual deals with her or his intentional states and experience, offering us a window to a person’s voice, position and perspective while participating in any culture. As a mode of thinking that accompanies participation, narratives must be cultivated and nurtured Bruner (1996) argues for two vital reasons - to help children find their identity in the culture they are growing up with, as also to bring forth the plurality of voices that are to fill democratic classrooms. Attention to narratives towards understanding how students come to know and make use of the mathematics they come to know, has been argued for by Burton (1999). Such an approach she says provides access to a child’s agency within socioculturally complex settings, wherein the product of educational activities and practices is a text that has to be authored by learners.

My interest in narratives is to understand another aspect of pedagogy that Olson (2003) draws attention to - the formation of joint intentionality between students and teachers so that it is students who take up responsibility for their own learning. I found the incidence of this in an earlier classroom study (Gade, 2006) to be a process that involved negotiation of the bridge between personal and propositional forms of societal knowledge. This bridge in terms of which Bruner (1996) conceived schooling is analogous with what Vygotsky (1978) saw as that between everyday and scientific concepts. Making either transition involved students actively utilising available cultural resources - enabling them to define themselves as individuals in the process. Analysis of the use of the cultural resource of language is informed in particular by the Bakhtinian construct of utterance (Holquist, 1990). As units of verbal communication as well as narratives, these are the stock in trade of teaching-learning. Bakhtin argues utterances resolve situated experiences and are realised materially in them. According to Holquist (1990) utterances are forms of communication that are particular to a person, relative to other speakers and never isolated from the context in which the speaker is situated. From such a perspective a student is constantly authoring her or his self in a process that transpires the length of teaching-learning. With a need to attend to narratives both to understand meanings that are being negotiated within schooling, as well as to grasp how students author themselves individually, I now turn to the possibilities afforded in utilising narrative as method of inquiry.

Narrative inquiry

The ubiquity of narratives is acknowledged by Barthes (1977) who states that they are simply found everywhere. Towards conducting narrative inquiry, I thus address what counts as a narrative before discussing what its pursuit can offer. With reference to the notion of truth Bruner (1991) argues that narratives being situated constructions cannot be verified and can only be judged by truth-likeness or verisimilitude. However and even though they are never faceless Riessmann (1993) argues that narratives can be sorted into categories allowing one to always ask why particular stories are told in particular ways. Ochs and Capps (2001) say in addition that narratives as stories may not occur in a finished form, as a narrator could well be in the middle of sorting out his or her experience. How then are we to distinguish everyday conversations and talk of students and teachers from narratives? In not being restricted to the medium of texts that I offer, Herman (2009) offers four characteristic features of a narrative: (1) a quality of being situated in a sociocultural setting; (2) a sequencing in time of events that take place; (3) a certain disruption of the world for a particular story to have been made or told; and (4) a concern for what it is like for the narrator to have the kind of experience that he or she is relating. Identifying narratives with these principal features I now turn to the conduct of narrative inquiry.

As a text that is both a process and a product, Clandinin (2007) points out that a narrative is a profoundly relational piece resting heavily on the ethical role of a researcher. As a method within qualitative research, Clough (2002) adds that the challenge of any narrative is in its being able to translate life as experienced - offering insight into events that seem familiar. In such a rendering the voice of the researcher, mine in this paper, not only offers the narratives of students but also the basis for their interpretation. My writing then is meaning made by me in retrospect, combined with a search for the distinct voices of students while learning mathematics in the sociocultural classroom I studied (Chase, 2005). The narratives that I offer are thus obtained by conducting what Mishler (1986) calls research interviewing wherein the context of my conversations are not relegated to the background, but a living part of my interactions with students and teachers. Either of them were collaborators with whom I shared control so that we could together understand what the story being made was about (Creswell, 2002). Working with narratives has advantage of working in a medium in which most teachers as practitioners work (Polkinghorne, 1988) making it possible to be a form with which to bridge a research-practice divide. The need for practice based empirical evidence within mathematics education is also argued by Silver (2009) with an objective of developing what he terms as a practice based professional development for teachers. With narratives having potential to inform a wide range of issues across educational practice, I now outline my role as researcher and the relevant aspects of the classroom in which I conducted my study.

The students in my study were taught mathematics by Lea in a class whose class teacher was Sofia (all names are pseudonyms). Lea consented to my being participant observer and took permission from the concerned parents, introducing me as a researcher at the University close by. Since I was not Swedish speaking everyone understood that I could learn some Swedish and they could learn some English. I observed Lea's conduct of instructional activities in a naturalistic manner. Lea eased me into her class with great sensitivity, which I hope to have reciprocated. After initial curiosity about my country India and if I could tell and spell their names correctly, Lea's students would approach my working table placed behind their desks. This table was used by Lea to work with students who sought her attention, though she also sat with them if necessary at their desks or with them in the adjacent group room. Over time students would ask her if they could work with me and Lea also asked if some students in particular could. Where some of my interactions with them were not in English I would ascertain the meaning and context of the Swedish used. An aspect of Lea’s practice that I found noteworthy was her having readily available, worksheets of mathematical problems copied from a book designed for such purposes (Rockström, 1996). I found students to attempt these on their own, as well as Lea to recommend specific ones depending on her judgement. In this manner Lea’s students always had some form of mathematics to pursue at all times. I offer in the next section, narratives of three students with whom I had opportunity for greater interaction than others. Drawing also upon observations made by their teachers, these were triangulated with observations I made while conducting problem solving sessions with student pairs and a year-end interview with both Sofia and Lea.

Alex, Felix and Kim

“Your work with Alex was a success” said his mother whom I met when he was in grade seven. Lea asked if I could work individually with Alex, who she said would ask her once again something she had mentioned to him a moment or two ago. During her meetings with Lea and Sofia, Alex's mother would agree that Alex needed to work additionally at his mathematics. Looking to explain his performance she would surmise “He is a humanist, like his father.” Lea wrote to her saying I would work with him. “I will also encourage him at home,” she replied. Alex had a sensitive command over English making it easy for me to work with him. He struggled with reducing fractions to their lowest terms. Alex wanted to reduce 24/72 at one go to get 1/3 which he saw other students being able to do. I suggested we simplify his fraction by reducing the numerator and denominator one factor at a time. Upon suggesting the factor of two, Alex was able to divide 24 but was hesitant and unsure of dealing with the carry over in 72. We soon spread out in the space of the group room and worked at building tables with appropriate number of rows and columns of coloured pencils. He was able to find patterns in the tables of 2, 4 and 8 as well as 3, 6, and 9. This was after finding more obvious patterns in the tables of 5 and 10. I encouraged him to find patterns in a 10 by 10 number grid over his mid-semester break but he said he lost the papers. During my conduct of problem solving he solved one problem easily and fluently in English, “Is it 12 [cm^2] ... Because if this is 6 [shaded star] and this is one star too and when you put them together you get 12.”

When asked if he enjoyed his problem solving session, Felix chose his words in English and said “This was special!” My interaction with Felix was mainly when he attempted a numerical crossword in one of Lea's worksheets. He worked through the clues to arrive at numbers that were to fill the crossword. Occasionally he turned to me to solve say 2 × 47 × 5. On blanking out 47 with my finger and asking him what 2 × 5 was, he saw the point of not multiplying 2 and 47 first. He turned to me again when the clue asked 124 + 89 – 123 where my finger on the number 89 was followed by his saying “90”. Working quietly he finished the crossword which then asked children to add all the digits that were entered. The instruction beneath read “If the sum is 108 you have worked correctly”. Felix now started working with a pencil and shifted to writing on his table. He wrote down the digits he had entered and added them in pairs obtaining get six secondary sums. Adding these sums two each at a time, he obtained three tertiary sums. Upon adding these three tertiary sums at one go he did not obtain the stated total of 108. But I was nearby. We then took a blue coloured pencil and tallied the entries on his worksheet. A few needed correction. What now about the many sums on his table. With a green pencil we now tallied and traced which one of those were incorrect and corrected the same. “Mycket tack [Many thanks] Sharada!” was his low cry when he obtained 108, hands on his head. “I am not too worried about Felix” Sofia would say, summarising him as a student at the end of the year, “His parents are patient and good with him.”

I found Kim waiting patiently at my table one day. I wondered why he was there yet soon got involved with his class work wherein he showed great facility with numbers. “You are good” I said and asked him a question a notch tougher. “You *are* good” I reacted. “But I am not good at my reading” he said, conversing in English. The next day, I took suitable material to evaluate his reading which he showed no problem with. “I’ve told my colleagues about you, but I did not tell them who you are.” I said. “Oh thank you” he said, happy to not be in the midst of limelight. What was his problem then? He showed me a page of text from a story book. “I am slow” he said. I shifted strategy and took him mathematical problems (Hagland et al., 2005). How many different kinds of ice creams combinations, each with two flavours, could be made with four different flavours? He paired his fingers and showed me six possible pairs on the diagram given. He went on to say that “*if* three flavours were to be chosen, *then* ….” Kim chose to next work with a problem related to the pieces of a tangram. If the area of all seven pieces together was 400 cm^2 what was the area of each piece? Kim obtained a ruler and measured the many sides and struggled. I introduced him to an algebraic approach in a while. “Which of the pieces look alike?” He pointed to the two largest triangles that made up one half of the square and said each had an area of 100 cm^2. We now found ourselves working with the other half. “Are there any more similar pieces, and, shall we give them a name?” He suggested “A” and labelled each of the two smaller right angled triangles. Some more silence with three more pieces to go - a square, a larger right angled triangle and a parallelogram. Proceeding intuitively he said that the square was “Two A’s” just as the larger right angled triangle was also “Two A’s”. The parallelogram he concluded “Must be two A's as well!” and went on to calculate the area of each of the pieces saying “I did not think this would be so easy!”

Voices and strategies

The three narratives I present were formulated against a textual backdrop of nineteen other classmates, some of whom I now sketch in brief. In their efforts at learning mathematics, I observed Emma to cut the net of a particular cube drawn on paper with scissors to be able to convince her partner Lisa as to why the given diagram would not form one in reality. I also noted Lisa to one day cut a net of five squares and tape the adjacent sides, in order to measure the volume of water that could be held. I found Casper to keenly observe Lisa in her attempts. Lea mentioned Casper to “think fast yet fell short in written work”. Casper solved a particular problem which awarded three negative points for an incorrect answer and five points for a correct one in no time. On inquiring how, he simply said “Oh! It just came to me.” The class also consisted of Elias and Ella a pair of twins who seemed to exhibit a reversal of gender stereotypes. While Ella was probably the most adept at football in the class, Elias was fond of art and puppets and on one occasion used matchsticks to explain his solution to his partner. Taken together with narratives of Alex, Felix and Kim, these thumb line sketches of their classmates, allow me to evidence not merely the presence in them of a plurality of voices but more crucially the presence in them of distinct strategies with which each were, as Burton (1999) argued, authoring themselves.

I crafted the three narratives I present towards the end of my study, by which time I had assembled in a historical manner for each student, utterances and anecdotes that I took down in my field notes. I now discuss how these differed greatly with reference to Herman's (2009) four features of a narrative: sociocultural setting, sequentiality of events, disruption of everyday routine and the individual experience of the students. While the sociocultural setting in Alex's case bridged his home environment and school, that in Felix's case was limited to a single problem in the classroom, where in my interaction with Kim both classroom and University were involved. The sequence of events from which I drew upon also differed in that my interaction spanned a few months with Alex, about an hour with Felix and the duration of my study with Kim. The disruption of routine events that allowed me access to the narratives also varied. In Alex's case this was initiated by Lea. My physical proximity to Felix led to my accessing his, and my presence as a researcher in the classroom led to Kim's. I point out that it was while eliciting the experiences of students *as* narratives, that their attempts at learning each portrayed a different story. While Alex's story is window into the nature of his struggles with respect to the mathematics being demanded of him, Felix's story is one of his ability to utilise assistance to diligently pursue his own goals. In this he revealed that he knew when he needed guidance, as well as how to utilise the guidance that was available near him. Unlike Alex and Felix, Kim's storyline of learning was different yet equally demanding. Kim both thought and demonstrated that he was adept and capable. It is possible that Kim was looking for challenges not available to him within classroom teaching-learning.

Five aspects of the conduct of narrative inquiry seem significant and worthy of a closer look here. Firstly, the utterances of students and teachers I drew upon in my study spoke for them in the material of language. Following Bakhtin (Holquist, 1990) they were the medium in which students participated in various classroom contexts in which they found themselves every day. Secondly, when formulated as narratives following Herman's (2009) criteria, one was able to highlight and discern the strategies of students from familiar talk that prevailed in the discursive space of the classroom. Third, in accessing the narrative mode of thinking that students were sharing and making public, one had insight into the nature of relationships that they were formulating with themselves, with one another as well as with the subject of mathematics. Following Bruner (1990) and even if vicariously, such access to their intersubjectivity offered a window into the world-making of students within which they were acting. Fourthly, my conduct of narrative inquiry offered means as argued by Olson (2007), with which to gain insight in anthropological terms into the manner in which classrooms realise the goals of schooling. Finally, in the form of situated narratives I find teacher, educator, researcher and policy maker to have the same basis with which to appreciate, though upon differing expertise and verisimilitude, the everyday actions of students in their classrooms.

A strategy of and for practice

Drawing attention to the need for understanding how the personal knowledge of teachers is shaped by the practical realities in which they work, Clandinin and Conelly (1996) argue for new ways with which to relate to the contextual nature of professional landscapes in schools. Towards meeting this objective I argue that the benefits of the conduct of narrative inquiry in classrooms deserve a closer look at two noteworthy levels – of practice as well as for practice.

At the level of practice, my study evidences how students utilised various cultural resources that were prevalent in their classroom, to demonstrate the sense they were making of the mathematics they were learning. For this they utilised scissors, paper, matchsticks, their language, the teacher, each other and even the researcher. In so doing they brought forth and shared their personal knowledge of mathematics, which Bruner (1996) and Vygotsky (1978) argued that education was to change into academic and scientific forms. The utilisation of various cultural resources by students in my study in turn informs another aspect of practice, the kind of teaching practice that Lea had established in her classroom. In her students being able to share in comfort and without hesitation their natural dispositions and understanding of the mathematics they were learning, Lea was able to inculcate as pointed out by Olson (2007), the adopting of responsibility by students of their own learning. I however argue that these very features of Lea's classroom also lie at the heart of a larger initiative of sustaining and strengthening teaching-learning in classrooms contexts. Whereas conceptualising education in terms of bridging personal and scientific forms of knowledge, and of students taking responsibility of their own learning may seem straightforward, the fact that Lea has to deal with the plurality of voices and strategies that her students demonstrated arguably makes very high personal and professional demands on her.

While Lea is seen to have considerable freedom to interpret her teaching on the basis of her students' experiences, interests and needs (Skolverket, 2008) in the light of my study it seems possible that this is the challenge that not just she but all of us who are keen on improving classroom practice face. Curriculum guidelines (Skolverket, 2006) seek that Lea stimulate her students, take as her starting point each student's needs and thinking, while also provide them scope for individual creative expression. I find both these guidelines well grounded in Lea's teaching practice. However, finding syllabus and curriculum guidelines well grounded in my study in no way diminishes the challenge that I contend Lea to face in her classroom day in and day out. The goals that Lea needs to achieve and strive for weigh in her classroom, alongside the expectations that Sofia as class teacher has about the performance of her students at National Tests. Though additional materials like worksheets were found to be of time-saving value in my study, I argue that we may need to think in terms of assisting specific areas of ongoing classroom teaching-learning or, for practice.

What can be done I ask, to strengthen Lea's teaching-learning practice so that she is not left feeling she that had no time for any of her students. And even if Lea had the time, I am not sure we are in a secure position to say in what way she is to cater to the plurality of voices that her students exhibited. I maintain that we may be encountering a problem that research needs to pursue - one which better understands the nature of learning strategies that students adopt, so as to gather these in more identifiable and perusable forms. With adequate number of cases pooled together, we may have the basis with which to contribute to and build a practice based professional development for teachers as argued by Silver (2009). The sketches and narratives of students at grade six for example, show the manner in which their mathematical abilities were embedded in the concrete as well as how students were both wanting and deserving of explicit attention from their teacher. As with Alex, one's personal ability at mathematics seemed inseparable from the disposition that one brought to one's learning as a student. Being able to discuss narratives in the form of such cases would, I argue, allow practising teachers the opportunity to draw from both personal as well as professional experience and offer reflections that could be beneficial when analysed and shared with one another.

Deploying narrative inquiry as a means with which to strengthen teaching-learning may have yet another advantage, that of communicating in a medium in which most teachers conduct teaching practice either with students, parents of students, fellow teachers or in various acts of mentorship. A narrative approach could thus afford teachers, implementable means with which to grasp one's own professional landscape, the benefits of which Stenhouse argued had potential to bring about change in schools. Present tacitly in a teacher's repertoire such an approach may however have to be recognised, legitimised and endorsed in robust terms of research. Such attempts would be able to inform two critical and crucial aspects of classroom practice – the gap between personal and scientific forms of knowledge that students are expected to bridge and the persisting gap between theory, research, policy and practice that teachers are left with bridging.

One final thought crosses my mind in the conduct of my study - that of the parallels I observe in the methodology of narrative inquiry with those forwarded by Vygotskian (1978) perspectives. In having potential to appreciate and understand human development within practical activities, in a medium and means that is simultaneously object, tool and result of the study, I believe the conduct of narrative inquiry to reflexively complement a central tenet of Activity Theory. Drawing upon a mode of thinking that Bruner (1985) differentiates from the logico-scientific mode - a mode synonymous with mathematics, I contend that attention to a narrative mode of thinking and situated narratives to be beneficial to the education of mathematics or mathematics education. Can a narrative approach as outlined in my study, not be an implementable strategy with which to bridge educational research, theory and practice within mathematics education?

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Achievement as a Matter of Choice?

Eva Jablonka, Maria Johansson and Mikaela Rohdin
Luleå University of Technology, Sweden

The contribution reports an initial analysis of the first mathematics lessons of two classrooms at the beginning of secondary school in Sweden from an ongoing research study. The comparison discusses similarities and some differences in the ways in which the teachers facilitate students' access to mathematical knowledge with a focus on episodes that refer to the transmission of the criteria for the production of legitimate contributions.

Introduction

Two classrooms at an upper secondary Swedish school are investigated in the context of an ongoing international study of classrooms in Germany, Canada and Sweden. We seek to identify discursive and interactional mechanisms that can explain emerging disparity in achievement in mathematics classrooms at the beginning of secondary school where the students and their teachers are together for the first time [1]. The Swedish data comprise video-footage of the first eight respectively nine mathematics lessons in the two classrooms, recordings of interviews with all students and one of the teachers, copies of a test from one of the classrooms, textbooks and other material used as well as information about the students' social, cultural and economic backgrounds. Both classes are beginning an obligatory mathematics course ("course A"). Completion of the course comprises a national test. The results are meant to inform the teacher's grading and are also reported to the school authority. Teachers also administer their own tests. The curriculum prescription for the course is outcome based and does not include recommendations for the order of topics, time allocation, pace and teaching methods. The grading system comprises four levels: not pass, pass, pass with distinction, and pass with special distinction.

Due to limited space we can only present some excerpts from the first lessons that deal with the transmission of the criteria for the production of legitimate contributions. Our discussion however also draws on additional information from what we have seen in the consecutive lessons as well as on a preliminary and rather cursory analysis of the textbooks and the interviews. We have not yet analysed in detail all the conversations between teacher and students that happen in both classrooms throughout the lessons when the teacher talks to individuals or to a group.

Theoretical background

In the following, we employ Bernstein's (1990, 2000) concepts of classification and framing for characterising the classroom practices. Classification generally refers to the strength of boundaries established between discourses, agencies, physical and social spaces, while framing refers to the principles that regulate how a discourse is to be transmitted and acquired in the pedagogic context. In our analysis, classification refers to categorizing areas of knowledge in the mathematics curriculum. Strong internal classification means that clear boundaries between mathematical sub-areas are maintained. Strong external classification indicates that few connections are made to other disciplines or everyday practices. At the micro level of pedagogic practice, framing refers to the options the students and the teacher have in the control over the selection of the communication, its sequencing, its pacing, the evaluation criteria for the knowledge, and the social base which makes access to the knowledge possible (Bernstein, 2000, p. 12-13). According to Bernstein, variations of classification and framing relate to differential access to institutionalised knowledge. Evidence for this theoretical claim has been produced by a range of empirical studies (e.g. Chouliaraki, 1996; Cooper & Dunne, 2000; Gellert & Jablonka, 2009; Hasan, 2001; Lubienski, 2000; Morais & Miranda, 1996). For being able to deliver a legitimate contribution, students not only need to have access to the classificatory principles of the knowledge that is transmitted, but also need to share the behaviour, aspirations, attitudes and values favourable to the context, that is, they need access to both the instructional and the regulative classroom rules. Specific instructional and regulative classroom rules can be seen as modifications of the underlying principles of classification and framing.

The case of the class enrolled in the Arts Programme

The 31 students in this classroom are enrolled in the Arts Programme (Estetiska programmet, short: ES) of the Gymnasium. They come from different comprehensive schools; most of them do not know each other. Mathematics is the only subject the teacher teaches in this class.

Choosing the suitable level of tasks

The first meeting between the students and the teacher was not planned to be a "lesson", but was devoted to organisational issues. One of the researchers happened to be there and made field-notes. After a personal introduction the teacher describes the mathematics course as basically a repetition from year nine. Then the teacher hands out the textbooks and explains:

Teacher: The book is grouped by levels and you will get a feeling which level suits you. [This is not a literal translation. In Swedish this sentence reads: Boken är nivågrupperad och ni känner själv vilken nivå som passar.]

After calling up the names of all students (three or four are absent) they are told:

Teacher: You will have to calculate a lot by yourself during the lessons.

Then follows an exchange of the students' previous experiences with school mathematics which leads to a short discussion about whether the "A tasks" in the book might be at the level of "pass". Throughout the book, all tasks are labelled (by colour and letters) as category A, B or C, some are labelled as "open". According to its title, this textbook is especially designed for the course A within the Arts Programme. On the first page there is a short explanation (addressed to the students as the readers) of the differences between the tasks:

> After the theory exposition follows a solved example that illuminates the theory. There are tasks at three different levels and of different character. Open tasks do not have one given answer and often require a mathematical discussion. A tasks are standard tasks that generally can be solved in one step, while B tasks often require a solution in several steps. C tasks are more complex in their character and for solving them you need to, amongst others, apply mathematical knowledge from several areas .

This description conveys that the labels A, B, C constitute a hierarchy of levels of difficulty, composed of "standard one-step tasks", "several-steps tasks" and "complex tasks". The status of the open tasks remains unclear. In the chapter the class deals with within the first lessons ("tables and diagrams") there are 25 A-tasks, 12 B-tasks and 2 C-tasks. There is only one open task (amongst the B-tasks) in this chapter. The textbook contains an answer key.

A possibility of s subsequent re-grouping

When the students come in to join the first lesson (and also in the consecutive ones) they are free to choose where they want to sit in the classroom. At the very beginning, the teacher hands out a working plan containing the page numbers of the textbook where the tasks to be dealt with can be found for the term and comments:

Teacher (01:20): Er, yes, we then can start by handing out a plan here... which runs to the break [autumn holiday]... ehm... why I did not do it for longer than until the break... because I will get to know you a little bit and see a test... roughly what level we are at... in order to/

Student: /I think this, too/

Teacher: /be able to do maybe a little re-grouping... maybe... because this is such a real big group... if all are here... because then you are actually thirty-two... be able to give some [students] permission that you go somewhere else and work for a while. In a group or so on... but I will have a little check-up on... and how fast we can go forward...and how fast I can push on the planning so to insert maybe more repetition time before the national test and so on... hence so we stop the planning until week forty-four.

By means of employing several hedges, the teacher seems to be mitigating the impact of the announcement of an upcoming test that will lead to tracking ("to know you a little bit", "roughly what level", "do a little re-grouping", "maybe", "a little check-up", "and so on"). The criteria for "getting permission to go somewhere else" remain hidden, achievement levels are not mentioned, but only a spatial separation of groups, argued by the actual group size.

Attempts to uncover the relation between the task levels and grade levels

Then the students start working with the textbook tasks. They can choose whether they talk with their peers, get help from the teacher or work on their own. After some minutes, a student sitting at the back calls the teacher by raising her hand. As can be seen from the video, some of the other students in the classroom stop working and listen to the conversation (all names are pseudonyms).

Teacher	(07:33): So.
Anna:	With these tasks... then... should one do A B or B C or only one of those
Teacher:	Ehm...
Anna:	So if we say that I have done type A tasks... will one then pass the test or does one need B tasks in order to get all tasks done [in the test]... because sometimes it is like this [?] tasks C... that is how it was in lower secondary...which come in the test... if I had done A I did not grasp what it was all about.
Teacher:	Nope... these are grouped by level of difficulty and if you go in the first place for solving A tasks and it works very well on the A tasks... then you probably don't need to solve all A tasks... but then you go to a B task which is a bit harder and take up a challenge.
Anna:	Yes/
Teacher:	/And the C tasks are of course a bit more tricky.
Anna:	Yes...
Teacher:	But the minimal requirement is that you have done A tasks to an extent where you feel that it works well with the A tasks.
Anna:	Well that is also a challenge.
Teacher:	Yes and then we will take it on [the challenge] together.

Anna wants to find out at which level the tasks in the test will be. It is not clear whether she means the teachers' or the national test. The teacher's first answer does not refer to the test. Instead the student gets a rather vague instruction of how to move through the tasks: "probably [you] don't need to solve all A tasks". It is left to the student to judge at which point she feels that "it works well with the A tasks". The teacher repeats this in the second comment when saying that she is supposed to have done A tasks "to an extent where you feel that it works well". Being able to solve A tasks seems to be a minimum requirement for passing the test. However, the teacher also wants to encourage the student to do more than just a minimum level and solve tasks that are "a bit harder" (B tasks) or even "a bit more tricky" (C tasks). The student insists that the A tasks are also challenging. The teacher agrees and offers help.

In the second lesson, the students again work individually on the textbook tasks while the teacher walks around between the desks talking to students. There is no exposition by the teacher. In the middle of the lesson a group of students calls the teacher:

Thomas (33:52): You [name of the teacher] I'm wondering about something.

Teacher:	Yes...
Valter:	No we are wondering about something.
Teacher:	Well we then will make a collective wondering/
Thomas:	/Does one have to/
Valter:	/Does one have to... it is A here and B and also C here does one have to do a task for all...
Teacher:	Yes you should if you feel that you succeed very well with the A tasks then you will have to get up a level and do B tasks yes of course.
Thomas:	But A counts as G [abbreviation for pass] B as VG [abbreviation for pass with distinction] and C as MVG [abbreviation for pass with special distinction]
Teacher:	Yes roughly it can indicate that it is roughly that level of difficulty for the somewhat more difficult B tasks but one stretches oneself up a little extra when sorting out the B tasks...
Hannes:	But [for the] B tasks it is unnatural.
Teacher:	No of course you should sort them out.

These students, again, inquire about the meaning of the task levels. The teacher provides the same instruction as to Anna in the previous lesson: If they feel that they succeed very well with the A tasks, then they should solve B tasks. These are now described as being "up one level" and as "somewhat more difficult". The students suggest a relation between the three task levels and the three pass grades. The teacher seems to agree as far as the first two levels are concerned but leaves the suggested relation between the C level and the pass with special distinction uncommented. It is unclear what Hannes means by "unnatural", but the teacher takes it as opposition to the suggestion of moving on to the B tasks.

Eventually, thirteen students do not pass the test announced in this lesson, although according to the teacher it would only contain A-level tasks.

The case of the class enrolled in International Baccalaureate

The class comprises ten students, who formally follow the same curriculum (course A) but aim at an international degree (IB). The language of instruction is English. They do not know each other as they come from different comprehensive schools and even from different areas of the country. The teacher only teaches mathematics in this class.

Choosing one's goals

At the start of the first lesson, the teacher hands out the English language version of the official course plan from the school authority. The 2-page document contains short outcome-based descriptions for a range of topics, phrased as the ability to use the named repertoire of mathematical procedures and concepts in different situations. Under the heading "evaluation criteria" the criteria for each of the three levels of "pass" are described. During the first 20 minutes of the lesson the teacher illuminates the descriptions of the topics and then moves on to the evaluation criteria. Some students look at the handout, some look uninvolved. They do not talk. In the interviews, some of them actually say that they did not pay much attention.

Teacher (22:15): What about the grades... have you looked at the grading criteria that you have... you have it at the back [of the page]... we have pass ...pass with distinction and pass with special distinction. If you are aiming for... pass... and I hope that is your least goal to get a pass...hopefully a higher grade and at least a pass should be your goal... then you are...or you must use appropriate concepts... learning what about what different things are called what different methods to use and how you solve problems. And for pass it's required that you can solve problems in one step...at least... and some oral and written reasoning of course... that is important that you can show your work both orally and in writing... it's difficult to know how students are reasoning sometimes if you don't see it... and then use of course mathematical terms and symbols and so on... and understand and know what that is. And that you also can differentiate between guesses and assumptions...when you are given facts you don't think you can solve... and some proof.

Teacher (24:18): To get pass with distinction...the biggest difference between pass and pass with distinction is that you can solve more types of problems... you can use... maybe use several methods to solve one problem... and you can connect different knowledge when you do your reasoning... and that you have a more deeper knowledge so that you can interpret different kinds of situations and when you solve your mathematical problems.

This expansion of the evaluation criteria matches largely what is stated in the text about pass and pass with distinction. The teacher basically reads out what the text says. According to the official interpretation of the Swedish grades the pass level should not be taken as the minimum threshold but as the outcome expected to be reached by all students. This might be the reason why the teacher focuses on the pass criteria, only shortly mentions the pass with distinction, and does not talk about the pass with special distinction. "Have you looked at the grading criteria?" has to be taken as a rhetorical question. Then the teacher suggests that the students already have made a choice about which grade level they intend to reach.

After the expansion on the criteria, the teacher again addresses the students' goals:

> Teacher (25:07): What is your goal... is that a question one is allowed to ask...have you thought about that...do you think about that now when you have started to take the different courses...what level do I want to achieve with my studies... do you think about that sometimes...would you...that is a good thing to think about because sometimes you have to choose...and think hard about what you want to achieve...or maybe I will put a pass with special distinction for everyone in the class...that would be nice... that would be good...mm...

"Is that a question one is allowed to ask" might be taken as an invitation to consider their goals more consciously and at the same time weakens the obligation to respond. The comment about everyone achieving pass with special distinction is to be taken as encouragement (and not as a form of sarcasm), as such an outcome is indeed an intended possibility within the framework of the grading system.

How to work with the book

In the following part of the lesson, the teacher introduces the book:

> Teacher (26:18): Let's move on then... and see what kind of a... book you are going to use... [the teacher hands out the books]... this one covers both A course and the B course and then some more... and I want you to sign for your textbooks of /I think I have one to many/

After a discussion about writing the names into the book and an expansion on general features of the book (such as lay-out, suitability for the course), the teacher advises the students to look at the sections with the worked examples and then start solving all odd numbered tasks in the category "on your own". Only for odd numbered tasks there is an answer key.

> Teacher (30:33):... the examples in the book are almost always very good to look at the examples... and then if you see... you have...all over in the text... you have... exercises... but we normally don't do them...not those exercises... you may of course if you want to but you don't have to... but when you get to page seven... then you will see you'll find exercises on your own... those are the exercises that we are going to work with... and we only do the odd ones...that does not

> mean the weird exercises... just the odd ones and you will do 1... 3.. .5... and so on... because that's what you have answers to... just the odd ones... and then that will be good enough...

That solving the odd numbered tasks "will be good enough" can be interpreted as referring to the pass level. The work set up in the subsequent lessons (in the first three weeks that we have observed), which consists most of the time in the students working with the tasks in the book, does not appear to contain opportunities to acquire all of the modes of work mentioned in the criteria (such as explaining the reasoning orally or in written form).

Discussion

Framing and classification: the implicitness of the criteria

Weak framing is apparent in both classrooms, although to a different degree. In both groups, the students can choose their own pace for working with the tasks. Framing over the communication is weaker in the ES lessons: When solving tasks, students can choose, for example, whether they talk with their peers, get help from the teacher or work on their own, and discussion with peers is explicitly encouraged. We do not see many efforts on the side of the teacher to control the students' participation. The option of openly discussing is usually not available in the IB lessons. There is stronger regulation about the selection in the IB group, as all students are told to work through the odd numbered tasks, although the teacher encourages the students to move on if they feel already familiar with some tasks. In the ES class, the students have to make their own decision about the amount and level of tasks.

In both groups, students have an apparent choice over the criteria as far as they can "choose" out of a given set of levels or grades to aim for, if they want to achieve more than pass. While in the ES group some students try to get their teacher to reveal the criteria, in the IB there is a prospective announcement. Being able "to differentiate between guesses and assumptions" and "some proof" (both at pass level) indicate a move into the esoteric domain of academic mathematics. The inclusion of these in the pass criteria reflects the intention to initiate all students into this domain. However, the criteria remain unspecific as they are stated independently of the context of acquisition of a particular mathematical topic. Further, the statements are hard to be interpreted by the students, as there is no relation of these to a mathematical practice which the students are already familiar with. Consequently, the prospective announcement of the criteria in the IB class cannot be considered to be of much help for the realisation of a legitimate contribution.

According to Dowling (1998, 2007), school mathematics text that contains formal mathematics expressed through mathematical symbols constitutes the esoteric domain. This is quite different from descriptions of everyday situations by means of non-specialised language in contextualised tasks. Such descriptions recontextualise domestic practices by casting a school mathematical gaze on them and constitute the public domain of action. The students get access to the esoteric through the public domain.

In both classrooms the textbooks operate mostly within the public domain of recontextualised domestic practices, with occasional insertions from the esoteric domain. The books also include hybrids between those: Descriptive domain text, where the expression is conventional mathematical language though its object of reference is not institutionalised mathematics, and expressive domain text, in which a mathematical concept, operation etc. is expressed via non-mathematical signifiers (cf. Dowling, 2007, p. 5).

The IB textbook contains more esoteric domain text, that is, the external classification of the content seems to be stronger than in the ES book. However, we see a mismatch between the criteria for the grades and the evaluation principles manifested in the solutions to the odd numbered tasks given in the answer key. The notion of proof is not specified in the textbook and the tasks do not invite alternative strategies and/or solutions. Our analysis of the ES textbook suggests that the grouping of the tasks does not reflect the grading criteria stated in the official curriculum documents. The explanation provided on the first page of the textbook about the task levels in terms of steps to be carried out applies to many of the A and B tasks, most of which are contextualised tasks. The tasks in the category C are different from the A and B tasks as some resemble more of a mathematical puzzle and can be classified as a recontextualisation of "recreational mathematics".

Consequently, in both classrooms the criteria for legitimate contributions at different grade levels remain hidden. All the students know is that they have to work through the A tasks as a minimum requirement for the pass level in the ES class, and through the odd numbered tasks in the IB class respectively. The students in the IB group might feel more sure about what to do in order to pass the course, although some might perhaps think that the even numbered tasks are more advanced. In the ES group some students might work from the hint that the B and C tasks are important for getting a higher grade.

While contextualised tasks dominate in the textbook of the ES class, the test focuses on mathematical notation (exponents, fractions) and operations (three out of eleven tasks were contextualised). The weakly classified and framed practice appears to have minimised the access to forms of school mathematics knowledge valued in the test and in higher education, in particular for the students who did not pass the test. The criteria remain implicit, also in the test, and, as can be inferred from the interviews, many students are not aware of the teacher's goal of identifying groups of students according to achievement. The choice for solving tasks that suit their own level turns out to amount to self-exclusion. The successful students must have acquired the classificatory principles of the practice they are starting to participate in somewhere else.

The teacher as coach: is this a pedagogic transmission of knowledge?
Theoretically, as the criteria are transmitted in the course of the practice into which the students are being introduced, continued weak framing over the criteria cannot lead to the establishment of strong classificatory principles. However, in these classrooms the teachers delegate the instructional discourse to the textbook that regulates the academic performance of the students. In the first lessons, both teachers set the rules for how to work with the book. When solving the textbook tasks, the students do not have many options over how the knowledge is to be communicated. There are only short written solutions, which can be looked up in the answer keys of the books.

In the lessons we have observed in the ES class there is virtually no whole class exposition from the teacher, in the IB there is some. In both classes the students spend most of the time working through the tasks in the book. The teacher is involved in the process usually only when students ask for help or have a question. This is typical of many Swedish classrooms, although the complete absence of teacher exposition in the ES group is an extreme case. In the two classrooms under study the size of the group makes a difference to what can be counted as public discourse. In the small group of the IB, the conversations of the teacher with single students are more likely to be audible for the rest. This is not always the case in the ES group and not all of the students are sufficiently alert to pay attention to such conversations, especially when they are preoccupied with solving tasks. So it is a consequence of the initiative of single students to get the teachers' comments on their work in order to have access to the criteria. In the IB classroom in such a case the other students more easily could profit from such interaction by attempting to listen.

As the teachers in these classrooms are there more as a resource for the students rather than as guarantors of knowledge transmission, it is the students' own decision how to make use of this resource. Especially in the ES lessons, there is no obvious sanctioning of a lack of participation. Success is likely to depend largely on study habits and behaviour. The personal characteristics of the students, such as organisation, concentration, confidence when facing difficulties and autonomy are perhaps the most important characteristics that create the teachers' expectations of the students' achievement in the IB classroom. In both classrooms, a "habitus" that represents favourable study habits is likely to account for successful participation and thus to (re)produce social identity and destiny (Bourdieu & Passeron, 1977).

Notes

1. For a broad description of the theoretical background and the methodology, as well as for a literature review we have to refer to the documents provided on the website (http://www.acadiau.ca/~cknippin/sd/index.html).

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Transcript conventions:

(01:20)	Time in minutes and seconds after start of the lesson
/	Cutting off speech of another person
... text	Pause of three or less seconds, respectively where it would be in the English translation
text ...	At the end of a turn, three dots indicate absence of fall in tone (often raising intonation)
[text]	Transcriber's or translator's comments

Investigating What Students Do Transfer: Implementation of the Actor-Oriented Transfer Framework

Gulden Karakok
Umeå University, Sweden

There is a rich body of research on transfer of learning with conflicting results. Some studies have concluded that students do not transfer, and yet several researchers have claimed that these findings contradict with everyday experiences in which most students do perform successfully. Some researchers even claimed that due to conflicts the topic of transfer should be avoided as a research construct. In this paper, a short overview of different approaches to transfer of learning will be shared. Then an example from a case study in which one of the contemporary approaches to transfer, the Actor-Oriented Transfer (AOT) framework, was implemented will be presented to share the differences between the traditional and contemporary approaches to transfer.

Introduction

There is a rich body of research from various fields including education and psychology on transfer of learning with a history of over 100 years (Thorndike, 1903; Lave, 1988; Detterman, 1993; Bransford & Schwartz, 1999; Lobato, 2003). One of the reasons that transfer of learning has been studied by education researchers is its direct relation to a main goal of education: providing learning experiences that can be generalized and used by the learner outside the initial learning situation (Bransford, Brown, & Cocking, 1999). To see if the goal was fulfilled, educators and researchers have been developing studies to assess the learners' ability to transfer knowledge and skills.

The literature review on transfer of learning however shows conflicting results. Some of the studies indicate that transfer is rare and only happens when two situations (the initial task and the transfer task) share similar features. Some studies suggest that people successfully transfer during their daily life. The purpose of this paper is first to provide an overview of the different approaches (traditional and contemporary views) to transfer, in order to explain the possible reasons for the conflicting results. Then, an example from a case study will be presented to share how one of the contemporary approaches, the Actor-Oriented Transfer (AOT) framework (Lobato, 2003), was implemented to investigate transfer and to discuss how the results would be different (or same) if more traditional approaches were implemented.

Traditional and contemporary views of transfer

There have been many approaches to investigate the issue of transfer of learning. In this section these approaches are divided into two parts according to the underlying learning theories. Such classification is chosen to present a brief summary of different approaches to highlight the existing varying definitions, methodologies and conclusions due to the underlying learning theories (see Karakok, 2009 for an in-depth summary). The contemporary views of transfer mentioned in this particular paper reflect one of the recent learning theories, namely the situated learning theory.

Transfer of learning has been traditionally defined as *the ability to apply knowledge learned in one context to new context* (Mestre, 2005). Early psychological views of transfer were based on the mental abilities of a person and these abilities were believed to become stronger by training them in different subject areas. Thus, the training of the basic mental functions was also thought to improve the person's ability to transfer ideas and skills to new situations. In other words, training was thought to have general effects that would transfer to new situations. The educational application of this *general effects view* was that all students were required to take courses such as Latin and geometry, which were believed to discipline students' minds by practicing logical thinking, and hence improve their abilities in other school subjects (Detterman, 1993). The general effects view could also be seen as training of the "thinking muscle" and its influence could be seen in the goals of some mathematics curricula (as mentioned in Niss, 1996, p. 23).

