SMDF **S**vensk Förening för **M**atematik**D**idaktisk **F**orskning
Swedish Society for Research in Mathematics Education

Perspectives on Mathematical Knowledge

Edited by

Christer Bergsten
Barbro Grevholm
Thomas Lingefjärd

Proceedings of ***MADIF 6***

The 6th Swedish Mathematics Education Research Seminar

Stockholm, January 29-30, 2008

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Preface

This volume contains the proceedings of *MADIF 6*, the Sixth Swedish Mathematics Education Research Seminar, with a short introduction by Barbro Grevholm. The seminar, which took place in Stockholm January 29-30, 2008, was arranged by *SMDF,* The Swedish Society for Research in Mathematics Education, in co-operation with Stockholm University. The members of the programme committee were Christer Bergsten, Barbro Grevholm, Kirsti Hemmi, Katarina Kjellström, and Thomas Lingefjärd. The local organiser was Katarina Kjellström at Stockholm University.

The programme included two plenary lectures (Eva Jablonka and Rosamund Sutherland), one plenary panel (Thomas Lingefjärd, John Mason, Anne Watson, and Paola Valero), ten paper presentations (Tomas Bergqvist et al., Elsa Foisack, Håkan Lennerstad, Cecilia Kilhamn, Lisbeth Lindberg, John Mason, Guri Nortvedt, Frode Rønning, Olov Viirman, and Anne Watson), and eight short oral presentations (Iiris Attorps, Iiris Attorps et al., Thomas Dahl et al., Hans Melén et al., Eva Norén, Maria Reis, Eva Taflin, and Magnus Österholm). In this volume one plenary address, nine papers, and all short presentations are included. We want to thank the authors for their interesting contributions. In addition to the pre-conference peer-review process, the revised final papers were submitted after the conference and re-reviewed by the editors. The authors are responsible for the content of their papers.

We wish to thank the members of the programme committee for their work to create an interesting programme for the conference, Kataria Kjellström for her valuable help with the preparation and administration of the seminar, and the special reactors to the papers for initiating stimulating discussions during the paper sessions. We also want to express our gratitude to the organiser of *Matematikbiennalen 2008* for its valuable financial support. Finally we want to thank all the participants at *MADIF 6* for creating such an open, positive and friendly atmosphere, contributing to the success of the conference.

Christer Bergsten

Content

Short presentations

Reflections on Perspectives on Mathematical Knowledge

Barbro Grevholm
University of Agder

’Mathematics is the sorrow kid of school’ was the title of a recent TV-programme in Sweden. Mathematical knowledge can look like that from one perspective. From another perspective it can be seen as a subject for joy, which was the theme of one of the biannual conferences in Sweden, ‘Matematik-biennalen’. At MADIF6, the 6th research seminar of SMDF (Swedish Society for Research in Mathematics Education), many other perspectives on mathematical knowledge were presented. In the welcome speech I reminded the participants, that it is 10 years since the seminar in Sundsvall took place, before 'Matematikbiennalen' in 1998, that led to the series of mathematics education research seminars called MADIF (an acronym for the Swedish expression MAtematikDIdaktisk Forskning, i.e. research in mathematics education). There a committee was appointed to prepare what was to become SMDF. This committee produced the constitution of SMDF, which was approved in January 1999 at a seminar in Stockholm, MADIF1. Over the years different themes have been in focus at the research seminars of SMDF (see references for the previous MADIF proceedings), and for this 6th seminar it is *Perspectives on Mathematical Knowledge*. At the seminar 10 research papers were presented and 8 short presentations took place in addition to two plenary addresses and a concluding plenary panel. In this book most of those papers are presented.

Eva Jablonka talked in her plenary address about *The everyday and the academic in the mathematics classroom: Confrontation or conciliation?* It is a common practice in mathematics teaching to include particular aspects of everyday out-of-school activities as resources for mathematical activities. But in doing so the tension between everyday and academic practices cannot be easily resolved. In research we find discussions and studies concerning the authenticity of contexts in word-problems, the function of horizontal and vertical mathematisation in ‘realistic mathematics education’ and the role of mathematical modelling. Although many argue for the inclusion of contexts from everyday practices, some are suspicious of the potential to create a bridge between everyday practices and formal, academic mathematics through the use of contextualised tasks. The paper draws on episodes from classroom discourse in different countries, which document the interactions between teachers and students. Jablonka claims that the students often face a dilemma when confronted with contextualised tasks: should they make extensive or only little reference to their everyday knowledge? These issues are discussed in terms of the re-contextualisation of out-of-school practices. The aim is to describe how some of the students' difficulties are related to the ways in which the re-contextualisation operates in mathematics classrooms.

Rosamund Sutherland talked in her plenary about *Integrating ICT into teaching and learning mathematics.* She discussed a model of professional development for mathematics teachers and linked it to a view of knowledge as being produced by engagement with and in practice. This professional enablement demands that practice is guided and legitimated by a blend of practitioner wisdom and researcher knowledge and credibility. Drawing on socio cultural theory she focused on the ways in which teaching and learning mathematics in schools is always situated in a particular cultural context, involving interactions with people and supported by the use of language and tools. She also emphasised the ways in which students bring informal perspectives on mathematics to any new learning situation, and that these inevitably influence what they learn in the classroom. Sutherland argued that teachers and teaching is a key to learning mathematics in schools. Sutherland's contribution is not presented here but interested readers can consult her book 'Teaching for learning mathematics' (Sutherland, 2008).

All the contributions in this book have been peer-reviewed before they were accepted to be presented at the seminar and then in the final round by members of the editing committee. We thank all the contributors for being willing to take part in the peer review process and read papers written by fellow contributors.

Podcasting in school is the theme of Tomas Bergqvist's paper. He claims that podcasting is a new phenomenon in Swedish schools. The paper describes a project where the main goal is to analyze if the students' interest in mathematics is affected, if mathematics is made accessible via podcasts and iPods. Teachers at eleven schools were encouraged to produce podcasts as a part of their mathematics teaching in grade eight. The results indicate that the possibility to explore mathematics wherever and whenever you want is an important aspect for the students. Other findings are that the technical difficulties for the teachers were underestimated in the project, and that teachers had problems in finding time for the production of podcasts.

Making sense of negative numbers through metaphorical reasoning is the title of Cecilia Kilham's paper. The concept of negative numbers is an abstract concept and it has been argued that it can only be understood through symbolic reasoning. Others argue that mathematical concepts are understood through metaphors. Kilhamn claims that previous research has identified three aspects of understanding negative numbers: direction and multitude, proficiency in arithmetic operations, and the meaning of the minus sign. Her study explores the theory of conceptual metaphors and metaphorical reasoning by investigating the use of models and metaphorical reasoning when dealing with negative numbers. The data consists of test results from 99 students in the teacher training program and follow-up group interviews. Kilhamn's results show that some students' difficulties seem to be a consequence of their use of metaphorical reasoning using a metaphor that is insufficient. Metaphorical reasoning seems to be helpful for students who are aware of the limitations of the metaphor. Enlightened use of metaphorical reasoning, i.e. being aware of the potentials and constraints of models and metaphors, could be described as a fourth important aspect of understanding negative numbers.

Håkan Lennerstad writes about *Spectrums of knowledge types: Mathematics, mathematics education and praxis knowledge*. He claims that, while mathematics is deductive and mathematical education is evidence based, practical knowledge is a type of knowledge that professionals in any profession develop by experience and by exchange with other professionals. Such knowledge, to a large extent is difficult to articulate, is also essential in important types of mathematical knowledge. Lennerstad argues for a more fluent cooperation between the paradigms, in which the advantages of all the different knowledge types may interact and become increasingly useful to each other. For such an idea to reach reality it is necessary for mathematicians, mathematics education researchers, mathematics teachers,, and others, to listen in depth to each other, and to have a dialogue. To achieve that, Lennerstad describes one alternative called the Dialogue Seminar.

Lisbeth Lindberg's paper is in Swedish and about the historical background to the importance of mathematics in vocational programmes in upper secondary school (*Historisk bakgrund till matematikens betydelse i yrkesprogrammen*). She investigates the development of vocational education in Sweden over 150 years and the parallel development of the role of mathematics in working life. The teacher's professional role in vocational education has changed from being the role model and practitioner to the theoretical representative. Teacher education has not been able to follow in this development but development projects have been carried out to help teachers work with the new task. The role of the students has changed as the education has grown more theoretical. Mathematics has changed from general calculations and vocational calculations to become a core subject that all students study. A political debate is going on about these changes and a new design of vocational education has been suggested for upper secondary school. This model builds on a tripartite school: a university preparatory way, a vocational way, and an apprenticeship education.

John Mason presented *A study of the movement of attention: The case of reconstructed calculation*. Mason's enquiries have convinced him that both what learners are attending to, and how they are attending to it play a central role in what learners are able to make sense of and eventually internalise. The paper illustrates that conclusion based on experiences of using a mathematical reconstruction task. Examples are given of different ways in which people's attention moves and shifts, sometimes voluntarily and sometimes involuntarily, in relation to variation theory, but Mason is also drawing on other theoretical discourses.

Guri Nordtvedt's paper is about *Reading word problems.* Students' work on word problems infers demands on comprehension and solving strategies. Students reading a word problem, construct a mental representation of the problem text that serves as the basis for solving the problem. But reading and solving word problems is not necessarily a linear process. Students might for instance reread the problem during solving or when evaluating answers. The first part of Nordtvedt's paper outlines a framework for investigating the connections between strategies for reading and solving word problems. Her last section concentrates on exemplifying by discussing some instances where one student rereads before tying the discussion of his competence to the suggested framework.

The paper by Frode Rønning is about *Early work with multiplicative structures*. The purpose of his paper is to discuss the way very young children handle problems connected to multiplication and division. The discussion is based on classroom observations from England and Norway. Rønning has linked this to a discussion about how Norwegian textbooks present the pupils' first encounter with division. In mathematics textbooks division is often regarded as the inverse operation of multiplication. Based on the classroom observations, Rønning argues that it could be worthwhile pursuing division as a process in its own right and postpone the strong link to multiplication until later.

Olov Viirman presents *Different views: Teacher and engineering students on the concept of function*. His study analyses what kind of conceptions teacher students and engineering students have about the function concept, and how these conceptions differ between the two groups. The study was conducted through questionnaires, and 34 students at a Swedish university participated. In the classifications of the function conceptions of the students he used modified versions of models presented by Vinner and Dreyfus and Sfard, DeMarois and Tall. Viirman claims that his study shows that the students primarily have operational conceptions and only a couple of students have structural conceptions. He also argues that distinct differences exist between prospective compulsory school teachers and engineering students, where the prospective teachers have less developed functional conceptions.

Anne Watson turned her interest to *Different versions of the 'same' task: Continuous being and discrete action.* In the paper she probes subtle differences in lessons which are based around similar tasks. This work is done by analysing the experiences the tasks afford students, and identifying what is available for students to construct from these experiences. This analysis provides a new lens for looking at mathematical activity in lessons, and at how teachers' own mathematical senses act out to afford different mathematical experiences for learners.

Prior to the seminar each paper was read by a reactor who during the paper session at the seminar contributed to the discussion with specially prepared critical questions to the authors. We thank all the reactors for their important and much appreciated contributions to the seminar: Ewa Bergqvist, Morten Blomhøj, Ola Helenius, Kirsti Hemmi, Ingemar Holgersson, Johan Häggström, Per Nilsson, Per-Eskil Persson, Astrid Pettersson, and Frode Rønning.

In the short presentations section of the book we find reports from Iiris Attorps, Iiris Attorps and Timo Tossavainen, Thomas Dahl and Thomas Biro, Hans Melén et al., Eva Norén, Maria Reis, Eva Taflin, and Magnus Österholm.

The plenary panel included the speakers Thomas Lingefjärd, John Mason, Paola Valero, and Ann Watson, with Christer Bergsten as chair. They shared their personal views on the theme of the seminar, *Perspectives on mathematical knowledge*, commented each other's presentations and engaged with questions from the audience. Different aspects were illustrated such as individual mathematical knowledge, socio-political views, cultural and contextual influences, habits to notice mathematical connections, and being mathematical and talking mathematics. The discussion highlighted the complexity of the theme and brought forth a wide spectrum of thought provoking examples and challenges.

The contributions in this volume illustrate that mathematical knowledge can be perceived from many different perspectives and it can be approached from different dimensions such as time, age, content, context, mediating tools, and expected aim for the knowledge. We hope that readers will find it interesting to inquire into the authors' different ways of approaching mathematical knowledge.

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The Everyday and the Academic in the Mathematics Classroom: Confrontation or Conciliation?

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Abstract: *Including particular aspects of everyday out-of-school activities as resources for mathematical activities is a common practice in mathematics teaching. The relationship between these two domains is a topic that has been extensively discussed in mathematics education. Discussions and research studies concern, for example, the authenticity of contexts in word-problems, the function of horizontal and vertical mathematisation in 'realistic mathematics education' and the role of mathematical modelling. Although many argue for including contexts from everyday practices, some are suspicious of the potential to create a bridge between everyday practices and formal, academic mathematics through the use of contextualised tasks. The students often face a dilemma when confronted with such tasks: should they make extensive or only little reference to their everyday knowledge? In the paper, these issues will be discussed in terms of the recontextualisation of out-of-school practices. The discussion will draw on episodes from classroom discourse in different countries, which document the interactions between teachers and students who are engaged in solving tasks that relate mathematics to everyday activities. The paper aims at describing how some of the students' difficulties are related to the ways in which the recontextualisation operates in mathematics classrooms.*

Introduction

The "everyday" in the title is used as an abbreviation for everyday practices that consist of common activities, in which people are involved in family, peer group, community or in some semi-skilled jobs that do not require specialised formal training. Some everyday practices include mathematical techniques, but it is not an established rule to use these, and no mathematical skills beyond those acquired in compulsory school are needed for successful participation. That means that the application of mathematical techniques is not institutionalised in those practices. Many of the studies classified as ethnomathematics are indeed devoted to identifying and studying mathematical activities that are not part of the practice in the conventional institutions where mathematics is taught and practiced.

Even though mathematical components might be identified from an observer's point of view, people who are engaged in the activities do not necessarily think that what they are doing involves mathematics. In line with other studies, Wedege's (2002) investigations of workplace mathematics show the invisibility to the workers of their own mathematical knowledge. The workers tend to "see" the mathematical components of a task only when they have difficulties in dealing with those components. A variety of studies of everyday mathematical practices show that children and adults engaged in those practices possess successful problem solving strategies (for example, Abreu, Bishop & Pompeu, 1997; Carraher, Carraher & Schliemann, 1985; Masingila, 1994; Saxe & Moylan, 1982). The mathematical strategies employed are not necessarily those learned at school. On the other hand, some studies indicate that the techniques learned at school are also likely to be used and connected with the strategies used by people who did not learn their strategies at school (Acioly & Schliemann, 1986; Knijnik, 2000).

The "academic" in the title is used as an abbreviation for academic mathematics, that is, the mathematical practice at higher academic institutions, in which mathematics is produced and acquired. This type of mathematical practice consists of highly specialised activities. The notion "activity" refers to a structure of relations and practices that regulates who can say or do or mean what (Dowling, 1998). The positions of the participants in an activity are not symmetric.

One shared goal of mathematics curricula for upper secondary education in many countries is to initiate students into academic mathematical activities. In the course of proceeding through a typical mathematics curriculum from primary through secondary towards higher education, the amount of traces of the everyday decreases. The non-mathematical contexts, if there are any, become more scientific or technological. Also the ways in which contexts are treated change. This can be seen from analysing textbooks and other curriculum materials (cf. Jablonka, 2002). In other words, in the course of learning mathematics, students move through a range of different mathematical practices, consisting of distinct activities, with their related discourses that represent different institutionalised forms of organising and expressing mathematical knowledge, ideas, and experience. In the end only a few successfully participate in the discourse of academic mathematics.

There are different purposes of including traces of the everyday in mathematics classrooms. The following functions are important for the subsequent discussion:

- Everyday as pretext for mathematics or as a springboard for developing school mathematical concepts and procedures; this is not only to attract the students' attention, but for establishing the meaning of mathematical expressions;

- everyday as a field for applying standard procedures or as being modelled by some school mathematics (which the students have already acquired); this is to teach useful mathematical strategies.

The classical study of Lave, Murtaugh and de la Roche (1984) and many other investigations of students' difficulties with contextualised school mathematics tasks show that a transfer of everyday mathematical activities to the school context is problematic. In any classroom, there is a process of confrontation and of translation of different discourses. The everyday discourse is confronted with that of school mathematics. Riesbeck (2008) recognises that the crossing over the boundaries between these two discourses is problematic, especially if it happens unknowingly, and points to the limiting effect of extended reference to the everyday with respect to the students' acquisition of school mathematical knowledge.

For investigating the relationship between everyday activities that include mathematical components and academic mathematics, it is essential to ask what happens to the *everyday* when it appears – in one or another form - in classroom mathematical activities. The question in the title of this presentation suggests that there is a discontinuity between the *everyday* and the *academic,* pointing to a disparity between these practices and their related discourses, which produces a tension. This tension permeates many aspects of mathematics education and of education in general.

Theoretical background

The theoretical background of the analysis presented here, are studies of re-contextualisation of practices and of the discourses that constitute these practices. Recontextualisation of a discourse means moving it from its original site in order to use it for a different purpose. Recontextualisation brings about the subordination of one discourse under the principles of the other:

> Pedagogic discourse is constructed by a recontextualizing principle which selectively appropriates, relocates, refocuses and relates other discourses to constitute its own order. In this sense, pedagogic discourse can never be identified with any of the discourses it has recontextualised. (Bernstein, 2000, p. 33)

Though it aims at enculturation into the practice of academic mathematics, school mathematics ("*math"*) is not academic mathematics:

> From one point of view pedagogic discourse appears to be a discourse without a discourse. It seems to have no discourse of its own. Pedagogic discourse is not physics, chemistry or psychology. Whatever it is, it cannot be identified with the discourses it transmits. (Bernstein, 2000, p. 32)

In which respect and to what extent, if at all, school mathematical practice resembles academic mathematical activities is not a simple question. From Bernstein's point of view there cannot be resemblance because discourses are specific to the contexts with their specific purposes. However, school mathematics not only recontextualises academic mathematics, but also outside-school practices. The traces of the activities from the everyday that appear in school classrooms can be, for example, solving a money problem when shopping, cooking, doing woodwork, buying a mobile phone etc. The transformation into school mathematical activities includes a sub-ordination of the everyday discourse under the school mathematical discourse. The everyday activities are viewed from the perspective of school mathematics. Viewing contextualised school mathematical tasks from the perspective of everyday discourse would amount to a critique of their authenticity.

From the perspective outlined here, contextualised mathematics tasks cannot be authentic. Solving a shopping task in a school classroom (however authentically the situation might be described) is neither an everyday nor an academic mathematical activity. It has to be acknowledged that school classrooms can be analysed as a distinct domain of practice. This raises the question: How can the students know in which discourse they are participating and what contribution they are supposed to produce?

The following examples are taken from different studies of classroom practice and are used in an illustrative way.

A challenge for the students

In the following example the everyday is used as a pretext for mathematics. The problem that the students are facing can be interpreted as stemming from too extensive reference to the everyday. The episode is from a year eight classroom (USA)[1]. The teacher introduces the example of sharing 10 candy bars. They have talked about the meaning of 10 divided by 1. The teacher writes '10/1 = 10' on the board and says: "Ten right? He gets all ten…So he's happy. He gets all ten pieces of candy. Now let me ask you this":

Teacher: *What happens in this case right here?* [He writes '10/0 = ' on the board.]
Teacher: *So- so if I divide it zero ways, is anybody getting anything?*
Students: *No.*
Student: *Not even you.*
Student: *It's your candy bar.*
Student: *I already told you.*

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[1] Transcript from *The Learner's Perspective Study* (for a description of the research design, see Clarke, 2000, or http://extranet.edfac.unimelb.edu.au/DSME/lps/)

Teacher: *Guys, sorry I- I can't multi-function, listen to both at the same time. I need one person at a time. I'm really trying.*
Student: *My other teacher can.*
Teacher: *Your other teacher? Pretty good. I haven't mastered it yet.*
Amiri: *It- It's like a hashy* [by 'hashy' she might mean 'minced into little pieces'] *bar cuz you don't have to ( ) little, uh little pieces you can tear and stuff.*
Student: *Squares.*
Amiri: *Uh, like you can give people pieces but- it, you're the only person that can eat the whole ( )*
Student: *But then that would be one, divided by one.*
Amelia: *Can you just give us the answer?*
Antoine: *Can't share.*
Student: *It's like you didn't buy a candy bar.*
Student: *You don't even have it.*
Student: *You never bought it yet.*
Antoine: *You can't ev- you might as well not even think of candy if you can't afford it.*
Teacher: *Okay. I'll tell you right now guys, um, division by zero ... is, what we call in math, is undefined.* [He writes "Division by zero is undefined."]

The two domains of practice and the related discourses at stake in this episode are the sharing of candy bars and ordinary real number arithmetic. The sharing activity is recontextualised from the perspective of real number arithmetic and used as a mediator between the everyday and the school mathematical domain. It is an interesting exercise to ask what the meaning of division by -5 or by 1/3 would be.

The episode shows the pitfall of the attempt to develop the fact that division by zero is undefined in real number arithmetic from the everyday activity of fair sharing (usually referred to as sharing in contrast to grouping as a model for division). The interesting part is that the episode starts by an exercise of translating mathematical expressions into everyday discourse. The teacher retranslates "10 divided by 1" into "He gets all ten…So he's happy".

Then a fragment of technical language is introduced ('10/0 = '), which cannot be easily translated into everyday anymore. But still "zero" has a meaning in the activity of sharing. It is, for example, possible to say, "you get absolutely zero". The students then try to translate the mathematical expression into some everyday meanings and produce some nice interpretations; they do this because this is exactly what they have been asked to do.

Amelia gets frustrated; the expected translations cannot be produced without knowing the principle. The conversation is reminiscent of a guessing game. Only one student tries to validate the contributions by a re-translation into mathematics. After Amiri's suggestion that "you're the only person that can eat the whole", this student observes that this "would be one, divided by one". He or she seems to know that the recontextualisation of sharing from the perspective of real number arithmetic is at issue and tries to work out the principle.

The teacher's attempt of developing the meaning of division by zero in arithmetic (that is: the fact that it has no meaning because it is undefined) from the everyday meaning of sharing is faced with a paradox: there is no fixed or reasonable everyday meaning of sharing among zero people and at the same time the mathematical meaning is unknown to the students. Consequently, both meanings have to be specified at the same time; one cannot be derived from the other. The problem can be seen as an attempt of expressing the meaning of a specialised discourse with well-defined, hierarchical meanings (real number arithmetic) into one that includes blurred or contradictory meanings (sharing). Translation is not possible without changing either the everyday or the school mathematical meanings.

In the example discussed above, the everyday is used as a springboard for developing school mathematical meanings. The second example that will be discussed below consists of an episode, in which the everyday is introduced as a field for usefully applying (or for being modelled by) some school mathematics. Many researchers argue that, with this type of tasks, there is a problem because students fail to link the mathematical results to their informal knowledge from everyday practices. The students' productions are often described as a "suspension of sense making" (e.g., Baruk, 1985; IREM, 1980; Silver, Shapiro & Deutsch, 1993; Schoenfeld, 1991; Verschaffel, Greer & DeCorte, 2000). Freudenthal (1982) challenges the interpretation of the students' reactions to the famous problem "Quel est l'âge du capitaine?" The students might interpret the text, and similar ones, as a story in a magic context, in which the relationships between the numbers indeed have a meaning. It is just in the everyday where these are meaningless.

On the other hand, the students' difficulties are often interpreted as stemming from too much reference to the everyday: Successful mathematical problem solvers abstract from the details and recognise the structural features (Suydam, 1980), attending to the contextual features referred to in a problem prevents the students from seeing analogies to other problems (Silver & Smith, 1980), and literally interpreting isolated phrases in word-problems in their everyday meanings causes errors (Cummins, 1991). Students are also found to fail in tests because of using too much knowledge from everyday practices when solving contextualised tasks (Cooper & Dunne, 1998; 2000). The question of referring too much or not enough to the everyday becomes a question of guessing the right dose, as epitomised in the following example from a grade-9 mathematics classroom in Catalonia (transcript from Gorgorió, Planas & Vilella, 2002).

The mathematical topic is proportionality. The teacher has asked the students to bring a cooking recipe to the next lesson. It is no surprise for the teacher that the students actually do not bring recipes. She has been anticipating that the students would not take this as a serious and important component of their normal mathematics lessons. Only Nadja, a 15-year old girl from Russia, has brought a recipe for a meat pie. The other students in the class think of Nadja that she is very clever [“She always gets it”].

The data in Nadja’s recipe are for 6 people. There are, for example, 250 g of meat needed. The teacher asks to find the amount of all ingredients “if you have to cook for 11 people”. The students are thinking a while for themselves. Then the following conversation starts:

Teacher: *Who wants to begin? Do we know how much meat we have to buy?*

Nadja [raises her hand]: *May I go to the blackboard?* [She goes and writes: “458,333333...” on the board]

Teacher: *Grams of meat?*

Nadja: *Shall I put the other ingredients?*

Teacher: *Wait, let us first finish with the meat. Are we going to buy fourhundretfiftyeight point three three three three three three dot dot dot grams of meat?*

Joel [shouting disgustedly]: *She is crazy!*

[Nadja, erases the 3’s and writes her result in this form: $458,\ \overline{3}$]

Joel: *And what is that thing over the three?*

Nadja: *You shut up!*

Teacher: *Wait Nadja. Let us hear what Joel wants to say. Joel, good manners, please. Could you please tell us what’s the matter?*

Joel: *She has never been shopping! We buy 500 grams and everybody eats a little more.*

Nadja: *But you are inventing a new problem, it is for 11 people, not for 12!*

The everyday activity to which the original task refers is cooking. In the course of the conversation, the activity is expanded and also includes shopping: The teacher initially asks how much is “needed”, and, before Nadja comes to the board, asks how much meat one has to “buy”. Joel interprets the original text (Nadja’s recipe for 6 people) as clearly belonging to the everyday domain. He sees no hints that this is not to be interpreted as such, and Joel produces an authentic solution: Being faced with the task of cooking for 11, he changes the constraints by planning for 12 people and the shopping problem is solved in an efficient way.

Nadja interprets the task as a problem that has to be re-contextualised from the school mathematical perspective of calculating exact proportions. Maybe she recognises that there is something unusual introduced: The number of 11 people is not commonly used in cooking activities, recipes in cooking books mostly refer to 2, 4 or 6 people or the number of the "eaters" is not given (if it is a cake, for example). Nadja knows that she is participating in a mathematics classroom and might be able to assume that asking for 11 people is a deliberate act of "de-authentisation" by the teacher. She understands the example not as cooking, but as cooking recontextualised from the perspective of real number arithmetic.

In this episode, it is as hard as in the previous episode about division by zero, to understand what the legitimate contribution consists of. Neither Nadja's nor Joel's versions seem to be accepted by the teacher. As in many similar examples, the students are expected to produce something in between an everyday strategy or a solution in real number arithmetic. How can the students know the demarcation line between the two discourses? Starting from Nadja's result, there is a series of questions that cannot be answered because of lack of criteria for a legitimate contribution: How many significant numbers should a rounded result have to inform the meat shopping? And then, does one have to buy the resulting amount? And if so, how? Are there only packages of pre-packed meat in the supermarket? Can the shopper buy more and make a soup from the rest? In everyday activities, efficient solutions often rely on changing the constraints. In school mathematical tasks this is not allowed, as Nadja insists.

The implicitness of the criteria

The following anecdote shows that the tension between the everyday and the academic is not restricted to solving contextualised tasks. In a grade-6 mathematics classroom in an Austrian high school in the early 1970ies, a girl (the best friend of the author) is called to the blackboard and asked to divide a line into three sections of equal length. She draws a line, takes one step backwards, looks for a while at the line, and then draws by eye a very accurate trisection of it. The teacher says: "Thank you, sit down. You have got a good visual judgement. You can become a tailor." The comment by the teacher shows how recontextualisation creates a hierarchy of knowledge and experience, favouring institutionalised knowledge over everyday knowledge. All three episodes have in common that the principle of the recontextualisation remains implicit. As a consequence of the implicitness, the students who are able to "guess" the criteria that are essential for producing a legitimate contribution are systematically advantaged over others. Thus school mathematics constructs hierarchies of positions of students as low-achievers and high-achievers. The mechanisms of classroom interaction that introduce students into the recontextualisation principle of school mathematical practice, and which at the same time, account for the emergence of disparity in achievement, need to be further described and analysed (see Knipping, Reid, Gellert & Jablonka, 2008; Jablonka, Gellert, Knipping & Reid, 2008).

