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Svensk Förening för **M**atematik**D**idaktisk **F**orskning
Swedish Society for Research in Mathematics Education

Challenges in Mathematics Education

Edited by

Christer Bergsten and Barbro Grevholm

Proceedings of MADIF 3

The 3rd Swedish Mathematics Education Research Seminar
Norrköping, January 23-25, 2002

Skrifter från *SMDF*, Nr 2

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Svensk Förening för Matematikdidaktisk Forskning
c/o Bergsten, Matematiska institutionen
Linköpings universitet
SE 58183 Linköping

Swedish Society for Research in Mathematics Education

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Preface

This volume contains the proceedings of *MADIF 3*, the Third Swedish Mathematics Education Research Seminar, with an introduction by Christer Bergsten. The seminar, which took place in Norrköping in January 23-25, 2002, was arranged by *SMDF*, The Swedish Society for Research in Mathematics Education, in co-operation with NCM, the National Center for Mathematics Education. The members of the programme committee were Christer Bergsten, Barbro Grevholm, Rolf Hedrén, Lisbeth Lindberg, and Anna Löthman. The local organisers were Carina Appelskog and Christer Bergsten at the Department of Mathematics at Linköpings universitet.

The programme included three plenary lectures, one plenary panel, two theme groups, eight paper presentations, and a special forum for young researchers. We want to thank the authors for their interesting contributions. The papers have been reviewed by the editors, and some minor editorial changes have been made without noticing the authors. The authors are responsible for the content of their papers.

We wish to thank the members of the programme committee for their work to create an interesting programme for the conference, and Carina Appelskog for her valuable help with the preparation and administration of the seminar. We also want to express our gratitude to the organiser of *Matematikbiennalen 2002* for its valuable financial support. Finally we want to thank all the participants at *MADIF 3* for creating such an open, positive and friendly atmosphere, contributing to the success of the conference.

Christer Bergsten, Barbro Grevholm
Editors

Content

Papers

Challenges in Mathematics Education

Christer Bergsten
Linköpings universitet

In his opening plenary lecture *Mathematics education in Sweden: A review of research and developmental work,* Ole Björkqvist reported from his overview of research in mathematics didactics in Sweden, based mainly on questionnaires sent to institutions of mathematics, pedagogy and teacher training at universities and university colleges in Sweden. The aim was to provide a picture of the current state of the field, its formal structure as well as the substance of research, including its strengths and weaknesses, and suggestions for improvement. New in Sweden is the increased participation of mathematics departments in this kind of research, which does not only help spreading the research efforts more evenly across the country, but also giving more focus to new areas of research. However, considering the need to build new research efforts on previous knowledge, it is important not to create gaps to the long and strong traditions of research at the pedagogy and teacher training departments. There is a risk that the maths departments do not develop this discipline to a specialty, and that departments of education in this process become more hesitant to specialise in the field. One strength in mathematics education in Sweden is the amount of developmental work taking place at different arenas, and Björkqvist stresses the importance that this work would channel itself into a developmental research status, using the paradigm of a design or construction science. Another strength is the noted willingness of cooperation between institutions. It is indeed a challenge for the Swedish mathematics education research community to find productive ways to channel the apparent strong interest and efforts as described and analysed by Björkqvist.

Jan de Lange contrasted, in his plenary presentation, the New Math movement with the Dutch Realistic Mathematics Education approach, linking it to the present focus on mathematical literacy or proficiency as one of the main general goals for mathematical education. In his paper *Evolving assessment practices: An international perspective,* the focus is on how to accomplish the difficult challenge of designing new ways to assess mathematics more as a constructive, reasoning and insightful activity than merely reproductive. Nine principles for assessment are outlined, based on the NCTM assessment standards, and three competency levels are identified for successful implementation of assessment designs. The first level concerns the handling of procedures and concepts, the second is about connections between different domains and representations of mathematics, and the highest level focus on mathematization, generalization and argumentation. An interesting model, called the Assessment Pyramid, integrates this three steps competency dimension with the dimensions of mathematical domains and of difficulty of questions posed. De Lange claims that a complete assessment program should cover all parts of this pyramid. He also gives examples of students' work with assessment questions within this framework.

A common view of looking at the learning and teaching of mathematics is that the conceptual content should be taught/understood before the standard symbolism connected to it is introduced and trained. In her plenary address, *There is more to discourse than meets the ears: Looking at thinking as communicating to learn about mathematical learning*, Anna Sfard challenged this view. Her focus is on the communicational approach to learning, according to which the process of learning mathematics is a process of developing a new discourse, i.e. new ways of communicating (with others or with oneself) about mathematical terms. By the use of examples of interview protocols, Sfard shows that for the student this means developing new vocabulary, new mediators for communicating, and new meta-discursive rules. For the educator it is crucial to know how such new discursive tools are created. The focus is then on the learner's use of these tools within the discourse. The rationale behind this view rests on the fact that to learn new mathematical concepts or procedures one by necessity must, in some way, communicate about these constructs. To overcome this kind of learning paradox that words or expressions not yet understood must be used in order to be understood, old discursive habits must be recycled in the light of new meta-discursive rules.

In the panel discussion on *The relationship between theory and practice in mathematics education research*, chaired by Rudolf Strässer, Morten Blomhøj chose mathematical modelling as an example of interplay between theory and practice, and identifies, from his experience of the practice of modelling work in education, key objectives for relating theory to practice and vice versa. For example, he notes that theories "need to be personalized and made concrete by the teacher to be of use in practice". Barbro Grevholm offers one example, on the use of a computer program in elementary arithmetic, of how practice can lead to theory, and one example of how theory can inform practice, on co-learning partnerships between different actors in teaching and research. As one reason why we need theories she mentions that out of our practice we have a need to reflect on our experiences in a systematic way. Thus, as an important object of study she identifies teachers' hidden theories on issues within mathematics education. John Mason, in describing his own practices as a researcher, offers a number of intriguing distinctions, principles and assumptions on the issue under debate, such as the danger of mistaking the jargon of a rich practice for theory, that a reactive-responsive practice is hard to communicate to others, and that detailed descriptions of practices are more likely to lead to reproduction of behaviour than to an awareness informing behaviour. In conclusion of the panel discussion, the difficulty of framing practice is seen as one of the reasons to why we need theory.

The seminar included two discussion groups in parallel sessions, one on the topic *Mathematical thinking and emotion*, chaired by Jeff Evans, the other on *Mathematical thinking and achievement*, chaired by Anne Watson.

In his introduction to the first topic, Jeff Evans notes that ideas of affect and emotion in mathematics education research have developed from a focus on individual characteristics to aspects of interactive processes of problem-solving, seeing emotions more as social phenomena. The background paper by Thomas Lingefjärd, *To study mathematics in an engineering program*, presents results from a study on how students comprehend a reformed linear algebra course, aimed at promoting collaboration and increased self monitoring by continuous examination and journal writing, with a focus on understanding. Using survey questionnaires and interviews as data, it was found, however, that fewer students after the course than during the course were convinced that the course improved their generic skills, and that the self confidence of students was low. Examples from interview data indicate disappointing results as regards emotions and attitudes towards the studies in the course.

In her background paper, *Mathematical thinking and achievement: Research issues*, Anne Watson explores the notions of mathematical achievement and mathematical thinking, their multi-meaning character and possible connections, in the light of recent research and debate. What are the necessary components of successful learners? Are the characteristics of advanced mathematical thinking related to all levels of doing/learning mathematics? The discussion leads to one major research question, to be studied within context and with careful definition: Does the development of mathematical thinking improve mathematics achievement?

The seminar had a special forum for young researchers, titled *New directions, problems and solutions*, chaired by Gilah Leder, who gives a short report from the activities that took place.

Eight submitted papers were presented and discussed in the two parallel paper sessions, where to each paper one reactor had read the paper in advance and opened up the discussions.

There has been an increased impact of semiotics on mathematics education research during the last decade. It is in this vein that Arne Engström in his paper *Semiotik och matematikdidaktik* (in Swedish*; Semiotics and the didactics of matehematics*) gives an outline of some basic components and challenging questions from the discipline of signs.

The results of mathematics teacher education is an area rarely studied in Sweden. In her presentation *Teachers' work in the mathematics classroom and their education - What is the connection?*, Barbro Grevholm offers a contribution to this field as a part of an ongoing longitudinal study. By following up what the former teacher students' actual teaching look like when they work as professional teachers, and relating this to their pre-service training, many interesting issues concerning the relationships teacher-pupil, educator-teacher, and researcher-educator-teacher are challenged. Examples from the study illustrate the kinds of problems new teachers face in complex classroom situations and how they sometimes manage and sometimes have problems to take advantage of the relevant parts from their pre-service teacher training.

In the presentation *The beginnings of algebraic thinking* by Milan Hejny and Graham Littler, a learning model for the arithmetic-algebra development is described and illustrated by data from their experimental teaching. Of crucial importance in this development are the steps from knowledge in action into knowledge in words, and from the latter into symbolic language knowledge. The model, which is based on a constructivist approach, emphasises that this development is a long term process, and that it is a challenge to mathematics educators to change established teachers' traditional views on the teaching of algebra.

Marj Horne and Doug Clarke report in *Making a difference in the early years* from the extensive three years research and developmental project ENRP (The Early Numeracy Research Project) in Australia, which involved 35 trial schools and 35 reference schools with students from pre-school to the second school year and their teachers. Focus was an a professional development program on student numeracy. A six-points growth scale was developed to assess students' numeracy competence within different domains and be used in the professional development of teachers, which took place on many different levels. Results showed a significant advantage to the trial classes on long term development of students' growth points, and a documented teacher change.

In his presentation *Numerical skills and arithmetic performance,* Bo Johansson investigates the relationship between the ability to correctly write numerals and perform arithmetic tasks. It is often stated that early practices to write numerals may interfere with the learning of the number concept and arithmetic skill. This view is questioned by Johansson, in agreement with some early Swedish mathematics educators, and tested empirically by a series of studies with young children. The results suggest that knowing to write numerals concur with the development of the number concept, numerals providing a means for handling numbers mentally.

The use of projects is challenging traditional mathematics teaching in dimensions of role of the teacher, student motivation, and independent student work developing learning strategies. The presentation *Projects as an educational strategy* by Marie Kubinova describes and discusses the rationale of project work and outcomes of three projects for grades 6, 8 and 9. Student work took place both inside and outside school, their spontaneous interest in the projects created a working climate favouring a constructivist approach to teaching.

One line of educational research has put the variation of the object of study as critical for learning. In her presentation *Learning velocity graphs - The case of Laura and Fiona,* Ulla Runesson illustrates how learning is dependent on the pattern of variation - such as what is varying, what is invariant, what is left out or taken for granted. For example, by changing one essential feature of a demonstration, the teacher may insert one necessary dimension of variation but take another one for granted. In the case shown in the paper the bi-directional property of a velocity was kept invariant but the variation of the horizontal-vertical dimension of movement was taken for granted, thus causing learning problems.

That the process of completing a substantial diploma thesis may be a crucial component of a teacher training program is well illustrated in the contribution by Nada Stehlíková and Darina Jirotková, *Process oriented research and its reflection in pre-service mathematics teacher education - A case of diploma thesis*. The authors challenge the traditional content oriented diploma thesis at their university by introducing a process-oriented way of working. The latter is characterised by a long-term elaboration, student's own experiments and introspection, creating a mutual influence between student and supervisor, involvement of supervisor in the topic, and a deliberate focus on the change of roles of pupil, teacher, researcher, and expert during the process. Two cases are described to illustrate the successful framework.

Aspects of problem solving in mathematics education such as its rationale and planning, from the teachers' point of view, students' conceptualisations and their opportunities to learn from problem solving, are less studied in research than student strategies and cognitive abilities in problem solving activities, but are in focus in a longitudinal study by Eva Taflin, Kerstin Hagland, and Rolf Hedrén. In their contribution *Vad lär lärare och elever i år 7-9 via rika problem?* (in Swedish; *What do teachers and grades 7-9 students learn from working on rich problems?*), examples of data from this study are presented and discussed, which indicate the crucial role of the teacher on the educational value of classroom work on working with rich problems. Critical factors seem to be teacher beliefs on the aims of problem solving, problem presentation and the follow-up of student solutions.

To learn, teach, and do mathematics is a complex human endeavour, as witnessed by many observed problems in mathematics education worldwide, as well as an abundance of joy and excitement. Mathematics education is a field with long and strong traditions, and it is important to build on those experiences that have proved viable. However, self criticism and change are key concerns for a field to grow along with the needs and development of society at large. For this purpose alternatives to common standards and views play a vital role. It is in this vein the different chapters of this volume challenge the ways to look at the learning, teaching, and assessment of mathematics, in general terms as well as within specific areas such as arithmetic, algebra, problem solving, and teacher education. We hope that this book will also inspire the readers to seek new challenges in their own thinking and practice in this rich field.

Mathematics Education in Sweden: A Review of Research and Developmental Work

Ole Björkqvist
Åbo Akademi, Vasa, Finland

I have been commissioned by the Swedish Committee for Mathematics Education and the National Center for Mathematics Education to register, analyse and describe the current state of Swedish research and development in mathematics didactics. An integral part of the task is putting emphasis on the strengths and weaknesses of the enterprise as well as providing suggestions for improvement. Let me start by stating that the task clearly would require insights that exceed what I have. I feel humble, and at the same time honored that my analysis, which must remain far from exhaustive, is valued by those who have given me the task.

This presentation describes some key characteristics of Swedish research in mathematics didactics as of today. As I consider it inseparable from traditional Swedish research and developmental work in mathematics education, I will try to avoid the pitfalls that come from too sharp a distinction between research in mathematics didactics and research in mathematics education. Let it suffice, at this moment, to see the use of the word *didactics* as primarily motivated by central European usage, and also as an indication of academic status, as compared with the broader connotations of the word education. Defining mathematics didactics may lead to slightly different results if you approach it from the discipline of pedagogy or from the discipline of mathematics.

I will try to give, at the same time, both national and international perspectives. Whenever there is a discussion of change – development of new patterns that can be discerned – the changes are seen with respect to the national circumstances. But Swedish research and developmental work in mathematics education is of course very much influenced by, and in continuous interaction with, the international community of researchers in mathematics education. From an international perspective, my position as a Nordic colleague will make me emphasize such characteristics of Swedish research as reflect shared Nordic values and needs. Sharing the Swedish language, too, I feel at the same time inside and outside that which I am to analyze.

There will be a follow-up seminar in Swedish during which there will be a chance to go into more detail than I do today. It is primarily intended as a chance for others to bring forward complementing views of the situation regarding research in mathematics education in Sweden today. These views will be carefully noted, and taken into account when my work goes into its next stage, documenting the findings in a written report. The final report will be presented later in the spring. I hope many of you will have a chance to participate in the seminar and in that way help me eliminate misunderstandings that I may have arrived at and add important facts that have escaped me to the final report. I would appreciate that kind of help in my work.

When I was given instructions by my commissioning agencies, it was agreed that a substantial part of the analysis would be based on the responses to two questionnaires that their representatives, Bengt Johansson and Gerd Brandell, and I worked out together. In addition, I was supplied with material collected by the National Center for Mathematics Education. I was not to collect research papers, nor to solicitate information beyond the scope of the questions in the questionnaires. Thus my analysis is largely based on self-reported facts, contrasted with material given to me and my previous general knowledge of Swedish research. The possibility of including a follow-up seminar was also visualized.

One of the questionnaires was directed to heads of academic departments, such as departments of teacher education, or similar departments within faculties of education, and departments of mathematics. Minor institutions of higher education, not having the status of university, also received the questionnaires. Lacking a specific academic department, such a regional college or institution of higher education still might have research or developmental work in mathematics education going on, and it was deemed important to learn as much as possible about its status within the college or institution through the questionnaire. It must be noted that the heads of department in most cases were people who did not work in mathematics education themselves. They were able to answer general questions regarding the present situation – if there are formal graduate programs, what kinds of financial support the department can offer, resources available, and plans for the future. They were not expected to answer more specific questions, and in some cases they seemed to have to rely completely on somebody else to answer most of the questions.

Another questionnaire was directed to individuals or groups of individuals within the departments, or individuals connected with the departments without being formally tied to them. This questionnaire probed the substance of research – themes and metods, the significance of it for society, national and international coworkers, etc. It was left open to the respondents to answer individually or to let the leader answer for the whole group.

In this way the two questionnaires, taken together, were expected to locate most of the people considering themselves as doing research, or planning to do research, in mathematics didactics. In addition, one would be able to identify what kind of networking there is within departments and between different departments in Sweden. This networking is sometimes described in terms of different "research environments" existing in parallel at different places in Sweden. These research environments may have come into existence through the influence of one or two prominent researchers, who have attracted newcomers to the field or made their colleagues interested in research. The word "research group" is not always applicable, as they vary in size and as within one research environment there might exist more than one group – and individuals only vaguely connected with each other. We might speak of groups of different order.

The questionnaires also give information about the degree of institutionalization of research in mathematics didactics – to what extent the departments explicitly state their intentions with respect to mathematics didactics, to what extent they plan to start graduate programs or participate in collaborations with other departments, etc. This aspect is of course central for mathematics didactics as an academic field, and it is something else than the existence of groups of researchers kept together by their specific research interests. Institutionalization based on financial support through public funds also is a sanction of the need for this particular kind of research, as society sees it.

There are difficulties connected with the interpretation of the answers to the questions in the questionnaires. The most important difference is that of scaling. What is seen as a research project by one respondent might be viewed as a minor study by another. Cooperation might mean anything from meeting three times a year to discussing common matters on a daily basis. People also differ in their estimates of what is possible to achieve. Overestimates might indicate optimism – in itself a positive trait that contributes to a good prognosis – but a coupling to realism seems necessary. My solution to these problems of interpretation has been recording the data as given by the respondents and using them as such. In summing up the situation for the whole country, there are adjustments as I implicitly give the responses different weight. Doing that is not a way of discrediting anybody´s answers – it is an attempt to rescale everything question by question, in order to increase the overall reliability of the analysis.

There is also a problem with cases of missing responses. It is easy for a questionnaire to get misplaced among the papers on the desk of a head of department, and it is also common for researchers to be so busy with their own work that they postpone their replies to questionnaires for a very long time. Reminders sometimes work. The last responses arrived as late as in December. And even so, there are a few institutions that are known to have important research in mathematics education which have not reported it themselves. In such cases, they have been taken into account via the other kinds of material available to me.

The most remarkable single fact, which in itself affects many of the responses, is the entry of the departments of mathematics into the field of research in mathematics didactics. Whereas didactics previously was, and still is, considered an integral part of the scientific discipline of pedagogy, those departments of mathematics who have entered the field of mathematics didactics see themselves able to do so without too much concern. This process has been speeded up significantly by a substantial grant from *Riksbankens Jubileumsfond* to start a national graduate school (or a "researcher school") in mathematics didactics. With money available to departments of mathematics, it was comparatively easy to find interest in the project, and the national graduate school is now well on its way. With some support from various other sources, added to the original grant to make it possible for even more students to participate, it will be an important factor in the next years to come.

Another source of public support for the whole sector of mathematics education is to be found in the establishment of the *National Center for Mathematics Education* in Göteborg. In this case it was not the result of a sudden turn of events, but rather a structure that was built upon previous work within the Göteborg mathematics education environment. My intention is not here to describe all the positive effects it is likely to have – but to emphasize that the image of mathematics didactics research in Sweden continues to be determined, at least partially, by the image teachers have of the results of the research. The National Center for Mathematics Education is an important agency in the dissemination of research results, and it also has the resources to bring in prominent foreign researchers with specialties not to be found in Sweden.

My point with respect to this is more general than that. On a national level, evaluating educational research cannot be separated completely from its possible connections to developmental work, and evaluating developmental work cannot be separated from the reality of the schools in the country. This goes for research in mathematics didactics, too. It has to find its channels for filtering into schools, and to do so it needs to be institutionalized in a way that establishes the necessary connections.

It is no secret that a great amount of research in mathematics education has been performed by academic teachers whose main occupation really is not research. The questionnaires bear out the fact that it continues to be so. Lecturers report doing research on a less than half-time basis, and there are very few positions that are principally intended for advanced research at the post-doctoral level. I see part-time research as no problem, provided the person is involved in other activities that promote the dissemination of research results or, e.g., acts as a mentor for younger researchers. In short, research in mathematics didactics cannot be a hobby that you do as long as you are specifically paid for it – it needs involvement of the kind that people have in most any other career. With many young people now preparing themselves in a systematic way, it is to be hoped that there actually will be careers available to them, when they reach their doctor´s degree. This is the time for some serious thinking about the need for professorships in mathematics didactics.

At the same time, I hope there will be no great reduction in the number of people doing research on a part-time basis. I motivate this with the previous argument – part-time involvement in research combined with part-time teacher education or part-time in-service training of teachers is a very strong combination. A country like Sweden will also need continuity in the production of textbooks for schools, and having researchers involved in that work is beneficial. All of this, taken together as a balance between actors in different roles, is a very well-functioning aspect of mathematics education in Sweden. In fact, the proportion of teachers coming into contact with modern research in mathematics didactics is probably higher in Sweden than in most any other country. The remarkable success of the Biennials in mathematics is a testimony to that – many of the lectures there are given by part-time researchers – and so is, particularly, the high proportion of elementary school teachers taking part in the Biennials. On the whole, if one is to mention one single area of strength in the Swedish mathematics education system, I would single out the outstanding participation in a common enterprise that involves both theory and practice. The challenge for research in mathematics didactics is to contribute to it in new ways without losing the power of the system as it is now.

I will mention one way of doing that. Turning the focus to developmental work, we notice that many individuals report developmental work as something they are involved in, and they are not really sure if it counts as research or not. This situation is found not only in Sweden. However, within the last fifteen years or so there have been formulated research paradigms that address exactly this problem (e.g. in the Netherlands, Germany and France). Typically this includes seeing mathematics didactics as a construction science, for which the products and their efficiency carry special importance. Adopting such a research paradigm retains the scientific respectability of the activity and justifies calling it developmental research.

The responses to the questionnaires also carefully avoid the term "matematikmetodik" (mathematics teaching methodology, as in mathematics methods courses). It seems evident that this nowadays is associated with an outdated approach to teacher education in mathematics, whereas mathematics didactics is the scientific approach. I see a problem with that. There is a certain risk that avoiding the term "methods" altogether might lead to a neglect of research that concerns teaching methods in mathematics. In fact, in the answers to the questionnaires there is no strong evidence for research that concerns the "how to do it" component of mathematics didactics, especially in situations that deal with typical large-size classrooms with a heterogeneous composition of students. That would be much needed research. But perhaps this is not a problem for Sweden alone. Research interests elsewhere follow trends, too, and presently there is an international emphasis on research focussing on, e.g., understanding the beliefs of students and teachers, individual behavior in problem solving situations, and descriptive studies of what is going on socially in classrooms.

The different themes for the research projects mentioned by the Swedish researchers in mathematics didactics vary quite a lot. Like Bergsten (2000), in a previous review of mathematics education research in Sweden, I find no clearly Swedish brand of mathematics didactics research. However, if one tries to take into account what is going on elsewhere in the world, it is comparatively easy to see that there are some areas in which Swedish researchers have reached some prominence. Among those I would like to mention the following

- Gender issues
- Qualities of learning under specific circumstances
- Phenomenographic analyses of mathematical conceptions

In addition, Swedish researchers have played a significant role in international collaborations concerning

- Mathematics education and democracy
- The history of mathematics and mathematics education

Some of these lines of research are still evident in the responses to the questionnaires. That kind of continuity is helpful to new researchers who want to reach the frontiers of research as quickly as possible. There are also indications that other fields of interests might be competitive in the near future. I would like to single out

- Computer-assisted learning and assessment
- Mathematical problem-solving behavior in upper secondary schools
- Symbol sense and the learning of the mathematical language

In accordance with my previous comments regarding the high quality of Swedish developmental work and the importance of developmental research there is also good reason to mention the development of

- New types of (national) tests in mathematics

The list of recent and promising areas of strength is by no means comprehensive. Beside the themes that have been listed one will find several lines of research pursued by individuals. They are doing high-quality work – not necessarily in cooperation with other researchers in Sweden, but rather in contact with people having similar interests abroad.

I would also like to mention that the former dominance of research on young children´s mathematics learning over secondary and tertiary mathematics learning appears to be changing. This is, of course, related to the kind of mathematical background that new researchers have. As long as this balance does not turn over completely, it is even to be welcomed. The diminished interest in research aimed at the lower ages is, however, a potential cause for some concern.

Among the reported themes of research, there are so far not many that take their starting point in a specific mathematical concept or procedure. The learning and teaching of mathematical topics that often cause difficulties is something one might expect research in mathematics didactics to deal with. This situation, in conjunction with my previous remark regarding the need for more research on the "how to do it" component of mathematics didactics, is an indication of an ever-so-slight tendency to avoid difficult research problems (even if they are important) in favor of research problems that are more easily tackled. This reminds me of an old piece of advice – if a problem is a real challenge, something that needs to be solved, it is always worth an effort, even if the tools that you have at your disposal to tackle it with are not yet as good as you wish they were.

I will now add some comments regarding the institutionalization of research in mathematics didactics in Sweden.

There were responses to the questionnaires from departments within the big universities, and at the other end there were reponses from very small regional institutions. In one or two cases the basic message was, "We have not developed mathematics didactics as a specialty within our department". Looking at the distribution over Sweden, the positive answers were approximately evenly distributed. This means, in itself, a northward shift, since earlier research in mathematics education generally has taken place farther south.

The effect of the grant from *Riksbankens Jubileumsfond* is clearly evident. The departments of mathematics within the universities prove to have the greatest readiness for doctoral programs in "mathematics with an orientation towards mathematics didactics". This should be contrasted with the the kind of departments that have produced dissertations in previous years – very often departments of pedagogy or departments of teacher education. Those departments now appear to be more hesitant to have mathematics didactics as a specialty. Yet that is something worth hoping for – at least one department of teacher education with a strong profile in mathematics didactics. This would make it possible for persons with a background in education to enter research in mathematics didactics in a natural way – perhaps with lower requirements in the form of courses in mathematics, but with a stronger background in appropriate research methods, and most likely with a research interest that relates to younger children.

It would be a pity if the departments that produce degrees in education are alienated from the process that is now taking place. To a very large extent they are the carriers of traditions, they are more knowledgeable with respect to methods of research, and they do have better resources in the form of libraries, which is evident from the answers to the questionnaires.

Another significant feature today is a remarkable willingness between departments to enter explicit collaborations in their development of mathematics didactics as a field of research. Even the big universities seem to be finding strength in this, and for the smaller regional colleges or institutions of higher education this is really the only way to become part of a formal doctoral program. The collaborations seem to be based partly on personal relationships, partly on geographical vicinity. A further kind of support system that several departments rely on is having foreign associated researcher colleagues.

In one of the most direct questions in the questionnaire, the departments were to rate their stage of development as a mathematics didactics environment. It is interesting to notice that the answers were very homogeneous. The majority of the responders chose the alternative "an environment that is being built up". This indicates that it might be appropriate to conclude that mathematics didactics in Sweden is indeed in a build-up stage, at least with respect to institutionalization.

The whole picture, however, is not that simple. My analysis has shown quite a bit of variety, the source of which is, of course, the individuals that do the work. I have seen a lot of enterprising spirit, a few cases of top-level international cooperation, but also departments without prospects of development within this particular field, at least at the moment.

From what I have brought forward, it must be clear that I do not fully accept the idea of a build-up stage that implies starting from nothing. Scientific knowledge always builds on previous knowledge, and part of the academic culture is acknowledging the wisdom that we use as guidance in our own search. Swedish research in mathematics didactics needs to do that, too. It should also constantly remember that a major condition for its survival is that it remains an organic part of the society that feeds it. That means being sensitive to the expectations of society and doing one´s best to meet those expectations.

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Evolving Assessment Practices: An International Perspective

Jan de Lange
Freudenthal Institute ,Utrecht

Introduction

We start with an assessment question for our readers. It is an extended response open-ended question format, and will be scored with partial credits by multiple marking: "Each country will have its own way of making the reform of introducing new material, of organizing the sequential study and of experimenting with possible programs. Channels should be provided for communicating the results of these programs and experiments between all of our countries, so as to enable us to use the best thinking of all countries in stimulating new ideas. The aim of all theses programs is twofold: firstly, to provide a better preparation for university study; secondly, to give to all students an instrument for use in daily life.

1. When was this recommendation made, and by whom?
2. Why did this recommendation lead to disaster?
3. What did we learn from this?

We will give only hints of answers, and as such not design a complete scoring rubric."