After many experiments, Thorndike and his colleagues challenged the existing belief of *general effects* and proposed an alternative idea, *the theory of identical elements* (Thorndike, 1903). Thorndike's work showed that even though people did well on a test of the specific content they had studied, this knowledge did not increase their learning in a new situation. They further concluded that transfer from one task to another happened only when two tasks shared identical elements.

Thorndike's studies did influence educational practice by including more skill drilling activities in mathematic curriculum, but they mostly influenced the transfer studies conducted later. Many researchers followed the same research paradigm: an initial learning task was followed by the target task created by researchers who thought that these two tasks shared similar features (e.g., Bassok, 1990; Gick & Holyoak, 1983). Researchers from the traditional paradigm were interested in the same research question: "Do students transfer?" Most of the studies conducted under this traditional paradigm reported failure of spontaneous transfer from one task to the next.

Judd stated that one of the possible reasons for failure could be the relationship between two tasks that was declared to be similar by the researchers (1939, as cited in Tuomi-Grohn & Engeström, 2003). He claimed that the learners might have a different opinion on the sameness and differences of two tasks. Bransford, Brown and Cocking (1999) also argued that the methodology that was followed during these experiments did not provide any insights for transfer, since the results from such experiments were binary: either transfer happened or did not happen.

It was not just the methodology that was criticized but also the traditional definition of transfer. Lave (1988) re-evaluated the traditional views of transfer and her main concern was with the definition of transfer. The derivative of the definition "the ability to apply knowledge or procedures learned in one context to new contexts" (Mestre, 2003, p. 3) suggests that transfer consists of measures of the proper use of previous learning in the new setting with the assumption that the settings (initial learning and the transfer) do not affect the learner's performance. In other words, the definition suggests that learning can be separated from the situations in which it is constructed instead of it being an overall connected experience (Lave, 1988; Lobato & Siebert, 2002).

Overall the researchers criticized the traditional definition of transfer, the research questions asked, and the methodologies used during the studies conducted by the traditional approaches. They suggested that new definitions should address that learning could not be separated from the environment that it was created in and it should capture the notion of transfer being an active process. The research questions should be posed in a way that more than binary results were achieved. Also, new methodologies were needed to identify what learners did during transfer experiments and how learners found tasks to be similar. In other words, there was a need to shift from the researcher's perspective to the learner's perspective during transfer investigations in order to understand the underlying mechanism of transfer. To overcome the shortcomings and also to complement the results of traditional approaches, new (contemporary) approaches to transfer were proposed.

Some of these contemporary approaches proposed were *Affordances and Constraints Approach* (Greeno, Smith, & Moore, 1993; Greeno, Collins, & Resnick, 1996), *Preparation for Future Learning* (Bransford & Schwartz, 1999), and the *Actor-Oriented Transfer* (Lobato, 2003). These approaches are all formed under a situated perspective and they use revised definitions of transfer together with modified research questions and methods to explore the issue of transfer. They also share the idea that what is transferred is not only the knowledge from task to task but also "patterns of participatory processes across situations" (Greeno, 1997).

These researchers, using a situated perspective, claimed that the findings from the earlier studies of transfer contradicted the everyday experiences of learners performing successfully in new situations by finding similarities to previous situations (Bransford & Schwartz, 1999; Mestre, 2005; Lobato, 2006). They noticed that earlier studies had only focused on the researchers' and experts' points of view, especially when developing the tasks and deciding on similarities of tasks. Moreover, the contemporary approaches view transfer as an active process, rather than a passive one. In other words, they believe that it is possible for students to transfer during the experiments (from the beginning of the experiment to the end of the experiment), and the researchers' job is to figure out what students transfer.

In this paper only one of the contemporary approaches, the Actor-Oriented Transfer (AOT) is discussed to provide more specific examples of the underlying mechanism of this approach (see Greeno, 1997, and Bransford & Schwartz, 1999, for information on the others). The example is chosen from a case study (Karakok, 2009) that implemented the AOT along with the traditional approaches.

The Actor-Oriented Transfer (AOT) defines transfer as the "personal construction of similarities between activities where the 'actors,' i.e. learners, see situations as being similar" (Lobato, 2003). The AOT focuses on how the actors (or learners) see the two contexts (tasks) as similar. The evidence for AOT is gathered by "scrutinizing a given activity by any indication of influence from previous activities and by examining how people construe situations as similar" (see Lobato & Siebert, 2002, p. 89, for details). Any indication of influence from previous tasks or experience on the given task is considered to be evidence for AOT. In other words, researcher should not decide or give a priority to what students should transfer "but rather adopt a student-centered perspective to find out what students do transfer and investigate the mediating factors" (Rebello et al., 2005, p. 219).

Lobato's (2003, 2006) AOT framework complements the previous studies that informed us only with success or failure of transfer of a particular knowledge. Lobato's (2003) AOT framework considers all possible previous experiences that a learner could connect to during transfer tasks. The construction of similarities between experiences helps the researchers to explore the question of "what counts as transfer" from the learner's perspective.

In the next section an example from a case study will be shared to point out the differences between the traditional approaches and the AOT framework.

An example

The example presented in this paper is taken from a case study that was conducted to describe and analyze a junior year physics student's (who will be called by the pseudonym Milo throughout this paper) transfer of learning of linear algebra topics (with a special focus on eigenvalues and eigenvectors) from courses to the interviews by implementing the AOT framework (for the details of the case study, see Karakok, 2009). This case study was conducted to explore the answers to the question, "What ideas *do* Milo transfer to the interviews (in which questions are related to linear algebra topics) from the courses he had taken or has been taking?" More specifically, the relations of similarities that Milo created between the interview tasks and other previous tasks were investigated in this study.

Milo was recruited from one of the junior-level physics courses offered at the beginning of the school year. He was selected to be in this study because he had taken one of the prerequisite linear algebra courses (the matrix and power series methods) prior to the study (his grade in the course was A). He was introduced to the concepts of eigenvalues and eigenvectors in the course and hence it was considered as the initial learning situation for Milo.

Three in-depth interviews were conducted with Milo for the case study. All the interviews were recorded and transcribed. Also, the classes that Milo took during the time of the case study were observed and videotaped by the researcher for the case study.

During the interviews Milo was asked to answer some questions and solve problems that were related to linear algebra. In the first interview questions and problems were worded differently than the ones found in a regular linear algebra course. For example, in a regular linear algebra course students usually are not asked to describe eigenvalues and eigenvectors and explain what they represent in a given linear transformation problem. During the second and third interviews, problems that could be defined as application of linear algebra were also asked. For example, Milo was asked to solve the eigenvalue problem for the differential equation $i\frac{d}{d\varphi}f(\varphi)-af(\varphi)=0$ with a subject of the condition $f(\varphi)=f(\varphi+2\pi)$. He was asked further to describe what eigenvalues and eigenvectors represent in this situation. Since the interview questions and problems were different than the questions that a student could experience in a regular linear algebra course, interviews were considered to be the transfer situations in the case study. The researcher's job was then to figure out what the student transferred from his previous experiences to the interviews. However, as mentioned earlier, the dynamic view of transfer indicates that transfer is an active process and could take place during the experiment. The example presented in this paper is chosen to highlight this aspect of transfer.

In this paper, the example is also investigated through more traditional approaches to transfer. In other words, the researcher asks the question, "Does Milo transfer his learning of the concepts of eigenvalues and eigenvectors from the initial learning situation to the interview task?"

Methods

All three interviews data were analyzed by implementing the actor-oriented transfer framework. The evidence for AOT was gathered by "scrutinizing a given activity by any indication of influence from previous activities and by examining how people construe situations as similar" (see Lobato & Siebert, 2002, p. 89). Any indication of influence from previous tasks or experience on the given task was considered to be evidence for actor-oriented transfer. In the case study, Milo's responses to the interview questions were first analyzed to see if there was any explicit reference to his previous experiences including the courses he took. To be more precise only the situations in which explicit reference were made to previous experiences were considered to have preliminary evidence of actor-oriented transfer. For example, in some cases Milo explicitly stated that the interview question reminded him of one of the activities he had done in one of the courses. Even though the AOT framework does not require for the explicit referencing to previous experiences, the researcher explored such instances first, since data from the student's previous experiences were limited to the classes observed by the researcher in the case study. It was very well possible that the student could have constructed similarities between the interview task and another course that the researcher did not observe. To eliminate such situation, first the explicit references were analyzed and considered to have preliminary evidence of AOT. All such situations having preliminary evidence of AOT were investigated further by first asking Milo to explain the experience in more details. Then, the researcher's field notes from observations and videos of the courses during which these experiences occurred were analyzed to identify similarities between the experiences mentioned by Milo.

The researcher also investigated the instances in which there was no explicit reference, by scrutinizing it with any indication of influence from the student's classes that were observed by the researcher. A short example from such investigation is also presented in this paper (see Karakok, 2009 for more examples and details).

Results and discussion

The following example is taken from the first interview with Milo. He was asked to describe eigenvalues and eigenvectors. The example is analyzed to answer the research question posed under the traditional views of transfer: Does Milo transfer his learning of the concepts of eigenvalues and eigenvectors from his initial learning to the interview task? The example is also analyzed by implementing the AOT framework to investigate the answers to "What does Milo *do transfer* to the interviews?"

Example

At the first interview, Milo was asked to describe eigenvalues and eigenvectors. (More precisely he was asked: What is an eigenvalue? What is an eigenvector? Tell me everything you know about them.) He stated that he "remember(s) a couple of general things" and he wrote the eigenvalue equation $Ax = \lambda x$; he explicitly said that A was a matrix and λ was the eigenvalue. He was asked to give an example after his description, and he then started to talk about the characteristic equation. He said there was something called the characteristic equation, but he could not recall it. He said that he had never got a "concrete feel" for what it really meant, and it was not attached to anything in his head. It was just "this thing that we had written down on a little formula sheet," and he had used it on the test of the matrix and power series methods course. He said that he had forgotten all about them.

When he was asked how the equation $Ax = \lambda x$ was used, he again mentioned the characteristic equation. When he was asked if the equation $Ax = \lambda x$ was the characteristic equation, he said, "No, there is something else". The researcher wanted to investigate further what else Milo had to say about the equation. It was observed that he initially did not recall that the variable x represented a vector. The researcher asked him what each variable represented, and then Milo said that it was "maybe" a vector. Milo decided to look at an example to check the presented idea where A was a two-by-two matrix and x was a two-by-one vector. He worked on both sides of the equation and concluded that both sides resulted in the same vector, so x must represent a vector. He was very pleased with his finding (that both sides of the equation ended up being the same vector) and said, "Huh, I taught something to myself". After his "discovery," he did not know what to do next. It was still unclear if he knew that x was an eigenvector of A.

To investigate further, he was asked to find the eigenvalues and eigenvectors of a linear transformation that reflected the vectors over x-axis. He found the matrix representation of the linear transformation and checked if it was the correct one by operating it on some vectors and drawing sketches of the vectors. Once he was convinced that the matrix representation was correct, he started to think about how to find eigenvalues and eigenvectors. He tried to implement his "discovery."

Milo: Well, an eigenvalue… oh wait. I am trying to remember what, what exactly you know from the eigenvectors. They are vectors made up from eigenvalues. There is usually more than one. I am trying to remember if the eigenvalues themselves become the components of a vector or…I don't remember. **From what I**

was trying to reason before, [pointing to the equation] an eigenvalue was just some constant, and so you would have to have a constant multiplied by the vector, sorry, the vector multiplied by the constant would be the same effect as multiplying it with matrix.

After working on this problem, he was asked to describe eigenvalues and eigenvectors.

Milo: Um, well I guess you can say that an eigenvalue is somehow a condensed version of a matrix, or a transformation, I suppose. […] It, the eigenvalue itself accomplishes the same thing that the transformation does. So you are finding a way to transform or alter a vector using a constant instead of a vector or a 1 by 1 matrix instead of an n by n.

It was noticed that Milo provided the algebraic interpretation of the eigenvalue equation by only focusing on the eigenvalue. He did not include eigenvectors in his description during the interview. Milo could not find the eigenvalues and eigenvectors of the given linear transformation. According to the researcher's perspective the two tasks (initial learning and the interview questions) did share similar features because in the linear algebra course that Milo took prior to the interview (the initial learning) the definitions of eigenvalues and eigenvectors were introduced and students found eigenvalues and eigenvectors of matrices and linear transformations. Even though the tasks shared similar features, it seemed that Milo could not successfully describe the concept of eigenvector and could not find eigenvalues and eigenvectors of a linear transformation. He only recalled the eigenvalue equation. Thus, from traditional views of transfer, one can conclude that Milo could not transfer his initial learning of the concepts of eigenvalues and eigenvectors to the interview task.

When this example was analyzed using the AOT framework, it was noted that Milo did transfer from his "discovery" experience to construct the algebraic interpretation of the concept of eigenvalue. He explicitly referred to his "discovery" while he was reasoning to construct the meaning of the eigenvalue. His construction of the algebraic interpretation of an eigenvalue was formed by his "discovery" of investigating both sides of the eigenvalue equation to figure out what the variable x represented. In other words, transfer took place during the "target task", not from a course to the interview. This particular student transferred an idea from the beginning of the interview, rather than an idea from a course. Since he was relating to his experience (discovery) in order to construct the algebraic interpretation of an eigenvalue, this example constitutes evidence of actor-oriented transfer.

This example also highlights one of the features of the AOT framework, which states that transfer is an active process and could occur even during the transfer task. The traditional views of transfer overlooked this feature of transfer, thus the traditional views did not inform us on the possibilities of what students do during transfer tasks.

This example was not the only instance in which transfer was observed. In the third interview Milo was asked the following question: “Let (1,1) be an eigenvector that associates with the eigenvalue of 1 and (1,-1) with the eigenvalue of -1 of an operator M. What can you tell me about this operator?" Milo first stated that M has two distinct eigenvalues and eigenvectors associated with them, so M is a two by two matrix. Then he said that eigenvectors were linearly independent because one vector could not be “get by the other” and these vectors were perpendicular to each other. From this explanation he jumped to the idea that M was not a rotation matrix, “[…] If you project one onto the other, you take the dot product of one on the other, you always get zero. I think M is not a rotation matrix. I think it would be a flippy guy.” The researcher took a note of Milo’s choice of words; similar words to “flippy guy” were used by other students in one of Milo’s classes “Spins and Quantum Measurements” during a group activity.

The researcher asked Milo how he knew that M was not a rotation matrix but a “flippy guy”.

Milo: Because the only way for both of these vectors to be changed only by a scalar, I think everything is flipped around one of them, so that one of them is totally unchanged the other one just changes by a negative. If it were a rotation, then they would both change direction and then they weren't eigenvectors anymore. I suppose there could be some stretching going along with the flipping.

Milo’s reasoning thought to be a construction of similarities between the interview task and his experience in the “Spin and Quatum Measurements” course he took 15 weeks prior to the third interview. The video recordings along with transcriptions were explored further to see if there were any indication of influence from activities in the classes. During one of the classes students were finding eigenvalues and eigenvectors of a rotation transformation in R^3 when the instructor asked students if they could guess what the eigenvectors would be without doing any calculation. As a class students discussed how they could identify eigenvectors and during that discussion the instructor mentioned a geometric interpretation of the eigenvalue equation. The instructor pointed out that eigenvectors were the vectors that do not change direction when they are operated on, so the rotation transformation would rotate all the vectors in R^3 and the eigenvectors of this operator would not be in R^3. Milo seemed to construct similarities between this particular interview question and the aforementioned classroom discussion. Even though the classroom discussion and topic was on rotation transformation within a physics context, Milo seemed to find these two tasks to be similar (see Karakok 2009 for the details of this particular example).

In this last example Milo did not explicitly mentioned a previous experience, however the language he used and the reasoning he presented had indications of influence from a previous observed class experience, which was investigated further. Milo's experience in class was within a physics context on rotation transformation and it seemed that he related to this experience during the interview. For this reason, this example also constitutes evidence of actor-oriented transfer.

Conclusion

The purpose of this study was first to provide an overview of different approaches to transfer. The previous studies that were conducted to explore transfer of learning had been under the influence of traditional transfer paradigms. Recently new studies have included new approaches to the research construct of transfer. One of the new approaches of investigating students' transfer has been proposed by Lobato (1996), the AOT, and in this study this new approach was implemented together with more traditional approaches.

The results of this study underline one of the most helpful features of the AOT framework. This framework could inform the researchers about the learning process rather than to merely observe the end result of learning. The AOT framework seems to focus a lens on how students connect their previous experiences (for example the experiences during teaching or ones within the interview) to new ones (for example the experiences in the interviews) as they find explicit or implicit similarities between the experiences. The analysis with the AOT framework provides an in-depth exploration of students' experiences that they seemed to connect to during interviews as seen in the example. The AOT framework seems to allow for an investigation of the learning process whereas the traditional approaches to transfer look only at the end product of learning. This framework informs us on what students do transfer rather than what students fail to transfer.

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Children's Perspectives on Mathematics Homework

Troels Lange and Tamsin Meaney
Charles Sturt University, Wagga Wagga, Australia

Homework is a world-wide phenomenon which provides advantages and disadvantages for children. In this paper, we want to widen the discussion about homework by analysing the views of some Year 4 children in a Danish folkeskole*. For these children, homework acted as an indicator of their ability in mathematics and was considered an unnecessary intrusion into their busy, out-of-school lives. This intrusion became a burden, if the children were unclear about what was required or if the level of work was too easy. Our analysis uses the work of Stephen Kemmis on practice architectures by considering homework to be an educational practice structured by culturally-discursive, material-economic and social-political structures and processes.*

Mathematics homework takes up the time of many school children across the world, yet in many ways it is under-researched as a function of children's lives. Generally, homework is not seen as a positive experience by participants (Trautwein, Lüdtke, Kastens, & Köller, 2006; Lange & Meaney, 2009). As well, Farrow, Tymms and Henderson (1999) found that primary school students who only received mathematics homework once a month performed better academically than students who received homework more often, as well as those who received no homework. On the whole they found that generally, "those pupils who did regular homework made less progress than those who did homework infrequently" (p. 331). Apart from concerns about the academic benefits of homework, there have been ongoing issues such as the impact on family lives, demands on parents and the time that homework takes up in children's outside school lives. As cited by Kohn (2006), the American Educational Research Association's (AERA) official policy on homework from the 1960's stated that homework was not good for children when it crowded out social, recreational, and creative activities and sleep.

On the other hand, homework is often given to students for reasons that are more socio-culturally, rather than cognitively, beneficial such as:

- create a firm partnership between parents and children in relation to children's learning;

• encourage parents (especially of young children) to spend time with their children;

• reinforce work covered in class or to practise or consolidate basic skills and knowledge (reading, spelling, multiplication tables);

• encourage children to develop perseverance, initiative and self-discipline through independent study;

• learn study skills and improve personal organisation

• make pupils more responsible; and

• prepare pupils for secondary school (OFSTED, 1996 cited in Farrow, Tymms, & Henderson, 1999, p. 325)

In discussions about mathematics homework, the views of children have not been heard. Yet, children are able to reflect not only about the product of the homework practice but also about the impact that this practice has on their lives. To make sense of children's stories about homework, we have drawn on Kemmis and Grootenboer's (2008) ideas about practice architectures that incorporated Schatzki's (2005) ideas about site ontologies. Practice architectures have not been used previously to understand the practices that students engage in, yet they appear relevant because children are the practioners involved in the practice of homework.

As illustrated in Figure 1, Kemmis (2009) perceived education, of which homework is one small part, as something that should contribute to the good of the individual engaged in it as well as to the good for humankind as a whole. This good is achieved through the interaction of individuals and collectives through society-constructed practices.

> [O]rganisations, institutions and settings, and the people in them, create practice architectures which prefigure practices, enabling and constraining particular kinds of sayings, doings and relatings among people within them, and in relation to others outside them. The way these practice architectures are constructed shapes practice in its cultural-discursive, social-political and material-economic dimensions, giving substance and form to what is and can be actually said and done by, with and for whom. (Kemmis & Grootenboer, 2008, pp. 57-58)

An individual's performance of a particular practice is constrained by the characteristics that have been historically embedded within the practice. However, options exist for alternative enactments of the practice that could contribute to the practice itself being reconfigured. Thus, it is within the enactment of the practice that both the individual and the practice can be affected. This recognition of a two-way change process is beneficial because an analysis using practice architecture provides indications of how the good of a practice could be increased without simplifying the actual reality of its enactment. When practices change, then there is also the possibility for societal change indicating how the practice acts as the mediator between the individual and the society.

	Individual and collective ***praxis*** constitutes, and is constituted in, action via	Dimension/medium	***Practice architectures*** constitute, and are constituted in, action via	
	Characteristic '***sayings***' (Kloosterman, 1996) and *thinking*)	The cultural-*discursive dimension* (*semantic space*) realised in the medium of *language*	Characteristic ***cultural-discursive*** orders and arrangements	
The individual: **Education and the good for each person** Education for living well	Characteristic ***'doings'*** (and ***'set-ups'*** of objects)	The *material-economic* dimension (*physical space*) realised in the medium of *activity* and *work*	Characteristic ***material-economic*** orders and arrangements	The world we share: **Education and the good for humankind** Education for a world worth living in
	Characteristic ***'relatings'***	The *social-political* dimension (*social space*) realised in the medium of *power*	Characteristic ***social-political*** orders and arrangements	
	which are bundled together in characteristic ways in ***teleoaffective structures (human projects)***		which are bundled together in characteristic ways in ***practice traditions***	

Figure 1: The individual and collective purposes of education constituted in praxis and practice architectures (reproduced from Kemmis, 2009).

Homework as a practice is a site of the social (Schatzki, 2005) that is composed of characteristic sayings, doings and relatings. It is the wider and the localised context as well as the practices that are recognisable by groups of people as homework. For example, the historical development of homework as a schooling practice is as much part of the site as is the page of sums that a child may take home. Considering homework as a practice means looking not just at the products that children produce (the 'doings' and material 'set-ups') but also how it affects and is affected by the wider societal discourse and the relationships between people.

In this paper, we explore how children's views could contribute to a discussion about homework. We were specifically interested to see how what was expected of them was constrained by the practice architecture that surrounds mathematics homework. Given that practice architecture had not previously been used to analysis children's perceptions of practices, we were also curious to see how this type of analysis would contribute to understanding of the socio-cultural-political environment in which mathematics homework operates.

Data Collection

The interview data in this paper comes from a larger study exploring children's perceptions about their mathematics education. The interviews were semi-structured life world interviews, i.e. interviews that "seek to obtain descriptions of the interviewees' lived world with respect to interpretation of the meaning of the described phenomena" (Kvale & Brinkmann, 2009, p. 27), in this case mathematics education. The interviewer's role was to be lead by the children, in that he introduced topics but allowed the children to guide what was discussed. Children aged 10-11 years in two Danish Year 4 classes were interviewed usually in pairs or small groups between 2005 and 2007 (Lange, 2009b). The *folkeskole* was in a regional centre in Jutland. Although some children were only interviewed once, others were interviewed on three separate occasions. In the interviews, children were invited to talk about their mathematics homework and many of them provided stories from their experiences. Extracts from the transcripts are provided in the original Danish with an English translation.

Data analysis

The children's stories were analysed for general themes and involved identifying the cultural-discursive ('sayings'), material-economic ('doings') and social-political ('relatings') dimensions. Although these dimensions appear as bundles to form practice architectures, we have separated them in the following section to make the discussion more coherent.

Sayings

Children are immersed in a sea of discourse that has a major impact on what they come to see as the norms, or the cultural expectations, of their world. For example, children perceived the setting of homework and the ensuring of its completion as something that teachers must do as part of their jobs. In the following extract from a group interview, Philip talked about the teacher's role in setting homework and her expectations of where it should be done.

Philip:	Jeg synes det er lidt ligesom alle andre fag. Det er kedeligt at have lektier for for nogle lærere de mener det gør sådan en STOR forskel at man laver dem derhjemme. Det er lidt stressende.	I think it is a bit like all other subjects. It is boring to have homework set because some teachers they think it makes such a BIG difference that you make them at home. It is a little stressing.
Troels:	Ok, så du synes, synes du ikke det gør nogen stor forskel om man laver det derhjemme?	Okay, so you think, you don't think it makes any big difference if you do it at home?

Philip:	Det er jo det samme om man laver det i skolen. Det er bare to forskellige steder. Man laver det samme.	After all it is the same if you do it at school. It is just two different places. You do the same.

Children knew that teachers thought homework was important because they got into trouble if it was not done. In the same group interview, Jacob described how his handball practice meant that he did not complete his homework. The result was that the teacher gave him a 'bollocking'. Whether or not they liked doing homework, children saw it as part of the cultural structure of schooling and an inevitable part of their lives. Philip found the pressure from having to do homework "a little stressing" which can be taken to mean "quite stressing" as understating in a characteristic feature of the Danish spoken in the part of Denmark where the children lived. Consistent with this interpretation is, that when asked what he would change about schooling, Philip stated that he would like teachers not to react as hard when he did not do his homework.

Children are not alone in realising that it is the role of the teacher to set homework (Farrow et al., 1999). The cultural context is accepted as the normal state of affairs by teachers, school authorities and parents. Children have an option not to comply, but then they must accept the consequences. As Kohn (2006) pointed out, very few people seem willing to query why homework is done, even if the academic benefits are known to be negligible. Children are no exceptions to this acceptance of these societal norms.

Doings

The work that is done as mathematics homework "is always already shaped by the material and economic resources made available for the task" (Kemmis, 2009, p. 6) which includes the physical space in which it is performed. The children were concerned with where homework should be done as well as the level of difficulty of the task that they were asked to do.

As indicated by Philip, teachers expected children to do homework at home. Nevertheless, several of the children told of thwarting this expectation by completing homework at school or other non-home settings. In this way, they could exert their influence over the structures they were immersed in. Kamal described how he often got his homework done during lessons whilst Dennis described how he did his during a break. Dennis also talked about practising times tables whilst riding his bike.

Troels:	Har I aldrig nogen siden oplevet at en lektie der var, der var god? - eller som det var sjovt at lave?	Have you never experienced good homework? – or that is was fun to do?
Both boys:	Jo, jo, jo	Yes, yes, yes
Dennis:	Tabellerne	The tables.
Troels:	Hvorfor tabellerne? Hvad ..	Why the tables? What ..
Dennis:	Da kan man jo, hvis man skal ned	Then you can, if you are going to a

	til en ven så kan man lige lave dem i hovedet	friend then you can just do them in your head
David:	Ja det er bare lige …	Yes, it is just …
Dennis:	Sidde og øve sig i dem	Sit and practice them
Troels:	Så det gode ved dem det er at det ikke tager så lang tid eller hvad, er det det?	So the good thing about them is that it does not take long time or what?
David:	Ja.	Yes
Dennis:	Nej, men man kan jo kø.., hvis jeg skal ned til ham, der er jo et godt stykke hvis jeg bor på enogtres og han bor nede i femten	No, but you see you can ri.., if I am going to him, there is a quite a way, you see, if I live in number sixty one and he lives down in fifteen
Troels:	Ja	Yes
Dennis:	Så kører jeg på min cykel, så kører jeg bare tabellerne imenst …	Then I ride on my bike, then I just run the tables all the while …

Altough Dennis was not doing his homework at school, he was not doing it at home either. The nature of practising times tables meant that it could be done in his head and thus required no other resources, including help from parents. Children often gave practising times tables as an example of the sort of homework that they did. A study in Canada found that children had the least negative attitude towards doing drill and practice than towards any other kind of homework (Cameron & Bartel, 2008).

In one class, the children could choose the times table that they were to learn and then the teacher would record how well they had learnt it the following day. Thus, they could see a reason for engaging in this kind of homework because the teacher's record showed what they had learnt (Lange, 2009a). Also, children could chose to take home a times table game to help them learn and so for some children homework became an enjoyable time with their parents (Lange & Meaney, 2009). No child complained about homework being too hard. However, some children did not appreciate homework that was too easy.

Kalila:	Jeg vil have lektier, det er.. altså hvis jeg får lektier, så er det ok, jeg kan godt lave det	I want homework, that is .. look if I get homework then it is okay, I am able do it
Troels:	Mmm	Mmm
Kalila:	Men det er ikke sådan at jeg vil have lektier for.	But it is not so that I want homework set.
Troels:	Men du kan ikke så godt lide for lette opgaver?	But you do not like too easy tasks?
Kalila:	Altså – hvordan skal jeg sige det?	Look – how can I say it?
Troels:	Hvis de er f..	If they are t..
Kalila:	Hvis jeg er.. jeg kan godt lide det jeg er god til, men ikke sådan, for	If I am, I like what I am good at, but not like, for example: one plus one

	eksempel: et plus et.	
Troels:	Nej, ok.	No, okay.
Kalila:	Eller to plus to, eller fem plus fem.	Or two plus two or five plus five.

Kalla wanted to be good at maths and having to do problems that she found too easy did not give her that experience (Lange, 2009a). Farrow et al. (1999) commented that the societal expectation that teachers set homework could result in children being given work that simply kept them busy rather than engaged them in learning. The children were aware that this could be the case and resented it. Homework was also seen as a punishment for not completing work in class. Unfortunately, this could result in those children who struggled with mathematics having more homework than their peers, on something that they continued to not fully understand. Sahra told of being able to do word problems only after her mother or older sister helped her find the numbers and operations.

The other aspect of homework that the children resented was the time that it took them away from other outside-school activities. Some of this is seen in the interview extract with Dennis and David and it resonates with AERA's 1960's policy on homework about not restricting children's leisure activities. Although the use of a material resource such as time is part of the doing structures, we discuss this aspect in the next section as it has much to do with teachers' position of control over children's out-of-school time.

Relatings

Relationships between people are already prefigured by the social and political situation in which they develop (Kemmis & Grootenboer, 2008). Thus, parents, teachers and children interact around homework in ways that reflect the social standings between each other. In the setting of homework and by ensuring that it is completed, teachers exercise their power to interfere not just with children's out-of-school time but also with enforcing how parents and children should spend time together. From her study of Irish school children's lives, Devine (2003) found that children felt that homework was "an unfair intrusion into their private lives and one over which they had little control" (p. 47). They saw it as a form of surveillance of their out-of-school time and a reflection of the powerful position that the teacher had. The children in our study, such as Philip, also saw homework as one way that the teacher tried to impose her authority. Jacob described homework as interfering with his leisure time. In the following extract, Maha described how homework interfered with her weekend.

Maha:	Og jeg hader lektier	And I hate homework.
Troels:	Og du hader lektier, hvorfor hader du lektier?	And you hate homework, why do you hate homework?
Maha:	Fordi at så skal man, hver weekend så får vi lektier for.	Because then you must, every weekend we are set homework.

Troels:	Ok – og så skal man sidde og bruge sin tid i weekenden, er det det?	Okay – and then you have to sit and use time in the weekend, is that it?
Maha:	Ja	Yes
Troels:	Hvad vil du hellere lave?	What would you rather do?
Maha:	Jeg vil hellere være ude eller være sammen med mine venner.	I would rather be outdoors or with my friends.

As previously mentioned, a societal understanding about homework is that it should be done with the support of parents. However, the children did not always have happy tales to tell about this support (Lange & Meaney, 2009). For many children of immigrant descent, such as Kalila and Maha, help came from older siblings rather than parents. Yet this was often not recognised or valued by the teacher (Lange, 2008). The powerful position of the teacher affected not just their own relationships with the children but also the relationships between the children and their parents.

Homework as a practice also influenced the relationships between the children themselves, as it indicated who was good at mathematics. Performance in mathematics has often been used to determine a student's general potential or ability (Davis, 1996). Thus, the societal norms that make important judgements about ability were reflected in the children's stories.

Troels:	Er det sådan status, er det sådan fedt at være på det bedste hold, eller?	Is it status, is it cool to be with the best group?
David:	Det synes jeg der er fedt fordi jeg ved … at jeg er en af de bedste	I think it is cool because I know ... that I am one of the best
Troels:	Mmm	Mmm
Dennis:	Det synes jeg ikke der er fedt, særlig fedt.	I don't think it is cool, not very cool.
Troels:	Hvorfor synes du ikke det?	Why don't you think so?
Dennis:	Fordi så får man flere lektier end de gør.	Because then you get more homework than they [the other group] do.

In the interview extract, Dennis described how being in the top group resulted in having more homework than children in other groups and how this to him outweighed the status connected to being in the top group. When homework consisted of work not completed in class, it also acted as a marker of the children's mathematical ability. For example, Sahra did not like to be seen to be slow because she felt that others would judge her as lacking in mathematical ability. Although the school ran a homework café and many of the girls attended it, only those children who did not go discussed it and named the children who went. The implications from having more or less homework were not universal. However, the localised context provided children with the necessary information to determine whether more homework meant that you were clever or not.

The social and political structure surrounding the practice of homework has an impact on the way that children relate to each other as well as to important adults such as teachers and parents. The structure shapes children's perspectives as well as being shaped by the wider context that sees mathematical performance as an indicator of a child's potential.

Discussion and conclusion

Analysing children's comments about homework using the dimensions of practice architectures showed that these children did not see homework as an isolated practice but rather as a series of related practices connected to schooling, such as understandings about the roles of teachers, parents and students within a Western society. The cultural-discursive, material-economic and social-political dimensions of homework were integrated into the wider and the local contexts in which children were operating. The bundling together of the different dimensions affected whether homework as part of the educational enterprise contributed to the good of the individual through the development of the children's identities.

> The identity of the practitioner who lives in and through familiar passages of practice is similarly shaped and formed by practice – the 'skin' of the practice is not external to the practitioner's identity but part of it. The practitioner is an agent and subject of the practice; her or his subjectivity is reflexively formed and transformed by living through both familiar passages and new and surprising ones that call for new ways of working or living within the practice (Kemmis, 2009, p. 11).

In the homework stories, children's identities appeared to be affected by the amount of homework being connected to ability and through their positioning as powerless students whose teachers controlled their out-of-school time. The discursive structures that made homework an accepted part of schooling practices coerced children to become participants in the hijacking of their out-of-school time through material-economic orders and arrangements, even if they were often very unwilling participants. Simultaneously, schools and their teachers were accepted as having a right to exercise power over their out-of-school lives. This emphasised to the children that as students they had limited power to control their own lives. Kemmis and Grootenboer (2008) suggest that self-understandings develop from sayings, but identity development seems to be clearly linked to how power is exerted through social relationships. Children's identities are thus formed through doing homework because of the way that power circulates through the sayings, doings and relatings that are connected to it.

In this paper, we explored children's perceptions of homework using Kemmis' (2009) ideas about practice architectures. This has been a useful tool for the analysis of children's stories because it enables the complexity of the practice to be explored without being overwhelmed by it. Completing homework has consequences for children and these consequences are not just those imagined and discussed by educators. They have an impact on children's identities and sense of the legitimacy in exerting their agency in regard to their learning. As can be seen in children's choosing to complete their homework in places other than their homes, children will exert their agency when they are able to. If education is about the good for each person, then homework needs to provide a benefit to children that they themselves can see and appreciate. Asking children about their views provides valuable information on the role of homework in their lives.

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Mathematical Maps
Students Working with Geography as a Global Metaphor for Mathematics

Håkan Lennerstad
Blekinge Institute of Technology, Sweden

In mathematical map making, a group of students discuss and formulate the mathematics they know in order to form an overall picture, usually as a geographical map. The design of the geography is designed by the student group to reflect mathematical ideas – their importance, character and connections with each other. It is possible to observe that students show a remarkable ease and energy in formulating mathematical ideas in this form. Conclusions may be that geography/landscape is a context that is extremely well known for everyone, and thus provides a rich source for metaphors. Geography is a global metaphor in the sense that it serves as a metaphoric scene for all of mathematics, providing countless "local" metaphors within this scene. Indeed, natural language already contains a large number of geographical metaphors for mental activity. Mathematical maps have been made at all levels, from kindergarten to university.

Introduction

Researchers have persistently emphasized the fundamental importance of mathematical student communication for conceptual understanding. Mentioning one example only, Jodie Hunter starts her paper (Hunter 2009) as follows:

> Developing student communication of productive mathematical reasoning has become a key objective for teachers in Western mathematics classrooms of the 21st century (Walshaw & Anthony, 2008). The pedagogical intent is that students are involved in learning communities in which all participants have opportunities to engage in productive mathematical discourse (Manoucheri & St John, 2006).

With this purpose, to develop student communication about mathematics, the activity of mathematical map making is presented in this paper. Thus, the following questions are central for this paper: How can we achieve active discussions about mathematical ideas among students? How can we make students aware of their own mathematical knowledge? How can students benefit from each other's mathematical knowledge?

This paper is describes mathematical map making, and how this activity possibly may provide partial answers to the questions above. I investigate the question: What is the effect of mathematical map making in mathematics education?

To this date and to my knowledge there exist about fifty mathematical maps. All are made during the last eight years, and all are made in Sweden, except one map: “Calculus in One Variable”. This map is a translation and completion of the first map from 2002, which appeared in the calculus text book *Envariabelanalys med dialoger* (Lennerstad, 2002). The paper *The scope of geography as a metaphor for mathematics – a case study* (submitted as short communication to MADIF 7) is devoted to this particular map, to describe the depth that is offered by the geographical metaphor to describe mathematics. These two maps cover a basic university calculus course in one variable. Since then, student groups have made maps in all levels of the educational system.

The organization of this paper is as follows. This introduction gives a brief description of intentions and practice of mathematical map making. After this presentation a comparison to previous work is much easier to do, which is Section 2. Section 3 presents empirical information about map making activity. Section 4 contains a discussion, and Section 5 describes further developments.

What is a mathematical map?

A mathematical map is usually a landscape with rivers, cities, mountains lakes and oceans, or a city with streets, buildings, industries, rivers and parks and so on. All names on the map are mathematical words, concepts, numbers or formulas. It may contain everything that appears on real maps, at the discretion of the group. The anatomy of the landscape or the city is chosen and designed by the group/person, in order to describe how mathematical concepts and mathematical activities to the group’s/person’s knowledge are related. Thus, mathematical issues are suggested, negotiated and drawn by the students during the activity. The learning that is implied in this activity is the core of map making. Examples of mathematical maps can be studied at the web site www.bth.se/matematikkartor (in Swedish).

This approach allows students to express their subjective experiences of mathematics in the same framework as pure mathematical relations. For example, mountain areas may represent troublesome calculations, high altitude for abstractness, and a cemetery for abandoned mathematics.

Global metaphors

Usually geography has played the role as a global metaphor in the activity that has occurred. By “global metaphor” we mean not only isolated metaphorical images for ideas in the subject (mathematics), but that *the subject is described within an entire metaphoric scene*. Thus, a large number of “local metaphors” appear within the “scene” provided by the global metaphor. An example of a local geographic metaphor, produced by students, is naming a highway as a

Calculator highway, due to the speed allowed. It is an example of a "local metaphor", being a part of the geographic global metaphor.

Another global metaphor, which is not geographic, has been presented in an art exhibition (Håkan Lennerstad: *Iconic Mathematics,* Blekinge Museum, Sweden, October 2008). Here the periodic system of the elements plays the role as a global metaphor for a selection of 100 of the most basic and important functions of one variable. Within this metaphor many important properties of functions are easily represented, such as odd, even, continuous, convex, concave, increasing, decreasing, maximums, minimums, inflection points, singularities, inverse functions, inverted values, derivatives, translates, dilates, Fourier transforms and Laplace transforms.

Overall purposes

The overall purpose of mathematical map making is to develop mathematical understanding by facilitating authentic mathematical dialogue, building a language for mathematical dialogue and increase students' metacognition in mathematics, all based on the students' own mathematical experiences. The purpose is to open a possibility, in the view of the students, for mathematical dialogue by allowing an extended language about mathematics to develop.

How has the activity been organized?

Most common is that the work has been done in the following phases.

1. *Mathematical words collecting*. Before forming groups, all students participate in suggesting mathematical words, which may be collected on the blackboard. This may require about half an hour or more, and may give from 20 to hundred words, including discussions about which words are mathematical. Certainly, words may appear and be included in the map later.

2. *Group forming*. In the next phase, students form groups. There have been groups of sizes from one to eight. In general, smaller groups are recommended for more active students. On the other hand, less active students may prefer smaller groups, and in some large groups all students have been very active, later in different parts of the map.

3. *Choice of overall structure*. This phase consists in deciding an overall organization of the map. It can be started by choosing subsets of the collection of mathematical words, a task that can be done in many ways, and then consider each subset as a country or other province. There are alternatives to geographic maps. One group has chosen bottles with mathematical labels standing in a shelf, while another choose a fruit tree. Furthermore, a geographical alternative can be carried out in many different ways, from maps of continents to one single apartment. The groups may be reformed after this stage if not everyone is in tune with the basic idea of the group.