The analysis of interactions between teachers and students while solving contextualised tasks, which has been presented here, suggests that the tension between the everyday and the academic does not go away, if the problems are changed, for example, by making them more “authentic”. Still, the students’ problems in these and similar examples can be conceptualised by asking:

- How can the students have access to the principles of recontextualisation that define what is to be viewed from which perspective, how the activities from outside school become fragmented, how the meanings change and how the relations between the meanings change?

Teachers employ different strategies in their interaction with students when handling the insertions of the everyday in classroom interaction (eg. Sethole, 2005; Chapman, 2006; Jablonka 2004):

- Accepting non-mathematical solutions and suggestions,
- treating the context as an aside, allowing a discussion before moving to a mathematical solution,
- contrasting and comparing the mathematical solutions with the students’ everyday experiences,
- allowing a critical discussion about the artificiality of the problems,
- pointing out the difference and independency of the mathematical structure,
- comparing problems with a similar mathematical structure, asking explicitly to judge the relevance of the everyday meanings,
- getting students to make their assumptions about the problem context explicit, comparing alternative meanings and solutions,
- deliberately making the context inauthentic (in opposition to maximising authenticity), as, for example, found in many classical word problems,
- getting the students to fragment the problem statement for translation into mathematics (often as “steps”).

These strategies differ in the extent to which they provide access to the principles of the recontextualisation of the everyday activities. All except the one listed last help focus on the difference between the discourses, and thus might assist in making the principle more explicit. The last strategy is more likely to camouflage it, by maintaining the fiction that abstraction from extra-mathematical contexts to mathematical concepts and structures is possible and straightforward (cf. Gellert & Jablonka, 2009 for a more theoretical elaboration of the problem). Introducing segments of everyday discourse, which is structured horizontally, for facilitating access to the vertically structured discourse of academic mathematics remains problematic. Bernstein (1999) describes a horizontal discourse as “a set of strategies which are local, segmentally organised, context specific and dependent, for maximising encounters with persons and habitat”, whereas vertical discourse takes the form of a “coherent, explicit, and systematically principled structure" (p. 159). Attempts of blurring the boundaries can make the transitions from horizontal to vertical discourse difficult.

Conclusion

In this paper the problem of the tension between the everyday and the academic has been conceptualised in terms of a relationship between domains of practice and their corresponding discourses. This relationship consists in the recontextualisation of out-of-school practices for didactical purposes, including a relocation and an appropriation of the corresponding discourses by the school mathematical discourse. From this perspective, the confrontation between the everyday and the academic is inescapable. Attempts of a conciliation through conceptualising the relationship between the everyday and the school mathematical discourse as a transformation between representational systems or as a translation between two languages are misleading: The mathematical structures projected into the everyday activities are not determined by the structure of these activities; the corresponding discourses differ in the ways in which the meanings are specified and organised. In the mathematics classroom, different versions of contextualised questions and tasks serve as mediators between the everyday and the academic. The challenge for the students, when confronted with this type of questions and tasks, consists in both recognising different forms of discourse and being able to produce a legitimate contribution. However, if the principle of the recontextualisation remains implicit, the students are unlikely to recognise the difference between the discourses, of which the academic is privileged with respect to successful participation. In addition, the institutionalisation of segments from everyday discourse within school mathematical discourse has a tendency to allocate the everyday insertions to marginalised groups. The distinctions between different types of knowledge structures (symbolic systems, tools, ways of acquiring them) are likely to be transformed into distinctions for categorising and labelling knowers (cf. Muller, 2006).

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Podcasting in School

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***Abstract**: Podcasting is a new phenomenon in Swedish schools. This paper describes a project where the main goal is to analyze if the students' interest in mathematics is affected if the mathematics is made accessible via podcasts and iPods. Teachers at eleven schools were encouraged to produce podcasts as a part of their mathematics teaching in school-year eight. The results indicate that the possibility to look at mathematics wherever and whenever you want was an important aspect for the students. We also found that the technical difficulties for the teachers were underestimated in the project, and that teachers had difficulties in finding time for the production of podcasts.*

Introduction

During the spring 2006 a pilot study was carried out (Gårdare, 2006) where a teacher, together with a media pedagogue, was introduced to the possibility to produce podcasts as a part of the teaching of mathematics in school year 8. Teaching mathematics is a difficult endeavour. The serious problems found in the Swedish school system connected to mathematics, for instance the large focus on rote learning, will not be avoided just by using podcasts. However, podcasts might offer opportunities to develop the teaching and address some of these problems, maybe problems related to the low interest and motivation among students.

Project description

PIS – Podcasting In School[2] was a project that involved 11 schools and 22 teachers of mathematics in lower secondary school in Sweden. One class of approximately 25 students from each school participated. The teachers were equipped with laptop computers and video cameras. After a one-day initial training they were encouraged to start testing on their own, and to produce short simple video recordings (podcasts) to be published on the web. Examples of podcasts in mathematics from the pilot study were presented, but no other instructions concerning the content were offered. The idea behind the low amount of instruction to the

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[2] The project was funded by Rektorsakademien, The Swedish National Agency for School Improvement, and Apple.

teachers had to do with issues of scalability, that is that support could be hard to offer in a large-scale implementation.

Each school in the project was supposed to involve one class and two teachers. The teachers should produce podcasts with a mathematical content, and the students would be offered to look at the podcasts on a computer and on a video iPod[3]. In some schools other subjects than mathematics were also considered. The iPods were provided to each class for half the project time, i.e. one term with an iPod and one term without.

The research part of the project was given the following overarching question: *Does the students' interest in learning increase when you introduce youth culture into the school?*

Youth culture is here represented by podcasting and ipods. The research design will look at affordances (action possibilities) and constraints (limitations) concerning podcasting in mathematics education, in connection to interest and motivation for learning.

One school in the project (here called School 11) was treated differently from the rest in that there was a close contact between the researchers (who are researchers in mathematics education) and the teachers. The ways in which the teachers produced podcasts were here developed in dialogue with the researchers. Issues concerning process goals, communication skills, problem solving and mathematical reasoning were raised to enhance the podcasts. The analysis of the data from School 11 will not be a central part of this paper, but some observations will be mentioned.

Research questions

Students' attitudes towards the learning of mathematics are central in many discussions of the problems found in the Swedish school. Of course it would be very interesting to be able to measure how students' performance in mathematics was affected by the use of iPods and podcasts very clearly. However, as in many other situations where a new tool or a new way of working is implemented, so many things are changing that it is as good as impossible to connect a change to a single variable. Therefore we will focus more on student attitudes to mathematics, since it is reasonable to believe that an increased student interest will result in better student performance.

The research questions we have formulated are the following:

1. How are student's attitudes towards mathematics, and the learning of mathematics, affected by the use of podcasts?
2. To what extent are the teachers producing podcasts, and what characterizes the podcasts in terms of what is presented? What are the affordances and constraints?

[3] Video iPod is registered trademark by Apple.

3. In what ways and to what extent are the students using the podcasts? What are the affordances and constraints?

The first question is very closely related to the overarching question. In order to reach clear answers to the first question, we need information concerning the podcasts as well as information on how the podcasts were used by both teachers and students.

It is also important to point out that this report deals with only the first half of the project, and that not all research questions will be addressed. For example, a discussion about the mathematical content of the podcasts will not be a part of this report.

Theoretical framework

The importance of attitudes for mathematics learning is well accepted among mathematics education researchers, especially in connection with motivation (Op't Eynde et al, 2006; Hannula, 2006). Motivation is often divided into extrinsic and intrinsic motivation (e.g. Ryan & Deci, 2000). Extrinsic motivation has to do with rewards of different kinds (grades, praise from parents etc.), while intrinsic motivation is connected to the learner's curiosity and wish to learn. The use of podcasting and iPods will in this study be discussed as important for both the students' extrinsic and their intrinsic motivation.

One example of a quality in the use of podcasting that might affect the students motivation is that it gives the students the possibility to decide when, and to what extent, they are to be exposed to mathematics:

> Rather than having the teacher make all instructional decisions, offering students control over the amount and sequence of instruction, including options for review, can result in higher achievement and improve student attitudes toward learning. iPod, with its virtually limitless opportunities for playback, literally places control in students' hands. (Pasnik, 2006)

This means that by giving the students control over their own learning environment, it is possible to achieve positive effects on the both extrinsic and intrinsic motivation.

We find the theory of affordances and constraints (Greeno, 1994) potentially helpful in the process of data analysis. In the case of the use of ICT, an affordance can be seen as a property of the particular application e.g. the use of digital media on the iPod. As such affordances are conceived as preconditions for activity and in particular for mathematical activity in the case of this project. On the other hand the affordances provided by a device or application may be seen as conditions for constraints. The existence of an affordance for some activity is not seen to imply that the intended mathematical activity will occur, although it contributes to the possibility that it will do so. The perception and motivation to engage in the activity on the part of the user becomes a key factor to consider.

Method

The research questions in this study demand a variety of data collecting methods and analyses. Interviews and observations of podcasts are the main methods in relation to the research questions. In order to understand what initial knowledge and experience the participants had, questionnaires, both to students and teachers, were used.

At the beginning of the project questionnaires were sent out to each school, one to the students and one to the teachers. We received answers from all teachers and near all students. The questions to the students centred around four areas: questions on attitudes towards mathematics, questions about previous experience of podcasts, questions about the use of mp3-players, and questions about expectations on the project.

The questions to the teachers also concerned, apart from standard background information, four areas: questions about the participation in the project, questions about the teacher's own goals with the project, questions about the teacher's normal teaching and lessons, and questions about expectations on the project. The main use of the questionnaires was as a background for the interviews.

In May we visited eight of the eleven schools. Interviews with teachers and students were carried out at each school. All interviews were audio recorded and selected parts were transcribed. The selection was based on the existence of passages considered important for answering the research questions. The interviews were of a semi-structured format using a pre-designed scheme.

The teacher interviews ranged from 30 to 90 minutes, and in most cases one teacher from each school took part. The interview scheme concerned attitudes to mathematics and the use of the podcasts. In the section when we discussed the teacher's views on the possibilities and limitations we connected the interview question to the each teacher's answer to the corresponding question in the questionnaire.

The student interviews ranged from 15 to 30 minutes, and three or four students from each school took part. We were also very careful to explain to the students that their answers are very important for the research, and that they are one of our main sources of information.

All interviews were transcribed in a condensed form, followed by a complete transcription of important passages, i.e. passages where discussions clearly related to the research questions took place. The complete transcriptions were used in the analysis.

In one school the teachers in the project had recurrent meetings with the researchers in the project. Their podcasts, both content and form, were discussed in a rather systematic way. The content was related to the syllabus and process goals of the education. Questions like "what process goal do we want to address with this podcast" and "what known difficulty in mathematics learning can be focus here?" were asked. Four meetings, in addition to the meetings for all participants in the project, took place during the spring. Each meeting lasted approximately 1.5 hours and the researcher took notes.

Analysis

The find answers to the first research questions in this study, concerning how students' attitudes towards mathematics, and motivation to learn is affected by the use of podcasts and iPods, we need information in how teachers and students have been using podcasts, i.e. the second and third research question. Therefore we will in this section first treat research questions two and three, followed by an analysis concerning the first research question.

In the analysis we have used the theory of affordances and constraints in order to find important issues that not only are interesting from the teachers' or students' perspective, but also indicate that it is the podcast or iPod in itself that gets attention.

Research question 2: Teachers and podcasts

Two issues were put forward by the teachers as central affordances in the use of iPods and podcasts. The first was the possibility for students to take part of content when they have been absent during a lesson. This was seen as one of the most important aspects. In relation to this it is not surprising that the podcasts to a large extent had the same content as classroom presentations, especially in the beginning of the project. In addition, this affordance can also become a constraint. If teachers are very convinced that this is the best way to use podcasts they may refrain from trying other types of podcasts.

The second affordance focused on by the teachers was the possibility to show out-of-school situations in order to present the setting for a problem, and also the problem itself. In some podcasts the problems were solved, in other the students were given the problem as homework. The use of podcasts to present problems was in many schools something that came after a few months of testing and trying out the technical issues in the production. It seems possible that the teachers after the initial period wanted to do something more than just copying the classroom. One might argue that presenting a problem in a podcast is no different from doing it on paper or any other media. However, some new possibilities occur, e.g. visualisations of problem situations or solutions. One example from a podcast in this study is when the concept of VAT[4] (25% in Sweden) is discussed and rectangular blocks are used to show that you need to remove 20% in order to get the price without VAT. All in all we must conclude that only a small number of podcasts in this study contained something that could not be done in the classroom.

[4] Swedish: MOMS.

One of the two constraints highlighted by all teachers was that there was too little time to plan, produce and publish the podcasts. Since few teachers in the study were compensated regarding their workload, the production of the podcasts had to take place in their free time. Most teachers had to carry out their normal assignment as well as taking part in the project, so it is not surprising that the production of podcasts declined radically for several teachers towards the end of the term. Some teachers could use the schools days for open-air activities to work with the podcasts, and some could use days specified for competence development, but on the whole, most teachers experienced the time issue as a significant problem.

The other constraint was that the teachers experienced a lot of technical difficulties, from the actual filming via the cutting and preparation of the podcasts, all the way to the final publication of the podcasts on the web. The most common problem had to do with the last part of the production, the uploading of the podcasts to the web, and the creation of an RSS-stream. Several teachers felt that they lacked a more thorough instruction concerning the last steps. In some school there were also difficulties originating in the fact that the project computers were Apple laptops (running Mac OSX), while the school (and the municipality) were Microsoft Windows environments only. The problems could for instance be that the Apple computers couldn't run Windows programs necessary to access the municipality intranet.

All teachers in the project had experienced a lot of positive feedback from students and parents concerning the project. This was regarded as important, both from a more general school perspective and for more individual reasons, like the possibility to get previously not so interested students to take part of the mathematics teaching to a larger extent.

Research question 3: Students and podcasts

In general, the students who had an iPod looked almost all podcasts. There were a significant difference between the group of schools *with* iPods and the schools *without* iPods. Few students without iPods had used the podcasts at all. They were all looking forward to the day when they would get their iPods. "The day the project really will start" was an often-heard comment, both from students and from teachers in these schools. The students with iPods the first period believed that the use of the podcasts would decrease when they have to do without the iPods and only look at the podcasts in iTunes. The importance of this affordance of the iPod is not so easy to value at this stage of the project. Maybe we can know more after the second half of the project when the iPods are switched between the schools. From the interviews we also believe that many students have looked at podcasts in addition to (and not instead of) their normal amount of mathematics outside school. This would indicate that the amount of mathematics the students have met during the project is larger that normal.

The most common perceived affordance discussed by the students was that the podcasts could be used for reviewing before a test and catching up when missing a lesson. The possibility to look at the podcasts anytime, anywhere appealed to all the students. "To be able to prepare for a test on the bus" and "during the lunch break we have no access to computers in school" were two statements that highlight the view that this affordance was important.

Interesting was also that in one class the students looked at a podcast before the teacher presented the content to the class. The students found it easier to follow the presentation and the teacher claimed that he got deeper questions from the students. Of course, this possibility exists in normal teaching also; the students can read the section in the book in advance. However, this is not common student behaviour according to the teacher. This affordance of the podcasts, that they might influence the students to look at mathematics in advance, is an important finding in this study, especially if we interpret this as a motivation to engage in mathematics to a larger extent than without podcasts.

Indications of another affordance were also found. Some students and several teachers reported on positive feedback from parents who had looked at some of the podcasts. After viewing the podcasts the parents suddenly could help their child in mathematics. It would be very interesting to interview some parents concerning this issue, but that is not a part of this study.

Among many of the students in the interviews, a perceived constraint of the podcasts was found, concerning the idea of putting a whole content area into podcasts. This had been tried in one school (not in mathematics, but in geography, concerning latitude and longitude) with very good result and positive opinions from the students. However, none of the interviewed students from other schools believed it to be a good idea. "You need someone to explain the content" was a common comment.

An interesting observation concerning the students' use of the podcasts is that most students were in agreement that it was rather easy to download the podcasts to iTunes and to the iPods. Some cases of technical difficulties were found, but nothing serious, and definitely not at all on the scale that the teachers reported concerning the production and publication of the podcasts. The students were very satisfied with the quality of the podcasts. "It's easy to see and the sound is very good".

Research question 1: Attitudes and interest in mathematics

It seems that the use of podcasts and iPods in this project have lead to an increased amount of mathematics the students meet in their learning of mathematics. Many students said that they spend more time with mathematics due to the fact that they have iPods and have podcasts to look at. It seems reasonable that increased time with mathematics is connected to motivational aspects of the learning. In that case, it indicates that the presence of iPods and podcasts have affected the students' attitudes and interest in mathematics.

Other reasons for the increased time with mathematics are the affordances described above. The possibility for a student to take part of the content after missing a lesson was a central aspect according to both teachers and students. The students also mentioned some other uses of the podcasts, for instance to practice before a test and to review a specific method. These affordances were often mentioned in connection to the affordance that the students' could look at the podcasts anywhere, e.g. on the bus or between classes.

All students we interviewed agreed that the use of iPods was a cool thing in mathematics. "Mathematics is more fun with the iPods" and "you can review in a more fun way". They also claimed that it was more fun and stimulating to look at a podcast compared to read in the book or review ones notes. They stated that mathematics had become a more popular subject in their school thanks to the project. Whether this was because the students could learn mathematics in a better way, or because they were the only class in their school to be part of a project where they got an iPod, was a question the students couldn't answer. Student voices concerning the content were also heard. One student was very clear on this: "If I won't benefit from the use I will not look at the podcasts." Similar statements, but maybe not so direct, was heard from several other students.

Concerning extrinsic and intrinsic motivation, all students agreed that the iPod is a very cool gadget that definitely leads them to spend more time on mathematics. They also stated that it is not enough with an iPod, you also need to benefit from the content. It was clear that the students to a very large extent talked about extrinsic motivation using expressions like "making the course", "getting good grades", "be prepared for higher studies". They rarely talked about intrinsic motivation, even if a few students mentioned things like "mathematics is more fun", which can be interpreted as such.

From the interviews with both the teachers and the students it seems clear that the iPod was a very important part of the project. When we compare the schools with and without iPods, there is a significant difference in the level of activity, both among the students and the teachers. This must of course be interpreted carefully. Since the iPods are considered as very 'cool gadgets' by the students, it is only natural that the students are affected and it is not necessarily so that the students' really are more interested in the learning of mathematics.

School number 11

The first podcasts produced at school 11 was similar to most other podcasts in the project in that they consisted of presentations of the same kind as in the classroom: introductions to a new area, presentations of a concept, presentations of a method, or presentations of problems.

The second group of podcasts was of a different type. Here the teachers went outdoors, interviewed people, and made things not possible in the classroom etc. The idea was now that the podcasts should be a complement to the classroom activities, not the same thing in another package. The teachers also instructed the students that it was mandatory to look at the podcasts in advance, and that they would discuss the content during the following class. Due to some practical circumstances this was not carried out fully, but the teachers are planning to use this strategy after the summer.

In one podcast the podcast consisted of a number of short interviews around large numbers and the use of prefixes. In the interviews we heard a lot of different ways of using prefixes, as well as some erroneous statements and concepts that are not prefixes. Here is one example of a passage.

Interviewer: How big is the hard disk in your computer?
Victim: Around… I would say… one cubic decimetre.
Interviewer: OK… and how much memory does it have?
Victim: Well, eighty gigabyte.

Here the victim uses two different prefixes and also the word "cubic". There are opportunities to discuss several aspects of the use of prefixes in everyday life starting from this short passage. Questions like "when do we use prefixes", "why do we use them" and "is 'cubic' a prefix" may trigger students interest in a way that might be different from a normal classroom situation.

Discussion

The results from this study indicate that students' attitudes towards mathematics are affected in a positive way in this project. There are of course several possibilities for this positive effect. The students all got a 'cool gadget', they were part of a research project, their teachers (in some cases) got extra time for the teaching of mathematics and so on. However, in the interviews some other reasons were found. The possibilities to take part of the content after missing a lesson and to review the content before a test were important aspects. Especially the possibility to look at mathematics anywhere and at any time, was highlighted by the students. This is also supported by Pasnik (2006), who means that giving the students control over the learning environment (to some extent) has a possibility to increase students' motivation. One might argue that the increased interest is a temporary situation, only while the situation is new, and that it will go back to normal after some time. One thing that supports such a belief is that the main part of the positive factors the students reported had to do with extrinsic motivation. On the other hand, several students pointed out that there had to be a gain in the learning of mathematics if they would use the podcast to any extent.

There were several effects found in the study that were caused by the specific use of podcasts, and not only by the fact that the students got a device or that they were part of a research project. One example is when the students looked at a podcast before the same content was presented by the teacher in the classroom. This lead, according to the teacher as well as the students, to better possibilities for the students to understand the content and to pose more accurate and profound questions. Another example is that several teachers reported on positive feedback from parents who, after looking at some of the podcasts, experienced increased possibilities with their child's homework. However, the most prominent affordance with the podcasts concerned the use of iPods that the students got an option to look at presentations of mathematical content whenever they wanted: in the bus, when waiting for a lesson to begin, during the lessons, at home etc.

One serious problem in the project was that many teachers experienced that they could not find enough time to produce podcasts the way they wanted. Almost all teachers raised the time issue. Of course, in a longer perspective, the production of podcasts in mathematics must be one part of the teachers' work, since we cannot assume that mathematics teachers suddenly will get more time for mathematics just because they use a new working model. One important question in a possible future use of podcasts is to find a balance between podcast production and other parts of the work as a mathematics teacher. What can be reduced and what can be made in more rational ways? What can be removed? If podcasts should be a normal part of the job as a mathematics teacher, without an increase in workload, something must change.

This paper has mainly discussed the availability of the podcasts, and the effect that student use of podcast might have on student attitudes. In the second part of the project we also want to include the quality of the podcasts. Why is one podcast interesting and another not? What is it in a podcast that catches a student and affects attitudes? Can we characterise what it is in a podcast that students find interesting?

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Making Sense of Negative Numbers Through Metaphorical Reasoning

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***Abstract**: The concept of negative numbers is such an abstract concept that it has been argued that it can only be understood through symbolic reasoning. However, others argue that all mathematical concepts are understood through metaphors. Previous research has identified three important aspects of understanding negative numbers: direction and multitude, proficiency in arithmetic operations, and the meaning of the minus sign. This study further explores the theory of conceptual metaphors and metaphorical reasoning by investigating the use of models and metaphorical reasoning when dealing with negative numbers. The data consists of test results from 99 students in the teacher training program and follow-up group interviews. The results show that some students' difficulties seem to be a consequence of their use of metaphorical reasoning using a metaphor that is insufficient. However, metaphorical reasoning also seems to be helpful for students who are aware of the limitations of the metaphor. Enlightened use of metaphorical reasoning, i.e. being aware of the potentials and constraints of models and metaphors, could therefore be described as a fourth important aspect of understanding negative numbers.*

Introduction and research question

Negative numbers are well known to be difficult to teach and to understand. Previous research has documented difficulties and dilemmas concerning both negative numbers as such and when they appear in algebra. Proficiency in calculating with negative numbers is a prerequisite for understanding algebra (Gallardo, 2001; Vlassis, 2002) and since algebra today is a part of school curricula (at least in the Swedish compulsory school) negative numbers are important to master. In the encounter with negative numbers students get important experiences of what mathematics is about; mathematics in a much broader sense than the art of counting and calculating quantities. Students often say they find negative numbers difficult because the negative numbers are so abstract and lack connections to the real world. The most common real world connection to negative numbers is the thermometer, at least in Sweden where we use the Celsius scale and have temperatures below zero every winter. A visual representation related to the thermometer is the number line. Although previous research and work on didactics of mathematics dating hundreds of years back have dealt with different models used when teaching negative numbers difficulties remain and teachers keep asking for better models. Instead of further exploring the models as such the models are here viewed through the theory of conceptual metaphors. It is therefore not the model itself but rather how the model is used that is focused upon. Reasoning about numbers and calculations in terms of actions on or with models or representations is here referred to as metaphorical reasoning.

The research question for the study reported here is: To what extent do students make use of representations/models and metaphorical reasoning when solving tasks with negative numbers and how does this relate to their solutions and their confidence? In the next section the theoretical framework used will be described.

Metaphorical reasoning

The theories that have developed around metaphorical reasoning and conceptual metaphors are in some ways an answer to the classical dilemma of how one can learn things about that which one does not know. It is assumed here that learning new things is about connecting new experiences to already known experiences. Metaphors serve as important links between prior knowledge and new concepts. Metaphors can be defined as understanding one conceptual domain (the target domain) in terms of another conceptual domain (the source domain). Lakoff and Núñez (2000) assert that most mathematical and abstract concepts are conceptualized in concrete terms, for example thinking of numbers as points on a line. There are, they say, two kinds of metaphorical mathematical ideas: grounding metaphors yielding basic ideas and linking metaphors yielding sophisticated or abstract ideas (p. 53). All metaphors have limitations since the target domain is never identical to the source domain. When we construct highly abstract concepts we build up whole metaphorical systems that together characterize the concept. Metaphors make sense of our experiences by providing coherent structure, and thus focusing attention on those aspects of the source and target domains that bear similarities. The theory of metaphorical concepts claims that different metaphors are used to structure different aspects of a concept (Lakoff & Johnson, 1980).

Since the words ‘model’ and ‘metaphor’ are used alternatively by many authors the two concepts are often confused and a clarification is called for. As shown in Figure 1; a conceptual metaphor can be described as a set of mappings from a source domain to a target domain (Kövecses, 2002).

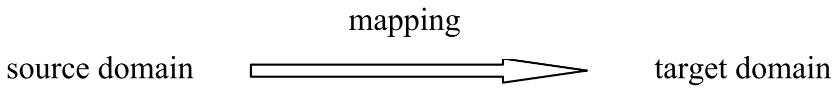


Figure 1: Illustration of a metaphor.

A ‘pedagogical model’ is an embodied experience or a visual representation chosen to represent a mathematical concept in order to reveal features of the mathematical concept and it is used as a conceptual metaphor by treating the model as a source domain and mapping the features inherent in the model onto the mathematical target domain. This use of the word model is not to be mixed up with 'mathematical modelling' where the metaphor is reversed. In mathematical modelling the source domain is mathematics and the target domain is reality. In a mathematical model we might investigate speed and time/distance relations in terms of functions, mapping features of functions onto reality, whereas saying "-2+5 is when the temperature is -2º and then goes up 5 degrees" is a way of using the thermometer model as a conceptual metaphor for addition, mapping experiences of temperature change and how that is shown on a thermometer onto arithmetic. Henceforth when the word model is used it refers to a pedagogical model, i.e. a visual representation or embodied experience chosen to represent a mathematical concept. The model is not a metaphor in itself; it supplies the metaphor with a source domain and can thereby be used as a metaphor. Several different metaphors can have different source domains but similar or even isomorphic mappings to the same target domain. Take for instance a model of an elevator going up and down in a building, temperatures rising and sinking on a thermometer or a glacier growing in winter and melting away in summer. They are all different but when these models are used as conceptual metaphors for arithmetic they have isomorphic mappings to the grounding metaphor 'Arithmetic as Motion Along a Path' (Lakoff & Núñez, 2000). The differences between the models lies in what is moving along the path and what kind of path it is, but the structure is the same.

Some models are already in themselves abstractions and as such removed from a child's everyday knowledge. A model has to be a *model of* something before it can be used as a *model for* something (Gravemeijer, 2005). Understanding of the source domain is a requirement if a model is to function as a conceptual metaphor. "*A metaphor can serve as a vehicle for understanding a concept only by virtue of its experiential basis*" (Lakoff & Johnson, 1980 p.18). Using a particular model as a conceptual metaphor by reasoning about a mathematical concept in terms of this model is what is here referred to as metaphorical reasoning.

Previous research concerning negative numbers

Researchers have previously identified three main aspects of understanding the concept of negative numbers. The first aspect is an understanding of the numerical system and the relative size of the numbers (direction and multitude) as well as an understanding of the number zero (Ball, 1993; Kullberg, 2006; Martínez, 2006). Ball shows that the absolute value aspect (the multitude) of negative numbers is very powerful. These are all different aspects of number sense (Reys & Reys, 1995).

A second aspect is how well the students understand the arithmetic operations (Chacón, 2005; Vlassis, 2004). The big problems when calculating with negative numbers arrive with the subtracting of a negative number (multiplication is even more problematic but is usually brought in later). Gallardo (1995) showed that it is of great importance whether the student understands subtraction only as an operation (taking away) or if they also have a structural understanding (as a comparison between two numbers). On the other hand Linchevski and Williams (1999) did an experimental study where 'subtraction as take away' was understood by the students to be 'the same as adding the opposite number' (through a dice-game and counting on a double abacus) and never used the structural aspect of subtraction. Sfard (1991) indicates that the interiorization of negative numbers is the stage when a person becomes skilful in performing subtractions.