The above sensible sounding recommendation was made a long time ago, in 1959 during the Royaumont seminar held by the Organization for European Economic Cooperation OEEC (Fehr, 1961). The recommendations had widespread effect all over the western world, in many cases leading to completely new curricula, called Modern Mathematics, based on set theory.

In retrospect, the Royaumant conference, dominated by research mathematicians (Bourbaki), was the starting point of a road leading to nowhere, as far as kids were concerned. The background philosophy, according to the Belgian mathematician Servais, was that we must "rebuild the whole edifice from the foundations and erected in accordance with modern ideas. The modern ideas are mainly set theory" (Servais, 1968).

The fact that pedagogical principles had been ignored, was recognized by the American Beberman in the early sixties already. "I think in some cases we have tried to answer questions that students never raise and to resolve doubts they never had, but in effect we have answered our own questions and resolved our own doubts as adults and teachers" (Beberman, 1962). Or as an ardent supporter of the New Math, the Dutch math educator Vredenduin reflected many years later: “A beautiful edifice, but I do not think there was one student who shared that opinion” (Goffree, 1985).

Well, Modern Mathematics or New Math is something of the past now, even in Belgium where its most thoughtful supporters dominated the curricula until very recently. New mathematics has been replaced by New New-Math or Fuzzy Math, and the discussion of the fifties and sixties has been and still is being carefully reconstructed, using the same words and arguments and ignoring history and reason.

Changing Goals

Let’s return to the 1959 recommendation. The aim of all theses programs is twofold: firstly, to provide a better preparation for university study; secondly, to give to all students an instrument for use in daily life.

Living in the zero’s (2000) we can notice at least the shift in the order of recommendations: firstly, to give all students an instrument for use in daily life; secondly, to provide a better preparation for university study.

But most countries are now following a track that was indicated by the Cockroft Report in 1982 (Cockroft, 1982): firstly to become an intelligent citizen (mathematical literacy); secondly to prepare for the workplace and future education; thirdly to understand mathematics as a discipline.

It is interesting to note that the whole New Math movement has forced many people to rethink mathematics education – and not only along the line of Beberman’s critique. In the Netherlands for instance the recommendations were not accepted: what students need is students’ mathematics and not mathematicians’ mathematics. As Freudenthal put it: “Mathematics should never be presented to the students as a ready-made product (‘the edifice’). The opposite of ready-made, ‘dehumanised’ mathematics is human mathematics in *statu nascendi*” (Freudenthal, 1973). According to Freudenthal’s view one should recognize that the learner is entitled to recapitulate the learning process of mankind. This means an instruction not starting with the formal system, which in fact is the final product, nor with the embodiments, nor with structural games.

This philosophy has resulted in a theory of realistic mathematics education (RME) that has as key characteristics:

1. Starting in the real world to develop mathematical concepts.
2. Using the real world also as an area of applications.
3. Broad attention to mathematization, schematization, situation models etc.
4. Student construction and re-invention.
5. Student interaction.
6. Intertwining and integration of learning strands.

But in many countries, after struggling with the pitfalls of New Math, similar developments took place. Especially it was clear that problem solving in a real world context was a desired and necessary component of any math curriculum. And the goal of mathematical literacy became more and more prominent. In 1999 the Organization for Economic Co-operation and Development (OECD) published 'Measuring Student Knowledge and Skills' (OECD, 1999). This is a publication as part of a wide study in which the main goal is the assessment of reading, and mathematical and scientific literacy.

Mathematical Literacy

It is indeed striking to see and compare the 1959 report and this present OECD publication, and to reconstruct the shift in thinking about mathematics education.

We will follow this publication in defining Mathematical Literacy and then move on to the problems and consequences related to measuring the competencies related to mathematical literacy.

By mathematical literacy we understand an individual's ability to identify, to understand, to exert well-founded judgment about, and to act towards the roles that mathematics plays in dealing with the world (i.e. nature, society and culture), as needed for that individual's current and future private life, occupational life, social life with peers and relatives, and life as a constructive, concerned, and reflective citizen.

It will be clear that the shift from reproduction skills to process- and production skills – essential for real world problem solving as part of mathematical literacy – makes assessment a key variable. And even a very complex one. This became very clear when the Netherlands adopted an application rich curriculum for secondary school students in the mid-eighties. The processes like mathematization and modeling, not to speak about reflection and communication, forced designers, teachers, students and decision makers to rethink their own mental framework for mathematics assessment. (See details in De Lange, 1987). This forced rethinking process made clear that assessment was less a trivial variable in the process than was commonly thought with the more traditional curricula. And more than anything else the integration and alignment of the teaching-learning process and assessment became the focal point.

The principle that the first and main purpose of testing is to improve learning (Gronlund, 1968; De Lange, 1987) is widely and easily underestimated in the teaching/learning process. The reasons are multiple: design of fair, rich, open and creative tasks is very difficult, the way the feedback mechanism operates, the organization and logistics of an opportunity rich classroom etc.

But the publication 'Assessment and Classroom Learning' (Black and Wiliam, 1998), a literature study on classroom, states very clearly that improvement in classroom assessment will make a strong contribution to the improvement of learning. Therefore careful attention to the development and design of appropriate tasks is essential.

Assessment Changes

The NCTM Standards (1989) had a great influence in the discussion on reform in the US, and the NCTM recognized that ‘Assessment Standards’ were needed as well (1995). But Standards will not be enough: a focus on Standards and accountability that ignores the processes of teaching and learning in classrooms will not provide the directions that teachers need in their quest to improve. Nevertheless the NCTM Assessment Standards offer an excellent starting point for a discussion on principles and standards in (classroom) assessment. The Standards are about the mathematics, the learning of mathematics, equity and opportunity, openness, inferences and coherence, and represent quite a bit of the present common thinking internationally:

> ***Standard 1. Mathematics***
> *Few would argue with the assertion that useful mathematics assessments must focus on important mathematics. Yet the trend toward broader conceptions of mathematics and mathematical abilities, raises serious questions about the appropriateness of the mathematics reflected in most traditional tests since that mathematics is generally far removed from the mathematics actually used in real-world problem solving. Nevertheless, there is still much debate over how to define important mathematics and who should be responsible for doing so.*

This, of course, is a key issue. School mathematics is defined by long traditions resulting in a set of separate and often disconnected sub-areas that have little relation with the phenomenology of mathematics. Not only is that subdivision in strands rather arbitrarily but also the timing of each of them in the learning process without any reasonable argument. Furthermore we do not attempt to give a full picture of mathematics by any standard but there is no discussion which subject in school mathematics should be covered: take the long discussion and the slow progress for instance on the introduction of discrete mathematics in school curricula. Traditional assessment practices have emphasized this compartmentalization of school-mathematics. Common features of teacher’s formative assessment focuses on superficial and rote learning, concentrating on recall of isolated details, usually items of knowledge that students soon forget (Crooks (1988) and Black (1993) as summarized by Black and Wiliam, 1998). It is for this reason that PISA (OECD, 1999) has chosen to focus on ‘Big Ideas’ in mathematics: a cluster of related fundamental mathematical concepts ignoring the school curricula compartmentalization and that we try to assess broader mathematical ideas and processes.

> ***Standard 2. Learning***
> *New views of assessment call for tasks that are embedded in the curriculum, the notion being that assessment should be an integral part of the learning*

> *process rather than an interruption of it. This raises the issue of who should be responsible for the development, implementation and interpretation of student assessments. Traditionally both standardized and classroom tests were designed using a psychometric model to be as objective as possible. By contrast, the alternative assessment movement affords teachers much more responsibility and subjectivity in the assessment process. It assumes that teachers know their students best because teachers have multiple, diverse opportunities for examining student work performed under various conditions and presented in a variety of modes. When teachers have more responsibility for assessment, assessment can truly become almost seamless with instruction (Lesh & Lamon, 1992).*

It will be clear that we see assessment as an integral part of the teaching/learning process. It is actually so trivial that one is surprised to see that the actual practice is so different. The main cause for this situation is the standardized test system. The ironic and unfortunate result of this system is that teachers resist formal evaluation of all kinds, given the intellectual sterility and rigidity of most generic, indirect and external testing systems. But because of that resistance, local assessment practices are increasingly unable to withstand technical scrutiny: teacher tests are rarely valid and reliable and 'assessment' is reduced to averaging scores out (Wiggins, 1992).

We should offer teachers a wide array of instruments and opportunities for examining work performed under various conditions. Teachers need to be aware about the connections between the tests tools and the curricular goals and how to generate relevant feedback from the test results.

> ***Standard 3. Equity and Opportunity***
> *Ideally, assessments should give every student optimal opportunity to demonstrate mathematical power. In practice, however, traditional standardized tests have sometimes been biased against students of particular backgrounds, socioeconomic classes, ethnic groups or gender. Equity becomes even more of an issue when assessment results are used to label students or deny them access to courses, programs or jobs. More teacher responsibility means more pressure on teachers to be evenhanded and unbiased in their judgment. Ironically, the trend toward more complex and realistic assessment tasks and more elaborated written responses can raise serious equity concerns, since reading comprehension, writing ability and familiarity with contexts may confound results for certain groups.*

It is clear that the teacher has a very complex task here. Because we do not assess objectively a person, but we assess how a person acts in a certain setting. Certain formats favor boys more than girls, others are more equal, boys do better under time pressure than girls (De Lange, 1987), girls seem to fare better when there is more language involved, certain contexts are more suited for boys, others for girls (Van den Heuvel & Vermeer, 1999), cultural differences should be taken into account. It is for these reasons that we need to discuss the role of context in some detail, the effect of different formats and the need to use at least several of them, that we should discuss the need for a variety of representations, that we advocate individual and group work and restricted timed test and unrestricted time test. Only if we offer that wide variety do we have a chance to have 'fair' classroom assessment.

> ***Standard 4. Openness***
> *Testing has traditionally been quite a secretive process, in that test questions and answers were carefully guarded and criteria for judging performance were generally set behind the scenes by unidentified authorities. By contrast, many today believe that students are best served by open and dynamic assessment – assessment where expectations and scoring procedures are openly discussed and jointly negotiated.*

This is an almost trivial standard for assessment. Students need to know what the teachers expect from them and teachers should have examples of all the different tests that are possible or to be expected with scoring rubrics and why these tests are given for what reasons. Again tradition and existing practice have done much damage. Secrecy was a key issue when testing: secrecy as to the questions being asked, secrecy as to how the questions will be chosen, secrecy as to how the results will be scored, secrecy as to what the scores mean, secrecy as to how the results will be used (Wiggins, 1992). According to Schwarz (1992) standardized tests can be done on a widespread scale only if secrecy can be maintained since this testing technology requires a very large number of questions that are expensive and difficult to generate. But according to Schwarz this is an undesirable situation. He proposes new approaches to the filing, indexing, and retrieving of previously used problems. Publicly available, richly indexed databases of problems and projects provide opportunity for scrutiny, discussion, and debate about the quality and correctness of questions and answers. It seems that we have a long way to go, but openness and clarity are prerequisites for any proper classroom assessment system.

> ***Standard 5. Inferences***
> *Changes in assessment have resulted in new ways of thinking about reliability and validity as they apply to mathematics assessment. For example, when assessment is embedded within instruction, it becomes unreasonable to expect a standard notion of reliability to apply (that a student's achievement on similar tasks at different points in time should be similar), since it is actually expected that students will learn throughout the assessment. Similarly, new forms of assessment prompt a re-examination of traditional notions of validity. Many argue that it is more appropriate to judge validity by examining the inferences made from an assessment than to view it as an inherent characteristic of the assessment itself. Nevertheless, it is difficult to know how new types of assessment (e.g. student projects or portfolios) can be used for*

decision making without either collapsing them into a single score (thereby losing all their conceptual richness) or leaving them in their raw, unsimplified, difficult-to-interpret form.

Reliability and validity are concepts from an era when psychometricians ruled the waves. They have gotten a specific and narrow meaning and have caused much damage to the students and society and more specifically have skewed the perception of what constitutes good school mathematics. More important, especially in classroom assessment, is authenticity of the tasks, that is: the performance is faithful to criterion situations. Authentic means that the problems are ‘worthy’ and relate to the real world, non-routine, have ‘construction’ possibilities for students, relates to clear criteria, ask for explanation of strategies, offer possibilities to discuss grading.

In order to do justice to the students (which means free of distortion and let the object speak (Smaling, 1992)) and in a sense add validity in the traditional sense we need a sample of authentic tasks to get a valid picture. And, indeed, reliability in the traditional sense is something to be avoided at all times if we really want assessment as part of the teaching learning process. If we offer the students the same tests at different moments we should note differences in levels of formality, different strategies, different answers even in some cases. If the tests would yield the same results (and thus be reliable) our teaching has failed. It is exactly for this reason that a longitudinal study on the effects of a new middle school curriculum has four different operationalizations of the ‘same’ problem to find out about students growth over time in grades 5, 6, 7 and 8. (Shafer and Romberg, 1999). Smaling (1992) defined reliability in a more ecological way: reliability refers to the absence of accidental errors and is often defined as reproducability. But here it means virtual replicability. The emphasis is on virtual because it is important that the result is reported in such a way that others can reconstruct it. What is meant by this is aptly expressed by the term ‘trackability’, which, according to Gravemeijer (1994), is highly compatible with Freudenthal’s conception of developmental research. This is because ‘trackability’ can be established by reporting on ‘failures and successes’, on the procedures followed, on the conceptual framework and on the reasons for the choices made.

Standard 6. Coherence

The coherence standard emphasizes the importance of ensuring that each assessment is appropriate for the purpose for which it is used. As noted earlier, assessment data can be used for monitoring student progress, making instructional decisions, evaluating achievement, or program evaluation. However the types of data appropriate for each purpose may be very different. Policy makers and assessment experts often disagree on this issue. The former may have multiple agendas in mind and expect that they can all be accomplished by using a single assessment, while the latter warn against using an assessment for purposes for which it was never intended.

Coherence in classroom assessment can be accomplished quite simple if the teaching/learning process is coherent and the assessment is an integral part of it. Teachers have a wide variety of techniques and tools to their disposal to ‘design’ their own classroom assessment system that fits with the didactical contract they have with the classroom. Depending on their teaching/learning practice and style, they will present the students with their ‘balance’ within the classroom assessment system. Coherence with colleagues will be achieved by sharing the same criteria and possibly by including the same ‘end-of-the-year test’. Thus ‘fairness’ for all students in the same year and over the years is ensured as the end-of-the-year tests are not secret, but just change over the years.

Principles for assessment

Reflecting on these Standards and the existing literature we make the following list of Principles for (Classroom) Assessment:

1. The main purpose of classroom assessment is to improve learning (Gronlund, 1968; De Lange, 1987; Black & Wiliam, 1998, and others).
2. The mathematics is embedded in worthwhile (engaging, educative, authentic) problems that are part of the students real world.
3. Methods of assessment should be such that they enable students to show what they know rather than what they do not know (Cockroft, 1982).
4. Multiple and varied opportunities (formats) for students to display and document their achievement (Wiggins, 1992).
5. Tasks should operationalize all the goals of a curriculum (not just the ‘lower’ ones). Helpful tools to achieve this are performance standards, including indications of the different levels of mathematical thinking (De Lange, 1987).
6. Grading criteria are published and consistently applied; including examples of earlier grading showing exemplary work and less than exemplary work.
7. Minimal secrecy in testing and grading.
8. Genuine feedback to students.
9. The quality of a task is not defined by its accessibility to objective scoring, reliability or validity in the traditional sense, but by its authenticity, fairness and meeting of the above principles (De Lange, 1987).

These principles form a ‘checklist’ for teachers taking their classroom assessment seriously. But from principles to practice can be a long way. So we will now turn to a discussion about several key issues in designing and implementing a classroom assessment system.

Competency Levels

One of the key issues that has been proven successful when implemented, is classification of competence levels in mathematics.

They were successfully operationalized in the National Dutch option of TIMMS (De Lange & Boertien, 1994; Kuiper, Bos & Plomp, 1997) and the ongoing longitudinal study on the effects of a middle school curriculum (Shafer & Romberg, 1999) and also have been adapted for the earlier mentioned OECD study. The three levels are:

- Reproduction, Procedures, Concepts and Definition.
- Connections and Integration for Problem Solving.
- Mathematization, Mathematical Thinking, Generalization and Insight.

We will elaborate on these levels next:

Level 1: Reproduction, Procedures, Concepts and Definition.

At this level we deal in the first place with the matter dealt with in many standardized tests, as well in comparative international studies, operationalized mainly in multiple-choice format. It concerns knowledge of facts, representing, recognizing equivalents, recalling mathematical objects and properties, performing routine procedures, applying standard algorithms and developing technical skills. Also the dealing and operating with statements and expressions containing symbols and formulas in 'standard' form relates to this level.

Not seldom items at this level are multiple-choice, fill in the blank, matching or (restricted) open ended questions format.

Level 2: Connections and Integration for Problem Solving.

At this level we start making connections between the different strands and domains in mathematics: integrating information in order to solve simple problems, where students have a choice of strategies and a choice in the use of mathematical tools. Although the problems are supposedly non-routine, they require relative minor mathematization.

At this level, students are also expected to handle different ways of representation, according to situation and purpose. The connections aspect also requires students to be able to distinguish and relate different statements like definitions, claims, examples, conditioned assertions and proof.

From the mathematical language point-of-view decoding and interpreting symbolic and formal language and understanding its relations to natural language forms another aspect at this level.

Items at this level are often placed within a context, and engage students in mathematical decision-making.

Level 3*: *Mathematization, Mathematical Thinking, Generalization and Insight.

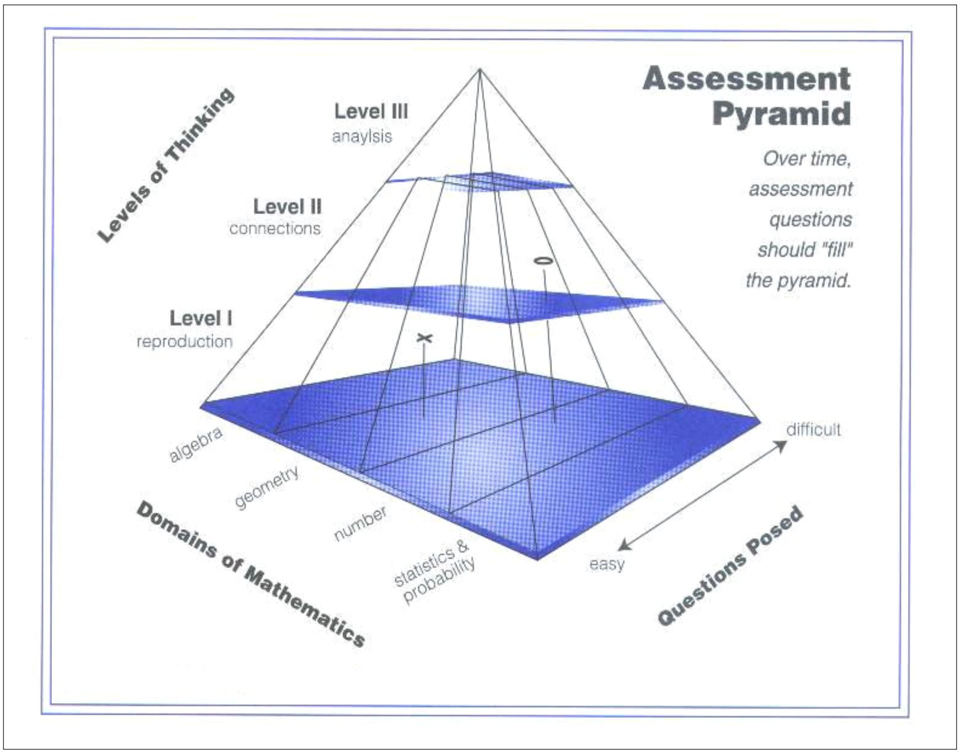
At this level students are asked to mathematize situations: recognize and extract the mathematics embedded in the situation and use mathematics to solve the problem. Analyze, interpret, develop own models and strategies, and make mathematical arguments, including proofs, and generalizations. These competencies include a critical component and analysis of the model and reflection on the process. Students should not only be able to solve problems but also to pose problems.

All these competencies function only well if the students are able to communicate properly in different ways: orally, written, visualizations etc. Communication is meant as a two-way process: students should also be able to understand communication with a mathematical component by others. Finally we would like to stress that students need also insight competencies. Insight in the nature of mathematics as a science, including the cultural historical aspect and understanding of the use of mathematics in other subjects as brought about through mathematical modeling.

The competencies at this level quite often incorporate skills and competencies at different levels. This is almost evident, like the fact that the whole exercise of defining the three levels is a somewhat arbitrary activity. There is no clear distinction between the different levels, and higher levels not seldom play out at different levels.

This level, which goes to the heart of mathematics and mathematical literacy, is difficult to test. Multiple-choice is definitely not the format of choice here. Extended response questions with multiple answers (with (super-) items or without increasing level of complexity) are more likely to be promising formats. But both the design and the judgment of student answers are very, if not extremely difficult.

The three levels can be visually represented in a pyramid (De Lange, 1995).



This pyramid has three dimensions or aspects: the first is the content or domains of mathematics, the second contains three levels of mathematical thinking and understanding (along the lines defined here fore) and the third aspect is from simple to complex. The dimensions are not meant to be orthogonal and the fact that the pyramid is used as a means of visualization, tries to indicate the relative number of items required to give a fair image of a student's understanding of mathematics. As we need only simple items for the lower levels, we can use more of them in a short amount of time. For the higher levels we need only a few items, as it will take some time for the students to solve the problems.

The simple to complex dimension can be extended to include a dimension that takes into account the variation from informal to formal.

All assessment questions can be located in the pyramid according to the level of thinking called for, mathematical content or big ideas domain, and for degree of difficulty. Because assessment needs to measure and describe a student's growth in all domains of mathematics and at all three levels of thinking, questions in a complete assessment program should fill the pyramid. There should be questions at all levels of thinking, of varying degrees of difficulty, and in all content domains.

As an example of a Competency Class 2 problem we show a released item of the PISA project (OECD, 2000):

> You are asked to design a new set of coins. All coins will be circular and colored silver, but of different diameters.
> Researchers have found out that an ideal coin system meets the following requirements:
>
> - Diameters of coins should not be smaller that 15 mm and not be larger than 45 mm.
> - Given a coin, the diameter of the next coin must be at least 30% larger.
> - The minting machinery can only produce coins with diameters of a whole number of millimeters (e.g. 17 mm is allowed, 17.3 is not).
>
> Design a set of coins that satisfy the above requirements. You should start with a 15 mm coin and your set should contain as many coins as possible.

According to the OECD publication this is a class 2 problem, that shows the constructive and finite use of mathematics. Some degree of modeling and argumentation, and symbolic, formal and technical skills are required, although the calculation skills required are definitely of the lowest competency level. Mathematization is called for in order to translate the problem from natural language into more mathematical language.

The format adapted is that of an open-constructed response item, a format that has become increasingly popular because of its possibilities for relative simple construction and accessibility to partial credit scoring and insight into students' reasoning.

The design of open-constructed response items (also called open-open questions) might be relatively simple in relation to more complex problem solving tasks; it is still a challenging and sensible process. We conclude this article with a concrete example of the design process.

The challenges of designing open-open assessment

(Examples from the MiC algebra unit Expressions and Formulas)

In the times of traditional mathematics instruction, designing a test was not a major challenge for a teacher. Look at the exercises in the textbook, take some simple, intermediate and difficult problems, change the coefficients and/or signs, and there you go. Those rules are not valid anymore in real world oriented, context rich mathematics instruction that builds on students' present knowledge and follows a path from informal to formal mathematical concepts. It is not always obvious for teachers or for students that the rules have changed.

In earlier experiments in The Netherlands, teachers were asked to design tests with a unit (on matrices) that was designed with the philosophy of Realistic Mathematics Education in mind (De Lange, 1987). It was noted that eighty percent of the problems designed, were either without context at all (and representing the traditional view on mathematics) or very similar to the problems in the unit of instruction, both in context and content. Only one out of every six problems was 'different' from the problems in the unit and tried to operationalize the more process-oriented goals and transfer from one context to another.

Sometimes the results are very confusing and deceiving when one follows the old 'rules' in the new situation, using a reformed curriculum. For example, one of the teachers used a beautiful problem on the growth of rat populations from the unit (to be solved with matrices) and changed it somewhat to make it into an assessment problem: the rats were changed into rabbits, and the numbers were changed slightly. The 'build-up' of the assessment problem was almost the same as in the original problem. Although the original rat problem was considered quite complex and difficult, the results on the test were astonishingly good: one student did not write down anything at all, one student got a six (out of ten) and all other students scored eight or higher, six students even a perfect ten.

Of course the meaning of these scores has to be considered marginal: the students just 'recognized' the problem and were able to 'reproduce' the solution, maybe even understanding their solution.

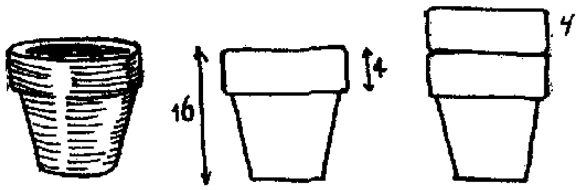
In a more recent experiment in an inner city school in the US, a teacher designed an assessment problem to go with the *Mathematics in Context*-unit 'Expressions and Formulas' (WCER & Freudenthal, 1997). The difference with the teachers in the Dutch experiments was that this teacher did not copy the 'build-up', but concentrated on the final, more formal questions. An assumption that the teacher made was that the information could be rather scant and that pictures could be left out, as the students would recognize the problem from similar problems from the unit. The results of the assessment problem were rather disastrous. Quite a number of the lower achievers did not even try to start solving the problem and only the very best were able to solve the assessment problem successfully. A classroom discussion afterward revealed the difficulties the weaker students (including ESL) had had with the assessment: they did not (try to) understand the problem since it was too mathematical and complex.

Following the disappointing experience described above the teacher, together with the designers of the curriculum, redesigned the assessment problem. The new problem can be characterized as follows: it consists of one single, rather simple question in plain English, supported and illustrated appropriately for those students that are more visually oriented or have problems with the English language. The new problem is shown in the following picture:

Name: ______________________________ Period: __________

Expressions and Formulas Test

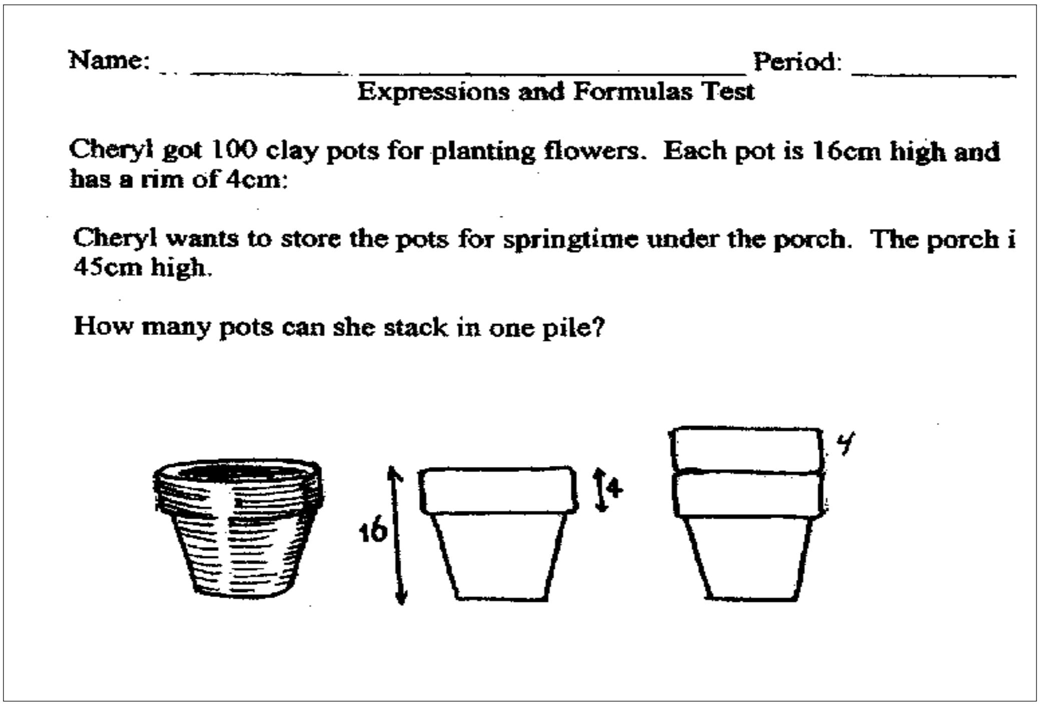
Cheryl got 100 clay pots for planting flowers. Each pot is 16cm high and has a rim of 4cm:



Cheryl wants to store the pots for springtime under the porch. The porch is 45cm high.