4. *Filling the structure with mathematical knowledge*. This is the main phase of the work. If the overall structure is well chosen and relevant for the persons involved, ideas usually flow, and very much of the mathematics that the students are aware of finds ways to be represented in connection to other topics. This is a negotiating activity about mathematical concepts. It is often done using post-it-notes, allowing flexibility.

5. *Drawing, painting and completing*. This part is mostly artistic, although mathematical completion or correction may occur also here, within the frames which now are rather rigid. Ideas of different mathematics maps, based on other grounds, may emerge, to be done later.

6. *Presentation to other groups*. The group presentations provide overall mathematical structure and details, seamlessly with the student's personal ativity, engagement and cooperation.

Although mathematical facts and relations are at the core in the activity, a mathematics map is necessarily also subjective. The aim is bridging subjectivity and mathematical objectivity. Therefore, the result is mathematical, but also a work of art, by the group.

Why geography?

How do the two domains mathematics and geography relate? We mention here three fundamental arguments for the use of geography as a metaphor for mathematics.

- *Geographical metaphors for ways of thought* are very common in natural language. We talk about *ways* to find a solution, of scientific *fields*, *stormy* relations and *surveys* of ideas. Stocks may *skyrocket*, people may face *oceans* of work. The first chapter in the volume *The mathematical experience* (Hirsch/Davies 1998) is indeed titled *The mathematical landscape*.
- Landscapes/geography is fundamental for human life, at least in the meaning of surroundings to the home. Maps may represent nature, geology, politics, energy, economics, demographics, and much more. We practice and develop knowledge of this context every time we transport ourselves. Therefore, *landscapes/geography is an extremely well known and flexible context, able to represent facts and relations within a vast spectrum of aspects*.
- Landscapes/geography is *very well known for children*. It provides a substantial part of their concrete knowledge of the world. Geography can for smaller children mean the home, the neighbourhood, playground and school, or the house or apartment where the child lives.

Here the reasons can be found for the observation that student seem to almost always find an interesting way to represent a mathematical idea geographically.

The two contexts are very different in many respects: mathematics being abstract, while geography is very concrete. This difference provides actually the goal of the activity, because *student's learning of mathematics is in need of bridging students' existing concrete experiences to the abstractness of mathematical ideas*. Mathematics in geographical form is abstractness in concrete form.

Concrete metaphors are here seen as preliminary tools for thinking about, handling and coming to terms with abstract concepts. It is important that the limits of each metaphor become visible, so that the understanding may mature by the aid of other metaphors into a fuller and in the end less metaphor-dependent mathematical knowledge.

Is map making shallow in the sense that after some work it is saturated – and more relevant mathematical knowledge cannot be represented? The companion paper *The scope of geography as a global metaphor for mathematics – a case study* (Lennerstad 2010) tries to answer that question by showing that the geographical metaphor can be deepened to represent rather advanced properties of mathematics.

Related concepts and research

One of the most famous approaches for representing theory in a graphical way is concept mapping, which was developed mainly by Joseph D. Novak (Novak 1985). Concept mapping is based on a constructivist view of learning, such as that of David Ausubel (Ausubel 1968), who stresses the importance of that students construct their knowledge, as well as the importance of prior knowledge:

> The most important single factor influencing learning is what the learner already knows. Ascertain this and teach accordingly.

A concept map is a flexible means to formulate basic relations in a subject. The anthology *Concept mapping in Mathematics* (Afamasaga-Fuata'i 2009) appears to be the first comprehensive book on concept mapping applied to mathematics learning. In (Grevholm 2005), work with concept mapping is described for teacher students' learning in mathematics and mathematics education. Note that geography plays no role in concept mapping.

Mind mapping (Buzan 2000) is related to concept mapping. It is usually non-geographic, less regulated/structured than concept maps, and thus perhaps more suitable for brainstorming where limiting prohibitions are temporarily avoided. A development of mind mapping for logic is logical graphs (Lennerstad 1996A and 1996B). Here a statement is never repeated, instead implication arrows are drawn from a single occurrence, which makes the logical structure of an argumentation obviously visible. The presentation form is consistent with the logic.

Geographical metaphors have been used in some contexts, such as *Tillvarons atlas* (von Swaaij & Klare, 2001), which depicts human life, and *Der atlas des managements* (Cairner, Derlove 2005), for management.

Importance of a global metaphor

Concept mapping, mind mapping and logical graphs can be very fruitful, but are rather abstract for students to work with. A main point of this paper is that, from students' point of view, the geographical metaphor provides energy and enthusiasm to the work. When students find a fitting local metaphor it is for them both an emotional and an intellectual event. This is very important for the emotional energy of their work.

Maria Selander was the first teacher that tried mathematical map making in a student group (Selander & Lennerstad 2004) in Strängnäs, for 16 year old students. The work took five weeks with about four hours work in school each week. To be able to finish the map, at least one group worked extensively also after school. She observed that students that do not have top grades often were most active, and that the map drawing students became more active during the "normal" mathematics classes. Initially the activity was time consuming for her as a teacher, but not after that phase.

Mathematical map making has taken place in the teachers' education in Linköping University and other places. In a high school in Åmål a very decorative mathematical map was made as a wall painting.

Classroom activities and teachers' experiences

This section contains ideas and conclusions by teachers. Some of the maps can be studied at the web site www.bth.se/matematikkartor (in Swedish). The majority of the maps have been done by groups of students, while a few are done by groups of teacher student's.

A different use is map construction by a group of teachers. Even more for teachers, it appears, map drawing is a practice in mathematical dialogue, challenging the ability to listen and compromising concerning mathematics.

Teachers' evaluations

Next follows sixteen teacher/ teacher groups that organized mathematical map making in their student groups. The teachers did the work as a part of a course in mathematics education. The excerpts are very short, focusing mainly how the work was started, main problems and overall evaluations by students and teachers.

Initially, the teachers listened to a lecture by Håkan Lennerstad, who described the idea. This meeting was three hours long and included an activity where the teachers in groups tried to find geographical metaphors for mathematical concepts. The meeting was compulsory for the teachers in their course. The following are translations of the teacher groups own words.

1. The task could be too large. The goal was to let the students, in 6^{th} grade, understand how much mathematics they understand. Some made a map over the Swedish language and some over mathematics. The task was fruitful both for teachers and students. It is a very good way for students to get to know their understanding in the subject. The students became engaged and many discussions about mathematics occurred spontaneously. Some students were frustrated over that the semester ended so they could not continue and develop their map.

2. We had some problems with the order in the class, and the students were not so motivated. There were interesting mathematical discussions, but we have that anyway. The students liked to draw, but they could likewise have done that in the art class.

3. The students immediately started to sketch their mathematics map. They used the math book to get ideas. The groups were allowed to make suggestions to maps to other groups. The cooperation in the groups has worked out very well. The students liked the project. They say that they have been able to get a better perspective for mathematics. I found the students very engaged in their work and I have seen that several students have grown during this work.

4. These students had some problems in their math learning. They made bottles with different mathematical concepts, and the amount of content reflected their knowledge of that concept. As a teacher I could observe that the student thought they knew more than they did, for example about common words as term, difference, etc. There were interesting discussions about the meaning of different mathematical words.

5. I thought a lot about how to introduce the idea to engage all of them without steering too much. They choose to make a mathematics map in the shape of a turtle, a town and a shop. Many debates appeared. For example, most of them thought that a ruler has nothing to do with mathematics, but had eventually to agree by a convincing argumentation by one girl. I got a deepened knowledge about how students think. Another large gain is that they have powerful argumentation to add something mathematical on the map. We can reach the students very much better if we are familiar with their mathematical thoughts.

6. Most of the student groups made landscapes, but one group, for example, made a mathematical soccer field. We think it worked well especially as a tool for the students to construct an overview and to concretize their own view of how different parts of mathematics are related.

7. We started the work by presenting the idea of map drawing and explaining that we for this need mathematical words. After the students had suggested words we grouped them in natural sets according to the students' opinion. The students became a bit impressed by the large number of mathematical words that they

understand. The work with the maps took four lessons, and almost all groups showed a miraculous imagination when they had understood what to do. Some groups, however, had difficulties in deciding over a common basic idea for the map.

8. Thirty 6th graders were involved in the project, and after having collected a large number of mathematical words the mathematical debates started. It was very interesting to walk around and listen. Thy really thought about the meaning of the mathematical words. Some groups had problems to cooperate. The artistic and the verbally fluent students liked the drawing of the map. They liked it, they learned the connections in mathematics, two of them said that they learned what algebra is, and a few found map making boring.

9. The groups were grades 4 and 5, and the work started by collecting mathematical words. They made one of five areas: geometry, the quattuor species, the positional system, weight and length, and finally time. The work was concluded with a presentation in a large group. The cooperation in the groups worked well, and the fantasy and energy was flowing. It was a very nice type of work for both teachers and students.

10. In a fourth grade we collected mathematical words, and the students started with their maps in groups of two in each. The student liked the work, but had difficulties in connecting the mathematical words with concepts, they did not have so much underlying idea behind their naming with mathematical words. However, they found several nice ideas in their maps. We all liked it and will perhaps try this again in a somewhat different way.

11. We tried to make mathematics maps in a group of ten students in eighth grade who need special support. The result was very mixed. It seems like mathematics maps do not fit so well since the students lack too many basic abilities in mathematics.

12. I tried to introduce mathematics maps in a seventh and one eight grade. I tried to imagine a walk through the woods in the area and the students need to pack abilities that one need during such a walk. I am not sure about the result.

13. We had only two students who decided to make a map about addition, subtraction, multiplication and division, one in each corner of the paper. Each was an island with smaller islands around connecting to other mathematical words. They discussed very much about meaning in mathematical words and when they use them in everyday life.

14. Students in fourth and sixth grade were very diverse in mathematical knowledge. We decided that most important would be that they talk mathematics. They did not draw maps but discussed which kind of mathematics that appeared at real maps that we gave them. The students discussed mathematics very much, so we succeeded with our goal.

15. We started the project in three classes, grades four, five and six. All liked the task and were very creative. They discussed mathematics in a logical way, which was the intention.

16. I had a group with six students with extra needs. To have mathematics, I said, we need to go to the mathematics shop and buy what we need, but I do not mean pencils and paper but words, signs and other that we use. What comes in your mind? A vivid dialogue started where many mathematical words came up on the white board. I am impressed by the number of concepts they put forward and their understanding of their connections. Particularly interesting was the discussion over numbers contra digits. The students liked the task. Mathematical map making is a good idea, I think. With the new students starting next year I intend to form groups working with mathematical maps.

Summary of teacher comments

Half of the teachers found that the students engaged in good cooperation or interesting discussion. Many also found that the students became more aware of their knowledge in mathematics and developed their perspective. The fantasy and energy were flowing, which is somewhat remarkable taking into account that it concerned inquiry in mathematics. A few groups had difficulties in finding a basic idea, while others became occupied in the drawing task only. Some groups where the students lack basic knowledge were successful, some were not. Note that the task was mandatory for the teachers, so some teachers were not motivated from start.

Conclusions and developments

Mathematical dialogue and discussion are of paramount importance for mathematics understanding, expertise in mathematics education do not hesitate on this. The dominant goal of mathematical activity in school is to solve mathematical problems by calculation. Tests consist only of solving mathematical problem by calculation, or are strongly dominated by it. Many students are therefore *extremely focused towards calculation*. With such a focus, discussion and dialogue is relevant only in case of problems during calculation. As soon as dialogue opens new doors for calculation, the dialogue is abandoned and calculation is reassumed. This gives *broken dialogues*. The goal is not understanding, the goal is calculation, which of course is important, but perhaps too limited.

Broken dialogues are not typical during mathematical map making sessions. Conceptual dialogues may continue following routes of discovery – they are not suddenly abandoned and replaced by calculation. The task is completed when most of the mathematics familiar to the students in the group is represented in the map in an acceptable way.

Is map making time consuming? As described in Section 3, the task may be given everything from one afternoon to several weeks. A mathematics map may grow gradually during the entire education. It may be time consuming initially for a teacher who has not tried the work before – it is a different mathematical activity. The students are usually driving the activity themselves, often outside of school hours.

Is map making useful for development of teacher skill? The map certainly reflects the prevalent mathematics views in the student group, and can be studied with this aspect by a teacher. Listening to students' mathematical dialogues is valuable. It is also possible to develop questions and deepen the mathematical content of the dialogue. This may sometimes be important since there always is a risk of a superficial map, for example only sorting words and concepts in groups/countries with no cross connections.

What does map making give for students? Students become more aware of the terminology in mathematics, they discuss meaning of mathematical words, and they form an overall picture of mathematics. They learn from each other and become more able in talking mathematics.

Many teachers approved of the ease with which students communicated conceptually about dialogue. Some teachers have remarked that they noticed a higher activity in normal mathematics classes for "map students". Surely a potential language for mathematical communication has been evoked, but how this is taken advantage of depends on the future activity in the classes.

Map making contributes clearly to the interest in mathematics and to the realization that mathematics contains questions that can be discussed in interesting ways. This make map making valuable in teachers' education. Further investigation is needed to evaluate long term effects.

Development 1 – conceptual study of students' dialogue: Students' mathematical conceptual dialogues can certainly be studied, developed and deepened by teachers, for example in cooperation with mathematicians and mathematics education researchers.

Development 2 – teachers' and mathematicians' maps: Experiences hint that it is not easy for a group of mathematics teachers and mathematics researchers to make a common map. Compromise about the mathematics picture is needed, which is a question rarely discussed. We here thus have a starting point for a conceptual understanding of mathematics.

Development 3 – subject maps: Mathematical maps may easily be generalized to "subject maps" – other subjects may be described with the geographical metaphor. Music map making has been made three times during a chamber music festival (Lyckå Kammarmusikfestival, 2007, 2008, 2009) by young musicians, aged 13 to 17. Here four music student groups with four students in each have completed a four maps in three hours with no preparation in advance. The students have expressed that they have found this work unexpectedly interesting.

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Mathematical Competencies: A Research Framework

Main Author: Johan Lithner
Co-authors: Ewa Bergqvist, Tomas Bergqvist, Jesper Boesen, Torulf Palm and Björn Palmberg
Umeå University, Sweden

This paper presents a summary of parts of a research framework constructed to be a basis for analyses of empirical data, with focus on students' opportunities to develop mathematical competencies. Six different competencies are defined and three competence related activities. International reform frameworks, mainly the NCTM Standards and the Danish KOM-project, inspire the framework. The six competencies are Problem solving ability, Reasoning ability, Applying procedures ability, Representation ability, Connection ability and Communication ability.

Background

An international trend concerning mathematics education is to describe knowledge in mathematics not only in terms of content goals (e.g. arithmetic, algebra, and statistics) but also in terms of goals that describe the processes and abilities that are involved in practicing mathematics (e.g. problem solving, reasoning, communication). This last type of goals is often called process goals, ability goals, or competence goals. We will not separate between these terms in this paper, but denote them 'competence goals'. An internationally influential (judged by the large number of references in the mathematics education literature) description of these types of goals is presented in the NCTM Principles and Standards (NCTM, 2000). Similar goal descriptions exist in e.g. the Danish KOM-project (Niss & Jensen, 2002), in "Adding it up" (Kilpatrick, Swafford, & Findell, 2001), and are also used in the international comparative studies TIMSS and PISA (Mullis, et al., 2003; OECD, 1999).

The main purpose of the first three frameworks mentioned is to communicate goals and means for educational development. In this perspective, it is not so problematic that the competence goals within these frameworks are largely overlapping and sometimes not precisely defined. However, since the purpose of the framework presented in this paper, the Mathematical Competency Research Framework (MCRF), is to be a tool for categorizing data (text, interview and observation) it is advantageous if the competencies are more specifically defined and less overlapping. We want to avoid classifying the same phenomenon in two different categories, something that is not a concern in the international frameworks. Niss states the following:

> The competencies are closely related - they form a continuum of overlapping clusters – yet they are distinct in the sense that their centres of gravity are clearly delineated and disjoint. (Niss, 2003, p.9)

The MCRF aims at taking the separation of the competencies even further. This does not mean that the competencies are independent, but it means that the focus of classification is different since the definitions below aim both at clarifying the notions and at separating the competencies. As a consequence, the MCRF is not identical to but still to a high degree inspired by the frameworks mentioned above, especially NCTM (2000). Some notions in the MCRF are directly taken from these frameworks, some are modified versions, others are merely inspired by the frameworks, and some notions have other origins.

The research project

The need to define the Mathematical Competency Research Framework (MCRF) comes from an ongoing research project at Umeå Research Centre for Mathematics Education. The project is called 'National tests in mathematics as a catalyst for implementing educational reforms' and the aim of this project is to clarify the role that the Swedish national tests in mathematics have in the schools' attempts to implement the competence goals of the syllabi. Within this project we therefore analyze (focusing on competence goals) national tests, teachers' interpretation of curricula documents and test items, teachers' intentions and plans, as well as how competence goals are present in the organized teaching activities.

We do not analyze what the students are actually learning since that would be too complicated within this project. Instead we base our research on the principle *opportunity to learn* since Hiebert (2003) argues that what the students learn is connected to the activities and processes they are engaged in. According to this principle the teachers give the students the opportunity to develop a certain competence when they provide the students with a good chance for practicing the specific processes involved in that competence. This means that students that never are engaged in e.g. problem solving during class, are not given the opportunity to learn problem solving (or, in other words, to develop their problem solving ability).

The structure of the competency definitions

Mathematical competence (in general) and six mathematical competencies

In this article we define the concept of general mathematical competence and specific mathematical competencies [1] according to the Danish KOM-project:

> To possess a competence (to be competent) in some domain of personal, professional or social life is to master (to a fair degree, modulo the conditions and circumstances) essential aspects of life in that domain. Mathematical competence then means the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role. (…) A mathematical competency is a clearly recognisable and distinct, major constituent of mathematical competence. (Niss, 2003, pp. 6-7.)

We have chosen this definition by Niss since it is one of the few explicit definitions presented in any of the frameworks mentioned above. We will define six mathematical competencies, each as the ability to handle a particular aspect of general mathematical competence. These competencies are Problem solving ability, Reasoning ability, Applying procedures ability, Representation ability, Connection ability and Communication ability. These are the same as the process goals of the NCTM Standards with the addition of the Applying procedures ability. We have chosen to base our framework on the NCTM Standards since it is (1) a widely spread description of this type of goals, and is (2) heavily influenced by research. The reason for including Applying procedures is that our research framework needs to be able to describe and characterize existing teaching. A possible reason that Applying procedures is not included in the NCTM Standards is that a reform framework rather aims at inspiring and developing teaching instead of describing it. We will for each competency define what aspect of general mathematical competence the particular competency concerns.

Since understanding is a concept often used when describing general mathematical competence, one possibility could be to include the ability to understand as one of the competences. All the above-mentioned frameworks avoid this possibility, perhaps because the concept of understanding is so vague and complicated to define. We instead argue that the Representation and Connection abilities describe central aspects of the concept of understanding, and are possible (but not easy) to define more clearly. Therefore we will not include 'understanding' as one of the main competency definitions, but instead use it in a more intuitive way when describing competence related activities below in a similar way that is done by Niss (2003) in the quote below.

The Competency-Related Activities (CRA)

In addition, Niss and Jensen (2002) argue that a mathematical competency is an "insightful readiness to act appropriately in situations which present a particular kind of mathematical challenges" (p. 44). Since the particular nature of this insightful skill in most cases does not follow directly from the definition of the competency, it is also necessary to specify what it means to master a competency. The starting point for our definitions of what we denote as *competency-related activities* (CRA) is the following quote from Niss (2003, p. 9):

> All competencies have a dual nature, as they have an analytical and a productive aspect. The analytical aspect of a competency focuses on understanding, interpreting, examining, and assessing mathematical phenomena and processes, such as, for instance, following and controlling a chain of mathematical arguments or understanding the nature and use of some mathematical representation, whereas the productive aspect focuses on the active construction or carrying out of processes, such as inventing a chain of arguments or activating and employing some mathematical representation in a given situation.

In addition to this definition, we have also taken into account what kinds of activities that are (often in rather unsystematic ways) mentioned in the international frameworks. This leads to a separation of the analytic aspect into two parts concerning: *understanding and interpreting* phenomena and processes and *meta-level considerations* such as judging and assessing them. One reason for this separation is that we want the meta-cognitive part (corresponding to Judge below) of the analytical aspect to be applicable also for the productive aspect.

The following categories of CRA are inspired by (mainly the terms but not the specifications) the definition of *mathematical competency* in Niss (2003). Some verbs that the international frameworks relate to particular CRA:s are given in square brackets. The idea is to use the CRA:s as classification subcategories on *all* competencies, and, if really motivated (since we are dealing with complex concepts), to add special categories to some competencies.

I) **Interpret** [build knowledge, understand, interpret, identify, recognize]. This CRA concerns *taking in information* in relation to the competencies. Since one purpose is to form specific definitions useful in characterizing data, the general and vague term 'knowledge building' is not included in the activity definitions.

II) **Do and use** [engage in task, pose, solve, use, respond, develop, argue, select, create, support, specify, apply, adapt, estimate]. This CRA is about using one's knowledge in order to solve tasks (in a wide sense). Our interpretation of the distinction between 'do' and 'use' is that 'do' concerns developing our knowledge within mathematics as a scientific discipline, and 'use' concerns applying this within and outside mathematics. We also consider two main versions of this activity: a) imitate and b) construct (see Lithner, 2008).

III) **Judge** [evaluate, monitor, reflect]. This CRA includes meta-level considerations and concerns evaluating, reflecting, and forming opinions and conclusions on mathematics and on the activities related to learning, understanding, doing and using mathematics.

The competency definitions

The definition of each competency will determine the particular aspect of mathematical competence that it concerns, i.e. a competency is the ability to handle *something* so we define what that something is. The definition will also include what it means to master the competency in terms of competency-related activities (CRAs). We will in each definition include all three types of CRAs. This might lead to that some parts of the text are repeated, but we choose to do this in order to make each competency definition complete and self-supporting.

Problem solving ability

Problem solving ability is *the ability to handle problem solving*. Here problem solving is defined as "engaging in a task for which the solution method is not known in advance." (NCTM, 2000, p. 51) This definition implies that in this perspective there are only two types of tasks: problems and non-problems (often denoted 'routine tasks'). Note that some aspects often included in similar definitions of problem solving are not included in the definition above, for example that the task is necessarily a challenge (Schoenfeld, 1985) or that the task requires exploration (Niss & Jensen, 2002).

What it means to master the problem solving ability

I) **Interpret.** Understand problem situations (e.g. verbal, visual, real), including understanding and recognizing the components of the problem. Also to understand the methods, tools and goals of problem solving.

II) **Do and use.** Use mathematics to solve different kind of problems that arise in mathematics and in other contexts. Apply and adapt a variety of appropriate problem solving strategies and methods. Posing and specifying different kinds of problems.

III) **Judge.** Judge and evaluate the validity of a solution. Monitor and reflect on the process of mathematical problem solving. General reflections on problem solving, e.g. concerning beliefs.

Reasoning ability

Reasoning ability is *the ability to handle reasoning*, to reason mathematically. Here we define reasoning as the explicit act of justifying choices and conclusions by mathematical arguments. This definition is based on a selected part of the definition of reasoning in NCTM (2000): "to develop and evaluate mathematical arguments and proofs" (p. 55). In addition it has a particular focus on the reasoning being explicit, in line with an idea from Niss and Jensen (2002), where the reasoning competency is intimately connected with problem solving and modeling as their so called juridical counterpart. We further define arguments to be mathematically founded if they, in the terms used by Lithner (2008), motivate why the conclusions are true or plausible and are anchored in intrinsic properties of the mathematical components (objects, transformations, and concepts) involved in the reasoning. A *proof* is a sequence of reasoning where the mathematical arguments are logically strict. Thus, according to the definition above, reasoning does not have to be logically strict but may, for example in line with Pólya (1954), be plausible.

What it means to master the reasoning ability

I) **Interpret.** Understand and interpret ones own and others' reasoning.

II) **Do and use.** Select and use (including both to imitate and create) informal and formal arguments that support choices and conclusions in conjectures, hypotheses, statements, task solutions and proofs. To use reasoning to construct interpretations of information (e.g. analyzing a difficult task formulation).

III) **Judge.** Judge and evaluate one's own and others' reasoning but also general reflections, e.g. on the role of reasoning or on qualities of valid reasoning. Meta-knowledge about reasoning: recognize reasoning and proof as fundamental aspects of mathematics; know what a mathematical proof is, and how it differs from other kinds of mathematical reasoning, e.g. heuristics.

Applying procedures ability

Applying procedures ability is *the ability to handle mathematical procedures*. Here we define a mathematical procedure as a sequence of mathematical actions that is an accepted way of solving a task. To apply a procedure is to carry out the sequence in order to solve the task. There are no definitions of *procedure* in the three frameworks mentioned above. We propose the above definition inspired by ordinary dictionaries (e.g. the Cambridge Dictionary). An additional component in our definition is that a procedure is, or can be reformulated as, an algorithm. "An algorithm is a finite sequence of executable instructions which allows one to find a definite result for a given class of problems" (Brousseau, 1997, p. 129).

What it means to master the applying procedures ability

I) **Interpret.** Understand and interpret one's own and others' procedures.

II) **Do and use.** Select and use procedures to reach conclusions, and to be able to do this fluently in order to make use of the benefits of procedures (cf. Kilpatrick et al., 2001).

III) **Judge.** Judge and evaluate the applications and outcomes of one's own and others' procedures (cf. Kilpatrick et al., 2001). General reflections, e.g. on the role and function of procedures.

Representation ability

Representation ability is *the ability to handle representations*. Mathematics is built on abstract mathematical entities [2] of different kinds, e.g. numbers, functions, geometrical objects, tasks, methods, principles, concepts, phenomena, and ideas, and their properties. When we do mathematics we have to think about these entities and their relations, or about some aspects of them. It is often impossible, or at least very difficult, to think about such an entity in abstract and/or fully general terms. Therefore it is usually necessary or at least advantageous to instead think about something more concrete (mental or real) that replaces the abstract entity, but still carries with it the, for the particular situation, useful and relevant aspects of the entity. Here we define representations to be the concrete replacements (substitutes), mental or real, of abstract mathematical entities.

What it means to master the representation ability

Translating and switching between representations are included in Representations in NCTM (2000) and Niss (2003). However, in the MCRF the ambition is to reach a clearer separation and distinction between the Representation and Connections competencies. Therefore relations between representations, including translating and switching between representations, are seen as connections (see the following section).

I) **Interpret.** Understand and interpret one's own and others' representations.

II) **Do and use.** Select and use (including imitate and create) representations to organize (e.g. in tables or graphs), record (for e.g. teachers or peers to see), solve problems, model and interpret physical, social and mathematical phenomena, and communicate mathematical ideas. Select and use interpretations of representations.

III) **Judge.** Judge and evaluate one's own and others' representations. General reflections, e.g. on the role and function of representations.

Connection ability

Connection ability is *the ability to connect between mathematical entities or representations of mathematical entities.* The concept *to connect* is defined differently but similarly in different dictionaries: to place or establish a relationship (Merriam-Webster online Dictionary); when something joins or is joined to something else, or the part or process that enables this (Cambridge online dictionary). Here *to connect* is therefore defined as the process *to use something that connects or makes a link between two things*, e.g. a relationship in fact or a causal or logical relation or sequence. Following the discussion about representation of entities, the 'something' that is connected in this framework is *a representation* or *an entity* (including a part or a sub aspect of a representation or entity). We define five different types of connections:

a) Between representations of different entities. Example: Between the number representing the interest rate in % and the number representing the amount of money to be paid.
b) Between different representations of the same entity. Example: Between the graph and a table representing a temperature as a function of time.

c) Between different parts of one representation. Example: Between x and y in a graph of a function (in a coordinate system with x- and y- axes).
d) Between different entities. Example: Between multiplication and addition, e.g. that multiplication can under certain conditions be seen as repeated addition.
e) Between different parts of one entity. Example: Between the edges and vertices of a cube.

Representations (including interpretations) consist of vertical relations in Figure 1, and Connections of horizontal relations.

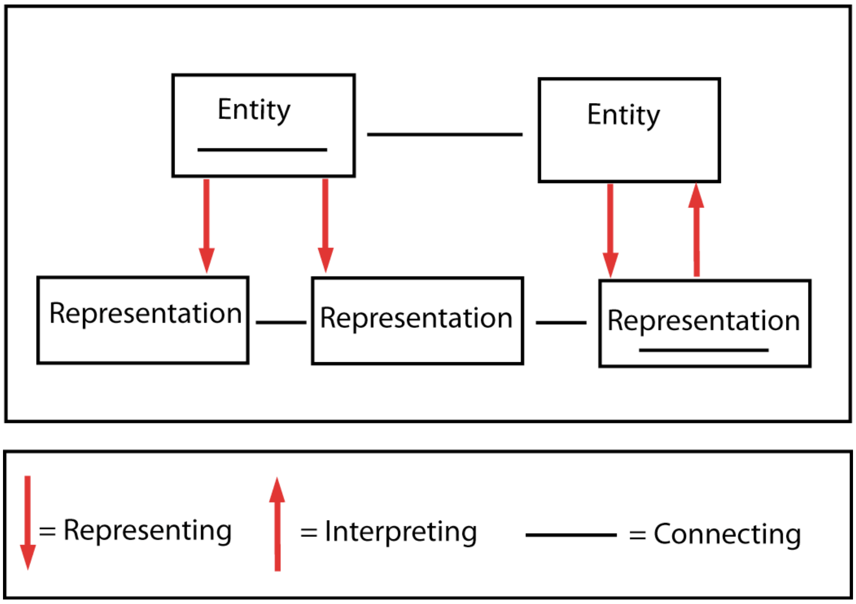

Figure 1: Relation between representations and connections.

What it means to master the connection ability

I) **Interpret.** Understand and interpret one's own and others' connections. Understand how mathematical ideas interconnect and build on one another to produce a coherent whole. Ability to see mathematical structure through connections. Recognize mathematics in the surrounding world, and the connections to mathematics (either through connections between entities or between representations).

II) **Do and Use.** Select and use (including imitate and create) connections to organize, solve problems, and model and interpret physical, social and mathematical phenomena. Apply mathematics in contexts outside of mathematics. In this use of connections, representations often have a central role.

III) **Judge.** Judge and evaluate one's own and others' connections. General reflections, e.g. on the role and function of connections.

The modelling cycle (forming a mathematical model of a real world phenomena, handling the situation mathematically, and interpreting the result back into the real world situation) can be specified using the definitions under Representations and Connections as follows. First, the entities of a real world situation are represented by mathematical representations. Secondly the situation is handled mathematically, using connections between representations. The outcome consists of representations of entities. Finally the outcome representations are interpreted and consequences for the real world situation are inferred.

Communication ability

Communication ability is *the ability to communicate*. Here *to communicate* is defined as to engage in a process where information is exchanged between individuals through a common system of symbols, signs, or behavior [3]. Communication therefore includes a sender and a receiver, and also a medium within which both can understand the communicated information. Within mathematics education, the sender is usually the teacher, the textbook author, or a student, the receiver is most often a student or a teacher, and the medium is usually auditory (e.g. speaking, listening) or physical (e.g. writing, gestures).

What it means to master the communication ability

I) **Interpret.** Understand and interpret information from a sender. To be able to interpret (note that this term is used in a more restricted sense in the Representation competency) the written, oral, and visual mathematical statements of others.

II) **Do and use.** Construct (including creating or imitating) and formulate information to a receiver. Students are successfully communicating when they are able to express themselves in different ways (in writing, orally, or visually) and at different levels of theoretical or technical precision in mathematical issues and for different categories of receivers. This means for example that they should be able to use ordinary language as well as more formal mathematics terminology when they speak or write mathematics. To be able to respond to both exemplary and problematic pieces of communication.

III) **Judge.** Judge and evaluate one's own and others' communication. General reflections, e.g. on the role and function of communication.

Applying the framework in analyses of empirical data

The summary of (parts of) the MCRF presented above defines the competencies in a way that is relatively specific and less overlapping than in the established frameworks mentioned in the introduction. However, the categorization of empirical data does by no means follow directly from the framework. Its operationalization in the research project mentioned above is based on sub-frameworks that depend on both the specific research tasks and the types of data.

One example is the ongoing analyses of beliefs related to mathematical competencies through surveys and in-depth interviews with 200 Swedish teachers. Here a sub-framework is formed with the aim to capture the teachers' competency-related knowledge, values and beliefs, and also their intentions with respect to helping their students develop mathematical competence. The main role of MCRF is here to distinguish oral or written responses that closely relate to the six competencies from data that relate to other types of activities, processes and goals. Of course, these analyses also require the application of other frameworks, for example for analyzing teachers' beliefs.

Another example concerns analyses of observations of the classrooms of the same 200 teachers. An analysis protocol has been developed that aims at capturing extent and qualities of the classroom activities. MCRF is here used when identifying activities that can be seen as providing opportunities for students to develop one or more of the six competencies.

A third example concerns discourse analyses of official curricula documents, aiming at clarifying the extent and weight by which the six competencies are communicated by these documents.

A final example is that textbook tasks and assessment tasks are classified according to if and in what ways the tasks require the students to activate one or more of the six competencies. Here, frameworks from earlier analyses of reasoning requirements in tasks (e.g. Lithner, 2004; Boesen, Lithner & Palm, in press) are extended and complemented to incorporate the competencies of MCRF.

The MCRF (and related sub-frameworks for data analyses) is far from final and still under development. There is a need for further clarification and specification of the MCRF itself. In addition, as the research project proceeds new aspects need to be included and new related frameworks need to be developed. One of the main challenges is to further develop a more holistic research framework, with MCRF as one component, which enables a synthesis of the different parts of the research project.

Notes

1. Note the difference between competence and competency/competencies as it is used by Niss (2003) and in this article.

2. Niss and Jensen (2002) use the term mathematical entity, but not the reasoning in this paragraph.

3. http://www.merriam-webster.com/, search "communication".

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The Missing Piece
An Interpretation of Mathematics Education Using Some Ideas from Žižek

Sverker Lundin
University of Gothenburg, Sweden

A sublime object is, according to the Slovenian philosopher Slavoj Žižek, something which seems to be the cause of pleasure and beauty or - to the contrary - of discord and error. This object does not really exist in itself though, but only by being assumed to exist. In this paper mathematics is interpreted as a sublime object. The interpretation is substantiated by reference to the contemporary discussion of mathematics education in Sweden, as well as to the history of mathematics education in Sweden, from the 18th century to the present.

Introduction

A sublime object is an object which seems to be missing where something is in discord. It appears to the mind as the one missing piece in an otherwise complete jigsaw puzzle: you can vividly imagine how it would complete the puzzle, you want it badly, and you do not have it. In this analogy, the pieces we have corresponds to reality as something given and ordered, in which we can orient ourselves and which makes sense – complete sense, had it not been for the missing tile. Talking about the puzzle, the shape of the missing tile is given by the ”hole” in the puzzle and we can infer what it probably looks like by the surrounding pieces. This is analogous to how we seem to know what is missing in reality by the structure of what is actually present.

The central idea of the sublime object though, is that it is the other way around. What we understand as the ”hole” gets its shape from our imagination of what the missing piece looks like and the same goes not only for the images of the surrounding pieces, but more or less for the whole puzzle as such: we think that reality almost makes sense, was it not for the missing piece, but to the contrary, it is only because we think we know what the missing piece looks like that reality makes any sense at all.

In this paper I will present some of the ideas that I develop in my dissertation (Lundin 2008). I will take three distinctive features of contemporary mathematics education as my points of departure: (1) the ambitiousness of the goals towards which it strives, (2) its theory of concept formation and (3) its seemingly very high level of critical reflexivity. I will try to show that mathematics education in all these three areas can be seen as a puzzle kept together by the missing piece of mathematics. [1]

Higher goals

”The importance of good knowledge in mathematics is indisputable” (Matematikdelegationen, 2004, p. 189; my translation). Why? Because: ”Mathematics is a precondition for large parts of the development of society and it permeates the whole society [...]” (Skolverket, 2000; my translation). Mathematics permeates society, therefore you need mathematical knowledge to understand it. Knowing mathematics is then not the same thing as being able to count, to have mastered a set of algorithms, memorized some formulae or having become an expert in e.g. finding primitive functions. Rather, knowing mathematics is to have acquired a set of general competencies for solving problems, representing, relating, reasoning and communicating (Skolinspektionen, 2009, p. 10). Knowledge of mathematics makes it possible to ”live and act in a democratic society”, it is supposed to lead to ”general education, economic growth and citizenship” (NCM, 2001, p. 81). [2]

Mathematics stands in a privileged relation to rational thinking as well as to social and physical reality. Mastery of mathematics is therefore a kind of *shortcut* – to thinking in general, to competent participation in society and to scientific understanding of reality. Mathematics represents, in a coherent and learnable form, an aspect of reality that is often ”invisible for the inexperienced viewer” (Skolverket, 2000). The task of mathematics education is therefore to take the students to a point of knowing from where they can see this hidden mathematical layer (c.f. Skovsmose, 1994), from which they understand what they see, and from which it is easy to learn new things, be they theoretical or practical. When looking at curricula, public reports and the discussion in the field of *matematikdidaktik* (the field of research in mathematics education) in Sweden during the last 30 years, this seems to be how many proponents of mathematics education understand their subject matter [3]. Let me now approach it from another direction.

Following the social anthropologist Jean Lave’s *Cognition in Practice* (1988) the shortcut-hypothesis can be said to have two reciprocally determining parts, one pertaining to the world one to the individual. As regards the world, it is supposed to be *sliceable* into ”knowledge domains” which cut across and create continuity between settings (Lave, 1988, p. 42). Mathematics is seen as one such domain: always present, only differing in thickness and closeness to the surface. In the individual, knowledge of mathematics has the form of *concepts*. These are supposed to be *formable* in specially designed settings (i.e. in school), containable in a more or less constant state in the individual, and thus transferrable between the settings of everyday and professional life. They supposedly give the individual access to the particular layer of reality pertaining to mathematics to an extent corresponding to the concepts' perfection and maturity.

Lave finds it perplexing that "learning transfer theory [has] endured for so long" in spite of empirical evidence that it is seriously mistaken, and hints that an important part of the answer to why this is so probably lies in the role of the education system in modern (democratic, capitalist) societies (Lave, 1988, p. 19). This is certainly correct, and it is something I will come back to. Here I will briefly suggest another, complementary answer, which I believe to be especially pertinent in the case of mathematics. It draws on the history of science. What you can see is namely how mathematics rises, from being a relatively minor science of computation in the 16th century, to becoming an integrated part of a new ontology during the 17th and 18th centuries.

It is well known that the world was "mathematized" with modern science. Not so clear though, is how and why this happened. What I want to point to is the close relationship between the rise of modern science and the transformation of Christianity. Drawing on recent studies in the history of science (Dear, 1995; Funkenstein, 1986; Gaukroger, 2006 and Gillespie, 2008) it can be claimed that the scientific revolution was to a large extent a Christian project, obviously and often explicitly related to belief in God [4]. This belief was, during the 17th century, intertwined with the belief that reality is, in its very essence, mathematical. Following Charles Taylor's *A Secular Age* (2006) it is clear that the rise of science cannot be seen as a history of simple "subtraction", where (religious) misunderstandings were removed to let reality shine through. Instead, what happened must be understood as a *transformation of belief* – of course completely inseparable from transformations on a social and material level. Coming back to Lave, this new belief is to a large extent a belief that general mathematical knowledge constitutes a scientific viewpoint from which the world in its entirety can be understood and mastered. This, I claim, is the very point to where mathematics education, according to its mainstream discourse, wants to bring the students, a point that, as Lave and many others have shown, is a fiction (see also Rorty 1979).

Coming back to my introductory analogy with the jigsaw puzzle, this fiction corresponds to the missing piece. What I want to make clear is how our conception of this piece, what we assume mathematical knowledge to be like, is seemingly *founded* on the corresponding pieces. The properties of mathematical knowledge obtain their plausibility from what the world seems to be like. But to the contrary, I claim that the world to be known is an object of belief deriving its structure and apparent obviousness from, among other things, our conception of mathematics.