The third identified important aspect is the meaning of the minus sign. The same sign is used both as a sign of operation and as a sign indicating a negative number, that is, indicating the nature of the number (Gallardo, 1995; Kilborn, 1979; Kullberg, 2006; Vlassis, 2004). In some ways it is unfortunate that the sign is the same, and there has been experimental research where different signs are used for the two different purposes (Ball, 1993). The different meanings of the minus sign could be described as the operational and the structural aspects of the sign, where the operational meaning is usually introduced long before the structural meaning. Many errors appear when the minus sign indicating a negative number is detached from the number (Herscovics & Linchevski, 1991; Vlassis, 2002).

Many textbooks use visual representations/models[1] (such as the number line, a scale, a time line) and everyday life representations/models (such as temperatures or money) to explain subtraction with negative numbers. According to Linchevski and Williams (1999) some researchers *"...argue against using the existing models for negative numbers [...] concludes that the topic of negative numbers should be taught only when the students are ready to cope with intramathematical justifications"* (p. 134). Contrary to this, through their experiment Linchevski and Williams draw the conclusion that at least subtraction with negative numbers can be understood through models; not a single model but a multiplicity of models. Gallardo (1995) suggests teaching negative numbers using discrete models where whole numbers represent objects of an opposing nature rather than using the number line. Kilborn (1979) points out that some teachers use several different models simultaneously during a lesson and that these models seem to confuse the students. Ball (1993), on the other hand, states that no representation captures all aspects of an idea and "*teachers need alternative models to compensate for imperfections and distortions in any given model"* (p. 384). She articulates a dilemma when she asks whether she confuses the chil-

[1] The word model is here used to be tantamount to a visual or experienced representation. Once seen, this visual representation can be referred to as a mental representation or mental model without being actually visible.

dren by letting them explore multiple dimensions of negative numbers by introducing different representations. Some work has been done on the use of metaphorical reasoning when dealing with negative numbers (Chiu, 2001; Stacey, Helme, & Steinle, 2001), which is also the focus of the study presented here.

Method

In this study a test was given to students (N=99) in the teacher training program prior to the topic being dealt with in their mathematics course. The test included calculation tasks on negative numbers with follow-up free text questions and self-estimate ratings. There were two groups of students who were enrolled in a one term course in Mathematics Education for young children. They were not obliged to have taken more mathematics than the basic mathematics course (mathematics A) before entering the course. More than 90 % of the students in the two groups were women. The test was given to the students at the end of an ordinary lecture on a randomly chosen day during the course without the students knowing beforehand about the test. The students attending the lecture that day therefore made up the respondent group. Participation in the test was voluntary but all the students chose to respond. The test took 10-15 minutes to complete and was anonymous. The students knew me as a teacher in some parts of the course, which might be considered a complication. However, the students had no reason to assume that this test would in any way influence their grades since the test data were analysed after their exam.

Solutions of the calculation tasks were categorised and counted. The follow-up questions were used to get hold of the students' reasoning and their answers were categorized according to similarities, and analysed with a special interest in the use of metaphorical reasoning and arithmetical (symbolic) reasoning.

As a means of triangulation (Bryman, 2004) three groups of students participated in video recorded group interviews some weeks after they finished the course where they discussed their answers and a few related tasks. Each of the three groups consisted of 2-5 students. Data from the group discussion interviews were used as a complement to achieve a more holistic view (Cohen, Manion, & Morrison, 2000, p.115).

There were three parts on the test. This paper reports on the results of the first part which dealt with subtraction of a negative number. The first question asked only for an answer to a calculation. The second question aimed at measuring the confidence of the student and in the third question the mathematical reasoning was sought[2].

1a) calculate: (-3) – (-8) = ________

1b) How sure are you that your answer is correct? (choose one answer)

__ very uncertain __ a bit uncertain __ rather confident __ very confident

[2] The questions were not numbered in the actual test but are so here for practical reasons.

1c) There are different ways of thinking to reach an answer to a question like this. Try to describe your way of thinking.

Results

As shown in table 1; 67 % of the students solved the subtraction task (1a) correctly and 33 % gave an incorrect answer. The most common incorrect answer was −11.

Table 1: Results from task 1a (percentage and number out of total 99).

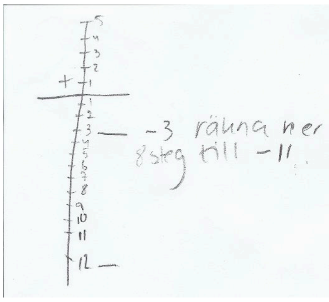
Answer	5 (correct)	−11 *or* −5	−11 *and* 5	Other / no	Total incorrect
total n= 99	67 % (n=66)	28 % (n=28)	2 % (n=2)	3 % (n=3)	33 % (n=33)

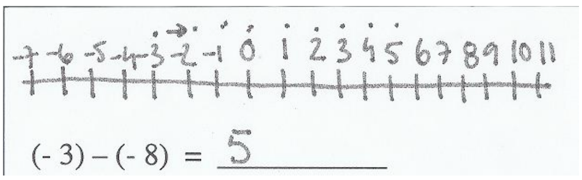
60 % of the students who gave incorrect answers on task 1a chose either 'rather confident' or 'very confident' on the next question, many of them referring to either a representation like the thermometer or to arithmetic rules. For those with correct answers the corresponding percentage was 85 %. The answers given to question 1c were categorized as either

- using metaphorical reasoning referring to some mental or visual representation of negative numbers (n=23)
- referring *only* to an arithmetic rule or a deductive argument (n=71)
- no or irrelevant answer (n=5)

Metaphorical reasoning on the first task included the thermometer, money debts and movements along the number line. The category using metaphorical reasoning (n=23) fell into two distinct subcategories: First; students who referred *only* to metaphorical reasoning on this task (n=14), *all* of which had arrived at a wrong answer, and second; students who used *both* metaphorical reasoning *and* an arithmetic rule (n=9), *all* of which had arrived at a correct answer (see table 2). One person gave two alternative answers; reference to the thermometer rendered the wrong answer and reference to an arithmetic rule rendered the right answer.

Table 2: Metaphorical reasoning using temperatures or number line models

Answers of the first subcategory: incorrect answer given (-11)
I think of the thermometer. It was minus 3º and then it got 8º colder.
On a number line or a thermometer I think of where -3 is and keep going -8 to get to -11.
 I add -3 and -8. That makes -11. If I have -3 on a thermometer. Take away another 8, that makes -11. *"-3 count down 8 steps to -11"*
I count up from -8, add 3 which in this case is -3. Think addition. I also have a vague idea that negative numbers sometimes become positive. I would show a thermometer.
It's just automatic. I have simply learned it. Just take a scale with negative numbers and back 8 steps from (-3)

Answers of the second subcategory: correct answer given (5)
I will start by simplifying the expression[3], take away as many brackets and signs as possible: I know that minus and minus makes + (that is in front of and inside the brackets) and then + and – makes – (1:st pair of brackets). Then what remains is just -3+8. (Which I in my head picture on a thermometer-scale and count up 8 steps from -3)
I think that two minus make a plus. I picture a thermometer.
I see a ladder/temperature scale & "look" where zero is & -3 steps from there and then +8.
From minus three you have to take away minus 8. That makes "+" 8 because it was minus to start off with. [*made a drawing of a vertical number line*]
Like my drawing. I remember that two minus cancel each other and makes plus, hence -3+8=5. 

[3] In Sweden the word "tal" is used both in the meaning of "number" but also in the meaning of "mathematical expression" or "mathematical task" This can be very confusing. In the translation I will use different words.

Summary of the results

The results and different categories can be summarised as in a figure 2.

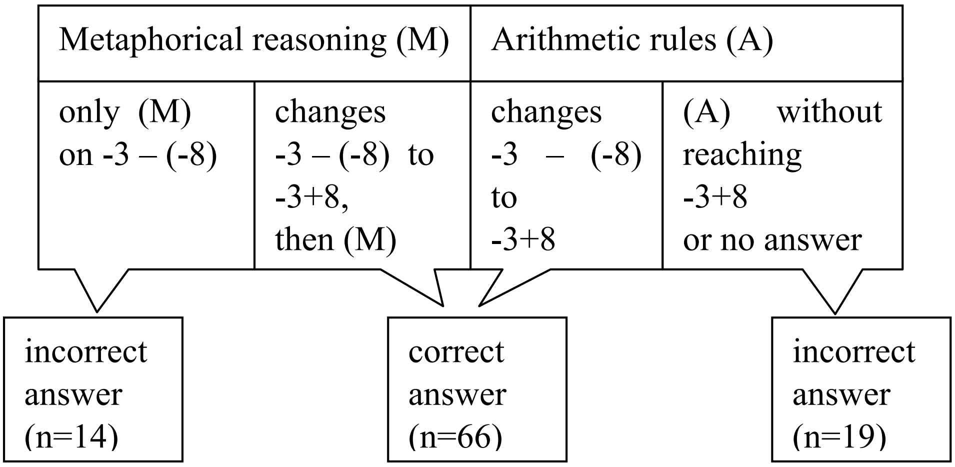


Figure 2: Overview of categories.

Discussion

It is interesting to note that many of the students, who failed the calculation task, neither seemed to understand these numbers well enough to solve the task, nor did they anticipate that they might be wrong. They seemed to believe that the metaphorical reasoning they used would give them a correct answer. Understanding can be defined in terms of connections between ideas, facts and procedures. "*…the mathematical idea is understood if its mental representation is part of a network of representations*." (Hiebert & Carpenter, 1992 p. 67). Of course it is desirable that this network of representations will help to produce mathematically correct answers to mathematical problems. Self-estimate ratings, which were used as a measure of how confident a student is that her answer is correct, are an indication of to what extent the mathematical idea is part of such networks, assuming that a tight network creates more confidence than loosely connected pieces of knowledge. Confidence with an incorrect solution would in that case indicate that the incorrect solution is based on ideas, facts and procedures that are part of a network of representations, but a network which is inconsistent with mathematics.

In this data a large group of incorrect answers (n=14 out of 33) was found to be those who *only* used metaphorical reasoning, and most of them declared that they were rather confident (n=6) or very confident (n=3) about their answer being correct. Another group of students (n=9) used metaphorical reasoning as a complement to symbolic transformation (changing –(-8) into +8). These students *all* arrived at a correct answer. It seems as if metaphorical reasoning is only helpful, when the student is aware of the constraints of the metaphor and is capable of treating numbers as entities without meaning in order to transform them into something that carries meaning and has similarities to the representation at hand. It is crucial that students become aware of the constraints of the metaphors that underlie their understanding. This is in line with the results of Chiu (2001), claiming that experts know the limitations of the metaphors they use and therefore learn when to use each metaphor.

The largest group of correct answers were found among students who transformed the operation (-3)-(-8) into -3+8 and then calculated the answer. It is possible, as seen in the group interviews, that these students implicitly used a mental number line, scale or thermometer or other representation when dealing with the operation -3+8. Those who explicitly did so arrived at the correct answer. Students who gave a wrong answer without referring to metaphorical reasoning gave more arguments about the number of minus signs or gave plus priority over minus and in general got all tangled up in arithmetic rules without meaning. A possible interpretation of this data is that metaphorical reasoning is essential in order to create meaning in the calculation and judge if the answer makes sense or not. This argument supports Chiu's (2001) claim that novices more often than experts use metaphorical reasoning to verify their results. An expert would have done so many similar calculations that she need not verify it anymore (as expressed by student B) whereas a novice would need some way of justification. In this respect the metaphor serves as scaffolding (Vygotskij, 1999). In addition to the three previously shown aspects of understanding negative numbers these results shed light on a fourth aspect that might create difficulties and cause incorrect calculations; relying on metaphorical reasoning using a metaphor that is insufficient for the purpose it is used. The results of this study suggest that teachers should make use of metaphors with experience based models and visual representations as source domains but emphasize their limitations concerning negative numbers. Knowing the potentials and constraints of a model is necessary if it is to function as a conceptual metaphor and for the learner to be creative in striving to understand. As a contribution to the body of research, these results suggest that the debate should not be concerned with which model to use and why one model is better than another but rather what are the consequences of our use of metaphors and how we deal with these consequences.

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Spectrums of Knowledge Types – Mathematics, Mathematics Education and Praxis Knowledge

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Abstract*: While mathematics is deductive and mathematical education is evidence based, practical knowledge is a type of knowledge that professionals in any profession develop by experience and by exchange with other professionals. It is based on experience more than on written text. It is well known that it to a large extent is difficult to articulate. Such knowledge is also essential in important types of mathematical knowledge. I would like to sketch a more fluent cooperation between these areas, in which the advantages of all the different knowledge types may interact and become increasingly useful to each other. For such an idea to reach reality it is necessary for mathematicians, mathematics education researchers, mathematics teachers, and others, to listen in depth to each other, and to have a dialogue. In this paper one alternative to achieve that is described: the Dialogue Seminar.*

Introduction

In order to improve mathematics education, are our efforts well spent? Are there possibly other ways of work that may lead to more significant improvements? This is an extremely basic issue for an applied science, which underlies this paper. It is a question that directly concerns how three professions form and perform their work: mathematics education research, teacher education, and teaching.

This paper tries to use a viewpoint of the teacher profession as a profession among professions. When comparing to non-teacher professions, one particularity of the teacher profession stands out: its relation to *knowledge*. It has particularly strong relations to knowledge in two ways. The first is that its main purpose is the learning of certain subject knowledge for other humans. The second is that there is a long knowledge tradition about how teaching and learning can be done: pedagogy and didactics – incorporated in educational science. Knowledge is essential in all professions, but for teachers it is the very material of work, and there is a lot of written knowledge on how this particular profession works – since the school is so important in society.

This strong knowledge tradition concerning the teacher profession has been developed in a positivist tradition – which was the dominant tradition in the previous century. In this tradition, knowledge is what can be formulated in words (Johannessen, 2006b, p. 273), a type of knowledge that can be called *proposi-*

tional knowledge. This viewpoint neglects much of practical knowledge, a fact that is not changed by propositional knowledge that has practical knowledge as focus (such as partly this article). Practice is different in essence from propositional knowledge. For example, it is very much possible to be an expert in every possible aspect of listening to dialogues, without being able to take part in one. Another person may be known to reoccurring valuable dialogues with students, but unable to describe what a dialogue is and how it works in depth.

Despite the strong knowledge tradition in the teacher profession, difficult-to-articulate practical knowledge is important in all professions – also in the teacher profession. Practical knowledge enables a professional to act in appropriate ways in unforeseeable teacher situations – it develops intuition. It is a knowledge that enables skill – the ability to *perform* the profession. This ability is distinct from knowing.

In terms of action research, for example, pedagogy and mathematical education are searching ways to handle teachers' reality and practical knowledge. However, it is not easy to allow teachers to fully express their views of their teaching situation if it contradicts established paradigms. This problem is one main focus of this article. The philosophical foundations of practical knowledge (Johannessen, 2006a) may also contribute to the understanding of propositional knowledge, both its limits and its values.

A different but related serious problem for research in mathematics education is that it serves two goals that are rather conflicting. One is to help teachers; the other is to fulfil the requirements for research. The second goal is a long term goal that makes results more reliable, but tends to make results inaccessible for teachers. In lack of time and resources, a researcher may have to choose between reaching teachers or reaching researchers. Some ambitious researchers have written popular versions of their research, more accessible for teachers and decision makers. This should be mandatory for an applied science. We need to leave behind the fact that "popular" writing sometimes is seen as negative among researchers.

The purpose of this paper is to find ways to discuss and illuminate the non-propositional components of mathematical and mathematics education knowledge, which take many different forms. It is also to suggest ways in which the knowledge traditions can cooperate, develop and stimulate each other. For this, the professional reality experienced by mathematics teachers need to be a well developed starting point, from which mathematics education research provides a resource. Since such cooperations are both extremely valuable and appear to be rather rare, I next describe one alternative in arranging such dialogue: the Dialogue Seminar.

Differences between paradigms, research questions

A Dialogue Seminar is an organized dialogue between professionals, to be more described later in this text. By taking part in Dialogue Seminars with other teachers, a rather obvious observation has become clear to me. It is that a teacher professionally faces a complex teaching situation with many different parts to be handled well. Such parts are subject knowledge, how to present subject knowledge, to understand students' present level of subject knowledge, how to respond to students questions and actions, correspondence to neighbouring subjects, how to plan future lessons, etc. etc. These parts need not only to be handled well; they also need to be balanced into a reasonable whole. I argue that from an epistemological viewpoint, mathematics education generally provides solid information about one or a few of the different parts at a time in order to be possible to base upon evidence, while it is more difficult to address the balancing act that a teacher needs to handle in such a way. On one hand, from the viewpoint of practical knowledge, the balancing act naturally attracts focus, since here active teachers formulate their needs and views. On the other hand, in the praxis paradigm solid evidence based results are rarely produced, as is more typical for mathematics education. Results in mathematics education are usually more solid and general. We have a trade-off between generality and authenticity.

These two paradigms have different and complementary roles to play. In research on praxis knowledge, a group of teachers play the main role in problem formulation as well as in reformulation and development. Mathematics education research results are founded in evidence and theory, which to some extent limits which problems may be studied. Mathematics education results tend to be more reliable, less dependent on local culture; however the depth typically means a larger distance from teachers' experiences. If we compare with mathematics, one may claim that the overwhelming reliability of mathematics lies in that here extremely limited problems are studied – so limited that they allow a very high degree of certainty.

Following a line of praxis knowledge, I argue that there are fundamental categories of knowledge that are vital for teachers and that cannot be accessed by traditional analytical approaches. One such category often mentioned by mathematics researchers as essential is intuition. The Dialogue Seminar can be seen as a serious and well developed means, mainly using analogy, metaphor and dialogue, often taking advantage of areas as history, philosophy, mathematics education, to put practical knowledge in motion that is particularly useful for teacher's education, and that provide an answer to the following fundamental question:

> *How can the sources of knowledge and skill that experienced teachers possess become available for student teachers, as described by*

experienced teachers themselves, and in ways that student teachers find valuable, supported by mathematics education results?

This is one of the main research questions in this paper. Related questions addressed here are:

Which types of knowledge are relevant in the mathematics teacher's profession? Which types of knowledge are important in subject knowledge in mathematics? How can the different knowledge types be handled in successful ways?

The purpose of the paper is to put forth underestimated types of knowledge, to give a general view of the epistemological landscape, and to suggest ways to design this landscape. This proposed design is to professionally take advantage of existing experience by allowing different knowledge traditions to meet systematically and constructively. Teachers will not acknowledge the value of their own resources of experience unless researchers emphasize these resources and try to find ways to develop them.

Propositional and professional knowledge

In Hudson (2002), Shulman's (1987) model of categories of teachers' knowledge is used. It contains the following categories, where I have added (in brackets) counterparts/characterizations relevant for the discussion in this paper. It illustrates well the balancing act that teachers constantly face in their profession:

- Knowledge of subject matter (mathematics)
- Pedagogical content knowledge (mathematics education)
- Knowledge of other content (knowledge of the educational program)
- Knowledge of the curriculum (course knowledge)
- Knowledge of learners and their characteristics (student culture)
- Knowledge of educational aims (political and school knowledge)
- Knowledge of the educational context (school culture)
- General pedagogical knowledge (pedagogy, classroom management)

Parts of these knowledge categories can be formulated in words, a knowledge type that may be called *propositional knowledge* (Göranzon, 2006, p. 19). Two types of knowledge that cannot easily be expressed in words are also part of most of these categories. These are *practical knowledge*, which is knowledge that contains experiences from having been active in a practice, and *knowledge of familiarity*, that is built by interaction with colleagues about examples of practice. These two are often called *professional knowledge*. This is knowledge with special properties, described by Kjell S. Johannessen as follows:

> Professional knowledge is essentially characterized by two basic traits: (a) It is acquired over a relatively long period of time by individuals; and (b) attempts

> as articulating it in some reasonably satisfactory way all fall short of even elementary standards of plain speech (Johannessen, 2006a, p. 229).

During scientific development new concepts become articulated, but there constantly seem to be new important kinds of knowledge to try to formulate. Can we expect this discovery process to stop so that finally everything that is relevant can be expressible in words?

These two properties cast doubt upon whether professional knowledge is knowledge. We have fundamental epistemological problems here: Is knowledge that is distinctly individual and cannot be shared knowledge? Is knowledge that cannot be articulated knowledge? Here are Johannessen's words (ibid.):

> Both of these traits stand out as inherently provocative to the adherent of the received and positivistically tinged view of knowledge that is predominant in our time. The first trait threatens to make knowledge dependent on individuals; and the second more than indicates that some kinds of genuine knowledge may in basic respects be resistant to verbal or notational articulation and thus be beyond the reach of language.

Is such professional knowledge (for examples, see Paul Ernest on vagueness below), difficult to formulate and perhaps to study, important for mathematics teachers? A famous and experienced research mathematician has once claimed that *logic is very important in mathematics, but it has never been used to <u>find</u> a proof.* Proofs are found by knowledge, experience, analogy, intuition and experiment, and later verified by logic. Mathematics students attempt to solve mathematical problems, which is a counterpart to mathematician's search for proofs. Inherent in the statement is the recognition of the vague concept of "intuition", which is addressed by Davis and Hersh (1995, p. 435) in the following way:

> (1) All the standard philosophical viewpoints rely in an essential way on some notion of intuition.
> (2) None of them even attempt to explain the nature and meaning of the intuition that they postulate.
> (3) A consideration of intuition as it is actually experienced leads to a notion which is difficult and complex, but it is not inexplicable or unanalyzable. A realistic analysis of mathematical intuition is a reasonable goal, and should become one of the central features of an adequate philosophy of mathematics.

These are strong words about the role in mathematics on something as vague and undefinable as "intuition". Personally, I look forward to the "realistic analysis of mathematical intuition". Intuition can be expected to require a metaphoric and poetic language, far from traditional mathematical language, and it would shed light on this very language.

Vagueness and knowledge

Epistomology and linguistics are firmly related to mathematics education, and have during later years found increasing attention. This is described by Paul Ernest in the preface of a book by Rowland (1999, p. x). He describes that the lack of attention to linguistics may depend partly on the focus of mathematical thought over talk. He continues to write that it may also depend on absolutist epistemology of mathematics, in the light of which language serves to describe absolute logic. Spoken mathematics is imprecise and has limited value in this perspective. This diverts the attention from students' actual mathematical thinking.

On the importance in mathematics of the opposite of preciseness, vagueness, Ernest writes in the same preface:

> Precision is the hallmark of mathematics and a central element in the "ideology of mathematics". Tim Rowland, however, comes to the startling conclusion that vagueness plays an essential role in mathematics talk. He shows that vagueness is not a disabling feature that detracts from precision in spoken mathematics, but is a subtle and versatile device which speakers deploy to make mathematical assertions with as much precision, accuracy and confidence as they judge the content and context warrant.

Thus, vagueness needs to be restored as a valuable complement to precision for good mathematics learning. Certainly, vague descriptions may lead to misinterpretations, but that is also possible for precise descriptions. Essential is that descriptions are to the point, and that the teacher has an idea of how students interpret. A better dialogue is required to understand each other's interpretations – vague or not.

A praxis paradigm

In the tradition of the Dialogue Seminar there is not much fear of vagueness. Instead, the unformulated knowledge that is possessed in a group of experienced professionals is focused. Unsuccessful computerizations of workplaces in the 80-ies were a starting point of this line of research. Reforms were related to a conviction that most or all of the relevant knowledge could be programmed in computers. Instead, groups of experienced professionals possess more knowledge than they are able to formulate in words. I have earlier described the knowledge categories *practical knowledge*, *knowledge of familiarity* and *propositional knowledge*. The two first develop while taking part in a professional practice, and interaction with colleagues, respectively. Such knowledge is not necessarily individual; it usually lives in a professional group. Sometimes the group is needed to find an appropriate action – for the knowledge to come alive.

The Dialogue Seminar is an arena for professional groups to find, formulate, characterize, stimulate and value their practical knowledge based upon experience, or in some sense (not necessarily with words) make it palpable or present. Analogue and example are important elements in this process. Historical texts are often rich in these respects. Musicians, engineers and others may participate in the same sessions. Meetings with other professions incur no rivalry, and appear surprisingly often to be fruitful for all parts. The sessions work with writing as a method of reflection. All members prepare actively each session along a certain theme with a text to be read aloud and reflected upon. Then, each member is invited to comment verbally upon each text that is read.

The invitation to reflect from experience is central. It makes the sources of experience increasingly visible for each bearer of that experience. These sources may grow into resources of knowledge that deliver more and more. Associations to other persons' experiences, which may be partly similar and partly different, is the tool for this discovery.

The dialogue seminar is an arena where mathematics teachers, mathematics education researchers, mathematics teacher educators and mathematicians can meet, listen to each other in depth, and learn from each other through dialogue. It appears as if mathematicians often experience dialogue with musicians as particularly valuable. This may come from the fact that intuition is important both in music and in mathematics, in somewhat similar meanings, as described above in Davis and Hersh (1995), while musicians may have advanced longer than mathematicians in formulating their intuition.

Göranzon and Hammarén (2006) describe the major goals of the dialogue seminar as follows:

> The dialogue seminar method is a method of working that aims to
> (i) create a practice of reflection
> (ii) formulate problems from the dilemma
> (iii) work up common language
> (iv) train the ability to listen.

I remark that the ability to listen does not follow automatically from the ability to talk. In academia, the ability to talk is trained much more than the ability to listen. From a very practical viewpoint, we learn in three ways: from reading, one-way-listening during lectures, and dialogue and reflection with others. Self-reflection is of course always a component. Reading is a form of listening, but without an opportunity of dialogue. The relative dominance of these learning modes in academia determines the corresponding degrees of training, and which abilities are developed.

Teaching and *Bildung*

Hudson also describes differences between the Anglo-American curriculum traditions versus the German. In the first case the teachers are employees of the school system which has a strong formal control of teachers (Westbury, 2000).

Professionalism is achieved by *training* and *certification*, to teach the curriculum. In the German tradition, the teacher is directed rather than controlled by the institutional framework. There is a larger professional autonomy for the teacher, for example in interpretation of the curriculum. This is related to the presence of the idea of *Bildung*. Klafki (1998) has specified three main elements of *Bildung*: (i) *self-determination*, to be enabled to make independent responsible decisions, (ii) *co-determination*, to be enabled to contribute together with others to the society, and (iii) *solidarity,* actions to help others.

Khalid El-Gaidi's doctoral thesis (El-Gaidi, 2007) *Teacher's professional knowledge – Bildung and reflective experiences* (my translation of the title, which was in Swedish only) was defended 2007 at the Royal University of Technology in Stockholm. Here teachers' skill at a technical university is examined. The dissertation starts with a case study in form of a Dialogue Seminar where university teachers participated actively in a series of meetings about their view of their praxis and of skill. The discussion on knowledge and skill based on this case study converged clearly towards *Bildung*. In this thesis *Bildung* is seen from many aspects, such as the ability to see the limits of the activity and to avoid misunderstandings. It is also seen as a standpoint involving judgement, sensus communis and taste (Gadamer, 2004), a way to view knowledge in that we need to recognize the different forms of cultures we live in, and as a way of seeing the whole picture, intuition and as rhythm of thought and communication.

Conclusions

Like in all professions, a large part of the relevant mathematics teacher knowledge is difficult-to-articulate practical knowledge. This is in striking parallel with mathematical subject knowledge in the sense that also this knowledge has important vague and tacit components. Such knowledge may be essential for the mathematics teacher profession, but is today largely neglected, for rather natural historical reasons. A positivist view of knowledge dominates, in which knowledge is what can be formulated in words, since the teacher profession has deep roots intertwined with positivist knowledge. Its particular purpose, different from other professions, is to induce learning of mathematics and other subjects, which at least in the previous century has a predominantly positivist view of knowledge.

Also, difficult-to-articulate knowledge comes into play mostly when the subject is verbalized. Mathematics is still a subject with a low degree of dialogue. It is a culture where written communication dominates over verbal communication.

Simultaneously, there is a division between mathematics teacher's culture and mathematics education researcher's culture that is dangerous from both perspectives: for the improvement of mathematics teaching, and for the relevance of mathematics education research. This division stems also from the differences in aims and history – on the view of knowledge. A dividing issue is the view of mathematics teachers' role. If they are not involved in formulating mathematics education research questions, dilemmas, there is a risk of the research to be difficult to apply. On the other hand, researchers may be able to add relevant aspects to teachers' view on teaching. Meeting points are needed where such exchanges are possible.

To bridge divisions, a complementary way to work is proposed in this paper. As a tool the Dialogue Seminar is proposed, which is designed and developed for the purpose of allowing experienced professionals to search and find difficult-to-articulated types knowledge – or at least to find tangible knowledge resonances. Such seminars are verbal in nature, but works with both reading and writing as reflection and individual preparation before the seminars.