How many pots can she stack in one pile?

Regrettably the teacher made a small error in the layout in the version as it was presented to the students: the illustration followed the text instead of being inserted in the text.


Name: __________ ____________________ Period: __________

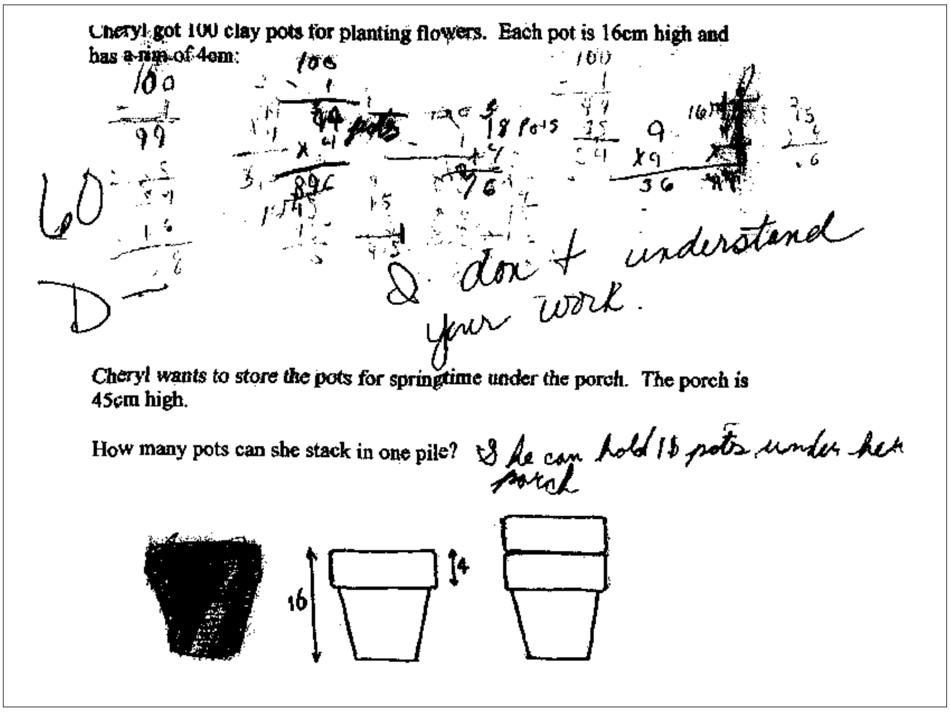
Expressions and Formulas Test

Cheryl got 100 clay pots for planting flowers. Each pot is 16cm high and has a rim of 4cm:

Cheryl wants to store the pots for springtime under the porch. The porch i 45cm high.

How many pots can she stack in one pile?

Although the error made the results of the assessment as such without much value, it shows how students are trained for tests. The moment they see an open space, many students start writing down answers even though no questions are asked at that point. An example of student work to illustrate this phenomenon is shown in the picture below.


Cheryl got 100 clay pots for planting flowers. Each pot is 16cm high and has a rim of 4cm:

60

D-

I don't understand your work.

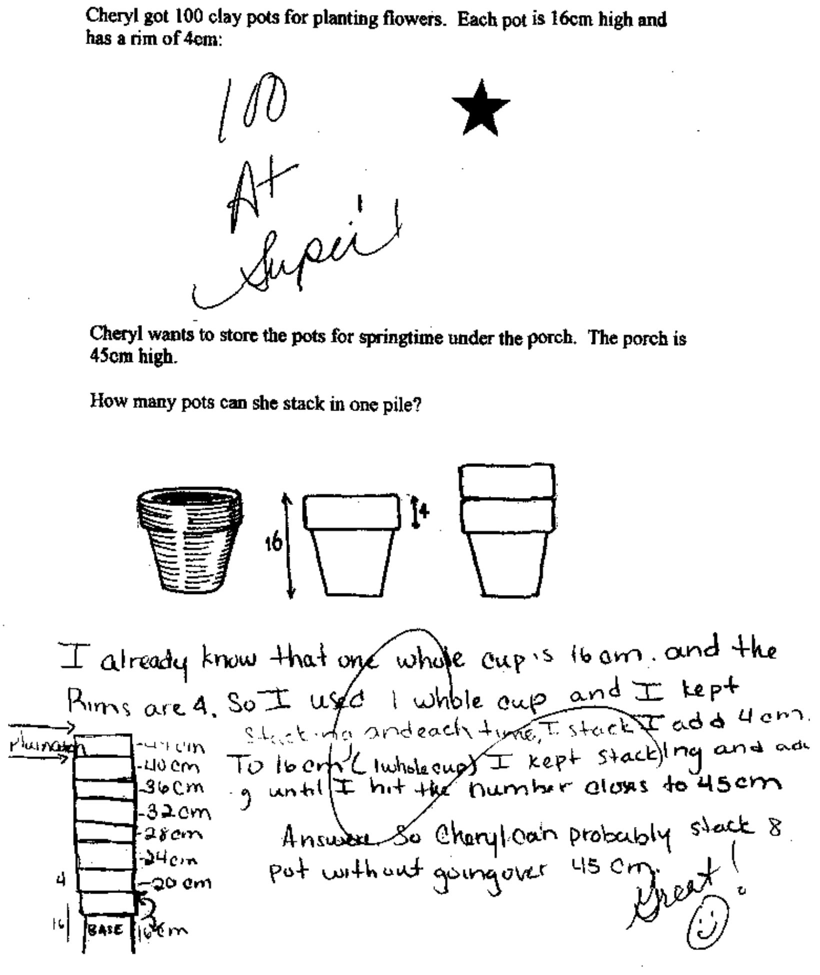
Cheryl wants to store the pots for springtime under the porch. The porch is 45cm high.

How many pots can she stack in one pile? She can hold 18 pots under her porch

It may come as no surprise that some of the students' work led to the teacher's reaction: "I don't understand your work." That is exactly what these students must have thought when looking at the problem.

Nevertheless, many students were successful in tackling the problem. All students at least tried to solve the problem (unless they were put on the wrong track by the layout error) the reasons for this being the use of plain English and the use of a supporting picture. It was also helpful that the picture provided was similar to pictures students had seen before in the unit. Although the assessment problem consisted of only one very straightforward question, students' solutions were not so straightforward.

Students who solved the problem more or less successfully demonstrated at least four distinct strategies. The most informal solution is based on drawing. A second strategy is of a very arithmetic nature. Finally students used two strategies that can be qualified as being of a pre-algebra nature: using arrow language and using the tree representation. Examples of each of these strategies are shown in the following figures.

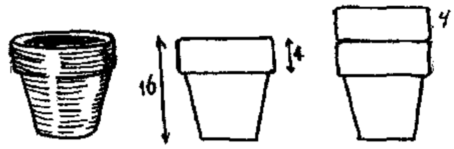


Cheryl got 100 clay pots for planting flowers. Each pot is 16cm high and has a rim of 4cm:

★ 90 A

Cheryl wants to store the pots for springtime under the porch. The porch is 45cm high.

How many pots can she stack in one pile?



She can only stack 7 pots in one pile. The way I got my answer was because I added . I added 16cm plus 4cm, 7 times and I came out with 44cm. Then I added 16 plus 4 again and I came out with 48cm. And the porc is only 45cm so that is to much.

What about the first pot with a height of 16cm?

Cheryl got 100 clay pots for planting flowers. Each pot is 16cm high and has a rim of 4cm:

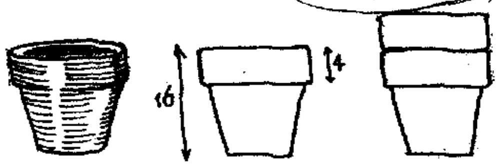
99 × 4 = 396

396 + 16 = 412 cm. high

95 A

Cheryl wants to store the pots for springtime under the porch. The porch is 45cm high.

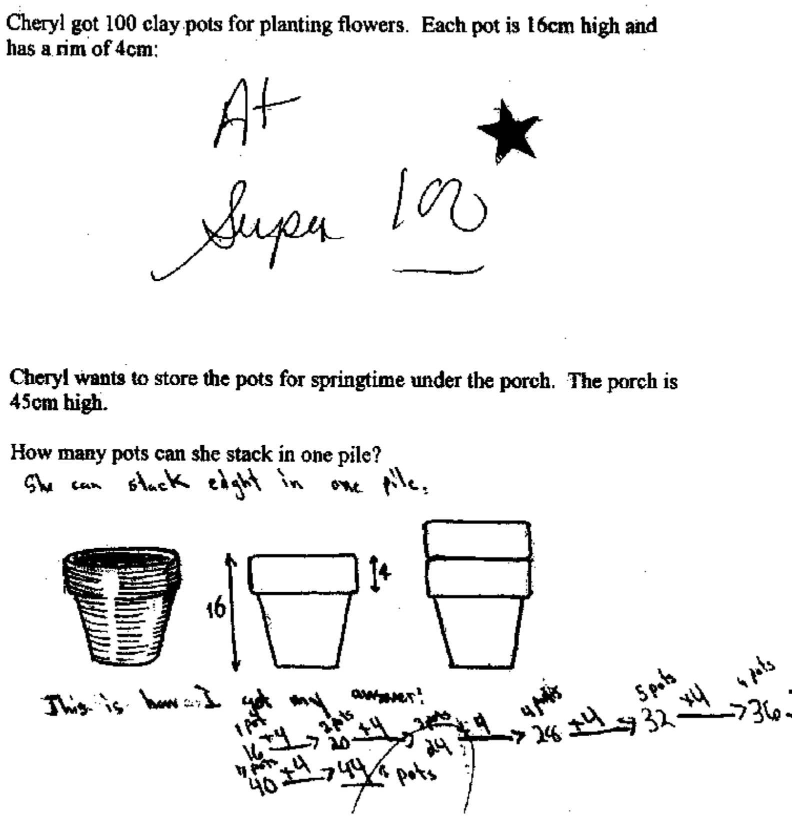
How many pots can she stack in one pile? 8 cups



7 × 4 = 28

28 + 16 = 44 cm. high

Good



The test was considered a success by designers, teacher and students alike. All students engaged in the problem, and the openness of the question resulted in a variety of successful strategies. These strategies gave rise to very valuable feedback to the students, which in turn supported and enriched their learning process. Reflecting on the design process, the strong points of the assessment problem are: plain English language; visual support; straightforward question and the possibility of solving by ‘common sense’ but not limited to common sense. On the contrary, interesting pre-algebra solutions gave rise to a new learning experience. Which shows how assessment can be a part of the students’ learning process.

Conclusion: for active, constructing assessment expertise the challenge is to develop judgment about what makes a test: accessible, understandable, fair in the use of representations and open to more strategies. (See also Feijs & De Lange, 2000).

Reflection

We started our short discussion in 1959 and compared an important and influential report with the present situation and trends that, in our opinion, are somewhat paradigmatically reflected in the 1999 OECD report on mathematical literacy. Goals have changed from mathematics as a discipline towards the functionality of mathematics in society, skills have moved from reproduction to construction, reasoning and (some) insight. These changes have put great emphasis on new ways to assess mathematics. The road has confronted us with major bumps and hurdles, but with even more exciting promises.

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There Is More to Discourse Than Meets the Ears:

Looking at Thinking as Communicating to Learn About Mathematical Learning[1]

Anna Sfard
The University of Haifa

Posing the question: What is it that changes when one learns mathematics?

In a weekly commentary that appeared in one of the American newspapers toward the opening of the current school year, the commentator, Crispin Sartwell, alerted parents to the dangers of the school, the most serious of them being the study of mathematics. He stated:

> "Mathematics is a sort of … pagan religion. It has no basis in fact or in theory. It is concerned entirely with entities of which it has no clear conception."

After this, his conclusion does not come as a surprise:

> "To expose kids to mathematics is the height of irresponsibility."

In this talk I hope to be able to show where beliefs such as those expressed by Sartwell come from and how a good teaching may help to show them wrong.

But all this in a due time. Let me first say a few general words on the topic of this talk. These days, the terms *discourse* and *communication* seem to be on everybody's lips. They feature prominently in research papers, they can be heard in teacher preparation courses, and they appear time and again in a variety of programmatic documents that purport to establish instructional policies (see e.g. *Principles and standards for school mathematics*, NCTM, 2000). All this could be interpreted as showing merely that we became as aware as ever of the importance of classroom conversation for the success of learning. In this talk, I will try to show that there is more to discourse than meets the ears, and that putting communication at the heart of education is likely to change not only the way we teach but also the way we think about learning. Above all, I will be arguing that communication should be viewed not as a mere aid to thinking, but

[1] This is a slightly modified version of the talk given at Psychology of Mathematical Education – North American Chapter 2001 annual meeting and published in Speiser, R., Maher, C.A., & Walter, C.N., Eds. (2001), Proceedings of the 23rd Annual Meeting of PME-N, vol. 1, 23-45. Columbus, OH: ERIC, Clearing House for Science, Mathematics, and Environmental Education. The modified text appears here with the permission of the PME-NA proceedings editors.

as almost tantamount to the thinking itself. The *communicational approach to cognition*, which is under scrutiny in this talk, is built around this basic theoretical principle.

To begin with, let us pay a brief visit to two classrooms where learning of a new mathematical topic has just started. The first class is being introduced to the concept of negative number. The teacher takes her place in the front of the group of twelve-year old seven graders and initiates the conversation.

Episode 1: The first lesson on negative numbers (from the study with Sharon Avgil)

[N1] Teacher:	Have you ever heard about negative numbers? Like in temperatures, for example?
[N2] Omri:	Minus!
[N3] Teacher:	What is minus?
[N4] Roi:	Below zero.
[N5] Teacher:	*Temperature* below zero?
[N6] Sophie:	Below zero… it can be minus five, minus seven… Any number.
[N7] Teacher:	Where else have you seen positive and negative numbers?
[N8] Omri:	In the bank.
[N9] Teacher:	And do you remember the subject “Altitude”? What is *sea level*?
[N10] Yaron:	Zero
[N11] Teacher:	And above sea level? More than zero?
[N12] Yaron:	From one meter up.

Since we are interested in learning, and learning means change, we may analyze this episode by trying to describe the modifications that have yet to occur in the children’s ways of dealing with the negative numbers. At the first sight, this future learning is not just a matter of a change; rather, it requires creating something completely new. The children, although not entirely ignorant of negative numbers, can do little more at the moment than associate the topic with certain characteristic terms, such as *minus* or *below zero*. It seems, therefore, that they will have to work on the subject almost from scratch. To put it in the traditional language, we may say that the children are yet to *acquire the concept of negative number* or *to construct this concept* for themselves.

Let me now turn to another episode, in which two first graders, Shira and Eynat, begin learning some basic geometry. The girls are first shown a number of geometrical figures, three of which are presented in Figure 1 below, and are asked by the teacher to mark those that can be called triangles. Once the task is completed, the following conversation between the girls and the teacher takes place[2].

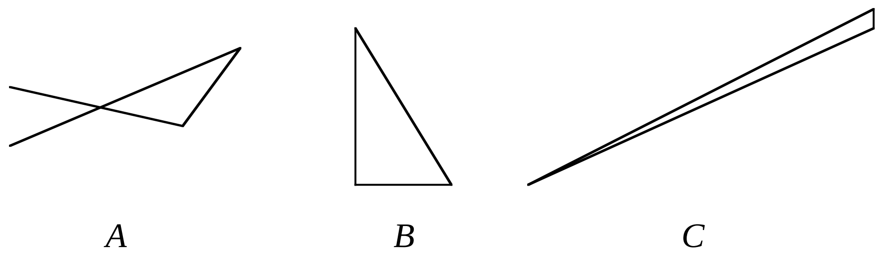

Figure 1. Which of these shapes is triangle? None of these shapes was marked by the girls as being a triangle.

Episode 2: The first meeting about triangles
(from the study conducted with Orit Shalit-Admoni and Pnina Shavit)

[T1] Eynat:	[*Pointing to shape A*] This is a triangle but it also has other lines.
[T2] Teacher:	Well, Eynat, how do you know that triangle is indeed a triangle?
[T3] Eynat:	Because it has three..aah…three… well.. lines.
…….	…….
[T22] Teacher:	[*Pointing to shape B*] This one also: one, two three..
[T23] The girls:	Yes
[T24] Teacher:	So, *is* it a triangle? Why didn't you mark it in the beginning?
[T25] Eynat:	'Cause then… I did not exactly see it.. I wasn't sure [*While saying this, Eynat starts putting a circle also around shape C*]
…….	……….
[T28] Shira:	[*Looking at shape C that Eynat is marking*] Hey, this is not a triangle. Triangle is wide and this one is thin.
[T29] Eynat:	So what? [*but while saying this, she stops drawing the circle*]
[T30] Teacher:	Why? Why isn't this a triangle [*points to shape C*]? Shira said it is too thin. But haven't we said…
[T31] Eynat:	There is no such thing as too thin. [*but while saying this, she erases the circle around shape C*]
[T32] Teacher:	Triangle -- must it be of a certain size?
[T33] Shira:	Hmmmm…Yes, a little bit… It must be wide. What's that? This is not like a triangle – this is a stick!

[2] These data are from a study conducted with Orit Shalit-Admoni and Pnina Shavit.

Here, unlike in the case of negative numbers, the students are already well acquainted with the mathematical objects in question, the triangles. And yet, neither the way they speak about these shapes nor the manner in which they act with them are fully satisfactory from the teacher's point of view. In her search for triangles, Shira disqualifies any shape that seems to her too thin. Eynat, even though apparently convinced that "there is no such thing as too thin", still cannot decide whether the stick-like shape in the picture is a triangle or not. The teacher will be eager to induce some changes in the ways the children think. Like before, we can describe this new change in terms of concept acquisition and conceptual change: We can say that the children face the formidable task of *overcoming their misconceptions about triangles*.[3]

In this talk, I reformulate this last statement after introducing a somewhat different way of talking about learning. My preference for the framework that will be called *communicational* stems mainly from the conviction that theories which conceptualize learning as personal acquisitions can tell us only so much about the complex phenomenon of learning. The acquisitionist approach relies heavily on the idea of cognitive invariants that cross cultural and situational borders. And yet, as has been convincingly argued by many scholars (e.g. Lave, 1988; Cole, 1996), human learning is too sensitive to its social, historical, cultural and situational context to be fully captured in a set of very general universal rules. In fact, my point of departure in this talk is that most of our *learning is nothing else than a special kind of social interaction, aimed at modification of other social interactions and practices*. Thus, rather than looking for those personal properties that can be held responsible for the constancy of an individual behavior, I am opting for a framework that allows me to stay tuned to the interactions from which the change arises. Let me add however, that my choice of the framework should not be interpreted as a rejection of the long-standing acquisition metaphor, but rather as an attempt to complement it while, at the same time, making explicit the metaphorical status of any theory.

Communicational approach to learning

Let me go back to the two episodes we have just seen in an attempt to be clearer about the change we expect to occur as a result of learning. While listening to the two brief conversations between the children and their teachers we had good reasons to wonder about the quality of the communication that was taking place. In the first scene, although it was obvious that the children were already familiar with the key term *negative number*, it was also clear that they could not say much about the topic of the exchange. They could not even formulate a proper sentence. We can say that at this point, the students could identify the discourse

[3] Alternatively, in this latter case we may say, inspired by Vygotsky (1987), that the teacher tries to help children in making the transition from *spontaneous* to *scientific* concept of triangle.

on negative numbers when they heard it, but they were not yet able to take a part in it. In the second episode the situation, although different, still asked for a change. True, the children eagerly participated in the discourse on triangles; and yet, the way they did this was unlike that of their teacher.

It is important to note that while introducing children to new ways of communicating seems to be the teacher's principal goal, the work never starts from zero. Whether the discourse to be taught is on negative numbers or on triangles, it will be developed out of the discourses in which the children are already fluent.

If so, we can define learning as the *process of changing one's discursive ways in a certain well-defined manner*. More specifically, a person who learns about triangles or negative numbers alters and extends her discursive skills so as to become able to communicate on these topics with expert interlocutors. The new discourse may be expected to make it possible to solve problems that could not be solved in the past.

At this point somebody may object and say that there is more to learning than modifying communication. Learning, the critic would say, is first and foremost about changing the ways we *think*, and the issue of how we communicate this thinking, although important, is still of only secondary significance. Let me then argue that thinking has not been excluded from my communicational account of learning. This point becomes immediately clear when we realize that the split between thinking and communicating is deceptive, and that *thinking is a special case of the activity of communicating*[4]. Indeed, a person who thinks can be seen as communicating with herself. This is true whether the thinking is in words or in images. Our thinking is clearly a dialogical endeavor, where we inform ourselves, we argue, we ask questions, and we wait for our own responses. If so, becoming a participant in a mathematical discourse is tantamount to learning to *think* in mathematical ways.

Asking what the children have yet to learn is now equivalent to inquiring how students' way of communicating should change if they are to become skilful participants of mathematical discourse on triangles and negative numbers. Clearly, our younger students Eynat and Shira have to modify their use of the keyword *triangle*. As I will be trying to show later, this seemingly superficial change is, in fact, quite profound and by no means easy to implement. The change of word use depends on modification of certain deeply rooted discursive habits.

In the case of negative numbers, even more far-reaching changes are required. The students will have to extend their vocabulary and to learn to operate with such new terms as "negative two" or "negative three and a half".

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[4] The communicational approach presented in this talk is similar to, although not identical with, the *discursive psychology* promoted, among others, by Harre & Gillett (1994) and by Edwards (1997).

Unlike in the case of triangles, where one can identify the object of talk with the help of pictures, the students will now need a new, specially designed visual means to mediate the communication. Some special symbols and geometric models will soon be introduced. Like in the case of triangles, a change will also be required at the meta-discursive level[5].

Let me generalize the above observations. The analysis of the two episodes has shown that in both cases the children's present discourse differs from typical school discourse along at least three dimensions:

- its *vocabulary*, that is, words and their discursive use
- *mediators*, that is the visual means by which the communication is mediated
- the *meta-discursive rules* that navigate the flow of communication and tacitly tell the participants what kind of discursive moves would count as suitable for this particular discourse, and in what situations this discourse would be applicable and helpful.

Thus, if learning mathematics is conceptualized as a development of a mathematical discourse, to investigate learning means getting to know the ways in which children modify their discursive actions in these three respects. In the rest of this talk I will be analyzing the manner in which the required change can take place. While doing this I hope to show that adoption of the communicational approach to cognition is not an idle intellectual game and that it influences both our understanding of what happens when children learn and our ideas about what could be done to help students in this endeavor.

How do we create new uses of words and mediators?

According to the popular, commonsensical vision of the sequence of events that take place in the course of learning, the student must first have an idea of a new mathematical object, then give this idea a name and, eventually, he or she must also practice its use. This picture of learning may well be the one that underlies the popular pedagogical belief in the primacy of conceptual understanding over symbolization and skill (see e.g. Hiebert & Carpenter, 1992).

Conceptualization of learning as an introduction to a discourse leads me to doubt this popular model and makes the case for a different course of learning.

[5] Note that my present description of the required change is quite similar – one can say isomorphic – to the one that could be given based on the van Hiele theory of the development of geometrical thinking (van Hiele 1985). Still, the two descriptions are put apart by their different epistemological/ontological underpinnings: While van Hiele's analysis, firmly rooted in the Piagetean framework, would produce a story of mental schemes, the present description is the description of students' ways of communicating. What makes the latter version qualitatively different from the former one is that it presents the development of a child's geometrical thinking as a part and parcel of the development of her communicational skills, and thus makes salient the principal role of language, of contextual factors and of social interaction.

Let us take the learning of negative numbers as an example. I will be arguing now that introduction of new names and new signifiers is the beginning rather than the end of the story. First, let me show the virtual impossibility of teaching a new discourse without actually speaking about its objects from the very first moment. Let us have a look at the way in which negative numbers are introduced in a school textbook.

> Let's choose a point on a straight line and name it "zero." Let's choose a segment and call it "the unit of length." Let's place the unit head-to-tail repeatedly on the line to the right of the point "zero." *The points made this way will be denoted by 1, 2, 3 and so on …*
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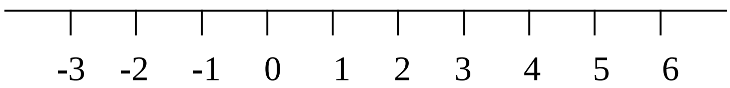
> 
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>
> To the left of the point "zero," we put the unit segment head-to-tail again and *denote the points obtained in this way with numbers -1, -2, -3,... The set of numbers created in this way is called the set of negative numbers*.

Figure 2. From a school textbook (Mashler, M., 1976, *Algebra for 7th grade*). Translated from Hebrew.

You don't have to read all that is written here to realize that the crux of this definition is in the interesting conceptual twist: points on the number line are marked with decimal numerals preceded by dash and, subsequently, they are called *negative numbers*. One may wonder how this verbal acrobatics – giving *new names* to *points* and saying these are *numbers* – can enable the child an access to a discourse on the negatives. At the first sight, the learning sequence that begins with giving a new name to an old thing seems somehow implausible. And yet, such order of things may be inevitable, and it may also be more effective than we tend to think.

It is *inevitable* because if we wish to initiate children to a discourse on new objects, we already have to use this discourse. The objects of the discourse must thus be identified in one way or another, in words or symbols. This is probably why the teacher in the first episode could not refrain from using words like "negative numbers", "minus two", etc. while introducing the topic for the first time. Clearly, she felt compelled to do it in spite of the fact that the children had little idea abut the uses to which these words can be put.

The proposed order of things in the process of learning is also more *effective* than we tend to think because of the simple fact that the new objects – the negatives – have been associated, and introduced with, the word *number*. The familiar notion evokes in the student expectations with respect to the possible uses of the new signifiers. For better or worse, the children seem to know quite a lot about this something to which they might have been exposed through a single sentence.

To see the workings of the former discursive experience in a new context, let us return to our seven graders learning about the negative numbers. In the new classroom scene that follows, we can see how the expectations evoked by he word *number* help the students find their way into the new discourse. At the present stage, three weeks and sixteen one-hour meetings later, the children already know how to add signed numbers and are trying to figure out for themselves how to multiply a positive by a negative. First, they do it in small groups. In one of these groups, the following exchange takes place after the teacher asked what 2·(-5) could be equal to:

Episode 3: The teacher asked what 2·(-5) could be equal to.

[N13] Sophie:	Positive two times negative five…
[N14] Adva:	Two times negative five..
[N15] Sophie:	Aha, hold on… hold on… It's as if you said negative five multiplied two times…. So, negative five multiplied two times – it's negative ten…

So far, so good. By projecting in a metaphorical manner from their former discursive experience, the children discovered for themselves the rule which is, indeed, generally accepted. I will now show that this is not always the case. During the classroom discussion that took place after the work in pairs was completed, the following exchange takes place in response to the same question as in episode 3:

Episode 4: In response to the question, "What 2·(-5) could be equal to?"

[N16] Roi:	Negative ten.
[N17] Teacher:	Why?
[N18] Roi:	We simply did… two times negative five equals negative ten because five is the bigger number, and thus… uhmm… It's like two times five is ten, but [it's] negative ten because it is negative five.
……	
[N42] Noah:	And if it was the positive seven instead of positive two?
[N43] Yoash:	Then it will be positive thirty five
[N44] Sophie:	Why?
[N45] Yoash:	Because the plus [the positive mutplier] is bigger.

On the first sight, Roi's idea may sound somehow surprising. On the closer look, it is as justified as the one proposed by Sophie: like the girl before him, Roi draws on previously developed discursive habits, except that this time the choice does not fit with the one made along history by the mathematical community.

Sophie's successful try: substitution into the discursive template

$$2 \cdot b = b + b$$

Roi's unsuccessful try: substitution into the discursive template

$$(+a) + (-b) = \begin{cases} |a - b| & \text{if } a > b \\ -\ |a - \mathrm{b}| & \text{if } a \leq b \end{cases}$$

in which *a* and *b* are "unsigned" and both + and – are substituted with ·

Figure 3. Recycling old discursive templates in the new context

Indeed, in the first case, the children substitute the new numbers for old numbers: The negatives slide into the slot of the second multiplier, occupied so far exclusively by unsigned numbers. In the second case, the students substitute operation for operation: The *multiplication* of signed numbers is obtained from the multiplication of unsigned number more or less in the way in which the *addition* of signed numbers was previously obtained from the subtraction of the unsigned. As already noted, while the choice of the first group may be deemed successful because it happens to adhere to what counts as proper in the official mathematical discourse, the choice of the other group fails to meet the standards. This difference notwithstanding, the two cases have a very important trait in common: in both episodes the students are trying to incorporate the newly encountered negatives into the discourse on numbers, and in both episodes they do it by using old discursive templates for the new signifiers.