Concept formation

”To learn mathematics is a lifelong project that starts already with the play and attempts of the infant. The child will experience form, number, order, relation, symmetry and pattern, and very soon there will emerge intuitive conceptions of many foundational mathematical concepts” (Matematikdelegationen, 2004, p. 87; my translation). Taking the intuitions and natural curiosity of the child as its point of departure, the task of mathematics education is, in this view, to provide a setting which gives the child opportunity to experience, discover and communicate mathematical ideas. Because of the inherently hierarchical structure of mathematics, teachers have to make sure that each student has fully mastered a given level of abstraction, before moving to the next. In sharp contrast to traditional teaching methods based on knowledge transfer, memorization and drill, the teacher is expected to support the formation of mathematical concepts through a constant sensitivity to the level of mathematical maturity of each individual student, providing them with properly chosen questions and exercises to promote their competence and understanding. This is how students are to be moved from where they stand entering school, to the point of mathematical knowledge. [5]

The contemporary mainstream idea about the formation of mathematical concepts is, I believe, the result of a historical process which has two beginnings: on the one hand, the mathematical science of antiquity, with Euclid's *Elements* at its core, and on the other, the arithmetical algorithms - often related to commerce - which from the 15th century onward were compiled and published as “the art of counting” (German: *Rechenkunst*). The mathematics of antiquity was, from the very start, a technique for forming the soul, and more specifically for turning it away from the sensibly given to higher realms of eternal truth (e.g. Burnyeat, 2000). It was revived in the 16th century and then in various ways incorporated into doctrines of religion and philosophy (Funkenstein, 1986). The art of counting was a set of techniques for handling practical affairs. They were, it seems, acquired by methods similar to those currently often described as traditional, i.e. by explanation, memorization and drill (see e.g. the instructions given in Andersson, 1830, pp. 85-86).

During the course of the 18th century (in Sweden) the mathematical science of antiquity and the art of counting merged to form the school subject we today indentify as mathematics education (Lundin, 2008, pp. 158-174). The idea of formation and turning of the soul was taken from the mathematical science of antiquity but changed to serve a completely different goal. Instead of turning the soul away from the sensibly given, mathematics was increasingly seen as a tool to harmonize the human intellect with reality. Mathematics was incorporated in a movement in which the idea of *bildning* (German: *Bildung*) was central, meaning the formation of the human as a whole, to correspond simultaneously to God and to God's creation, i.e. physical and social reality [6]. It was in this phase that the teaching methods of the art of counting was forcefully rejected.

While this happened in schools which would in our time approximately correspond to secondary or late primary level, something else happened in relation to younger children, which also fits this general picture. Near the end of the 18th century, *åskådlighet* (German: *Anschaulichkeit*, approximately meaning "visual perspicuity") was given a central place in emerging practices of education. The idea was similar to the one proposed in relation to older students: that mathematical concepts corresponding to the inner (Godly) truth of reality were to take form in the children by their confrontation with reality itself, cleverly arranged so as to make this inner truth inescapably perceptible (Lundin 2008, p. 239-269).

Since the last decades of the 19th century, use-value and scientific understanding has replaced *bildning* as the primary goal of mathematics education. However, *this did not lead to any dramatic changes in teaching practice* (even though other mechanisms did). As regards the quite obvious shift in discourse following the formation of the scientific fields of developmental psychology and eventually also *matematikdidaktik*, it is surprisingly easy to retranslate the new terminology back into the wordings of the 19th century [7]. The remaining and central idea is that the only path to mathematical knowledge goes through sustained confrontation with cleverly arranged reality – represented physically or in the form of pseudo-realistic exercises.

One might wonder where this path leads if, as I claimed in the previous section, the scientific point of view towards which it is aimed does not exist. My answer is that it leads to belief: in the purposefulness of the path itself and in the existence of the kind of generally applicable mathematical knowledge to which it purports to lead. How this belief is in practice produced has been studied by the British sociologist Paul Dowling, who talks about it in terms of *myths*, primarily the two myths of *reference* and *participation*. Simplifying, these myths basically say that you need mathematics to *understand* and *master* reality, respectively (Dowling, 1998). The structure and production of this belief also stand at the center of my dissertation (Lundin, 2008, p. 43-80). [8]

What I wanted to show in this section is how our conception of mathematics is intertwined with the actual practice of mathematics education. In the next section I will show that further, mathematics education only makes sense as an *imperfect* realization of the promise of mathematics. The jigsaw puzzle makes sense only as compromised by a "hole", near its center, supposedly corresponding to a missing piece containing quite a lot of crucial detail.

Critical reflexivity

According to the mainstream discourse of mathematics education, the importance of mathematical knowledge is indisputable. This knowledge has the form of mathematical concepts. These are related to general competencies which are in turn related to the higher goals of society. The one and only route to mathematical knowledge is through the time-consuming, revelatory and at the same time creative practice of concept formation. Unfortunately though, mathematics education does not in its current state lead all, or even most, of its students to mathematical knowledge. To the contrary, many students find the subject of mathematics boring and irrelevant. On an individual level mathematics education causes frustration, anxiety and contempt for mathematics. On the social level it contributes to the reproduction of hierarchical segregation.

Following the account given by proponents of mathematics education there are two reasons for this state of affairs. First, mathematics education was originally founded on a double misconception: one of mathematics and one of the child. It saw mathematics as a set of facts, procedures, formulae and proofs; it saw the child as a passive container that could be filled up with this knowledge through listening, memorization and drill. This origin constitutes a *tradition* which mathematics education has ever since tried to transcend. Second, the society of which mathematics education is a part has never been of much help: the resources provided for the work of change are always scarce and the history of mathematics education is full of misconceived political interventions hindering progress. What mathematics education needs, and in the extension what society needs as a whole, is thus competent and resourceful experts on mathematics education which, through hard and sustained work on the large social body of the education system, can finally make real change possible (for sources see Lundin, 2008, pp. 29-34).

Let me now approach this from another angle, and begin by coming back to what I have stated previously as regards the *goal* and *means* respectively of mathematics education: conceptual knowledge and concept formation. In the account above it becomes apparent that neither of them should be understood as *actualities* of mathematics education but rather as *potentialities*. Because of the properties of mathematics, i.e. the relation between conceptual knowledge and the structure of social and physical reality, mathematics education can *potentially* help society reach its higher goals. But this potential is not (yet) *actual* because something is *preventing* its actualization. And what is preventing it is, you could say, the imperturbable mechanism of the social: of mathematics education and of society as a whole.

My central concern here is precisely this potential around which mathematics education moves. It is, quite literally, a *matter of belief* and it is important to note that it is the comparison with this matter that makes the actuality of mathematics education appear as essentially defect. The potential of mathematics defines what mathematics education *should be able to be*. It defines an agenda for change. Just as important to note is that this also works the other way around: mathematics appears as a potential against the background of the failure of mathematics education. The two reciprocally determine each other – in a way which I claim fits perfectly with how Žižek describes the function of the sublime object of ideology. I would now like to point out what I believe to be an important social function of its dynamics.

Until the first part of the 20th century, schooling in Sweden carried out two quite distinct social functions (beside the always explicitly intended goals of teaching students how to count and making them think better). While the *folkskola* (public education for the lower classes) which emerged in the 19th century enforced social stability by naturalizing the dominated position of its pupils, the *läroverk* (public education mainly for higher social strata) did the same by naturalizing dominating positions. These functions were effectuated by the teaching of religion in the *folkskola* and classical languages in the *läroverk* respectively. The introduction of what we today call mathematics in both of these school-forms, from the middle of the 19th century, did not challenge this separation very much, since the subject was itself divided into basic arithmetic and geometry, with geometry, conceived as the only "consecrating" part of mathematics, being almost exclusively reserved for the *läroverk*.

This changed during the first part of the 20th century, as arithmetic and geometry were increasingly seen as mere aspects of the homogenous subject of mathematics. During the same period the two types of school, the *läroverk* and the *folkskola*, were merged to form, around 1970, the Swedish *grundskola* and *gymnasium* (institutions for primary and secondary education respectively). Similar social functions as those previously performed by different subjects in different school-forms were now to be performed with all students in the same type of school studying the same subjects (c.f. Rose, 1985, especially p. 128). This fact is, I claim, closely intertwined with the increasingly clear image, during the 20th century, of the education system as *essentially malfunctioning*, i.e. as an unfortunate actuality contrasted with the potential of its subject matters. It is through this posture of imperfection that it explains why the *usual* consequence of schooling is preclusion, despite the potential inherent in the subjects taught.

My point is that the reflexive critique of mathematics education contributes to the very constitution of the failed students lack of mathematical knowledge. It is not talked *about* but *made*, as an individual cause of their dominated position. This socially effective, and thus in a sense *present absence*, is exactly the sublime object of mathematics education. [9]

Conclusion

To the theory presented in the introduction has to be added a distinction as regards our belief in the missing piece. On one level, there is certainly a fiction of mathematics, fantastic at times, which plays the role of the missing piece. Important to note though is that many people, maybe most, do not believe in this fiction in the sense that they would, if asked as a personal question, assert that it was really ”the truth”. To the contrary, the normal state of affairs contains great amounts of personal doubt as regards the purported properties of mathematics. It is necessary to acknowledge a kind of double negation of the actually present: not only do we consider the most important piece of the puzzle to be wanting – we also tend to assume a skeptical distance towards any specific properties explicitly conferred to it. We are generally, in a word, *cynics* (Sloterdijk, 1988).

The point of the argument here is that the piece exists and has consequences nonetheless, because it is anchored in institutionalized practice. People act as if the fiction was true, at the same time as they on a personal level think that they know ”better”, whatever it may be that they think is closer to the truth. What they do not realize, is that the reality which they perceive as given, the jigsaw puzzle, is *framed* as one could say, by the fiction they believe themselves to, on a personal level, transcend. They do not realize how the very structure and sense of that reality is connected to it. One would here have to speak of a kind of unconscious belief, a fundamental socially instituted fantasy anchored in practice, supported by laws and regulations, that is largely independent of what anyone on a personal level thinks he or she ”believes in” [10]. Ideas of mathematics function as a sort of structuring principle of reality, not because it was there all along, but because we unknowingly put it there. We do not necessarily acknowledge it by what we say, but by what we do when we do what is expected of us, seemingly without any choice of doing otherwise.

Following Žižek, this common feeling of having to conform to a system that you can not really believe in is not, as one could think, a threat to its proper functioning. To the contrary, mathematics education could not function without our cynical distance towards it, because this is what puts our belief in the “proper” system – a system not quite as the one present but very importantly not too dissimilar from it either – at safe distance from actual experience. Our “insight” that whatever we hear and see, this is not *it* but just the stupid way things happen to be, functions as a kind of protective belt around our fundamental assumptions regarding the structure of reality. Thus, if we remove one present error of mathematics education, another will surface and that this process can go on indefinitely.

Coming back to the jigsaw puzzle, this means that *there is no missing piece*. Even though the term “mathematics” refers to an enormously rich variety of ideas, methods, algorithms, techniques and, if you like, institutions and practices, *it* does not contain the answer to the *these* problems: the problems of mathematics education and, on a deeper level, problems regarding e.g. democracy, economic growth, poverty and segregation. It is not that the erroneous idea has got hold of society that its problems can somehow be solved by mathematics. Rather, the very idea of mathematics as commonly conceived, should be understood as a *symptom* of the society which believes in it. The idea of a missing piece helps us make sense of our incomplete puzzle. Simultaneously it makes serious rearrangement of the pieces seem unnecessary.

Notes

1. I draw here mainly on Žižek (1989 and 1999) but see also (Laclau and Mouffe, 1985; Hacking, 2000; Sloterdijk, 1988, and Castoriadis, 1987). Other sources can be found in Lundin (2008, pp. 43-80). There are many ways to characterize sublime objects, and I do not here aim at generality but at clarification of my own use of the term.

2. Matematikdelegationen is a delegation that in 2003 was commissioned by the Swedish government to strengthen the subject of mathematics and the teaching of mathematics. Skolverket is the Swedish national board of education. Skolinspektionen is the Swedish schools inspectorate. NCM is a Swedish national resource centre for mathematics education, its main task being to support the development of mathematics education in preschool, school, and adult education. Quotations from these sources are translated by me.

3. Of course there are also many proponents of mathematics education that would disagree. My point here, though, is not that the given description fits what any specific individual mathematics education thinks, but that it (more or less) is what must be assumed to be true for the social institution of mathematics education to make sense. This is what I try to bring out in the last section of this paper, and it is theoretically related to the concept of the Big Other of Lacanian psychoanalysis, i.e. which "does not exist" except as "presumed to exist", which in the end means that it does in fact exist after all in the sense that it is socially effective (Žižek, 1999, pp. 56-57).

4. Gaukroger (2008, p. 3) writes: "Indeed, a distinctive feature of the Scientific Revolution is that, unlike other earlier scientific programmes and cultures, it is driven, often explicitly, by religious considerations [...]". Funkenstein (1986, p. 3) states in his introduction: "Never before or after [the 16th and 17th centuries] were science, philosophy and theology seen as almost one and the same occupation".

5. C.f. *Hög tid för matematik* by NCM (2001, pp. 46-47). Again it is necessary to point out that I do not think that this is what all or even most mathematics educators believe to be the case.

6. See (Taylor 1975, pp. 3-51) for a general account of this philosophical movement.

7. Compare Dewey (1980, pp. 46-47) and Piaget (1972, pp. 38, 54, 57, 63) with the references in Lundin (2008, pp. 239-269).

8. See also Walkerdine (1984 and 1988).

9. I would here like to mention the argument of Skovsmose (1994) as an illustration of the kind of positing of absence that I am talking about (Lundin, 2008, pp. 54-56). Skovsmose basically says three things. First that mathematics has a “formatting power” which makes it reasonable to conceive of material reality (in some cases and to some extent) as “frozen mathematics” (p. 43). Second, Skovsmose claims that you have to acknowledge this presence of mathematics to be able to understand and master physical and social reality (p. 26). Third, Skovsmose says that this presence of mathematics is “hidden” to anyone who lacks the proper training. This puts mathematics education in exactly the position I am talking about, as the institution which *should* provide students with knowledge that would put them in connection with the essence of reality, but which *does in fact not do this* (and therefore, of course, needs to reformed). What remains are students essentially *lacking* means to properly understand and master the world of which they are a part.

10. “It is here, at this point, that the distinction between symptom and fantasy must be introduced in order to show how the idea that we live in a post-ideological society proceeds a little too quickly: cynical reason, with all its ironic detachment, leaves untouched the fundamental level of ideological fantasy, the level on which ideology structures the social reality itself.” (Žižek, 1989, p. 30).

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An Immigrant Student's Identity Formation in a Swedish Bilingual Mathematics Classroom

Eva Norén
Stockholm University, Sweden

This paper challenges current understandings of multicultural and bilingual students in mathematics classrooms in Sweden. Bilingual students are often pre-defined as disadvantaged and bilingualism is constructed as an obstacle. But students' identity formation can be effects of agency and of participation in a variety of competing discourses available in a classroom. In a discourse where bilingualism is encouraged an immigrant student's ability to positively build upon opportunities in the mathematics classroom seems to enhance.

Introduction

One of the complex relations in multicultural mathematics classroom practices will be addressed in this paper, namely students' identity formation. Conceptions of identity can mainly be described from three perspectives; the psychological/ developmental with a focus on the individual; the socio-cultural with a focus on interactions between the individual, culture and society; and the post structural perspective where identity formation is described as a dynamic and unstable process of becoming, and neither as an individual nor a social phenomenon (Grootenboer, Smith & Lowrie, 2006). In this paper, I define identity in line with a post structural approach, as "created at the intersection of a multiplicity discourses, always crisscrossing each other" (Walshaw, 2007, p. 81).

Often multicultural and bilingual students in Swedish classrooms are constructed as disadvantaged and bilingualism is constructed as an obstacle in the classroom. Lack of Swedish cultural capital and lack of Swedish language competence are usual explanations to multicultural students' low achievements in mathematics (OECD, 2006). Runfors (2003) calls it lack of "Swedishness" where deficiencies within the students themselves or within their families are the main explanatory factor to students' low performances in school. Other views are that individual students' participation and performance in the classroom may be empowered or restricted by teachers (Toohey, 2000) and peer students (Sahlström & Lindblad, 1998). Both empowering and restricting processes can be viewed as effects of institutionalized discourse (Foucault, 1971; 1984). In a Swedish context such discourses are for example discourses about bilingualism, discourses about multicultural students and teaching of mathematics discourses. For multicultural and bilinguals students discourses about bilingualism may have an influence on their identity formation since their identities as bilinguals can be both empowered and restricted in school mathematics practices. And by the exercise of deficiency discourses teachers' may have low expectations on multicultural students' performances in mathematics. Stentoft (2007) formulates it: "In educational discourse students are often placed in predefined identities which are used to determine or forecast their performance in school and higher education and predict obstacles on the way" (p. 1597).

Aim for the paper and the setting of the scene

The aim for this paper is to explore how competing discourses may influence students' identity formation in bilingual mathematics classrooms. To do this I will report on two examples from a case study (or micro ethnography) within a larger four years (2004-2008) ethnographic study (Heath & Street, 2008). A male student, Amir, is involved in the examples, which are chosen because of the explanatory possibilities rising from interviews, observations, audio- and video recordings in one particular bilingual mathematics classroom. The analysis is set in relation to the larger four year study, where themes have been categorised and related to wider societal contexts such as public, traditional and intended discourses (Gee, 1999/2005).

Amir is participating in a group of ten students, eight girls and two boys. He had the same teacher in mathematics from sixth to ninth grade; I followed him and the group from eight to ninth grade (2005-2007) in compulsory school. In sixth and seventh grade Amir was taught mathematics in only Swedish, in eighth and ninth grade he was taught in Arabic and Swedish. Both Amir and his teacher have an origin from Iraq. The teacher has an Iraqi engineer degree and a Swedish mathematics and technology teacher degree.

Theoretical considerations

Boaler & Greeno (2000) note that "Learning of mathematics has traditionally been regarded as an individual, cognitive activity" (p 171), but, Boaler & Greeno continue, "students' learning of mathematics can be considered as a trajectory of participation in the practices of mathematical discourse and thinking /…/ participation in social practices is what learning mathematics is" (p 172). However learning mathematics can be viewed as more than participation; it also relies on students' self-positioning in the classroom (Solomon, 2007) and on how students identify themselves as mathematics learners. Therefore I depart from a Foucaultian perspective on identity. Foucault (1982) rejected the view on identity as a person's internal or set essence, he viewed a true self as a fiction. Instead the self is constituted by a continuing discourse in a shifting communication of oneself to others. The individual is constituted in and through a culture and as a person you are constantly changing. Identity is then a relational concept and a result of social interactions, negotiations and power relations (Selander & Aamotsbakken, 2009).

In other words, identity – or rather identities – is something we communicate and experience in interactions with others, it is not a fixed entity or a set of qualities, and it is a temporary and shifting construction. Identity formation is influenced by discourse but also by students' agency and their active participation in different discourses in the classroom. Students are constantly contributing to negotiations about what it means to be a mathematics learner, and to know and to do mathematics in a classroom. The discourses operating in a classroom are connected to networks of power relations and wider societal contexts which make possible different understandings of learners and of what learning mathematics can be.

According to Skovsmose (1994) students' "intentions-in-learning" is their reason for being involved or not in classroom activities. He sees students' foreground, not their background, as a resource for bringing intentions into learning. In this paper, I perceive students' intention-in-learning as part of their identity formation.

Competing discourses as analytical framework

A variety of competing discourses is available to a learner in a mathematics classroom, and can be resisted or complied. Lerman (2001) elaborates on how different positions "can be adopted by participants, an extreme being resistance, especially in 'coercive' practices such as schooling" (p. 104). The first example of pairs of competing discourses in the bilingual mathematics classroom are those who give voice to taken for granted discourse about bilingualism and those who give voice to research reported discourse about bilingualism. According to Lindberg (2002) there are many myths about bilingualism. The discourse is what people take for granted and is often reflected through media. Such discourse is that researchers do not agree on the advantages of using students' mother tongue in educational situations in a second language learning environment and that the use of mother tongue should have a negative influence on the learning of a second language. Contrary researchers agree on the significance of mother tongue for second language learning and the importance of mother tongue for bilingual students' achievement in school (Lindberg, 2002). It is also shown that a distinction between additive and subtractive second language learning mark the importance of the sociocultural conditions that characterize bilingual children's upbringing. Additive bilingualism represents a discourse where the languages complement and support each other; subtractive bilingualism represents a discourse where the second language is learned on the expense of the first language. The discourses give different implications for bilingual students' identity formation and learning. Moreover researchers report on bilingualism and its positive effects for cognition (Lee, 1996).

The second pair of competing discourses is traditional school mathematics discourse and intended school mathematics discourse (Persson, 2009; Björklund Boistrup & Selander, 2009). The traditional discourse that has been dominant for many years in Swedish schools is when a teacher is tutoring at the students' desks while the students are working individually in their textbooks. Students spend most of their time in mathematics classes working at their desks with mathematical tasks (Mellin-Olsen, 1991; Pettersson, 1993; Sahlström & Lindblad, 1998; Persson, 2009). The intended national mathematics curriculum on the other hand supports teachers to use inquiry teaching and laboratory type teaching in the school mathematics practices. The Swedish national curriculum is "reform-oriented" (Boaler, 2002) as it also encourages students' learning via communication and participation in mathematical discussions.

A discourse of social relations and a mathematical discourse are the third pair of competing discourses in the classroom. A mathematical discourse is operating when communicating mathematics, i.e. using the mathematical register in the classroom to be able to agree on the meaning of mathematical statements, as well as mathematical concepts and ideas. The social relations discourse is manifested through social interactions in the classroom. An example is when teacher and students discuss other things than mathematics. The discourse brings attention to what usually is considered as "noise" or "impossibilities" (Biesta, 2005; Valero & Stentoft, 2009) in studies of mathematics classroom interaction. Discourses taking the identity of multicultural students for granted reflected through media and peoples everyday practices may also be such that position students as "noisy immigrant students" or "noisy male immigrant students" in school contexts.

Instead of taking the identity of a student as a learner of mathematics for granted in the ongoing process of constructing identities, intentions for engaging in learning mathematics may go hand in hand with social relations to the mathematics teacher and peer students. When the male student that I focus on in this paper, Amir, was late to mathematics class the following interaction took place. The third pair of competing discourses was used in the analysis of the interaction.

"I am an engaged mathematics learner"

Amir enters the classroom a minute after the teacher has started the group working with algebra, she is writing on the white board. The nine students are listening while writing in their books. By Amir's entrance and agency the situation calls up an identity and positions Amir exercising a discourse of social relations with the teacher. Amir starts by saying in Swedish [1]:

Amir:	The physical education teacher is a racist! [the other students look at Amir]
Teacher:	And why do you think he is a racist? [in an ordinary voice]
Amir:	He will not let me pass […] I will not get a grade in sports! [very upset]
Teacher:	How come? Why don't you get a degree in sports? You like sports.

The teacher does not interpret Amir's lateness as an obstruction to learn mathematics and she responds to the comments from Amir, though the lesson on algebra has already started. She challenges his statement about the PE teacher being a racist. From earlier experiences she knows that Amir probably wouldn't drop the "subject" if she just ignores him. The teachers point is that she cannot let Amir call Swedish teachers for racists because he doesn't get the grades he wants (from interview with the teacher). When they have discussed it further for a short moment, in both Arabic and Swedish, Amir comes to the conclusion that the PE teacher cannot give him a pass grade, as he has not attended the sports classes that he should. The teacher then asks Amir why he calls the PE teacher a racist when the grade or not has to do with Amir's own decisions, not participating in the sports classes. Amir then says he hopes he can talk with the PE teacher about it and that he really not is a racist. He goes on (Arabic in italics [2]):

> I was just so mad when I understood I was not going to get a grade that I called him a racist! I will talk with him and maybe we can agree on me doing additional work. /…/ [saying something to himself, but inaudible,] */…/ I know I will get a good grade in mathematics though /…/ what are we doing today*?

Amir said he did not articulate directly to the PE teacher that he was a racist. "It is what I say about him to you", he told the mathematics teacher.

Amir's ability to deal with a problematic situation he has put himself into was handled by himself when he was giving a suggestion to solve the situation. The last utterance gives an indication of Amir positioning himself in the mathematics classroom as an engaged mathematics learner and becoming a good mathematician, as he does at other occasions as well. He chooses to engage himself in the mathematical discourse in the classroom by showing his intentions in learning mathematics. The rest of the lesson Amir answered questions from the teacher and put questions to the teacher himself in a mathematical discourse. He also showed his intentions by working intensively together with two girls in the group on worksheets repeating algebra, to be prepared for a test next week.

It is not possible to tell what would have happened without the teacher's response to Amir, but in this situation it seemed to be a response that supported Amir's aspirations to be good in mathematics. At this occasion Amir, with interactive support from his teacher, chose to be in a social relations discourse for a short while, and then moving on to a mathematical discourse. The teacher responded to Amir both as a listening adult caring about what he was saying and about his behaviour, calling a colleague of her a racist. She also enacted a discourse of becoming a certain kind of person, and Amir becoming a responsible young citizen taking responsibility for his own actions, a discourse promoted by the Swedish curriculum as a teacher should ”clarify and discuss with the pupils the basic values of Swedish society and their consequences in terms of individuals action” (Skolverket/National Agency of Education, 1994 p. 9). Also the school “should strive to ensure that all students develop a confidence in their own ability” (Skolverket, 1994 p. 9).

In the next section, the first and second pair of competing discourses was also used in the analysis of classroom interactions.

Social relations wherein bilingualism is not an obstacle

In a Swedish context bilingualism in school is categorized as an obstacle (see for example Runfors, 2003; Haglund, 2005), except when English is involved (Lim Falk, 2008). Though the official discourse promotes bilingualism, the dominant public discourse denies it. In the classrooms in my larger study where mathematics was taught bilingual, bilingualism was contrary not an obstacle. Two languages were used for teaching and learning mathematics in a discourse promoting additive bilingualism, as the students activated and learned mathematical concepts and procedures in both Arabic and Swedish. In the particular classroom addressed in this paper, the teacher and the students at the end of ninth grade were evaluating their four years together.

Amir asks the teacher if she remembers sixth grade and tell her that nowadays “you do a lot more tutoring to the whole class, then in sixth grade /…/ and we work a lot more together and talk about [mathematical] stuff. You explain when we ask about it /…/ even though you know we know”. He says he thinks it is because they all speak the same languages, Arabic and Swedish. “It is more relaxing to be able to use Arabic as well, to use both languages”. The teacher agrees with Amir and says it was not easy to teach in two languages when they started up doing it. The students have “helped” her along the way, she says, by asking more and more questions, interacting and showing their confidence in the classroom. She was also hesitating to use both languages to start with, as she was “worried about their [the students’] improvement in Swedish”, she says. Amir says the teacher has become a better teacher over the years, “though she was good from the start as well”. To Amir the most important is that “everybody can say what he or she wants in this group and that we learn from each other”.

The teacher's hesitation to use Arabic demonstrates the competition between the institutionalized discourse that works normalizing towards "Swedish only" and "Swedishness" (Runfors, 2003), a taken for granted discourse about bilingualism and a discourse that promotes bilingualism (Lindberg, 2002). There have also been struggles between the institutionalized traditional school mathematical discourse, when students work a lot by themselves at their desks, and the more reform-oriented school mathematical discourse, promoted by research and the curriculum. The acceptance of bilingualism as a resource for the teaching and learning of mathematics seems to have enhanced the reform-oriented school mathematical discourse in this classroom. The acceptance of bilingualism and the social relationships between the teacher and Amir as well as the reform-oriented school mathematical discourse seem to assign him a positive position in the classroom. Though he from time to time performs like a "noisy" student he positions himself as an engaged mathematics learner and positively builds upon the opportunities in the classroom.

Students' identities

Amir's alternation between the identities as a "noisy multicultural student" calling a Swedish teacher a racist, and an "engaged mathematics learner" are influenced by the crisscrossing of competing discourses in the mathematics classroom. As there is space for social relations and negotiations between teacher and students in this particular classroom Amir has the possibility to negotiate about his identities. In contrast to earlier research in multicultural classrooms in Sweden, where institutionalized classroom discourse often implicitly works normalizing towards "Swedish only" and "Swedishness", Amir is not normalized towards Swedish only and Swedishness. In contrast to Parszyk's (1999) study where multicultural students experienced school was not for them but for others [the Swedish students], Amir out of his experiences, acknowledged school was for him. Through the discourses available in this particular classroom he has the possibility to show his intentions-in-learning and to format an identity as an engaged mathematics learner and an identity as bilingual too.

However Amir is also, from time to time, a "noisy" student in the mathematics classroom. It is not a deficiency within Amir, his family or his cultural background that makes him a "noisy" student, but the discourses available to him in school. Through the discourses available he might feel "comfortable" in the mathematics classroom, but we do not know about how discourses work in other school locations, except from what Amir is saying and doing in the mathematics classroom, in informal talks and in interviews. An assumption based on his absence in PE classes, as well as on a formal interview and informal talks between Amir and me, could be that Amir is taking a resisting position in the PE classroom, towards certain institutionalized discourse normalizing multicultural students towards “Swedishness”.

Concluding reflections

Discourses shape and organise what teachers and students can say and do in the mathematics classroom. At the same time discourses are not static; they change and are not lasting over time. There is space for teachers’ and students’ agency in this classroom, and their agency exercise different discourses. The variety of discourses available to Amir is part of social structures of the classroom. The ways in which Amir is accepting or opposing those discourses format his identity of becoming an engaged mathematics learner, a bilingual individual, or a “noisy” immigrant student. His attempts to construct his identity as an engaged mathematics learner is in opposition to discourses which determine immigrant students’ school performance as low. Amir’s ability to positively build on the opportunities in the bilingual mathematics classroom activates a foreground of Amir. His image of himself as an engaged mathematics learner and his intentions-in-learning mathematics are interwoven with his actual learning of mathematics.

The advantages of speaking the same languages and having the same cultural experience in a learning situation, is one but not a single explanatory factor for Amir's potential to become and identify himself as an engaged mathematics learner. This is in a discourse that promotes bilingualism.

In Sweden, the dominant discourse has been to normalize bilingual students towards “Swedish only” and “Swedishness”, to “make” the students Swedish and to “take away” the disadvantage of not knowing Swedish and not having “the right” or “enough” of the “right” cultural capital. The normalizing strategy is also in line with an assumption that it is easier to teach and learn when teachers and students speak the same language. But this is usually done in a normalizing discourse that promotes monolingualism – Swedish only. What is lost in a monolingual instructed classroom is that language and identity are closely related to identity formation (Toohey, 2000). Multicultural and bilingual students speak and use more than one language, and belong to more than one culture. Those conditions are acknowledged in the bilingual mathematics classroom, but have to become acknowledged also in monolingually instructed mathematics classrooms. The dominant discourses in Sweden about bilingualism may counteract that fact.

Multicultural students in the mathematics classroom can no longer be placed in pre-defined identities (Stentoft, 2007) as “immigrant students” which are used to decide their performance in school or predict obstacles on the way.

Notes

1. My translation into English.

2. A bilingual teacher helped me translate from Arabic to Swedish, my translation into English.

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Broadening the Role of Theory in Mathematics Education Research

Alexandre Pais, Diana Stentoft and Paola Valero
Aalborg University, Denmark

We intend to engage in a theoretical discussion on the limitations that the concept of learning imposes to mathematics education. After a brief analysis based on recent key publications dealing with the issue of theory in mathematics education research, we show how the concept of learning configures the theoretical framework of the field. This leads us to develop a critique that aims at highlighting the dimensions that are lost when we think of education exclusively as a process of learning. As a result of this critique, we argue for a broader theoretical approach to mathematics education that takes into account the social and political dimensions involved in that process. [1]

Introduction

The issue of theory in mathematics education research is currently on the agenda. At the *11th International Congress on Mathematics Education* (ICME11) one of the survey teams was responsible for developing a study on the notion and role of theory in mathematics education research. This survey team had the task of identifying, surveying, and analysing different notions and roles of 'theory', as well as providing an account of the origin, nature, uses, and implications of specific theories pertaining to different types of research in mathematics education. The *Second handbook on mathematics teaching and learning* (Lester, 2007) contains two articles dealing with the issue of theory: "Putting philosophy to work: Coping with multiple theoretical perspectives" (Cobb, 2007), and "Theory in mathematics education scholarship" (Silver & Herbst, 2007). In the Congress of the European Research in Mathematics Education (CERME) there has been a working group dealing with the problem of linking, contrasting and comparing the wide variety of theoretical approaches found in the field to tackle the teaching and learning of mathematics. The second number of the fortieth volume of the international journal *ZDM* was published in 2008, collecting some of the results of the work of this group. Finally this year the theme of the *33rd Conference of the International Group for the Psychology of Mathematics Education* (PME) is "In search for theories in Mathematics Education". There seems to be a widespread desire for understanding the role of theory in mathematics education research:

> The moment seems propitious for a serious examination of the role that theory plays and could play in the formulations of problems, in the design and methods employed, and in the interpretation of findings in education research. (Silver & Herbst, 2007, p. 41)

We wish to make a modest contribution to this discussion by engaging in a critical discussion on the way theory is used in mathematics education research. The educational sciences are generally construed around concerns of providing research that informs practices of learning and teaching in educational institutions. This research shows a propensity to emphasize the "technical" aspects of education, being primarily concerned with providing solutions to practical problems (Biesta, 2005). We will argue that mathematics education as a research field is not an exception. Thus we wish to understand in more detail how research perspectives in general and theoretical perspectives in particular construct and/or ignore particular discourses and, in this, our possibilities for addressing these basic yet powerful questions. In what follows we briefly present the groundings of our analysis. We will then address the texts mentioned above and show how they construct issues pertaining to theory in mathematics education research. Then we will refer to the problems that this type of construction leaves unattended. Drawing inspiration from the work of the philosopher of education Gert Biesta and on current socio-political research in mathematics education, we argue for an opening in the understandings of theory in the field.

Our theoretical and methodological standpoint

As the "linguistic turn" in the social sciences has touched mathematics education research (Lerman, 2000), it appears increasingly important to pay attention to the discourses that mathematics education research constructs about itself and the contributions and limitations of these constructions. By discourses here we understand the ways of naming and phrasing the ideas, values and norms that emerge from the constant and complex interactions among human beings while engaged in social practices. Academic fields construct particular discourses about themselves and their objects of study. Such discourses constitute systems of reason that regulate what is possible to think and do in a given field (Popkewitz, 2004). Discourses thus both generate a space for possibilities as well as limitations of what we can imagine and construct as alternatives to existing orders.

Mathematics education as a field of research is not an exception. As researchers engage in studying the field, they not only define what is characterizing legitimate practices of mathematics education. They also define the ways in which it is valid and legitimate to research those practices. We have elsewhere engaged in examinations of the discourses generated in and by the field of mathematics education research, such as the idea of mathematics education being "powerful" (Christensen, Stentoft, & Valero, 2008), the conceptions of students as mathematics learners (Valero, 2004), the concept of learners' identity in mathematics (Stentoft & Valero, 2009) and the concept and view of ethno-mathematics (Domite & Pais, 2009). In this paper we examine the construction of "theory" and its implications for mathematics education research. In our investigation we point to some of the blind spots of theoretical construction and try to broaden the theoretical horizon as to embrace aspects of mathematics education which we are constantly witnessing as fundamental parts of the every-day practices of teachers, parents, students and school leaders in real schools. In addressing these blind spots we intend to enlarge the research gaze of mathematics education to embracing the "noises" that are often ignored in mathematics education research, in a search for new possibilities for our field of study and for the educational practices in mathematics (Stentoft & Valero, in press). This we see as the role of a critical, socio-political approach to research in mathematics education.

We are aware that the "social turn" (Lerman, 2000) in mathematics education brought to the field new concerns and new theories that progressively de-emphasise cognitive psychology as the only interpretative framework and instead favour socio-cultural theories. In this we have witnessed a move from an understanding of children's learning focused on the individual subject and his/her cognition to an understanding that perceives learning as a product of social activity, where not only the cognition of the subject is at stake but also his/her relations with other individuals and their shared discourses. Some of the research bearing social, cultural and political connotations, has opened up the field of mathematics education by conceiving theory as more than "theory for learning", and posing questions that do not imply a "technical" response or solution but rather an intellectual and philosophical reflection.

However, despite this invigorating openness, we argue that a significant part of research in mathematics education labelled socio-cultural-political research shows a tendency to understand mathematics education in a didactical sense and to aim primarily to address questions of how: How to teach in multicultural classrooms? How to teach for social justice? How to educate teachers for social justice? How to integrate immigrant students in the learning of mathematics? How the socio-cultural contexts of students influence the learning of the concepts of chance and probability? These questions were found in the proceedings of the Mathematics Education and Society, MES conference in Albufeira, Portugal in 2008 (Matos, Valero, & Yasukawa, 2008), and shows how even in a research environment where the emphasis is on the political, the research persists on the question addressing the technicalities of the field. In Pais, Stentoft and Valero (2010) we developed an analysis that shows how even in a research environment characterized by a social, cultural and political approach, a concern with the technicalities of education, namely with "learning", persists.

In this paper we will rather focus on the mainstream research in mathematics education. Therefore we selected for our analysis some recent and mainstream publications in mathematics education research addressing the issue of theory, the collection of contributions emerging from the ICME11 survey team and the *Second handbook on mathematics teaching and learning*. We will show that the central idea in these texts is that the understanding of theory in mathematics education is mainly reduced to that of learning theory. This trend is not exclusive to the field of mathematics education research, but has over the last two decades also proliferated in broader discourses of education. The language of education has largely been replaced by a technical language of learning (Biesta, 2005). The contradictions on the role of school and the goals of education that fuelled part of the educational debate during the last century seem to have been surpassed. We seem to have reached a consensus on the benefits of schooling, we need to make it more effective and, therefore, we live an apparent consensus in what concerns education. The problems with schooling and school subjects are not anymore to be political or ideological, but have become primarily technical or didactical. In most cases, solutions to educational problems are being reduced to better methods and techniques to teach and learn, to improve the use of technology, to assess students' performance, etc. Education has progressively been reduced to be a controllable, designable, engineerable and operational framework for the individual's cognitive change. Although the prevalence of theory as "learning theory" has allowed us to gain deeper knowledge on the processes of teaching and learning mathematics, we suggest that it has left important problems faced by the educational communities in their everyday practices unaddressed. We will argue that in order to bring these problems seriously into the gaze of research, we need a broader theoretical frame which allows us to understand theory not just as "theory of learning", but also as "theory of education".

Mathematics education as a learning science

There seems to be a consensus that the main concern of mathematics education research is to improve students' performance in mathematics. Niss (2007) is very clear when answering the question of why do we do research in mathematics education:

> We do research on the teaching and learning of mathematics because there are far too many students of mathematics, from kindergarten to university, who gets much less out of their mathematical education than would be desirable for them and for society. (Niss, 2007, p. 1293)

If this is the main concern of mathematics education research, it is not surprising that the field has been designed as a space for researching "the problems of practice" (Silver & Herbst, p. 45), defined as problems relating to teaching and learning, in a systematic, scientific way. According to Boero (in press) "this is a rather obvious widely shared position" (p. 1). In this framework, the work of mathematics educators is:

> To identify important teaching and learning problems, considerer different existing theories and try to understand the potential and limitations of the tools provided by these theories. (Boero, in press, p. 1)

Cobb (2007) addresses the issue of philosophy in mathematics education. The author makes it clear that although the invitation to write the article was to focus on the philosophical issues of mathematics education, he decided to engage on a personal perspective rather than developing an exhaustive overview of currently philosophical ideas in the field. However, the fact that it is the article dealing with philosophy in one of the most significant publications in mathematics education is, in our opinion, symptomatic on how the majority of researchers understand the role of philosophy in relation to mathematics education. Cobb suggests that mathematics education should be understood as a "design science" (2007, p. 7), and provides as an example the NCTM standards. By design science Cobb understands "the collective mission which involves developing, testing, and revising conjectured designs for supporting envisioned learning process" (p. 7). The ultimate goal of a science designed this way will be to "support the improvement of students' mathematical learning" (p. 8). Under the pragmatic realism philosophy adopted by Cobb a substantial part of his article focuses on the comparison between four significant theoretical perspectives used in mathematics education research (experimental psychology, cognitive psychology, socio-cultural theory and distributed cognition). The discussion revolves around how these theoretical perspectives could help improving students' learning of mathematics. We can research at the level of the national educational system, school or classroom, but the ultimate goal remains the same. Theory is understood as a tool to give insight and understanding into learning processes.

An alluring analogy made by Silver and Herbst (2007) between mathematics education and medicine helps us to better understand the meaning of theory as "theory for learning". The authors place mathematics education as a science of treatment, and by understanding the symptoms that characterizes the difficulties of students' learning of mathematics we can propose the proper treatment:

> The evolving understanding of the *logic of errors* has helped support the design of better instructional treatments, in much the same way that the evolving understanding of the *logic of diseases* has helped the design of better medical treatments. (Silver & Herbst, 2007, p. 63)

In this perspective, students are seen as patients in need of treatment, and the role of mathematics education is to understand students' problems and elaborate designs that threat those learning diseases.

Problems left behind

After describing the decadent state and lack of minimal conditions (broken windows, holes in the roof, full exposure to climate change) in many South African classrooms serving mainly black African students, Skovsmose (2004) asks a very innocent question: "What seems to be the most obvious learning obstacle to the children in this school: their colour of skin, their dominant father, or the hole in the roof?" (p. 35). According to Skovsmose, the hole in the roof, absent from the majority of mathematics education research, calls any deficiency theory of the child into question. Which kind of treatment "for learning" should mathematics education and mathematics education research deploy in this case?