The proposal is to apply this method for the benefit of mathematics education. Mathematics education research texts are natural starting points for the participants, but the dialogue may focus dilemmas that mathematics teachers find crucial in their profession. Since it is a very open way to meet, it offers an opportunity for mathematicians, mathematics education researchers, mathematics teachers and teacher students, and others, to give their view of practical mathematics teaching problems, listen in depth to different experiences and conclusions, and to have a dialogue.

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Historisk bakgrund till matematikens betydelse i yrkesprogrammen

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***Abstract**: Under de senaste 150 åren har yrkesutbildningen förändrats drastiskt i Sverige samtidigt som matematikens roll i yrkeslivet har utvecklats. Vi har gått från skråväsendet till ett modernt gymnasieprogram där alla elever studerar matematik. Skolreformer har avlöst varandra. Lärarnas yrkesroll inom yrkesutbildningen har förändrats från att vara förebilden och praktikern till att även vara teoretikern. Även yrkesläraren ska verka som matematiklärare och integrera yrkesämnet med matematiken. Lärarutbildningen har inte hängt med i denna utveckling, men utvecklingsprojekt har bedrivits för att hjälpa lärarna ute på skolorna för att genomföra det nya uppdraget. Elevens roll har förändrats allt eftersom utbildningen har teoretiserats. Matematikens roll har förändrats från allmän räkning och yrkesräkning till ett kärnämne som alla elever studerar. Just detta är något som ifrågasätts i den politiska debatten idag. Vi väntar dessutom på förslag om en reformerad tredelad gymnasieskola som skall formas med tre utbildningsvägar, en högskoleförberedande, en yrkesförberedande och en lärlingsutbildning.*

Inledning

Syftet med denna artikel är att visa hur dagens situation för matematiken i yrkesutbildningarna har sina rötter i den historiska utveckling som föregått dagens skola och hur detta påverkar möjligheterna till en framtida utveckling. Stort utrymme ges därför åt en historisk tillbakablick på vad som kan anses vara yrkesskolans ursprung. Källorna är utvald litteratur om yrkesutbildning särskilt med matematiken i fokus och officiella dokument samt styrdokument för utbildning. Flertalet av dessa källor är utvalda i dialog med professor Lennart Nilsson, Göteborgs Universitet, som har varit framträdande inom fältet yrkesutbildning under flera decennier och som disputerade 1981 med sin avhandling 'Yrkesutbildning i nutidshistoriskt perspektiv' samt Anne Outram Mott som har varit delaktig i utredningsuppdrag inom fältet och som nu är verksam vid Université de Géneve. Genom att fördjupa mig i sådan utvald litteratur (Elmgren, 1952; Marklund, 1980; Marklund, 1981; Mellin-Olsen, 1984; Lindberg, 2003) och genom att studera relevanta dokument utfärdade av regering och riksdag (SOU 1981:96,1981; Ds 1995:56, 1995; 1996; SOU 1997a, 1997b) har jag följt utvecklingen fram till de yrkesinriktade programmen i dagens gymnasieskola.

Sedan 1991 är alla kommuner i Sverige enligt lag skyldiga att erbjuda alla elever som avslutat grundskolan en gymnasieutbildning. Den svenska gymnasieskolan skall ge grundläggande kunskaper i yrkesliv - och samhällsliv. Skolans uppdrag är bland annat att utbilda för yrket. Sedan 1994 studerar alla elever på gymnasiet i varje program en kurs i matematik. Eftersom innehållet i denna artikel fokuserar på yrkesutbildning är det viktigt att definiera vad yrkesutbildning är. Den år 1963 tillsatta Yrkesutbildningsberedningen fick i uppdrag att ge en definition av begreppet yrkesutbildning. Resultatet lyder:

> Med yrkesutbildning avses såväl meddelande som inhämtande av kunskaper och färdigheter vars syfte är att utbilda och förbereda för arbetsuppgifter varav utövaren kan vinna sin utkomst (SCB 1984:2, s 17).

Historisk tillbakablick på yrkesutbildning med särskilt fokus på matematiken

Under medeltiden var lärlings- och hantverksutbildningen organiserad i mycket bestämda former. Från den svenska historien känner vi också till den stränga ordningen med de tre stadierna lärpojke, gesäll och mästare. Gesällarbetets kvalitet bedömdes av mästare och mästarna utfärdade mästerbrev som var internationellt gångbara. En mästare var inte enbart en erkänt skicklig yrkesman utan skulle också vara pedagog och lärare. Rätten att utöva sitt hantverk var förenad med skyldigheten att utbilda hantverkets utövare. Den sammankopplingen mellan yrkesutövning och yrkesutbildning stod kvar ända till mitten av artonhundratalet, då skråväsendet avskaffades och näringsfrihet stadgades.

Genom 1649 års skolförordning fastställdes att det skulle finnas tre slag av läroanstalter, nämligen trivialskolor, gymnasier och akademier. Genom skolordningen av år 1611 ägde yrkesutbildning i mera skolmässig form rum i skriv- och räkneklasser i gymnasiet. Innehållet i dessa räkneklasser skulle i första hand tillgodose det behov som de som inte skulle fortsätta till högre studier hade. Det handlade om att tillgodose borgarklassens barn. Det som främst nämns är handelsräkning. Innehållet är matematik som är direkt användbart i ett handelsyrke. Första gången gymnasium används som officiell benämning i Sverige är i ett donationsbrev år 1623. Genom ett beslut 1649 inrättades den så kallade apologistklassen som var en direkt fortsättning på skriv- och räkneklassen. Denna klass byggde på skolans första klass. (Varje klass omfattade två år). Genom inrättandet av apologistklassen fick man en kortare gymnasieutbildning för de borgerliga yrkena. Det blev en utbildning för de elever som inte ville fortsätta den lärda banan som var tänkt att leda till universitetsstudier.

Förra seklet

Yrkesskolor i form av fristående inrättningar ledda av kommuner och landsting hör huvudsakligen 1900-talet till. Så kallade praktiska ungdomsskolor fanns från 1918. Dessa var kommunala. De grundläggande bestämmelserna för yrkesskolorna var 1918 och 1921 års yrkesskolstadgar. Man skilde där på olika typer av yrkesskolor, nämligen centrala verkstadsskolor och centrala yrkesskolor, kommunala yrkesskolor, enskilda yrkesskolor samt företagsskolor och industriskolor. På 1930-talet kom de centrala verkstadsskolorna i landstingskommunal regi. I dessa skolor varierade matematikinnehållet och företrädare för ämnet var verkstadslärarna, dvs. yrkeslärarna. Yrkes- och affärsräkning för metallarbetare sjunde utgåvan (1939) som tillkom på uppdrag av Kungliga skolöverstyrelsen innehåller geometrisk räkning, teknisk räkning och affärsräkning. I förordet står:

> I samlingen har medtagits ett avsevärt större antal uppgifter än vad som i allmänhet torde medhinnas, varigenom läraren har tillfälle att åt eleverna välja ut exempel han anser lämpliga, på samma gång som mera intresserade elever kunna finna material för självständigt hemarbete (1939, s.5).

Uppgifterna är insamlade från yrkeslärare med början från 1923 och insamlade av Yrkespedagogiska Centralanstalten. Syftet var att i första hand utarbeta exempelsamlingar för lärlingsskolorna.

På 1940-talet inrättades allt fler kommunala yrkesskolor, främst i städerna. Huvuddelen av utbildningen gavs till en början som kvällskurser, i betydande utsträckning i form av lärlingsskolor inom företag eller vid särskilda företagsskolor och industriskolor. I Göteborg startade företag och industrier skolor av vilka några fortfarande existerar. Den stora expansionsperioden inföll efter 1950 års riksdagsbeslut om skolväsendets utveckling.

Det dröjde ända till efterkrigstiden, innan man kunde skönja ett brett och stort intresse för yrkesskolan och dess ställning inom skolväsendet i stort. Kännetecknande för yrkesutbildningen har nämligen tidigare varit, att man sett den som helt fristående från den allmänna utbildningen. Till detta kom att man tidigare såg föga samhörighet mellan olika slag av yrkesutbildning. Yrkesskolan som ett samlat begrepp var därför ännu på 1940-talet en ny företeelse. I Yrkesräkning för industri och hantverk (1961) kommenterar författaren att i den inledande matematikundervisningen i yrkesskolorna är det av erfarenhet nödvändigt att repetera tidigare i grundskolan genomgången kurs för att ge en bredare bas för de följande lärlingsårens kommande yrkesräkning. I läromedlet presenteras grundläggande matematikinnehåll såsom enheter, allmänna bråk, procenträkning, överslagsräkning och medelvärde. Enklare ekvationslösning med hjälp av prövning presenteras. Även grafisk framställning och grunder i algebra presenteras.

Handelsgymnasiets förebild

Förebilderna till handelsgymnasiet utgjordes dels av Göteborgs handelsinstitut, senare Levgrenska Gymnasiet, grundat 1826, dels av Schartaus Handelsinstitut i Stockholm, som började sin verksamhet 1865. Båda fick statsbidrag från 1894. Nya handelsinstitut grundades från sekelskiftet även i Malmö, Helsingborg och Gävle samt strax efter sekelskiftet i Örebro och Nyköping. I 1913 års riksdag antogs ett förslag om att inrätta handelsgymnasier. Nya handelsgymnasier tillkom i Örebro och Norrköping 1914 och 1915. Handelsgymnasierna omfattade ursprungligen en tvåårig normalkurs som byggde på avslutad sexårig kurs från allmänt läroverk. 1957 bestämde överstyrelsen för yrkesutbildning att handelsgymnasiernas normalkurser skulle vara treåriga. Den matematik som var i fokus var handelsräkning och handels- och ekonomilärarna undervisade i detta ämne.

Den gymnasiereform som trädde i kraft 1966 innebar att handelsgymnasiet inlemmades i det gymnasium som fick fem linjer. I och med denna reform tillkom ämnet matematik som undervisades av matematikutbildade lärare. Den ekonomiska linjen ersatte det treåriga handelsgymnasiet. Den blev kvar till 1994.

Tekniska gymnasier växer fram

De tekniska gymnasiernas föregångare var de tekniska elementarskolorna som upprättades vid mitten av 1800-talet. Den första tekniska elementarskolan började sin verksamhet i Malmö i oktober 1853 efter principbeslut vid 1850-1851 års riksdag. Målet för utbildningen var att den skulle utgöra dels en allmän förberedelse för tekniska yrken, dels en grund för fortsatta högre tekniska studier. Matematikundervisningen präglas av den tekniska inriktningen och innehåller mycket aritmetik och huvudräkningsstrategier samt geometri, där enhetsbyten och mätteknik finns med.

År 1872 tillsattes en kommitté att utreda dessa skolors uppgift och organisation. Utredningens förslag godtogs i stort av riksdagen 1877. Skolorna blev ettåriga med en differentiering på en mekanisk, en kemisk och en byggnadsteknisk linje. År 1874 föreslog man att allmänna ämnen skulle tas bort och man skulle endast inhämta kunskaper som hade omedelbar praktisk användning. Genom ändring 1901 bortföll de tekniska elementarskolornas roll att förbereda för högre tekniska studier. Nya tekniska skolor upprättades nu i Stockholm, Eskilstuna, Göteborg och Härnösand. Under åren 1907-1912 utreddes den lägre tekniska utbildningen av två statliga kommittéer. På grundval av deras förslag fattade 1918 års riksdag beslut om treåriga tekniska gymnasier och tvååriga tekniska fackskolor. Då sådana fanns på samma ort, skulle de utgöra ett tekniskt läroverk. Beslut fattades också om krav på förpraktik och om ny linjedelning. Förpraktiken skulle vara minst 2 år och minimiåldern för tillträde 17 år. De vanligaste linjerna var maskinteknisk, byggnadsteknisk, elektroteknisk och kemisk-teknisk linje. Den stadga som utfärdades 1919 gällde i stora delar ända fram till den 1 juli 1962. I 1948-års tekniska skolutredning som presenterades 1955 fastslog man att de tekniska läroverken skulle indelas i två skolformer, dvs. gymnasium och fackskola. Detta skedde för att man skulle kunna få en allsidig rekrytering till ingenjörskåren. De tekniska gymnasierna avskaffades 1967.

Fortsättningsskola

År 1918 blev en fortsättningsskola efter folkskolan obligatorisk för elever i Sverige. Den hade två huvudformer, en allmän och en yrkesbestämd. Uppgifterna om elevtalen för utbildningen vid yrkesskolorna är ofullständiga och även svårtolkade. Flertalet elever i den utbildningen gick deltidskurser eller heltidskurser som var kortare än fem månader. Antalet elever i heltidskurser om minst fem månaders längd var ännu vid 1940-talets slut bara hälften av antalet elever vid gymnasierna och endast tiondelen av antalet realskol- och flickskoleelever. Det fanns inte specifika matematikkurser, utan matematikinnehållet var integrerat i tillämpningarna.

Yrkesskolväsendet i Sverige och dess huvudmän

Överstyrelsen för yrkesutbildning (KÖY) inrättades vid 1943 års riksdag och övertog den 1 januari 1944 skolöverstyrelsens befattning av yrkesskolväsendet. Kommerskollegiet hade varit tillsynsmyndighet och huvudman för lärlingsutbildningen hos hantverksmästare. Dessa uppdrag övertogs också av KÖY den 1 juli 1944.

Överstyrelsen för yrkesutbildning utövade den centrala ledningen av yrkesutbildningen för industri, handel och husligt arbete.

De anstalter som stod under överstyrelsens ledning var högre tekniska läroverk (gymnasier och fackskolor), handelsgymnasier, centrala verkstadsskolor, kommunala och statsunderstödda enskilda yrkesskolor, avseende industri, hantverk, handel och husligt arbete. Tekniska skolan i Stockholm, textilinstituten i Borås och Norrköping, bergsskolan i Filipstad samt Grafiska institutet i Stockholm låg under överstyrelsens ansvar. Övriga utbildningsanstalter för yrkesutbildning underställdes också överstyrelsen på grund av särskilt beslut av Kunglig Majestät.

Den centrala statliga myndigheten på yrkesundervisningens och yrkesutbildningens område, nämligen överstyrelsen för yrkesutbildning, utövade högsta insyn över yrkesutbildningen i riket liksom även över den yrkesutbildning, som med bidrag av statsmedel bedrevs inom näringslivet. Överstyrelsen var organiserad på fem byråer, varav den femte hade hand om den av överstyrelsen bedrivna lärarutbildningen och pedagogisk reformverksamhet. En yrkesskolstadga antogs den 30 juni 1955 och gällde för statsunderstödda yrkesskolor. Denna stadga nämner inget om matematikinnehåll för yrkesskolan.

Skolkommissionen 1946

1946 års skolkommission huvudsakliga arbete kom att bedrivas i ett antal delegationer. En anledning till dess tillkomst var de svaga resultaten i matematik i dåtidens realskolor (Nilsson, 1992).

Delegationen för lärarutbildningsfrågor bestod av 10 personer. Kursplanedelegationens ordförande var Alva Myrdal. Yrkesutbildningsdelegationens ordförande var Emil Näsström. Den delegationen skulle arbeta med den lägre yrkesutbildningens ställning i skolsystemet och dess framtida utformning. I och med att huvudbetänkandet avlämnades var kommissionens egentliga uppgift slutförd. Yrkesutbildningsdelegationen inom kommissionens avlämnade i oktober 1949 ett mindre betänkande med förslag till bestämmelser om tillsyn över privata skolor för yrkesutbildning. Ett år senare, i november 1950, avlämnade samma delegation ytterligare ett betänkande med förslag till statsbidragsgrunder för det kommunala och enskilda yrkesskolväsendet. Yrkesutbildningsdelegationen ingick också tillsammans med skolöverstyrelsen, yrkesöverstyrelsen och arbetsmarknadsstyrelsen i en samarbetsdelegation för utredning av yrkesvägledningen inom skola. Denna samarbetsdelegation gav sitt betänkande i juli 1952.

De skolproblem man stod inför 1950 var den obligatoriska skolans eftersläpning, de många parallella organisationsformerna på real- och gymnasienivå, yrkesutbildningens otillräcklighet, den klara dualismen med ett lägre och ett högre utbildningssystem osv. 1946-års skolkommission tog avstånd från den gängse uppdelningen av eleverna i ”teoretiska ” och ”praktiska” och stödde sig därvidlag på forskning av pedagogikprofessor John Elmgren. Han fann att de båda begåvningarna hade ett starkt samband (1952). I en skrift från 1967 föreslog skolöverstyrelsen att bemöta flykten mot g-sidan, dvs. den del av enhetsskolan som var gymnasieförberedande, genom att göra hela högstadiet mera teoretiskt inriktat och mindre inställt på praktiska ämnen och yrkesutbildning.

Ny beredningsgrupp 1963

1963 bildades Yrkesutbildningsberedningen för att göra en översyn av yrkesutbildningens uppgifter, innehåll och organisation. Hösten 1964 fattade regeringen ett beslut om reformering av de gymnasiala skolorna som fick stor betydelse för yrkesutbildningen. Från den 1 juli 1966 infördes dels en kommunal gymnasial skola, gymnasiet, och dels en parallell skolform, nämligen fackskolan. Beredningsgruppen erinrade om tankar vid 1964 års riksdag då man diskuterade att de tre gymnasiala skolformerna gymnasium, fackskola och yrkesskola skulle likställas och i möjligaste mån integreras. Den föreslog därför att *en* skolform skulle bildas med de tre ingående delarna och att den skulle kallas gymnasieskola. Denna skolform infördes från den 1 juli 1971.

Utvecklingen står inte still

I regeringens proposition 1983/84:116 Gymnasieskola i utveckling under Lena Hjelm-Walléns tid som utbildningsminister står följande:

> Förändringarna mot en mer kunskapsintensiv produktion, som jag tidigare pekar på, kommer att ytterligare höja kunskapskraven för inträde på arbetsmarknaden och därigenom starkare understryka behovet av någon form av gymnasial utbildning för alla. Ungdomar med enbart grundskoleutbildning kommer enligt min bedömning också fortsättningsvis att ha stora svårigheter på arbetsmarknaden.
>
> Gymnasieskolan och de mer arbetsmarknadsinriktade insatserna inom uppföljningsansvaret måste därför ses som en helhet. Om inte gymnasieskolan utformas så att den står öppen för samtliga ungdomar kommer kraven i stället att öka på insatser inom uppföljningsansvaret…… Samtliga insatser för ungdomar

> måste således samordnas och sättas in i ett gemensamt utbildningspolitiskt sammanhang. (1983/84:116, s.11)

Vidare skriver man:

> Den största nackdelen är den stora åtskillnaden mellan teoretiska (studieförberedande) och yrkesinriktade utbildningar. Detta är i sig en spegling av utbildningssystemets tidigare uppdelning i dels gymnasier och senare också fackskolor, vilka framför allt skulle ge förberedelser för universitets- och högskolestudier, dels yrkesskolor som skulle ge direkt användbara yrkesfärdigheter. Denna uppdelning svarar varken mot arbetslivets behov eller mot den syn på livslångt lärande som jag tidigare redovisat. (1983/84:116, s.15)

Betänkanden från utbildningsdepartementet som senare har presenterats (1989, 1996, 1997) uttalar hela tiden en vilja att fortsätta integrationen av gymnasieskolan och att följa upp de läroplaner som började gälla när linjerna har ersatts av program. I programgymnasiet har eleverna möjlighet att komponera en utbildning utifrån givna kurser. Detta har emellertid bidragit till att det Individuella programmet är bland de största inom gymnasieskolan.

Läroplan för de frivilliga skolformerna 1994, Lpf94

Gymnasieskolan följer läroplanen från 1 juli 1994 (Lpf94). Styrdokumentet anger 16 olika nationella program där samtliga är treåriga. De avses ge en bred basutbildning och grundläggande behörighet för att kunna studera på universitet eller högskola. De nationella programmen har åtta kärnämnen, nämligen engelska, estetisk verksamhet, idrott och hälsa, matematik, naturkunskap, samhällskunskap, svenska (alternativt svenska som andraspråk) och religionskunskap. I varje program ingår den obligatoriska matematikkursen, Kurs A, som omfattar 100 gymnasiepoäng. Kurs A innehåller aritmetik, geometri, beskrivande statistik, algebra, datoranvändning och funktionslära och kursen kan närmast ses som en sammanfattning och avrundning av grundskolan kurs i matematik. Avsikten är att matematiken i kurs A ska exemplifieras med material från elevernas valda karaktärsämnen. Ämnet ”skall därför knytas till vald studieinriktning på sådant sätt att det berikar både matematikämnet och karaktärsämnena. Kunskaper i matematik är ofta en förutsättning för att målen för många av karaktärsämnena skall uppnås” (Lpf 94). Varje program får sin inriktning genom sina karaktärsämnen (fackämnen). Fjorton av programmen har yrkesämnen och ska omfatta minst femton veckor på en arbetsplats utanför skolan, s.k. arbetsplatsförlagd utbildning. De flesta programmen är uppdelade i olika grenar under år 2 och 3. Genom att kombinera karaktärsämnen från olika program kan en kommun inrätta specialutformade program. Dessa skall tillgodose lokala och regionala behov. Även individuella program kan inrättas och ha olika längd och innehåll och bestämmas av den enskilde elevens behov. Inom individuellt program finns också möjlighet att kombinera yrkesutbildning, som anställd på ett företag, med studier. Sådan s.k. lärlingsutbildning omfattar tre årskurser.

1995 beslutade riksdagen att ta bort beteckningarna studie- och yrkesförberedande utbildningar. Förändringen är motiverad av att man önskar öppna vägen för vidare studier för elever från alla program.

Cirkeln är sluten

En försöksverksamhet startade hösten 1998 i form av en modern lärlingsutbildning. Det finns redan konst- och hantverksskolor inom de fristående gymnasieskolorna. Detta för tankarna till det som skedde för cirka 500 år sedan.

De allt fler fristående gymnasieskolorna motsvarar den kommunala gymnasieskolan, så till vida att de erbjuder olika gymnasieprogram och får kommunala bidrag. Skolverket beslutar om en fristående gymnasieskola får inrättas eller inte.

En annan form av fristående skola som erbjuder utbildning över grundskolenivå är de ”kompletterande skolorna”. Till dessa utbildningar hör t ex vissa konst- och hantverksskolor.

Lärarutbildning

Det kan ligga nära till hands att numera se den tidigare beskrivna mångfalden av skolor vid 1940-talets slut som en tillgång, ett rikt smörgåsbord att fritt ta för sig från. Mångfalden var dock klart delad i en folkskoledel och en läroverksdel. Gränsen mellan dem var mycket påtaglig. Folkskolor, fortsättningsskolor och vissa slag av högre folkskolor var helt skilda från realskolor, flickskolor och läroverk.

Lärarna var klart uppdelade i två läger, man kunde rent av tala om två lärarkårer. I ett ingenmansland mellan dem fanns facklärarna i de så kallade övningsämnena. I den mån de valde sida var det som regel för folkskoledelen de var hänvisade till, de hade ingen koppling till läroverkssidan i skolan omkring 1950.

I början av 1960-talet bestod yrkeslärarutbildningen av en 5-veckorskurs i Kungliga yrkesöverstyrelsens (KÖY) regi. Yrkeslärarutbildningen var tänkt för den som tidigare arbetat med industri och hantverk, handel och merkantila ämnen. Den förutsatte och byggde nämligen på en god föregående yrkesutbildning. Kurserna utökades och organiserades från 1964 i fem yrkespedagogiska institut, som senare inordnades i lärarhögskolorna som ettåriga yrkeslärarlinjer, innefattande även en praktiktermin. I dessa kurser finns inte matematik med annat än i tillämningar och undervisningen sköttes ej av matematikutbildade lärare.

Lärarfortbildning - ändrad lärarutbildning

I SOU 1996:1 diskuteras de svårigheter med gemensamma kärnämnen som redovisas av lärare och i den allmänna debatten. Det finns lärare och skolledare som anser att man inte kan ha samma krav på elever som går på program med yrkesämnen och elever som går samhällsvetenskapligt program och naturvetenskapligt program. Detta anser man trots att utbildningarna numera är lika långa och har samma kärnämnen med samma kursplaner. Det man argumenterar för är att elever på yrkesprogram bara behöver yrkesmatematik, yrkesengelska osv. Det gemensamma för dessa synpunkter är att kraven bör vara lägre i kärnämnena för dessa elever än för andra.

Följande citat utgör en tänkvärd kommentar:

> Praktiskt taget alla ungdomar (98 %) går direkt från grundskolan till gymnasieskolan. Detta innebär en stor omställning för lärare som har sin bild av gymnasieskolan från den tid de hade hela sin tjänstgöring på teoretiska linjer och gymnasieskolan fortfarande var en urvalsskola.
>
> Det finns också lärare som har omprövat sina arbetsmetoder och som i samband med det fått intresserade, framgångsrika elever i kärnämnena och på program med yrkesämnen. (SOU 1996:1,s. 42).

Inom yrkesutbildningen i den gamla gymnasieskolan dominerade de tvååriga yrkesinriktade linjerna. Där ägnades cirka 80 % av undervisningstiden åt linjespecifika yrkesämnen. Matematiken upptog cirka 10 procent av undervisningstiden. På många linjer svarade en enda yrkeslärare för hela undervisningen och satte endast ett terminsbetyg i yrkesämnet. I utvärderingen av försöksverksamhet med en 3-årig yrkesinriktad utbildning i gymnasieskolan som skedde från 1988 under tre år, påpekas att fortbildningsbehovet är mycket stort bland lärare och handledare (SOU 1989:90, s 79). Den nya gymnasieskolan ställer nya och andra krav på många av karaktärsämneslärarna.

Förslag till fortbildningssatsning i 'Den nya gymnasieskolan – Hur går det?' ges på sidan 74 av kommittén för gymnasieskolans utveckling.

> Kommittén vill i detta sammanhang också betona vikten av att yrkeslärarna får en breddad kompetens. Det är viktigt att yrkeslärarna har bättre kunskaper än sina elever både i yrkesämnen och i t ex matematik. Kommittén har i flera sammanhang framhållit betydelsen av att karaktärsämneslärare och kärnämneslärare samarbetar för att eleverna skall kunna få sammanhang i sitt lärande. För många elever på program med yrkesämnena är det av avgörande betydelse att yrkeslärarna stöder undervisningen i kärnämnen och därmed ger legitimitet åt elevens hela studieprogram. Det är också viktigt att kärnämneslärarna har förståelse för undervisningen i yrkesämnena. Därför bör fortbildningen för lärarna i kärnämnena och lärarna i yrkesämnena i så stor utsträckning som möjligt ske gemensamt.

Här pekar man alltså på behovet av att integrera matematiken och karaktärsämnena. Integrationstanken har funnits länge och för flera behov. Mellin-Olsen skriver i Eleven, Matematikken og Samfundet (på sidan 124):

> Yrkesskolene har lang tradisjon i å integrere arbeid og teori i forbindelse med norskfaget og matematikkfaget (”trekke praksis inn i teorien”).

Det indikerar parallella traditioner i Sverige och Norge när det gäller yrkesutbildningen och behovet att integrera matematiken i yrkesämnen för att vinna elevernas intresse.

Början av det nya milleniet

Elever som väljer program med yrkesämnen får första delen av en utbildning till sitt yrke. Yrkets kännetecken skall synas i hela utbildningen. Det gäller både i kärnämnena och i karaktärsämnena.

Allt fler friskolor har bildats, flera av dem med direkt knytning till industrin. Troligtvis kommer denna trend att fortsätta. Det är nu och i framtiden viktigt att ha samarbete med branschorganisationer för att hålla skolan informerad om vad som är modern yrkeskunskap.

En genomtänkt satsning på fortbildning av karaktärsämneslärarna behövs för att hålla lärarna a jour med förändringar i yrket. Det bör ges möjlighet till gemensamma fortbildningar av karaktärsämneslärare och kärnämneslärare med god pedagogik och yrkesdidaktik för att förbättra elevernas möjlighet till en relevant utbildning för yrket och möjlighet till fortsatta studier och ett livslångt lärande.

Skolverket stödde flera utvecklingsprojekt där man ville uppnå en god undervisning för eleverna på yrkesprogram. Ett av dessa projekt var KAM-projektet (Grevholm, Lindberg & Maerker, 2001, 2002) som fick stöd för att öka samverkan mellan karaktärsämneslärare och kärnämneslärare när det gäller de innehållsliga aspekterna. Speciellt gällde det matematikämnet och karaktärsämnen inom fordonsprogrammet. Resultatet från projektet visar på goda möjligheter att kunna förändra undervisningen så att matematikinnehållet kan lyftas fram och synliggöras för eleverna. Projektet pekade också på svårigheter att nå resultat på kort tid eftersom det kräver stöd från många aktörer inte minst inom skolans område. En del av resultaten kan härledas till det faktum att lärarna har helt olika utbildning och bakgrund och olika syn och värderingar när det gäller mål och genom förande (Lindberg, 1998).