Turning to old discursive habits may be the only way to deal with the somewhat paradoxical nature of learning. At a closer look, the process of extending the discourse turns out to be inherently circular: If new objects, such as negative numbers, are discursive constructions, we have to talk about them in order to bring them into being. On the other hand, how can we talk about something that does not yet exist for us? (See also Sfard, 2000a, 2000b, 2000c)

This circularity, a distant cousin of some other, better known quandaries, such as hermeneutic circle or Meno's learning paradox, may well be the reason for yet another complaint by the journalist whom I quoted at the beginning of this talk:.

> "Perhaps you are thinking that you know quite clearly what seven means, and you are even now in a rather irritated way counting out fingers or something. However, it is easy to see that seven does not actually refer to anything in the world."

How do we create new meta-discursive rules and turn them our own?

So far, we have been focusing on discursive changes that take place following a change in vocabulary or in mediators. We will now see that along with all these, yet another change, this time on the meta-discursive level, has to take place.

Such change must certainly happen in children's discourse on numbers if they are to be able to decide which of the two ways of multiplying positive by negative – the one offered by Sophie or the one designed by Roi – should be accepted as the proper one. Without going into details, let me state that the only justification of the rules that govern the operations on negative numbers is these operations' consistency with certain properties of the sets of numbers that is being extended. And yet, this argument is far removed from anything that counts as convincing in everyday discourse, which is the only discourse the students know so far: Instead of pointing to mind-independent, extra-discursive reasons, as is the case with colloquial discourses, the justification rests, this time, on the assumption that all that counts is the inner consistency of the discourse itself. One thus cannot expect students to accept this justification easily, let alone to reinvent it for themselves.

This inherent difficulty may well be the reason why the teaching of negative numbers has been grounded for ages in the didactic principle epitomized in this unforgettable rhyme: "Minus times minus is plus, the reason for this we need not discuss" (W.H. Auden, quoted in Cline, 1980).

The teacher of our seven-graders did not, however, listen to this advice and, in the spirit of learning with understanding, did venture a discussion of the rule. To see the outcomes, let us go back to the class whom we left puzzling over the question what should be the result of multiplying a positive by negative. The debate went on for two full periods and the class got eventually convinced by Roi who claimed that the sign should be like that of the multiplier with the bigger absolute value. The teacher seemed quite desperate.

Episode 5: Why choose one template rather than another?

[N46] Teacher:	You keep repeating what Roi said and I want to know *why*.
[N47] Yoaz:	Because this is what Roi said.
[N48] Teacher:	But Roi himself didn't explain why it is the magnitude that counts.
[N49] Roi:	Because there must be a law, one rule or another.
[N50] Teacher:	There must be some rule, fine. But does this mean that we should do it according to the magnitude?
[N51] Leegal:	The bigger is the one to decide.
………..	
[N83] Teacher:	Six times negative two is negative twelve – is this too complicated?
[N84] Roi:	But I am more charismatic… I managed to influence them all.

The picture we get from here may be described as follows: The children know that if they deal with numbers, there must be rules; and yet, they have no idea where these rules should come from. Their helplessness finds its expression in Roi's truly postmodern declaration that popularity and consensus are as good a reason for the acceptance of a mathematical definition as any other.

If you think about it, there is nothing rational about these meta-rules and the children can only arrive at the proper rules by interacting with an expert participant, at least part of the time. This didactic suggestion sounds pretty straightforward. And yet, if we now go back to the first graders who are learning about triangles, we will see that children do not easily accept changes in meta-discursive rules even if the initiative comes from a very determined teacher. In the following episode, the long debate on the status of the stick-like shape reaches its climax.

Episode 6: Trying to convince Shira that shape C is a triangle

[T35] Teacher:	But you told me, and Shira agreed, that in triangle there must be three lines, right?
[T36] Eynat:	Right.
[T37] Teacher:	So, come on, tell me how many lines do we have here? [*points to shape C*]
[T38] Shira:	One, two, three….
[T39] Teacher:	So, maybe this *is* a triangle? Here you said this one *is* a triangle [*shows another, more "canonic" triangle*].
[T40] Shira:	Because this one is wide and is like a triangle. It is not thin like a stick [*illustrates with hand movements and laughs*]
[T41] Teacher:	How do we know that a triangle… whether a shape is triangle? What did we say? What do we need in order to say that a shape is a triangle?
[T42] Shira:	Three points… three vertices… and…
[T43] Teacher:	Three vertices and…?
[T44] Shira:	Three sides.
[T45] Teacher:	And three sides. Good. If so, this triangle *[!-points to shape C*] fits. Look, one side… and here I have one long side, and here I have another long side. So, we have a triangle here.
[T46] Shira:	And one vertex, and a second vertex, and a… point?!
[T47] Teacher:	Look here: one vertex, second vertex, third vertex
[T48] Shira:	So it *is* a triangle?

Let me analyze the brief conversation while trying to answer a number of questions.

How do the meta-rules of the children's discourse have yet to change?
The first meta-rule that has to change is the one that regulates children's activity of giving names to geometrical shapes. At present, Eynat and Shira perform the naming task unreflectively, on the basis of their previous visual experiences. They recognize triangles and squares the way they recognize people's faces, that is, in a direct, non-linguistic way and without giving reasons for their choices even to themselves. From now on, they will be requested to communicate to others not only their decisions but also the way these decisions were made. They will now have to tell their interlocutors how a shape should be scanned before the decision regarding its name is made. The scanning procedures are mediated by, and documented in, language. In fact, they are only possible as a part of verbal communication. When we check whether a shape is a triangle, we have to count its sides. The counting is a linguistic act and the result of counting is a word (*three*, in the case of triangles). The new way of making decisions about the names of geometrical figures will thus be done *by analyzing and comparing words associated with shapes*.

This new meta-discursive rule entails a change of yet another meta-discursive principle. So far, giving names has been an act of splitting the world into *disjoint* sets of objects: This means the meta-rule according to which one cannot call a shape both *triangle* and *stick*, or both *square* and *rectangle*. This will have to change once the naming decisions are based on the results of linguistically mediated scanning procedures. These verbally mediated procedures can be ordered according to the relation of inclusion. The hierarchical organization of the scanning procedures becomes, in turn, a basis for the hierarchical categorization of geometrical shapes.

How does the teacher try to induce this change?
The transition from the old to new meta-discursive rules must clearly take place before Eynat and Shira become fully convinced that the stick-like shape is a triangle. Impatient to see the transition happening, the teacher repeatedly reminds the criterion which should be used in deciding. Over and over again she initiates scanning the shapes and counting their elements. Invariably, the words "one, two, three" are followed with the telling "So.." and, eventually, with the statement asserting that the shape is a triangle. The word "so" is very effective in suggesting that whatever comes next is an inevitable entailment of the "one, two, three" sequence.

How successful is the teacher's effort?
On the face of it, the teacher's method of repeated use of a certain discursive sequence works: Shira soon learns to complete the procedure of counting to three

with the words “So, this is a triangle”. And yet, the fact that in the case of the stick she utters this conclusion as a question rather than as a firm assertion signals that she may be declaring a surrender rather than a true conviction. The lack of certainty can be felt also in Eynat’s contributions. The ultimate evidence for the fact that old meta-discursive habits die hard will come some time later, when the children are asked to distinguish between rectangles and other polygons. Both girls will then adamantly reject the teacher’s suggestion that a square can also be called rectangle and they will adhere to their version for a long time in spite of the teacher’s insistence.

One could conjecture that in the case we have been analyzing, the slowness of learning resulted not so much from the stubbornness of the old discursive habits as from the ineffectiveness of the teaching method. Moreover, since this method was based on demonstrating the application of the new meta-rules rather than on arguing for them explicitly, some people may criticize the teacher for violating the principle of learning with understanding. This is thus the proper place to remind ourselves that unlike the object-level rules of mathematics, each of which is logically connected to all the others, the meta-rules are not dictated by logical necessity. In consequence, one cannot justify them in a truly convincing, rational way. The children, if they wish to communicate with others, will have to accept these rules just because they regulate the game played by more experienced players. They will have to become participants of the new discourse before they can fully appreciate its advantages. This may be why the mathematician von Neumann has been heard saying to a journalist “One does not understand mathematics, young man, one just gets used to it”.

Final remarks: How does all this affect the practice?

It is now time for a summary. This can best be done by trying to justify the title of this talk: “There is more to discourse than meets the ears: Looking at thinking as communicating to learn about mathematical learning”.

There is more to discourse than meets the ears. Indeed, discourses are activities in which people do much more than can be heard. Among others, they follow sets of tacit rules that make the communication possible and of which most of the time they are not aware. By defining thinking as a case of communication and school learning as extending and modifying discourses, I have declared that much more goes into learning than has been taken into account until recently in the traditional cognitive research. Becoming skillful in modifying tacit meta-discursive rules is one of the aspects of learning that has yet to be studied.

I also believe that *looking at thinking as communicating helps us learn more about learning*, and have been trying to show it throughout this talk. Here are some of the points I tried to make about the activity of extending and modifying discourses:

- First, learning mathematics means new word uses, new mediators, and new meta-discursive rules;
- Second, the process of discursive change is circular: to be used, new signifiers must be meaningful, but to be meaningful they have to be used;
- Third, the way out of this circularity leads through recycling old discursive habits;
- Fourth, there is nothing rational about most meta-rules, and it is thus not easy to change them.

Although teaching doesn't appear in my title, we did talk about it as well. In the last episode we saw that a teacher may induce changes in a discourse by creating communicational conflict, that is by exposing children to word uses that are different from their own. The term *communicational conflict* brings to mind the well known idea of *cognitive conflict*. And yet, there is a difference: Communicational conflict is a clash between the ways two interlocutors use the same words and not, as in the case of cognitive conflict, between one persons' own contradictory beliefs about the world.

But this is not all. I would like now to claim that the change in the conceptualization of learning proposed in this talk is bound to affect some of the most common beliefs on teaching. I could say quite a lot about this, if I had time. Since I don't, I will just list some of such beliefs. I am perfectly aware that without a proper justification, my claims may sound more provocative than I intend. On the other hand, why not to end the talk with a bit of provocation?

- The first pedagogical belief that, perhaps, must undergo a revision is the nowadays popular principle of *learning with understanding.* This principle stresses the primacy of conceptual understanding over symbolization and skill. The foregoing analysis, which has shown the circularity of the learning process, casts doubt at the very possibility of this order of things. I am not trying to say that we should compromise understanding. Rather, I would suggest that we look for a new understanding of understanding and for a new vision of its underlying mechanisms. Communicational framework seems to have much to offer in this respect.
- Another belief that warrants scrutiny is that in learning mathematics, formalism and skill should be de-emphasized (see e.g. Devlin, 1997). And yet, as I was trying to show, it is a mistake to think of symbolizing as giving a new "expression" to the "old thought" and of proficiency as a matter of "finishing touches". Rather, this is the very fabric of the mathematical communication. Thus, the decision that must be made is not whether to teach mathematics without symbolism and skills, but rather whether to teach mathematics at all.

- Finally, let me mention the famous constructivist principle according to which the students are the builders of their own knowledge. In the communicational language, this would mean that the children are creators of the discourses in which they are supposed to participate throughout their lives. And yet, discursive meta-rules are products of traditions rather than of any extra-discursive need or of logical necessity. If so, learning these rules cannot occur without an interaction with an expert participant.

I cannot end this talk without getting back to the journalist whom I quoted in the beginning. It's time to ask whether I managed to answer the challenge with which he faced us, the educational community. Just to remind you, Sartwell questioned the idea of school learning, with mathematics being, in his opinion, the most problematic of subjects. Whether to teach mathematics in school or not was not the question I was trying to address in this talk. This is a political issue that must be left in the hands of professional decision makers. I did, however, try to show Sartwell wrong in the argument he gave for his provocative proposal. As you may recall, Sartwell's answer to the question "what is mathematics" was, more or less, like the one Hamlet gave to Polonius when asked what he was reading: "Words, words, words."

Being an adherent of the communicational vision of cognition, I certainly agree with this. And yet, as I was arguing in this talk, the words of mathematics, even if not immediately clear, are not doomed to remain empty. More often than not, what in the beginning is but a word, would eventually be made flesh. In the end, the words will also become powerful and will do things for us. Our aim is to fathom the way it happens.

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Panel Discussion on

The Relationship between Theory and Practice in Mathematics Education Research

Morten Blomhøj, Barbro Grevholm, John Mason, Rudolf Sträßer

Introduction

Rudolf Sträßer, IDM University of Bielefeld

Mathematics education as a research activity must be distinguished from the actual practice of teaching and learning mathematics. As every research activity, its fundamental goal is to create, develop and advance knowledge about its object of study - which can be defined as the teaching and learning of mathematics in all its respects. This obviously includes research into individual learning processes of mathematics as well as research into organised learning & teaching of mathematics in school, at work and in the society at large.

As with every scientific activity, the didactics of mathematics (we use "mathematics education research" or didactics of mathematics as synonyms) tends to create its own **theory** on the subject under study. I offer two well known examples: The modelling cycle (starting from reality → "real" model → mathematical model → mathematical treatment/solution → interpretation in "real" model/reality) is widely used to better understand the application of mathematics in various contexts - and its teaching and learning (see also the second section below). The "didactical transposition" (Chevallard, 1985) provides a way to describe and understand how and why (scientific or other) knowledge is transformed into teachable knowledge (by means of compartmentalisation, sequentialisation ...). More generally: Theory can be thought of as a textual and/or graphical description in order to better understand activities in the practice of research or the teaching and learning of mathematics. Scientific theory "normally" is or is part of a net of inter-related concepts and relations used to describe a certain object under study (ad-hoc "definition").

As with every scientific activity, the didactics of mathematics tends to create its own **methodology of research** or uses methods from other disciplines. I offer just two extreme examples: Comparative, large scale statistical analyses yield information about the state of the art of teaching & learning mathematics in a specific grade, type of school, nation, type of society etc. with hypotheses to be rejected and/or "confirmed" by means of formal test procedures. Generalisation is controlled by formal, statistical practices (for example see the recent PISA-study). Exemplary case studies in an ethno-methodological, participatory style in order to (better) understand the special situation under study. Generalisation (if ever) is only aimed at by means of heuristical argumentation. Hypotheses may come out of this methodology only "ex post", after the completion of the study.

This description of didactics implies deep links to other scientific disciplines such as Psychology, Pedagogy (i.e.: the science of education), Sociology, History and Epistemology/Philosophy – to name only the most important "neighbouring" disciplines. The didactics of mathematics should develop in close co-operation with didactics of other subjects – such as the didactics of science and the didactics of the mother tongue.

Mathematical modelling – interplay between theory and practice

Morten Blomhøj, IMFUFA Roskilde University

During the last fifteen years I have worked with mathematical modelling in a number of different connections. I have been involved in writing textbooks that introduce mathematical modelling in the mathematics curriculum at gymnasium level. Developing and testing courses in mathematical modelling for grades 8 and 9 in close co-operation with mathematics teachers was a central part of my Ph.D.-study. In recent years supervising students' projects in mathematical modelling has been the dominant part of my teaching at Roskilde University. The study format at Roskilde Uuniversity is project organised and our master programme includes as a substantial element projects where students are building, analysing and acting out critique of mathematical models (Niss 2002). For the time being I am involved in the development and teaching of a new course in mathematical modelling which is offered to first year students in the natural science study programme with the aim of developing the students' mathematical modelling competence (Blomhøj & Højgaard 2002, p. 6). Beside these teaching related activities I have used the idea of the "theory" of mathematical modelling as a tool for analysing the role and functioning of mathematical models in different contexts in society.

In these activities – which partly belong to the practice of mathematics teaching and partly to mathematics education research – the cyclic process of mathematical modelling and its possible functioning in mathematics teaching has played the role of a basic theory. Of course this "theory" has developed considerably during the years. This is true both for my personal understanding of the modelling process and for what could be considered as the shared understanding of this "theory" in the mathematics education research community.

To illustrate what the "theory" is about I refer to our latest (graphical) model of the cyclic process of mathematical modelling (Blomhøj & Højgaard 2002; see figure 1 below). Here mathematical modelling is depicted as a cyclic process between six sub-processes, which could be connected in many possible ways. It should be emphasised that the "theory" is much more than just this model of the modelling process. It also includes the justification of mathematical modelling as a central element in mathematics teaching at different levels of the educational system, description of the different competencies involved in mathematical modelling, understanding of the fact that these competencies cannot be acquired independently – in order to learn mathematical modelling you need to practice full scale modelling, and as a special but important element the theory also includes the idea that competence to act out critique of a modelling process does not automatically follow the competence of performing mathematical modelling.

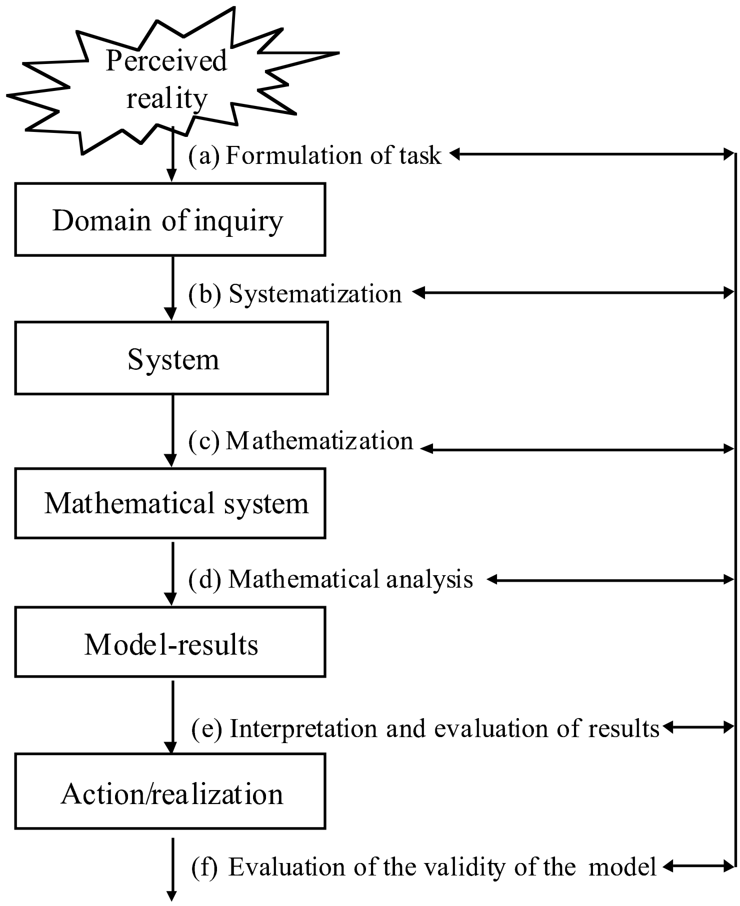

Figure 1: A graphic model of the cyclic process of mathematical modelling

Until now I have placed theory in inverted commas (“ ”) to indicate that there is a separate discussion on criteria to constitute a theory in mathematics education and to what degree mathematical modelling theory meets this criteria. However, in this text I assume that it make sense to talk about the theory of mathematical modelling.

Summarizing, the theory of mathematical modelling can be used as a tool for:

* analysing authentic modelling processes retrospectively
* identifying the different competencies involved in modelling

* designing tasks and teaching episodes that challenge these competencies
* communicating and discussing the aim of a modelling course with students & teachers
* teaching the idea of mathematical modelling to students
* analysing students modelling activities

Both in the research process and in the process of developing teaching practice the interplay between theory and practice is characterised by complicated mutual developments in our understanding of both theory and practice. But for analytical purposes I will analyse the relationships of theory → practice and of practice → theory separately.

Theory → Practice

How can or how should theory influence the development of practice?

- The ultimate reason for the development of theory is to improve the practice of teaching.

Theories tend to live their own life. And so does the theory of mathematical modelling. Special conferences are organised (ICTMA's) and special research questions – which do not always seem relevant for the practices of mathematics teaching - are addressed (Niss, 1989). However, these activities are important for the development of the theory, and eventually it may therefore also be of relevance for the development of practice. Therefore, research should not be tied too closely to the development of practice. But what is really important is that researchers feel obliged to reflect on the possible relevance or use of their theories in relation to the practice of mathematics teaching. In my experience mathematical modelling theory does generally include reflections on the relevance for practice.

- Theories can explain (some of) the teaching and learning difficulties that appear in practice and thereby support design and teaching of courses that enhances students' learning.

A result of a conceptual analysis of the mathematical modelling process is that the challenges that the students meet vary with the degree of pre-structuring the tasks and situations that form the context for the students' activity. It is necessary to spend time on developing the students' competences related to the inner parts of the mathematical modelling process (i.e. sub-processes (c), (d) and (e) in figure 1). Pre-structured tasks that give the students the feeling of "*knowing what the goal is without knowing how to achieve it*" are appropriate for this purpose. In order to develop mathematical modelling competence it is also necessary to challenge the students on the outer sub-processes and for this purpose one needs open-ended tasks placing the students in a situation where they feel "*perplexity* *due to too many roads to take and no compass given".* The balancing of tasks in this span is a key issue when developing a mathematical modelling course.

- Theories need to be personalized and made concrete by the teacher to be of use in practice.

Mathematical modelling theory cannot be directly transformed into teaching practice. In order to teach mathematical modelling properly you need to believe in the theory. And for many teachers this requires experience in their own teaching showing that it is actually possible and worthwhile to get students involved in modelling activities. This aspect of the theory-practice relation does actually form a serious obstacle for the development of the practice of teaching mathematical modelling and for the development of mathematics teaching in general.

- Theories are often "soft" and very difficult to falsify through practice.

This could of course be considered a weakness of a theory. But I think this weakness is unavoidable. Theories in mathematics education should not be expected to meet the same criteria as theories in natural science. Theories in mathematics education must consider the complexity of the teaching and learning process. Theories in mathematics education should be validated through their long-term influence on research and of the practices on mathematics teaching.

Practice → Theory

How is practice influencing theory? (or how should practice influence theory?)

- The key object of mathematics education research is the actual and potential practices of mathematics teaching.

In order for research to develop theory, relevant for the development of practice, it must – although not always – be based on problems appearing in practice or in the excellency of a particular form of practice. Research must be sensitive to the development of practice. In my opinion the theory of mathematical modelling is an example of a theory in mathematics education that has developed on the basis of reflections on different forms of practice in mathematics teaching.

- Practice may give new meaning to theory and thereby support new theoretical developments.

The importance of practice for further development of theory ought to be recognised in mathematics education research. This is really important for the possibility of forming fruitful forms of co-operation between researchers and teachers. Of course the practice of mathematics teaching is not always relevant for the development of theory.

- Practice sometimes needs to be arranged in order to be of relevance for the development of theory.

The practices of mathematics teaching are bound by official regulations and by tradition. Therefore it is really important that research in mathematics education can arrange teaching situations where boundaries are crossed. In connection with mathematical modelling it is often necessary to establish special teaching conditions in order to study the students' work with the entire process of mathematical modelling.

- Developing practice and theory may be considered as two sides of the same process.

However, it is also important to recognise that the practices of mathematics teaching and mathematics education research are embedded in different sociological contexts. That is one explanation of why some teachers who have been very enthusiastic about teaching mathematical modelling in developmental programmes return to more traditional ways of teaching after ending such projects.

The relationship between theory and practice in mathematics education research

Barbro Grevholm, Luleå University of Technology

In his book *Vetenskapsfilosofi* (Philosophy of science) Molander gives a definition of theory: A system of propositions, some of which are seen as laws, that in a unified and coherent (consistent) way describes and explains phenomena in a certain area of investigations. (Translated from Molander, 1988)

I would rather express it like this: A theory is a model that can be used on specific phenomena or areas to interpret, explain or understand the phenomenon or area and to be able to forecast or predict what can be expected in certain situations where the phenomenon can take place or in the area in question.

What is practice in mathematics education research? Presumably most teachers and researchers would refer to the actual practice in the mathematics classroom or practice of teaching and learning. The object of research often is the interactions in the classrooms, what teachers do, what pupils do and the cooperation between them. But it can also be a single pupil in a learning situation or one or several teachers in an instructional situation or discussion. The actual practice of research could also be referred to, that is what the researcher undertakes in his studies.

My example of a situation where practice leads to theory is about *Counting cave*, a computer program with the aim of improving the pupils' number sense (see Grevholm, 1989; 1991a/b). In this program the pupil takes the role of a number and investigates a mysterious cave that contains number rooms. In a developmental study I investigated how pupils use this program. Observing, videoing and tape-recording pupils finally led to a model of how pupils build up and use their knowledge about number, such as number facts and operations. As I observed the pupils working with the program I imagined the elements of knowledge as points in a multidimensional spider web and the associations pupils used to develop new knowledge as the web itself. It was obvious how pupils could take departure in one element of number facts and step by step from this walk along the web and create new knowledge.

A conversation like this could take place:

- How do we get 72 by the help of 6?
- I know that it is not 9 because 9 times 6 is 54.
- But I know that 9 times 4 is 36.
- Yes and 36 is also 6 times 6.
- If we take 36 two times we get 72.
- So then we know that 2 times 6 times 6 is 72 and we have 12 times 6 is 72. It is 12 we are looking for.

Figure 2: Counting cave

In my imagination I saw how pupils jumped from one point to another in the spider web of number knowledge and then ended up with the result of new knowledge that could be included in the web. Many years later I found out that other researchers had described similar models for how structures of knowledge develop and are organised. Novak (1998) among others have made this picture concrete through his concept maps.

The second example is about how theory can inform practice. Jaworski (2002, p. 52) introduced the diagrammatic picture of a co-learning partnership shown in figure 1, page 105 in this volume. In her theoretical model I can understand the research study I have been carrying through since 1996 with a group of student-teachers who are now working as teachers of mathematics and science. The model explains the dependence between students, teachers and researchers that I have experienced in the practice of my research and made me more aware of the learning that takes place in all of the groups included and especially of my own learning as researcher.

The relationship between theory and practice is a complex one and has to be problematised for many reasons. First of all we have a gap between researchers and practitioners, teachers. According to Mogens Niss this gap seems to be widening:

> One observation that a mathematics educator can hardly avoid to make is that there is a widening gap between researchers and practitioners in mathematics education. The very existence of such a gap is neither surprising nor worrying in itself. The cause for concern lies in the fact that it is widening.[...]..what we can do is to reduce the sociological gap between teachers and researchers. This can be done by creating meeting points and fora for concrete collaboration between teachers and researchers, by making the demarcation lines between the two professions less rigid, by giving teachers opportunities to take part in research from time to time, and by having researchers never be completely out of touch with the practice of teaching mathematics (cf. Niss in press).

Do we need theories? Do teachers need theories? John Mason has pointed out that not all theories are practice based. This then depends on what we mean by practice.

Teachers would probably expect the theories they find useful to be related to their practice. They would also try to understand the theories in the light of the experience they have in practice. Carlgren and Marton (2001) claim that the professional object of teachers is learning. I would claim that the professional object is teaching with the aim of creating a situation for student learning. This necessarily includes that the teacher becomes a learner herself (cf. Jaworski, 2002).

Why do we need theories then? One reason is that we have practice and to be able to overlook it and systematise our experiences from it we need theories. An interesting object of study would be to find out what open or hidden theories teachers have about mathematics education, about what knowledge is, about how pupils learn, about how teaching should take place and about how and what learning can take place in specific situations and under given conditions. Even if we have some findings about these questions most of it is still unexplored. Such research could enlighten the complex relationship between theory and practice.

Theory and Practice

John Mason, Open University Milton Keynes UK

> Theory and Practice have been in tension ever since humanoids started to think about and reflect upon what they were doing. Each has an important role to play in the transformation of individuals as they mature and grow wise, and in the transformation of societies as they evolve. With two impulses in tension there is the potential for release of energy. There is also the potential for

> acceding to desire to reduce tension by migrating to one extreme or the other. But the potential is held by balancing the two, and by transcending them through action, that is, through permitting a third impulse to mediate between and to hold the two together. I hope that my examples will illustrate this at least to some extent, for as Newton said, … since skills are more easily learned by example than precept …
> (Isaac Newton: see Whiteside 1972 p. 129-157).

Examples: Practice leading To theory

- I try to preach what I practice, not the other way round, which itself is a practice which contributes to my theories about how professional development is most effectively supported.