Schmitz (2006) analyzed the way six mathematics teachers developed and interpreted their practice in a typical Brazilian school. They represent what we can consider traditional, unqualified teachers. They use the textbook as the main resource in their classes; confine the teaching to the specific content of the official curriculum; do not incorporate students' culture in the learning process (at best they mention anecdotal examples with the purpose of illustrating the true knowledge of the curriculum); and perform the traditional way of teaching mathematics. Students are on their seats listening, and the teacher at the black-board speaking. They go through lots of exercises. The teachers do not invest in their classes. In an interview one of the teachers drops the veil: she would like to use the textbook less, and build her own materials; but unfortunately she has to work in two different schools in order to have two salaries to pay all her needs. Working more than twelve hours a day to earn less than three thousands *reais* [2] per month does not leave time or energy to prepare her classes. How can the research in mathematics education address the problems of these teachers? [3]

These are only two examples of the problems that are predominately un-addressed by research in mathematics education. A reason for this lack of attention to these problems could be found in the language theories offer to the research field, namely a language concentrating on issues of learning and lacking possibilities to properly formulate questions to addressing socio-political aspects of mathematics education. They are what we call, inspired by the work of Valero (2004), socio-political problems emerging by the way in which the social practices of mathematics education are part of the larger social, political, economic, cultural and historical frameworks for education, schools and classrooms. To understand these problems we need to start questioning not just how students learn mathematics, but also why do students learn mathematics. And we need to construct a language through theory that allows us to move beyond research into processes of teaching and learning mathematics.

So, why do students learn mathematics? Vinner (2007) argues that students have very good reasons to study mathematics. However, these reasons are not related to the common aims associated with mathematics education (utility, professional, or to educate critical citizens in an increasingly *mathematized* world). The good reasons students have to learn mathematics are rooted in "the selection role that mathematics has in all stages of our educational system" (p. 3). Thus, the predominant reason for students to learn mathematics is to become winners in a world where mathematics is a gatekeeper in accessing further studies and well paid jobs. Indeed, they need mathematics to participate in this world, not directly with the knowledge of school mathematics, but in the form of a diploma, as a schooling valorisation, first, and then professional. They need to learn mathematics to become economically relevant and revealed. This seems to be the case when students of different socio-economic backgrounds are asked about their reasons to engage in the learning of mathematics (i.e., Skovsmose et al., 2008a, 2008b).

The past twenty years have seen a remarkable rise of the issue of measurement and testing in education. The mass scale comparative studies as the *Trends in International Mathematics and Science Study* (TIMSS) and the OECD's *Programme for International Student Assessment* (PISA) represent the most prominent manifestation of this phenomenon. These international, comparative, measurement studies are to an increasing extent brought into the political sphere placing pressure on national governments to regulate their educational systems according to the standards stipulated by those tests (Biesta, 2009; Wilson, 2007). This is what has been happening in the last eight years in very many developed countries where education tends to be transformed, by the pressure of politicians' demands for accountability, into an evidence-based profession. Consequently, political measures contribute to formatting teaching and learning of mathematics in a clear and crude way. Teachers tend to tailor their instructional practices to the format of the test out of concern that if they design their teaching differently, their students will fail. Although they might know all the didactical novels tricks and methods to promote learning in a way meaningful to the students, if what counts is to pass the test, that is how they will 'educate' their students (Wilson, 2007; Lerman, 1998).

The idea that we should take seriously the political and social context in which teachers work and students learn has been growing in the last years. Covaleskie (1993) argued that the institutional arrangements, in ways no one quite seems able to pin down, make even the most able and intellectually capable of teachers to tone down their teaching to the level of the approved curriculum materials. De Freitas (2004), drawing on the work of Michael Apple, Basil Bernstein and Thomas Popkewitz, made it clear that any attempt to reform teacher education programs will dash against the "insidious structural elements operating normatively in maintaining 'common sense' practices within schools" (p. 259). The question remains whether it pertains to mathematics education research to address these issues, and how to construct a theoretical language that goes beyond theories of learning, which offers a scope for this kind of research.

Broadening the horizons of theory: From learning back to education

As we argued at the beginning of this article, the ultimate goal for mathematics education appears to be improving students' mathematical learning. The idea of mathematics education as a therapy or a design science described previously constructs education as a technological endeavour, where mathematics education is understood as a technical engineering of students' mathematical thinking and learning. We acknowledge the contributions that this learnification has brought to our understanding of what happens in a mathematical classroom at a micro-scale. Nevertheless, we argue that reducing the possible meaning of "mathematics education" to "mathematical learning" can narrow our perspectives. And thus it becomes impossible to think and act in ways that could open spaces of possibilities inside and outside mathematics education research. Cobb (2007) is well aware of this. When referring to the theory that informs the researcher he mentions that "the constraints on what is thinkable and possible are typically invisible" (2007, p. 7). This awareness also emerges strongly in much research and it is clear that addressing mathematics education from a narrow perspective pointed out here, reconfirms the fact that "if we look strictly at events as they occur in the classroom, without consideration of the complex forces that helped to shape those learning conditions, our understanding is only partial [and] the solutions to the problem [are] ineffectual" (Rousseau & Tate, 2008). Very few researchers, however, have addressed these limitations in serious ways. In the remaining lines we wish to sketch a modest contribution to understand theory differently in mathematics education, by focusing on the field of teacher education.

Part of the problems felt by teachers in their profession are not only problems related to finding better ways to improve students' mathematical learning from a micro-didactical perspective. They are sociopolitical problems that constitute part of the embeddedness of teachers' practices in a socio-political context. These problems are not present in the "cognitive subject" (Valero, 2004) or in the "prototypical classrooms" (Skovsmose, 2005) that dominate discourses of mathematics education research. These problems are obliterated, and made invisible. They are excluded through the application of 'orderly' research methods and theories that need "sanitized" environments (Vithal & Valero, 2003). But those are problems that real teachers face in their real everyday activity. The problems can sometimes make their teaching skills seem obsolete. We therefore appeal for a teacher education that makes these problems visible to the teacher so he can act and react appropriately, critically and responsibly. As Vinner (2007) pointed out, we as researchers and educators responsible for teacher education have to tell our teacher-students the truth about the system in which they operate. This implies confronting the future teachers with more than desirable and visionary goals of mathematics thinking and learning, by making the real importance and the role of mathematics in school explicit. School cannot be taken for granted. It needs to be critically questioned as an institution capable of promoting new opportunities as well as stratifying children in social hierarchies by stipulating who is capable, who will have access to the best university courses, and who will enjoy well paid jobs. This implies going beyond the idea that mathematics teacher education consists only on giving teachers the didactical tools necessary to enhance students' mathematical learning. Teachers need to know about the social and political context in which this enhancement occurs, to understand which their contribution in the school is and to their students and which are the limits of their ability.

In order to address these problems, teachers and researchers need alternative discourses. If the main discourse in mathematics education is a didactical one, it is not possible to formulate these problems adequately. The discourse of mathematics education from a critical and socio-political perspective can offer an alternative. Any discourse will of course have blind spots. So our point is not to simply replace a discourse of learning for a discourse of education, but rather to make them both available as competing and complementing discourses available to researchers and practitioners to engage and explore the otherwise limited possibilities that the dominance of only one may offer.

Notes

1. This paper is part of Alexandre Pais's PhD study, supported by the Foundation for Science and Technology of Portugal, grant SFRH/BD/38231/2007.

2. Brazilian currency.

3. The research developed by Schmitz, contrary to what it may seem, doesn't take these aspects into consideration. They are treated as marginal political vicissitudes of teachers' life.

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Konflikt mellan vision och möjlighet hos blivande lärare i matematik

Hanna Palmér
Linnéuniversitetet, Växjö, Sweden

This paper proceeds from a research project investigating the professional identity development of novice primary mathematics teachers. The focus of this paper is an expected conflict between visions and possibilities that the primary teachers express just before graduation. The conflict is illustrated with quotes from two primary teachers in the study and questions are raised regarding how this expected conflict will affect their identity development as primary mathematics teachers.

Inledning

Detta paper utgår ifrån en studie med syftet att undersöka hur nyexaminerade lärare för de tidiga skolåren utvecklar läraridentitet med fokus på matematikundervisning. Det finns studier där uppfattningar, kunskaper och attityder gällande matematik och matematikundervisning hos lärarstudenter (t.ex. Frykholm, 1999; Gellert, 2007; Mcleod, 1994; Pajares, 1992; Persson, 2006) och verksamma lärare (Mcleod, 1994; Pajares, 1992) har kartlagts och klassificerats. Likaså finns studier där relationer mellan uppfattningar, kunskaper, attityder och undervisning har undersökts (t.ex. Hill m.fl, 2008; Mcleod, 1994; Pajares, 1992). Gemensamt för dessa studier är att de främst är framställda i ett observatörsperspektiv i syfte att kategorisera, förutsäga eller förklara. Andra studier har sökt förståelse för individers uppfattningar, kunskaper, attityder och handlande utifrån individens perspektiv (t.ex. Ensor, 2001; Skott, 2001) och det är ur detta perspektiv som studien i detta paper genomförts.

En primär forskningsfråga i studien som detta paper utgår ifrån är vilka aspekter som är centrala i identitetsutveckling som matematiklärare. Den första datainsamlingen genomfördes via intervjuer strax före lärarnas examen. I dessa intervjuer uttryckte de en förväntad konflikt mellan visioner och möjligheter som nyexaminerade lärare. Denna konflikt och dess betydelse för identitetsutveckling som lärare i matematik är fokus i detta paper. I första delen presenteras det perspektiv på identitet och identitetsutveckling som används i studien. I andra delen fokuseras den uttryckta konflikten.

Identitet och identitetsutveckling

Det finns inte en enhetlig definition av begreppet identitet och till viss del förklaras olikheterna av definitionernas olika ontologiska utgångspunkt. Både Lave och Wenger (1991), Wenger (1998) och Sfard (2008) placerar identitet i skärningspunkten mellan det sociala och det individuella och menar att identitet är ”man-made and […] constantly created and re-created in interactions between people (Sfard & Prusak, 2005, p. 15)”. Denna placering av identitet används i studien. Att studera identitet utifrån sociala termer innebär inte att det individuella förnekas utan att individualitet ses som en del av en praktikgemenskap (community of practice) (Wenger, 1998).

I studien behandlas både identitet och identitetsutveckling som matematiklärare. Enligt Sfard och Prusak (2005), Lave och Wenger (1991) och Wenger (1998) är lärande och identitet tätt sammanbundna med varandra. Lärande handlar om att konstituera identitet och identitetsutveckling innebär lärande då lärande formar vår identitet genom att förändra vår förmåga att delta, tillhöra och förhandla mening i världen. Därmed är identitetsutveckling ett sätt att tala om hur lärande förändrar vilka vi är och den vi blir genom deltagande i olika praktikgemenskaper.

Wenger (1998) beskriver en praktikgemenskap som relationer mellan personer, aktiviteter och världen, en delad lärandehistoria. En praktikgemenskap behöver inte innebära en väldefinierad grupp eller socialt synliga band mellan deltagarna i praktikgemenskapen. Istället innebär praktikgemenskaper ett deltagande i ett aktivitetssystem kring vilket deltagarna har sin förståelse rörande vad de gör och vad det innebär för deras liv och denna praktikgemenskap. En praktikgemenskap existerar över tid och olika praktikgemenskaper överlappar varandra. Eftersom en individ deltar i, alternativt utesluts ur eller tar avstånd från, olika praktikgemenskaper innebär identitetsutveckling också att förlika dessa olika medlemskap (memberships). En nyutbildad lärare kan tillhöra flera olika praktikgemenskaper inom sin lärarroll, tillexempel olika grupperingar inom lärarutbildningen med lärare och studiekamrater i olika ämnen men också praktikgemenskaper med verksamma lärare på olika grundskolor där de har gjort VFU, sökt eller fått sina första arbeten. Dessa olika tillhörigheter kan ge upphov till spänningar inom individernas identitet vilket gör att de får förhandla sin identitet mellan praktikgemenskaperna. Att göra detta kan enligt Wenger (1998) vara den svåraste utmaningen för en individ som flyttar mellan olika praktikgemenskaper.

Att examineras som lärare och börja arbeta innebär inte enbart en förhandling mellan olika praktikgemenskaper utan även en förändring inom praktikgemenskaper. Lave och Wenger (1991) menar att en praktikgemenskap genomgår utveckling när nykomlingar får tillträde och utvecklar fullvärdigt medlemskap vilket handlar om att få tillgång till innehållet i praktikgemenskapen. På så vis pågår en ständig utveckling både hos individen och i praktikgemenskapen. Att ge nykomlingar legitimt perifert deltagande i en praktikgemenskap innebär på så vis en spänning mellan kontinuitet och förändring i praktikgemenskapen.

För att kunna undersöka tidigarelärarnas identitet och identitetsutveckling empiriskt, ur deras perspektiv, måste begreppen göras operationella. Sfard och Prusak (2005) kopplar identitet till kommunikation inom vilken de även inkluderar egen-dialog, dvs tänkande. Utifrån denna koppling definierar de identitet ”*as a set of reifying, significant, endorsable stories about a person* (p.14). Förtingligandet i berättelserna innebär att vi, när vi pratar om oss själva, har en tendens att prata i faktiska tillstånd snarare än i handlingar. I berättelserna fryser vi intryck gällande aktiviteter i olika praktikgemenskaper vilket gör att vi får ihop flödet av förändring i en samlad berättelse om oss själva. De möjligheter som tidigare fanns i handling blir istället möjligheter hos den som handlar. På så vis är berättelserna, identiteten, skapande av handlingar i flera olika praktikgemenskaper och de förändras utifrån författarens och mottagarens perception och behov. De berättelser som enligt Sfard och Prusak (2005) är identifierande berättelser och som används i denna studie ska också vara signifikanta och trovärdiga utifrån individens perspektiv.

Sfard och Prusak (2005) urskiljer olika typer av identitetsberättelser utifrån vem som berättar dem och för vem de är avsedda att berättas. Första-persons-berättelser är identitetsberättelser berättade av individen själv. Andra-persons-berättelser är identitetsberättelser berättade till individen av någon annan. Tredje-persons-berättelser är identitetsberättelser om individen berättade av en andra part för en tredje part. Det är enbart första-persons-berättelser som kvalificerar sig som de förtingligande, signifikanta och trovärdiga berättelser som används i denna studie. Eftersom dessa berättelser är kollektivt skapade i olika praktikgemenskaper är de dock påverkade av andra-persons- och tredje-persons-berättelser. De identitetsberättelser som lärarna berättar i studien är påverkade av identitetsberättelser de hört av andra, både om dem själva och om andra individer i deras situation.

Identitetsutveckling i studien är kopplat till första-persons-identitets-berättelser av två olika slag, berättelser som uttrycker det nuvarande tillståndet och berättelser som uttrycker det tillstånd som förväntas bli. Sfard och Prusak (2005) benämner dessa för nuvarande identitet respektive designerad identitet och dessa benämningar används även i denna studie. I individens berättelser uttrycks nuvarande identitet i presensform medan designerad identitet uttrycks i framtidsform eller som önskningar, åtaganden, skyldigheter eller nödvändigheter. Den designerade identiteten behöver inte innebära en enhetlig sammanhållen berättelse utan innebär olika identitetsberättelser kring önskningar, åtaganden, skyldigheter eller nödvändigheter. Den designerade identiteten är inte alltid önskvärd men ses ibland som bindande och den ger riktning åt individens handlingar och har inflytande på individens behov. Identitetsutveckling i studien innebär de handlingar och det lärande som sker i syfte att minska glappet mellan nuvarande identitet och designerad identitet. Samtidigt avgör den nuvarande identiteten och den designerade identiteten vad som görs och vad som lärs för att minska detta glapp.

Datainsamling

Studien är en fallstudie där åtta lärare med inriktning mot tidiga skolår följs under sina första år som verksamma lärare. Resultaten i detta paper, den förväntade konflikten mellan vision och möjlighet, är ett fall som vuxit fram gradvis under analysen av det empiriska materialet och som sträcker sig tematiskt över de olika individernas fall.

Det empiriska materialet till detta paper består av de första intervjuerna som gjordes strax före lärarnas examen. Intervjuerna innehöll öppna frågor om matematikundervisning, lärarrollen i allmänhet och i matematik i synnerhet, tankar kring att börja arbeta mm. Förutom dessa öppna frågor innehöll intervjuerna även elevlösningar och elev-lärar-dialoger som de blivande lärarna fick kommentera. Intervjuerna berörde dels hur de uppfattade frågorna, lösningarna och dialogerna just nu men även hur de trodde sig möta liknande situationer i sin kommande yrkesverksamhet och hur de själva tänkte sig som verksamma matematiklärare. Dessa olika berättelser ses i studien som deras, vid detta tillfälle, nuvarande identitet (snart nyexaminerad tidigarelärare) respektive designerade identitet (verksam tidigarelärare). För enkelhetens skull kommer informanterna i fortsättningen benämnas lärarna, även om de vid intervjutillfället hade några veckor kvar till examen. I detta paper ges utdrag från två av intervjuerna, de med Jenny och Barbro. Jenny och Barbro är valda då de representerar de två olika utgångspunkter för den förväntade konflikten som blivit synliga i det empiriska materialet.

Konflikt mellan vision och möjlighet

Något som samtliga lärarna lyfter fram i sin designerade identitet inför examen är att de som verksamma lärare ska bedriva en annan matematikundervisning än den som idag, enligt dem, förekommer ute i verksamheten. I deras identitetsberättelser av designerad identitet ger de sig själva en roll som reformatorer som ska förändra matematikundervisning i skolan. Dessa identitetsberättelser har byggts upp utifrån deras nuvarande identitet innehållande kunskaper och uppfattningar om matematik och matematikundervisning. I deras nuvarande identitet framkom vid intervjutillfället dock uttryck för faktorer som, enligt lärarna, kommer att begränsa deras möjligheter att utforma sin kommande matematikundervisning på det sätt de skulle önska enligt sin designerade identitet. Det är dessa faktorer som kommer att fokuseras här.

Vid ett tillfälle i intervjun pratade Jenny om att man måste passa in på skolan och när jag frågar henne vad hon menar med det säger hon:

Jenny: Alltså man har ju den där bilden av en lärare. Hur den ska vara och hur den ska verka. Å som ny kanske man just får väja sig lite för alla storsinta idéer man har. Men samtidigt så vill man ju också visa sina framfötter, jag vet inte. Det känns ibland som att vissa lärare på fältet, om man säger så, har en tanke om en att man ska vara för storsint eller. Ja, nu är du nyutbildad men du får inte ta för dig för mycket. Ja, jag har ju, ja man ska ju ha kött på benen för det man vill å det man vill hålla på med. Å då så tänker jag att det, det kan vara svårt att komma med nya idéer då liksom. Att den här mallen eller som man tänker, det är ingen mall heller för den ändrar sig hela tiden. Samhället vill ju ha nya grejer av lärarstudenter om man säger så. Ut med det gamla.

Jenny har i sin nuvarande identitet en bild av hur en lärare ska vara och denna bild formar hennes designerade identitet som lärare. Hon uttrycker att ”man” har en bild av hur en lärare ska vara vilket kan indikera att hon inte anser sig vara ensam om denna bild. Detta understryks ytterligare av att hon benämner bilden som en ”mall”, en mall som ändrar sig. När jag frågar henne om vilka som vill ha ut ”det gamla” säger hon:

Jenny: Alltså, media säger det tycker jag. Alltså det känns som om dom vill ha nått nytt, nått fräscht så, men sen säger verkligheten en annan sak. För media känns inte som en verklighet.

I utdragen uttrycker hon att hon av samhället och media känner ett uppdrag att tillföra skolan ”nya grejer”. Att vara nyexaminerad lärare innebär bland annat att vara en reformator som tillför ”nya grejer” till skolan. Rollen som reformator uttrycker Jenny kommer utifrån, från samhället och media, det är de som står för mallen. Denna roll är inte helt enkelt enligt Jenny eftersom ”verkligheten säger en annan sak”. En nyutbildad lärare får inte vara för ”storsint” eller ”ta för” sig för mycket. Detta kan även ses som ett uttryck för en förväntad spänning mellan kontinuitet och förändring likt den Lave och Wenger (1991) beskriver uppstår när nykomlingar får tillträde till en praktikgemenskap. Jennys uttryck för svårigheterna gällande att komma med nya idéer indikerar att hennes bild av hur en lärare ska vara inte ligger i linje med de lärare hon tänker att hon kommer att möta ute i verksamheten vilket hon understryker ytterligare när hon säger:

Jenny: […] jag kan ju komma med massa idéer men att dom kanske ”vad gör du nu?” liksom. Eller ”Vad händer? Nej men så här jobbar inte vi.

Även om hennes designerade identitet innebär att reformera matematikundervisning ser hon begränsningar för hur hon kan genomföra detta då hon inte får ”ta för” sig för mycket. Detta kan tyda på att hon vill tillhöra en praktikgemenskap med lärare ute i verksamheten och att hon har en idé om att hon inte får sticka ut för mycket för att få vara med. Jenny ser som ett av sina uppdrag att förändra skolan men för att kunna göra detta måste hon först bli medlem i de praktikgemenskaper hon vill förändra. I samband med att Jenny uttryckte att media ville ha nya saker ifrån nyutbildade lärare men att media inte var verkligheten frågade jag henne vad verkligheten säger.

Jenny: Ja, det ska inte va, visst du får komma och krypa in typ, alltså det, du får visa dina å vad du kan och vi vill gärna ta del av det men sen, sen liksom får du hålla dig på din plats liksom. Det är inte, sen kan jag säkert tänka mig att det finns vissa arbetsplatser som är paradis för vissa, där man verkligen får spela ut, där du verkligen får ta för dig.

Även om Jenny uttrycker att hon som nyexaminerad får ”krypa in” öppnar hon upp för att olika skolor kommer vara olika öppna för de nyheter hon kommer med. Enligt Wenger (1998) kan en förflyttning mellan praktikgemenskaper kräva förhandling gällande individens identitet. Olika praktikgemenskaper kommer troligen att kräva olika förhandling för lärarna, både beroende på vad de själva har med sig och beroende på hur praktikgemenskaperna ser ut. Enligt Lave och Wenger (1991) sker en ömsesidig påverkan mellan praktikgemenskap och individ när individen får tillgång till praktikgemenskapen. Hur denna ömsesidighet påverkar lärarnas identitet och identitetsutveckling kommer vara ett fokus i den fortsatta datainsamlingen. Syftet är att nå denna process ur lärarnas perspektiv vilket gör att praktikgemenskaperna måste definieras utifrån deras perspektiv. Hur får de tillgång till de praktikgemenskaper som de ser sig som reformatorer av och vilka praktikgemenskaper identifierar de sig med för att få stöd och förebilder för sin designerade identitet?

Jenny uttrycker själv i det första intervjuutdraget att hon behöver ”kött på benen” för att kunna uppnå designerade identitet, ett uttryck för behov av kunskaper och belägg för de förändringar hon vill åstadkomma. Vid ett senare tillfälle i intervjun reflekterar hon över sina egna kunskaper.

Jenny: Och just hur jag ska lägga upp det i rätt ordning för elevernas bästa liksom. Det, jag har jättemycket idéer så det är inte det å jag kan säkert spruta ut dom när som helst ja, när jag väl kommer på den punkten. Men just det att få, fånga eleverna i rätt ordning, att ge dom liksom från början och sen väx där liksom. […] lärarbiten liksom. Å sen är det å bli accepterad där man hamnar så att säga. Jag vet ju inte ens var jag hamnar.

I utdraget uttrycker Jenny en osäkerhet kring hur hon ska lägga upp sin undervisning i matematik i ”rätt ordning”. Hon har idéer för undervisning men uttrycker osäkerhet gällande hur hon ska organisera undervisningen kring dessa idéer. Återigen kommer hon in på vikten av att bli accepterad på skolan och hon uttrycker osäkerhet över att inte veta var hon ska arbeta efter examen. Nästa utdrag kommer från Barbro som även hon pratar om den framtida arbetsplatsen.

> Barbro: Sen beror det alldeles på vart man kommer, till vilket skola man kommer och hur öppna dom är för att ta emot. Men sen får man inte komma med för mycket nytt heller i början, man får väl smyga in det lite om det är gammalt och ingrott men, det känns ändå som, jag har varit på vissa skolor med som där dom har kommit långt i sitt matematiktänkande så det är inte alltid som det är, ..nattsvart.

Barbro uttrycker skolan som ett kollektiv, (vilket kan ses som en praktikgemenskap), och att hennes möjligheter som lärare beror på hur denna praktikgemenskap ser ut där hon får arbete efter examen. Liksom Jenny uttrycker Barbro att hon kommer med ”nytt” och att skolan behöver vara öppen för att ta emot detta nya. Hon uttrycker också, precis som Jenny, att man som nyexaminerad lärare inte får komma med för mycket nytt i början. ”Man får väl smyga in det” tyder på att visionen finns men att hon ser begränsningar även om det inte alltid är ”nattsvart”. Även Barbro reflekterar i intervjun över vad som kan komma att bli svårt när hon ska börja arbeta som lärare.

> Barbro: Svårast kommer nog va att, man kommer in med ganska så lite erfarenhet, att man kanske inte riktigt vet vad man gör om man säger så. Man tror att man, att man inte riktigt når dit man vill nå, kan jag tänka mig. Det tar ju ett tag innan man inser vad som funkar och inte funkar. Sen kanske det kan bli lätt att skulle man misslyckas så går man tillbaka till det som har funkat på skolan innan. Så man kanske kommer tillbaka och fastnar i läromedels…strukturen.

Även om Barbro har en idé om att hon ska komma med nytt uttrycker hon en osäkerhet gällande bristen på egen erfarenhet. Vid misslyckande uttrycker hon att det kan bli att man går ”tillbaka till det som har funkat på skolan innan”. Uttrycket ”går man tillbaka” kan vara en tidsangivelse men även ett uttryck för att hon har en föreställning om hur undervisningen ser ut idag och att den i stor utsträckning är kopplad till läromedel samt att undervisningen är ett steg ”tillbaka” från den undervisning hon vill bedriva. Uttrycket att gå ”tillbaka” kan också sättas i relation till hennes tidigare uttryck om skolor som ”kommit långt” och båda ger sken av att det finns ett ideal, ett mål för hennes sätt att undervisa vilket kan kopplas till den ”mall” som Jenny tidigare pratade om. Jennys ”mall” för matematikundervisning kom dock utifrån medan Barbro genomgående utgår ifrån sig själv i sina visioner gällande sin kommande matematikundervisning. Pajares (1992) och Mcleod (1994) framhåller i sina sammanfattningar av forskning om uppfattningar och attityder till matematik och matematikundervisning att lärarstudenters tidigare uppfattningar om lärare och elever påverkar hur de tar till sig innehållet i lärarutbildningen. Individer byter ogärna uppfattningar utan försöker istället tolka in det som sker i sina gamla uppfattningar. Ju tidigare dessa uppfattningar har uppstått ju starkare är de då de i sin tur har filtrerat hur man tagit till sig nya erfarenheter. Nyare uppfattningar är mer sårbara och påverkbara. Detta ligger i linje med Gellert (2007) som har studerat lärare som vill reformera matematikundervisning. Han menar att nyexaminerade lärare som har arbetat framgångsrikt med reformerande ansatser under utbildningen och som har långtgående reflektioner om förändrad matematikundervisning ofta faller tillbaka in i ”traditionell” undervisning strax efter anställning.

Barbro har dock inte mött den lärare hon önskar bli i sin designerade identitet. När hon i intervjun ombeds att beskriva en bra matematiklärare och att hon gärna får utgå ifrån någon hon mött uttrycker hon:

> Barbro: Jag kan inte komma på någon för jag har nog inte mött någon som jag ser det är en jättebra matematiklärare. Sen kan det ju va matematiklärare som har bra kunskaper, som kan matematik, men det betyder inte att man kan lära ut matematik.

Barbro föreställning om en bra matematiklärare kommer alltså inte ifrån egna eleverfarenheter. Enligt Frykholm (1999) har VFU-lärare [1] det största inflytandet på lärarstudenters föreställningar om matematikundervisning och på deras sätt att undervisa. I Barbros fall verkar det dock som om bilden av en bra lärare och bra undervisning i matematik skiljer sig från det som hon själv har erfarit. Medan Frykholms lärarstudenter verkar ta avstamp i erfarenheterna uttrycker lärarna i denna studie att de vill ta avstamp mot erfarenheterna. Jenny och Barbro uttrycker förändrade uppfattningar om matematikundervisning vilka troligen är nyare uppfattningar vilket, enligt Pajares (1992) och Gellert, (2007) gör dem sårbara och påverkbara.

Barbro skiljer i utdraget ovan på kunskaper i matematik och förmågan att lära ut matematik. När hon i intervjun pratar om den undervisning som hon själv vill bedriva i matematik säger hon:

> Barbro: Jag tror ju mycket på att man ska experimentera kunskapen så att man verkligen, och att man. Men sen vet jag ju att man har ju sina mål och man ska hinna dit och dit och tid men man kanske måste, ta avstamp. Alltså att man vågar att inte, att det gör kanske inte så mycket om man ligger efter ett kapitel där utan att man verkligen har förstått det som var innan. […] Men just att man vågar variera och vågar gå utanför läromedlet lite. Samtidigt som det är ett jättebra stöd att ha ett läromedel givetvis, man ska inte gå ifrån det helt kanske men att man kanske gör lite experiment vid sidan om.

Även här framhåller Barbro läroboken som något hon vill ”våga” gå utanför även om läromedlet samtidigt uttrycks vara ett bra stöd. I uttrycket ”man har ju sina mål” är en fråga vem ”man” och ”sina” är, vems mål syftar hon på? Målen uttrycks dock som en begränsande faktor för den experimenterande undervisning hon vill bedriva vilket tyder på att det är mål som definieras utanför henne men som hon måste förhålla sig till. Uttrycket ”man ska hinna dit och dit i tid” tyder på ytterligare en begränsande faktor som hon anser sig behöva förhålla sig till. Att avvika från dessa begränsande faktorer uttrycker hon i termerna av att ”våga” vilket tyder på att det inte är en självklarhet.

Sammanfattande diskussion

Utdragen från intervjuerna med Jenny och Barbro utgör två exempel på innehåll som finns i alla intervjuerna. Lärarna har en föreställning om hur bra matematikundervisning går till, en föreställning som är en del av både deras nuvarande och designerade identitet. Denna undervisning skiljer sig ifrån den matematikundervisning de själva erfarit och de ger uttryck för att de ska förändra matematikundervisning. I deras identitetsberättelser av designerad identitet ger de sig själva en roll som reformatorer som ska förändra matematikundervisning i skolan. Dessa identitetsberättelser har byggts upp utifrån deras nuvarande identitet innehållande kunskaper och uppfattningar om matematik och matematikundervisning. Jennys uttrycker att visionerna i hennes designerade identitet som lärare till stor del handlar om krav utifrån medan Barbro uttrycker visionerna som sina egna. Dessa två olika utgångspunkter för den förväntade konflikten är de som blivit synliga i det empiriska materialet. Oavsett om visionerna i den designerade identiteten uttrycks som egna visioner eller som samhällets visioner uttrycker både Jenny och Barbro förväntningar om en kommande konflikt. Mellan deras nuvarande identitet, (*nästan nyexaminerad lärare med visioner*), och deras designerade identitet (*verksam lärare som reformerar matematikundervisningen i skolan genom att förverkliga visionerna*) uttrycker de olika begränsande faktorer. Dessa begränsande faktorer är av både yttre och inre karaktär.

Begränsande faktorer av yttre karaktär är tjänst, traditioner, lektionsmönster, läroböcker, tid, lokala mål, material, resurser och personal på skolan. Alla dessa faktorer är på olika sätt delar av praktikgemenskaper i skolans verksamhet. Faktorerna begränsar, enligt lärarna, de möjligheter som de anser sig ha när de ska börja arbeta som lärare. De har en klar idé om vad man kan och inte kan göra som nyexaminerad lärare på en skola där det man kan göra inte ligger i linje med vad de vill göra. De uttrycker att faktorerna begränsar möjligheterna gällande den egna matematikundervisningen eftersom bra matematikundervisning, enligt dem, är undervisning som på olika sätt går utanför eller utmanar de nämnda begränsande faktorerna. På så vis kan konflikten mellan vision och möjlighet ses som begränsande faktorer som ska överskridas för att minska glappet mellan nuvarande och designerad identitet. Begränsande faktorer av inre karaktär är brist erfarenhet och kunskaper. I intervjuerna framhåller lärarna kunskap som en resurs de behöver ha och utveckla ytterligare för att överskrida de yttre begränsande faktorerna. För att kunna utveckla erfarenhet måste de dock få arbete och tillgång till en eller flera praktikgemenskaper på skolor vilka i sin tur, enligt ovan, kan vara en yttre begränsning för deras visioner.

En framtida forskningsfråga är hur identitetsutvecklingen för dessa lärare kommer att se ut i de olika praktikgemenskaper de tillhör och kommer att tillhöra. Hur ter sig den förväntade konflikten i praktiken, hur gör de för att uppnå sin designerade identitet och hur påverkar det deras identitet som lärare i de tidiga skolåren med fokus på matematik? Den fortsatta datainsamlingen genomförs genom observationer, intervjuer och audiodagböcker under deras första år som verksamma lärare. Ett scenario är att det inte finns någon konflikt mellan vision och möjlighet, ett annat är att den designerade identiteten förändras och konflikten försvinner, ett tredje att de hittar sätt att bemästra konflikten. De begränsande faktorerna skulle också kunna vara ett uttryck för osäkerhet. Genom att måla upp en bild av yttre begränsande faktorer behöver lärarna inte utmana sig själv att nå designerad identitet utan kan arbeta enligt den matematikundervisning som (enligt dem) råder på skolorna utan att förlora bilden av sig själv som en bra matematiklärare enligt den designerade identiteten.

Anmärkning

1. Lärare verksamma på skolor. Dessa lärare handleder lärarstudenter under den verksamhetsförlagda del som ingår i lärarutbildningen.

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University Students' Contextualisations of Threshold Concepts in Calculus

Kerstin Pettersson
Stockholm University, Sweden

Two empirical studies were carried out to explore students' understandings of threshold concepts in calculus. The studies investigated engineering and mathematics students' understandings of the concepts of function, limit, derivative and integral. The engineering students expressed their understanding in an algorithmic context, in which procedural knowledge was predominant. The students in the mathematics programme expressed their understanding in a formal context where also intuitive ideas played an important role. The results also point out the importance of contextual shifts in the development of conceptual understanding of function, limit, derivative and integral as threshold concepts.

Introduction

Research on students' conceptions in calculus has shown that the learning of some concepts is problematic for students (e.g. Artigue, Batanero, & Kent, 2007). Several studies have been carried out categorizing students' misconceptions and these studies give a rather depressing picture of students' understanding (e.g. Artigue et al., 2007). However, some researchers argue that a misconception of a concept may be a first step on the way to develop an understanding of the concept (e.g. Berger, 2004). The point of departure taken in the studies presented in this paper is a perspective that focuses on the potential of students' conceptions. As part of a doctoral thesis (Pettersson, 2008a), two empirical studies were carried out to explore students' understandings of concepts in calculus. The concepts considered in this paper are function, limit, derivative and integral. The aim is to explore how university students use their conceptions of these concepts when working with mathematical tasks in calculus. How do students contextualise concepts in calculus? What interplay among different contextualisations occurs?

Theoretical framework

From a broad constructivist stance student learning in higher education can be seen as involving simultaneous processes of approximation and feedback. Students try out interpretations of the learning material, and on the basis of the response they get from teachers and other significant influences of the learning environment they develop ideas about the meaning of the learning material and of what it means to study and learn a particular subject. Research on learning and conceptual change has described this belief formation process as a process of *contextualisation* (Halldén, Scheja, & Haglund, 2008) through which students develop individual understandings of learning material by putting it in a particular context or framework where it makes sense for the learners in the perceived circumstances. It is through such contextualisation processes that students gradually may learn to differentiate the preconditions for developing conceptual understanding in various subject areas. From this perspective, it follows that learning is dependent on the students' ability to interpret information presented in the teaching in ways that are in accord with the ways of thinking endorsed in that particular teaching setting. By contextualising the subject matter—by actively and cognitively structuring the learning material in an individual context to make sense of it for themselves—students may either succeed in accommodating the demands of the learning environment, or end up struggling with their understanding. The research questions of this paper, which have been presented in the introduction, are focusing the contextualisations made by the students. The data analysis end up indentifying the contextualisations and the interplay among different contextualisations.

Mathematical activities can be looked upon from many different perspectives and there are many possible ways to categorize and structuring the learning material into an individual context. Fischbein (1994) pointed out three basic components of mathematics as a human activity: the *formal*, the *algorithmic*, and the *intuitive* component. The interaction between these components is, according to Fischbein, very complex. Several researchers have discussed the interplay and the tension between these aspects of mathematics (e.g. Bergsten, 2008; Lithner, 2004). In the present studies these three components have been used as a framework for the analysis of the contextualisations.

Another distinction, useful when analysing students' conceptions, is the distinction between procedural and conceptual knowledge. Hiebert (1986) introduced a definition of these types of knowledge which was later developed by Star (2005). Star defined *procedural knowledge* as knowledge about rules and procedures, that is how to do things and how to solve problems, and *conceptual knowledge* was defined as knowledge about concepts and principles. Baroody, Feil and Johnson (2007) have taken these definitions further. They describe development of knowledge and understanding as a process which increases knowledge of these kinds, creating richer connections and ending up with well connected procedural and conceptual knowledge. Moreover, these two types of knowledge are seen as supporting each other in the learning process. In accordance with this line of thinking understanding is defined as including both procedural and conceptual knowledge with the level of understanding depending on the richness and connectedness of the knowledge. In the present studies the definitions from Baroody, Feil and Johnson (2007) have been used in the analysis of the students' conceptions.

Data analysis

Two empirical studies have been carried out to explore students' understanding of concepts in calculus. In the analysis of the data I drew on the *intentional approach* to learning and meaning making developed in research on learning and conceptual change in school and higher education (Halldén et al., 2008). An analysis based on such an intentional approach focuses on the students' activities in terms of intentional action. As outside observers, we do not have direct access to the students' intentions so we have to infer them by viewing the observed activities from a perspective which renders them meaningful. To adopt such an intentional perspective is to view social and communicative behaviour in terms of some purpose of the acting person to achieve a goal. By analysing what the students say or do in a particular situation and by focusing on how they approach and understand a particular learning task or a piece of information, we gain a picture of the students' perceptions of the situation at hand and what they are trying to achieve. Intentional analysis, by virtue of its focus on individual students' ways of reasoning in relation to a particular task or a particular learning environment, helps us to understand what the students take for granted, what they hold true or commit themselves to.

The analyses elaborate on the meaning of the students' utterances by constructing narratives of what the students' utterances were communicating in terms of understanding. These narratives, foregrounding the intentionality of the students' accounts, naturally meant that some parts of the interviews were brought to the fore while others receded into the background. To ensure that the narratives were consistent with what was being said in the whole data material, repeated readings of the transcripts were carried out and checked against the narratives. The next step of the analysis involved constructing models of the sorts of understanding reflected in the narratives. This process of interpreting the students' accounts into contextualisations involved identifying not just what the students were talking about, but also identifying the particular context in which the students were reflecting on the concepts. In this analysis the three components of mathematics, the formal, the algorithmic and the intuitive, as pointed out by Fischbein (1994), are used when interpreting the contexts. The analysis of the data from the two studies in this way ended up with interpretations of the students' accounts into students' contextualisations of the concepts.

Limit and integral in an algorithmic context

The first study investigated the nature of students' understandings of the concepts of limit and integral (Pettersson & Scheja, 2008; Scheja & Pettersson, 2010). The study included 20 engineering students taking a course in calculus. The students were in their first year of a three year engineering programme. They have only taken an introductory course in mathematics before this nearly completed calculus course, only the examination was remaining. The students were asked to reflect in writing on the meaning of the concepts limit and integral. Four of these students, two women and two men, volunteered to take part in a subsequent interview. The interviews involved questions probing the students' understanding of the two concepts.

An analysis of the students' written and oral accounts reveals that the students' conceptual understanding of integral and limit may be described as fragmented; it emerges as a loosely woven structure of descriptive and explanatory accounts, lacking in definition and with few, if any, connections being made between the calculus concepts involved. The students' understandings can also be characterised by a pronounced procedural focus on solving tasks. As Victor pronounces it: '*I [focus on] how to solve problems […] I try to understand and apply. I don't go around mulling over why it's one way or the other.*'

This emphasis on how the concepts are used "to do" something, to achieve something, was repeatedly brought to the fore in the interviews. Setting such procedural accounts in a wider framework of individual learning it was clear that the students were seeing these concepts from within a particular *algorithmic contextualisation*. The students described the concepts as tools or operations to be used algorithmically, following stepwise procedures for coping with typical calculus problems. Despite the fact that such an approach to understanding the two concepts meant putting more emphasis on procedural than conceptual aspects of the concepts, it seemed to be highly functional for the students, who reported having so far been successful in their studies and also expressed confidence in their understanding of limit and integral.