Några slutsatser om matematikens betydelse i yrkesprogrammen

Utifrån innehållet i denna artikel där den historiska utvecklingen och matematikens roll diskuterats från ett yrkesutbildningsperspektiv kan man utan vidare påstå att historien har i hög grad påverkat var vi står idag. Matematikinnehållet har inte lyfts fram utan har varit en otydligt integrerad del av yrkesämnet och dess tillämpning. Dessutom har yrkeseleverna fram till läroplan Lpf 94 i obetydlig grad undervisats i matematik av matematiklärare. Debatten om huruvida yrkeselever ska erbjudas yrkesräkning eller matematik verkar inte ha fått ett klart slut. Fortfarande diskuteras hur matematiken ska kunna tjäna som verktyg för yrkesämnena. Dilemmat att eleverna dels behöver matematik för att klara sin yrkesutbildning och det framtida yrket och dels behöver det som en teoretisk grund för fortsatt livslångt lärande är inte löst. Klyftan mellan hur yrkeslärare ser på matematiken och hur matematiklärarna ser på ämnet har ej heller överbryggats. De skilda historiska traditionerna dröjer kvar och de två lärargrupperna verkar tala olika språk till eleverna när det gäller matematik (Grevholm, Lindberg & Maerker, 2002). Försöken att integrera matematiken och yrkesämnena verkar inte ha lett till framgång för eleverna. Lärarutbildningen har inte gjort några försök att närma de två lärartraditionerna till varandra så att eleverna uppfattar att lärarna leder dem åt samma håll.

Vad pågår nu?

I aktuell samhällsdebatt har frågan om teori och praktik i utbildningen kommit tillbaka. Matematikens roll kommer med all sannolikt att diskuteras i samband med presentationen utifrån direktiven för reformering av gymnasieskolan (Dir 2007:8). Där talas om tre utbildningsvägar: en högskoleförberedande, en yrkesförberedande och en lärlingsutbildning Den nuvarande skolministern har pekat på behovet av kvalificerad yrkesutbildning på postgymnasial nivå. Ett argument i debatten är att inte alla ungdomar är intresserade av teoretiska studier. Men frågan är om de kategorier av praktisk och teoretisk, som fördes fram bland annat av 1946 års skolkommission, kan appliceras i dagens samhälle. Är inte de flesta av dagens yrken med den datorisering och specialisering som sker relativt teoretiska? Kan inte matematik tillämpad i yrket vara i hög grad praktiskt? Behöver vi andra kategorier än teoretisk och praktisk för att komma till kloka slutsatser om hur matematiken ska vara utformad i professionsutbildningar? Ett förslag för framtiden är att kursplaner i matematik för olika yrkesutbildningar i högre grad bör vara grundade på vad forskning om användning av matematik i yrkena kräver och på synen att utbildning idag måste vara livslång.

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A Study of the Movement of Attention: The Case of a Reconstructed Calculation

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Abstract: *My enquiries have convinced me that both what learners are attending to, and how they are attending to it play a central role in what learners are able to make sense of and eventually internalise. This paper illustrates this based on experiences of using a mathematical reconstruction task. Examples are given of different ways in which people's attention moves and shifts, sometimes voluntarily and sometimes involuntarily, in relation to variation theory, but also drawing on other theoretical discourses.*

Background

The construct of attention has ancient roots, and has been the subject of attention (*sic*!) by researchers in various periods in the past. William James (1890) is a notable example of someone who recognised the central role of attention in human experience, whereas some of his immediate precursors saw attention as an aspect of emotion linked to sensation. Recently there has been an upsurge of interest in attention as people try to find somatic correlates for psychological constructs.

My own interest stems from multiple experiences of being asked, at the end of a workshop, where my accent is from. This led me to realise that for some or much of the time, these people had been attending to my accent rather than (or perhaps in addition to) what I had been saying and what they had been experiencing. I was led to ask first, "what are learners attending to?" in classes, and then, because of my experiential methods, to "what am I attending to?". I soon discovered that what matters is not only what is being attended to, but how that attention is structured. By observing movements of my own attention from this perspective, I have been led to observations which impact directly on teaching and learning, and which go some way to explaining why teaching mathematical reasoning is so difficult.

Methods

My methods are experiential. I pay close attention (*sic*!) to my own experience. I find myself making distinctions in order to recognise incidents as instances of phenomena. I then seek tasks which I can offer others which serve to highlight distinction that I have found useful, in order to seek resonance in the experience of others: do they notice what I notice, and do they find such sensitisation to notice informative in their future practice? My methods are based on what I refer to as *the discipline of noticing* (Mason, 2002), which provides both methods and foundational justification for those methods, for researching your own experience, or *researching from the inside*.

This paper is a distillation of my experience of using a particular mathematical task with hundreds of people over several years. It is an empirical enquiry in the sense that I look for resonance and recognition by others, but it is not empirical in the sense of analysing systematic observations of individuals or small groups 'doing' the task. In that sense, my approach is both phenomenological and phenomenographic (Marton, 1981). The data I present, in the first instance, is what you the reader notice in the movement of your attention as you work on the task. My comments and analyses will make sense only to the extent that they resonate with your recent experience or illuminate your past experience, and they will prove useful only if you find that they inform some of your choices in the future.

Analytic frames

I shall use my experience of working on this task with others to develop a collection of distinctions that I have already described elsewhere (Mason, 2003, 2004; Mason *et al*., 2005), under the heading of *the structure of attention*: holding wholes, discerning details, recognising relationships, perceiving properties and reasoning on the basis of agreed properties. These distinctions are closely related to other ways of speaking about understanding in mathematics such as the van Hiele levels in geometric thinking (Usiskin, 1982), the onion model of understanding developed by Pirie and Kieren (1989, 1994), the SOLO taxonomy (Biggs & Collis, 1982).

I shall also call upon *variation theory* (Johansson, Marton & Svensson, 1985; Marton & Booth, 1997; Runesson, 1999; Marton & Tsui, 2004; Marton & Pang, 2007). The essence of variation theory is that in order to distinguish some feature or aspect of something, that is, in order for it to come to attention, it is necessary that there has previously been some experience of variation in that aspect. For example, the familiar adage "if you want to know about water, don't ask a fish" refers to this, because the fish, being immersed in water, knows nothing else. Similarly, if everything were a single colour, there would be no such thing as 'colour' because without variation there is no discernment. Marton goes further of course, suggesting that learning is a matter of becoming sensitised to discerning or distinguishing aspects not previously discerned. Explorations have also demonstrated two basic principles: variation needs to take place within a short period of time in order that it be experienced *as* variation, and if too much or too little is varied at one time, it may not be detected as variation (*op cit*), but merely as noise (Skemp, 1969, p. 28).

The Task

In the 18th and 19th centuries particularly, children destined to become clerks were required to keep a copybook 'in best writing' in which they recorded the contents of their lessons. These copybooks included laying out arithmetical computations in formats inherited from the abbacist traditions of the 13th and 14th centuries (Grattan-Guiness, 1997, p. 140).

Diamond

Try to trap movements of your attention as you make sense of the following calculation.

```
   79645
   64789
  -------
     30
    2420
   361635
  54242840
 4236423245
  28634836
   497254
    5681
     63
 ----------
 5160119905
```

Throughout the following comments, reference is made to 'some people', because no observation is universal, highlighting one of the difficulties in researching attention extra-spectively (Mason, 2002).

Some people find themselves gazing at the whole, even, in some cases, transfixed by the complexity. There is a sense of an array of numbers, and of course individual numerals are noticed but not really attended to. They do not become focal.

Some people manage to overcome or put to one side any emotional and cognitive obstacles triggered by the array. For them it is then possible to focus for a moment on the top two rows of numerals and to see them both *as* numbers, and as made up of digits: there may be a simultaneous awareness of individual digits and of a number (strictly, a numeral) made up of those digits. Attention may slide down and become aware of the diamond shape in the middle, and then the large number (or the multiplicity of digits) in the bottom row.

Familiarity with number calculations leads some people to expect, and so to look for, some numerical relationship between the top two rows and the bottom row via the intermediate diamond. Sometimes they are explicitly aware, and sometimes only implicitly as a *theorem-in-action* (Vergnaud, 1997) of a conjecture that this is probably multiplication, perhaps with some degree of caution, but in some cases with a good deal of certainty. People not familiar with long multiplication sometimes do not experience this coming to mind, so they have more work to do to try to make sense of the whole.

Either in the context of expecting multiplication, or by attending to specifics, attention turns to details. Often people report that the top or bottom of the diamond attracted their attention, and so they searched for a source for the 30 or the 63. Other people report focusing on the middle row, or one end of one of the rows. Others start with the units digits in the top rows (5 and 9) and look for the product, 45. Alternatively, the 45 may trigger a search for a 9 and a 5, and so reinforce the conjecture that this is a long multiplication. The point is that different details are distinguished and treated as entities (30 is seen as thirty not as three and zero).

At this point, some people experience a state summarised by “Oh, it’s just long multiplication rearranged”. This labelling of immediate experience wraps the display up into a ‘thing’, an ‘it’, and *assimilates* it into a well established schema of long multiplication. Labelling can be beneficial, reducing the need to dwell in particulars because of assimilation, but it can also be maleficent in wrapping the whole into a bundle and so reducing any impulse to look more closely. The labelling reifies the calculation while at the same time adding an affective component of ‘no need to look further’. Where individuals are working on the task, they may be inclined to stop at this point. Where small groups are working on the task, the presence of multiple perspectives is likely to mean that different people are in different states, and so someone may call upon others to explain or elaborate, and this may induce the group to probe more deeply.

For example, long multiplication includes ‘carrying’, but there is no evidence of carrying in this display. Attention is often attracted to the 45 at the extreme right of the diamond. Notice that the digits are now being perceived as pairs, so the second row of the diamond is seen as twenty-four and twenty not as two thousand four hundred and twenty. The 45 is readily related to the 9 and the 5 in the units places of the top two rows. This relationship stands out for many people, and draws them to look at the 36 below the 45 (more often than, for example, the 40 above it). The 36 is readily seen as related to the 9 and the 4. When people find themselves, for example, expecting to see the 9 (in the units place) and the 6 in the hundreds place of the top row related to a 36 their experience can usefully be described as ‘making a delicate shift from recognising relationships to perceiving properties’. An almost automatic conjecture accompanies or is integral to this shift from relationship to property, and most people do not even notice the shift until it is described and labelled. The diagonal movement is related to multiplication of 9 by the entries in the top row, confirming the conjectured perception of a property that holds for several of the relationships recognised so far.

Again some might be satisfied that they have plumbed the calculation. Others want to be sure, and so other details are discerned and checked. Perhaps the rising right hand diagonal of the diamond, although this can obscure structural awareness because now it is the 5 in the upper units place that is being held invariant while the entries in the second row are changing. Alternatively, the diagonal parallel to the bottom right diagonal might be inspected, under the conjecture that it will be formed by multiplying by the 8 in the tens position of the second row. But to check this requires some careful discerning of details, isolating through focused attention the relevant pairs of digits.

Some people inspect the middle row, and discover how it relates to the products of the digits in the units, tens, hundreds, etc. places in the two top rows. Again a conjecture accompanies the perception of a property which is instantiated in this particular calculation.

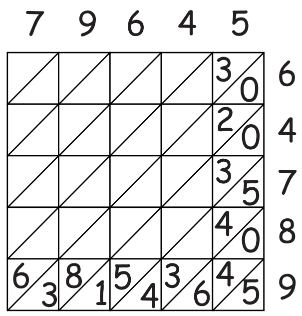
There is often an expressed sense of satisfaction as people shift from recognising how the various pairs of digits are related to products of digits in the top two rows. However it usually takes some outside force to shift the affective component from satisfaction that it can be explained, to describing to yourself, then to someone else, 'how to do diamond multiplication'. Some people experience this as a necessity for themselves, because they want to justify the diamond layout in terms of the long multiplication format with which they are familiar. For others it requires some outside stimulus.

In the process of bringing to articulation 'how to do diamond multiplication', it may occur to some people that there is an un-instantiated variation: what if the product of two of the digits is only one digit rather than two? Of course the answer arises immediately, illustrating the instantiation of a property already perceived but not previously challenged or instantiated: you use a leading 0, so 3 x 2 = 06. Justifying this calls upon making use of properties of place value which are more intuitive than formal, more like a *theorem-in-action* (Vergnaud, 1997) than explicitly articulated. It is an expression of an appreciation of structure which, if challenged, might cause some stumbling, even some *babbling* (Malara & Navarra, 2003) before settling down to a succinct and clear explanation.

One of the best ways to show that you 'know how to do diamond multiplication' is to construct an example which displays all of the things that can happen. How big it needs to be, and what digits to use involve a combination of mathematical and pedagogical aesthetic (Dreyfus & Eisenberg, 1986; Sinclair 2005). Mathematically, there is the issue of how many digits are needed in the multiplier and the multiplicand in order to provoke the general shape of the diamond, and pedagogically there is the issue of which digits to use (including at least one pair that have a single digit answer).

Some people are open to making more connections with the more familiar long multiplication format via a version of *gelosia*, *lattice*, or *grid multiplication*.

The diamond looked at from the bottom right, with the paired digits placed in cells divided by a diagonal line between the digits looks very much like grid multiplication.



Even without seeing a connection with grid multiplication, the question may arise as to why the layout 'works' in the sense of getting the digits into the correct place value column. To justify the layout's correctness requires, among other things, another shift of attention from recognising relationships between digits and their products, to the place value of those digits. Again details are discerned as special cases to 'check'. For example, in the diamond it could be tempting to read down the middle columns the numbers 30, 42, 16, 42, 42, 34, 72, 68, 63, isolating adjacent pairs of digits. If discerning these details of the diamond remains unconnected to the awareness that the digits in the diamond belong in pairs throughout each row, then the person is likely to become rather stuck, as it is impossible to make sense of these numbers. Juxtaposing the awareness of digits to be taken in consecutive pairs in each row contradicts the selection of some of these pairs. This could serve to reinforce recognition of the relationship of lying on diagonals which almost immediately becomes an instantiated property.

Roles of the teacher

The teacher plays several roles in work on the task.

Task choice

First there is the choice of the task itself. Here the task was chosen because it has proved fruitful in many sessions with many different groups of people concerned with the teaching and learning of mathematics. It serves to provide a taste of the ways in which attention darts around and of the different ways of attending to the same details. A teacher in school might choose the task to introduce formal long multiplication (anticipating inviting the class to modify the layout to make it more efficient, or to relate it to *gelosian*, *lattice* or *grid multiplication* with which they may already be familiar).

In the language of Ainley and Pratt (2002), there is potential *utility* for some learners who have trouble remembering to 'carry' and who might choose to do long multiplication this way, as well as *utility* in a way of working with examples. There is potential *purpose* for the learner in trying to work out what children of a similar age were doing over 100 years previously and so comparing themselves with those children ("if they could do it, we can surely work out what they were doing!"). Thus there is some potential motivation in addition to the simple challenge of working out what is going on. The teacher may be using it to provoke rehearsal of single digit multiplication facts and further experience of the role of place value.

In terms of the *Structure of a Topic* (Mason & Johnston-Wilder, 2004a), the need for massive numbers of clerks to run the Indian civil service as well as to record business transactions provides a source for the problem of laying out multiplication so that they could be checked easily, and also a context in which it was useful at the time.

There is also purpose from the teacher's point of view, in evoking learners' powers to attend in different ways, to make use of their powers to imagine and to express, to conjecture and convince (Mason & Johnston-Wilder, 2004b), and to encounter the mathematical themes of *invariance in the midst of change* and *doing and undoing* (here is the answer, what was the question, or how was it done?). The teacher can choose how much of this to make explicit during work on the task, how much to refer to later during reflection and reconstruction, and how much to leave unspoken as a contribution to enriching experience.

Task presentation

Secondly there is the choice of how to present the task. It could be presented in silence, as a phenomenon to be construed. It could be introduced with a story about children in the 18th century being trained as clerks and having to write neatly and without any blots. It could be presented in an animated form, though this would alter the affordances considerably, by using a 'this is how' format rather than a reconstruction format. It could be presented with some entries missing or incorrect, with a challenge to complete or correct it, justifying any proposals. Having initiated activity, it is necessary to sustain relevant activity so that there is sufficient experience, including both sufficient variation and sufficient opportunity, to get-a-sense-of what is going on and to bring that to articulation (Mason & Johnston-Wilder, 2004a).

Task involvement

Sustaining relevant activity involves redirecting attention when it is deemed necessary, such as when learners think they have it all worked out, and including asking learners to justify their conjectures. It may mean proposing supplementary tasks such as constructing your own example, or finding a different layout, or even an interesting layout for long division. It is during the activity phase of task work, when learners are experiencing some of the intended *inner aspects* of the task as envisaged that the core of teaching takes place (Tahta, 1981). Here is where much of the teachers' values are manifested in terms of what constitutes an adequate justification, how conjectures are dealt with, how learners interact, and how justification resides in mathematical structure not in opinions. If learners are kept in a situation long enough to rehearse and refine the language patterns associated with a topic (a component of the *structure of a topic*) then they may internalise the actions together with *inner incantations* (another component of the structure of a topic) which accompany those actions and which can be used to reconstruct the actions in the future if required.

Task conclusion

Doing things is not the same as construing. What learners make of a task and the subsequent activity can be highly varied. Because ‘a succession of experiences does not add up to an experience of that succession’, it is usually necessary to initiate some reflection so that something is actually learned from the experience. As Watson (private communication) has observed, reflection as a geometrical transformation can only be manifested as a rotation by moving up into an extra dimension. This supports the view that reflection as a pedagogical strategy requires more than ‘thinking back over what has happened’. It involves drawing back from the action and getting-a-sense of the whole, of the choices made, the obstacles encountered, and the actions which overcame or otherwise dealt with those obstacles.

In the present case, the drawing back is being done by the author of the paper, intended as a stimulus to the reader to recognise possibilities for themselves in terms of what might be learned and what was possible to experience. In particular there is the principal intention of elucidating in experience distinctions concerning the structure of attention. The teacher also engages in personal reflection and on the basis of experience, imagines doing something a little differently in the future.

Structure of attention

Sometimes people gaze at something, only vaguely aware of constituent details if at all. This is a useful state to employ when doing geometry: staring at a diagram and allowing it to speak, as it were, but this can also be usefully done with complicated algebraic expressions. In order to make use of alliteration as an aid to memory, it is useful to refer to gazing as *holding wholes*. Attention can suddenly switch to *discerning details* thereby creating new sub-wholes for gazing at. More usually, and perhaps more naturally, scanning different details in quick succession leads to *recognising relationships* between different details. Sometimes you can catch yourself seeking out relationships, and sometimes you become aware of recognising relationships. These relationships all concern the specific details discerned in what ever is being contemplated. But suddenly you can become aware of a relationship as an instance of a property. *Perceiving properties* is often so natural and quick that it is hard to trap, and yet it represents a major hurdle for many learners of mathematics. Properties can be seen as recognising relationships between particular relationships, and it can happen at many different levels or even in a sort of fractal form, where relationships between relationships between relationships between … are eventually perceived as properties at some meta-level. What perhaps distinguishes mathematics from some other disciplined modes of enquiry is that mathematics also involves the formalisation of properties and their relationships. Some properties are chosen as basic building blocks (often referred to as axioms or assumptions) and then other properties are deduced from these (using agreed properties of logic known as reasoning). It is the agreement to use only agreed properties and certain modes of reasoning which constitute mathematical reasoning, and it may be that mathematical reasoning is so difficult to teach mainly because it depends on property perception which itself involves a delicate shift from recognising relationships, which itself depends on discernment of details or aspects as entities.

Variation theory

On the surface, it may seem that this single example of diamond multiplication might form a counter-example to variation theory: there is only one example. However the reader brings to the situation multiple examples of calculations laid out in one way or another. Furthermore, once the digits in the diamond have been discerned as pairs of digits forming products of single digit pairs, there is considerable variation (though not in the aspect of the number of digits in the product). It is precisely the variation available in the construction of the diamond that enables a reader to re-construct diamond multiplication: there are plenty of instances from which, through becoming aware of what is varying (digits used in product, location of the product in the diamond in relation to the position of the multiplied digits) to recognise relationships, and for this to shift to perception of a property which is instantiated in the specific relationships.

The reference to mathematical and pedagogic aesthetic can usefully be re-interpreted in terms of variation theory. When constructing your own 'example' (Watson & Mason, 2005) it is quite natural to want to vary the digits used. Under an aesthetic of making it a bit challenging, you might be led to include some digits more than once, as in the case presented here. Under an aesthetic of wanting it to be as simple as possible while still being paradigmatic as an example, you might want to reduce the number of digits in the numbers being multiplied, perhaps to four. Under an aesthetic of sufficient variation, perhaps you might choose to have at least two instances of single digit answers to single digit multiplications, though you could also argue that one instance should be enough if the reader brings an already developing sense of single digit multiplication, because it would simply confirm to a reader their own conjecture.

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Reading Word Problems

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Students' work on word problems place demands on both comprehension and solving strategies. When reading a word problem, students construct a mental representation of the problem text that serves as the basis for solving the problem. Still reading and solving word problems is not necessarily a linear process. Students might for instance reread the problem during solving or when evaluating answers. The first part of this paper outlines a framework for investigating how strategies for reading and solving word problems are connected. The last section concentrates on exemplifying by discussing some instances where the student Boy1 rereads before tying the discussion of his competence to the suggested framework.

Introduction

Word problems are not a novelty in school curricula. Different aspects of student work on word problems are extensively researched, still many students, even some teachers, find word problems challenging. Much research is conducted within a cognitive framework, researching different aspects of reading or solving word problems (Cook, 2006; Cummins, Kintsch, Reusser & Weimer, 1988; Reed, 1999; Verschaffel, Greer, & De Corte, 2000). In the present study I am concerned with students' strategy use while working on word problems. The overall aim of the study is to describe different levels in strategy use and domain competence. This paper outlines a framework for investigating students' strategy use while reading and solving word problems. In the last section of the paper data on one student, exemplifying rereading during different phases of solving word problems, are presented. The full range of this student's competence for working on word problems is however not presented, but his use of the strategy rereading is discussed in relation to other measures of competence and the suggested framework. As the title suggests, the main focus is on reading the word problems, on understanding what a word problem is about.

Prior research on students' comprehension of word problems

In my study understanding is connected to comprehension of word problems: To understand means to create a mental representation of the problem situation to an extent that enables you to solve the problem (see for instance Thevenot, Devidal, Barrouillet, & Fayol, 2007).

Word problems are defined as verbal descriptions of problem situations by Semadeni (1995). Each problem embeds one or more questions that can only be answered by first constructing an understanding of the mathematical relationships in the text. They are traditionally associated with a school setting wherein the student is being asked to solve the problem in connection to a mathematics lesson, a test situation or as homework (Verschaffel et al., 2000).

While students' competence traditionally is evaluated through inspecting calculations and explanations, causes for student' difficulties or errors could also lie in shortcomings in text comprehension or difficulties in constructing a mental representation of the underlying mathematical situation (Reed, 1999). Cummins et al. (1988) found that solution performance is related to comprehension of the word problem text and to the language in the text. When students make errors and arrive at an erred solution, it is often the correct solution of the word problem the student thinks (s)he is solving. In other words they solve correctly the problem as it is constructed in their mental representation of the problem text.

My research interest lies in investigating students' strategy use when reading and solving word problems. This is not necessarily a linear process; students might reread problem text while planning or solving (Hegarty, Mayer, & Monk, 1995). Findings in a pilot study also suggest that reading and solving strategies are intertwined: Strategies for reading[1] were visible also when students were working on solving the problems (Nortvedt, manuscript). Word problems are traditionally short and condensed texts. The pilot study revealed that texts are often *so* short that students keep necessary numbers in working memory and move directly on to solving the problem. This was the case both for comprehended as well as for misunderstood problems. When working on their solutions, students revealed several general reading strategies such as rereading, looking for key words, clarifying etc (ibid.).

Nathan, Kintsch & Young (1992) suggest that when reading word problems, students construct a qualitative situation model representing the social context of the word problem and a quantitative model containing the algebraic structure or schema of the text. Thevenot, Devidal, Barrouillet and Fayol (2007) argue that this mental model is a situation model containing the mathematical relationships, not an activated schema. This suggestion is based on findings tied to increases in success rate for word problems when the question is placed in front of the body text versus after as in traditional word problems. Increases are significantly higher for low achieving students and not as expected for high achieving students that supposedly have more schemas accessible for use. The situation model is a "*temporary structure stored in working memory that contains in addition to the mathematical information necessary to solve the problem, nonmathematical information that is related to the context in which the situation described by the*

[1] This refers to reading comprehension, not decoding

problem takes place. A situation model is therefore more qualitative and less formal than a schema" (ibid., p. 45).

Addition and subtraction problems can be placed in different categories according to the semantic structures in the text. Making content more explicit without changing the semantic structure raises the number of students solving the problem successfully (De Corte & Verschaffel, 1991). Errors are often due to misconceptions of the problem situation grounded in insufficient understanding of the semantic schemes, also "*word problems that can be solved by the same arithmetic operation but differ with respect to their underlying semantic structure have very different degrees of difficulty*" (ibid., p. 119).

Cook (2006) found that college and elementary school students used the same strategies to discriminate between relevant and irrelevant information in word problems. The same strategies were also found when students did not succeed in solving the word problem. Transition strategies or multiple strategy use were more frequent for incorrect word problem solutions. Littlefield and Rieser (1993) found that when successful students based their discrimination on a feature analysis of the text, less successful students were more likely to use surface-level aspects like positioning of numbers or number grabbing. Such results were also found by Brekke (1995) in the Norwegian KIM-study, while Cook and Rieser (2005) found that mathematically disabled students attempted to apply the same strategies as successful students, only were not able to implement the strategies effectively. Misinterpretations or surface-based strategies might lead students to automatically consider a word problem an addition problem if they encounter the word *all together* in the text (Cummins et al., 1988; Reed, 1999). Cook (2006) on the other hand suggests that when students do not succeed in constructing a fitting mental model, this could be due to lack of mathematical knowledge (domain knowledge) relevant in the given situation.

The model of domain learning – a framework for describing student competence

The model of domain learning is an alternative perspective on expertise (Alexander, 2003). While research on expertise within cognitive science has given rich profiles of experts and contrasted novice behaviour by expert behaviour, the model of domain learning describes characteristics of competence at *acclimation, competence* and *proficient* levels within academic domains (Alexander, Buehl, Sperl, Fives, & Chiu, 2004). A novice student typically has little and fragmented domain knowledge accompanied with few and rigid strategies and can be described as being at an acclimation level, while an expert will have reached the level of proficiency (Alexander, 1997). The goal of compulsory schooling is to bring students to the level of competence (Alexander, Graham & Harris, 1998).

The model of domain learning is a stage theory, defining competence as consisting of three interrelated parts: domain knowledge, strategic behaviour and interest (Alexander, 1997). Differences in domain knowledge and strategy use within the mathematical domain are well documented (see for instance Ostad & Sørensen, 2007 or Schoenfeld, 1992). Collins, Brown & Newman (1989) claim that the strategic knowledge experts possess usually is tacit knowledge and thereby difficult to verbalise and make explicit and observable to others. Experts have a broad range of strategies that they apply in a flexible manner, allowing them to perform different activities necessary to work on unknown or unfamiliar problems (Alexander, 1997). Often strategies are defined as goal directed and non-obligatory actions. Collins, Brown and Newman (1989) claim that strategies can consist of both automatic and controlled processes. Within the model of domain learning "*strategies are defined as intentional and effortful actions taken when individuals perceive some problem or gap in understanding*" (Alexander et al., 2004, p. 547). Alexander et al. (1998, p. 132) stress that "*principled knowledge is the hallmark of academic competence, and that strategies are necessary to achieve such principled understanding*".

Researching strategy use

Verbal protocols are traditionally used in research studies where researchers aim at investigating thinking or strategy use. Such protocols in the form of either retrospective or concurrent think aloud protocols, are collections of students' talk when conducting an activity (Pressley & Afflerbach, 1995). While retrospective reports are generally considered too reflective to reliably tap into only cognitive and not also meta-cognitive strategies and self-reflection, concurrent think aloud protocols are considered appropriate tools for building theory about cognition. The challenges of validity are lesser than with retrospective reports since think aloud protocols are reports of the latest contents of short term memory (Ericsson & Simon, 1993).