Having found myself working with experienced school teachers and not having that same experience in schools, I based my approach on speaking to their experience by concentrating on and being aware of my own experience, but cast in a form which others could immediately enter, through mathematical activities and through speaking from my experience to their experience.

- Whenever I am faced with a question in mathematics education I begin by trying to locate an example (possibly analogous) in my own experience, and to generate a new similar experience. I then try to construct a task-exercise for others in order to generate similar experience.

This is a component of the use of the Discipline of Noticing in what I call *researching from the inside* (Mason 1992, 2002). It arises as a necessity from the first example, and provides the basis for that practice to be effective.

Examples: Theory used in practice

- The theoretical position that learners learn better when they are actively engaged cognitively, affectively, and physically (even if virtually mentally), and specifically, that asking learners to construct their own examples of mathematical objects meeting certain constraints entices them into active engagement with mathematics, informs my choice of tasks in workshops that I lead.

My wife, Anne Watson and I are assembling everything we can about this particular practice, building a theory about it, and using that theory to refine the practices we have encountered and use ourselves. I currently make this a feature of most task-exercises I suggest to participants in workshops.

- The theoretical construct of *transposition didactique* informs my reading of research reports which claim that certain tasks 'worked' with certain learners.

It reminds me to ask questions about possible differences between the intended, activated and experienced task, and about the inner and outer aspects of the task (Tahta, 1981; Mason, 1993). These theoretical ‘frameworks’ consisting of potent distinctions have proved fruitful in the past and continue to prove fruitful in task design and in reading research reports.

- The theoretical notion of *distinction making* as the basis for sensitivity and noticing, lies at the heart of my 'theoretical' description of how I, and I suspect others, actually function.

The notion of *distinction making* is richly embedded in my functioning, triggering access to a variety of distinctions which then inform my practice (Mason 2002). It is related to ideas concerning the role of disturbance and distinction making in learning to be found in many places such as Varela, Thompson & Rosch (1992), van Hiele (1986), Maturana & Varela (1972), Bennett (1966), Heidegger (1927), Festinger (1957) and at least implicitly in Plato.

Principles & Assumptions

In preparing for this panel, I found myself led to ennunciate some principles in case there might be opportunity to discuss them and others related to them.

• Every rich practice (e.g. didactics of mathematics, teaching of mathematics in a given culture at a specified level) develops its own technical language, which outsiders call jargon and which can be mistaken for, or may in fact indicate, a theoretical basis.

• The test for empty, obscurantist jargon is whether the same thing can actually be said in other words but at greater length so that others can recognise what is being said and test it in their own experience (a modified Popperianism), or whether more words merely compound abstractions and obscurity.

• Whenever an abstraction or generality is encountered (such as this one), it is natural and useful to test it within one's own experience (past, present and future), that is, to specialise the generality through making use of the power of mental imagery and the resonance mechanisms of memory, and to seek confirmation or contradiction in future experience.

• Practice which is entirely reactive-responsive without being abstracted or generalised is fine for the individual, but is not easily communicated to others (simply showing has all sorts of pitfalls when the learner mistakes what is irrelevant for what is important, and vice versa) and is not easily developed; practice which is formalised through explicit description is unlikely to be fruitful for others, since the more explicitly and precisely the practice is stated, the more likely it is that people will try to reproduce the behaviour, without the awareness which informs and directs that behaviour (see for example Mason 1998).

• Not all theory is practice based! Is all practice theory-based, even if not explicitly articulated? Some argue that it must be, but others argue that theory derived from observing the practice of others is as much an indicator of the describer's sensitivity and need to encapsulate in language as it is a theory which drives behaviour. Indeed some would argue that behaviour is not driven by theory but by habit.

• Theoretical studies can locate useful distinctions which can sensitise practitioners to notice what they previously did not discern; they can develop practices which practitioners do not have the time and energy to refine. I am less certain that they can validate distinctions and practices, since these must be in ecological balance with the psycho-socio-cultural propensities of the individual and their context.

• Since the only thing I can actually change is myself, a suitable focus of attention for study is myself: myself (including my claimed past experience) as litmus for testing assertions; myself as locus of significant action (converting some habitual reactions into sensitively considered responses); my own awareness as constituting the world in which I operate; my own practices as constituting my presence in the material world which we all apparently share.

• The more precisely I try to describe some incident concerned with teaching and learning, the more I learn about the describer (an educational-researcher Heisenberg theorem); the more tightly I define details of a practice, the less useful that practice becomes and the less likely I am to locate an opportunity to use it (an educational-practitioner Heisenbergian theorem).

Concluding Remarks

M. Blomhøj, B. Grevholm, J. Mason & R. Sträßer

Looking back to what was presented during the panel presentations and discussion, some salient features emerge. The most obvious observation seems to: Actual practice is not usually as well analysed and understood as the theory. The introduction focused totally on theory - and the panelists had less trouble describing the role of theory than coping with the diversities of practice. Practice may not be describable in words. Closer inspection reveals that one can even distinguish two types of practice: There is the practice of the researcher – and John Mason in his presentation gave an elaborated description of how he himself sees his practice as a researcher trying to co-operate with practitioners such as teachers, based on his study of his practices in learning and doing mathematics. On the other hand, there is the practice of teaching and learning mathematics which is the unquestioned object of study of didactics of mathematics (or research in mathematics education). In her presentation, Barbro Grevholm pointed to a theoretical perspective on the co-operation and co-learning of researchers, teachers and students. In addition to that, she gave a nice example of how a developmental practice gave rise to a piece of theory without being noticed as theory at first. It was only later and with the explicit identification as theory by a different researcher that she came to realise that practice made her develop a piece of theory.

Morten Blomhøj reminded us of the tension inherent in these two "cultures" of theory and practice: while he sees development of the practice of teaching and learning as the ultimate goal of didactical theory, he nevertheless spoke of the (relative?) autonomy of theory which sometimes should not be tied too closely to the development of practice and must be developed in not too close a relation to the actual practice of teaching and learning. John Mason (in his "principles & assumptions") added that "not all theory is practice based". One must not go as far as taking a philosophical perspective (like platonism or Kantian a-priori) to defend the possibility, if not necessity of a theoretical stance to do research in mathematics education. This must not stop a researcher from realising that – as Barbro Grevholm put it – we have practice and theory enables us to reflect upon it and systematise our experiences. Consequently, there are good practical reasons for why we need theories.

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Mathematical Thinking and Emotion

Jeff Evans
Middlesex University, London

An examination of the development of concepts of affect in mathematics education can begin usefully with McLeod's (1992) spectrum of forms of affect, from beliefs (more stable, 'cooler') over to emotions (more transient, 'hotter'), with attitudes intermediate. Early research focused on more stable aspects of affect, using surveys to study dimensionality, and correlations with performance. Further, 'mathematics anxiety' was used to provide a non-cognitive explanation for any gender differences in performance. Recent research also focuses on emotions, using process-oriented research methods (e.g. semi-structured interviews). Thus, over time, the conceptions of affect and emotions have changed from those of individual 'traits', to aspects of an interactive process of problem-solving. Developments using discursive perspectives emphasise language use (e.g. metaphor) and display emotions as cultural and social phenomena.

The discussion in this session picked up on Thomas Lingefjärd's emphasis of the urgency of studying the affective and emotional reactions of students towards mathematics, because of the pressing problem of dropout of students from science and engineering courses, even in Sweden's best Universities. This problem was echoed by other participants, from other institutions and other specialisms. It is also pressing in general, in other countries, such as the UK (Furedi, 2002).

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To Study Mathematics in an Engineering Program

Thomas Lingefjärd
Chalmers University of Technology & Göteborg University

Introduction

During the last 10-15 years, there has been an increasing interest in studying how students experience, learn, understand or demonstrate advanced mathematical thinking, a categorization of thinking that most often refer to upper secondary or university students' learning of mathematics. This is a report from a study of how first year students in the Chalmers University of Technology program of Mechanical engineering comprehend linear algebra and real analysis, but there are many similarities to how younger students learn mathematics as well.

Humankind is a learning creature; it is one of her most dominant characteristics. But learning is not possible to define in one definite way or another; it is a multidimensional process. We learn collectively and individually, alone and together with others, inside and outside school or institutions, at work and at home. Contrary to past views of learning, the cognitive psychology of today (Marton & Booth, 1997; Säljö, 2000) suggests that learning is not linear but proceeds in many directions at once and at an uneven pace. People of all ages and ability levels constantly use and refine concepts. Furthermore, there is tremendous variety in the modes and speed with which people acquire knowledge, in the attention and memory capabilities they can apply to knowledge acquisition and performance, and in the ways in which they can demonstrate the personal meaning they have created.

Several authors have been trying to categorize advanced mathematical thinking. Robert and Schwarzenberger (1991) describe in the following statement how learning in advanced mathematics is different from learning in elementary mathematics.

> There is a quantitative change: more concepts, less time, the need for greater power of reflection, greater abstraction, fewer meaningful problems, more emphasis on proof, greater need for versatile learning, greater need for personal control over learning. The confusion caused by new definitions coincides with the need for more abstract deductive thought. Taken together these quantitative changes engender a qualitative change, which characterizes the transition to advanced mathematical thinking. (p. 133)

It is unclear from this statement how much of the difference that is due to the mathematical content itself rather than to the way the courses are taught. So far, there seems to be no agreement in the community of mathematics and mathematics education on what advanced or elementary mathematical thinking really is. In *The Nature of Mathematical Thinking* (Sternberg & Ben-Zeev, 1996), Sternberg wrote in the culminating chapter, "In reading through the chapters of this volume, it becomes clear that there is no consensus on what mathematical thinking is, nor even on the abilities or predispositions that underlie it" (p. 303).

The algebra course

The algebra course in the Mechanical engineering program is built around the common core of linear algebra, such as concepts like vectors, matrixes, determinants, complex numbers, polynomials and algebraic equations. The course is organized in what is called "theme weeks" as follows:

- Day 1: An introductory lecture for 2 hours with an introduction to the theme, the area of the following week, objectives and goals, examples, important theorems and relations.
- Day 2 & 3: The students work in small groups of 4 with a total "class" of about 30 students and with one teaching assistant. The teaching assistant serves as a coach but can also demonstrate further examples within the content area for the whole class. So-called huge questions are left for the after-lecture.
- Day 4: The students are examined on that week's work.
- Day 5: A concluding after-lecture for all students.
- Note: Day 1 is a Thursday throughout the algebra course.

An important idea with the organization is that the student should work in groups of 4, thereby allowing discussions and according to the examiner prepare for "learning by explaining for someone else" but also for stimulating group discussions. Another important idea is every week's 2 assignments, larger problems on which the students are examined day 4, both orally and in written form. The students are also encouraged to write a journal over each theme week in order to reflect over their learning. The journal is also valued in the examination. The 14 homework problems and the 6 journal writings can altogether give maximum 20 points, and every student need to have at least 12 points from this part of the examination to pass. In addition, the students need to get at least 12 points of the final exam's 30 points.

As a part of the coursework assessment can play different roles and fulfill a range of purposes. Purposes of assessment could either be *formative* or *summative*. Formative assessment could be seen as a help to form and develop student learning and summative assessment as a way to sum up what has already been achieved. When the two components are well integrated, it is also more likely to prompt deep and relevant learning (Morgan & O'Reilly, 1999). If the teaching and assessment are structured so that one assignment builds upon the next, with formative feedback from the first promoting to the next, and so on, it could be a strategically important way to maximize learning. All graded assignments taken together can form a final grade but also an end-of-the-course examination might be included, the last one with a summative function only. Unfortunately, this mixture of an ongoing examination with different types of assignments from the first week and a final exam at the last week was hard to see through and master for the students.

Students

In the fall of 2001, 189 students were admitted to the program of Mechanical Engineering at Chalmers. They ranged in age from 19 to 33, with a median age of 22 and a mode of 20. The median value of their gymnasium grade was VG (well passed) or 4. Thirty-two were women, and 157 were men. They all began their studies in mathematics with the algebra course, which started with an introduction at September 4 and ended with the final examination at October 25.

Method

The study was conducted with the help of surveys and interviews. The survey instrument was based on an indicator instrument from Australia, called Course Experience Questionnaire (CEQ). The CEQ survey is used annually in Australia to evaluate how people who have graduated from university programs value the program afterwards. The instrument uses 25 questions to measure the following factors or indicators:

- Quality of teaching
- Clear goals and standards
- Appropriate assessment
- Appropriate workload
- Development of generic skills
- Overall satisfaction

The CEQ defines generic skills as if one's performed studies might lead to improvement of skills that are useful in a wider context than just university courses. The generic skills are measured by six questions. The quality factor corresponds to one question. The remaining 18 questions are of an apparent process character, which means that the students are asked to judge whether the teaching has had certain qualities or not. The questions have been copied from the report of the investigation in 1995 (Johnson, Ainley & Long, 1996).

Following the example of Lander and Larson (1997), some questions and factors were added in order to improve the survey. Finally, a survey with 44 different questions was constructed, all questions formulated as statements with which the students could agree or not on a five graded scale. See Appendix for surveys that were used during and after the algebra course. A number of 141 students answered to the first survey, while 120 students responded to the second one. More than 100 students responded to both surveys.

The interviews were all conducted with voluntary students, who responded to a general request regarding students who would like to sit in together with a researcher for an interview regarding their studies in mathematics. Fifteen students contacted the researcher and came to interviews, one by the time. The interviews all started with information about the CDIO project, then with questions about the reasons for studying at Chalmers, the time spent on studies, the way the student experienced studying mathematics at this level in general, and so forth. The interview also included conceptual and technical questions about linear algebra and finally we talked about the algebra course in a more detailed way. One supposedly new experience for the students was the writing of a mathematical journal.

For many different subjects, in school as well as at universities, the importance of fostering the students to write as part of the learning experience is natural. Subjects like for example English, social science and natural science regularly use the writing process as part of their teaching. For the last 10 or 15 years there has been a growing concern about how to create and implement relevant writing assignments also throughout the mathematics curricula. Different reasons such as an increasingly advanced and available technology among learners of mathematics, a growing interest among teachers of mathematics at all levels to learn more about what and how their students learn, and a likewise growing certainty among researchers in mathematics education that assessing knowledge in mathematics is much more than just a written test at the end of a course, have all contributed to a strong interest about the use of writing assignments in mathematics.

Results

The total report with the full set of surveys and responses to each item, interviews and students results is planned to be available at the end of 2002. I have chosen to present the responses to the category *Generic skills* in the survey in this paper. The following questions are interpreted as measuring what the students think of the course's effect on their Generic skills, which Johnson, Ainley and Long (1996) together with Lander and Larson (1997) define as a quality or characteristic that corresponds to the students' general ability to sustain and succeed as university students. When the figures in a line do not add up to the sum of 141, it depends on the fact that some students did not answer to all the questions in the survey.

Survey 1

The first survey was given in the middle of the algebra course, which was early October (n=141).

Table 1

Question	**Meaning**	**Responses** Agree totally	Agree partly	Don't know	Agree hardly	Agree not at all
2	Problems-solving skills	44	77	16	4	
5	Analytical skills	30	79	29	2	
10	Ability to solve new problems	8	55	61	15	1
11	Ability to communicate in written form	14	47	59	14	6
21	Ability to plan one's own work	29	75	29	5	2
9	Ability to work in team	34	74	24	7	2
37	Ability to reason and discuss	32	73	29	4	2
41	Ability to make good reports	22	66	37	11	2
		18,88%	48,40%	25,18%	5,50%	1,33%

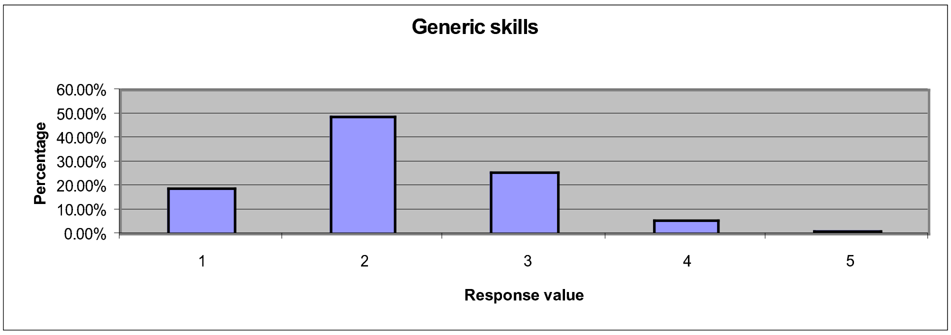

Figure 1. About 30% were not convinced that their generic skills would improve during the course.

Survey 2

The second survey was given after the algebra course was finished, with the same set of 44 questions (n = 120).

Table 2

Question	**Meaning**	**Responses** Agree totally	Agree partly	Don't know	Agree hardly	Agree not at all
2	Problems-solving skills	26	76	16	1	1
5	Analytical skills	14	68	33	3	
10	Ability to solve new problems	9	61	43	5	2
11	Ability to communicate in written form	10	46	36	25	3
21	Ability to plan one's own work	18	49	46	4	3
9	Ability to work in team	12	58	39	9	2
37	Ability to reason and discuss	13	59	36	7	3
41	Ability to make good reports	11	50	31	20	6
		11,77%	48,65%	29,17%	7,71%	2,08%

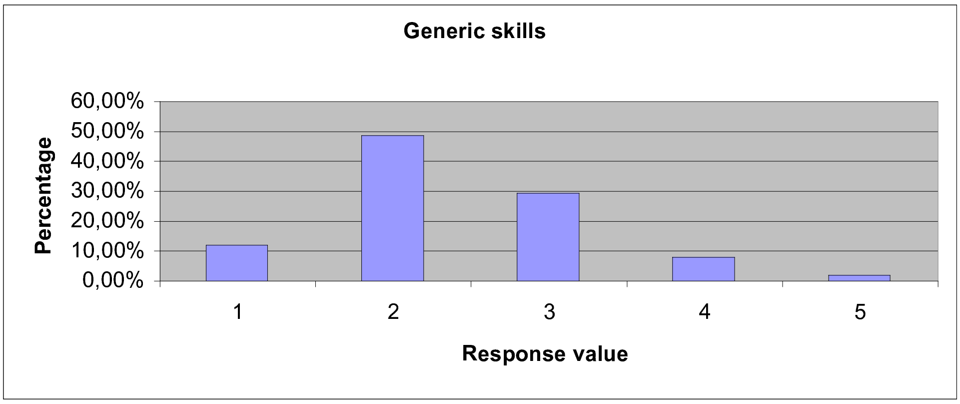

Figure 2: After the course, the number of students who were not convinced about the improvement of their generic skills had increased to 40%.

All the results in the study should be seen in relation to each other, but as a single and isolated result, the decreasing trend concerning the students' belief in their generic skills is interesting. Wouldn't a course with all this emphasis on learning, on different examinations methods, on group work on larger projects, and so forth, more likely lead to an increase of the generic skills? Well, maybe it does? We have to remember that it is just the students' attitudes, emotions, and feelings that we are measuring in a survey like this. Nevertheless, the students experience themselves as less competent problem solvers after the course, which can be seen as a serious damage to their self-confidence.

Another interesting aspect to measure is the students' opinion about the reflective writing (questions 26 & 36). During the course there was a polarization of students' opinions about the usefulness of reflective writing. After the course, a larger number and a larger percentage of the students, expressed the opinion that they had no benefit whatsoever of reflective writing when learning mathematics. Since the results of the surveys in several respects contradict what the teachers and planners of the course expected, the next step was to perform interviews with a group of students.

Interviews

The first interviews took place after the algebra course was finished and after the final exam and had three different objectives; the social and educational situation the students come across, their mathematical learning (e. g. do they know what a singular matrix is?), and how the course had affected them in terms such as *emotion*, *attitude*, and *confidence*.

One goal of all mathematics education should be for students to take responsibility for their own learning. This means empowering the students to read, write and discuss mathematics intelligently and with self-confidence. Engineering students will also be entering into fields where they will be doing technical reading, and reading mathematics can help them learn to read difficult written material. Writing is also the most likely way in which most engineers will display their work for public exposition. Discussion is probably the most common and fastest way of transmitting information between two people, and so having the ability to describe orally one's thoughts and concerns becomes a very important quality, especially when describing technical terms or concepts.

Another very important question for anyone who teaches mathematics is: "What do I want my students to feel?" Do I want my students to be confident in their knowledge of the concepts I am teaching? Do I want my students to feel free to ask questions in class or free to come to my office hours? Do I want my students to experience the good feeling after having solved a difficult problem? Several studies have considered the impact of reform efforts on student attitudes (Tucker & Leitzel, 1995), and Douglas (1987) mentions wanting students to feel ownership over the material, but little is mentioned in these works about relieving student anxiety or increasing a students' pleasure in doing mathematics.

The affective part of the interviews revealed many surprising findings, together with some more expected ones. As with the surveys there is no room here for reporting from all the interviews. I have selected the following quotations from the students to illustrate some of the differences between the course objectives and the students' experiences. I like to underline that we did not speak explicitly about confidence, emotion or attitude during the interviews; they are simply artifacts of my selection of different interviews for this particular paper.

Confidence

Student 1: *Confidence – I don't know.., well you know, when you are in 9th grade you are the best, and when you are in the gymnasium you are one of the best, but here I'm just one in the crowd of students who doesn't understand what the lecture is about…*

Student 2: *I used to be very open and in the gymnasium class I asked directly when I didn't understand but here I'm in a lecture hall with almost 200 students and I don't know more than 20 of them and I don't dare to ask questions that maybe are stupid…I wait and ask someone of my friends after the lecture*

Student 2: *My project group consists of three girls and four boys and two of the boys are very smart and you can bet that they have the solutions to next week's project problem already the first day and it really kills my confidence that they have done it before the rest of us even have understood what it's all about…*

Emotion

Student: *In the beginning of the course people were more relaxed. Now it is like everybody is so tensed and irritated…you really don't ask anyone for a solution of a problem directly any longer, you have to take it slow…*

Interviewer: Could you elaborate on that, please?

Student: *Well, you know… there is like 10 – 15 students who really can solve the project problems on their own now and they know it and we know it. So there is a hierarchy that everybody is more or less aware of…but if you* *take it slow and act friendly and properly humble, well then you can get the solution in a couple of hours…*

Interviewer: Then what?

Student: *Then you share it with the rest of your group… and then we try to understand the solution in all details and prepare to write our own report…*

Attitude

Student: *I really don't emphasis on deep understanding any more if you know what I mean, it has to wait… at first I tried to understand, really to understand what was going on. Now I focus on technical expertise instead.*

Student: *The first one and a half or maybe even the first two years are just something you have to live through, like something necessary evil – then you can start study what you really are interested in. I'm really not interested in or fond of mathematics so I just have to live through it and "survive" until the fun starts in the third year.*

Student: *I'm really not interested in the overarching goals of the course, I expect that the teachers or whoever in charge know what they are doing and I just keep doing my tasks so to speak…*

The interviews also revealed rather surprising gaps in the students' knowledge of linear algebra. Only one third of the students could for instance describe the concept of rank, while no one was ready to give an answer to the somewhat challenging questions: "*What is linear algebra*?" After some time I usually added the questions: "Is it a formal game?", "Is it a set of abstract structures?", "Is it a language?", "Is it a tool with which to investigate natural phenomena?" A vast majority of the students selected the last alternative, but I find it intriguing how these students miss the opportunity to see an overall picture and merely choose to study in order to learn facts and procedures.

Conclusions

With all the emphasis from mathematicians and mathematics educators to change this specific algebra course to a course with more focus on understanding, the result was of course a real disappointment. Linear algebra is hard to learn, it has been known for many years as one of the many obstacles that exist within introductory university mathematics.

> During the sixties, at a conference in Zürich, I made the acquaintance of a charming old man who was none other than Plancherel – of Plancherel's theorem – and who, during a very interesting conversation, insisted on the fact that of all the teaching he had done that of linear algebra seemed to be by far the most difficult for students to understand. Thirty years later the situation

> does not seem to have changed very much and we can assure Plancherel that he is in good company. (Revuz, 200, p. xv)

Nevertheless, this is not a satisfactory state of art and we must continuously keep working on the improvement of teaching, learning, and assessment of linear algebra to increase its accessibility to more students.

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Appendix

Chalmers Technical University / Göteborg University
Course evaluation 1 Algebra M1/TD1 2001/2002

Please mark with an X that corresponds with your reaction to the suggestions below. Some of the questions are similar, which is done with intent to increase the validity. Thank you for your assist.

In general – how do you encounter the course?

(It often says ’the teachers’ but think ’the teacher’ if that is better)	Agree totally	Agree partly	Don’t know	Agree hardly	Agree not at all
1. It is always easy to know the standards of work expected	❑	❑	❑	❑	❑
2. The course helps me to develop my problem-solving skill	❑	❑	❑	❑	❑
3. The teaching staff in the course motivates me to do my best	❑	❑	❑	❑	❑
4. The workload is too heavy	❑	❑	❑	❑	❑
5. This course sharpen my analytical skills	❑	❑	❑	❑	❑
6. I usually have a clear idea of where I am going and what is expected of me in this course	❑	❑	❑	❑	❑
7. The staff put a lot of time into commenting on my work	❑	❑	❑	❑	❑
8. You only need a good memory to do well on this course	❑	❑	❑	❑	❑
9. The course help me develop my ability to work as a team-member	❑	❑	❑	❑	❑
10. As a result of this course, I feel more confident about tackling unfamiliar problems	❑	❑	❑	❑	❑
11. The course improves my skills in written communication	❑	❑	❑	❑	❑
12. The staff seems more interested in testing what I have memorized than what I had understood	❑	❑	❑	❑	❑
13. It is often hard to discover what is expected of me in this course	❑	❑	❑	❑	❑
14. I am generally given enough time to understand the things I have to learn	❑	❑	❑	❑	❑
15. The staff makes a real effort to understand difficulties I might be having in my work	❑	❑	❑	❑	❑
16. The staff normally gives me helpful feedback on how I am going	❑	❑	❑	❑	❑
17. The teachers are very good in explaining things	❑	❑	❑	❑	❑
18. There are too many examination tasks on plain facts	❑	❑	❑	❑	❑
19. The teachers work hard to make the subject interesting	❑	❑	❑	❑	❑
20. I feel a strong pressure to do well in this course	❑	❑	❑	❑	❑
21. The course helps me to develop my ability to plan my own work	❑	❑	❑	❑	❑

In general – how do you encounter the course?

(It often says 'the teachers' but think 'the teacher' if that is better)	Agree totally	Agree partly	Don't know	Agree hardly	not at all
22. It is so much to cover in the course, that there is no way that it all could be thoroughly comprehended	❑	❑	❑	❑	❑
23. The teachers made it clear from right the start what they expected from the students	❑	❑	❑	❑	❑
24. Overall, I am satisfied with the quality of the course	❑	❑	❑	❑	❑
25. The examination helps me to understand the content better	❑	❑	❑	❑	❑
26. To write a journal is of great importance for my learning of mathematics	❑	❑	❑	❑	❑
27. The teachers encourage us to try our own ideas	❑	❑	❑	❑	❑
28. The course is far too burdensome	❑	❑	❑	❑	❑
29. The teachers notice what the students need to get further explained in the content	❑	❑	❑	❑	❑
30. The lectures are effective and clear	❑	❑	❑	❑	❑
31. The teachers actively try to find why certain topics are difficult for us	❑	❑	❑	❑	❑
32. We are encouraged to find our own solutions to the problems	❑	❑	❑	❑	❑
33. The lectures are clear and distinct	❑	❑	❑	❑	❑
34. The teachers discusses with us about how we think about the problems	❑	❑	❑	❑	❑
35. The teachers adjust their teaching according to what the students find difficult	❑	❑	❑	❑	❑
36. The journal writing is mainly about information	❑	❑	❑	❑	❑
37. The course strengthens my ability to discuss with others in a trustworthy and reasonable way	❑	❑	❑	❑	❑
38. The teachers ask us to recapitulate the content and to highlight the significance it has	❑	❑	❑	❑	❑
39. In the examination I am expected not only to show what I have learnt, but also to apply my knowledge theoretically or practically	❑	❑	❑	❑	❑
40. The teachers have useful comments on my work in the course	❑	❑	❑	❑	❑
41. The course helps me to become better in explaining to others	❑	❑	❑	❑	❑
42. The teachers encourage us to use our own ideas	❑	❑	❑	❑	❑
43. The teachers want the examination to show if I can generalize my knowledge into new situations	❑	❑	❑	❑	❑
44. I think this is an interesting and rewarding course	❑	❑	❑	❑	❑

THANK YOU VERY MUCH FOR YOUR HELP! Please write your own comments below:

Appendix

Chalmers Technical University / Göteborg University
Course evaluation 2 Algebra M1/TD1 2001/2002

Please mark with an X that corresponds with your reaction to the suggestions below. Some of the questions are similar, which is done with intent to increase the validity. Thank you for your assist.