However, a closer look at the interview data, and in particular on how the students dealt with questions posed in the interview, makes it clear that this initial picture of the students' understandings as exclusively fragmented, merits a more nuanced commentary. More specifically, there are instances in the interviews where the students, prompted by probing questions from the interviewers, touch on issues which seem to lead them to think about topics that they previously have not considered. For instance, Philip was asked about whether all integrals could be seen as areas and vice versa. He said he did not know for sure, and that was why he had refrained from mentioning anything about this issue in his written account. However, he subsequently started to think about this topic in a way that indicated he found it quite challenging:

I:	Are all integrals areas and are all areas integrals?
Philip:	No.
I:	Can you say a bit more about it? When, for instance, is it impossible to use an integral to compute the area…?
Philip:	It's when the integral…to take an example, if you…or yeah, maybe so. If you have a …if you have, like, a broken graph or something… But then again, you could calculate with that…and find the different areas and then add them together. I have to think about whether you can do that for all integrals. I've never thought about that before...

Although Philip did not pursue this line of reasoning it was clear that the topic brought up by the interviewer sparked ideas about the relationship between area and integral which previously Philip had not considered and which made him reflect on this issue from a new angle. At first, Philip adamantly replies that integral and area are not the same thing. His utterance *'It's when the integral…'* suggest the view that not all areas can be computed with the help of integrals. When he goes on *'to take an example, if you…or yeah, maybe so. If you have a broken graph or something … But then again, you could calculate with that…'* it is clear that Philip, unsuccessfully, tries to find examples illustrating his point. After a couple of attempts he suggests the notion of *'a broken graph'*, but in trying to explain that the integral of such a function cannot be computed, he comes up with a counter argument: *'you could calculate with that…and find the different areas and then add them together'*. Philip moves from adamant to wavering in his position with regards to the relationship between area and integral. He realises that he has not previously thought about the possibilities and limitations of the use of integrals. The interviewer's probing questions pose a challenge for his procedural understanding of the concepts in question, and it is clear from his way of reasoning that the discussion has sparked his interest in the topic and so has opened up an area for conceptual reflection.

As mentioned before, the analysis of the students' written and oral accounts reveals that the students were expressing their understanding of the concepts within an *algorithmic context* in which the operations of these concepts were seen as defining features and a basis for understanding those concepts. The algorithmic context was predominant but when probing questions were put to them addressing topics that the students previously had not considered, the students were prompted to think about these unfamiliar topics. The students' ways of reasoning revealed an awareness of conceptual areas in need of further elaboration and reflection and hinted at a potential for developing conceptual understanding. The interviews made it clear that the students' algorithmic contextualisation of the concepts limit and integral and the understanding thus developed were not fixed and final. Faced with probing questions they appeared to shift to a contextualisation foregrounding ideas relating to the conceptual dimensions of calculus. Such contextual shifts, allowing the development of conceptions at different levels of abstraction, may interact to shape the students' awareness of the ways of thinking and practising in mathematics (Entwistle, 2009). The students' understanding within an algorithmic context can be seen as a stepping stone towards a more complete understanding of calculus.

The function and the derivative in a formal context

In the second study a group of four students in their first year of a mathematics programme was working with a challenging task including the concepts of function and derivative (Pettersson, 2008b). The problem was to decide how many zeros there exist when you know that the derivative of order n is nonzero. The task also included proof by induction:

> Let f be a function defined on all of **R**.
>
> A) How many zeros at most can the function have if $f'(x) \neq 0$ for all x?
>
> B) If instead $f''(x) \neq 0$, what can you say about the number of zeros of the function?
>
> C) If we have $f^{(n)}(x) \neq 0$, what can be said about the number of zeros of the function? Use induction to prove your statement.

The group was not given any maximum time for their work and as it turned out they spent about two hours on working with it. The group work was videotaped and notes from the students were collected and also used in the interpretation of the group work.

The group started to discuss subtask A. Beth and Diana said that the function is increasing, or decreasing, on all of **R** and the group concluded that the function can have one zero at most. In the same way they solved subtask B and then turned to subtask C. Swiftly, without any further discussion, Carl formulated a statement: *'About zeros, n zeros. That's what we believe, isn't it?'* Subtask C requires of them to compose a formal proof of the statement. Their intuitive ideas, good enough to produce answers to subtasks A and B, suddenly seemed insufficient. They remembered the formal pattern for proof by induction but they did not see how to match this task with that pattern. Carl stated an inductive hypothesis: *'Our assumption is that this will hold for some unspecified p, for this unspecified p we have $f^{(p)} \neq 0$. And then it holds that has p zeros. This feels dead dangerous to me, but this is the way it works.'*

The students sometimes use a language lacking in precision. For example, Carl's statement above leaves out the specification that *f* has *p* zeros at most. This should not, I suggest, be interpreted as a sign of a lack of understanding or of misuse of the mathematical language but rather as a reduction of the complexity of the problem. These specifications are, as can be seen later in the discussion, accessible to the students and are referred to when they are needed.

After a while Diana presented an important idea: *'We can differentiate this [$f^{(p)}$], then we will get that [$f^{(p+1)}$].'* The idea of induction is that a statement is reduced to, or brought back to, the preceding statements. To make the induction procedure possible, there must be some kind of relation between the statements. In this case the key is that $f^{(p+1)}$ is the derivative of $f^{(p)}$. As soon as this key is found the rest of the proof is often straightforward, even if computational technicalities can remain. However, in this task further ideas must be introduced to obtain a proof. At this time the students went over to make an inventory about what they really know about functions and derivatives. Diana said: *'If it [$f^{(p)}$] is non-zero, what does that tell about the derivative? And if the function is non-zero, what does that tell us? That doesn't tell us anything, or, that tells us…'* Why did Diana turn over to talk about the function? One possible interpretation is that she wanted to look at the special case where $p = 0$. Another possibility is that she gave a new name to $f^{(p)}$. This step, to give a new name, makes it possible to use intuitive ideas about the derivative. By giving a new name, Diana got a new intuitive basis that could be useful in the further work with the proof. Giving a new name also reduces the complexity of the task at hand.

Alex went on: *'If $f^{(p)}$ has one zero, then I think it [the function] could have p+1, or couldn't it?'* and Diana filled in: *'What should be shown, I think, is that if f' has so many zeros at most then f can have so many zeros plus one more, at most.'* A direct consequence of this was used for the proof later on:

Carl:	If $f^{(p)}$ has *n* zeros and, its primitive has *n*+1 zeros at most, if we should know that and we know that the *n*, or *p*, derivative, if it has no zero, then we know in that case that the next…
Diana:	... has maximum one.
Carl:	…and then it must be possible to go on with the induction saying that if it has at most one then that has at most two. That must be pure induction.
Diana:	Yes, you know that it will imply **that** which in turn implies **that** which implies **that** and so on.

Here the students have a proof that can match the ordinary pattern for such a proof by induction, but the students did not see that. To make the proof match the ordinary pattern a more general inductive hypothesis has to be used. The students left the task with a feeling of having a proof but not a proof that will pass for the requirements of a proof by induction.

This short presentation of the students' work with the problem points out that the students created a proof by induction but did not themselves regard it as fitting the ordinary pattern for such proof by induction as they remembered it from textbooks and teaching. The students both had and used intuitive ideas relevant to the concepts brought to the fore by the task. The students also used formal reasoning in the problem solving process. The students' interpretations of the task were carried out in a *formal context* where also intuitive ideas played an important role. Looking across the whole data material, presented more extensively in Pettersson (2008b), the results also show that the students used formal reasoning and intuitive ideas in a dynamic interplay. The analysis of the data also indicates that this interplay had several functions: to control intuitive conceptions, to offer a new basis for reasoning, to reduce the complexity of the problem and to push the problem solving process forward.

Conclusions and discussion

The results reveal that the students in the first study displayed fragmented conceptions with a procedural focus, a result which confirms previous research results (cf. Engelbrecht, Bergsten, & Kågesten, 2009). There were few connections between different concepts and the operations were seen as defining features of the concepts. The students' interpretations were carried out in an *algorithmic context*, in which the very operations were seen by the students as crucial to their understanding of the concepts. However, faced with probing questions the students appear to shift to a contextualisation foregrounding conceptual knowledge. In the second study the students' interpretations of the mathematical task were of a different kind. The task was interpreted in a *formal context* where also intuitive ideas played an important role.

The two empirical studies included different student groups. The results suggest that differences in time and study programme may explain the differences in the students' contextualisations of the tasks presented to them. However, there are also differences in the contextualisations made by different individuals regardless of programme. Even if the algorithmic context was predominant among the engineering students, probing questions opened up the possibility for the students to contextualise the task in a different way. Probing discussions also initiated shifts between formal reasoning and intuitive ideas among the students who worked in a predominantly formal context. The students used this interplay in ways familiar to professional mathematicians (Burton, 1999).

It is interesting to notice that the concepts investigated in the two studies can all be regarded as *threshold concepts* (Meyer & Land, 2003). In students' efforts to understand the subject some concepts may be more crucial. A threshold concept can be seen as a 'portal' or a 'conceptual gateway' that leads to a previously inaccessible, and initially troublesome, way of thinking about something. A new way of understanding may thus emerge – a transformed view of the subject. Threshold concepts are defined by its characteristics, they are characterised as initially troublesome, transformative (occasioning qualitative changes in students' conceptions), integrative (integrating pieces into a conceptual whole) and irreversible (unlearned only through considerable effort). Research about how students in higher education understand concepts in different subjects, such as computer science and engineering, have been carried out in the last years (Land, Meyer, & Smith, 2008) but there is a lot more to do, especially in relation to mathematics education. However, limit is given as an example of threshold concept (Meyer & Land, 2003) and previous research about students conceptions of the other concepts considered in this paper, function, derivative and integral, reveals that these concepts also fulfil the criteria defining threshold concepts (for an overview of previous research see e.g. Artigue et al., 2007).

By using a theory of contextualisation to model the students' conceptions of these threshold concepts it is made clear that the students' interpretations of a given task are not merely due to cognitive shortcomings, but rather ways of dealing with a learning situation at hand. The results indicate that students may have a potential for developing a formal understanding of a mathematical concept previously viewed within an algorithmic context. The fact that students sometimes tend to rely on techniques that serve their immediate interest of coping with a particular learning task, or passing an exam, does not necessarily prevent them from developing a solid conceptual understanding. To focus solely on the procedural aspects of a concept might, of course, delay a more complete conceptual understanding but there is reason to believe that conceptual understanding of a subject-matter comes gradually (Baroody et al., 2007). Students who develop their understanding of calculus using an algorithmic context of interpretation do so, not because of a misconception, but because it is functional for them and enables them to deal pragmatically and often successfully with learning tasks that they are confronted with in teaching and in exams (Lithner, 2004).

One of the characteristics for threshold concepts is to be transformative (Meyer & Land, 2003). That is, coming to understand the threshold concept includes qualitative changes in students' conceptions. The shifts in contextualisations displayed in the empirical studies presented in this paper can be seen as a clarification of the transformative aspects involved in understanding threshold concepts. The students' ways of contextualising the issues brought to the fore set the frames for developing conceptions of the concepts. Such conceptions may differ, for example, in terms of being focused on procedural or conceptual aspects. But in response to perceived demands of the situation learners may well begin to change the way in which they contextualise subject matter, thus allowing alternative conceptions to come into play which can influence the development of understanding. For instance, by being presented with questions that indirectly demanded a contextualisation that emphasised not only procedural but also conceptual aspects of those concepts, the students made a contextual shift to try to adapt to such a way of thinking. So, transformation in relation to developing an understanding can be described in terms of changes in students' contextualisation of those threshold concepts. It is through such contextual shifts students will be able to differentiate the role that different contextualisations have in solving and providing insights into mathematical problems (cf. Halldén et al., 2008). Through such contextual shifts the students will also gradually traverse different modes of variation and be made aware of the boundaries of the discipline thus gradually building an awareness of the ways of thinking and practising endorsed within mathematics (Entwistle, 2009).

Of course it has to be recognised that the results presented here are based on a rather limited set of empirical data and thus more substantial evidence is clearly needed to explore these findings further. Further research on threshold concepts is required if we want to gain a fuller picture of the role that such concepts play in teaching and learning, and of the potential such concepts offer to the understanding of the relationship between contextual shifts and conceptual development. Furthermore, such an understanding would be crucial to teachers' possibilities to help students approach ways of thinking and practising which are fundamental to mathematics.

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Augmented Reality as Support for Designing a Learning Activity Concerning the Mathematical Concept of Scale

Håkan Sollervall, Per Nilsson and Daniel Spikol
Linnaeus University, Växjö, Sweden

This paper describes the design of an ICT-supported learning activity developed in collaboration between teachers, researchers and technical developers. Inspired by an existing design of a teaching activity based on advanced use of ICT-tools, we have reduced the technological complexity to design a new activity aimed at stimulating learners' thinking and reasoning in relation to the concept of scale, where we make use of augmented reality, that is, mixing real-world images with computer-generated images. Using the methodology of Design Experiments as a foundation for the design process, we argue that constructs from the analytical framework of contextualization in prospective and reflective analyses may be used to support the control aspect of the design process.

Introduction

The authors of this paper are involved in an on-going project regarding design of an ICT-supported learning activity which is developed in collaboration with mathematics teachers and technical developers. The goal for the project is to investigate how augmented reality – that is, a real-world projection enhanced with computer-generated images – may be used to stimulate individual learners' thinking regarding the mathematical concept of scale, with special attention put on strategic thinking and decision-making during their interaction with the designed activity. During the design cycle, we put focus on merits and limitations of different teaching arrangements that involve augmented reality as a central part of the design.

In the current paper, we discuss the initial stages of the project and argue for a particular choice of common concepts of reference, with the purpose to enhance the discussions within the whole development team and support the design process. Our aim is to show how involvement of ICT-tools in the activity may provide unique opportunities for stimulating learners' strategic thinking and decision-making, by offering referential settings that are not possible to realize within a traditional learning environment.

Methodological considerations for the on-going project

The members of the development team may be described as having complementing competencies regarding the task at hand. The learning activity described in this paper makes use of ICT-tools, including a specific kind of computer-based technology for visualizing real-world projections and computer-generated images merged together (Milgram & Kishino, 1994), and is developed in collaboration between researchers in mathematics education, researchers and developers in media technology, and high school teachers.

We recall Kaput (1992): ”major limitations of computer use in the coming decades are likely to be less a result of technological limitations than a result of limited human imagination and the constraints of old habits and social structures” (p. 515). As a starting point, we assume that experience from teaching mathematics and conducting research in mathematics education supports innovative design of mathematical activities. Furthermore, technological experience and expertise is needed both to carry out the teachers’ ideas about design and to show the teachers what possibilities new technologies can offer for the learning of mathematics. It is also reasonable to assume that collaboration between stakeholders with varied backgrounds and complementing competencies may reduce the influence of social and cultural constraints and stimulates thinking in innovative and productive directions.

The methodology used in this project is founded on the principles of Design experiments (Cobb et al., 2003). The first feature of Design experiments (DE), *develop theories*, is followed by *control*: “The intent is to investigate the possibilities for educational improvement by bringing about new forms of learning in order to study them” (p. 10). Control implies the need for *prospective* and *reflective analyses*. This paper puts focus on the prospective (a priori) analysis. The next step of the project will consist of carrying out the activity with students and the following reflective work will be based on observations of students’ actions. The prospective and reflective aspects come together in a fourth characteristic of DE, *iterative design*. Iterations are carried out with the aim at enhancing learners’ thinking and reasoning in relation to the designed activity. The fifth feature refers to the *pragmatic roots* of DE. As schoolteachers take active part in the design process, we feel confident that the activities are relevant for teachers’ practice. Inspired by Design experiments we have used a slightly modified version as described by Nilsson, Sollervall and Milrad (in press). The main difference between the two approaches is that Cobb et al. (2003) emphasize a thorough theoretical base concerning understanding processes of learning and the means that are designed to support that learning, while we adopt a strategy that to a greater extent relies on combined competencies of the mathematics teachers from high school and university. This methodology is in line with a co-design approach that shares similar affinities with Design experiments (Roschelle

& Penuel, 2006). Co-design can be defined as a highly facilitated team-based process in which teachers, researchers, and developers work together to design an educational innovation (Penuel, Roschelle, & Shechtman, 2007). The benefits of co-design are the direct involvement of stakeholders helping to ensure that the concerns and values of the users are kept in focus. Our approach is motivated by the aim of Design experiments with the goal of creating a tangible innovation through iterative cycles. The ambition in later stages of our project is to let the preliminary observations guide us toward relevant learning theories, which will support further enhancement of the design.

Actions, contextual resources, and contexts

We will now focus our discussion on the control aspect of Design experiments. As the design of the task is a common project for stakeholders with different backgrounds, it is necessary to apply methods for control that are accessible and seem reasonable for all within the development team, independent of individual expertise and previous experience. From a teacher's point of view, it is natural to observe activities that students initiate and engage in while they explore and interact with a given task, in particular, how they reason and negotiate between possible strategies. For the purpose of evaluating students' performance, the teacher will be able to observe how students make use of their mathematical knowledge and competences as well as tools that are available in the classroom or, specifically, within the designed activity.

Regarding the reflective analysis, we will draw on the *actions* of the students, while the prospective analysis is based on *hypothetical actions*. These actions will be described from an observer's perspective with focus on the *resources* students make use of. Both actions and resources are shaped by personal interpretations of the activity. From a theoretical point of view, we will apply a constructivist perspective regarding our theoretical constructs. In particular, we will relate to the concepts of actions and resources from the analytical framework of contextualization (Halldén, 1999; Wistedt & Brattström, 2005; Nilsson, 2009). Within this framework, the researcher's aim is to understand students' meaning-making processes of complex phenomena in situations where opportunities are provided for them to integrate different ways of reasoning. Such meaning-making will involve not only knowledge and beliefs about mathematical concepts and constructs, but also beliefs about discursively oriented element such as beliefs about learning mathematics and what is allowed to make use of in the given situation.

As mentioned above, our analysis will be based on observations of students' actions. The actions will be described in terms of what resources students make use of. These resources may concern for example knowledge, conceptions, and objects that students make use of in the situation. Such a resource becomes a *contextual resource* only if the student takes it into consideration when negotiating further initiatives and strategies within the situation. The contextual resources may hence be interpreted as determinants for action. A specific object, such as a pencil, may be used as a contextual resource in several different ways. For example, the pencil can be used to take notes, but it can also be used to mark a distance on a map or as a unit measure. The description of the pencil as a contextual resource will be focused on how the user applies it in the situation.

To clarify how different contextual resources may interact and serve as points of reference in the learners' processes of contextualization, we can delineate at least three types of analytical contexts (Halldén, 1999). First there is the *conceptual* context denoting personal constructions of concepts and subject matter-structures brought to the fore in a study situation. Second, there is the *situational* context, which refers to interpretations made in the interaction between the individual and the immediate surroundings, including interpretations of figurative material, possible actions and directly observable sensations. This context may have a more prominent role in an explorative learning environment than in a static one (cf. Tiberghien, 1994). Third is the *cultural* context, referring to constructions of discursive rules, conventions, patterns of behavior and other social aspects of the environment (Halldén, 1999).

Reflections on the use of actions and contextual resources as common references

The actions and resources may serve as common references for all members (teachers, researchers, developers) of the development team. The overall aim of the team is to design a task that supports students' learning processes. However, the different stakeholders may adopt different strategies for achieving this goal. The researchers may be focused on understanding students' meaning-making processes, where the theoretical descriptions of actions in terms of resources will serve as a means to communicate their understanding to the teachers. The teachers may, based on their experience, be able to both strengthen the researchers' interpretations of students' actions and take part in forming strategies for improving the design. Discussions and negotiations with the technical developers concerning both conceptually oriented strategies and technical functionality will result in strategies for technical development. Technical implementation will be supported by the fact that descriptions of conceptually oriented strategies in terms of actions and use of resources may be understood also by the technical developers.

Furthermore, we would like to point out that although contextual resources may be of common interest for teachers and researchers, the may be so for different reasons. The objective for the researcher is to use the resources primarily for describing processes of contextualization, while the teacher may reflect on students' resources in relation to supporting their learning strategies.

Design of the activity

From previous experience of working with ICT-support in mathematics and designing innovative mathematical activities (Nilsson, Sollervall, & Milrad, in press) we are aware of the extensive work required to design technology supported material to be used successfully in mathematics teaching. When invited to join the local GeM-project, an on-going collaboration between university representatives from media technology and a group of high school mathematics teachers, we saw this invitation as an opportunity to develop an interesting and potentially mathematically productive teaching activity. We were asked to join the project based on earlier collaboration, expertise in mathematics and mathematics education, with the purpose to enhance the mathematical content of the activity. The initial focus of cooperation has however been on developing a new activity. This may be viewed as a natural feature concerning collaborative processes, where new stakeholders may propose new initiatives instead of aligning their work with previous suggestions.

The GeM-project is an ongoing project on school geometry making use of mobile technology in an outdoor environment, followed up by indoor activities. For our purposes, we have made use of a selection of available technological tools from these indoor activities in the design of a new activity intended for the learning of mathematics. In this process, our ambition has been to follow up on the mathematical intentions of the school teachers that were involved in the GeM-project and are also involved in the current study.

Based on demonstrations at a GeM-project meeting, the university mathematics teachers proposed a mathematical activity focusing the concept of scale. The activity, described below, may be described as a low tech version of the activities designed within the GeM-project. The design makes use of an ordinary computer with webcam and a Cleverboard, which is commonly available technology in Swedish classrooms. (Projector and Whiteboard would be sufficient substitute for the Cleverboard.) What needs to be added is software supporting the technique of augmented reality.

The physical set-up of the activity may be described as follows. A photograph of a landscape is laid down horizontally on a table. The webcam projects a picture on the vertical board through a projector. The webcam is placed just slightly above the table so that the map is projected from a close to horizontal perspective. A movable small tag is placed on the table. Software supports showing a building (or any kind of picture) on top of the tag on the board, so called augmented reality.

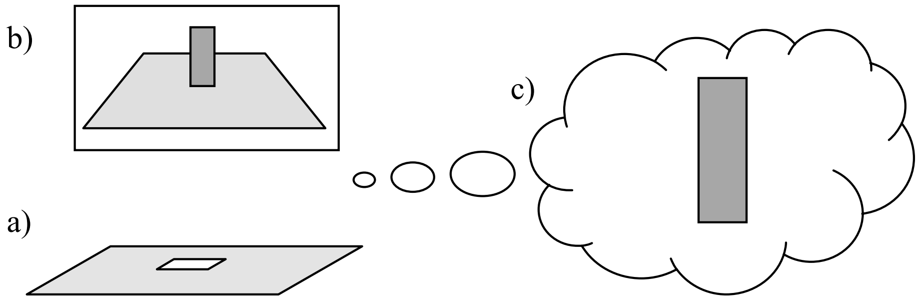

Figure 1: a) Tag on top of a photograph. Building cannot be seen, just imagined; b) Projection of the photograph, showing a building on top of the tag; c) Building imagined in reality.

The task for the students is to figure out the building's height in reality. The students are free to use tools and take measurements both on the screen and on the table. Available physical tools should include a whiteboard pen, magnetic dots and a ruler. It would be desirable to allow the students to draw and erase directly on the photograph.

Didactical gains of the design

We note that augmented reality makes it possible to provide three distinct referential settings (photograph, projection, reality) intended to support learners' reasoning and decision-making. If, for example, the projection instead would have been based on objects physically present on the table, the task could be handled by only referring to the real objects on the photograph thus giving less stimulation for learners to coordinate between the photograph and the projection.

The surprise effect of a building unexpectedly showing on the board – but not on the table – not only motivates learners to engage in the task but may also stimulate them to compare and negotiate the two settings.

A specific feature of didactical value is that the length scale differs considerably from front to back in the projected picture, so that movable images appear to grow when moved from back to front. This feature provides opportunity for learners to reflect on the fact that measurements made in the projected picture do not readily translate to real-world measures, and certainly not through direct application of the scale on the map.

Prospective analysis regarding hypothetical actions

The main feature of attention will certainly be the building showing on the board but not on the table. A priori, we may suspect that the students will use either of the following initial strategies (Fig. 2): they may place a ruler or object raised vertically from the table to measure the height of the building in the scale of the photograph, or, they may measure the height of the building on the screen and relate to a corresponding horizontal measurement on the screen that can be related to a corresponding measurement on the photograph.

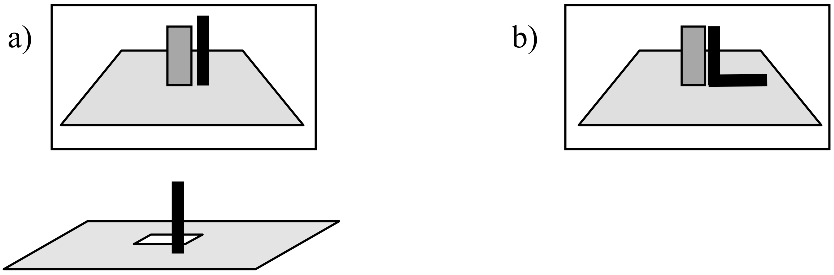

Figure 2: Examples of possible strategies: a) A real object is put on the table. The object is projected on the board. b) Students measure directly on the board.

Once the students have either of these measurements they may find the corresponding measurements for the real building by using either the concept of scale or by comparing with objects in the photograph, such as cars or people, together with principles of similarity (proportions, Regula de Tri). We elaborate these ideas in the next section.

Prospective analysis regarding hypothetical use of contextual resources

At this stage of the design, two features are not yet determined. First, it should be negotiated whether a scale should be shown next to the photograph. Second, it should be decided what picture the photograph should show, in particular, which measurable objects that should be included.

What mathematical concepts, methods and strategies may the students bring to use in their personal context for interpreting a building's height in reality? Starting with the strategy a) described in Fig. 2, it will be natural to relate the acquired measure to distances on the photograph. These distances may either be estimated by relating to objects in the photograph, such as cars or buildings. One possibility may be that these objects are used as units, so that the height of the building may be expressed for example in terms of cars whereafter the length of a car may be found either on the Internet or by going out to a parking lot to measure. Another possibility is that object measures are converted into a scale. For example, after measuring a 2,3 cm long car in the photograph which is 3,5 m long in reality, the students may conclude (after performing the division 350/2,3) that the approximate scale is 1:152 (or, after dividing 3,5 by 2,3, that 1 cm in the picture corresponds to 1,52 m in reality). This strategy may be used even if the scale is given on (or next to) the photograph. After finding out the real measure corresponding to 1 cm in the photograph, the students may readily find the real height of the building by multiplying the measured distance by the scale. Even if they have not performed the division, they may figure out how many 2,3's that fit in the measured distance (either by reasoning or drawing pictures).

If strategy b) is applied, the students can either continue working on the board or relate to the corresponding distance on the table. In the first case, the students must take into account that the scale changes with the depth of the projected picture. As a consequence, they can compare only with objects that are placed on a horizontal line from the building. If this is done carefully, the whole problem can be solved directly on the board using a strategy similar to the one described above. If objects are chosen off the horizontal line, they have to negotiate the change of scale that is due to the projection on the board. However, if they choose to represent the distance on the photograph on the table, they may choose any object as reference (or relating to a given scale).

Remark on technical complexity

In the research literature, regarding design of teaching activities using ICT, there is a strong focus on the use of advanced technology. For example, the well known geometric construction tool Construct3D (Kaufmann & Schmalstieg, 2002) utilizes special equipment computer hardware and software that is not readily available to the average mathematics teacher. Alternatively, Scarlatos (2006) works from the other direction, bringing physical interaction to virtual spaces through tangible user interfaces (TUI). This project presents tangible math that explores how data and representation can be manipulated using alternative input devices that combine moving physical objects with computer graphics (Scarlatos, 2006). Similarly, the GeM project has developed advanced software for mobile applications that is used together with other technologically advanced tools. These tools combine outdoor work with mobile devices and indoor work using 3D software and mixed reality tools.

While research that pushes the technological frontiers is of great interest also to the community of educators, the specific designs are not easily integrated in an ordinary teaching environment. Especially concerning the pragmatic roots of our approach, we feel that there is extensive unexploited potential in educational activities using less advanced technology that is available in most classrooms such as video projectors and PCs with webcams.

Next step of the project

The immediate next step of the project is to try out and discuss the outcome of the activity, first within the development team and subsequently with students of different ages. Our ambition is to involve several groups of students in grades 5-9 (ages 10-15). At each instance, students will be chosen from the same class and their specific tasks will be appropriated in collaboration with their mathematics teacher. The students will be videotaped and a reflective analysis will be performed based on their discussions and actions. Some results may need further investigations in research literature and may also initiate new research questions. The ambition of the development team is to enhance the design of the activity and the tasks based on the outcomes of students' interactions with the activity and the appropriated tasks.

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The Transition from Secondary School to University: Learning and Understanding Mathematics from a Student Perspective

Erika Stadler
Linnaeus University, Växjö, Sweden

This paper deals with some result in my Ph.D. thesis. The aim of the thesis was to understand the transition between mathematics studies at upper secondary school and university from a student perspective. In this paper I present the central results form the thesis. Firstly, three categories can be used to describe and analyse students' learning of mathematics. Secondly, the transition is about small, sometimes implicit and hidden changes in the relations between these categories.

Background

Why do some students get problems with university studies in mathematics? Research concerning the transition between upper secondary school to university in mathematics education shows that this is a complex issue, which include an individual, a socio-cultural and a situated perspective on learning and understanding mathematics. Previous research which directly addresses the transition point at insufficient pre-knowledge in mathematics, a new learning environment and changes of the mathematical learning content.

From an individual perspective of the transition, students' concept understanding and concept development of university related mathematics have been examined (Juter, 2006; Lithner, 2003). From a socio-cultural perspective, the transition has been understood as students' enculturation in a new setting with new discursive tools and artefacts (Guedet, 2008; Wood, 2001). From a situated perspective, newcomers' entrance to university studies in mathematics can be regarded as peripheral participation in a community of practice (de Abreu, Bishop & Presmeg, 2002; Watson & Winbourne, 2008).

To choose a theoretical perspective for a study in advance also involves presumptions about how the transition should be understood. However, it was an end in it self to gain understanding of the fundamental character of the transition from a student perspective. This required an open methodological approach that would result in a theoretical description of essential aspects of learning mathematics in a new setting in the light of pervious experiences. A pre-defined theoretical perspective for the study would have contradicted the aim of the study, which is to gain understanding of the transition from a student perspective. Thus, instead of using an existing theoretical framework or formulating research questions that implicate a specific theoretical approach, I have chosen to use an operational definition of the transition as students learning mathematics in a new setting according to their previous experiences as a point of departure for my research project. Through collecting empirical data from situations where the transition, according to the definition, takes place, it has been possible to work in reverse order, i.e. moving from real world situations and empirical data to generate descriptions of the transition in terms of a theoretical model.

Aim and research questions

The aim of this paper is to contribute to an understanding of the transition between secondary school and university in mathematics education from a student perspective. The following questions will be addressed:

1. What constitutes significant aspects of learning and understanding mathematics from a student perspective?
2. Which aspects are remaining and which aspects are changing by the transition?

Method

Five student teachers in mathematics have participated. They belonged to a group of 15 mathematics student teachers during the first semester of their education. Before the courses in mathematics had begun, the students were requested to answer a questionnaire about their previous experiences of learning mathematics at upper secondary school, their view of mathematics and their expectations on their future university studies in mathematics. They were also asked if they wanted to participate in the study by being interviewed and observed during lessons and tutorials. Five students volunteered; three females and two males.

In the first phase of data collection students' previous experiences of learning mathematics at upper secondary school were elucidated. The preferable way to do this would have been to make observations of the same students there and then follow them to university. However, for practical reasons, this has not been possible. Instead students have been interviewed about their mathematics studies at upper secondary school. The main difference between observing students in action and interviewing them about their previous experience is that in the first case the researcher's interpretation of the observations is central. In students' own stories it is possible to capture their views of their experiences; how they perceive their earlier experiences and their view of themselves accordingly. In Sweden, teacher education starts with a common semester when all student teachers with different subjects study more general teacher courses about what it means to learn, children's intellectual and logical development, school values and ecological and environmental issues. The pre-interviews were conducted at the end of this course before the students had begun their studies in mathematics.

During the second phase, data concerning students' learning of mathematics in a new setting were collected, consisting of observations of students attending mathematics lessons and individual work with mathematics. I attended lectures and took notes about how the teacher presented the mathematical content. The main aim with observing lectures was to become familiar with the new learning setting for the students. I also observed students when they were working with exercises during lessons and outside scheduled teaching. The teacher helped students when they tried to solve exercises individually and in pairs. When students were working alone I initiated conversations with them by asking them what they were working on. These discussions were sometimes focusing the specific tasks that the student was working on, but sometimes we had discussions about more general issues about studying mathematics at university.

From a theoretical and methodological point of view, a central issue that has characterized the whole study is that the transition is a complex phenomenon. To choose a theoretical approach or framework from the beginning would also decide how to characterize the transition and which aspects are crucial from students' point of view. Instead, the starting point has been my operational definition of the transition that has implied the data collection. To capture the core of the transition from a students' perspective, I have analysed data with methods inspired by *Grounded theory* (Charmaz, 2006). From a systematic analysis of transcriptions, categories have been discerned that from a student perspective seem to be crucial for their learning of mathematics in a new setting. The core issue in a grounded theory-approach is the emergence of theory from empirical data. Instead of gathering data to test a hypothesis or use an existing theory, the researcher approaches data without any pre-defined theoretical framework (Strauss & Corbin, 1998).

Grounded theory is often used in studies about humans and their actions. By focusing some "real world" operational issues or questions, the researcher can start collecting data. There are no specific rules or recommendations for which kind of data that can be used with grounded theory. It is the aim of the study and the availability of empirical situations and sources that determine which kind of data should be gathered. What is more important is the character of the data. Suitable data for grounded theory should consist of rich descriptions. Rich descriptions make it possible to study human phenomena beneath the surface or behind the visible when it comes to social and subjective aspects of life, for example feelings, intentions and approaches as well as contexts and structures of humans and their life (Strauss & Corbin, 1998; Charmaz, 2006). An important task for a researcher who uses grounded theory as a method for analyzing data, is to structure and organize data into conceptual categories according to their characteristics and dimensions (Strauss & Corbin, 1998). Operational entities that from a more general perspective seem equal are grouped into categories. The overall goal is to generate categories and/or concepts that relate to each other in a logical and systematic way, which ends up in theoretical descriptions of the phenomenon that the researcher is interested in (Charmaz, 2006). A grounded theory approach starts with empirical data and ends up with theories, constructed from data. Thus, using these research methods makes it possible to theorizing complex "real world" entities by using empirical data as the starting point. To analyze empirical data from classrooms observations of novice students working with mathematics has been one way to research the transition according to its operational definition. Instead of defining how the transition should be regarded in advance, the underlying processes and experiences can be disclosed from data.

Students' learning of mathematics – three categories

In this section, I will present the main results from the pre-interviews consisting of three categories, which I have named Mathematical learning objects, Mathematical resources, and Students as learners. A detailed description of the origin of these categories and how they have been created from data can be found in the thesis (Stadler, 2009).

Mathematical learning objects refer to the students' idea of the overall purpose of learning mathematics. It captures students' view of what mathematics is and what learning mathematics is all about. There is an important distinction between Mathematical learning objects that refer to a mathematical content and meta-Mathematical learning objects dealing with how to learn mathematics, what to do to learn mathematics and which mathematical content you should focus while learning mathematics. Thus, Mathematical learning objects can be derivatives or vectors, but it can also consist of mathematical processes and what in everyday language often is referred to as understanding mathematics. Some examples of meta-Mathematical learning objects are how to use the textbook, the teacher and peers when working with exercises and how to interpret a given answer to an exercise as compared to one's own results. There is an important dividing line between Mathematical learning objects that mainly can be regarded as algorithmic understanding versus learning and understanding of mathematics which is of relational nature. I have chosen to define the former as a procedural Mathematical learning object, while the latter is considered as a structural Mathematical learning object.

Textbooks, teachers, peers, mathematical pre-knowledge and logical thinking are some examples of entities that can constitute *Mathematical resources*. However, it is not the textbook or the peers themselves which are Mathematical resources. They become Mathematical resources when they are used by the students for learning mathematics. For example, a teacher can explain something for a student that is not at all helpful for the student and the student does not use the explanation in his or her work with mathematics. Even though there is an interaction between teacher and student, the teacher and his or her explanation may not constitute a Mathematical resource. Thus, the definition of Mathematical resources is made on a relational basis. It is in the interplay between a student and a potential Mathematical resource that it actually becomes a Mathematical resource. Another way to express the specific character of Mathematical resources is that it is not the object that is a Mathematical resource, but the object is used as a Mathematical resource by the student.

Students as learners is the third category that has been discerned from the pre-interviews. Which Mathematical learning objects students are focusing on and what Mathematical resources students are using in their work with learning mathematics are shown in their actions. Thus, Students as learners include students' actions with the overall intention to learn mathematics where Mathematical learning objects and Mathematical resources are exposed in their actions and statements. There seems to be a dialectic relationship between Mathematical learning objects and Mathematical resources. Students use Mathematical resources that they evaluate as useful with respect to a specific Mathematical learning object, but on the other hand the availability of different potential Mathematical resources determines which Mathematical learning objects students focus on.

Classroom observations

The three categories that have been discerned from the pre-interviews have constituted a scaffold for the analysis of classroom observations and further interviews. The transition from a student perspective can be described in terms of changes of the relations between these categories. The following episodes offer some examples of these changes. They contain interview extracts and observations of the students when they are working with exercises during mathematic lessons. The students are working individually or in spontaneous groups, while the teacher is available for giving explanations.

Episode 1

One student, we can call her Sara, is interviewed during a mathematics lesson.

Erika: How is it going?

Sara: I think it is okay. It is harder than it was at upper secondary school, but I think it is going fairly well.

Erika: In what way is it harder?

Sara: Well, I think you have to read more. I think that is tiresome.

Erika: Read?

Sara:	You know, read in the book about different… about how you are supposed to do and such things. Before it has been, before I haven't had to read very much. Then I understood the lecture and then you just counted on. But it, well, it has turned out well anyway.
Erika:	How come that it feels like it turns out well?
Sara:	Well, you know, I do understand in the end. Even if it may take longer time. But I do understand. It only takes longer time.
	[…]
Sara:	I start with exercises. But if I don't manage I read afterwards.
Erika:	What do you read then?
Sara:	Then I read that chapter in the book where the exercises are. So it is about that.
Erika:	Do you read both examples and theory?
Sara:	Yes, I do.
Erika:	What is most rewarding to read?
Sara:	In some cases the examples show you how to do, but then it is often more general explanations afterwards. So if you begin with looking at examples, then you may understand the current exercise and when you read the more general explanations you can also learn what to do in other cases. So I think you need a bit of both, because it might have been hard to understand the general explanations without an example.

Analysis

From Sara's statements it can be concluded that she considers it to be a connection between solving exercises and understanding. One possible interpretation is that she considers understanding as equivalent with successfully solving exercises. In this case understanding equals "to understand how she should do to solve an exercise". However, it is also possible that she thinks that to gain understanding one has to solve exercises successfully. Thus, in the first case, solving exercises is a Mathematical learning object in itself, which in some sense walks hand in hand with understanding. In the second case exercises can rather be viewed as a Mathematical resource to achieve the Mathematical learning object of understanding.

From the dialogue, it can be concluded that whatever Mathematical learning object Sara is working towards, she regards it as achievable. The crucial change between upper secondary school and university seems to be which Mathematical resources she needs to use to be able to gain different Mathematical learning objects. Sara has partly changed strategies for that. During upper secondary school, the teacher's demonstration of some examples was a sufficient Mathematical resource for her to be able to start with solving exercises. At university Sara has to use several Mathematical resources to be able to solve exercises. This change of working route illustrates the need of supplementing the Mathematical resources. Schematically it can be described as:

> Upper secondary school: teacher's demonstration [MR] → solve exercises [ML]
>
> University: lecture by the teacher [MR] → individual reading [MR] → solve exercises [MR/ML]

Another change is in what way a Mathematical resource should be used. Sara says that at upper secondary, the teacher's demonstration was sufficient to be able to work with exercises. Now she has to read in the book and study different examples. One explanation to this difference is that the lectures simply have different aim and functions. Whereas the aim with the teacher's demonstration at upper secondary school was to show "how to do", observations show that the intention with the lectures at university rather is to present and discuss more general mathematical ideas. Many examples can be regarded as generic, instead of a usable formula for how to produce a correct solution. Thus, the word "understanding" slightly changes in connection to the transition. Instead of "understand what to do" the students have to "understand how it is" and how this knowledge can be used to solve exercises.