When students think aloud while performing an activity, there is no delay in time between conducting the activity and reporting. Such reports can be termed level 1 reports and are considered to be actual reports of the content of working memory and hence strategies in use (Ericsson & Simon, 1993; Pressley & Afflerbach, 1995). However such verbalisations are difficult to children, and they also some times forget to verbalise their thinking. At such occasions the student should be reminded to think aloud (Ericsson & Simon, 1993). If time spans are too long, students might be giving retrospective reports and thereby expose meta-cognitive understanding or reflections (level 3 protocols) instead of cognitive strategies in use. Still meta-cognitive and cognitive strategies are an equally important part of students' strategy repertoire. Of special interest are strategies in consideration to self-monitoring or students' strategies to evaluate answers (Alexander et al., 1998). Considerable empirical evidence exist that demonstrate that giving verbal protocols in the form of thinking aloud while engaging in higher order thinking activities does not affect students' success-rates or test scores (Leow & Morgan-Short, 2004; Norris, 1990).

Ginsburg, Jacobs and Lopes (1993) suggest that think aloud protocols are a potentially strong tool in order to assess thinking. Verbal protocols applied with scaffolding open up for diagnosing students' zones of proximal development. Scaffolding offered by a more knowledgeable other, adjusts for and is sensitive to the student's difficulties in consideration to a specific task. Scaffolding can be described as "*controlling those elements of the task that are initially beyond the learner's capacity, thus permitting him to concentrate upon and complete only those elements that are within his range of competence*" (Wood, Bruner & Ross, 1976, p. 9). Hence a larger portion of students' strategy use is displayed than when left alone and unable to solve the task. This difference between what students can do when working independently and what they can do with scaffolding is what is in a student's zone of proximal development (Vygotsky, 1978).

Investigating student strategy use when working on word problems

The main part of the study *Understanding and working on word problems* consists of a protocol analysis. Students have given verbal protocols while working on a collection of word problems. While the verbal protocols give rich information about student strategy use, only limited knowledge about students' domain knowledge can be derived from them. For this reason test scores on national tests in numeracy[2] have been collected to give individual measures of domain knowledge, while test scores on national tests in reading serve as individual measures for reading comprehension. No separate measure for students' interest has been applied. While giving protocols some very limited knowledge of students' likings or dislikings, motivation or interest has been demonstrated, but not to an extent that would validate using these mere outbursts as measures of interest.

The decision to introduce scaffolding resulted in protocols that are in part students' concurrent reports of thinking while reading and solving word problems (level 1 protocols) and in part scaffolding conversations between student and researcher (level 3 protocols). While the level 1 protocols are appropriate for investigate students' strategy use, protocols including scaffolding talks are more appropriate for diagnosing what students can do in their zone of proximal development.

Data collection

19 students[3] have given verbal protocols while solving eight word problems and one practice problem. Protocols were given in an interview setting. Students were

[2] 'Regning' in Norwegian.

[3] The students come from two different schools situated in a major city in Norway. Both schools are combined primary and lower secondary schools. Protocols were collected during the last

asked to read and think out aloud while working through the problems as similar to what they would do if the problem was assigned in a lesson or for homework as possible. The word problems were age appropriate, collected from national tests, exams and text books, some challenging even to very competent students. The collection consisted of both multiple choice (3) and open ended questions (5) representing different levels of complexity. To these students all word problems demanded more than one "step" of calculations in order to be solved correctly. Each problem was printed on top of a separate page (A4), allowing more than sufficient space for taking notes, drawing or performing calculations. Students would work through the word problems in their own pace[4]. They would decide for themselves when a problem was solved and when to move on to the next problem. Scaffolding was offered when students got stuck.

Analysis

The analyses of level 1 protocols are mainly based on a priori developed categories, representing findings in other laboratory studies where the unit of analysis has been students' strategy use. The category *rereading* is applied when students reread a part or all of the word problem text before, during or after solving the problem. This category is used both when students are successful and when students fail to approach an understanding of the text. Analysis is still ongoing, which is why only examples of one student's strategies is used to exemplify the present discussion. The presented examples are parts of level 1 protocols.

Exemplification: Boy1's use of rereading

In this section of the paper examples of how Boy1 interacts with the text while working on the word problems are discussed in the light of the suggested framework. The strategy of interest is *rereading.* Also his comprehension of the word problems in the form of manifestations of mental representations will be touched upon. Only parts where the student rereads will be discussed, focus will not be on solving in itself. Boy1 can be termed a student at acclimation level. His scores on the national tests are well within the score interval for the 20th percentile for both reading comprehension and numeracy. WP3[5] is the only one of the eight word problems where he arrives at a correct solution (se transcript note). Boy1 rereads for several reasons and during different phases of solving of the word problems. He rereads *before solving* for the practice problem and WP1, 2, 3 and 4, he rereads *during solving* for the practice problem and WP1, 2, 5 and 6. He only rereads *after solving* for WP3. For the two last word problems he never rereads.

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week of teaching in grade seven or during the first fourteen weeks of grade 8. A total of 22 students participated. 19 students have reminded in the study.

[4] Interviews lasted between 25 and 60 minutes.

[5] I use WP1 for word problem 1 etc, referring to the word problems used for collecting the verbal protocols

Example 1: Boy1 WP3 – Anita' wages – rereading before solving.[6]

Boy1:	*Anita works in a store after school every Tuesday and Thursday. She makes 80 kroner per hour. When she turns 16, she will get a raise in her wage for five percent an hour. How much will she make an hour when she turns 16?*	Initial reading, reads through WP text once
	After school every Tuesday and Thursday.	Rereads
	80 kroner, I think she earns… Five percent. Five percent. How much is five percent more an hour?	Elaborates
	How much will she make an hour when she turns 16?	Rereads
	Five percent more	Identifies relevant information
	Then it is 85.	Suggests solution
I:	*Mm*	
Boy1:	*I think it will be 85.*	Confirms suggestion
I:	*Mm*	
Boy1:	*I need to find out. I need to. I must try to…*	Questions model

There are several reasons for solvers to reread. Rereading can for instance be aimed at discriminating between different text elements or at elaborating on the content of the text (Pressley & Afflerbach, 1995). Before they can start out to solve a word problem, students need to comprehend the text of the word problem in order to arrive at a mental representation (Thevenot et al., 2007). If texts are short or easy to comprehend, it is possible to hold the whole text in working memory after reading through once (Pressley & Afflerbach, 1995), but often it is necessary to reread to be able to discriminate between relevant and irrelevant information (Cook, 2006). In example 1 Boy1 is working on WP3, a word problem that contains both relevant and irrelevant information. When he rereads the part of the first sentence that states when Anita works in the store (irrelevant information), this is not elaborated on further, and he moves on to elaborate on more relevant information. Students need strategies to discriminate between relevant and irrelevant information in the word problem text. Many students employ a question orientated strategy to do this (Cook, 2006).

Boy1's elaborations in the transcript is guided towards understanding the problem text, that is creating a mental representation and identifying the relevant

[6] After questioning his initial understanding of the word problem, Boy1 decides that he needs to work out how much is 5 % of 80 kroner. When he reaches the total of 84 kroner for the new wage, he very much doubts that this is the correct answer. He goes back to 85, and after a scaffolding discussion and rereading the segment *after school every Tuesday and Thursday* once again, he decides 84 is the correct answer, because, *after all 5 % is not very much.*

information needed to solve the problems as represented. His elaborations could be viewed as either repeating or rewording text elements for elaboration purposes in order to explain parts of the text (Pressley & Afflerbach, 1995). He also focuses on the word problem question in relation to the rest of the text. Such elaboration can be viewed as being about understanding the social situation, as in understanding the context. According to Thevenot et al. (2007) the mental representation is to the main part a more qualitative social model that the mathematical structure of the problem lies within. Boy1's initial structure suggests adding 80 and 5. It is not clear however whether this means 85 kroner or 85%. Other students adding 80 and 5 demonstrate two different surface strategies: Number grabbing or use of key word. Alexander et al. (2004) labels such strategies text base strategies. In the Norwegian version of WP3 the term "more" is used in the problem text about the raise in wages. This could suggest that adding is the appropriate mathematical action to students focusing at key words (Reed, 1999). However use of key word can be a success strategy in situations where key words give indications as to the mathematical structure and prototypical character of word problems (ibid.), but then the use of key words is a result of deep processing (Alexander et al., 2004).

Students who fail to form an appropriate mental representation often reread the whole text several times. According to Cook (2006) these students could either not know how to implement rereading in order to discriminate between relevant and irrelevant information, or they could lack necessary domain knowledge. WP1 asks for students to calculate how many bikes having two and three wheels respectively you can have if you have 19 wheels. This word problem has three correct solutions, and students are asked to produce as many as possible. When working on WP1 Boy1 rereads both before and during solving. Two different reasons for his failure to understand this word problem can be identified from his work. Partly he is not able to make meaning of the sentence "Find as many solutions as possible" and partly he fails to understand that he has to find a combination of 2's and 3's that form 19. His attempt to make meaning consists of reading the whole text without elaborating on the meaning, just repeated reading. This could be termed a surface-strategy (Alexander et al., 2004) or his efforts could be termed using a transition strategy, meaning that he does not recognise how rereading can be used to get to the meaning of the text (Cook, 2006).

Rereading can also be part of monitoring (Alexander et al. 2004, Pressley & Afflerbach, 1995). Monitoring your own process, to keep on track, is an important part of a student's domain competence (Schoenfeld, 1992; Teong, 2003). Students for instance reread to check if what they are calculating is an expected answer to the problem question, to adjust their mental representation. Boy1 demonstrates the ability to use the word problem text to check if he is on track for one of the more traditional two-step word problems (WP2). When he arrives at an erred solution for this problem, it is because he does not manage to employ standard algorithms well enough, not because he fails in employing the reading strategy.

Concluding remarks

According to Alexander's model, to be mathematically competent means different "things" to different people; acclimation students also know something even though it is fragmented and not flexible in respect to domain knowledge and strategy use (Alexander et al., 2004). Viewing domain knowledge and strategy use as two parts of students' domain competence, as suggested, seems to be useful in order to understand what Boy1 accomplishes when he uses the strategy *rereading*. The word problems in the examples are rather trivial problems to competent grade 8 students. To students at an acclimation level they can however be challenging. To solve them correctly students need to master appropriate domain knowledge (multiplication, division, subtraction and percentage) as well as reading strategies aiming at comprehending the text of the word problems, adjusting the mental representation, monitoring or keeping on track. To some extent Boy1 has knowledge of the strategy in question. However, unlike in the transcript, the use of text-based strategies rather than deep processing strategies was found to be more frequent. But while test scores suggest that Boy1 might be considered to be at acclimation level, the protocol data reveals that some strategic knowledge was applied in company with some domain knowledge even though some strategies were transitions strategies. In a wider perspective, to understand how he uses strategies and what he accomplishes might be a key to understand how he might move towards competence.

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Children's Early Work with Multiplication and Division

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***Abstract**: The purpose of this paper is to discuss the way very young children handle problems connected to multiplication and division. The discussion is based on classroom observations from England and Norway, and it is linked to a discussion of how Norwegian textbooks present the first encounter with division. In textbooks division is often simply regarded as reverse multiplication. Based on the classroom observations I will argue that it could be worthwhile pursuing division as a process in its own right and postpone the strong link to multiplication until later.*

Introduction

Following a social constructivist view on learning (Ernest, 1998) I take the stance that knowledge is developed by the learner in interplay with teachers, fellow learners and the teaching material. A constructivist view entails that knowledge cannot be regarded as detached from the knower but it is actively built up in a process where the learner organises his/her experiences in relation to previous knowledge (von Glasersfeld, 1995). When it comes to teaching, this means that it is of vital importance to establish strong links between the children's own ways of thinking and the input provided by the teacher and the teaching material. Along with the development of a basic number concept children will also develop ways of solving problems involving calculations, both in everyday life and in school contexts. Algorithms that children develop for solving tasks they meet in school is referred to by Steffe (1994) as child generated algorithms. These algorithms usually differ from the standard algorithms taught in school and are also often quite inefficient and insufficient. Nevertheless they are important, and for the teacher the challenge will be to create links between the child generated algorithms and other, more efficient algorithms.

Fischbein, Deri, Nello, and Marino (1985) write that there seems to be certain natural, intuitive models for multiplication and division. The intuitive multiplication model is based on equal grouping and is often referred to as repeated addition. For division the basic model is that of sharing equally between a given number of persons, often called partitive division. Alternatively, division may be regarded as sharing out in groups with prescribed size, and this is called quotitive division. Although the equal grouping model for addition and the partitive model for division will be insufficient for modelling situations that the students encounter at a later stage, they have a strong influence on how students perceive the conceptual field of multiplication and division.

An examination of current Norwegian textbooks indicates that the main approach to division is through the partitive model, and the quotitive model is more or less tacitly introduced, if it is at all mentioned. In addition the study of the textbooks indicates that division already in the very beginning is presented as the reverse operation of multiplication and that in this context commutativity of multiplication is taken for granted. Based on observations of children at an early stage of working with multiplication and division problems I will discuss whether it might be desirable to a larger extent to work with division as an operation in its own right, and I will discuss what considerations should be made when the link between division and multiplication is introduced. I will also discuss how commutativity of multiplication is intuitively investigated and justified. This could be formulated as the following research questions: *What are children's intuitive ways of approaching division problems, and how do these approaches fit with the way division is introduced in textbooks? How will commutativity of multiplication be justified in an experiential situation which is not in its nature commutative?*

Theoretical framework

Children's first encounter with multiplication is usually connected to situations involving a number of groups of objects having the same number in each group (Greer, 1992). Later other models for multiplication will appear but the equal grouping model (repeated addition) is recognised as the basic intuitive model for multiplication (Fischbein et al., 1985). For division Fischbein et al. discuss both partitive (or sharing) division and quotitive (or measurement) division. However, they claim that "*there is only one intuitive primitive model for division problems* – the partitive model" (p. 14, emphasis in original) and that the quotitive model is acquired later as a result of instruction. According to Fischbein et al. these intuitive models for multiplication and division reflect the way the concept was initially taught at school and they correspond to features of human mental behaviour that are primary, natural and basic.

Following Steffe (1994, p. 7) a child's solution process can be described in three steps. First there is an experiential situation as perceived by the child. Then there is the child's procedure to deal with the situation, and finally there is the result. These steps may involve different challenges. To get from the situation to the procedure the child has to determine which arithmetic operation should be used. In this step familiarity with a diversity of models for each arithmetic operation is expected to be helpful. After the operation has been determined the task is to carry out the actual calculations, and this step may be quite independent of the chosen models for the situation. Here the child may use different strategies irrespective of how the actual situation is modelled. In early learning these steps are not clearly separated.

Neuman (1999) has studied small children's work with division problems. She distinguishes between the situational aspect, meaning the modelling of the practical situation, and the computational aspect, meaning the procedure to obtain the answer. These two aspects correspond to Steffe's (1994) three steps in the sense that the situational aspect corresponds to the experiential situation and the computational aspect corresponds to the procedure and the result. In solving division problems by dealing out one at a time there is hardly any difference between the situational and the computational aspect.

The most basic strategy for division is dealing out objects one at a time by direct counting, and Mulligan and Mitchelmore (1997) write that direct counting "achieve the aim of creating equal-sized groups, but the calculation procedure does not reflect this structure" (p. 318). Repeated subtraction can be applied directly in quotitive division where the required number in each group is taken away repeatedly until the initial set is empty. In partitive division however, the number in each group either has to be guessed or, if the process is carried out by sharing out one item at a time, the required number is seen only when the process is completed. Multiplication is usually regarded as a binary operation, that is a mapping from $A \times A \to A$, where A denotes the number set one is working within (integers, rationals, etc.). In many situations involving multiplication the multiplier and the multiplicand will have different roles. Then it might be natural to look at multiplication as a unary operation, that is a mapping from $A \to A$, where the multiplicand is the number that is operated on, and the multiplier is the operator. Vergnaud (1983) discusses two ways to look at multiplication as a unary operation – the scalar and the function operator – using the example "if the price of one cake is *a*, what is the cost of *b* cakes?" Here *b* is called the scalar operator and *a* is called the function operator. In the language of Vergnaud, partitive division amounts to reversing the scalar operator and quotitive division amounts to reversing the function operator. Neuman found that in partitive situations, dealing out 28 marbles to seven boys, the children sometimes constructed a quotitive model by looking at the number *seven* as the number of marbles being dealt out in each round. The answer *four* will then represent the number of rounds (Neuman, 1999, p. 113). Hence, although the situation is partitive the computation may be quotitive in the sense that it measures how many 7s there are in 28. To make this change the number 7 has to be reinterpreted from representing boys to representing marbles.

A model which is quite different from both the partitive and the quotitive model is the splitting model suggested by Confrey (1994). This model is based on a repeated halving procedure, and Confrey suggests that splitting should be taken as a primitive action complementary to counting, which is the basis for the partitive and the quotitive model. The splitting model exhibits a structure like exponential growth.

Method

I will discuss three episodes that are based on observations made in a class consisting of 18 1st grade pupils (6-7 years old) in a small English school. I was an observer in this class in all of the mathematics lessons for three weeks. In the whole class teaching I was passively observing and after having been in the class a few times, I videotaped the sessions. When the pupils were working in groups or individually I interacted with them in ways that a teacher would do, and I video- and/or audiotaped also some of these sessions. I had no influence over the topics the class worked with, or the ways in which the topics were handled. However, the teacher said that she made some adaptations due to my presence. The episodes that are included in this paper are based on video and audio recordings. All three episodes take place on the same day but in different class-room settings, partly in a whole class situation and partly when the children are working in small groups.

Furthermore I will discuss the relation between multiplication and division by analysing excerpts from two Norwegian textbooks on the level where division is first introduced as a formal arithmetic operation. The textbooks that are chosen are all recent editions adapted to the current Norwegian national curriculum, LK06 (Kunnskapsdepartementet, 2006).

Finally I will discuss a classroom episode from a Norwegian school where three 2nd grade pupils (eight years old) are working on a specific problem. This episode is also videotaped.

The classroom episodes on division

Episode 1

This is a whole class situation and the class is working with halving. First they have been rehearsing with numbers by saying out statements like “half of eight is four”. Afterwards the teacher picks out 10 plastic cubes from a box and asks two children to come to the board. Following the earlier activity they establish that “half of ten is five”. The teacher then shares out the cubes in a “one for you – one for you” manner and when all the cubes are shared out, the two children count their cubes and confirm that they have five each. This lays the foundation for the perception of division as ‘sharing equally’ and that sharing between two is the same as halving. Next the teacher picks out eight plastic cubes from the box and asks two children to come to the board. A third child is given the cubes and is asked to share the cubes equally between the two others. After the sharing process is completed the two children find that they have four cubes each and the teacher prepares the following sentence on the board

___ shared equally between ___ is ____

Here the children are expected to suggest what should be filled into the blank fields. After having completed the sentence the teacher writes with mathematical symbols $8 \div 2 = 4$, and says out while writing: “Eight divided by two is four”. The teacher states that the sign ÷ is “another way of saying divide by or share between”. After another example of the same kind the situation is changed into sharing between five children. First, Kyle, who is sharing out, is given ten cubes. He gives one cube at a time to each of the five children until there are no more cubes left. The children now verify that they have two cubes each and $10 \div 5 = 2$ is written on the board. Next Kyle gets 20 cubes to share between the same five children and the following conversation takes place while Kyle is doing the sharing.

Teacher: Does anyone think they know the answer already?
Amy: Four.
Teacher: Why do you think the answer is going to be four?
Amy: When they were going round with ten they had two, and then they just had to double that.

In the dialogue above Amy shows that she is able to find the answer by generalising. She knows that 10 divided by 5 is 2, and from this she infers that 20 divided by 5 must be 4. It seems that she is seeing a general pattern in the sense that when the amount to be shared is doubled, the outcome for each person will also be doubled. I support this claim by the fact that she is actually using the word ‘double’ in her utterance, hence she is not just going from 10 to 20 but from 10 to “double 10”.

Episode 2

Here the children are sitting in groups of four and they are playing a game in pairs that involves solving division tasks. The tasks are given with numbers and symbols only, for example $8 \div 4 =$ __ . They have centicubes available, and a calculator that they use to check their answers. At the bottom of the task sheet is written “Knowing division facts up to 25 ÷ 5”. Amy and Alice are playing together and when they get to 20 ÷ 4 they are stuck. They sit for some time without coming up with any suggestion about how to solve the task and I decide to intervene. I suggest that they leave that particular task for a while and instead try with some other numbers. I ask then “what is eight divided by two?” The answer “four” comes immediately. Then I ask “what is eight divided by four?” Again the answer comes immediately, “two”. I continue by asking “what is twelve divided by two?" The answer "six" comes without hesitating. My next question is "what is twelve divided by four?" and then there is no answer.

In Episode 1 Amy observed that if the amount to be shared was doubled the outcome would be doubled. Here I had hoped that she or Alice would make the observation that if the number to share between was doubled the outcome would be halved. They do not seem to make that observation. I continue to try to help them by inventing a situation and the following conversation takes place.

Frode:	Imagine that you have twelve apples and you are going to share between you and Amy, then you would get six each … right? You just said that. But what if you had the same twelve apples and also Jack and Jamie [sitting opposite them at the table] should have a share.
Alice:	We would share all.
Frode:	Yes, and what would happen with your lot of apples then?
Amy:	I would give three of mine to Jamie and she would give three of hers to Jack.
Frode:	Then you have done twelve shared out between how many?
Amy:	Four.
Frode:	And how many are you left with?
Alice:	Three.
Frode:	Now imagine that you have twenty apples and you are going to share between yourself and Amy. If you have twenty apples.
Alice:	So it's ten each.
Frode:	Ten each.
Amy:	Five each, between all of us.

In this dialogue the girls, probably supported by my way of presenting the situation, develop a way of dividing by four by successive halving. In the final part of the dialogue, with 20 apples, it seems that both girls envisage the imaginary sharing process going on at the table. Alice makes a stop after the first halving, saying "so it's ten each". Amy, however, says out almost simultaneously "five each, between all of us" indicating that she has done the two successive halvings in her mind before stating the final result. The model that is used here is the splitting model (Confrey, 1994).

Episode 3

In the last episode Amy wants to solve 15 divided by 5. She cannot come up with an immediate answer and after a short while of thinking she says "I'm going to use the cubes". She then counts 15 cubes which she distributes one by one into five heaps. When she has finished she counts the cubes in each heap and states "Fifteen divided by five is three".

In this case none of the strategies that she had developed and used earlier in the lesson were applicable so she falls back on the basic strategy of sharing out one by one and then counting the result.

Textbook presentations of division

In school, multiplication is usually presented and worked with before division but problems involving multiplication and division are solved by children long before they have been exposed to any formal teaching on the subject (Mulligan & Mitchelmore, 1997; Neuman, 1991). In a Norwegian textbook for 4th grade (Solem, Jakobson, & Marand, 2006) the first formal encounter of division is through the following problems (p. 5[1]).

> Three children eat two pizza slices each. How many do they eat altogether?
> $3 \cdot 2 =$ __ . They eat __ pizza slices altogether.
> There are six sweets left. Tom and Eric share them. How many do they get each?
> $6 : 2 =$ __. They get __ sweets each.

At the bottom of the same page is written: "Multiplication and division are inverse arithmetic operations. $6 : 2 = 3$ and $3 \cdot 2 = 6$."

The first situation in this example will most naturally be modelled as $2 + 2 + 2 = 6$ whereas the second situation will be $3 + 3 = 6$, or $2 \cdot 3 = 6$. The interpretation $2 + 2 + 2 = 6$ is meaningless in the situation with the sweets. Despite of this the book links it to $3 \cdot 2 = 6$ which elsewhere in the book is taken to mean three lots of two. This example can be linked to the following example discussed by Vergnaud (1988, p. 144).

> *Connie wants to buy 4 plastic cars. They cost 5 dollars each. How much does she have to pay?*
>
> *a)* $5 + 5 + 5 + 5 = 20$
>
> *b)* $4 \cdot 5 = 20$
>
> *c)* $5 \cdot 4 = 20$
>
> *d)* $4 + 4 + 4 + 4 + 4 = 20$

Commenting on the four procedures a) – d) Vergnaud writes: "Procedure d is meaningless in terms of cars and costs. […] Young students apparently are aware of this and never use procedure d. So there is a strong asymmetry between procedures b and c" (p. 146). Therefore the multiplicative situations in the

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[1] All excerpts from textbooks are originally written in Norwegian and translated by me.

textbook are not conceptually the same, and to see that they are mathematically equivalent requires knowledge of commutativity of multiplication.

In another book (Alseth, Kirkegaard, Nordberg, & Røsseland, 2006) the authors seem to be conscious about presenting situations giving rise both to partitive and quotitive division. On page 104, under the heading "We practice division", there are three monkeys holding four bananas each. Below the picture is written "We write 12 : 3 = 4". On the side of the picture there are two fantasy creatures, and one of them says "Now I have shared equally. That gave 4 to each". The other one says "Yes, that is correct because 4 · 3 = 12". On page 105 is another example presented by the text "20 carrots are shared out between a number of zebras. Each zebra gets 5 carrots. How many zebras will get carrots?" A picture with four bunches of carrots, five carrots in each, is shown, and the two fantasy creatures are saying: "Look, it is enough for four zebras. That is correct because 4 · 5 = 20". Also here it seems that commutativity of multiplication is tacitly assumed because the situation with the monkeys and the bananas really should have been modelled by 3 · 4 = 12.

The situational aspect (Neuman, 1999) is clearly different in these two examples. In the example with the monkeys a partitive situation is modelled, and in the example with the zebras a quotitive situation is modelled. However, when the answer is tested using multiplication both examples use a model where the dividend is measured with the divisor. To do this in the partitive situation (12 : 3 = 4) would require that the number 3 is taken to represent the number of bananas in each round in a similar way as in the example with the 28 marbles from Neuman mentioned earlier.

Commutativity of multiplication

It is important for many purposes that the children develop an understanding of the commutativity of multiplication. In some of the textbook examples discussed above it may seem that commutativity is tacitly assumed in order to see multiplication and division as reverse operations. Also for making calculations easier it is important to use commutativity, in particular if one of the factors is large and the other is small. Through working with problems connected to models for division I have become interested in the process through which children develop understanding for the commutativity of multiplication. If multiplication is regarded as a unary operation, which is natural for young children (Vergnaud, 1983), the two factors will have different roles, and therefore the situation is not commutative (three lots of two is not the same situation as two lots of three). It is only later, when models involving area or number of combinations are introduced, that the roles of the factors are symmetric, and multiplication will be viewed as a binary operation. I will give an example to show how a child constructs his own model of thinking to justify commutativity in a situation which is not at the outset a commutative situation. This example is taken from a Norwegian classroom.

Three boys, age eight, are sitting together working on the following problem.

Today is Kenneth's birthday. He is eight years old. There are 20 children in the class and each child will light eight firecrackers to celebrate Kenneth's birthday. How many firecrackers do we need altogether?

Kenneth, himself being one of the three boys, starts by counting "two, four, six … ", and he suggests that the answer will be 160. He supports his thinking by saying "We can take every twenty. In a way every twoeth[2] just that we take a tenner." Harry objects, saying "it is not the tenners we are counting, it is eight." Then Kenneth says "I know that but every twenty because we were twenty children." Kenneth continues to talk and he also starts writing on a sheet of paper: 20 40 80 100 120 140 160 180. He counts the numbers to make sure that he has eight entries and seems a bit puzzled by the answer 180 since he got 160 the first time.

At the same time Brian is sitting quietly writing on his sheet of paper: 8 = 16 = 24 = 32 = 40 = 48 = 56 = 64 = and so on. Obviously he is adding eight at a time and he keeps track of how many times he does this. Kenneth walks over to him, looks at what he has written, and counts the number of entries. He counts to 19 and says that one more is needed. Adding one more eight Brian arrives at 152. (The error is due to the fact that at two instances he has added four instead of eight.) Now there are three different solutions. Kenneth seems to think that both his and Brian's way of thinking should lead to the same answer, and the fact that they have come out with different answers must be due to some computational error. Kenneth says "Eight times twenty, eight twenty times or twenty times eight, twenty eight times, 'cause this is the same you know."

These children have not been formally taught multiplication so I interpret what is happening here to express how they intuitively perceive the situation. All three boys are making composite units iterable (Steffe, 1994) which is essential for establishing a multiplicative situation. Brian and Harry are counting eights, which might be the most natural in this case, but Kenneth is counting 20s. Hence they iterate on different composite units, and at least Kenneth expects that the two approaches should give the same result. This amounts to establishing

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[2] Here Kenneth is not using the usual Norwegian words for 'every second' but a constructed word ('hvert toende') which I translate to 'every twoeth', a construction corresponding to 'every twentieth'.

commutativity of multiplication. In the beginning Kenneth cannot justify his thoughts very clearly but suddenly he says:

> *Yes, it is like this, 'cause, twenty, for each child, everybody sends up, this (pointing to the numbers on the paper) is one firecracker for each child, two from each child, three from each child, four from each child, five from each child, six from each child, seven from each child, eight from each child. I am eight years old.*

The expression on his face and the tone of his voice indicate that at this point he made a discovery and was able to express in words why it would work to take 20 eight times.