In general – how did you encounter the course?

(It often says ’the teachers’ but think ’the teacher’ if that is better)	Agree totally	Agree partly	Don’t know	Agree hardly	Agree not at all
1. It was always easy to know the standards of work expected	❑	❑	❑	❑	❑
2. The course helped me to develop my problem-solving skill	❑	❑	❑	❑	❑
3. The teaching staff in the course motivated me to do my best	❑	❑	❑	❑	❑
4. The workload was too heavy	❑	❑	❑	❑	❑
5. This course sharpened my analytical skills	❑	❑	❑	❑	❑
6. I usually had a clear idea of where I was going and what was expected of me in the course	❑	❑	❑	❑	❑
7. The staff put a lot of time into commenting on my work	❑	❑	❑	❑	❑
8. You only needed a good memory to do well in the course	❑	❑	❑	❑	❑
9. The course helped me develop my ability to work as a team-member	❑	❑	❑	❑	❑
10. As a result of the course, I feel more confident about tackling unfamiliar problems	❑	❑	❑	❑	❑
11. The course improved my skills in written communication	❑	❑	❑	❑	❑
12. The staff seemed more interested in testing what I had memorized than what I had understood	❑	❑	❑	❑	❑
13. It was often hard to discover what was expected of me in the course	❑	❑	❑	❑	❑
14. I was generally given enough time to understand the things I had to learn	❑	❑	❑	❑	❑
15. The staff made a real effort to understand the difficulties I might have in my work	❑	❑	❑	❑	❑
16. The teaching staff normally gave me helpful feedback on how I was performing	❑	❑	❑	❑	❑
17. The teachers was very good in explaining things	❑	❑	❑	❑	❑
18. There were too many examination tasks on plain facts	❑	❑	❑	❑	❑
19. The teachers worked hard to make the subject interesting	❑	❑	❑	❑	❑
20. I felt a strong pressure to do well in the course	❑	❑	❑	❑	❑
21. The course helped me to develop my ability to plan my own work	❑	❑	❑	❑	❑

In general – how did you encounter the course?

(It often says ’the teachers’ but think ’the teacher’ if that is better)	Agree totally	Agree partly	Don’t know	Agree hardly	not at all
22. It was so much to cover in the course, that there was no way that it all could be thoroughly comprehended	❑	❑	❑	❑	❑
23. The teachers made it clear from right the start what they expected from the students	❑	❑	❑	❑	❑
24. Overall, I am satisfied with the quality of the course	❑	❑	❑	❑	❑
25. The examination helped me to understand the content better	❑	❑	❑	❑	❑
26. To write a journal is of great importance for my learning of mathematics	❑	❑	❑	❑	❑
27. The teachers encouraged us to try our own ideas	❑	❑	❑	❑	❑
28. The course was far too burdensome	❑	❑	❑	❑	❑
29. The teachers noticed what of the content the students needed to get further explained	❑	❑	❑	❑	❑
30. The lectures were effective and clear	❑	❑	❑	❑	❑
31. The teachers actively tried to find why certain topics were difficult for us	❑	❑	❑	❑	❑
32. We were encouraged to find our own solutions to the problems	❑	❑	❑	❑	❑
33. The lectures were clear and distinct	❑	❑	❑	❑	❑
34. The teachers discussed with us about how we thought about the problems	❑	❑	❑	❑	❑
35. The teachers adjusted their teaching according to what the students found difficult	❑	❑	❑	❑	❑
36. The journal writing was mainly about information	❑	❑	❑	❑	❑
37. The course strengthened my ability to discuss with others in a trustworthy and reasonable way	❑	❑	❑	❑	❑
38. The teachers asked us to recapitulate the content and to highlight the significance it has	❑	❑	❑	❑	❑
39. In the examination I was expected to show what I had learnt, and also to apply this knowledge theoretically or practically	❑	❑	❑	❑	❑
40. The teachers gave useful comments on my work in the course	❑	❑	❑	❑	❑
41. The course helped me to become better in explaining to others	❑	❑	❑	❑	❑
42. The teachers encouraged us to use our own ideas	❑	❑	❑	❑	❑
43. The teachers wanted the examination to show if I can generalize my knowledge into new situations	❑	❑	❑	❑	❑
44. I think it was an interesting and rewarding course	❑	❑	❑	❑	❑

THANK YOU VERY MUCH FOR YOUR HELP! Please write your own comments below:

Mathematical Thinking and Mathematical Achievement: Research Issues

Anne Watson
University of Oxford

In this presentation we will identify and discuss some research issues which arise from considering relationships between mathematical thinking and mathematical achievement.

During the 80s and 90s several countries reorganised their mathematics curricula and assessment systems to encourage and reward more creative thinking, deeper understanding, applications and a problem-solving approach to mathematics teaching and learning. These developments were only briefly in place when the TIMMS report triggered something of a backlash towards a 'back to basics' approach (notably in California and other parts of the USA) or a rather prescriptive 'numeracy' curriculum (in the UK). Nevertheless there remains the desire that learners should understand their mathematics enough to be able to use it flexibly in a variety of situations, so the phrase 'mathematical thinking' is significant in the education discourse of many countries, alongside the desire for skills outcomes. In addition, there is also a view (illustrated in the teaching methods of Phoenix Park school in Boaler's work (1997)) that if learners are encouraged to think mathematically they will become better learners of all mathematics, including the 'basic' skills.

Mathematical achievement

Governments' need for tests and test results with which to monitor achievement means that, from their point of view, any teaching methods need to be accountable in some way. Methods which do not lead to improved results, or which may even depress results, cannot be sustained politically. Hence, if a test of mathematics achievement involves groupwork on an unseen, unfamiliar problem but the teaching has been focusing on the rote-learning of replicable, familiar techniques then something has to change. What is more usual, of course, is that the test is about selection and performance of techniques and the teaching has mainly been about preparing for such questions. Unsurprisingly, many learners become confused by all the different things they have to learn to do, and may not recognise when it is appropriate to do them anyway. Further, accumulation of such confusion means that many lack confidence to approach problems in any way other than finding and using a semi-recalled technique. Year after year the same kinds of question lead to the same kinds of wrong answer, and examiners' reports make the same kinds of comments about the weaknesses of candidates.

So a first research issue might be the questions:

What is mathematical achievement? How is it measured?

We also need to decide whether to describe what happens currently, or whether instead to pursue the questions philosophically, epistemologically or ideologically.

Values

Inevitably we will be talking about value, with the attendant question: value to whom? One tension we might discuss is whether education is for developing individuals, introducing them to the most worthwhile things humans can do, getting them excited about future possibilities and thus helping them become good citizens, or is about ensuring future workforces have appropriately marketable skills and can do the things society expects of them, and thus become good citizens! We could also discuss what values politicians express when they panic, as most of them did, about international maths comparisons. Also, what values do university mathematicians bring into play when they complain about falling standards.

Mathematical thinking

Dalia Aralas (2001), a mathematics educator from Malaysia, points out that many different kinds of activity can be described as mathematical thinking and there is no universally agreed definition. She points out that 'thinking' could mean: reasoning, understanding, competence, skills, knowledge, behaviour, intelligence, development, problem solving or disposition.

Typically, researchers refer to Polya (1962), Mason et al (1982) or Schoenfeld (1985) to find definitions. However, while Schoenfeld describes a detailed heuristic for mathematical problem-solving, Polya and Mason write more about their experience of doing mathematics which might result in no solutions at all but rather a lot of further questions! This sense of mathematical experience can also be found in the work of Hadamard (1945), and Davis and Hersh (1981), but with rather less attention to practical application. It is the naming of aspects of mathematical thinking which makes them available to be discussed and thought about and, hence, for pedagogical questions to be posed about them.

An appropriate question at this point would be:

How can the personal experiences of successful mathematicians inform us in general about mathematical thinking?

Successful learners

A further source of descriptions of mathematical thought is Krutetskii (1976), whose detailed research into the features of successful learners of mathematics produces the following list of common abilities to:

- Extract formal structures from problems
- Generalise
- Operate symbolically
- Work with spatial concepts
- Reason logically
- Use shortcuts
- Be flexible between approaches
- Reverse chains of thought
- Achieve clarity and economy in argument
- Memorise mathematical knowledge

Of course, we need to ask ourselves whether there is anything absolute about his description. In other words, was his identification of successful mathematicians something we would all agree with, or did the system select those who had these characteristics and call them successful mathematicians? I am not directing this question solely at the old Soviet Union. I can also ask it of any author who says things like: "the strongest mathematicians were able to self-correct their work". Is the relationship causal, and if so which way and how does it operate, or does our definition of 'strongest mathematician' *necessarily include* the propensity to self-correct?

If we identify mathematical thinking by looking at those who have achieved in mathematics, we should also look at what is implied by the word 'achieve'? In the UK, students are taken into University on the basis of results in examinations which can be the result of detailed and intensive examination practice. Then lecturers complain that they 'cannot think for themselves' and cannot bring knowledge into play in other contexts. The way many of these learners 'think mathematically' is to look for a similarity with something already done, and thus simplify what they are asked to do. Both Polya and Schoenfeld give this as a strategy, but when searching for templates is used as the *main* strategy of performance it offers little insight into the ability of learners to work with structures and meanings, and offers them little insight into mathematics.

A question emerges:

What ways of thinking allow learners to achieve in particular high-stakes assessments? What mathematical thinking skills could we expect those who are successful to have?

Advanced mathematical thinking

The final source of descriptions of mathematical thinking for this paper is the analysis of mathematics itself, and the identification of the mental constructions and reconstructions necessary to understand and develop it. For this, we have to have a view of what mathematics is. Writers in this area see it as a recursive structure of successive abstractions, in which learners experience ranges of examples which are then reflexively abstracted (Dubinsky, 1991) and reified (Sfard, 1991) in order to become objects for the next layer of work, or tools with which to develop more complex mathematics (Douady, 1986). Hence, their descriptions of mathematical thinking tend to be about being able to generalise from special cases (Mason, 1998), exemplify from generalities (Watson and Mason, forthcoming), change between representations (Dreyfus, 1991), habitually see abstractions (Tall, 1991) and so on. Those who would have mathematics only as a descriptive device and tool for engineering, science, or everyday living will find little to excite them here. The significant text (Tall, 1991) is supposed to refer to the ways one needs to think about advanced mathematics. However, the title 'Advanced Mathematical Thinking' is ambiguous, and this leads to some further questions:

> *Can the characteristics of mathematical thinking described as 'advanced' be applicable to learners at all stages? Can teachers encourage learners to think in advanced ways about simple mathematics?*

Thinking and achievement

In spite of their success in the TIMSS report, several high-achieving countries are trying to move away from an approach to mathematics teaching which focuses mainly on performance of skills (for example, Goh, 1997). Instead, broader definitions of mathematical achievement relating to the reform movements of the 80s and 90s are replacing traditional agendas. There is recognition that developing mathematical thinking skills might lead to achievement in mathematics which is more meaningful than traditional approaches have produced. That is, it may lead to deep learning of mathematics (Marton and Säljo, 1997). Further, students who have been explicitly encouraged to think deeply and creatively about mathematics seem to achieve more in very ordinary kinds of mathematics test (Boaler, 1997; Adhami et al, 1998) as well as in tests which relate directly to mathematical thinking. Boaler's results are, however, only from one school and Adhami's results are, so far, only from schools involved in a supported project. On a larger scale we could point to typical teaching methods in Japan and Hungary, both of which use problem-solving and discussion approaches in the classroom, and say that they produce strong and successful mathematicians.

The use of variety of examples in Japan is particularly interesting, since learners naturally seek out what is the same and what is different. Thus being given a variety of raw material for that process is useful (Becker and Shimada, 1997; Marton and Trigwell, 2000).

Finally, the big question is, of course:

Does the development of mathematical thinking improve mathematics achievement?

But, as we have seen, this has to be answered within context and with careful definition.

Future research

Research in this domain is patchy and, as I have pointed out, may be based on assumptions about language and meaning which cannot be taken-as-shared. However, if there are going to be any convincing answers to questions about teaching methods, teaching foci and the development of creative, autonomous mathematics students this would be a fruitful, if difficult, area to study.

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Forum for Young Researchers

A **Forum for Young Researchers**, titled *New Directions, Problems and Solutions* was held in conjunction with the MADIF 3 conference. The Forum was chaired by **Gilah Leder** (La Trobe University, Australia), and consisted of two parts:

- presentations by doctoral students, most of whom were part of the newly formed *graduate school*, and
- a workshop lead by **John Mason** (the Open University, UK) and **Anne Watson** (Oxford University, UK). The activities selected challenged participants to adopt researchers' perspectives rather than the more familiar roles as learners or teacher-educators.

As part of their presentation the doctoral students were asked to:

- give a brief overview of their project (or interesting paper they had selected for discussion), and
- identify a stumbling block or difficulty they anticipated or had already experienced in their research.

A group discussion about the problem(s) identified and possible means for resolving the difficulties raised followed each presentation.

The following students gave a presentation:

Andreas Andersson – Students´ Understanding of Discrete Mathematics in Higher Education

Jesper Boesen – A Short Presentation Concerning My Research Interest in Testing and Assessment

Sivbritt Dumbrajs – Collaboration and Communication as Part of Mathematics Instruction

Torbjörn Fransson – To What Extent Are Students Able to Create a Formation of Concepts by Problem Solving?

Monica Johansson – Textbooks in Mathematics (for Fifth Graders)

Per Nilsson – Experimentation as a Tool for Discovering Advanced Mathematical Concepts

Constanta Olteanu – Students' Development of Algebraic Ability and Understanding in Upper Secondary School Project Description

Semiotik och matematikdidaktik

Arne Engström
Örebro universitet

Inledning

Ett växande och intressant område inom det internationella matematikdidaktiska forskningsfältet utgörs av tvärsnittet mellan semiotik och matematikdidaktik. Föreliggande paper presenterar några grundläggande frågeställningar inom detta område. Jag har tidigare delvis behandlat detta område i Engström (2001), där jag försökte tematisera förhållandet mellan *rationalitet* och *intersubjektivitet.* Avsikten var att förstå hur elever utvecklar en förståelse för nödvändig kunskap genom en social interaktion med andra. Matematiken ges därmed en kommunikativ karaktär.

I detta inledande avsnitt ska jag ta en utgångspunkt i den matematiska notationen för att belysa några matematikdidaktiska frågeställningar som berör semiotiken. Semiotik betyder teckenlära. Från en matematikdidaktisk synvinkel är det av intresse att studera den mening vi tilldelar matematiska tecken, som vi använder i olika sammanhang.

Matematisk notation

Matematik är intimt förknippat med symboler och formler av olika slag. För den oinvigde ter sig dessa många gånger som något mystiskt, för att inte säga magiskt, något som tillhör en förtrollad värld bara några få förunnade att känna till.

Under historiens gång har den matematiska notationen tagit sig olika uttryck allt från några regelbundet grupperade streck inkarvade i vargben, sumerernas kilskrift inpräglade i lertavlor, egypternas hieroglyfer på papyrusrullar, kinesers, japaners, indiers, arabers och andra kulturers olika teckensystem till våra dagars närmast standardiserade matematiska notation. Men matematisk notation är inte bara symboler och formler utan också text. I detta paper ska allt detta benämnas som *tecken*.

Till skillnad från matematiken själv, med dess karaktär av *nödvändighet*, är dess beteckningar *sociala konventioner* och därmed arbiträra (godtyckliga). Det finns ingen nödvändighet i att vi låter representera ett visst tal med t.ex. ”3” eller ”4–1”. Men har vi väl kommit överens om att beteckna ett visst tal på ett visst sätt så följer därav ett antal konsekvenser.

Vissa tecken som vi använder oss av inom matematiken har en intressant historia. Kring flera av våra vanliga matematiska tecken har det stått en lång strid om hur det ska se ut.

Matematiska tecken står *för* något, dvs. *representerar* en idé eller tanke. Människor uttrycker matematiska tankar med hjälp av tecken av olika slag. Vi använder tecken till att tänka med, uttrycka och kommunicera våra tankar med andra människor.

Tecken att tänka med

För varje teori med anspråk att söka förklara lärande blir det av intresse att studera vilken mening vi tilldelar olika tecken när vi använder dem för att tänka och kommunicera vårt tänkande med. För att göra oss förstådda bland andra måste det finnas en viss kongruens i denna meningstilldelning av tecknen. Ibland förefaller oss användandet av tecken så självklart, framför allt i matematik, att vi kanske inte funderar så mycket över tolkningen av tecknen. Semiotik handlar om *tolkning av tecken*.

Den klassiska matematiken var retorisk, dvs. den framställdes med ord eller ordförkortningar. Under medeltiden skedde en betydelsefull förändring, mot en högre grad av abstraktion av matematiken, den s.k. symboliska abstraktionen, genom mötet mellan den orientalisk-medeltida och den grekiska matematiken vilket resulterade i algebrans födelse i början av 1600-talet (Thompson, 1991). Med den symboliska abstraktionen, den analytiska geometrin samt infinitesimalkalkylen inleddes den moderna matematiken.

De viktiga abstraktionssteg som togs kan följas i utvecklingen av den matematiska notationen, exempelvis genom införandet av särskilda tecken för ”intet” (nollan), det ”okända” (x,y,z), ”det ospecificerade” (a,b,c), samt ”det förmodade lilla” inom infinitesimalkalkylen (Sällström, 1991). Utvecklingen av den matematiska notationen motsvarar en allt högre grad av abstraktion, vilket var en förutsättning för den snabba utvecklingen av matematikens tillämpning inom naturvetenskapen. Det handlar inte främst om en abstraktion av den direkta verkligheten, utan om en kedja av abstraktioner, *abstraktioner av abstraktioner*. I varje abstraktions- eller generaliseringssteg har vi att göra med tecken eller symboler, vilka representerar (generella abstrakta) objekt, som härstammar från en abstraktion och en generalisering.

Matematik kan uppfattas som en generalisering av tecken eller representationssystem, framhåller Hoffmann och Plöger (2000). En väsentlig del av den moderna matematiken handlar om *symbolisering*.

Semiotik

Semiotik betyder läran om tecken. Själva termen semiotik används på lite olika sätt. Den kan dels uppfattas som en *metod*, som ett sätt att närma sig ett undersökningsobjekt, dels som en *särskild vetenskap*. En företrädare för denna senare uppfattning är Sonesson (1989, 1992).

Sonesson utvidgar semiotiken till en *betydelselära*, som studerar hur mening uppstår generellt, hur tecken och betydelser fungerar i allmänhet och ser därmed semiotiken som en nomotetisk (lagsökande) vetenskap.

Teckenmodeller

Semiotik utgjorde hos de antika filosoferna en del av filosofin. Numera framstår den alltmer som en egen disciplin. Det är framför allt två pionjärer, schweizaren Ferdinand de Saussure, grundare av den moderna strukturella lingvistiken, och pragmatismens grundare, den amerikanske filosofen och matematikern Charles S. Peirce, som kommit att stå för två huvudinriktningar inom semiotiken. Intresset för både semiotiken och Peirces arbeten inom fältet har kraftigt accentuerats under 1990-talet.

Ferdinand de Saussure

En av de viktigaste teckenmodellerna utgörs av de Saussures distinktion mellan uttryck och innehåll, mellan den abstrakta ljudbilden *signifiant* (betecknande) och begreppet *signifié* (betecknat). I språket är uttrycket (en föreställning om) ett ljud, medan innehållet är ett begrepp, en idé eller (föreställningar om) personer, föremål och andra. Tecknet förhåller sig relativt godtyckligt till verkligheten. Tecknet uppstår först i och med att något betecknas och betecknandet är en viljeakt.

Charles S. Peirce

Peirce har sina filosofiska rötter hos Kant. Ett gemensamt drag hos dem är att verkligheten framstår som organiserad i några grundläggande kategorier inom ramen för vilka den först blir uppfattbar. Vi kan enligt Kant inte veta något om tinget *i sig*. Hos Peirce är världen endast tillgänglig för oss genom en förmedling via tecken. Föremål har ingen betydelse i sig. För tolkare av världen finns inga föremål, bara tecken.

Tecken har en föremålslig sida, en materiell komponent (t.ex. symboler på ett papper). En annan sida hänför sig till att tecken har en kulturhistorisk framvuxen och i en social interaktion manifesterad betydelse

Peirce arbetar med tre fundamentala kategorier, etthet *firstness*, tvåhet *secondness* och trehet *thirdness*. Alla fenomen i verkligheten uppfattas antingen som enskilda eller också som delade på två eller tre element. Ettheten bildar utgångspunkt. Det är något i sig själv. Tvåheten är något som står i relation till det förra och treheten är en förmedling av de båda andra. Tecknet är ett exempel på trehet och består av tre led: *representamen, objekt* och *interpretant*; eller uttryck, innehåll och tillämpningens särart.

Varje tecken kan kategoriseras i enlighet med de tre sätt på vilka var och en av de tre leden kan skifta: efter uttryckets egen natur, det slags relation som sammanbinder uttryck och innehåll och tillämpningens särart (Sonesson, 1992). Vi kan beskriva detta med figuren nedan.

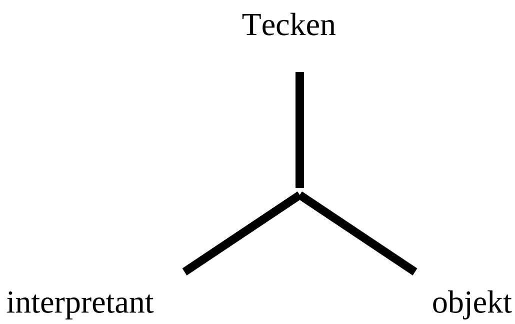
Trichotomy / Category	I. of the representamen	II. of relation of object	III. of relation to interpretant
Firstness	qualisign	icon	rheme
Secondness	sinsign	index	dicent
Thirdness	legisign	symbol	argument

Figur 1.1. Peirces teckenklasser (Nöth, 2000).

För Peirce råder det ett *triadiskt* förhållande mellan tecken, objekt och interpretant.

> Ett tecken är något som för någon står för något i viss bemärkelse eller kapacitet. Det är riktat till någon det vill säga skapar ett motsvarande tecken i personens medvetande, eller möjligen ett mer utvecklat tecken. Tecknet som det skapar kallar jag *interpretanten* av det första tecknet. Tecknet står för något dess *objekt* (Peirce, citerat i Fiske, 1990, s. 63).

Följande figur kan illustrera detta förhållande:



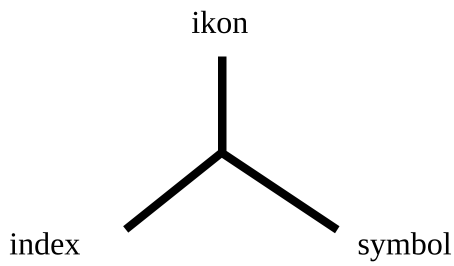
Figur 1.2. Förhållandet mellan tecken, objekt och interpretant hos Peirce.

Interpretanten är inte tecknets användare utan syftar på ”den egentliga betecknande effekten” (Peirce, citerat i Fiske, 1990, s. 63). Interpretanten av tecknet är resultatet av användarens erfarenhet av tecknet och de sammanhang där detta ingår. Gränserna sätts av sociala konventioner. Variationerna inom dessa är en fråga om sociala och psykologiska skillnader mellan användarna (Fiske, 1990).

Den triadiska strukturen i Peirces teckenbegrepp motsvaras för vår del av spänningsfältet mellan det matematiska sakförhållandet, representationen och tolkningen.

Peirce behandlade också tre *teckenkategorier,* dvs. olika förhållanden mellan tecknet och objektet det hänvisar till. En *ikon* liknar objektet på något sätt; i ett *index* finns ett direkt samband i verkligheten mellan tecknet och objektet; i en *symbol* saknas både likhet och samband. Ett fotografi är en ikon, rök ett index som är kopplat till (indikerar) eld och ett ord är en symbol (Fiske, 1990).

Även teckentyperna kan visas i en triadisk modell.



Figur 1.3. Förhållandet mellan ikon, index och symbol enligt Peirce.

Förmågan till varseblivning och tolkning av tecknen är inget givet, utan kan vara ganska olika. Det ger varseblivningen och tolkningen en hypotetisk karaktär. Varje varseblivning och tolkning uppfattas som bildandet av en hypotes, eller som resultatet av en *abduktion* (se nedan).

Semiotik och matematikdidaktik

I detta avsnitt ska semiotikens betydelse för matematikdidaktiken diskuteras, varvid framför allt tre dimensioner, det matematiska sakförhållandet, representationen och tolkningen av det, kommer att behandlas.

Matematikdidaktik och meningsskapande

Elevers meningsskapande processer har traditionellt studerats genom olika samtals- och diskursanalytiska studier. Semiotikens klara fördelar gentemot dessa studier är att *alla* uttryck, språkliga som icke-språkliga, t.ex. de för matematikdidaktiken så centrala matematiska symboler, representationer av olika slag (diagram, tabeller, etc), åskådningsmateriel och hjälpmedel av olika slag, räknas in (se t.ex. Seeger, 2000).

Semiotiken berör några viktiga dimensioner av matematikdidaktiken, nämligen

- objektet, eller sakförhållandet
- representation (interna och externa) av detta objekt eller sakförhållande
- tolkningen av det.

Ett objekt eller sakförhållande kan bara förstås och kommuniceras när det representeras och tolkas i någon form. Spänningsfältet mellan sakförhållande, representation och tolkning gör därför semiotiken mycket intressant för matematikdidaktisk forskning. Det handlar om vilken mening människor tillskriver de olika tecken som de använder sig av inom matematiken. Lärande uppfattas som en *teckenprocess* (Hoffman & Plöger, 2000).

Sakförhållandet

Sakförhållandet, eller det matematiska objektet, är ett abstrakt eller idealt matematiskt objekt eller ett problem uppkommet ur vardagserfarenheter. Det bör poängteras att semiotiken inte förutsätter en reell existens av ett objekt, dvs. platonism. Som filosofisk position är platonismen högst problematisk (se Engström, 2001).

Hypostasering innehåller både generaliserande och abstraherande processer i en utveckling till ett mentalt objekt, som hanteras av matematiker *som om* det existerade.

Frågan om hur matematisk kunskap utvecklas har traditionellt hanterats mellan de två ytterligheterna rationalism och empirism. Det finns en paradox i matematikfilosofin som avser hur man kan avvisa empirismen som grund och samtidigt förklara matematikens stora tillämpbarhet på verkligheten. Kants väg ut ur detta dilemma var konstruktivism.

Jean Piaget och Charles Peirce har båda sina rötter i den Kantska filosofin. Det finns en intressant parallell i hur de båda försöker ersätta de aristoteliska begreppen om abstraktion och generalisering i sina respektive diskussioner om en matematikens epistemologi.

Piaget överskrider Kant genom att visa att de Kantska åskådningsformerna och kategorierna inte är aprioriska utan konstrueras genom *reflekterande abstraktion* och *konstruktiv generalisering* (se Piaget, 1985, 2001). Piaget gör en distinktion mellan *empirisk* och *reflekterande abstraktion* och förklarar matematikens tillämpningar med att den matematiska kunskapen har en grund i konkreta handlingar som att ordna, gruppera, föra samman etc. Dessa utvecklas till reversibla operationer genom en reflekterande abstraktion och konstruktiv generalisering, vilka sedan i sin tur utgör utgångspunkt för vidare abstraktioner och generaliseringar.

Peirce diskuterar ett tredje alternativ till slutledning vid sidan av deduktion, att följa en ”regel” på ett enskilt fall för att uppnå ett ”resultat” och induktion, en omvänd slutledning från ett enskilt fall och resultat till en regel. Induktionen är visserligen användbar i många sammanhang, men är ingen logiskt giltig slutledningsform. Peirce kallar sin tredje slutledningsform för *abduktion*. Enligt Peirce är abduktionen vardagens slutledningsform som leder oss till den mest sannolika förklaringen till ett fenomen som väcker vår förvåning. Peirce kallar sitt begrepp *hypostaserande abstraktion*. Abduktionen innebär hypotesbildning och sanningsantagande av denna hypotes.