When it comes to understanding, a common way to describe students' understanding of mathematics and mathematical concepts is in terms of dichotomies. Algorithmic, superficial and instrumental understanding have been used as opposite to systematically, deep and relational understanding (Skemp, 1987). However, to categorize mathematical understanding in terms of opposites does not seem sufficient to describe Sara's Mathematical learning object. Instead, I describe her Mathematical learning object as functional understanding, which can be regarded as something in between instrumental and relational understanding. Sara is obviously aware of that learning mathematics at university is more than "know what to do". On the other hand, exercises still play a central role in her work to learn mathematics. To be able to solve exercises she has to attend this new way of working; to use new Mathematical resources and work with familiar Mathematical resources in a slightly new way, i.e. complementing Mathematical resources, to gain an understanding that is functional in the new learning setting that the university constitutes.

Episode 2

Jenny asks the teacher for help with the following exercise:

> Does the graph of the function $y = x + \sin x$ have any horizontal tangents in the interval $0 \leq x \leq 2\pi$? If so, where? If not, why not?

Teacher: All right, what have you done and how have you been thinking?

Jenny:	What I have been thinking? I have… differentiated.
Teacher:	Yes, and it sure was a good start. Why did you differentiate?
Jenny:	To find the tangent.
Teacher:	Yes, that's right. And then you want to find the points where the tangent is horizontal.
Jenny:	Yes.
Teacher:	So, what does that mean for the derivative?
Jenny:	Eh… that it is positive…? Zero?
Teacher:	Yes, zero.
Jenny:	Yes, zero.
Teacher:	You know, at a point where the tangent is horizontal, you know, parallel with the x-axis, the direction coefficient of the tangent is equal to zero. And the derivative at that point is the direction coefficient, so…
Jenny:	So it should be zero then?
Teacher:	Yes. Do you agree on that?
Jenny:	Mm…
Teacher:	Well, you did differentiate, so it seems like you knew what you were supposed to do.
Jenny:	But differentiation is what I have been doing this whole chapter!
Teacher:	So why not continue with that! But do you agree why you are supposed to differentiate? Well, simply to find the points where the derivative equals zero. You could re-formulate the question in that way. Find the points where the derivative equals to zero. It is the same thing as to find points where the tangent is horizontal.
Jenny:	So, I shall solve that one?! Is that the answer then?
Teacher:	No. You have done some work here, but now you must solve that equation and find the x:es. You know, you are to find out if there are any horizontal tangents in the interval. And if there are, where? And where, that is to give the x-coordinates for example.

Analysis

The dialogue between Jenny and her teacher contains an example of a discrepancy between Jenny's Mathematical learning object and the teacher as a potential Mathematical resource. Jenny is focusing a Mathematical learning object of a procedural character, when she seeks an algorithmic understanding of how the exercise should be solved. However, the teacher directs his explanation towards a more general understanding of the connection between the tangent of the function and the derivative. With such an understanding, Jenny would be able to draw her own conclusions of how to solve the exercise. Instead, Jenny's intention with using the teacher as a Mathematical resource is to get instructions of what to do. The explanation she gets from the teacher does not correspond to her Mathematical learning object, which makes the teacher less useful for her. Thus, when the teacher is treating the mathematics in a university manner, Jenny tries to make sense of the exercise according to the local situation and her experiences and beliefs about what the teacher as a Mathematical resource can and should offer. To be confronted with this discrepancy between Mathematical learning objects and Mathematical resources is a crucial step in the transition from a student perspective.

The transition from a student perspective

The short episodes shown above offer samples of the empirical data that the second part of the results is based upon. This consists of a description of the transition from a student perspective, based on changes of the character of and the relations between the categories Mathematical learning objects, Mathematical resources and Students as learners. One important aspect of the transition is that learning mathematics at university demands *complementary use of Mathematical resources*. This involves both using new Mathematical resources, but also to use familiar Mathematical resources in a partly new manner. This is well illustrated in Episode 1, where Sara has to use the textbook as a Mathematical resource before she can start working with exercises. The lecture, given by the teacher, is not a sufficient Mathematical resource for the Mathematical learning object, which in this case can be both process and product oriented.

Episode 2 is one example of how the transition results in an *increasing gap between Mathematical learning objects and Mathematical resources*. This increasing discrepancy is also manifested in students' intentions to use more generic examples for supporting a more process orientated Mathematical learning object, and in students' completion of Mathematical resources to obtain more meta-Mathematical learning objects (Stadler, 2009). Both episodes show the first implications of students' reorientation of Mathematical learning focus towards more structural understanding and how to use different Mathematical resources in a more rewarding manner.

The categories Mathematical learning objects, Mathematical resources and Students as learners can be used to describe, illustrate and understand central aspects of learning and understanding mathematics from a student perspective. It is important to emphasise that the categories do not offer descriptions of new phenomena concerning students' learning and understanding of mathematics, but new ways of describing familiar situations. Thus, my research contribution is not the discovery of the transition as a new phenomenon. Instead I offer a description of a transition, which many students and teachers have empirical experiences of, with new theoretical concepts that makes it possible to gain new insight and knowledge of this complex real world issue.

The changes and transformations of the categories and especially the relations between them constitute a theoretical description of crucial aspects of the transition, as seen from a student perspective. Instead of using a pre-defined theoretical perspective, these descriptions contain individual, cultural and situated elements of learning and understanding mathematics in a new setting. An additional potential with these results is the possibility to make an aposteriori analysis of the relations between these theoretical perspectives. Thus, in addition to generating a theoretical description of the transition, an analysis of the more general theoretical character of the transition can be done. Instead of beginning with adopting a theoretical perspective on the transition, an analysis of whether the character of the transition should be regarded as mainly an individual, sociocultural or situated entity can be done through further analysis of the empirical data.

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Relationships Between Epistemological Beliefs and Properties of Discourse: Some Empirical Explorations

Magnus Österholm
Umeå University, Sweden

In this paper, I investigate what types of epistemologies are conveyed through properties of mathematical discourse in two lectures. A main purpose is to develop and explore methods for a type of analysis for this investigation. The analysis focuses on the types of statements and types of argumentations used in explicit argumentations in the lectures. This type of analysis proves to be useful when characterizing epistemological aspects of lectures. However, some limitations are also noted, in particular that it was common to use more implicit types of argumentations in the lectures, which was not included as data in the present analysis.

Introduction

There seems to be an agreement in educational research about the importance of beliefs for understanding the processes of teaching and learning. For example, there exist plenty of quantitative empirical studies showing a connection between variation in students' epistemological beliefs and variation in different kinds of ability or comprehension (Schommer, 1990). Results about connections between beliefs and teaching seem to be more tentative, which have highlighted some problematic theoretical and methodological issues (Pajares, 1992; Skott, 2005; Speer, 2005). A general problem with this kind of research is the focus on such a "large" construct as teaching practice since this practice can be influenced by many factors, in particular that it is not only beliefs that influence the decisions a teacher makes during lessons (Skott, 2005).

Thus, there is a need to study the relationship between beliefs and teaching at a more detailed level. My choice in this endeavor is to focus on epistemological beliefs and on some aspects of communication in teaching situations. Epistemological belief here refers to belief *about knowledge* (what knowledge is) and *about knowing* (how knowledge is acquired), which is sometimes referred to using different notions such as "personal epistemology, epistemological beliefs or theories, ways of knowing, or epistemic cognition" (Hofer, 2002, p. 3).

A study of relations between epistemological beliefs and communication can include how beliefs can affect, or be affected by, communication, including how one expresses oneself or how one interprets something expressed by someone else (e.g. in writing or orally). Another perspective on the relations between beliefs and communication is not to see them as two separate "objects" that can affect each other, but as more integrated aspects of cognition and/or behavior. These different perspectives on relationships between beliefs and communication are discussed more in the section Theoretical perspectives.

Purpose

In this paper, I examine the types of epistemological beliefs that are highlighted, more or less implicitly, through how one talk about mathematics. Thus, an objective is to examine what types of epistemologies are conveyed through properties of mathematical discourse. In particular, mathematical lectures are analyzed in this paper.

This study is a first attempt at empirically analyzing connections between epistemological beliefs and communication. Therefore, a main purpose is to develop and explore methods for this type of analysis. What is meant by "highlighted" is therefore, at least for now, defined through the types of analyses I use in the present study.

Theoretical perspectives

The overall interest in this paper is to study the relationship between epistemological beliefs and communication. In the present study I have an empirical focus, but in my previous studies the focus was on theoretical aspects. For example, theories describing epistemological beliefs and communication have been compared (Österholm, 2009). This comparison revealed some differences between theories regarding assumptions about cognition and discourse, but it also revealed some possibilities to develop existing theories in order to create a coherent framework for forthcoming studies about relationships between epistemological beliefs and communication. However, in the present paper I will not use a specific theory, but in the following I argue that the type of analysis performed here is relevant for different types of theories, including theories with a cognitive perspective and theories with a discourse perspective.

In general, the students taking part in a lecture are (or could be) influenced in some way by what is presented. This influence can include aspects of epistemology, through what types of epistemologies are highlighted in the presentation, implicitly or explicitly.

From a cognitive perspective, a lecture could be described using the notions of sender and receiver. The lecturer's beliefs can be seen as a cause for how s/he presents the mathematics, for example that epistemological beliefs are a basis for how it is argued that one knows something. How the lecturer presents the mathematics is then influencing how the students think about mathematics, including epistemological aspects. In this perspective, focus is on cause and effect, where the study of properties of discourse in the lecture can be relevant both as a sign of the lecturer's beliefs and also as a potential cause for students' beliefs. The focus on cause and effect seem common in educational research about beliefs, for example when studying teachers' *attributed* beliefs (Speer, 2005) or when *explaining* students' differences in performance through differences in beliefs (Schommer, 1990).

From a discourse perspective, or a social perspective, a lecture could be described using the notions of participation and enculturation. The lecturer's statements are then not seen as a reflection of some cognitive structure, but as being constitutive themselves, as a part of the social situation (Skott, 2009). In this perspective, beliefs could be defined in terms of discourse practices (Edwards & Potter, 2005). The study of properties of discourse in the lecture can then be relevant both in the process of defining belief in this manner and also by seeing it as a part of students' enculturation (including becoming familiar with the discourse on epistemological aspects of mathematics).

Method

Two mathematics lectures at university level are analyzed in this study. At this point, the main purpose is not to compare the lectures or lecturers, but to develop and test some analytical tools for the analysis of epistemological properties of mathematical discourse (i.e. the analysis of what types of epistemologies are conveyed). For this purpose, the chosen lectures have different lecturers and different types of mathematical content, in order to have more differentiated discourses for analyses. One lecture is part of a course in calculus and this particular lecture is about improper integrals, while the other lecture is part of a course in statistics for natural scientists and this particular lecture is about some examples of discrete probability distributions. Both lectures are approximately two times 45 minutes long, but in this paper only the first part of each lecture is analyzed, since the plan is to use the other half for some other type of analysis for comparison. Only the lecturers' activity is analyzed, in order to focus on one type of discourse; the one used in lecturing, and not in for example dialogue. The lectures were recorded with audio and video, but students' statements are not audible in the recordings and the camera is always focusing on the lecturer's activity at the whiteboard.

The analysis in this paper focuses on the lecturers' auditory communication, and the lectures were transcribed from the audio recording, but using the video recording in case of doubt in the process of transcription and in case of unclear references in the lecturers' statements (e.g. referring to "this" or "that" when pointing to something on the whiteboard).

Basis for data analysis

The type of analysis used in this paper is somewhat inspired by the framework of epistemological resources (Hammer & Elby, 2002), which utilizes a bottom-up type of analysis when observing children's behavior in situations when they decide how they know something. Other frameworks seem to have a more top-down perspective, when describing categories of epistemological beliefs that can be discerned theoretically or philosophically (e.g. Schommer, 1990). Such types of categories seem difficult to apply to the type of data used in the present paper. In addition, my previous analyses show that the bottom-up perspective seems to be the best starting point in the study of relationships between epistemological beliefs and communication (Österholm, 2009). Therefore, I create my own structure for how to analyze epistemological aspects of discourse in mathematics lectures, but relate to other relevant frameworks in the creation of this structure.

Two central aspects of epistemology are the nature of knowledge (what knowledge is) and the nature of knowing (how knowledge is acquired). The *types of statements* used in a lecture could highlight the first epistemological aspect, regarding the nature of knowledge, and the *types of argumentations* used could highlight the other aspect, regarding the nature of knowing. Instead of analyzing all statements in a lecture, I choose to focus on those statements that are part of an explicit argumentation, that is, statements that are used when (at least) one statement is explicitly given as an argument for another statement. For example, when stating that "function f looks the same to the left as it does to the right *since* it is an even function", the second statement is given as an explicit argument for the first statement through the use of 'since'. The main reason for the choice to limit the analysis to these explicit argumentations is to have a clear focus in the type of data I use and also that both epistemological aspects can be included in the analysis. However, it should be noted that this choice excludes some aspects of verbal communication as well as other forms of communication that could be relevant from an epistemological perspective, but for now this choice is seen as suitable in order to have a clear focus regarding type of data.

When focusing on the *types* of statements and the *types* of argumentations, the analysis does not focus on the mathematical content of the statements or the argumentations, and the purpose of the analysis is not didactical, in the sense that the focus is not on aspects of teaching and learning the mathematical content nor on the teaching and learning of argumentation or proving. Instead, the analysis of types of statements and types of argumentations in the lectures is used in order to draw conclusions about what is conveyed *about* mathematics, in particular regarding epistemological aspects.

In order to create an a priori categorization of *types of statements* relevant from an epistemological perspective, I relate to a central distinction in mathematics education regarding different aspects of knowledge; conceptual and procedural knowledge (Hiebert, 1986). Thus, in the analysis of statements used in a lecture I separate two main types; statements about the use of mathematics objects, labeled *use-statements* (related to procedural knowledge) and statements about properties of mathematical objects, labeled *object-statements* (related to conceptual knowledge). Here I choose to use the general notion 'mathematical object', which can refer to concepts as well as procedures. The difference between the two types of statements is therefore that they describe either properties of objects or the use of such objects, which is seen as a central aspect regarding the difference between conceptual and procedural knowledge. For example, the statement "The derivative of ln x is one over x" is an object-statement while the statement "When you take the derivative of ln x you get one over x" is a use-statement.

Regarding the *types of argumentations*, you could use some elaborate framework for the analysis, such as that of Toulmin (1958). However, for the more exploratory type of purpose in this paper, I choose to use a more simplified structure for my analysis, consisting of a conclusion that is drawn (or a *claim*, using Toulmin's vocabulary) together with statement(s) used as argument for this conclusion. In order to locate the argument, the words or wordings used to make explicit the argumentative relationship between statements are of great importance (e.g. words such as 'therefore' and 'since'). As an abbreviation, these words or wordings are labeled *connect-words*, and the analysis in this paper will focus on these types of words.

From these main aspects of my intended analysis, some more specific areas of interest can be outlined as a guide for the exploration of the results from the analysis:

- Regarding the types of statements, focusing on the use- and object-statements
    - If there is a tendency to use different types of statements in the different lectures, which could highlight properties of different areas of mathematics or of purposes of different types of courses.
    - If statements of one kind easily could be re-formulated to become a statement of the other kind, which highlights the possibility to choose how to express yourself.
- Regarding the types of argumentations, focusing on the connect-words
    - What types of connect-words that are used and how they are used, e.g. if they are used in a consistent and clear way.
    - How chains of arguments are created, i.e. argumentations consisting not only of the relationship between two statements.

Procedure of data analysis

Statements from the lectures are analyzed in several steps, in order to create a clear structure in the analysis and also to make certain that only relevant statements are analyzed. However, it is not as certain that *all* relevant statements are analyzed, but the purpose of this paper is not to create a complete picture of each lecture or lecturer. Instead, the focus of this paper is on the creation and exploration of the method of analysis.

The first step in the analysis is to mark use- and object-statements, and also connect-words in the transcription. Each coherent section of the transcription is then extracted from the transcription, for further analysis. A coherent section refers to a set of statements that are connected through the use of connect-words. Such a section can for example be only one conclusion together with an argument, as in the following example from the lecture in calculus, where the connect-words are in italics: “The derivative of ln x is one over x, which is larger than zero, *which means that* it grows all the time”. Note that there is actually a linguistic ambiguity about exactly what the word ‘which’ in ‘which means’ refers to, but logically all information given before the conclusion is needed, and this full statement is therefore regarded as the argument. A section can also include several argumentations, as in the following example from the lecture in statistics:

> And you can show that the expected value is one over p. This can be seen as. Yes if we imagine that for example p is zero point two. Then this means that we will succeed on average each fifth time. And this also means that the expected value then becomes one over zero point two, which is five. So it is exactly that we will have to do on average five tries in this case.

A next step in the analysis, which is mostly relevant for sections that do not consist of a single argumentation, is to extract the relevant statements from the excerpt and arrange them in a structured manner, which for the latest example can be done in the following way, where the connect-words are in italics:

1. You can show that the expected value is one over p.
2. P is zero point two.
3. *Means that*: We will succeed on average each fifth time.
4. *Means that*: The expected value is one over zero point two, which is five.
5. *So*: We have to do on average five tries.

From this structure it is easier to analyze how the statements are related according to the connect-words, although the analysis has to include some considerations to what is reasonable, as was done in the previous example about the derivative, since it is not always clear exactly what is referred to as being the argument for the conclusion. In such situations, the logically necessary statements previously stated are listed as included in the argument. In this example, we see that line 3 is only based on line 2 as an argument, while line 4 cannot only be based on the previous line, although the exact same type of connect-words are used. The result of this type of analysis is then summarized in a three column table with a conclusion, the argument(s) for this conclusion, and the connect-words used in the argumentation. From the example above, one line in the table thus becomes:

We will succeed on average each fifth time	P is zero point two	Means that

An exploratory analysis can then be performed on the content of these tables, one for each lecture, in relation to the areas of interest outlined at the end of the previous section.

Results from data analysis

Before discussing the areas of interest in the exploration of the results from the data analysis, it can be noted that many statements from the lectures are not part of this exploration since they do not have an explicit connection to another statement in an argumentative way. For example, there are many statements in the process of formal calculations for which no explicit connect-words are used, but where one can assume that everybody knows that one step in the process is seen as the argument for the next step. However, there are also statements that are not part of such a process but still are not explicitly used in an argumentation. This is the case in situations when statements are listed one after the other, where perhaps it is meant that the second statement is a conclusion based on the first, like the following example from the calculus lecture: "f is an even function, it looks the same to the left as it does to the right". In this example there is no explicit connection between these two statements, and a reason for this might be that they are seen as synonymous, but logically the argumentation could go in any direction between these statements.

Through the described procedure of analysis, the produced tables for each lecture consist of 39 lines for the calculus lecture and 43 lines for the statistics lecture. Each line in the table corresponds to one argumentation, which consists of a conclusion, the statement(s) used as argument for the conclusion, and the connect-words.

Use- and object-statements

When studying how common the different types of statements are in the two lectures, a clear difference between these lectures appears: In the calculus lecture, use-statements appear as a conclusion on four lines in the table (10 %) and as an argument on two lines (5 %), while in the statistics lecture, use-statements appear as a conclusion on 22 lines (51 %) and as an argument on 18 lines (42 %). Thus, in the calculus lecture object-statements are most common, while in the statistics lecture the two types of statements are about equally common.

There are several examples of statements of one type that can easily be reformulated in order to turn it into a statement of the other type. For example, in the statistics lecture there is the use-statement "if you add a constant to all values of the function, this will not change the variation", which can be reformulated into a statement saying that a property (the variation) of two functions is the same (i.e. an object-statement). An example of the opposite type of reformulation is taken from the calculus lecture, where there is the object-statement "(the graph of) one over x has a similar appearance (as the graph of one over x squared)", which can be reformulated into "if we sketch the graph of one over x, the result is similar as when we sketch the graph of one over x squared" (i.e. a use-statement). At the moment no more in-depth analysis has been made regarding this aspect of the relationship between these types of statements, mainly because there is vagueness in the "easiness" of reformulation. The easiness has so far been seen as a sort of reasonable type of reformulation in the sense that you could imagine someone using this new formulation, so that it is not merely something that is grammatically correct but could be seen as part of mathematical discourse.

Connect-words

A common connect-word is 'so'; it is used in about half of all argumentations. In the performed analysis this word has been interpreted as a word that can signal an argumentative relation. However, there are also several examples where this word is primarily used as a sort of temporal transition in the monolog. This observation creates some doubts whether there are any real differences between to use this kind of connect-word and to only give statements without any connect-word, where it could be meant or assumed that one statement follows from a previous one.

There are several occasions where it is unclear which statement(s) is/are referred to as argument for a specific conclusion, of the same kind presented through an example in the section Procedure of data analysis. This type of uncertainty occurs in chains of arguments, where one statement is part of several single argumentations within a section of a lecture. I have found only one example where the lecturer talks more explicitly about the relationships between statements in a chain of arguments, while all other arguments are signaled by more simple connect-words, if signaled explicitly at all. This one example is from the statistics lecture, where the lecturer refers to what they did recently (they told what type of random variable x was) and to what assumptions have been made (that a certain probability equals 0.7) as arguments for what parameters the random variable has.

Conclusions

The study of use- and object-statements shows a potential difference between the two lectures regarding some epistemological aspects. However, since many statements seem easy to reformulate into the other type of statement, there is some arbitrariness, but not necessarily randomness, regarding what type of statement is used. These observations highlight the questions if or how these properties of discourse can be seen as tied to the individual, to the mathematical content, to the type of course, or to other aspects of the situation. Such questions seem possible to examine in more detail using the type of method for data analysis presented in this paper. However, it is also necessary to relate to a theoretical perspective since the interpretation of the results from the data analysis depends on a chosen theory. For example, differences in discourse can be seen as mainly caused by "properties" of the lecturer or as constitutive in the situation (see the section about theoretical perspectives).

The analysis in this paper has focused on explicit argumentative connections between statements in the discourse of mathematics lectures. The fact that many statements in the lectures are not part of this analysis together with the unclear uses of connect-words that have been observed show that more implicit types of argumentation seem common, at least in the lectures studied here. Other types of analyses are therefore needed in order to characterize these implicit types of argumentation. More generally, as Duval (1999) points out; "argumentation cannot actually be reduced to the use of a single argument". Thus, in order to capture also more implicit aspects of argumentation, the analysis cannot focus only on linguistic aspects but needs to take into account for example contextual aspects, including "the position of the person being spoken to relative to the arguer […], the motivation of the argumentation […] and its objective" (Duval, 1999).

If students are mostly exposed to the more implicit types of argumentation it would be interesting to examine how students interpret these and how they handle situations where more explicit argumentations are demanded from them, which might be the case in exams. In addition, it would be interesting to compare epistemological characterizations of different settings (not only lectures) for the same person and of communication of different persons (in particular to include also students) within one setting.

In conclusion, on the one hand I have shown the usefulness of the type of analysis presented in this paper, but on the other hand I have also noted a need for other types of analyses in order to better characterize epistemological aspects of mathematical discourse.

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A Study of Problem Centered Approach in Mathematics

Yukiko Asami Johansson
Linköping university and University of Gävle, Sweden

This paper presents a study of the use of a Japanese mathematics teaching method in the Swedish classroom discourse. In Japan, classroom discourse centered on problem solving was developed in a way that was distinctive compared to that in the United States. Through the process of solving mathematical problems, it aims to foster students' creative attitude towards mathematical thinking. Meanwhile in the U. S. mathematics lessons foster and develop students' abilities to solve mathematical problems (Nagasaki, 2007). Hiebert, Stigler, and Manaster (1999) analysed the differences in teaching methods and classroom discourse between Japan, Germany and the US, by studying video films from the TIMSS research material. They state that Japanese teachers emphasize mathematical thinking rather than mathematical skills as a goal of the lesson and "select problems to begin the lesson that can be solved by modifying methods that have been developed during the previous lesson" (p. 200) and that their students are encouraged to develop the methods themselves. Kazuhiko Souma is one of the pioneers who has introduced and practiced such classroom discourse. He calls his method "problem centered approach (PCA) - method" (author's translation; "mondaikaiketu no jugyou", in Japanese). I have two main intentions with my project. First, to describe and analyse the PCA method, in the relation to other Japanese methods in the same tradition, second, to analyse the possibility of adopting this method to Swedish classrooms in order to develop the mathematical discourse focusing on foster students' mathematical thinking.

The PCA method stresses that the teacher should present partial problems in order to encourage a group discussion concerning alternatives. In particular, the teacher should allow the students to guess answers, to make conjectures and to reach solutions by discussion before advancing to methods and definitions. It emphasizes that the teacher carefully needs to prepare problems and subjects, so that students can relate them to and distinguish them from previous knowledge and so that these problems can lead to a multitude of answers and thoughts. It appears that the PCA- method has many common points with socio-cultural and social constructivist theories: Through communicating mathematical ideas and by formulating reasoning, the learning subject constructs and assimilates his or her thoughts (Sfard, 2008). Students' individual mathematical experience is linked to verbal communication (Björkvist, 1993). The processing of solving well-crafted problems as regular classroom activities supports the developing students' mathematical thinking (Silver, Kilpatrick, & Schlesinger, 1990).

Before using this method there are some requirements that need to be addressed. First, the teacher must be confident in his/her mathematical knowledge and thus able to detect equivalence in mathematical methods and representations of mathematical objects (Wood, 1993). Second, the teacher must establish a social classroom norm, so that the students can feel totally safe to express and justify their actual thought. It is important that the students have a positive attitude towards listening to their peers' explanations of mathematical topics (Yackel & Cobb, 1996; Wood, 1993).

In this presentation I will present a short qualitative empirical study of a Swedish class in an upper secondary school, using the problem applying method. The study is based on video recordings of Swedish and Japanese classes and interviews with teachers. The content area is algebra and the solving of equations and I am focusing on how the method influences the students' level of activity in the classroom and how the teacher uses it to foster their attitude towards mathematical discourse in the classroom.

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Improving Undergraduate Mathematics Teaching

Learning Study - The Definite Integral Concept

Iiris Attorps, Kjell Björk and Mirko Radic
University of Gävle, Sweden

This paper reports on a design of a learning study regarding the definite integral concept in undergraduate mathematics teaching. The concept of the definite integral belongs to the core of calculus of a real variable. In Sweden, as in many other countries, the concept is introduced to students at their age of 17–18, during the last two years of upper secondary school.

Several studies have highlighted difficulties that students encounter with the integral concept (Artigue 2001; Rasslan & Tall 2001). In these studies it has been found that students' technical ability could be quite strong, despite their minimal understanding of the concept. Transformation from procedural to conceptual understanding of the notion of the definite integral requires gradual reconstructions of students' perceptions. Earlier research has, however, documented the limitations of standard teaching methods, showing that students have become reasonably successful on standard tasks and procedures but have difficulties in developing a solid conceptual understanding of the topic itself (Artigue, 2001).

The experiment was carried out at a Swedish university by using the cyclic Learning Study model, which is based on Variation Theory (Marton, Runesson, & Tsui 2004). There are essentially two fundamentals in Variation Theory. The first one is that learning always has an object, in our case the definite integral concept. The second one is that the object of learning is experienced and conceptualized by learners in different ways. In our experiment we started with a pre-test in order to diagnostisize the students' pre-knowledge. Using the results in the pre-test, we planned the first lesson which was conducted by one of the authors. Students' learning outcomes were then evaluated in a post-test. If the results with respect to the goals were not satisfactory, we revised the lecture and carried it out again.

Along with Variation Theory we applied in our experiment the theory of concept definition and concept image (Tall & Vinner, 1981). In the latter perspective, the most important goal for a teacher in mathematics ought to be to change the students' conceptions from trivial (e.g. integral = area) to mathematical perceptions, i.e. to change the students' personal knowledge of the mathematical concepts to the ideas that correspond to mathematical definitions. This goal can only be achieved if the students are given varying problems which cannot be solved correctly by referring just to the concept image.

The overall research question in this study is: Is it possible to use the Learning Study model and the Variation Theory when developing the teaching of mathematics at an undergraduate level? What are the critical aspects for students' learning?

The data consist of the documents and observations of four lessons together with students' interviews and answers to pre- and post-tests. Both engineering and teacher students participated. In the study both quantitative and qualitative analysis methods were applied. The preliminary results in our experiment indicate that the students' conceptions of the definite integral can be enhanced significantly by using the Learning Study model.

Furthermore the research results indicate that a majority of the students' understanding of the definite integral is at an operational level (Asiala et al., 1997) and that they cannot describe meaningfully the Fundamental Theorem of Calculus.

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Klassrumsbedömning och betygsättning i gymnasiematematiken

Semir Becevic
Linköpings Universitet, Sweden

Introduktion

Det finns flera skäl att vi ska titta närmare på hur lärare resonerar om bedömning och betygssättning i gymnasieskolans matematik.

Trots att det redan finns mängder av styr- och policydokument som på ett eller annat sätt förklarar hur det ska gå till råder en allmän känsla på skolorna att lärare sätter olikt varandra betyg på olika ställen, både på den egna skolan och mellan skolorna. Lärarnas grundresonemang om egen betygsättning och bedömning med funderingar om vilka moment vägs in i betyget samt varför är oerhört viktiga i sammanhanget.

Behoven av att se en helhet och samspel mellan planering av undervisningen som ska bedrivas och bedömningen/betygsättningen som ska ingå ges här som en självklarhet.

Att få igång lärardiskussioner angående betygsättning och tolkningar av målen och kriterierna inom den egna skolan eller utanför den, att försöka få fram vad som anses fördelaktigt respektive bristfälligt med dagens bedömning och betygsättning och att fokusera på relationen mellan de kvantitativa ingredienserna i målen och de kvalitativa delarna i betygskriterierna exemplifierar enbart ytterligare fler anledningar.

Syfte och metod

Syftet med studien är att belysa hur lärare resonerar om bedömning och betygsättning i matematik på gymnasienivån. Forskningsfrågan är:

> *Hur beskriver matematiklärare i gymnasieskolan sina strategier beträffande bedömning och betygsättning samt hur legitimerar de dessa?*

Den empiriska delen av studien består av 15 halv-strukturerade individuella lärarintervjuer och 7 gruppinspelade lärardiskussioner. Lärarna har i de individuella intervjuerna gett sina egna bilder av hur det går till när de sätter kursbetyg, hur undervisning och bedömning/betygsättning hänger ihop och vilka nackdelar/fördelar de stöter på. I de ljudinspelade gruppdiskussionerna har lärarna fått möjligheten att fritt diskutera bedömning och betygsättning över skolgränserna.

Teori och analys

Studiens angreppssätt är både diskurspsykologiskt och antropologiskt. Diskursanalys (Jørgensen & Phillips, 2000; Potter & Wetherell, 1987) varvas ihop med den antropologiska teorin (Bosch & Gascón, 2006; Chevallard, 1992) inom den matematikdidaktiska ramen. Medan diskursanalysens roll är att identifiera och analysera lärarnas diskursiva konstruktioner syftar den antropologiska teorin att institutionalisera och lokalisera de förekommande diskurserna.

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Val av arbetsmetoder och diskursiva strategier i syfte att främja klassrumskommunikationen om matematik

Marie Bergholm
Linköpings universitet, Sweden

Svenska elevers genomsnittliga matematikkunskaper har försämras, även i jämförelse med elever i andra länder (Skolverket, 2008, 2009). Styrdokumentens krav på individualisering har inte inneburit en individualisering efter elevens behov, utan istället i bemärkelsen att eleverna i allt högre grad får arbeta med uppgifterna i läroboken efter egen förmåga (Skolverket, 2008, 2009; SOU 2004:97). Behov finns att bedriva vidare forskning och fråga inte bara i vilken omfattning, utan också hur och varför, läroboken används i undervisningen (Johansson, 2006). Samuelsson (2008) framhåller att varierande undervisningsmetoder krävs för att olika matematiska kompetenser ska kunna lyftas fram. Han betonar också vikten av att läraren använder strategier för att skapa ett stödjande klassrumsklimat, vilket bidrar till att eleverna utvecklar en positiv inställning till matematik. Elever som i undervisningen erbjuds möjlighet att lyssna till och delge varandra kunskaper, presterar bättre (Boaler, 2002). I PISA 2003 redogörs för att emotionella faktorer påverkar motivationen och förmågan att urskilja och tillägna sig lärandemål (OECD, 2004).

Denna studie utgår från det kommognitiva (eng. *commognitive*) teoretiska ramverket, som tar fäste i ett sociokulturellt perspektiv med rötter hos Vygotsky (Sfard, 1008). Lärande definieras här som den process där eleverna utvecklar, modifierar och förbättrar sin diskursiva repertoar i matematik. Den matematiska diskursen följer karakteristiska mönster av rutiner, ord, visuella mediatorer och berättelser om matematiska objekt. Dessa återfinns även på en metanivå, då diskursen expanderar med nya ord och rutiner (Sfard, 2008). Utifrån detta perspektiv påverkas elevernas lärande i hög grad av den befintliga klassrumsdiskursen och lärarnas val av kommunicerande arbetsformer och strategier.

Syftet i studien är att utforska vilka undervisningsstrategier lärarna beskriver gynnar kommunikationen om matematik i klassrummet. Mer specifikt fokuseras på att identifiera diskursiva strategier och arbetssätt lärarna anser framgångsrika för att utveckla elevernas lärande och olika matematiska kompetenser. Forskningsfrågor: Vilken betydelse tillskriver lärarna den matematiska kommunikationens roll för elevernas lärande? Vilka diskursiva strategier och arbetssätt beskriver lärarna främjar kommunikationen om matematik? Vilka är de förväntade matematiska kompetenser lärarna avser att eleverna ska utveckla?

Metodvalet är semistrukturerade intervjuer av 25 gymnasielärare, där inspelade samtal dokumenterar lärarnas reflektioner omkring sin praktik (Gudmundsdottir, 1991). De intervjuade lärarna exemplifierar sin praktik utifrån skiftande kursinnehåll och nivåer, varför analysen av diskursiva strategier och arbetsmetoder kopplats till de kompetensrelaterade aktiviteter lärarna uppger sig sträva efter. Med inspiration hämtad från två kompetensramverk har följande kompetenser definierats: problemlösnings-, resonemangs- och procedurhanteringskompetens, kompetensen att förstå olika representationer och deras samband, förmåga till en positiv attityd till ämnet, samt att kommunicera med ett matematiskt språk (Lithner et.al.,2010; Kilpatrich, Swafford & Findell, 2001). Ur ett commognitivt perspektiv kan kommunikationskompetens betraktas som en paraplykompetens, delvis överordnad utvecklingen av flertalet av övriga kompetenser, vilket är avsikten att belysa under presentationen.

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Making the Mathematics Visible in Children's Free Activities in Preschool - Challenges for the Teaching Profession

Kerstin Bäckman
Åbo Akademi University, Finland and University of Gävle, Sweden

This paper reports on how preschool teachers from two preschools work with four years old preschool children's mathematical learning and expressions. In this study The Learning Study Model is applied, in which mathematical content is focused and a variation in the learning activities is arranged (Marton, Runesson, & Tsui, 2004; Runesson 2006). The Learning Study Model is based on Variation Theory. In this theory necessary conditions for learning are the experience of discernment, simultaneity and variation. In the preschool model of Learning Study used in this study play and children's experience of critical aspects are important. The teachers have chosen to work with the concept of numbers. Variation should be done in the context of practical activities and through variation different aspects of relations within numbers and between numbers can be discerned.

The present study reports from one part of my ongoing study, namely teachers' opportunities to catch children's mathematical learning and expressions in the preschool context. In the study, play and children's various way of thinking on mathematical tasks play an important role.

Children experience mathematics in different ways in their everyday life, for example through play in the sandboxes, climbing in the woods, building decks, at lunchtime and in conversation with friends. They play games, listen to stories and they get challenges in their logical thinking through problem solving and in conversation with playmates and adults. In the interaction with the outside world children experience mathematic physically and mentally, which in turn enables them to create representations of various mathematical concepts and meanings. In order to grasp the children's views there must be adults who listen, follow up, ask questions and challenge their thinking.

The overall research question in the study is: How can a preschool teacher catch mathematics and challenge learning in children's activities in the preschool context?

In the study I have used a video camera in interview situations, in planned activities and in children's play in the preschool environment. The data consist of video observations from children's interviews, from planned activities and from children's play. In the study qualitative analysis methods are applied (Lindahl, 2003; Marton et al., 2004; Runesson 2006). The research results so far indicate that teachers do not consciously use mathematical language in connection with everyday language in spontaneous learning situations.

The data analysis suggests that children think and reason, explain and draw conclusions, sometimes they use mathematical language but it is also common that they are quiet when they are exploring things together. The preschool environment gives children opportunities to use numeracy; explore shape, size and pattern during block play and imaginative play when they play inside and outside but children often play with themselves without adults. Children explore their environment and communicate with their bodies and minds not always with words. When teachers are included in children's activities they can help children to identify critical aspects of a phenomenon. In this study, the teachers use variation to make relationships within and between numbers visible in children's activities in order to help children to discern critical aspects and thereby learn. Therefore teachers who can see the mathematics in children's activities may have opportunities to discern critical aspects for children's learning and thereby "catch a learning moment".

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Stimulering av matematiska förmågor i en matematisk aktivitet

Henrik Carlsson
Linnéuniversitet, Växjö, Sweden

Bakgrund

Studien som berörs här är en del av ett större forskningsprojekt som syftar till att undersöka hur undervisningen kan organiseras för elever på gymnasiet med fallenhet och intresse för matematik. Syftet med studien är att utforma en undervisning som stimulerar två utvalda matematiska förmågor: förmågan att *minnas matematisk information* och att *begränsa ett matematiskt resonemang*. Förmågan att minnas matematisk information är väsentlig redan vid inledningen av lösningsprocessen, dvs. när eleverna börjar generalisera den matematiska informationen i en problemlösningssituation (Krutetskii, 1976). Detta i sin tur följs direkt av förmågan att begränsa sitt arbete genom ett matematiskt resonemang (ibid., s. 269-270).

Eleverna som deltar i studien går på gymnasiets naturvetenskapliga och tekniska program. Dessa elever observeras i en longitudinell fallstudie när de arbetar med uppgifter inom talteori individuellt och i grupp. Intressanta frågor för studien är hur dessa elevers lärprocesser kan stimuleras och uppmuntras och vilken betydelse den aktuella aktivitetens utformning har som stöd för resonemangs- och minnesförmågan.

Matematiska förmågor

Definitioner av matematiska förmågor presenterades 1976 av den ryske psykologen V.A. Krutetskii och dessa utgör grunden i det teoretiska ramverket i den aktuella studien. En förutsättning i Krutetskiis studie är att matematiska förmågor endast kan identifieras och utvecklas när en individ ägnar sig åt en matematisk aktivitet.

Förmågan att begränsa ett matematiskt resonemang

Förmågan att begränsa ett matematiskt resonemang synliggörs och stimuleras när eleven arbetar med att bryta ner och förenkla en uppgift, dvs. när ett resonemang förkortas på olika sätt beroende på uppgiftens struktur och svårighetsgrad. Krutetskii (1976) visar hur elever med fallenhet för matematik har en tendens till snabba och radikala förkortningar av resonemang när de löser matematiska uppgifter. Med en snabb och radikal förkortning menas den tid det tar för en elev att lösa uppgiften, oftast eleganta lösningar, och den tid som tillbringas för nödvändiga beräkningar. (Ibid*.,* s. 264-275)

Förmågan att kunna bevara matematisk information
Elever som har förmågan att ekonomisera sina resonemang minns vanligtvis inte överflödig information som t.ex. numeriska värden som har förts in i en uppgift. Vid observationer noteras vad eleverna minns från tidigare möten med ett matematiskt stoff. För att studera minnesförmågan behöver vi alltså följa elevernas arbete genom en serie uppgifter av liknande karaktär. (Ibid*.,* s. 295)

Metod

Studien är upplagd som en serie interventioner där elever följs genom tre sekvenser som pågår över sex lektioner. I den första sekvensen inhämtar eleverna kunskap vid genomgångar och diskussioner med läraren och med övriga elever. I sekvens två och tre genomför eleverna olika problemlösningsuppgifter och då undersöks hur de uttrycker de förmågor som fokuseras i studien och hur dessa förmågor utvecklats genom undervisningen i de olika sekvenserna. Sekvens tre genomförs en eller ett par månader efter den andra sekvensen och eleverna bör då få en känsla av att *denna uppgift* har gjorts tidigare (uppgiften är inte densamma, men uppgifternas struktur är liknande) och att de då kan använda sig av samma generaliserings- och lösningsmetoder som tidigare.

Eleverna kommer att ha tillgång till digitala pennor som de använder när de individuellt löser uppgifterna och vid redovisningen laddar läraren upp elevernas lösningar på en interaktiv tavla, följt av en diskussion och ett argumenterande för respektive lösningsmetoder. Studien är i ett inledningsskede och det material som hittills har samlats in har ännu inte analyserats.