Afterwards Kenneth starts to scrutinise Brian's calculations. He does not finish it but goes back to his own paper where he writes 20 + 20 + 20 + 20 + 20 + 20 + 20 + 20 = . He and Brian go through this together and find the answer 160. Now the teacher intervenes and asks them to look at Kenneth's first attempt and the erroneous step from 40 to 80 is soon discovered. It takes more effort to discover the errors in Brian's calculations but with the help of the teacher the two instances where four is added instead of eight are found and they all agree that the correct result is 160.

Kenneth's calculations correspond to procedure d) in the example from Vergnaud (1988) discussed before, and Brian's calculations correspond to procedure a). It could be argued that Kenneth's procedure is meaningless in the sense that it seems as if he is adding children to obtain firecrackers. My interpretation of what he is doing is that he tries to get around this and he struggles to find a way of justifying his thoughts. This leads to interpreting the number 20 as one firecracker for each child. In this way he is really adding firecrackers, 20 for each year of his age, and there is consistency between the left and the right hand side of the calculations. Compare this example to Neuman's (1991) example 28 : 7 = 4 where the 7 is reinterpreted to mean marbles per round instead of boys. In the same way Kenneth reinterprets the number 20 to mean the number of firecrackers for each year of his age instead of children.

In the example with the firecrackers it is clearly much easier to count eight 20s than twenty 8s. Hence Kenneth's strategy is more efficient, which probably is the reason why he chooses it in the first place. Commutativity is not obvious for these children, and the two numbers (8 and 20) have different roles, 8 represents firecrackers (or years) and 20 represents children. I find it interesting to observe that Kenneth is working with two different unary operations at the same time, and that he is able to justify that both operations will give the solution to the given problem. I interpret his expression "this [the number 20] is one firecracker for each child" that he is describing the function operator (Vergnaud, 1983) where he takes the number of firecrackers from the whole group per year of his age (20) times the total number of years (8). Brian uses the scalar operator, thinking that 20 children will send up 20 times as many firecrackers as one child.

Final comments

In the development of multiplicative thinking there is a goal that children should see multiplication and division as inverse operations. It could be argued that this is advantageous for both their conceptual and procedural knowledge (Hiebert & Lefevre, 1986) of multiplication and division. In terms of conceptual knowledge the argument is that pupils should develop the understanding that dividing by a is the same as multiplying with $1/a$, thereby developing understanding for the multiplicative inverse. In terms of procedural knowledge it will make computations more efficient if one can solve division problems by evoking knowledge about the multiplication table, in particular when doing long division. I will argue however, that in children's first encounter with division, important conceptual development may be lost if the children are not given time to investigate division as a process in its own right. In children's first encounter with the concept of division, which commonly is in situations involving sharing equally, the multiplicative structure is not readily apparent. As the classroom examples in this paper, and also a number of other studies, show, children treat division as an independent process without linking it to multiplication. Despite the fact that the class where I made my observations about division recently worked with multiplication there is no sign of employing multiplication facts in solving the problems with division. This is coherent with the findings of Neuman (1991, 1999) who worked with children of about the same age as I did. Commenting on the view on division as reverse multiplication Marton and Neuman (1996) write "[I]t was not with division of this type that most of the children in Neuman's investigation addressed the problems" (p. 319).

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Different Views – Teacher and Engineering Students on the Concept of Function

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***Abstract**: This study analyses what kind of conceptions student teachers and engineering students have about the function concept, and how these conceptions differ between the two groups. The study was conducted through questionnaires, and 34 students at a Swedish university participated. The function conceptions of the students have been classified according to modified versions of models presented by Vinner and Dreyfus, Sfard, and DeMarois and Tall. The study shows that the students primarily have operational conceptions, with only a couple of students having structural conceptions. The study also shows distinct differences between prospective compulsory school teachers and engineering students, where the former have less developed functional conceptions.*

Theoretical framework

Different approaches have been developed to explain the mechanisms governing concept acquisition. For example, in mathematics education there has been considerable discussion concerning the distinction between concept definition and concept image, the concept definition being the formal mathematical definition, while the concept image is a much wider concept, representing "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes." (Tall & Vinner, 1981, p. 152).

Regarding the concepts themselves, Sfard (1991, 1992) speaks of the duality of mathematical concepts, in that they can be regarded both as processes and as objects. While "there is a deep ontological gap between operational and structural conceptions" (Sfard, 1991, p. 4), the two are not mutually exclusive, but rather complementary. Sfard (1991) has also formulated an influential theory of concept formation. According to this model, concept formation consists of three consecutive stages: interiorization, where you get acquainted with the processes behind the concept by performing operations on already familiar mathematical objects; condensation, where you get more familiar with the concept, gaining increasing capability to switch between different representations of it; and reification, where you gain the ability to view the concept as an object in its own right. For Sfard, this last step is qualitatively different from the first two.

Previous results on the concept of function

The process of reification is by no means an easy one. Several studies (e.g. Hansson, 2006; Norman, 1992; Sfard, 1992; Even, 1990; Vinner & Dreyfus, 1989) show that even students who have come far in their studies, and in some cases practicing teachers, do not have a reified, but rather a process-oriented, view of the function concept. Sfard claims that this needs to have consequences for the teaching of mathematics. One should not introduce structural descriptions of concepts until they are needed, and, more specifically, one should never introduce new concepts in structural terms (Sfard, 1992).

When students first encounter the definition of a concept, more often than not they already have concept images, which may be more or less developed. Of course the concept definition will influence the concept image, but when the concept is used in practice, it is almost always the concept image which is evoked. (See e.g. Attorps, 2006; Hansson, 2006). Earlier studies (e.g. Akkoç & Tall, 2002; Tall & Bakar, 1991) indicate that prototypes, that is, standard examples of the concept used for a pedagogical purpose, tend to contribute strongly to the concept image, even though they are often chosen in order to highlight just one particular aspect of the concept. Hence different aspects of the concept image may very well be contradictory, since different aspects of the concept image are used in different contexts. This is called compartmentalization, and has been detected in several studies (e.g. Eisenberg, 1992; Vinner, 1992; Vinner & Dreyfus, 1989). Moreover, many studies (e.g. Akkoç & Tall, 2002; Meel, 2000; Vinner & Dreyfus, 1989) have shown considerable discrepancies between students' concept definitions and concept images.

The study

Research questions

Although students' understanding of the function concept has been studied by quite a number of researchers internationally (e.g. Akkoç & Tall, 2002; Meel, 2000; Even, 1993; Tall & Bakar, 1991; Vinner & Dreyfus, 1989), not that much research on the subject has been done in Sweden. Therefore, one of the aims of this study is to investigate what the participating students' conceptions of the function concept look like, and, if possible, to compare this with the results of studies conducted elsewhere. Hence, the first research question posed in this study is: What is the students' understanding of the function concept? More specifically: How do the students define the concept of function? What do their concept images for the function concept look like?

In my opinion a good conceptual understanding of mathematics is of special importance for prospective teachers. Therefore I have chosen to conduct my study on a group of student teachers, and to compare these students with a group of engineering students. These students also study quite a lot of mathematics, but their goals are different, and more aimed at the use of mathematics in a practical setting. The second research question posed in this study is: What are the differences in the understanding of the function concept between the teacher students and engineering students in the study?

Methodology

The study was conducted at a Swedish university, and the participants were student teachers currently attending a course in calculus (14 students), and first-semester 5-year engineering students, also attending a course in calculus (20 students). The student teachers had taken more than one semester of mathematics (except for the 3 students aiming at upper secondary school, who had only taken a course in algebra), while the engineering students had only taken a course in algebra.

The data were gathered by questionnaires. The students were asked to associate freely regarding the concept of function, and to construct a "mind map". They were then presented with a number of mathematical expressions and figures, and were asked to determine which of these represented functions and to rate the degree of certainty of their answers. Furthermore, they were asked for their opinion on the possibility of constructing a function with certain given characteristics, and finally they were asked to state their own definition of the concept of function. When classifying the answers, use has been made of categorizations presented by Vinner and Dreyfus (1989), Sfard (1991), and DeMarois and Tall (1996).

Results

The first research question deals firstly with the students' definitions of the function concept. Classifying the definitions according to a modified version of a categorization developed by Vinner and Dreyfus (1989), it was found that most students gave definitions that could be described as process-oriented, and that only a small minority gave structural definitions. Furthermore, nearly a third of the students failed to provide any meaningful definition whatsoever.

The classification makes use of the following eight categories, of which category 3 is not used by Vinner and Dreyfus. One is a "no answer"-category, and the order of the other seven categories more or less traces the historical development of the function concept (see e.g Kleiner, 1989). The categories are the following (each followed by an example from the questionnaires, where T refers to student teachers and E to engineering students):

1. ***Correspondence***. *A function is any correspondence between two sets that assigns to each element in the first set exactly one element in the other set.*

   A function always gives just one value when you insert a value. If you have one set which is the domain and insert one of those values into the function you get one of the values in the range. (T2)

2. ***Dependence relation.*** *A function is a dependence relation between two variables.*

A function depends on a variable. Depending on what value the variable has you get a unique value of the function. (E11)

3. **Machine.** *A function is a "machine" that transforms variables (which need not be numbers) into new variables. In this case no explicit mention of domain and range is made.*

   A 'machine' which to any input-variable assigns a specific number or something similar. (E5)

4. **Rule.** *A function is a rule. The difference from 3. is that a regular behaviour is expected, whereas the machine could conceivably perform totally different transformations of different elements.*

   A description of a pattern, which varies depending on different variables. (E7)

5: ***Operation.*** *A function is an operation or manipulation. Here the input values are assumed to be numbers, on which mathematical operations are performed to yield the output value.*

   A set of operations giving the same result if you insert the same value. (E17)

6: ***Formula.*** *A function is a formula, an algebraic expression or an equation.*

   A function is a formula for which value y assumes for any given value of x. (T1)

7: ***Representation.*** *The function is identified, in a possibly meaningless way, with one of its representations.*

   A curve where one x-value has one y-value. (T3)

8: ***No answer*** *or a meaningless answer.*

   A function is an explanation of how something works. (E4)

It is worth noting here, that the definition given in the textbook used by the student teachers (Rodhe & Sigstam, 2000, p. 88) is of category 2, while the textbook used by the engineering students (Adams, 2006, p. 24) gives a definition of type 4 (but with explicit mention of domain and range).

Table 1. The number of students' answers in the eight categories

Category	1	2	3	4	5	6	7	8
Number of students	1	1	6	2	8	6	6	4

Two students fall into categories 1 or 2, the categories that resemble the structural definition of function, while 10 students end up in categories 7 or 8, failing to give a useful definition.

The second part of the first research question concerns the students' concept images for the function concept. According to Even (1990) the essential features of the concept of function in the modern sense are arbitrariness and univalence. Arbitrariness means that the value of a function at any given point is independent of the value at other points, but also that the domain and range can be arbitrary sets; specifically they need not be sets of numbers. Univalence simply means that for each element in the domain there is a unique element in the range.

In classifying the students' conceptions of the function concept, a model constructed using elements from the classifications of Sfard (1991) and DeMarois and Tall (1996) has been used. The students' conceptions of the function concept have been ordered into pre-operational, operational and structural conceptions. A student with a pre-operational conception has a rudimentary and inconsistent concept image. A students' conception of function is operational if she clearly views a function as a process, and structural if she is also able to view the function as an object in its own right. Using this classification it was found that 12 students had pre-operational and that 20 students had operational conceptions of the function concept. Two students had something resembling a structural conception. In the following, some interesting aspects of the answers to the questionnaire, and their implications regarding the concept images of the students, will be noted.

The most common concept to appear in the mind maps (20 students) was the concept of graph or curve. Yet only one student mentions the vertical line test. Common are also such calculus concepts as derivative and integral, as well as the function machine and terms like formula, expression and operation. As for the essential features mentioned above, 8 students mention domain/range, and 4 mention univalence. Notable by their absence are such concepts as inverse function and composite function, as well as examples of standard functions. Only a handful of students mention any of these concepts in their maps. The students were also asked to determine whether a number of expressions and graphs could be said to represent y as a function of x. Some of these expressions, together with the distribution of Yes and No answers, are presented in Table 2 below.

Table 2. The distribution of students' Yes and No answers concerning certain of the expressions

Expression	$x^2 + y^2 = 4$	$xy^2 = 5$	$x = 3$	$y = 3$	$f(x) = 3$	$y = \begin{cases} -3 & x < 0 \\ e^x & x \geq 0 \end{cases}$	$y = \begin{cases} 1 & x \text{ rat.} \\ 0 & x \text{ irr.} \end{cases}$
Yes	24	23	4	13	22	32	**24**
No	**10**	**10**	**29**	**20**	**12**	**2**	**10**

We see that a majority of the students consider both of the first two expressions as being functions $y(x)$, despite the fact that such “functions” would not be univalent. Also, a substantial number of students reject constant functions. Finally, an overwhelming majority of the students accept split domain functions. Here, some interesting inconsistencies appear. For example, the function

$$y = \begin{cases} -3 & x < 0 \\ e^x & x \geq 0 \end{cases}$$

is constant on part of its domain, but is still accepted as a function by more than twice as many students as the function $y = 3$. It is also interesting to compare this with another question in the questionnaire, where the students were asked about the possibility of constructing a function which is integer-valued for all non-integers, and non-integer-valued for all integers (this example was found in Vinner & Dreyfus, 1989). About half of the students accept the existence of such a function, and 12 students construct one. But quite a few of those who reject it, do so based on an assumption that a function must be defined by one formula on the whole of its domain, despite having had no problem accepting the piecewise defined function above. On the other hand, of the students who accept this type of function, only two reject the Dirichlet function, so the students who have grasped the idea of arbitrariness appear to have done so in a consistent manner.

The second research question concerns the differences in the understanding of the function concept between engineering and teacher students in the study. The following table (Table 3) shows the distribution of the students' definitions, divided according to student category.

Table 3. The number of students' answers in the eight categories, split according to student category (teachers: 14 students; engineers: 20 students)

Category	1	2	3	4	5	6	7	8
Student teachers	1		1		1	3	6	2
Engineering students		1	5	2	7	3		2

We note that of the student teachers, 8 end up in the two last categories, that is, fail to provide a useful definition. This percentage becomes even larger if the three student teachers aiming for upper secondary school are discounted. They end up one in category 1 and two in category 6. Hence 8 out of 11 prospective compulsory school teachers cannot give a useful definition of the function concept. Of the engineering students, only two fail at this. Conversely, only two student teachers end up in the three operational categories 3, 4 and 5, whereas 14 of the 20 engineering students do so. However, when it comes to applying the function concept, few obvious differences between the two groups can be seen. The student teachers tend to be less confident about their answers, and there are those among the student teachers who give incorrect answers even to the most straightforward examples. But on certain examples, for instance those concerning univalence, the student teachers perform better than the engineering students.

Instead, the most striking difference is seen in how the students handle the construction of the integer/non-integer function mentioned above. None of the prospective compulsory school teachers are able to give an answer. Indeed, only one student even tries. The rest just answer *"I don't know",* and several claim not to have understood the question. On the other hand, all of the prospective upper secondary school teachers, and most of the engineering students, have given a correct construction, and even those who believe no such function can exist have provided some kind of argument in favour of this view.

Finally, looking at the classification of the students' conceptions of the function concept, obvious differences are seen. Of the student teachers, one has a structural, 4 operational and 9 pre-operational conceptions of function. Among the engineering students one has a function conception which is approaching the structural, 16 have operational and only 3 have pre-operational conceptions of the function concept. If we discount the prospective upper secondary school teachers, the tendency is even clearer. Among the prospective compulsory school teachers, 9 have pre-operational and only 2 have operational conceptions. So it seems fair to say, that the prospective compulsory school teachers in the study have less developed conceptions of the function concept than the engineering students. There also appears to be a difference between different types of teacher students, but the number of prospective upper secondary school teachers participating in the study is too small for me to dare draw any such conclusions. What can be said, however, is that while even the prospective compulsory school teachers with the most developed conceptions of the function concept still have rather inadequate conceptions, the function conceptions of the prospective upper secondary school teachers are among the richest in the study.

Discussion

This study shows that the participating students primarily have operational, and in some cases pre-operational conceptions of function. This agrees well with earlier research on the subject, which has indicated that a reified concept of function is rare among students of mathematics. But, contrary to several earlier studies (e.g. Akkoç & Tall, 2005; Meel, 2000; Vinner & Dreyfus, 1989), the students in this study show no great discrepancies between their definitions and concept images of the function concept. A probable reason for this is the definitions they have encountered during their studies. It is explicitly stated in (Akkoç & Tall, 2005), and implied in (Vinner & Dreyfus, 1989), that in Turkey and Israel (where the respective studies were conducted) the structural Bourbaki definition of function is used in schools, something which is not at all the case in Sweden.

But even though the students' concept images tend to agree rather well with their concept definitions, their concept images are not very rich, something which agrees with (Hansson, 2006), where it is shown that the function concept is not so well integrated into the general conceptual structure of the students. The lack of more specific concepts, and examples of standard functions, mentioned earlier could be seen as contradicting earlier results regarding the importance of prototypes on the formation of concept images (e.g. Akkoç & Tall, 2002; Tall & Bakar, 1991), but it could just as well reflect an attempt at generality on the part of the students. Furthermore, several examples of compartmentalization were found. For example almost all students stated that the diagram showing a curve with a loop did not represent a function, but at the same time a majority of the students claimed that it represented a function. Here it should also be noted that I make no claim to generality regarding my results. I am well aware that the validity of my study could be greatly enhanced by for example increasing the number of participating students, and also by including interviews with students.

As noted above, several distinct differences between the engineering students and the prospective compulsory school teachers in the study were found, regarding both the function conceptions and the answers to certain of the questions in the questionnaire. Before beginning the study, I had some preconceptions about this. Since mathematically interested and gifted students in Sweden tend to study to become engineers rather than teachers, I had expected differences in mathematical ability. But I had expected the student teachers to show greater interest and ability regarding conceptual understanding and expressing mathematical ideas in words, since these are important abilities for the teaching of mathematics. When it turned out that most of the student teachers had taken quite a lot of mathematics at a university level, my hopes were raised further.

But, contrary to these expectations, the conceptual understanding of the prospective compulsory school teachers was less developed than that of the engineering students. Also, their general attitude and low self confidence is cause for concern. A few of them include words like "hard" and "difficult" in their mind maps, and rate their level of certainty below average on all statements. Asked to define the function concept, one student writes: "Is part of a graph over a coordinate system. Eeeh… I can't explain it." (T6) This is a problematic answer, coming from a prospective teacher. This uncertainty was most apparent in their answers to the question about the integer/non-integer function. Almost none of the prospective compulsory school teachers even tried to answer this question. Among the answers were these: "Firstly, I had to read the question about five times before I understood a little. Then, when I understood a little, I couldn't picture this function in my head." (T4) and "I have no idea. I won't even think about it, since I don't intend to study functions in any detail." (T13) This last answer I find especially troubling, since it displays an attitude which I have a hard time reconciling with wanting to become a teacher.

One last point I want to make concerns the difference between the prospective upper secondary school and compulsory school teachers. Although the number of prospective upper secondary school teachers participating was very small, and no real conclusions may therefore be drawn about them, I still find the difference between them and the rest of the prospective teachers in the study striking. One thing worth noting is that, at the university where the study was conducted, they take the same classes as mathematics students and engineering students, while the rest of the prospective teachers take classes specially designed for student teachers. One would expect such classes to be more focused on conceptual understanding, for example, but in the light of this study one has to wonder whether these classes are appropriately designed. I find it problematic that future teachers, having taken more than half of the mathematics classes required, have such low mathematical self confidence, such undeveloped conceptual understanding, and such a hard time expressing themselves mathematically.

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Different Versions of the 'Same' Task: Continuous Being and Discrete Action

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Introduction

In this paper I probe subtle differences in lessons which are based around similar tasks, by analysing the experiences they afford students, and identifying what is available for students to construct from these experiences. This provides a new lens for looking at mathematical activity in lessons, and at how teachers' own mathematical senses act out to afford different mathematical experiences for learners.

There is a resurgence of interest in task design as an important factor in mathematics teaching (Sierpinska, 2004; Burkhardt & Schoenfeld, 2003). An international society has been founded and a number of publications show that design has to be taken seriously not only for extended, multi-stage, authentic and assessment tasks (as described by, for example, Wittman, 1998), but also for the very ordinary things we ask students to do day-to-day in classrooms (e.g. Swan, 2006; Mason & Johnston-Wilder, 2006; Karp, 2007). Runesson (1999), Emanuelsson (2001) and their colleagues offer mathematical variation as a critical feature of the process of task design while others (e.g. Watson 2004) focus on the affordances of tasks as structures for potential mathematical activity. Variation informs us about affordances.

However, over-reliance on task design as a vehicle for improvement in mathematics teaching is known to be a flawed approach on its own. There is a steady history of research which shows how the designers' intentions become altered as their tasks are taken up and used in classrooms when teachers apply their own perspectives and local purposes (Stein, Grover, & Henningsen, 1996; Stylianides & Stylianides, 2008; Palhares, Gomeros, Carvalho, & Cebolo, 2008). Emphasis on the ensuing activity – what is actually done, talked about, learnt, and how this takes place – places pedagogy and culture alongside tasks as equally important factors.

Analysing lessons

For about ten years I have been developing descriptions of the mathematical and pedagogical choices which make a difference to learners' mathematical experiences. Various attempts to do this have been published, and yet whenever I think a particular approach is 'finished' another arises from my reflections on classroom observations, and my own mathematical activity, alone or with others. Analysis which starts with questions and prompts (Watson & Mason, 1998; Watson, 2007) focuses on what learners are asked to do; analysis which starts with examining variation among presented examples (Watson and Mason, 2006) focuses on the material from which learners can generalise. Both of these are different from analysis which starts from comparing mathematical possibilities offered through the teacher's role in interactive sequences (Watson, 2004; David & Watson, 2007); this latter approach to analysis tends to focus on how learners' engagement can be mathematically shaped. There are other methods of analysis around, such as categorising different epistemological aspects of mathematics which are emphasised during a lesson (e.g. Andrews & Sayers, 2005). All these approaches are valuable, but none completely capture the full sense of how one lesson is mathematically different from another. Mathematical activity appears to be fractal, and to unfold differently depending on the starting focus (see also Davis & Sumara, 2006). By unfolding layers of activity from one perspective, the folds hide other aspects which might be equally important.

The nature of mathematical activity

For this paper I have returned to mathematics itself, and its assumed structures, concepts and definitions, to think about differences in lessons. In the TIMSS seven-nation video study 'mathematical quality' was 'measured' by a team of well-qualified mathematicians (Hiebert, Gallimore, Garnier, Givvin, Hollingsworth, Jacobs et al., 2003) and used as a comparative characteristic. The categorisations were very vague, it being assumed that people with strong mathematical qualifications can make such judgements. Something more informative is needed for teachers and teacher educators whose interpretations of 'mathematical quality' are necessarily limited by their own mathematical experiences. Variation theory (Marton, Runesson, & Tsui, 2004) provides a tool for doing this to some extent, but I am going to show that there is more. I do this by looking at 'the same' task taught by different teachers.

Rather than looking at tasks to predict activity (using the distinction developed by Christiansen & Walther, 1986), I am going to look at the nature of public mathematical activity to find a new lens for seeing task implementation.

My method is to use classroom observation and video to reflect on the nature of classroom mathematics. To focus on 'public activity' means to ask the questions: 'What is the class supposed to be doing right now? What are they supposed to be thinking about? What is being said and done, and by whom, that is shaping and is shaped by the activity?'

Using the 'same' tasks
Several teachers in the same school were teaching groups of 12 year-olds who were more or less similar in previous attainment.[1]. All teachers agreed to use a similar approach to teaching loci using a combination of straight-edge-and-compass constructions and the physical whole class activity of acting out loci by following instructions to 'find a place to stand so that ….' (e.g. 'find a place to stand so that you are the same distance from these two points'; or '… all two metres from this point'; etc.). All classes constructed circles, perpendicular bisectors of line segments, and angle bisectors and some other loci. An indoor open space was available to do the physical task, and teachers chose to do this at different points during their lessons. Students were intended to relate their physical experience of standing according to such rules to the processes of geometrical construction. This kind of connection, enabling shifts between three very different representations (words, actions and diagrams), is one of the characteristics noticed by Krutetskii (1976) as typical in gifted mathematics students. The problem for teachers is, therefore: how can all students be helped to make the connections that the highest achieving students are expected to make for themselves? This problem is exacerbated by the affordances of the physical task: it is possible to take a 'gap-filling' role without constructing a personal interpretation of the instructions, and hence not to have an experience of being a point in relation to other points to refer to when reproducing the locus on paper. It is also worth mentioning that these students had little experience of geometry beyond some knowledge of angles, and the naming of polygons.

The five lessons were compared qualitatively in a variety of ways. I looked at the nature and amount of variation offered in the task, the questions and prompts used by teachers, whether teachers worked with whole classes or small groups, interaction patterns, combinations of 'doing' and 'thinking' prompts, the emphasis on reasoning, and whether teachers simplified their questioning when students could not, at first, answer. These foci for analysis were selected on an underlying theory that students can only respond to what is made available to them in the words, actions and artefacts of the lesson. In other words, the mediational devices and instructions used by the teacher and other students, whether intentional or not, shape the learners' experience of the lesson. In these five lessons, this shaping turned out to be mathematically different even though the actions and artefacts were similar. The details of the data are omitted here in order to focus more quickly on what I

[1] The data on which this analysis is based was collected by my colleague Els De Geest during work on a joint project funded by Esmee Fairbairn Foundation (05-1638); the analysis is my own responsibility and the school context is disguised.

claim to be essential differences, these having been arrived at by comparing features of lessons under the categories described above.

There were strong similarities between the lessons: all teachers used a mixture of asking, prompting, telling, showing, referring students to other students' work and so on. All teachers focused on getting students to explain their choices and actions. All students had to work out as much as they could themselves about how to do the constructions, either by reasoning or by listening to others' reasons in whole class discussion. The tasks were presented in remarkably similar ways and, in variation theory terms, offered similar variation in similar ways due to the mathematical structures being taught and the choice of loci with which to work, which had been agreed by the team. Teachers' intentions were similar, and all of them praised accuracy and sought for reasoned action. Written work was similar, and students might report similar experiences after the lessons. Analyses in terms of variation and affordances and constraints, and situational norms, and the nature of questions and prompts, and the kinds of demands made on learners provided very similar results. None of the dichotomies used in the literature to compare lessons superficially (open/closed; teacher-centred/learner-centred; traditional/reform) were helpful in identifying difference, yet as a mathematical observer I know that the mathematical affordances of the lessons varied. They provided different kinds of intellectual and mathematical engagement. The components of the tasks were offered in different orders by different teachers; teachers said different things to students at different times; there was a range of different patterns of participation for individual students in each lesson; the various possible constructions were offered in different orders.

To express these different kinds of engagement I shall draw on the five observed lessons to present phenomenographic constructions of three possible lessons to show that different mathematical learning experiences can arise from lessons which are very similar. As a way of presenting research this is valid because everything that is included is from an actual lesson, and hence has authenticity and credibility in the field.

Lesson one

The lesson started with students working as a class, with guidance from the teacher, working out how to use a pair of compasses to construct circles, a locus with constant distance from a straight-line segment, perpendicular bisectors and angle bisectors. The teacher repeatedly referred to compasses as the tool for reproducing equal lengths: he said this himself, and also asked students 'what can we use to get equal lengths?' and 'what do compasses do for us?' and 'why would I use the compasses?' Students were then asked to compare the perpendicular bisector and angle bisector constructions, and to identify the role of compasses within these. The words ‘same distance’ and ’equidistant’ were used frequently throughout the lesson. The teacher invited students to demonstrate their ideas on the board, and also used the strategy of placing ‘wrong’ points to encourage students to understand the role of constraints. The physical activity took place at the end of the lesson, and was treated briefly as a summary of the rest of the lesson.

The focus on the power of the tool was reinforced by comparing its role in constructing the two different bisectors, so that students were looking at the positions of, and relationships between, the equal lengths are in the constructions. By taking this approach, learners were able to talk about relationships within the diagrams as if they were caused by the equal lengths, rather than equal lengths merely being a drawing method. It was made possible for them, by this focus, to get a sense of classical geometrical tradition. The physical activity used the same language of ‘equal lengths’ in instructions and descriptively where necessary and offered no further public engagement of mathematical thinking, merely rehearsal in a different context, using a different representation, with nothing said about how to make equal lengths in physical action.