Det finns en annan intressant parallell mellan Peirce och Piaget. Deras teorier om hypostaserande respektive reflekterande abstraktion är ett svar på frågan om hur ny kunskap utvecklas. Platon hävdade i sin dialog Menon att idén om ny kunskap var en paradox. I stället handlar det om en återerinring. Genom den speciella samtalstekniken, frågor och prövningar av svaren på frågor, låter Platon Sokrates utveckla sin majevtiska metod, dvs. förlösa vetande som redan finns.

Frågan om hur ny kunskap uppstår i matematik kan förstås utifrån begreppen W*issensbegründung* och *Wissensentwicklung* (Jahnke, 1978; Steinbring, 2000), dvs. *grundläggande* av vetandet och *utveckling* av vetandet.

Å ena sidan är matematiskt vetande logiskt konsistent och hierarkiskt ordnat; varje ny kunskap kan deduceras från den gamla och därmed är den på basis av den logiska strukturen inte ny. Å den andra sidan uppstår det faktiskt ny och hittills okända insikter i matematik, t ex genom lösningar av problem och genom bevisning av förmodanden (t ex är existensen av oändligt många primtal en ny insikt för den lärande, men en formell slutsats från talbegreppets axiom).

Matematisk kunskap kan förstås på två sätt:

- *logisk struktur* – tautologisk, logiskt konsistent, strukturellt nätverk
- *matematiska objekt* – i varje struktur kan man identifiera och konstruera kunskapselement och begrepp, vilka öppnar nya frågor och problem, vilka ännu inte är inordnade i den matematiska kunskapsstrukturen (problem som ännu inte är ”lösta”).

Representationens problem

Ett matematiskt sakförhållande inte bara *kan* representeras på olika sätt, utan *måste* representeras på något sätt för att vi ska kunna operera eller handla (vanligtvis räkna) och därigenom omvandla det på något sätt. Genom att representera ett sakförhållande, tolkar vi det, dvs. *tilldelar det en mening*. Ett matematiskt sakförhållande framställs (representeras externt) genom ett aritmetiskt uttryck, en ekvation eller olikhet, en integral eller differentialekvation. Genom ett regelstyrt opererande, dvs. vi utför de aritmetiska operationerna (addition, subtraktion, etc.), löser ekvationen, beräknar integralen, leder detta till nya framställningar av sakförhållandet, vars tolkning i den kontext vari sakförhållandet ges ger ny information om detta sakförhållande. Nedanstående framställning bygger i huvudsak på Pescheck (2000).

Vid de flesta matematiska aktiviteter sker vid sidan av en regelstyrd omformning av symboliska framställningar också en ”översättning” mellan olika framställningar. Att bedriva matematik innebär väsentligen också en interaktion mellan människa och framställningsform (på ett papper eller en bildskärm).

Därvid uppkommer ett antal frågor: Vad är det egentligen som framställs och hur framställs det inom matematiken? Normalt är det *abstrakta*, dvs. något som inte är direkt tillgängligt för våra sinnen, *relationer*, t.ex. ett tal som en kvantitativ relation, en ekvation som en relation mellan två variabla storheter, en funktion som en relation mellan elementen i två mängder. Dessa abstrakta relationer framställs genom skrivna symboler, eller kan numera också materialiseras som förnimbara objekt på en datorskärm. Abstrakta relationer kan också materialiseras genom skriftspråk.

Är de redan specifikt abstrakta, som vi materialiserar i matematiska symboler, eller blir de specifikt matematiska genom att vi materialiserar dem på ett specifikt, matematiskt, sätt?

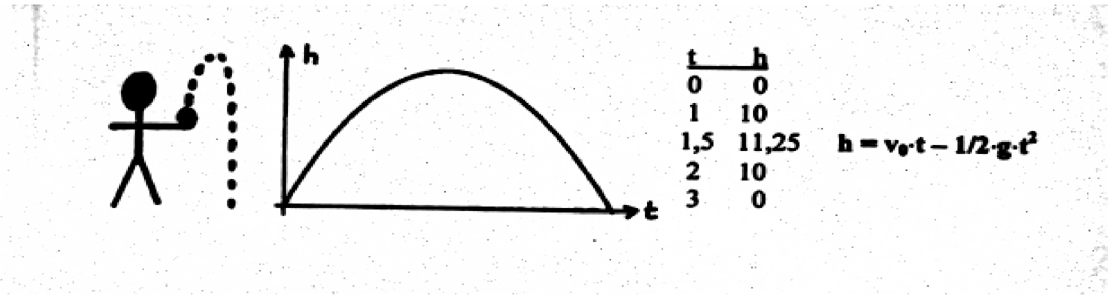
Det kan vara ekonomiskt att använda en och samma framställning för olika matematiska sakförhållanden, men varför använder man olika framställningsformer för att materialisera ett och detsamma matematiska sakförhållande?

Vad är det för mening att skilja mellan objekt och dess representation, alltså anta existensen av ett objektområde bortom dess materiella eller kognitiva framställningsformer?

Olika framställningsformer

Man kan språkligt beskriva hur någon kastar upp en sten i luften, hur stenen allt långsammare stiger, verkar stanna i luften och sedan börjar att falla, allt snabbare och slutligen faller på marken.

Denna framställning kan också visas *visuellt*. Man kan här skilja mellan ikonisk, schematisk och symbolisk framställning.



Figur 2.1. Ikonisk, schematisk och symbolisk framställning (Pescheck, 2000).

Den *ikoniska framställningen* gör en närmast kvasianalogisk beskrivning av sakförhållandet, i form av ett förenklat ”fotografi”.

Den *schematiska framställning* fokuserar på relationen mellan tid och höjd. Tabellen gör det på ett diskret sätt och den grafiska framställningen på ett kontinuerligt sätt. Båda framställningarna abstraherar från bestämda aspekter av sakförhållandet, något som fortfarande kan ses i den ikoniska framställningen, att ett föremål kastas upp och vilket föremål det handlar om och av vem det kastas.

Den *symboliska framställningen* går ett stycke längre. Den visar att förhållandet mellan tid och höjd är kvadratiskt, samt på vilket sätt höjden är beroende av utgångshastigheten och gravitationskraften. Den ger också möjlighet som de andra framställningsformerna saknar att genom en regelstyrd omformning få fram utgångshastigheten ur tid och höjd.

Det som skiljer de olika framställningsformer åt är:

Ikonisk framställningsform hänför sig till reellt synbara omständigheter och kan ge ett första inblick i ett sakförhållande, stimulera associationer och vara utgångspunkt för utvecklingen av schematiska framställningar, men tillhör egentligen inte matematikens område. Ikoniska framställningar används sällan i matematiska texter.

Schematiska och *symboliska framställningsformer* finner man däremot i varje lärobok i matematik, varför man måste förutsätta att de vid sidan av språkliga framställningar är särskilt viktiga för matematik och lärandet i matematik. Genom dessa framställningsformer görs vissa aspekter av ett sakförhållande mer intressanta än andra. Man måste ”kunna läsa” sådana framställningsformer, särskilt måste man veta vad man abstraherar från.

Medan schematiska framställningar ofta refererar till förnimbara mönster i referenskontexten och abstrakta relationer framställs med hjälp av förbindningslinjer, pilar osv., så framställs sådana relationer i symboliska framställningar i formaliserad form, dvs. med symboler. Betydelsen av symboler måste överenskommas eller förhandlas. Symbolerna görs ofta generella genom att *en* bestämd symbol får står för *alla* reella tal, *alla* punkter, eller *alla* integrerbara funktioner. Övergången till detta symboliska plan kan beskrivas som ett uttryck för *generalisering*.

En annan skillnad är att med symboliseringen följer en ökad rörlighetsgrad. Ytterligare en skillnad är att medan ikoniska och schematiska framställningar lämnar ett större utrymme för skilda tolkningar, så är innebörden i symboliska framställningar mer fastlagda, men också mer villkorade (jag kallar denna funktion g – det kan ingen förbjuda mig.)

För matematikens del är de båda framställningsformerna viktiga, framför allt karakteriseras matematiskt arbete av en växling mellan schematiska och symboliska framställningar.

> Das Wechselspiel zwischen schematischer und symbolischer Darstellung entspricht etwa dem Wechselspiel zwischen Intuition, Einsicht, Ideengewinnung einerseits und syntaktischer Rechtfertigung, Kontrolle, Auswertung andrerseits (Fischer, 1984, s. 157, citerat i Peschek, 2000, s. 8).

Den ovanstående diskussionen leder till en rad frågor om förhållandet mellan sakförhållandet och framställning:

- Hur kan man bortom representationen föreställa sig ett sakförhållande?
- Finns det över huvud taget ett matematiskt sakförhållande, en abstrakt relation, utan representation och hur är det i så fall ”närvarande”.
- ”Existerar” inte ett matematiskt sakförhållande först genom dess representation, vilket innebär att det är meningslöst att tala om representation *av* något abstrakt. Är inte framställningar mer representation *för* något abstrakt?

- Om man, på goda grunder, vill undvika att likställa abstrakta relationer och dess kognitiva eller materiella representationer, måste man formulera det annorlunda:
- Konstituerar sig inte en abstrakt relation, dvs. ett matematiskt objekt, först genom handhavandet med representationen och skapar sig en existens också genom detta?

Av stort intresse för denna diskussion är användningen av datorer. Datoranvändning ger för det första en specifik form av framställning och materialisering av matematik, och för det andra kräver och producerar den vissa framställningar av matematiska sakförhållanden. Förstärker datorer problemet med förhållandet mellan sakförhållandet och framställningen?

Man betonar ofta möjligheten av att kunna experimentera med olika framställningar och simulera olika utfall, dvs. framställningar. Den snabba tillgängligheten till och växlingar mellan olika framställningar anses av många stimulera och befrämja begreppsförståelsen och matematisk kreativitet. Ännu måste detta uppfattas som en hypotes att undersöka.

Representation som föreställning

Att förstå någonting innebär att kunna representera detta någonting, internt (föreställning) och externt. Varje kunskap yttrar sig eller förmedlas genom en intern representation, som föreställningar och betydelser. Tecken kan förstås både som intern och extern representation. Man kan från semiotisk synvinkel förstå lärandet som *en process av tillägnelse och utveckling av representationssystem* (Hoffmann & Seeger, 2000).

Tolkning

För att förstå och kommunicera ett matematiskt sakförhållande måste det representeras på något sätt, både internt och externt. Att representera innebär att tolka. Vi tilldelar de tecken vi använder för att representera sakförhållandet en viss mening. För att vi ska kunna förstå varandra, dvs. att kommunikationen ska bli framgångsrik, rationell, måste det finnas en viss *ömsesidighet* och *kongruens* i denna meningstilldelning.

Att studera tolkningsprocessen är att fråga sig hur matematik kunskap, som är både nödvändig och universell, kan utvecklas; hur utvecklas en kunskap som är både *intersubjektiv*, dvs. alla kan utveckla och därmed äga den, och *självidentisk*, dvs. det är samma kunskap som förvärvas av alla dem som förvärvar den. Ur matematikdidaktisk synvinkel blir det intressant att kunna modellera denna meningstilldelning semiotiskt.

Avslutning

Tecken och representationer spelar en avgörande roll inte bara i matematikundervisningen utan också inom matematiken. Genom att studera kommunikationsprocessen ur ett semiotiskt perspektiv överskrids de begränsningar som traditionella kommunikationsstudier utgör där man fokuserar på språkanvändningen.

Den matematiska praktiken liksom det matematiska tänkandet är huvudsakligen *icke-språkligt*. Det blir uppenbart när man tänker på förmågan att strukturera matematiska problem eller sakförhållanden. Spatialt, diagrammatiskt och relationellt tänkande, vilka kännetecknar den matematiska tankeprocessen, kan inte representeras språkligt.

Att betrakta matematikdidaktiska frågeställningar semiotiskt har därför klara fördelar. Därmed kan representationer av matematiska sakförhållanden, tolkningen och hur dessa kommuniceras tematiseras och därmed grundläggande frågor kring rationalitet och intersubjektivitet.

Matematikdidaktiken behöver utveckla semiotiska modeller för att förstå hur rationalitet och intersubjektivitet uppkommer i kommunikationsprocessen.

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Teachers' Work in the Mathematics Classroom and Their Education – What Is the Connection?

Barbro Grevholm
Kristianstad University and Luleå University of Technology

In the final phase of a longitudinal study on teacher students' development of concepts in mathematics and mathematics education the teachers' work in the classroom is investigated. The question is what is the connection between what the students have experienced and learnt during their mathematics teacher education and their interactions with the pupils in the classroom? I report here on work in progress and give some tentative interpretations and results.

The studied group and methods of the study

The investigated group consists of six teachers (and their pupils), with whom deep interviews have been carried through since 1996, when they started their teacher training. In earlier parts of the study their learning in mathematics and mathematics education has been studied, documented and analysed (Grevholm, 1998, 2000, 2002, in press).

The methods used in this part of the study are questionnaires and interviews with the teachers before and after observing and videotaping their mathematics lessons. The cameraman follows the teacher very closely and the microphone makes it possible to catch all the conversations the teacher is involved in. The videotapes are being analysed and interpreted. Evidence of influences from earlier learning during the teacher education will be of interest. The focus is on the teachers interactions with the pupils. A triangulation will be used to secure the interpretation. This means that a research colleague will make interpretations independently and comparisons and adjustments will be made. After the analysis of the videotapes the teachers will be interviewed after seeing them and able to confirm or reject my interpretations of what is documented from the lessons.

Theoretical framework

The student teachers' cognitive development has been investigated in earlier phases of the study applying Ausubel's theory of meaningful learning (1963). The perspective used is close to social constructivism (Björkqvist, 1993). Teachers' instructions in the classroom have to be understood through their actions and their interactions with the pupils including conversations. The observations are made by the researcher as a participating observer with an ethnographic standpoint. Jaworski (1991) quoting Eisenhart states that ”the researcher must be involved in the activity as an insider and able to reflect on it as an outsider”. Jaworski refers to Blumer speaking of symbolic interactionism as a methodological approach where the researcher has to take the role of the actor and see the world from his standpoint. She sees this as ”an unresolvable paradox” to the researcher. Jaworski claims that at best the researcher can interpret what she experiences as honestly as possible with every attempt to verify interpretations. I share Jaworski’s standpoint that it is not possible for me to take the role of the actor, the teacher, but I will ascertain clear observations that can confirm the interpretations honestly. To my view teacher knowledge has to be knowledge that has a potential to be used in interactions during instruction. Knowledge that cannot influence the teachers’ actions and interactions in relation to the classroom is not relevant or active teacher knowledge.

Sierpinska and Lerman (1996) state: ”Knowledge, in relation to a theory of instruction, should be regarded as a ‘potential of action developed through experience’. The orientation of an epistemology can be descriptive; a theory of instruction must be action-directed, or didactic.” (p. 865). Their view is obviously close to my position that relevant and active teacher knowledge is knowledge that could influence the interactions with students.

Different perspectives can be used while studying teaching of mathematics such as constructivist, socio-cultural, interactionist, or variation-theoretical. For a discussion see Runesson (1999). My position when it comes to epistemology is social constructivism and complementary to that in the interpretations of teacher instruction I am an interactionist in the sense mentioned above (compare Bauersfeld, 1994, pp. 138-139). I observe the teacher interacting with the students and try to interpret what knowledge these interactions is based on.

Some early results from the lesson studies

The six new teachers got a letter of invitation to be videotaped with a first questionnaire shortly after their graduation. It was not easy for them to find a suitable occasion for me to visit the class. After a year only three of them had been visited in spite of many reminders and phone-calls. After three semesters still two of them have not suggested a suitable time for a visit. From letters and phone-calls the reason seems to be that they are hesitating to show their work at any time. They find it uncomfortable to be observed. Two of the teachers express in the interviews that they almost cannot take the burden it means to work as teachers. One of the teachers expresses great pleasure with the working situation and is deeply involved in development work in his school. What kind of evidence based on observations and interviews have been found in the classrooms so far?

Ways of working

The studied teachers express that they in class prefer short common presentations of the content of the lesson. Contrary to this there is evidence that many of the more experienced teachers nowadays avoid whole class exposure in mathematics. In the presentations our subjects carry out a conversation with the pupils with questions and answers that lead forward during the lesson. After the exposure the teacher assists pupils when they work on their own or in pairs or groups. In this phase the problem of how to be able to help all pupils that need assistance is obvious. New teachers express that they have problems judging how much they have to tell the pupil and how to do it. Some conversations with one specific pupil are far too long and other pupils get very frustrated waiting for help. The difficulty seems to be for the teacher to know what pre-knowledge the pupil has. In classes with many pupils this situation creates discipline problems. An illustration of this is given below.

Use of textbooks

New teachers use the textbook and point out that they are governed by the textbook. In the interviews they express a need for help to make a choice among too many problems in the books and help to know what is the most important content for the students to learn when time is running out. They seem to be too uncritical towards given tasks or problems in the books. One of the episodes below will illustrate this.

Working load

The observations show that teachers spend a lot of energy in the classroom and try to be friendly with their pupils. It is a problem for them when they have to raise their voice to silence the class to get a good working atmosphere. It creates a conflict between their ideal of a teacher and reality. It is obvious from the interviews that these teachers spend much time preparing their lessons and that this part takes up so much of their time out of classroom that they can hardly cope with it.

General situation for a newcomer in school

When the new teachers got their first position in a school they were promised special conditions to assist them in the unknown situation. Examples of such things can be a lower teaching load or a mentor that can help out in the process of learning how to cope as a teacher. These conditions have in several cases just lasted for a short time in the beginning. Then difficulties in the staffing of the schools have created changed situations that have taken away the assistance for the new teachers.

The lack of qualified colleagues

In some cases the new teacher finds himself to be the most educated mathematics teacher in the school. In several cases the teachers were given special tasks as responsibility for a subject or a class. Another consequence is that the teacher has to take care of many or all classes with older students. He has no colleague to discuss with when some questions or problems occur. There is no one around that is familiar with for example assessment and marking of pupils' work in the higher grades. Especially questions about assessment seem to worry the teachers. They express the awareness of their influence on the pupils' future by the grading. Watson (2000) has found that teachers' practices, when acting as assessors of pupils' mathematics, are complex and intimately related to every aspect of teaching and learning. She claims that even teachers who have undergone some assessment training may underestimate the role of interpretation of evidence. The teachers in our study seem to intuitively realise that the limited experience of classroom interactions makes the assessment a crucial activity for them.

The need for competence development

The new teachers express that they need to know more about how to select the most important content to work with, how to create problems for diagnoses and assessment, how to judge textbooks and make choices from them, how to plan their work over longer and shorter periods, and how to find relevant working materials. In the area of general schooling problems they mention how to be able to motivate students, get them to school regularly, make them work enough in relation to ability and capacity, show respect to comrades, and maintain discipline.

The connection to the earlier teacher education

Some of the teachers' actions that have been observed show clear connection to earlier studies: The use of concept maps, working with games, working in pairs or groups, posing questions to pupils in supervision, using experimental material, aiming at learning for deeper understanding, and communicating mathematics through conversation or written work. Of course any new teacher might do such things without having taken the courses these teachers have, but at least the interactions do correspond to the teachers own learning experiences during education.

Short presentations in whole class settings are preferred in spite of a commonly held negative view of ”traditional desk teaching”. The interviews after the teachers have seen the videotapes will confirm or change these interpretations. I will comment on some of these observations.

The use of concept maps

One of the observed teachers had used concept maps in his class in the introduction of geometry in year 7. The teacher had a brain storm with the class as a starting point and let the pupils come up with ideas about all they already knew about shape and area. The teacher and the class used all knowledge that was written on the blackboard after this discussion to draw a concept map of geometry. The teacher said he did it this way because he himself had experienced the use of concept maps in his education as powerful for learning (Grevholm, in press). He says he will continue to use concept maps.

Working with games

One of the teachers worked with a smaller group of pupils that had difficulties with mathematics. It was Friday and the pupils were not concentrated during the lecture. The teacher said to the pupils that if they work well they will play a game with him for the final ten minutes. Obviously this helped to motivate some pupils to concentrate even if there was a lot of uneasiness during the lecture. When the teacher finally said to the pupils that it is time for game everybody was alert and extremely concentrated. It was a game where the teacher throws a dice and reads the result aloud. In front of them the pupils have a sheet of paper with three times three squares and can choose to put the number in any of the nine positions. Winner is the one who can add up the three three-digit numbers to a sum as close as possible to one thousand. The game was played some times and all students took part with eager. There was almost complete silence during the game and everyone made the additions quickly. There was a complete trust that everyone was reporting her correct sum. For some of the pupils obviously more calculations were made during these ten minutes than during the earlier part of the lesson.

During teacher training this teacher had lectures on how to use games in instruction. This kind of games was one of the examples he met there. Having problems to get the pupils motivated he took advantage of the game. The game was also an opportunity for him to get variation in work forms during the lesson.

Working in pairs or groups

In one of the classes the students were sitting around bigger tables in groups of four of five. In another class they worked in pairs. A third class had the desks arranged as single tables in four rows and consequently most pupils worked alone with some exceptional pairs created. In all three cases the teachers asked the pupils to discuss problems with a neighbour first before asking the teacher for help. This seemed to work rather well. The teacher that was most stressed by pupils queuing for help was the one mentioned in third case. She seemed to get stuck with one pupil and in the mean time many other pupils had problems and raised their hands for help.

During training the teachers had experienced working in pairs and groups and found this useful. The way they use this experience seems to be depending on the conditions in the school. In one of the schools the room and the desks were not so well fitted to arrange for group work. My interpretation is that this caused the teacher unnecessary stress. An illustration will be given below.

Using experimental material

In the class working with geometry the teacher was preparing the next lesson to be an experiment. He was making them curious about this during the lesson observed. The pupils were going to use a lot of ”round shapes” and measure perimeter and diameter. The aim was of course to introduce π. Another teacher was preparing the pupils for a lesson that was going to take place in the school kitchen. They were going to measure and use different units and get some experience of prefixes in connection with these units. Pupils had difficulties with g, hg, kg, ml, cl, dl, and l. Better understood were m, dm, cm, and mm. In the exposure during the lecture observed the teacher tried to get the pupils to understand that the prefix meant the same thing used with different units. The meaning of each prefix was written on the blackboard. There was an intention that pupils should understand the prefixes and not just learn by heart how many grams are one kilogram and so on. She planned to have this illustrated and made concrete in an experiment lesson.

The teacher training for these teachers included many laborative investigations where they experienced manipulatives, concrete material and experiments that could be used in class. It seems that learning by doing is something that the teachers have included in their collaboration with pupils. The concrete activities are prepared carefully in class and carried through by the students in independent experimental pair work.

The aim expressed by the teachers was that pupils should be offered an opportunity to understand and not just learn mathematics as rules. Such an attitude towards mathematics was promoted during their teacher training.

Examples of conversations in class

I will try to outline a few examples of conversations that were observed. The examples chosen might be useful starting points in my follow up interviews with the respective teachers.

A first episode

A boy calls for help and tells the teacher he does not know what to do with a problem in the textbook. The teacher starts to read the problem aloud. You are going to buy boxes of potato chips for a party. Each box contains 25 g of chips. You want to have 3 kg. How many boxes do you need?

The teacher asks the boy what is the problem. He does not know at all what to do. The teacher starts asking him questions to get started. How many boxes do you need to get 100 g? Two, is the answer. How much is the weight of two boxes if each is 25 g? Well, 50. How many for 100 g then? Well, four. The teacher gave the boy time to think before he gave answers. How many more do you need for a kilogram then? The boy answers somewhat unclear something about nine. The teacher ignores this and asks how many times 100 g is needed to get a kilogram. After some confusion they agree about ten times. The boy then says that he told her that before. He says he is now tired and he does not mind and he will just take a lot of boxes. It does not matter. The teacher asks him to continue just a little bit. We are almost there, she claims. If you need 40 boxes for one kilogram how many do you need for 3 kilograms? The boy shows that he is tired of the problem and starts to pack his things. The teacher tells him the result.

This conversation took a long time and in the meantime the other pupils started to get uneasy and the background noise raised. Many other pupils were waving their hands for help. When the teacher looked up from the long conversation with the boy she had to raise her voice and silence the class.

I could notice a feeling of unhappiness after this episode. Reading body language I could see that the boy seemed to be unhappy and upset, the teacher looked disappointed and the class was not satisfied.

The boy expressed that he was upset because the teacher did not understand that what he meant was that when they had taken 4 boxes with a 100 g they needed to do this again nine times to have one kilogram. The teacher could not follow his line of thought there. If she had been able to do that maybe the conversation had ended with a little less bad feelings.

The teacher probably knew that it was no use to ask this boy to divide 3000 by 25. It was necessary to funnel him by breaking down the problem into smaller steps. She did this without telling him how she intended to work. She led him blindly so to say. According to my interpretation, this made the boy confused and lost. How could he know what the teacher was aiming at?

The boy also expressed that he thought the problem was irrelevant by telling her that he would just take a lot of boxes. Pseudo problem is a label used for this kind of problems (Bratt, Grevholm & Nilsson, 1987), where real life problems are turned upside down. In reality it is more plausible that we know how many guests will come to the party and then bye one or two boxes for each guest. It is a sign of health when pupils react to these problems. If the aim is that education should result in independent mathematical thinkers with ability to criticise we should accept the boy’s reaction.

The teacher did not ask herself if this boy should actually work with this problem. She trusted the textbook. I asked another newly educated teacher what she would have done in this situation. She came up with the idea to choose other values such as boxes of 50 g and how many are needed for 250 g.

She also suggested that the teacher could have asked the boy to make a table showing number of boxes in one column and total weight in another. The fact that the interaction ended up in mental calculations and memorisation of partly results made it more demanding for the boy.

Another idea would be to let the boy change the problem into other similar problems that he could handle. The teacher alternatively could have created a series of problems that finally led to the starting problem here. I am looking forward to seeing the video together with the observed teacher and listen to her reflections about alternatives. A conjecture is that she will see alternatives when she looks upon herself from the outside. In her situation with a cameraman next to her and me observing it was probably hard to change content for the boy by an instantaneous decision. Reflecting on your own interaction during instruction is a demanding activity for a new teacher. However the teacher showed a lot of patience and with more experience she will probably develop a higher aptitude to make decisions about change of course during instruction.

A second episode

In the smaller group with pupils having difficulties in mathematics (mentioned above) a girl was asking for help. Normally she wanted to sit all by herself in a sheltered part of the room but because of the videotaping she was now sitting with the group. She asked for help from the teacher although the boy next to her tried to assist her. The teacher reads the problem aloud. A triangle is given with all three sides. Create a square with the same perimeter as the triangle. What will be the length of the side of the square? The teacher asks the girl to calculate the perimeter. She finds that the perimeter is 26 cm. What will you do now, asks the teacher. I do not know, says the girl. How many sides has a square? Well, four. What do you do then? The girl hesitates but after a little while they end up with the fact that she wants to divide 26 by four. The teacher asks her to write down 26/4. And what do you do then? I don't know. How many times does 4 go into 26? I don't know. Well, then we have to take the four times table. Two times four is…? Well, eight. Three times four is…? Eh, eh, … 12. Four times four is….? Eh, eh … 14, no… no.. 15, no,.. no..16. Five times four is…? Eh, eh…19, no 20. Six times four is…? Eh, eh… 23, no 24. Seven times four is…? Eh, … 27, no 28. Well, how many times then goes four into 26? Seven. No, that is too much. Well, 6 then. And what then? How much is left? Well, two. And what do you do then? Put a decimal sign and see how many times four goes into 20? I do not know. You said that just before. Five time four is…? Well, five. So what is the result? 6.5. What? What unit? 6.5 cm. That is the length of the side in the square with the same perimeter as this triangle.

The teacher walks away to another pupil and the girl writes down the answer in her book. What did she learn? Did she learn that she rarely gave a correct answer to the questions? The teacher was persistent and went on asking questions. Was it effective learning? Effective teaching? What could he have done instead? Did his teacher training offer him any alternatives? The training did offer him alternatives, but they obviously did not come to his mind in the actual situation. In the interview after the lesson we talked about the opportunity to work with concrete material and he saw that he could have offered the girl a piece of string, let her cut off 26 cm and do the partition in reality. She might then have realised that dividing by four can be done by halving twice? Would it have changed her learning? She would maybe still not know her four tables?