Ytterligare frågor som kan behöva belysas är hur det matematiskt minnet ska definieras och hur det kan testas samt vad som ska stimuleras och uppmuntras och vilka som ska medverka i processen.

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CAS-calculators in the Classroom

Patrik Erixon
Linköping university, Sweden

Introduction

Advanced calculators with CAS (Computer Algebra Systems) are able to do algebraic calculations in addition to numerical and graphical calculations that can be made with scientific and graphical calculators. CAS-calculators have been available for many years but they are still quite rare in Swedish upper secondary school. Since autumn 2007, CAS-calculators are allowed in the national exams in Swedish upper secondary school, leading to an increased use. Internationally the use of CAS in mathematics education has varied a lot and the research around CAS has, among other things, described obstacles and different categories and difficulties in the process of instrumental genesis (Drijvers, 2002; Guin & Trouche, 1999). See Figure 1.

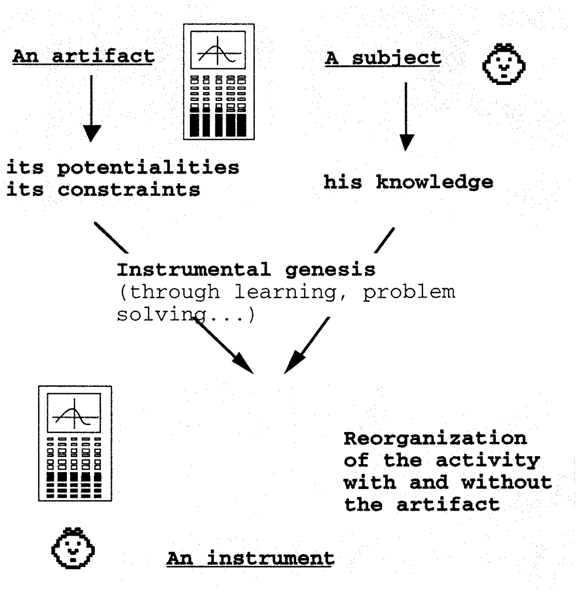

Figure 1: The process of instrumental genesis (Guin &Trouche, 1999, p. 202).

More examples can be found in a recent literature review of Persson (2009). My own background is that I during a long time have been interested in these tools and have several years of experience of teaching upper secondary science classes with CAS-calculators. I´m now studying for a licentiate degree in the research school Lic-FontD at the University of Linköping. In Sweden little research is done about CAS-calculators and a lot of the research internationally is done with designed teaching or in special courses. The aim of this research is to investigate how students use calculators with CAS in two ordinary classes in a Swedish upper secondary school. The goal is to identify critical didactical aspects, contribute to a description of current status and suggest implications for teaching and further research. The general research question is “What types of activities can be identified in students’ work with CAS-calculators?” Furthermore, in what way do these activities depend on student’s attitude, technical skills and type of exercises and how does the calculator influence student´s communication?

Method

The planned study will be made in two different upper secondary classes. A small pilot study is followed by questionnaires, classroom observations and interviews of groups of students working with selected tasks (see Figure 2). The theoretical framework for the qualitative data analysis is planned to be The Instrumental Approach (Guin, Trouche, & Ruthven, 2005). In alignment with research question and aim, the framework offers possibilities to categorize and describe different profiles for students work, development of usage, calculators’ constraints and different activities with calculators.

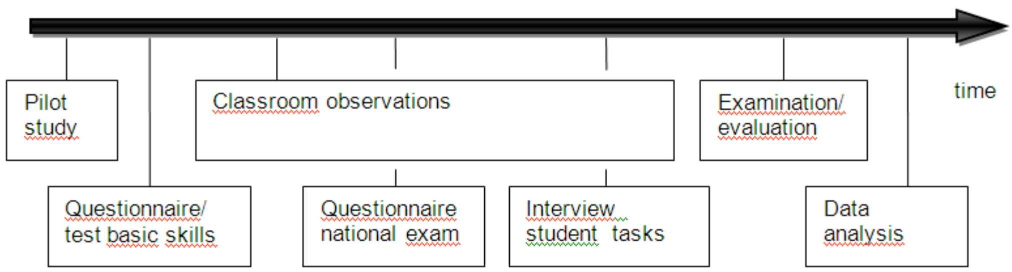

Figure 2: Time and activity plan for planned research study.

Expected results for this study are implications for teaching and research but no results are available yet since data still is being collected and analyzed.

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Communication of Solutions of Mathematical Problems Using Computer Algebra Systems

Gunnar Gjone
University of Oslo, Norway and Karlstad University, Sweden

School mathematics is changing because of technological tools being used. In some countries, e.g. Norway and Denmark, the use of CAS at national secondary exams has been allowed (in Norway in 60% of the exam time). This should change the way problems are posed and how we regard the solutions by students.

The tradition in Norway, and in many other countries, when you write up an answer it should be as complete as possible, showing the reasoning behind the solution as well as numerical calculations. For some examples see Brown (2007). The use of CAS naturally introduces new forms of solutions to problems. The following problem was given to students at the Norwegian national exam (spring 2009) in the second year of upper secondary education:

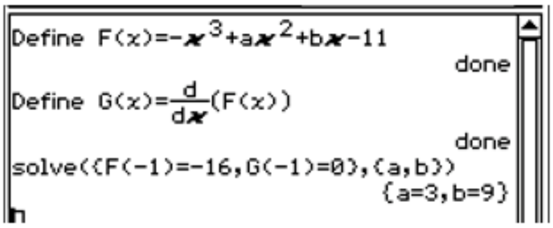
> Find the exact solution to the equation ”by calculation” (show the steps in the solution) $(\ln x)^2 + \ln x^2 = 3$

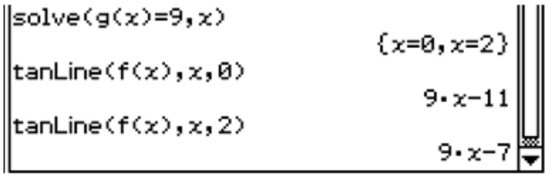
The traditional way to solve this is by writing $(\ln x)^2$ as $2 \cdot \ln x$ and solve the equation as a second degree equation in $\ln x$ and then transform the solution to find the value of x. With the use of CAS the equation can be solved directly, or almost directly. It should be noted that some mathematical software, e.g. *Microsoft Math*, will also be able – to some extent – to show steps in a calculation. One can argue that the students with the help of CAS get the answer, and then they should fill in the details.

In the same exam in Norway the students were asked to find the equation for a tangent line at a certain point to a curve, again this is a task that could be almost automatically done by a CAS tool as shown (Problem 4 (I), my translation):

> The function *f* is given by $f(x) = -x^3 + ax^2 + bx - 11$.
>
> The graph of the function *f* has a local minimum at (–1, –16).
>
> a) Show that $a = 3$ and $b = 9$.
>
> …
>
> d) Find the equations for the tangents with slope 9.

The two figures above show how these questions can be solved using CASIO ClassPad (f and g are the functions F and G with $a = 3$ and $b = 9$):





It is important to discuss what should be required of a solution, what is "natural" by using CAS. Should we be satisfied with just the answer: $a = 3$ and $b = 9$? Would it be sufficient to write down the expressions that could be used by the software? Should we require the students to write down the two conditions (in the example: two equations with two unknowns)? Or – is the problem formulation not suitable? The situation is further complicated by the fact that CAS-tools might in some cases give wrong or incomplete answers. Some such examples are presented in (Gjone, 2009).

Use of CAS extends the limits what the tools can do in solving mathematical problems, but I will argue that the requirements for presenting the solutions should follow traditional mathematical usage.

The school authorities in Norway (Utdanningsdirektoratet – UDIR) wanted an evaluation of the exams where CAS was allowed, and both qualitative and quantitative evaluations were carried out. In the quantitative evaluation surveys were given to students and teachers, in the qualitative evaluation documents and research literature were studied, and teachers and school administrators were interviewed (UDIR, 2009). Our recommendation was to postpone the introduction of CAS. The authorities in Norway (UDIR) has not – in my opinion – taken sufficiently into account, the challenges introduced by CAS, as outlined by Roger G. Brown (2007), for example assuring that the current examination structure is appropriate. In Norway it is the two graders of an exam paper who will determine the grade, and it seems that UDIR is careful not to direct the grading process too detailed. The situation calls for an international effort to formulate standards for problem solutions using CAS.

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School Mathematics: An Initiation into What?

Maria Johansson
Luleå University of Technology, Sweden

This is a short report from an ongoing study of two Swedish mathematics classrooms in the beginning of their first year in upper secondary school (Gymnasiet). Two classes were videotaped for about three weeks and students and teachers were interviewed. One class (32 students), is from an Arts Programme (Estetiska programmet, short: ES), the other (10 students), is from a track for the International Baccalaureat (IB). The data are generated within a larger study of the emergence of disparity in achievement in mathematics classrooms [1]. My initial aim was to study a phenomenon, which I described as “switch” between (school) mathematical discourse and everyday or non-specialised discourse. It is a “switch” rather than a “translation” because these discourses are fundamentally different. Reference to everyday discourse is frequently made through contextualised tasks, but also through metaphorical expressions and images. Research has shown that this is problematic and that the insertions from the everyday are likely to be misinterpreted by certain groups of students (cf., e.g., Jablonka, 2008). They have a problem in knowing to what extent they are expected to ignore the everyday meanings or not. Consequently, I wanted to see how the students in the classrooms under study managed these “switches” and how the teachers dealt with the issue in classroom interaction, in particular whether the switches between the discourses were made explicit or not. This question is important to ask, because eventually it is specialised academic mathematical knowledge that is a privileged base for success in further mathematics.

Knipping, Reid and Gellert (2009) analyse two lessons from Canada and Germany (from the same project) with a view on whether and how the everyday, familiar disappears. They find that in both lessons the familiar vanishes in an implicit way and only some students recognise that reference to the everyday is no longer legitimate. These are the successful students.

In contrast, in the two Swedish classrooms it is not obvious whether the everyday vanishes at all. The textbooks consist mainly of contextualised tasks. In the class from the ES, the tasks are divided into three “levels” (and some special tasks at the end of each chapter). It is not clear what these levels reflect. However, the teacher indicates that mastery of the different levels of tasks is linked to the grades the students will be likely to achieve. The grades are: not pass (U), pass (G), pass with distinction (VG) and pass with special distinction (MVG). In the IB class, grades are not explicitly linked to mastery of task levels. The tasks are numbered consecutively. Hence it is important to ask: Do the task levels (in the ES) or the series of tasks (in the IB) reflect any progression of knowledge from the "public domain" (of recontextualised everyday activities) towards the "esoteric domain" (cf. Dowling, 2007)? And if there is no progression towards the esoteric, then towards what does the curriculum progress? What is the legitimate discourse for each grade level and how can the students recognise it?

From the student interviews, one difference between the two classes emerged. We asked the students to pick a team from their class to join an imaginary mathematical contest. In the ES class, most students had difficulties in recognising a hierarchy in terms of the mathematical achievement amongst their classmates. In the other classroom from the IB the students were able to pick two students they considered to have the potential of being successful in a mathematical contest. But also in this classroom, which formally follows the same curriculum but uses English as a language of instruction because it aims at an international degree, the legitimate discourse is hard to recognise.

I will further illuminate the question of how a hierarchy of discourses is translated into a hierarchy of the students (in terms of grades) in these two classrooms by an analysis of the textbooks and the student-teacher interaction in the lessons. If it is not possible to differentiate domains of knowledge, on which grounds are the students then marked? And how does this relate to the production of educational (dis)advantage?

Notes

1. See Knipping et al. (2008) for a description of the research design; see also http://www.acadiau.ca/~cknippin/sd/index.html

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Developing a Tool for Analysing Upper Secondary School Textbook Tasks About Proportion and Proportionality

Anna L. V. Lundberg
Linköping University, Sweden

Most of the mathematics education studies conducted about proportion shed light on the teaching and learning of the notion of proportion in primary and lower secondary grades but we have very little knowledge about what happens during the upper secondary school (Lamon, 2007). The purpose of my study is to investigate what possibilities Swedish upper secondary school textbook tasks offer students to develop their understanding of proportion and proportionality during the first course in mathematics. What cognitive challenges do textbook tasks offer to Swedish students? What types of proportional reasoning are required to solve the tasks?

I chose to study textbooks because they are important artefacts in the teaching of mathematics. There are no established tools for analyses of textbook tasks involving proportional reasoning and proportionality. I developed such a tool and used it in a pilot study on the textbook Matematik 4000 kurs A blå (Alfredsson, Brolin, Erixon, Heikne, & Ristamäki, 2008). This textbook was chosen because it is the most commonly used textbook (in course A) in upper secondary school classes in my region (three municipals) (Lundberg & Hemmi, 2009). The data for the pilot study comes from the proportion section. Later, several sections of the book, as well as other textbooks will be analysed and finally, proportion tasks in national exams will be compared with those in the textbooks.

The textbook analysis tool was inspired by the PISA assessment framework (OECD, 2003), which divides the cognitive demands into three clusters: *reproduction*, *connection* and *reflection*. All three clusters were used in my data analysis and I also applied PISA's assessment components concerning the context of the tasks. They are *intra mathematical*, *personal*, *educational/occupational*, *public* and *scientific*.

According to several researchers there are three main types of proportional reasoning tasks: *missing value*, *numerical comparison*, and *qualitative prediction & comparison* (e.g. Lesh, Post, & Behr, 1988). These types were developed to study mathematics in lower grades. But in upper secondary school, students meet proportion also in connection to graphs, so a new category called *decide k* was also included in the tool. Decide k is a variant of missing value tasks where one has to decide the proportional constant out of one pair. The tasks in the study of Lesh et al. (1988) involve only direct proportionality whereas I include also inverse proportionality, square proportionality and inverse square root proportionality in my study. Finally, the openness of the tasks was analysed.

To test the reliability of the analytical tool I let a research fellow analyse the tasks in the textbook. Most of the tasks were categorised in a similar way by both of us but there were some differences concerning the connection category and the open/closed category. As a result, the open/closed category was included in the category of cognitive demand because the openness of the task can be related to the level of reproduction and reflection. The connection category still needs a refinement and will be developed in the following analyses of several textbook sections as well as other textbooks.

The preliminary analysis of the other chapters of the textbook shows that more categories of proportional reasoning can be found there than in the pure proportion chapter. For a detailed description of the categories developed in the pilot study as well as the report of the results, see Lundberg and Hemmi (2009).

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Scaffolding Students Solving Multistep Arithmetic Word Problems

Guri A. Nortvedt
University of Oslo, Norway

When students attempt to solve multistep arithmetic word problems they often encounter difficulties, either with understanding or solving a problem. However, with scaffolding that is sensitive to their difficulties they might complete the problem, even though it would otherwise be outside their competence. This paper will report on a study where we used task-based interviews to investigate students' competence for solving such problems. The aim is to present a framework for coding scaffolding prompts, and to discuss some preliminary findings considering differences in scaffolding prompts and patterns found in the interview dialogues.

Task-based interviews are designed in a structured way to elicit more information about students' problem solving processes (Goldin, 2000). An important part of the design is the scaffolding prompts that provide data otherwise not available when considering students' problem solving processes. We draw on the definitions by Rogoff (1990) and Wood, Bruner and Ross (1976) when we understand scaffolding as taking away the parts of the process that are too challenging to the student, allowing him or her to thereby complete the other parts. Also scaffolding is the support provided by the researcher to help children "extend current skills and knowledge to a higher level of competence" (Rogoff, 1990, p.116).

The sample consisted of nineteen grade 8 students from two combined primary and secondary schools in Oslo. They worked on eight multistep arithmetic word problems. When students got stuck or when they asked for it, scaffolding was introduced. The scaffolding was offered by the researcher and was aimed at adjusting for and being sensitive to the individual student's difficulties with a specific word problem. Hence, protocols consisted of both independent and scaffolded work. A neutral prompt (heuristic level 1) would first be tried out before more direct scaffolds (level 2 or 3) were introduced. The framework for coding scaffolding prompts has been developed using Wood, Bruner and Ross (1976), Roehler and Cantlon (1997), and Goldin (2000) (see Table 1). One aim of the session was to introduce as little scaffolding as possible so as to be able to judge whether causes for student difficulties could be found in the comprehension of the problem or in the execution of the necessary mathematical operations. As such, scaffolding was at times exploratory in the sense that during the scaffolding conversations the researcher needed to be sensitive to student feedback, to determine whether comprehension or execution represented the difficulty. Based on this, the researcher had to decide how then to scaffold the student.

Table 1: Scaffolding prompts.

Overall label	*Action*	*Heuristic level*
Showing	Direct intervention	3
Telling	Verbal correction, Reminding, Suggesting, Explaining	2 - 3
Clarifying	Asking	1 or above
Supporting	Emotions, Process, Feed forward	1 - 2

A preliminary analysis suggests that many of the scaffolding prompts fell into the "Telling" (verbal corrections, reminding) category. The analysis also suggests that students need many supportive scaffolds, mainly for monitoring progress and processes, but also for emotional support. Few instances of "Showing" can be found among proficient students, while more can be found with struggling students. However, many students found it hard to retrieve number facts from memory and used different counting strategies to produce such facts. Occasionally the researcher would provide the student with a number (direct intervention, "Shoving") in order to allow the student to focus on the overall activity of solving the problems or executing a basic operation. With proficient students, scaffolding would to a larger extent be short prompts that would support independent work. Many of the prompts were directed towards helping the student monitor his own processes, or towards reminding the student of previous actions. However, for students with less proficiency in numeracy, the scaffolding had a more dialogic nature and the researcher would be more present in the students' problem solving. Much of this scaffolding was aimed at supporting students' execution of basic operations, both when they used mental strategies and when they tried to apply basic algorithms.

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What Is Valued in Effective Mathematics Lessons: Preliminary Findings from a Swedish Lower Secondary School

Aihui Peng and Mikaela Nyroos
Umeå University, Sweden

Introduction

The present study is part of a large regional collaborative project, which adopts the socio-cultural perspective, to investigate how effective mathematics education might be facilitated through an understanding of what teachers and students value in different social contexts. This paper reports the preliminary findings from one of the twelve participating regions, Umeå.

Effective mathematics teaching and learning

Effective mathematics teaching and learning could be seen as one of the major aims of educational research in many countries. The various studies generally arrive at similar conclusions that effective mathematics teaching is more about responding to and valuing the socio-cultural aspect of the learning environment than it is about adopting particular teaching methods (Seah, 2007). Thus, instead of defining effectiveness in the study, it is premised that the various teaching methods identified are reflective of a group of values.

Values related to mathematics education

"Values are the principles, standards and qualities explicitly or implicitly considered worthwhile or desirable by the participants of a distinct social practice" (Jablonka & Keitel, 2006). Values related to mathematics education operate at different levels, and they can be classified as mathematical value, mathematics educational value, general educational value, institutional/ organizational value (Seah, 2007). These categories are used in the data analysis in this study, the aim of which is to examine what are valued in effective mathematics lessons from the perspective of Swedish students and their teachers.

Methodology

Two teachers and their students in grade 7 and 8, respectively a regular group and a special group who has difficulties in mathematics participated in this study. Data collection included teachers' own reflective journals, lesson observations, students focus group interviews and teacher interviews. Structured interview questions were respectively based on students' recall of moments in the class when they feel that they are learning mathematics particularly well and drawn upon each teacher's reflective thoughts relating to the lessons observed. Validity of research findings was enhanced through triangulation of data.

Results

There are qualities highly valued in effective mathematics lessons by students from regular group (SRG) and special group (SSG), teachers from regular group (TRG) and special group (TSG). Table 1 shows the results.

Valuing of…(SRG)	Valuing of…(SSG)	Valuing of…(TRG)	Valuing of…(TSG)
personalized help explanation quietness collaboration sharing strictness concentration	explanation independence relaxation quietness fun personalized help	explanation whole-class interaction quietness communication group work experiment hands-on outdoor learning	interests communication visualization quietness explanation authenticity

Table 1: Qualities highly valued in effective mathematics lessons**.**

Conclusions

The preliminary findings reveal that both the teachers and the students share some commonalities in what they both value in the shaping of effective mathematics lesson. These include teacher's clear and detailed explanations, and the classroom atmosphere with quietness.

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A PhD-project in Multiculturalism: Ethno-mathematics, Language or Educational Systems

Jöran Petersson
Stockholm University, Sweden

What could a PhD-project within a multicultural perspective on mathematics education comprise? So far a literature survey suggests three possible aspects, namely ethno-mathematics, multilingualism or educational systems.

Ethnomathematics

Gerdes (1996) sees D'Ambrosio as the founder of the research area ethno-mathematics though there are forerunners. One of several definitions is "how mathematics is practiced in some clearly specified community". This opens up for the community to be an ethnic group, some vocational group or some societal community such as school mathematics. Examples of the latter could be e.g. subject matter didactics and real world vs. school world problem solving. For Swedish circumstances, some of the PhD-theses presented in Sweden, as listed at the web page of the National Center for Mathematics Education (NCM; see http://ncm.gu.se/node/171), fit aspects of ethno-mathematics.

Multilingualism

Barwell (2009) gives a research overview exemplified by classrooms in Africa, Asia, Europe and North America where usually English is a second language. He considers a mathematics classroom to be multilingual if two or more languages are present in the classroom or if the students *could* (my italics) use two or more languages to do mathematics. The main result is that mathematics education in a second language is difficult though there are also suggestions how to tackle this challenge. Barwell discusses and exemplifies three kinds of tensions: (1) mathematics and languages; (2) formal and informal language; (3) home language and school language. Barwell adds that there might be other tensions as well. For Swedish classrooms, the work of Norén and Ramsfeldt (2008) and Lim Falk (2008) could be fit into one or more of these tensions.

Educational systems

A third aspect of multiculturalism in mathematics education is the educational systems of a country as suggested in the following quotation.

> 'Multiculturalism´ as a pedagogical concept, that is, 'multicultural education', may be defined as considerations concerning the contents of a curriculum or more broadly, as reflections on how the educational system should relate to different actors and groups' possibilities and conditions within the educational system and process (Buchardt, Kampmann & Moldenhawer 2006:5)

Some examples of concerned actors and educational systems are

- Student focus: Change due to transition between primary, secondary and tertiary education, as focused in e.g. Stadler (2009). I have not yet found Swedish research on mathematics educational system change due to migration.
- Generation focus: Parents (Swedish or immigrants) with experience from one educational system helping their children, in another educational system, with homework (a common theme for practitioners arranging parental meetings). See e.g. Abreu, Bishop and Presmeg (2002).
- Teacher focus: Teachers' transition between different educational system due to school reforms or due to migration. The former was studied in some Swedish PhD-theses in the 1960's and 1970's (see the list at http://ncm.gu.se/node/171) and there is a new reform at hand, but on the case of migration I have not found Swedish research.

Conclusions

We can conclude that there is Swedish research in all the three aspects of multiculturalism although very few put the focus on migrated students or teachers.

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The Emergence of Disparity in Mathematics Performance: Early Results from the Swedish Part of a Project

Mikaela Rohdin
Luleå University of Technology, Sweden

The aim of the research project *The emergence of disparity in mathematics performance* (a joint project with researchers in Germany and Canada, see http://www.acadiau.ca/~cknippin/sd/index.html), is to study to what extent mathematical achievement is due to the social codes and rules of the mathematics classroom and students' understanding of these. Students and teachers quickly discover who is or is not 'good' at mathematics, and such impressions often remain throughout the course of study. Thus it is important to understand what the basis is for such impressions. Another issue is the differences between countries in terms of the achievement gap between the highest and lowest achieving students (see Knipping et al., 2008, for an outline of the project).

In the two Swedish classrooms – in the first year of the gymnasium (upper secondary school) – in the study there was a strong emphasis on working from the book. In one of the classes cooperation between students was actively encouraged, in the other it was not. In one of the classes there was no whole class exposition from the teacher, in the other there was some. But in both classes the students spent most of the time working through the exercises in the book and checking their answers against those given in the answer key. The teacher was involved when students asked questions. This is typical of many Swedish classrooms, and makes the situation different from one where the teacher has a more "obvious" role. One point to note is that in the class with no whole class exposition but encouraged cooperation between students many found it difficult to say which students were 'best' in mathematics, whereas in the class with some whole class exposition from the teacher but no encouraged cooperation between students most could pick out some of the students as the 'best'.

The invisibility of criteria creates a difficulty. As long as the students can do the exercises, are they fulfilling the grading criteria? Does the teacher have some other ways of evaluating the students' performance? In that case, what are they, and how do the students access these "hidden" criteria? Are the students even aware that there may be such criteria?

All students – nearly 40 – and both teachers were interviewed, and the lessons of the first three to four weeks were observed and videotaped.

The study is now in the early phase of analysing the material. For this presentation, the focus was student interviews and beginning to analyse them.

The student interview answers presented in tables 1 and 2 are examples of how some of the students in the class with no whole class exposition interpreted the situation.

Table 1: Interview answers to the question "Is this different from mathematics lessons in grundskolan (compulsory school)?", translated into English.

Student	*Is it different from grundskolan?*
A	Not much. We've done a lot of the maths, we're well prepared.
B	A lot. We worked a lot with understanding, not so much in the book. Now one has to learn stuff.
C	Not yet, it's mostly repetition. It's new people, we talk about different things.
D	It's a lot quieter, not so much noise.

Table 2: Interview answers to the question "Do you understand what is expected of you in the mathematics lessons?", translated into English.

Student	*Do you understand what is expected?*
A	Sometimes not, it's a bit too quick, I can't keep up.
B	Not always, the teacher doesn't say. It's not so easy to understand in the book.
C	To learn what I'm doing. I guess that's a good expectation.
D	Well, we haven't had a test yet.

Many students seemed to struggle with the question of whether they understood what was expected of them. The student referred to as *B* in the tables was one of the few who did not seem to find the question strange. This could be because the need to try to understand a different kind of mathematics classroom had increased the student's awareness of different kinds of expectation. This student also said that

> I'm not so good at maths, I still have the thinking that one should understand, but I'm getting into that one should work fast.

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Discourse Protection Against Manyology

Allan Tarp
MATHeCADEMY.net, Denmark

Social theory describes two kinds of social systems using education to enlighten or patronize its people. As an educational discourse, does mathematics enlighten or patronize? Full paper version at www.MATHeCADEMY.net.

Manyology - a science about the natural fact many

To survive, humans must deal with Many. To do so we count and add Many.

Counting by 1st order places sticks in icons so that there are five sticks in the five-icon 5, etc. Ten is written without an icon as 10: 1 bundle and 0 unbundled.

2nd order counting uses bundling and stacking in icon-bundles: T= 3 4s= 3*4.

3rd order counting bundles and stacks in ten-bundles: T = 3 tens = 3*ten.

2nd order counting results in a double-stack: a stack of bundled and a stack of un-bundled. Two cups, a left cup for the bundled and a right cup for the unbundled, can represent this. And since 4 1s is the same as 1 4s, one 4-bundle can be represented by 1 stick in the bundle-cup.

I I I I I I I I I -> IIII IIII I -> IIII IIII) I) -> II) I = 2)1) = 2.1 4s

Thus 2nd order counting always results in a decimal number carrying a unit where the decimal point separates the bundles from the unbundled.

The counting result can be predicted on a calculator by using two formulas:

A 're-bundle' or 're-count' formula	A 're-stack' formula
IIIIIIII -> II II II II	IIIIIIII -> IIIIII II
8 = (8/2)*2 or T = (T/b)*b	8 = (8–2)+2 or T = (T–b)+b

Here '8–2' means 'from 8 take away 2'; '8/2' means 'from 8 take away 2s'.

Here + means 'added next-to'; and * means 'added on-top' several times.

Thus re-counting 4 8s in 6s can be predicted by a calculator as:

T = 4 8s = (4*8)/6 * 6 = 5.? *6 , T = (4*8 – 5*6) + 5*6 = 2 + 5*6 = 5.2 6s

Re-counting means changing units, also occurring when changing physical units as \$ and £ and kg. Double-counting a thing in 4\$ and 5kg creates a per-number 4\$/5kg or 4/5 \$/kg. The questions '9kg = ? \$' and '7\$ = ? kg' are answered by recounting the 9kgs in 5s, and recounting the 7\$ in 4s:

9kg = (9/5)*5kg = (9/5)*4\$ = 7.2\$, 7\$ = (7/4)*4\$ = (7/4)*5kg = 8.75kg

Unknown bundle-sizes give equations: 9 = 2.1 xs = 2*x + 1. Re-stacking and re-bundling solve an equation, which gives the rule: Moving numbers across reversing their calculation signs solves equations:

Re-stacking 9: Re-bundling 8: Result:	2*x + 1 = 9 = (9-1) + 1 = 8+1 2*x = 8 = (8/2)*2 x = 8/2	2*x+1 = 9 2*x = 9–1 = 8 x = 8/2 = 4

Thus counting leads to decimal-numbers with units, to changing units and to solving equations (1digit math). Once counted, Many can be added or split.

Adding can take place on-top or next-to. To add on-top, the units must be changed to be the same. Multiplication is repeated adding on-top. Adding next-to is integration integrating the bundle-sizes; reversed integration is differentiation.

Adding on-top may lead to overloads as 7.3 5s. Here the 7 5s can be re-counted to 1.2 5s thus giving a bundle of bundles, 5 5s. With one bundle of bundles = 1 bundle-bundle, we introduce a new cup for the bundles of bundles:

T = 7 5s = IIIIIII) III) = IIIII II) III) = I) II) III) = 12.3 5s

Overloads give many different results: 9.2 3s = 16.2 3s = 23.2 3s = 100.2 3s. Splitting uses overloads: 5.2 7s = 1.3 7s + ?. But 5.2 7s = 4.9 7s = 1.3 7s + 3.6 7s.

Foucault discourse protection silences competing discourses as manyology

Traditional mathematics and Manyology represent two competing discourses about Many. Manyology is a physical science investigating the natural fact Many by defining its concepts ‘from below’ as abstractions from examples. Defining its concepts ‘from above’ as examples of abstractions makes traditional mathematics a metaphysical science investigating the consequences of ungrounded axioms.

Traditional mathematics only does 3rd order counting in tens, and rejects decimal numbers with units: 3.2 tens IS 32. Numbers are added without the units: 1+2 IS 3 and 1/2 + 2/3 IS 7/6 in spite of the fact that 1week+2days = 9days; and 1/2 of 2 bottles and 2/3 of 3 bottles is 3/5 of 5 bottles. + IS on-top, not next-to. A calculation as 3+4 IS a number-name. An equation IS an equivalence relation to be transformed by doing identical operations to both sides. Shifting units IS a homomorphism, i.e. a linear function. A function IS a subset of a set-product where first component identity implies second component identity. Integration is restricted to functions only and defined as an example of a limit. The number 1 and the follower-principle construct the natural numbers thus making 10 the follower of 9 in spite of 10 being the follower of 6 when counting in 7s. + is defined as repeating the follower principle, multiplication as repeating addition.

So, Manyology enlightens Many; traditional mathematics patronizes people.

Since the creation of the two Enlightenment democracies, social theory has focused on hidden patronization. To keep block-organized enlightening from spreading, Germany invented line-organized Bildung to induce nationalism into the people and to sort out its elite for the strong central administration.

Naturally line-organized Bildung prefers patronizing mathematics. Does block-organized enlightening prefer enlightening Manyology? No, because of discourse protection, described by Foucault to disciple both itself and its subjects.

The KOM Project and Adding It Up – Through the Lens of a Learning Situation

Jorryt van Bommel, Yvonne Liljekvist and Cecilia Ottersten Nylund
Karlstad University, Sweden

In spring 2009 we attended a PhD course with the aim to understand two different frameworks; the Danish KOM project (*KOM*) (Niss, 2003, Niss et al, 2002) and the American report Adding It Up (*AiU*) (Kilpatrick et al, 2001). Application of the frameworks on a videotaped learning situation enabled us to compare *KOM* and *AiU*. *KOM* describes eight co-dependent competencies, meaning that students cannot possess one competency without another: Thinking mathematically, Posing and solving mathematical problems, Modelling mathematically, Reasoning mathematically, Representing mathematical entities, Handling mathematical symbols and formalisms, Communicating in, with, and about mathematics, Making use of aids and tools. *AiU* however shows critical strands for developng mathematical proficient students through five interrelated proficiencies: Conceptual understanding, Procedural fluency, Strategic competence, Adaptive reasoning, and Productive disposition. The aim of this study is to determine whether *KOM* and *AiU* are complement or duplicate of the principals of learning mathematics.

The empirical data used in this study was collected within a research project spring 2009 and consists of a voluntary group of four pre-service teachers working for 25 minutes on a mathematical task, given as part of their teacher training program. The analysis was based on transcribed data were each sentence was categorized in terms of proficiencies and competencies. These categories gave us a tool for comparing the two models described above and five patterns became clear.

A first pattern –'Flow'– puts a focus on the process, where we found that the competencies (*KOM)* describe processes more than the proficiencies (*AiU)*. The five strands seem to be more static and although they are interwoven we do not see dynamics there, whereas there seems to be a dynamic process going on between the eight competencies. The data show switches between mathematical activities and the competencies seem to capture this better. Two other patterns – 'Symbols' and 'Tools'– point toward an existing intersection of the two frameworks but also point out the differences. Since symbols and tools are described as two different competencies (Representing mathematical entities, Handling mathematical symbols and formalism) and could be seen as part of two strands (Productive disposition, Procedural fluency), these four are closely related. The pattern showed however that there are no implicit connections between either one of the strands and competencies. A fourth pattern –'Problem or not'– indicates that one strand (Procedural fluency) covers one of the competencies (Posing and solving mathematical problems) however other conclusions might occur if we would have access to other data containing problem solving activities. Finally a fifth pattern –'Students'– showed that the two frameworks can give different results when used for judging students' capabilities within mathematics. Both committees got their charge from their respective Department of Education. With a similar background (e.g. questions about the decreasing levels in students' achievements, and how to improve teaching in mathematics), both reports strive to describe a framework, useful in a "for all" perspective. The intention was also to "overcome" the divergence between goals and curricula, and the present forms of teaching and learning in classrooms (Lester, 2007). However we can see some distinctions in how the aims and conclusions are described. Where *KOM* puts the focus on describing students' capabilities, *AiU* focuses more on describing a successful way of instruction.

In conclusion, we started wondering if *AiU* and *KOM* were disjoint or overlapping frameworks. Through the data used we can give examples of intersection and of complements. In our last step using the patterns observed, we could reflect on the frameworks again and we noticed a difference in focus between the two frameworks; *KOM* has an underlying tone to "provide ideas and give inspiration" (Niss, 2003, p 6), and it therefore ends up with a next-to perspective being more student centred, while *AiU* has to fulfil "guide to best practice" (Kilpatrick et al, p 3), so it ends up with a top-down perspective being more instruction centred. The next step is, analyzing another group of students to verify our patterns found and see if other patterns might arise.

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Undergraduate Mathematics Teaching Through a Commognitive Lens – The Case of Functions

Olov Viirman
University of Gävle and Karlstad University, Sweden

This short communication reports on an ongoing research project concerning the practice of university mathematics teachers, focusing on how they act in order to promote conceptual development in their students, primarily regarding functions. I believe that there is a need for studies of the actual practice of mathematics teaching at Swedish universities, gaining knowledge that might then be used to improve said practice. The empirical material, which is still being collected, will include videotaped lectures and lessons by 6-8 mathematics teachers at three universities, as well as interviews with the teachers.

During the last decade or so, Anna Sfard has written extensively of the acquisition and participation metaphors as basic metaphors underlying theories of learning (cf. Sfard, 1998). Through the acquisition metaphor, learning is described in terms of bringing to mind the accumulation of material goods. The acquisition metaphor governs traditional cognitivist approaches describing learning in terms of mental entities such as concept images (Tall & Vinner, 1981), or the schemes of Piagetian constructivism. The participation metaphor, on the other hand, describes learning as the process of becoming a member of a certain community. As a prime example of a participationist theory one can cite Lave and Wenger's (1991) theory of learning as *legitimate peripheral participation* in communities of practice.

In recent years Sfard has developed a participationist theory which she calls the *commognitive* framework (Sfard, 2008), drawing on the work of Vygotsky and Wittgenstein. From a commognitive standpoint, thinking is viewed as the individualized version of interpersonal communication. Different types of communication are called discourses, and these discourses then become the unit of analysis. Four characteristics can be used to describe different discourses (Sfard 2008, pp. 133-134):

- *word use* - words specific to the discourse or common words used in discourse-specific ways;
- *visual mediators* - visual objects operated upon as a part of the discursive process; examples from mathematical discourse could be diagrams and special symbols;
- *narratives* - sequences of utterances speaking of objects, relations between and/or processes upon objects, subject to endorsement or rejection within

the discourse; mathematical examples could be theorems, definitions and equations;

- *routines* – repetitive patterns characteristic of the discourse; typical mathematical routines are for instance methods of proof, of performing calculations, and so on.

In my own project I now plan to use this theory to describe the discourses of function presented by the different teachers in the study, and then using some additional analytical tools, perhaps variation theory (Marton, Runesson, & Tsui, 2004), to investigate what possibilities for learning are offered by these discourses. A tentative analysis of a small part of the material I have gathered so far (one teacher only, in an introductory course) suggests for instance that common visual mediators are formulas and graphs. Common narratives are definitions and presentations of functions as formulas, while there are almost no theorems or proofs. Routines include the sketching of graphs given formulas of functions, identifying the domain and range of functions given their formulas, etc. From a variation theoretical standpoint, it can be noted that the critical aspect of univalence of functions is made possible to discern, while differences in domain and range are not made visible at all, since all examples of functions are real and real-valued.

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E-mail addresses of the contributors

Yukiko Asami Johansson	yukiko.johansson@norr.johnbauer.nu
Iiris Attorps	iiris.attorps@hig.se
Semir Becevic	semir.becevic@hh.se
Marie Bergholm	marie.bergholm@soderkoping.se
Ewa Bergqvist	ewa.bergqvist@math.umu.se
Tomas Bergqvist	tomas.bergqvist@educ.umu.se
Christer Bergsten	christer.bergsten@liu.se
Kjell Björk	kbk@hig.se
Lisa Björklund Boistrup	lisa.bjorklund@mnd.su.se
Jesper Boesen	jesper.boesen@ncm.gu.se
Kerstin Bäckman	kerstin.backman@hig.se
Henrik Carlsson	henrik.carlsson@skola.ljungby.se
Paul Dowling	p.dowling@ioe.ac.uk
Patrik Erixon	patrik.erixon@utb.karlshamn.se
Peter Frejd	pefre@mai.liu.se
Sharada Gade	sharada.gade@matnv.umu.se
Gunnar Gjone	gunnar.gjone@ils.uio.no
Eva Jablonka	eva.jablonka@ltu.se
Maria Johansson	maria.l.johansson@ltu.se
Gulden Karakok	gulden.karakok@math.umu.se
Anna Klisinska	anna.klisinska@ltu.se
Troels Lange	tlange@csu.edu.au
Håkan Lennerstad	hln@bth.se
Yvonne Liljekvist	yvonne.liljekvist@kau.se
Johan Lithner	johan.lithner@matnv.umu.se
Anna L. V. Lundberg	anlun@mai.liu.se

Tamsin Meaney	tmeaney@csu.edu.au
Per Nilsson	per.nilsson@vxu.se
Sverker Lundin	sverker.lundin@gmail.com
Eva Norén	eva.noren@mnd.su.se
Guri A. Nortvedt	g.a.nortvedt@isp.uio.no
Mikaela Nyroos	mikaela.nyroos@educ.umu.se
Cecilia Ottersten Nylund	cecilia.ottersten-nylund@hv.se
Alexandre Pais	xande@learning.aau.dk
Torulf Palm	torulf.palm@matnv.umu.se
Björn Palmberg	bjorn.palmberg@math.umu.se
Hanna Palmér	hanna.palmer@vxu.se
Aihui Peng	aihuipeng@gmail.com
Jöran Petersson	joran.petersson@mnd.su.se
Kerstin Pettersson	kerstin.pettersson@mnd.su.se
Mirko Radic	mrc@hig.se
Mikaela Rohdin	mikaela.rohdin@ltu.se
Håkan Sollervall	hakan.sollervall@vxu.se
Daniel Spikol	daniel.spikol@vxu.se
Erika Stadler	erika.stadler@vxu.se
Diana Stentoft	dsr@learning.aau.dk
Allan Tarp	at@grenaa-gym.dk
Paola Valero	paola@learning.aau.dk
Jorryt van Bommel	jorryt.vanbommel@kau.se
Tine Wedege	tine.wedege@mah.se
Olov Viirman	olov.viirman@hig.se
Jonas Bergman Ärlebäck	jober@mai.liu.se
Magnus Österholm	magnus.osterholm@matnv.umu.se

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