Lesson two

In this lesson, students were asked to locate points which fulfilled certain rules: points which are all the same distance from another point, two points, two lines and so on. This was done on the whiteboard with discussion, and also on paper, the initial approach being consisting of rough diagram and reasoning, and compasses being introduced later as a way of joining up the points for the circle and locating particular points. The emphasis was on how to use them, rather than why they worked. The predominant language was about points which ‘obey rules’. The word ‘locus’ was introduced during discussion of where all such points would be. During the second part of the lesson students took part in the physical representation of loci in response to the same language as was used to find points. Students were expected to link the different parts of the lesson (pencil and paper construction, whiteboard drawing and physical activity) through the use of the same language to express the same ‘rules’ for placing points and people, and an increasing use of the word ‘locus’. One student called out: ‘this is what we have just done!’

In each case the emphasis was on collections of points, each of which has a particular property, and on joining up the points. The role of the compasses was not emphasised; they were treated as a means to join up points which are equidistant from other points. Verbal instructions about finding individual points with properties were the most repeated sound of the lesson.

Lesson three

In the third lesson, the physical activity took place first, and the teacher offered a story to encourage visualisation: standing the same distance from two trees; steering a ship between two icebergs. Students then returned to the classroom and were asked to construct the same loci.

The physical activity happened first so that students were expected to have some memory to draw on when they came to make constructions in pencil and paper. No public instructions for constructing were given, instead students were asked to work out how to do them using their memories. The teacher worked hard with small groups of students asking them what they remembered and how they could reproduce it. In general she said ‘you can use the compasses’ when equal lengths were needed, sometimes showing them how to do it and then asking them to do it again for themselves. A significant amount of time was given at the end of the lesson to the task of developing statements that linked the physical activity to the pencil-and-paper constructions. Students had to express the isomorphisms between the situations.

Discussion

The three lesson possibilities are likely to have left different traces in students' minds about what the key ideas were about loci:

- trajectories derived from relationships between equal lengths;
- sets of points which have certain properties;
- reproductive constructions of physical situations.

From a mathematical viewpoint these are equivalent in terms of relationships and properties, but in terms of learning experience they are different and memory of the lesson content is likely to be triggered by different stimuli in future. I am reluctant to arrange these in any sort of hierarchy of mathematical challenge: each invites learners to shift from obvious, intuitive visual and physical responses to the more formal, ‘scientific’, responses required for mathematics. In each of these lessons there are emphases on relationships between variables, properties, reasoning about properties and relationships among properties, so available hierarchies based on assumptions about cognitive challenge and ways of seeing (e.g. van Hiele 1959) do not identify difference – and yet different mathematics is learnt – or at least the ‘same’ mathematics seen, described, and triggered in different ways. It is important to sustain the delicacy of these differences in mathematical terms, rather than to dive into pedagogic differences between the lessons (e.g. how much groupwork, what sort of questions, patterns if interaction etc.) which will give less information about mathematical didactic structure.

Task differences

In each of the lessons above, interpretations of the task have been made by individual teachers, after team planning. In these lessons we do not see any reduction of challenge as is reported about adoption of published tasks. The teachers have discussed common approaches, which loci should be used, and how lesson should be resourced in terms of space and equipment. Overt activity is similar; a more casual observer might say they were the same lesson. The effects of tool use on drawn diagrams were the same, although the sense of appropriation might be different, and the mathematical content was equivalent.

What differed was what was emphasised by the teacher, but I am not saying that this was merely talk. Rather, the difference was, I claim, due to the underlying general relationships within which the teacher saw the task as being embedded. Because teachers see these differently they therefore use different language, different sequencing and different emphases so that different comparisons and connections can be made – yet all of these are equally mathematical.

In each lesson such differences were continuous. Each lesson was coherent throughout in the relationships among its tasks, language, emphases, prompts and other components. Each lesson was an expression of how the teachers saw the links between the tasks, tools and learners within their understanding of what loci generally entailed.

This realisation releases me from attempts to describe good mathematics teaching as a collection of actions, utterances, tasks, and examples, and instead leads me to look for the continuous threads of mathematical awareness the teacher is revealing by her/his actions and decisions. We can then see teaching mathematics as the more-or-less fluent expression of an understanding of a mathematical context for the current work.

The implications of this insight are that we can see mathematics teaching as a way of being mathematical, and the education of mathematics teachers as a mathematical experience.

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Ongoing Research in Mathematics Education in School Context

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This paper discusses the ongoing project in mathematics education at University of Gävle. The aim of the project is to develop and test methods that refer to improve learning in mathematics from school to university level. Furthermore, the aim is to develop a research environment for learning in mathematics that can serve as a platform for the development of teachers´ competence in mathematics education. Fundamental for this project is that it combines as well student's learning, teacher's learning as researcher's learning.

The project will promote the research question, which has to do both with the theory and practice. The overall research question is: Does the teacher's way to handle the object of learning (e.g. the fraction concept, equation concept, function concept, etc.) in the classroom influence on pupils'/students' learning? What are the critical aspects for pupils'/students' learning?

As a theoretical framework we use the Learning Study model designed by Marton and Tsui (2004). The model is based on the variation theory (ibid.), which originates from the phenomenographic research tradition carried out in Sweden in the later 1960s and early 1970s. The variation theory has two fundamentals; learning always has an object and the object of learning is experienced and apprehended on different ways. According to this theory the most powerful factor concerning pupils' learning is how the object of learning is handled in teaching situation; what aspects are in focus - what aspects are variant and what are invariant.

Our ongoing study is three years research project. During the spring time 2007 we have done a preliminary investigation in a compulsory school. Our study deals with the fraction concept and how it is handled in the lesson. In this study one compulsory school teacher together with university teachers planned two lessons of the fraction concept for two classes in grade eight. The pupils were tested before and after the lessons and the lessons were video-recorded. The test-results for each class, before and after the lesson, were compared. The results from our preliminary investigation indicate that pupils have poor-developed conceptions about fraction concept. Especially, they have difficulties with addition and subtraction of fractions with different denominators. In order to increase the pupils' understanding of fractions, we propose a number of actions for future planning of learning studies. Autumn 2007 we have started our learning studies in upper secondary school and preschool. Two upper secondary teachers and one student teacher and one preschool teacher are involved. Year 2008 we are going to extend our project to other schools and districts. We have also planned to test Learning Study-model on studies on university level and to develop methods for distance learning in mathematics. The whole project will be evaluated and documented during 2009. Based on experiences from the study a new plan for future research is created.

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On the Equivalence Relation in Students' Concept Image of Equation

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The concept of equation is, in principle, very elementary in mathematics. One can say that any mathematical expression of the form $A = B$, where A and B are relevant mathematical objects belonging to the same category, is an equation. However, a large variety of vague and incorrect conceptions about equation exist among mathematics teachers and student teachers. Only a minority of students can state a mathematically satisfactory definition for equation (Attorps & Tossavainen, 2007a, 2007b).

Seen from the point of view of mathematics and the language of mathematics, a student's concept definition of equation can be wrong basically for three different reasons: failure of understanding the equivalence relation $=$; misconceptions related to the truth value of the statement including the equality sign; or the confusion about choosing A and B from incompatible mathematical categories. With respect to mathematics at school, the latest reason is, nevertheless, only marginal.

We have reported elsewhere on our preliminary results on how teachers' and students' misconceptions related to the understanding of the properties of the equivalence relation are related to the misconceptions that teachers and students possess about equation and that the belief that the equation must always convey a true statement strongly affects how students themselves define equation (Attorps & Tossavainen, 2007a, 2007b).

Since our original questionnaire did not completely reveal the relationship between the understanding of the mathematical properties of the equality relation and the concept of equation, we have collected new and larger data from Finland and Sweden (N=64) using a newly developed questionnaire to study this relationship and further to understand what kind of concept definitions students possess about equation. As in the previous cases, we use a phenomenographic research method in our analysis of the data (e.g. Marton & Booth, 1997). We also acknowledge the dual nature of mathematical concepts (e.g. Sfard 1991), the distinction of mathematical knowledge to the procedural and conceptual components (Haapasalo & Kadijevich, 2000), and the APOS theory (e.g. Asiala et al., 1997) when we classify students' concept definitions of equation and estimate how mature they are in a mathematical sense.

Students' conceptions of the notion of equation are often based on the existence of a variable to be solved out and, generally, dominated by the operational/ procedural view of the concept. Also, the expectation that an equation must always be a true statement was clearly revealed from our data. These conclusions are indicated e.g. by the fact that even 88% of the students claimed that $1+2=5$ is not an equation.

By our analysis, it appears that at least one third of the students do not understand the reflexivity of the equality. For example, 39% of students claimed that $x=x$ is not an equation. Also the failure of understanding symmetry of the equality is common. Half of all the students think that $x=2$ is not an equation but merely "an answer to an equation", e.g. to $2=x$. The classification of equations and their answers to different categories raises an immediate question: how well do these students understand the logic and the language of mathematics if they write $2=x \Leftrightarrow x=2$? The same phenomenon appears with the transitive property and with the similar generality: For example, 55% of the students claimed that $a=b=c$ is an equation and only a few of those who answered correctly motivated their answers by pointing out that there are several equations in the expression.

All in all, the misconceptions about equations which are related to the properties of the equivalence relation are surprisingly common among mathematics student teachers.

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Detecting Mathematical Abilities in Students' Solutions of Mathematical Problems

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Mathematical talent appears to be constituted by a spectrum of abilities that each individual displays in different degrees. Identifying and classifying the mathematical abilities is a major task, and V.A. Krutetskii has done a thorough study in this field in the Soviet Union in the middle of the last century (Krutetskii, 1976). Krutetskii claimed that mathematical ability could be looked upon as divided into seven categories. Those abilities are shown to be more abundant in capable students than in others. His technique was to design mathematical test items addressing these abilities one by one, and use factor analysis on the students' responses to justify his claim.

While Krutetskii uses different sets of problems to address the specific abilities and thereby justifying his system of abilities, we will do it the other way around. Our purpose is to give the student a set of rich mathematical problems, and by analysis of written solutions and interviews of the students, we try to detect what abilities the students display.

The design of problems is of crucial importance for our study. To meet Krutetskii's categories of abilities our problems must apply to the students' creativity and flexibility in thinking, and also give them good opportunities to express generality in their reasoning. This leads us to look at so called rich mathematical problems as candidates for our test problems. Different researchers have used the term "rich mathematical problems" in a partly different meaning, but we have found Hedrén's et al. definition close to our demands (see Hedrén, Hagland, & Taflin, 2005; Taflin, 2007). Tasks that are supposed to stimulate students to formulate generalizations can be found in various works treating mathematical problem solving. One such work is Mason, Burton, and Stacey (1985).

To collect data, an extended task sheet with three or four selected problems is used. These problems are of two types:

1. Problems to be worked out individually by the students in the classroom during a limited amount of time.
2. More complicated problems to be mulled over during a period of two or three weeks.

To promote the reliability of the investigation, data from this latter type of problems is supplemented by a clinical interview with the problem solver.

Classroom tasks have been made for full classes on five occasions, both with upper secondary school students and first year mathematics student teachers. A few of these students also have completed the long-term tasks, but this far no interviews have been conducted. We expect, however, more students to complete the long-term tasks in the near future.

The research question in focus is:

By looking at the outcome from students' solving of rich mathematical problems, which of Krutetskii's mathematical abilities can be identified, and how are these abilities revealed?

In the analysis of the results we first look at each problem in the light of the Krutetskian scheme and make our own interpretation of that in the context of that specific problem. We then look at the students' solutions in order to detect which of these abilities they use in solving the problem.

It is a well-known fact that working with challenging problems foremost attracts high achieving students. For these students problem-solving activities might meet their needs of more adequate education in mathematics. We believe, however, that with properly designed tasks, it is possible to fruitfully work with problem solving in a mixed-ability classroom.

Knowledge about the abilities that a certain problem might reveal can be used as a parameter when classifying problems into some kind of taxonomy. It may also increase our understanding of the nature of mathematical talent.

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En bro mellan forskning och skolvärld för vuxnas matematiklärande – ett nordiskt samarbetsprojekt

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Hur förs didaktiska forskningsresultat ut till skolvärlden? Hur kommer de den vuxenstuderande till godo? Hur kan lärare synliggöra för praktiken intressanta forskningsområden? Hur kan dialogen mellan lärare och forskare utvecklas?

Den matematikdidaktiska forskningen i Norden är i kraftig utveckling men dialogen mellan forskare och praktiker är, enligt våra erfarenheter, begränsad. Detta har vi, projektdeltagarna, upplevt i egenskap av lärare, forskarstuderande, forskare, skolledare och studievägledare. Därför kartlägger vi vilka etablerade kontakter, vilka behov och vilka utvecklingsmöjligheter som finns. Vår utgångspunkt är alltså personliga erfarenheter från våra olika yrkeskategorier i Sverige, Norge och Danmark.

Vi undersöker hur matematikdidaktiska forskare för ut sina resultat och rön till matematiklärare inom vuxenutbildningen. Det är i detta sammanhang av intresse att undersöka om detta sker på ett sådant sätt att de påverkar undervisningspraktiken. Vidare undersöker vi huruvida kommunikationen även sker i motsatt riktning, dvs. att lärare formulerar undervisningsproblem som bör beforskas. Är det möjligt att formulera forskningsprojekt där både forskare och problemformulerande lärare samarbetar?

Som ett hjälpmedel i vår kartläggning har vi en webbplats[1] för fortlöpande dokumentation och kommunikation med organisationer och aktörer utanför projektet. Vår metod utgörs i huvudsak av att vi använder enkäter, riktade till främst lärare, forskare och skolledare. Ytterligare viktiga inslag är artiklar och seminarier, där vi har för avsikt att öka förutsättningarna för erfarenhetsutbyte och informationsspridning. Exempelvis planerar vi ett stort seminarium i Oslo i början av juni.

Projektet är inte ett forskningsprojekt i formell mening utan snarare ett kartläggningsprojekt med en kvalitativ utgångspunkt. Med detta menas att vi inte är ute efter att med statistisk säkerhet presentera representativa slutsatser. Snarare vill vi fånga upp intressanta tankar och idéer och, i förlängningen, utgöra en utgångspunkt för framtida projekt.

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[1] Se http://www.cormea.org/

I den här föreläsningen presenterade vi projektets visioner och delresultat av vår kartläggning. Projektet (Connecting Research and Mathematics Education for Adults, CORMEA) finansieras med hjälp av Nordplus Voksen och är ett samarbete mellan:

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Mathematics Education: Discourse, Knowledge and Power

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I am working on a doctoral thesis concerning multilingual students and mathematics education. This is a short presentation discussing the possibility of using certain post-modern theories in mathematics education research. In my analyses of the empirical data I whish to try to use Foucault's conceptions of knowledge and, power, but also his concept of discourse according to Gee, can it be possible?

Introduction

I am working on a study concerning *minority students in bilingual mathematics classrooms* in Sweden. For almost two years I have followed bilingual teachers and students in mathematics classrooms. The mathematics teaching and learning were going on in Swedish and Arabic. In this short presentation I would like to consider theoretical perspectives, how and what usefulness they may have. It also brings methodological consequences. A theory I wish to try as a thinking lens derives from Michel Foucault. Foucault's work is by some commentators looked upon as a paradigmatic example of 'post modern' thought. Mathematics education and postmodernism yet have rarely addressed each other (Walshaw, 2004).

Theoretical perspectives

Within my study sociocultural influences are important, but as I find discourses (Gee, 1999), and structures of power as well as concepts of knowledge present in the bilingual mathematics classrooms context I look for different ways of analysing empirical data. Foucault's thinking includes three key concepts: discourse, power and knowledge (Walshaw, 2007). With his theory it might be possible to track historical, cultural and social circumstances as a way of understanding events in the mathematics classrooms.

Thoughts about my study

Students' differences in mathematics achievement have become wider and there are concerns about minority students' marginal performance in mathematics in Swedish mathematics classrooms (PISA, 2006; Skolverket, 2007). Often students' low performances in mathematics refer to deficiencies that call for remediation in the students, in their languages or cultural backgrounds. One example is their "lack of Swedish-ness" (Parszyk, 1999; Runfors, 2003).

Instead of looking at deficiencies it is possible to view the languages and cultural backgrounds of the students as resources for learning mathematics and as a potential for their future lives. Apple (1996) said about education:

> Education is deeply implicated in the politics of culture. The curriculum is never simply a neutral assemblage of knowledge, somehow appearing in the texts and classrooms of a notion. It is always part of a selective tradition, someone's selection, some group's vision of legitimate knowledge. It is produced out of the cultural, political, and economic conflicts, tensions, and compromises that organize and disorganize a people (p. 22).

If mathematical knowledge is to be looked upon as socially constructed rather than found, mathematical knowledge and meanings has to be located in social practices – in discoursers of mathematical communities as in a mathematics classroom. Learning then occurs collaboratively in the context of shared events and as each ones experiences, languages and cultural backgrounds are valued as resources for learning mathematics – students become empowered (Cummins, 2000). Walshaw (2007) refers to Foucault's concept of power and says it is constituted through discourses and that power circulates in practices. She writes:

> In the course of Foucault's work, power came to be considered as something quite different from coercion, prohibition, or domination over others by an individual or a group. He took issue with analyses that express power merely in centralised and institutionalised forms in which an individual or group deliberately imposes will on others. … As it turns out, Foucault maintained that power underlies all social relations from the institutional to the intersubjective (p. 20, 22).

Knowledge and power

It is difficult to separate power from knowledge – there is no power relation without a field of knowledge being constituted. Foucault is redefining power as coextensive with knowledge (Walshaw, 2007).

In one of her articles Setati (2005), a South African Mathematics Education researcher, claims that language is always political, and language is a symbolic resource in educational as well as social and employment markets (Bourdieu, 1991). Setati asked what language and discourse practices teachers used in multilingual mathematics classrooms. She found that two categories of mathematical discourser emerged; procedural and conceptual. At a micro level of the classroom interaction language was political, as power relations existed in the classroom. The mathematics teacher projected herself as a certain kind of person in a certain kind of activity. English was used in procedural discourse, which according to Setati highlighted the political tension in the multilingual classroom. English was also used as the language of regulation and assessment - the language of authority. Students' (and teachers') mother tongue – Setswana – was used as a language of solidarity and contextual discourse. Setswana also functioned as the language of conceptual discourse. The teacher's personal experience made her struggle with the tensions between her identities as an African and as a mathematics teacher. It was evident in the way she used languages but also in the discourses she used during teaching. She was "aware of the political role of language during apartheid in South Africa and the power of English in enabling learners to gain access to educational and socioeconomic resources in South Africa" (p. 460).

Remarks

It is not possible to simply transform South African research findings to a Swedish context. But it might be useful to analyse empirical data from multilingual mathematics classrooms with inspiration from theories as that of "Knowledge can't be separated from power", "language is always political" and "power relations exist in multilingual mathematics classrooms".

Sjögren (1997), referring to Hyltenstam (1996), wrote:

> In Sweden the principle of home-language teaching as a way to provide students with an academically sound bilingualism has been accepted for more than twenty years, but still has great difficulties in becoming incorporated as a basic element into the school curriculum. (p. 7)

Since 1997 the view on students home-languages or mother tongues have changed. There has also been a shift in official policy which language to use for instruction in mathematics education. The Stockholm *Mother-tongue teaching of mathematics project* enacts an example. But still there are hesitations among teachers, politicians and administrators. Sjögren (2002) writes:

> It's not so much Swedes themselves who are 'Swedish,' but institutions-the Swedish schools, parliament, police, press, and so on. And being institutions, they are extremely slow to change. They support the existing ideology and way of thinking. (p. 16)

Concluding questions

This takes me back to Foucault. Is it possible to use his theory, connecting knowledge, power and discourses, as a way of understanding teaching and learning in bilingual mathematics classrooms in Sweden? What does it implicate to be a bilingual student in a bilingual mathematics classroom in Sweden? What Discourses, mathematical and others are used? How do bilingual teachers use the languages, in what Discourses?

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Förskolebarns utveckling av ”pre-matematik”

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Denna presentation fokuserar mitt avhandlingsarbete. Avsikten är att problematisera analysen av en videofilmad episod samt diskutera möjligheter och hinder med studien. Som verksam förskollärare och lärarutbildare är jag intresserad av hur yngre barn (1-3 år) lär matematik och på olika sätt löser matematiska problem. Mitt forskningsarbete syftar till att synliggöra yngre barns matematiska aktiviteter i interaktion och socialt samspel med kamrater och vuxna. En möjlig utveckling av studien är att också studera något äldre barn (5-7 år) för att kunna införa en kontrast och genom en tvärsnittsstudie belysa barns utveckling av tidig matematik. Det övergripande syftet med studien är att försöka beskriva *hur barn i förskolan utvecklar begynnande matematisk förståelse*.

Som en grund för min avhandlingsstudie finns följande frågeställning: Hur kan de yngsta barnens utvecklande av begynnande matematisk förståelse i lek och i eget valda eller styrda aktiviteter identifieras, beskrivas och förstås?

Forskningsfrågorna utgår ifrån antagandet att yngre barn utvecklar och lär sig matematik genom eget agerande i lek och andra situationer.

Avsikten med studien är att studera variationen av barns agerande och lärande samt att synliggöra variation av de pre-matematiska aspekter som barn möter och urskiljer. Forskningsansatsen som inspirerar mig är variationsteorin (Marton & Booth, 1997; Marton, Runesson, & Tsui, 2004).

Hur barn lär matematik är ett område som är välstuderat. Många forskare har tagit utgångspunkt i Piagets teorier och försökt att utveckla eller motbevisa hans forskning om hur barn skapar egen matematisk förståelse och hur barns begreppsbildning utvecklas (t.ex. Gelman & Gallistel, 1978; Baroody, 1987; Fuson, 1992; Gelman & Meck, 1992).

Jag använder följande tentativa definition av ”pre-matematik”[1]: barns strukturerande och systematiserande och försök att erhålla symmetri/asymmetri eller annan ordning utifrån de ”matematiska normer” som vi har skapat i vår kultur.

Yngre barn kan sägas lösa dilemman med matematisk innebörd utifrån de meningserbjudanden/handlingserbjudanden (Gibson & Pick, 2000) de själva/ andra barn, materialet/aktiviteten och sammanhanget erbjuder dem. Barns utveckling av pre-matematiskt kunnande är en aktiv, skapande process som beror på egna erfarenheter och vad vuxna genom kultur och sociala normer definierar

[1] Detta är min egen definition.

som matematik. Barn möter till exempel föremål som kan beskrivas med geometriska uttryck, antal eller symboler. De agerar på olika sätt beroende på situationen eller sammanhanget och sin egen och kamraters förmåga.

Med utgångspunkt i ovanstående kommer jag att redogöra för en första analys av en filmsekvens där tre barn (mellan 2 år och 2 år, 3 mån.) interagerar med varandra samt med en så kallad ”plockbox”[2] i byggrummet på en förskola. Episoden är tagen från en pilotstudie till avhandlingsarbetet. Aspekter som jag vill diskutera och synliggöra i min analys kan relateras till pre-matematisk förståelse/lärande, interaktion och socialt samspel.

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[2] En burk med hål i locket i vilket man kan stoppa olika objekt (i detta fall kuber, cylindrar, trekanter och stjärnformade klossar). Plockboxen ger ett visst handlings/meningserbjudande då det finns ett ”rätt” sätt att använda den. Barnen ska strukturera och systematisera sitt handlande utifrån de geometriska objektens egenskaper och de hål/mönster som finns i locket.

Mathematical Problems in School Context Research – a Teacher Perspective

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The purpose of this paper is to discuss the teachers' reflections when taking part in a research project. Research methods have included video- and audio recordings, stimulated recall with the teachers and interviews. One result was that the teachers found that all pupils could work with the problem and achieved better than expected. At the same time they displayed job satisfaction. Another result was that the teachers started to formulate their own research questions as a consequence of taking part in the research project. A third result is that even if the problems implied another way of teaching, the teachers intend to continue as usual.

Introduction

This study is a part of a larger project that is called RIMA (Rich Mathematical Problems), where 4 classes (pupils aged 13 – 16 years) and their teachers worked with 10 rich problems during their lessons in mathematics for three years. ´Rich problems' are defined as problems which are especially constructed for mathematics education in a school context. Seven specific criteria for ´rich problems' will also be formulated.

Background and aim

In a research project, Cooney (1999) states that teachers are traditional in their instruction and have difficulties formulating more complex questions. Boaler (2003) describes the connection between theory and practice: "*What it means to have broad conception of knowing – for research and for mathematics*". She stresses the necessity to develop new knowledge about the practice of school. In an earlier study Boaler (1997) points out that pupils acquire poor conceptual understanding in traditional teacher directed instruction. Teachers have few possibilities to develop their practice into research. One reason for this might be that there is a great difference between teaching and research (Jaworski 2003).

From the perspective of this background, the present project set up to study the following research question: *What influences the teachers when working with a RIMA problem?*

Result and discussion

The teachers gained knowledge by taking part in the research project. One teacher said: "*Then it would be interesting, too, to compare with a common* *lesson ... and see on what occasion they learn best*”. He continued with the following thoughts, as his own instruction with the text book was compared with the RIMA problems: *“If this is not better our way […] to have a great problem? Then it is better, anyhow, to do a little variation”.* One teacher said at the post interview: *“what activity, you know, it is quite incredible*”. The same teacher explained in the post interview: *“You know, I have, I follow my book pretty well and I'll continue to do that […] and now I have some tasks that I know I can run”.*

If we want teachers to do research, courses implying that the teachers do their own studies in their own practice, in the way Jaworski (2003) describes it, will be necessary. Several teachers thought that they worked in another way on the problem solving occasions. But these teachers, too, considered working with the textbook to be the way in which their pupils learn mathematics. The teachers formulated their own research questions as a consequence of taking part in this research project. Even if the RIMA problems implied another way of teaching, the teachers intend to continue as usual.

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Epistemological Beliefs and Communication in Mathematics Education at Upper Secondary and University Levels

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This short report presents the outline for a project that will commence in 2008.

Beliefs of many kinds, and perhaps especially epistemological beliefs, are often described as an important factor in relation to learning – both from a more general perspective and also in particular when it comes to learning mathematics. However, the study of people's beliefs is not a trivial matter. McLeod and McLeod (2002) note several different types of definitions of the term belief that are being used within the research community, but they also see a "general agreement on the core commonalities of the construct" (p. 115). In order to discuss methodological problems one needs a more in-depth discussion about the definition of beliefs. As an example, if we limit ourselves to a cognitive perspective, we can on the one hand distinguish between knowledge and beliefs (Abelson, 1979), where beliefs are of a more subjective nature, for example that you are aware that different persons can have different beliefs about the same matter, while knowledge is something that is more collectively in common. On the other hand, we can focus on similarities between beliefs and knowledge, for example that both can affect how you express yourself when communicating with others or how you interpret situations you are faced with. Using this latter perspective, my study about students' interpretations of mathematical texts yielded complex relationships between beliefs, prior knowledge and reading comprehension, where beliefs did not have a clear and independent effect on reading comprehension (Österholm, 2006).

This project will study the communication in mathematics education at upper secondary and tertiary levels, where focus is on epistemological beliefs. Thereby, the mathematical content in itself is not primarily in focus, but the questions focus on how the mathematical content is treated, from an epistemological perspective. Regarding epistemological beliefs and communication, the following perspectives will be studied:

- How epistemological beliefs can be seen as a part of communication; what types of beliefs are mediated in different situations?

- How epistemological beliefs can affect communication; how can beliefs affect how you express yourself and how you interpret what others have expressed?
- How epistemological beliefs can be affected by communication; how can beliefs be affected by how someone expresses oneself and how you interpret this?

Existing differences between upper secondary level and university level regarding mathematics education have been attended to in research, both internationally and also specifically for Sweden. Thunberg and Filipsson (2005) noted a gap between the content that is covered at the different levels in Sweden, and also a kind of cultural gap was noted (e.g., regarding the use of calculators). Whether such differences stem from differences in epistemological beliefs is unclear, and also if and how these differences affect the students' epistemological beliefs.

Regarding mathematics education at the university level, the teacher education could be of special interest to study, since student teachers not only go through the transition from upper secondary to university level, but also go through a transition from being a student to becoming a teacher. Therefore, student teachers can be exposed to different epistemological perspectives through different kinds of courses; content courses focusing on the students' own learning of mathematics and didactical courses focusing on their development as future teachers of mathematics.

The perspectives and questions mentioned above will be studied at different educational levels:

- At university level, the variation of communicational situations that the students face will be studied; the communication between students and other persons (such as lecturers, teachers, and tutors) and the communication in different types of courses within teacher education (such as content courses and didactical courses).
- Upper secondary level is studied for comparison with university level.

This project aims at producing results that are of interest from different perspectives:

- Theory: To deepen the knowledge about beliefs, in order to create more in-depth models about how beliefs can affect or are affected by different educational situations.
- Methodology: To develop existing methods for studying beliefs.
- Practice: To deepen the knowledge about possible differences between upper secondary and university levels. To gain knowledge about possible variations within teacher education programmes and how these can affect students.

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