Reflections on the episodes

What is actually going on in the study? It involves students, teachers, educators and researchers. All four groups are trying to learn something. The pupils are learning mathematics. The teacher is learning how to teach mathematics and how students learn mathematics. The teacher educator is trying to learn how teacher training influences the teacher. The researcher is learning about the interactions between the student and the teacher in the mathematics classroom. They are all depending on each other. Barbara Jaworski (2002) has suggested a diagrammatic representation of participants, concepts and relationships for what she calls a co-learning partnership (figure 1 below). She derives it from Jon Wagner's idea about co-learning agreements as one style when he analyses the relationship between researchers and practitioners. Wagner says that in a co-learning agreement researchers and practitioners are participants in processes of education and systems of schooling. They are both engaged in action and reflection. Each might learn something about the world of the other by working together but they may also learn more about their own world and how it relates to institutions and schooling. Jaworski extends the notion to relationships between educators, teachers and students. She underlines the responsibility of the learner to be an agent of inquiry. Thus she considers all participants in the co-learning partnership as researchers.

For this study I find her picture of the co-learning community thought-provoking. It gives a full picture where all the fragments in the study can be fitted into its place. The concept of didactic tension that Jaworski has brought into the picture is relevant both for the level pupil-teacher and the level teacher-educator and for researcher-educator-teacher. How can we create better learning and understanding through substance and not form?

The video recorded lessons give raise to questions and issues that should be addressed by the teachers and the educators in the future work in a co-learning partnership. Learning can take place only if the partners agree to participate in the study and to make inquiries into the actions and interactions that take place.

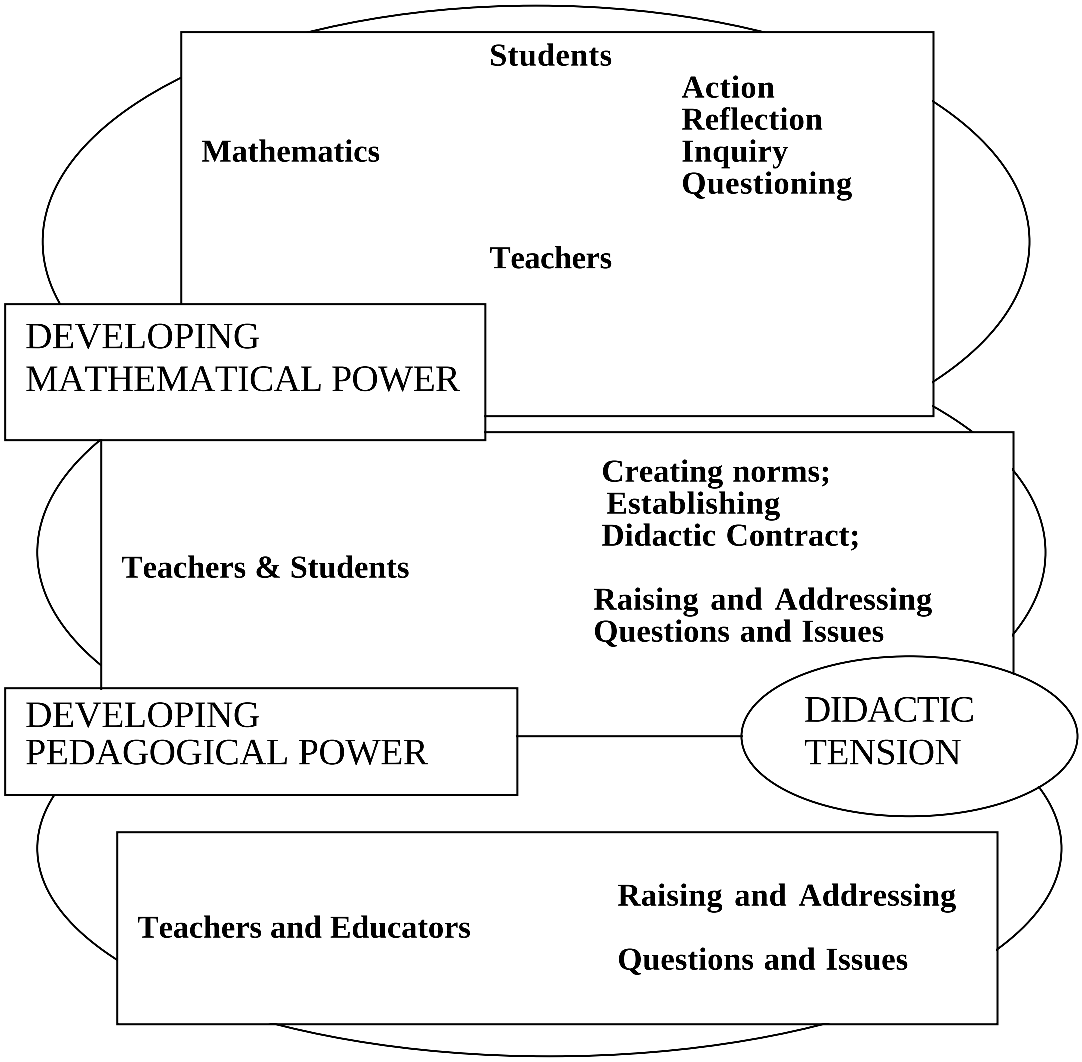

Figure 1. A co-learning community in developing mathematical learning and teaching (Jaworski, 2002, p.52).

Can the study inform future teacher education?

During the interviews in connection with the observations the teachers spontaneously bring up the question about where their education has given them a solid base to work from and where they feel more knowledge is needed. Carefully listening to the teachers' reflections during the study could give inspiration to development in pre-service and in-service mathematics teacher education. Participation in the classroom will give the researcher a conception of the conditions and limitations for learning that are at hand for new teachers, for students and for teacher educators and researchers. The quality of mathematics teacher education is rarely investigated and secured by exploring the outcomes of it but this study could contribute to this.

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The Beginnings of Algebraic Thinking

Milan Hejny, Charles University, Prague[1]
Graham Littler, University of Derby

Introduction

We believe that the way in which a child is able to use a piece of knowledge depends profoundly on the way the child's mind has grasped the piece of knowledge. This research is therefore based on the premise that *constructive* rather than *transmissive* methods of teaching allow the child to use each piece of knowledge with understanding rather than as a skill. These methods also allow the easier joining of pieces of knowledge to form into linkages in the mind.

The research is aimed at a critical analysis of the manipulative skill oriented strategy in algebra and at looking for more effective educational strategies in this topic. Thus the target of our research is to understand the development of arithmetical thinking and the very first period of algebraic thinking, that is the period that can be named *pre-algebraic*. We are going to analyse this pre-algebraic thinking to look for more effective teaching strategies for this topic. Our belief is that algebra must be rooted in arithmetical experiences. The nature of these experiences arises from three distinct kinds of situations, which we will label unknown, parameter and variable.

In their research, the authors began by having comparative discussions of their extensive experiences of experimental work and teaching, which resulted in the consolidation of these shared analysed ideas. A main outcome of this was an arithmetic-algebraic developmental model.

Arithmetic-algebraic developmental model

The model is described by ten stages of learning, from a child's early experiences with number to the pupil's confident handling of abstract algebraic expressions and equations.

(1) Early experiences with number, words and rhymes.
(2) The understanding of small numbers (words, symbols, patterns) and the structured recognition of symbols.
(3) The understanding of place value in two-and three-digit numbers.
(4) The active usage of arithmetic symbolic representation. A child uses with understanding symbols >, <, =, +, -, x, and : (here 'x' means the multiplication sign).

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(5) The ability to grasp and analyse simple word problems and arithmetical schemas.
(6) The appearance of the symbol for the unknown number (blank place _ or ,
or symbols such as ?, x). Using this symbol when solving word problem or arithmetical situation.
(7) Inventing a pattern which is hidden in a set of similar situations. This knowledge in action (see Mason and Burton, 1994) enables students to solve tasks which are beyond the manipulative horizon.
(8) Extending the gained knowledge in action into the knowledge in words. This is the ability to formulate verbally relations between magnitudes or numbers in situations and schemas.
(9) Extending the gained knowledge in words into symbolic language. Letters appear in the function of parameters.
(10) Ability to use algebra in modelling/analysing/generalising situations.

This sequence of stages gives the framework for the research and should not be understood to be a strict description of the development for each child. In some cases the order of the stages described must be preserved, e.g. stage 2 must come before stage 4. In some cases stages might be parallel, or even in a reverse order; e.g. stages 5 and 6.

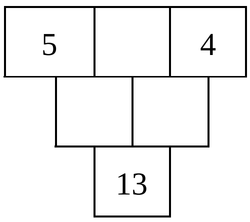
The first four stages focus on arithmetic. Our interest concerns the pre-algebraic and algebraic thinking. Therefore we will start with stage 5.

The problem situation – Stage 5

From the point of view of the context, problem situations can be divided into: arithmetical, mathematical but not arithmetical (i.e. geometrical, combinatorial,…), and real life problems. Five examples will illustrate these terms.

Task 1. (Arithmetical) The sum of three consecutive integers is 33. Find these numbers

Task 2. (Arithmetical) Find the three missing numbers in the 'adding triangle'. In any 'adding triangle' the number under any pair of neighbouring numbers is the sum of this pair. (Solution: 5, 2, 4 first row; 7, 6 – second row; 13 the bottom number.



Task 3. (Geometrical) The perimeter of a square is 12. Find its area.

Task 4. (Combinatorial) How many rectangles can you find in 3 x 3 chessboard?

Task 5. (Real life) Ann is 3 years old. When she will be as old as Ben is today, he will be 15. How old is Ben today? How old will Ann be?

We are interested in the solving process used for these problems rather than the problems themselves. When we consider problem situations we will divide the solving processes used into direct and indirect solving processes.

By *direct solving processes* of a problem situation we mean that the solver is familiar with the structure of the solution of the problem, and immediately knows the strategy for the solution. In such a case we will say that the problem situation is direct for the solver. For example, in Task 1, such a direct strategy might be described by the statement 'If I divide 33 by 3 and then I will obtain the middle number of the triple I am looking for'.

Looking at a pupil's direct solving process we have to ask the fundament question: 'How did the solving strategy come into the pupils mind?' If it was committed to his/her memory by transition from a teacher's or a friend's mind as an instruction, then this solving process is of a low educational and diagnostic value. It simply shows the ability of the pupil to memorise and recall an instruction but gives no evidence about his/her mathematical knowledge. If the knowledge of the solving strategy is the result of the pupil's previous experiments and analysis, or if it is the result of the pupil's insight into the given problem, this process is of a high educational and diagnostic value.

By *indirect solving processes* for a problem situation we mean that the solver is not familiar with the structure of the solution of the problem and has to start the solving process by analysing the problem situation to get an insight into it. In such a case we will say that the problem situation is *indirect for the solver* since for another solver the problem might be a direct problem. Two of the most frequent indirect solving strategies will be illustrated in the solving of Task 1.

First strategy. A solver starts by guessing the triple to provide the solution, say 5, 6, 7. Since the result 18 is too small the solver will try different triples such as triple 8, 9, 10 and then the triple 10, 11, 12, this last one giving the required result. This solving process has used the *trial and error strategy*. The experience gained by the solver during this solving process will give him/her a better understanding of the analysed situation and the next problem of this type will be solved more easily.

Second strategy. Using the previous experience of adding consecutive numbers a solver sees that all three numbers are nearly the same and then alike. Therefore (s)he take a third of 33 as the input number and reasons that since 11 + 11 + 11 = 33, then (11-1) + 11 + (11+1) = 33. Hence the three consecutive numbers are 10, 11, 12. This solving process involved the *insight strategy*.

The symbol for the unknown number – Stage 6

In stage 5 we discussed direct and indirect problem solving. So far we did not use a symbol for the unknown number. The next stage introduces this symbol.

Unknown

Mathematicians use this single word for the phrase 'unknown number' or 'unknown magnitude'. The 'definition' of the term is given in its name. It is a *number*, but we *do not know its value* directly. We are given some information about this number and our task is to use this information to find out the hidden value. To get the answer we need to determine a solving process. If it is successful, an unknown number is found. It is no longer an unknown, it is known. In many schools pupils are taught algorithms for these solving processes. This teaching approach develops skills for the pupils but neither understanding nor the ability to grasp the problem and what is behind the situation.

Task 6: The perimeter of a rectangle is 20 cm and its length is 7 cm, find its width.

We are going to compare two different solving processes. The first one without the usage of the symbol x, the second with the help of algebra.

The first solving process consists of three phases.

First phase – A pupil will draw a rectangle putting 7 on both its long sides.
Second phase – The pupil adds 7 + 7 = 14 and subtracts this from 20. He/she gets the answer 6 and then finds half of it getting 3 as the result.
Third phase – The pupil writes the width of the rectangle as 3 on his figure and checks to see whether the total perimeter is 20 cm. Notice that in phases 1, 2, and the first part of 3, the pupil is working with numbers and not magnitudes. The picture helps him/her to translate magnitude to number. It is only when he checks the total perimeter that he returns to magnitudes.

The second solving process consists of three phases.

First phase – A pupil will draw a rectangle putting 7 on both its long sides and x on both its short sides.
Second phase – The pupil adds $7 + x + 7 + x = 20$. So (s)he created an algebraic model of the geometrical situation.
Third phase – The solving process comprises three purely algebraic steps:
$7 + x + 7 + x = 20 \ \rightarrow\ 14 + 2x = 20 \ \rightarrow 2x = 20 - 14 = 6 \rightarrow x = 3$.

Fourth stage – the solver puts the 3s into the picture and then will proceed as in first solving process.

The first solving process depends profoundly on the geometrical situation. Each solving step came from the geometrical situation. The second solving process is based on the possibility of transferring the geometrical situation to an algebraic equation. The main part of the solving process is done inside the algebraic world. Only the last step, the geometrical interpretation of the numerical result, returns the process back to the geometry.

Here the equation is an algebraic model of the given situation. This is the most frequent way of how an unknown number ‘*x*’ is used to find a solution. However there are also problems in which the unknown number ‘*x*’ penetrates deeply into the starting situation. An example of such case is given in the following solving process for task 2.

5	x	4
$5+x$	$x+4$	
13		

Solving process. Let us denote the central number in the top row by x. Then numbers in the second row are $5 + x$ and x + 4 and hence the bottom number is $9 + 2x$. Finally we get the equation $13 = 9 + 2x$, therefore $x = 2$.

In this solution x is involved in the first line and then carried through using the triangle rule to the second line. Finally the equation is determined by using the rule in lines two and three and is just the last step of the solving process.

Inventing a pattern – Stages 7 to 9

So far we have been using a letter in the sense of an unknown. From now on two other interpretations of a letter will be used, namely a letter used as a *parameter* and a letter used as a *variable*. Both are closely linked to inventing a pattern. In fact they allowed us to express the pattern in a dense form of algebraic language.

Parameter

If following several outcomes of an experiment they exhibit a pattern which can be set down as a generalisation, then those part(s) of the generalisation which can take different values is/are the parameter(s). Each parameter is taking the place of a number.

Task 7a Find and record the length and width of all rectangles with integer sides whose perimeter is a) 20 cm and b) 200 cm. (Pupils already solved Task 6)

Solving process
a) The pupils record in tabular form all nine rectangles.

Length	9	8	7	6	5	4	3	2	1
Width	1	2	3	4	5	6	7	8	9

Two patterns are obvious: the numbers in the top row are decreasing, whilst those in the bottom row are increasing. At this stage the third pattern, that the sum of two numbers in each column is 10, is hidden.

Solving process b): The experience gained already will be generalised into a long table; because of its length if written out fully only the first few columns of the table will be completed. The teacher then asks the question ‘Can you tell me the number which is under 47?’ which forces the pupils to look for the hidden pattern: length + width = 100. If the words ‘length’, ‘width’ are abbreviated by letters l, w these will be understood as parameters. In this case the pattern is expressed by $l + w = 100$ and the question rephrased to 'Can you tell me what the width of the rectangle is when its length is 47 cm?'.

Task 7b Suppose the length of a rectangle with perimeter 200 cm is known, find its width.

Solving process: The already found formula is used and rearranged as $w = 100 - l$. This will give the solution for any suitable value of l.

You will notice that there are two different mental usages of parameter; the first is to generalise from a series of outcomes of an experiment and *describe* these in one formula, the second is to use the formula to *analyse* that situation and find particular solutions.

Variable

Variable is used to analyse the *dependency* between two parameters where one is regarded as the input and the other as the output of the relation. From this it can be seen that variable is a very important sub-set within the concept of parameter.

Task 7c Using the result of Task 7b to explain what is meant by 'suitable value of l'.

Solving process: This is best done within a classroom discussion situation. From our experiences working with different pupils at Grade 5, (already familiar with fractions and some negative numbers) different approaches are made:

a) For variable l we can take any number;
b) Only integers can be used;
c) Only positive numbers can be used;
d) Only natural numbers can be used;
e) Only natural numbers less than 100 can be used

These are some examples of pupil's ideas. None of these ideas is important in itself but the discussion which takes place around the idea is extremely valuable. Idea a) grasps the formula just within the world of arithmetic and does not link it to the previous geometrical context. This is an acceptable approach. It will be rejected by many Grade 5 pupils and at a later stage of geometrical development following the introduction of orientation into geometry, negative lengths and perimeters could be considered. A similar analysis could take place with all the other ideas.

The described process of inventing patterns will now be discussed in more detail.

Inventing patterns as knowledge in action – Stage 7

Pupils usually invent, through knowledge in action, patterns which are hidden in a set of similar situations (see Mason and Burton, 1994). This knowledge enables students to solve tasks which are beyond the manipulative horizon.

The following problem is given to second graders.

Task 8a How many matches are needed to construct (a) a square 1x1, (b) a rectangle 1x2, (c) a rectangle 1x3, (d) a rectangle 1x4?

Each of these cases would be illustrated for the pupils. The pupils will do these tasks in order and produce a series of results. A teacher draws a table and pupils then put the results (in italic) into it.

length	1	2	3	4	
matches	*4*	*6*	*8*		

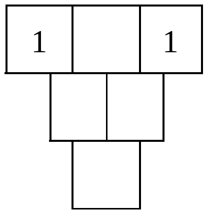
Pupils will find that the pattern in the top line increases by one and in the second line by two.

This enables pupils to find the last number in the table without creating a 1x4 triangle. This invention enables pupils to solve the problem for a rectangle 1x*n* with a larger *n*, e.g. *n* = 20. Matches are no longer used, the pattern of the table will be applied instead. Pupils will use the extension of the table to obtain the result 42. If a teacher will set, say *n* = 100, the extension of the table will be tiring. However some pupils might find the relation between the first and the second row and use this new pattern to find the number 202. The teacher must not force this invention on the children but to continue the problem in grade three with the larger numbers.

From knowledge in action into knowledge in words – Stage 8

A lot of our everyday knowledge is knowledge in action which is not knowledge in words. We know how to tie our shoe laces, but we are not able to express this knowledge by words. It is because of the lack of words needed for such a description. The same problem arises for a pupil if (s)he would like to express some mathematical idea (s)he knows in action and has not got the mathematical vocabulary. Let us give one example from our experimental teaching.

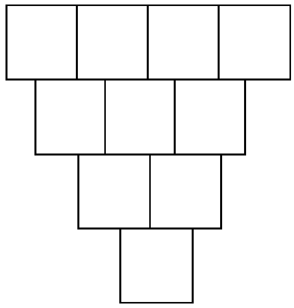
Michael invented the relation between the first and the second row of the table above. A teacher asks him to explain his finding to the class. He said: ‘to get the number of matches take the length twice and add two matches’. Eva re-formulated the result: ‘take the top number twice, add two and you have the bottom number’. This shift in grasping the analysed situation is a step towards abstraction, since the table is more general than the particular case we analysed. For example the same table arises from the following problem.



Task 8b In the adding triangle two numbers are known. Find how the bottom number of the triangle depends on the missing number of the first row. (It is our intention not to denote these numbers by '*m*' and '*b*'.)

In this case the first row of the table represents 'the missing number in the first row of the triangle' and the second row for 'the bottom number of the triangle'.

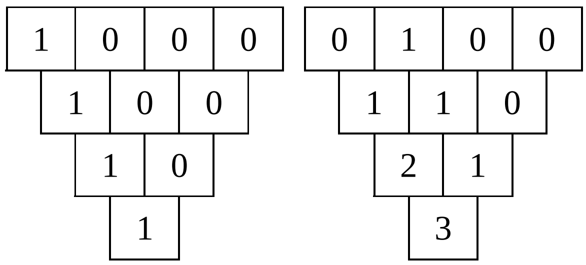
When discussing this table with our class two different rules for finding the relation between the top and the bottom numbers appeared. In addition to Eva's relation defined above we also have 'add one to the top number and multiply this result by two'. For some pupils it was surprising that two different rules yield the same result. Later on this experience will be the beginning of the understanding of the Distributive Law.



Task 9 Fill in four numbers 1, 2, 4, 5 into four windows of the first row of a triangle to get the bottom number as big as possible. Find all solutions.

Grade 3 pupils found the general rule: put the big numbers into the inside windows and the small numbers into the outside ones.

The teacher asked the class 'Why is this?' One boy immediately began to answer the question: "it is because, … because… " and he could not find the words. Obviously the boy was sure he knew the explanation, since the idea was in his mind. However, when he tried to articulate this knowledge in action he recognised that he was not able to put it into words.



Several days later the boy returned with the perfect idea. He brought two triangles and pointing to the first one said "this number only slips down, but this one (pointing to the second triangle) spreads all over these windows". This perfect idea, which is pre-inventing of the concept of the base of a vector space was more than an explanation, it was nearly a proof.

From knowledge in words into symbolic language knowledge –Stage 9

In the previous paragraph we saw two solutions of tasks 8a and b, both expressed in words. If pupils deal with such rules more frequently, they start to simplify the wording of rules by using abbreviations of words. For example the rule 'take the top number twice and add two' might be re-formulated as 'bottom number = 2 x top number + 2' and later on even '$b = 2t + 2$'. This is in fact the algebraic language, but still the letters b and t are linked to the objects they denote. The next and the last developmental step to understanding algebraic language is to get rid of the linkage between the letters and the objects they stay for. A pupil familiar with the algebraic language on this most developed level would express the rule by '$m = 2n + 2$, where n above can be any natural number.

Here it is worth recalling the famous students and professors problem:

> *'Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university." Use S for the number of students and P for the number of professors'.* See Rosnick & Clement (1980, p. 4).

Rosnick and Clement proved that there is a serious danger of misinterpreting symbols. The phrase 'six times as many students as' is often written down using the literal translation by '$6S =$'. This misinterpretation is a consequence of the transmissive teaching approach to algebra and formal knowledge of a student. When solving word tasks a student does not analyse the given situation but uses the *signal strategy* instead. This can be described as 'find in the text some word(s) or symbol(s) which is (are) associated with a particular algorithm stored in the memory; then apply it'.

The signal strategy is a strong diagnosing tool which indicates the formal knowledge of a student. The following examples will give more illustrations.

Fourth graders solve the task '*Tom loses 10 crowns. Now he has 23. How many crowns did he start with?'* Nearly one third out of 58 pupils answered '13'. The word *losses* served as the signal since this word is associated with subtraction.

Task 5 was given to sixth graders. Five out of 22 students gave the answer: Ben is 12. In the following interview all these students gave the same argument 'Ben will be 15; now he is younger; so he is 15 – 3 = 12'. In this case the solvers did some analyses of the text, but this was not deep enough.

Eighth graders solved the task '*find the length of the side c of the right-angled triangle with the hypotenuse a = 13 cm and second side b = 5 cm (on the figure all these lengths were shown).* Six out of 22 students wrote the Pythagorean formula in the standard way $a^2 + b^2 = c^2$ and found $c = \sqrt{194}$ cm. For these students the standard notation served as the signal.

Window to symbol manipulation

Usually teaching algebra starts with manipulating symbols and is a pure skill training. We believe that the constructivist approach has to be grounded in semantic understanding of symbols as shown above. We also believe that manipulating symbols has to start with semantic anchored situations to give students confidence and help with their understanding of the process. This approach will be illustrated using Task 8c as an example.

Task 8c Consider the addition triangle in which two numbers are known and four are unknown as shown in the Task 8b figure above. If one of the four unknown numbers is given, find the rules which determine the other three unknowns in each case.

When solving this task students will start with the known a, and find that $b = a + 1$ and $c = 1 + a$. From this fact they can observe that $b = c$. This observation is the first finding by using the language of algebra. This observation may also be made from the symmetry of the triangle. These two different roots to this observation support in the students' mind the structural linkage between the real situation and its algebraic model. The rules $d = 2(a + 1)$ and $d = 2a + 2$ have already been found in Task 8b and the result for d can be found by transferring this previous knowledge. However, pupils in grade 4 or 5 do not use this approach. The idea of transfer of knowledge is too difficult for them at this stage. These pupils will in the future probably use the procedure for getting the number below two known numbers, namely $d = b + c = a + 1 + 1 + a = 2a + 2$.

If d is known a can be calculated from the equations above, but this will not be carried out in practice. Students will repeat the methodology of the table finding that d must be even otherwise there is no solution. If d is even then $a = d/2 – 1$ or $a = (d – 2)/2$. We know that these two functions a$\rightarrow d$ and $d \rightarrow a$ are inverse to each other but for pupils it is not so evident. This experience may serve for some pupils as the first model of the phenomena of the inverse function. Later on when fractions are introduced this task may be discussed and the result extended to all natural d and not just to the even d.

Conclusion

The whole process elaborated above concerns not just algebraic language but also the cognitive development of a pupil, since it creates new processes, concepts and structures. All these need a considerable amount of time for their development and therefore we have to start stage 5 of this process early in the first grade by means of word problems. The teacher should not hurry this process, they must give the pupils time to construct and assimilate this new knowledge. The teaching approaches to this work should be constructive, and crucial to this is the class discussion, which allows pupils to penetrate into the more abstract level of the problem, using his/her classmate's understandings.

The word problems given to the pupils should be drawn up using the pupil's experiences from both inside and outside the school environment. There should be a graduation of the problems to suit the stages of development expressed above. The teachers should motivate their pupils to grasp the problem situation using a variety of 'languages' namely, pictorial representations, tables, schemas and dramatisation.

Our experiences of this approach through long-term teaching are promising. We have been less successful when we have been trying to motivate teachers to implement such approaches in their classes. The difficulties found in changing teachers' established ways of working is well known. It requires changes in their hierarchy of pedagogical values. It is incumbent on mathematics educators to continue to attempt to make these changes.

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Making a Difference in the Early Years

Marj Horne & **Doug Clarke**
Australian Catholic University

The Early Numeracy Research Project (ENRP) is researching effective approaches to numeracy learning in the first three years of school. Established in 1999 as a three year long project, it has been a collaborative venture between Australian Catholic University, Monash University, the Victorian Department of Employment, Education and Training, the Catholic Education Office (Melbourne), and the Association of Independent Schools Victoria. All teachers of grades Prep – 2 (0-2) in 35 Victorian schools have actively been participating in the project. As well as these 35 "trial" schools there are 35 "reference" schools matched in factors such as language background, socioeconomic environment, region and type of school and size (for details see Clarke, 1999, 2000; Clarke, Sullivan, Cheeseman, & Clarke, 2000). One of the project's aims is to evaluate the effect of the key design elements and the professional development program on student numeracy outcomes. These key design elements were proposed by Hill and Crevola (1998) as part of the Early Literacy Research Project. These key design elements include leadership and coordination; standards and targets, monitoring and assessment, classroom teaching programs; professional learning teams; school and class organisation; intervention and special assistance; home, school and community partnerships; and beliefs and understandings.

The ENRP has a major professional development component, with teachers meeting with project staff for statewide, regional cluster, and local in-service programs. Important differences from the literacy project, from which the key design elements arose, included the need for development of a comprehensive and appropriate learning and assessment framework for early numeracy (such frameworks were well established for reading), and the need to address the personal confidence with and understanding of mathematics of many primary teachers.

Measuring mathematics learning

The impetus for the project was a desire to improve mathematics learning and so it was necessary to quantify such improvement. It would not have been adequate to describe, for example, the effectiveness of the professional development in terms of teachers' professional growth, or the children's engagement, or even to produce some success stories. A search for appropriate assessment that covered the range of mathematics and was suitable for the age range of children did not produce an already established measure or set of measures. It was decided to create a framework of key “growth points” in numeracy learning to use both as a framework for assessment and to provide a framework that could be easily used by teachers in their planning and implementation of curriculum.

The project team studied available research on key “stages” or “levels” in young children’s numeracy learning (e.g., Boulton-Lewis, 1996; Fuson, 1992, McIntosh, Bana, & Farrell, 1995; Mulligan & Mitchelmore, 1995, 1996; Pearn & Merrifield, 1992), as well as some frameworks developed by other authors and groups to describe learning (see, e.g., Bobis & Gould, 1999; Wright, 1998).

In developing the ENRP framework, it was intended that the framework would

- reflect the findings of relevant research in mathematics education from Australia and overseas;
- emphasise the “big ideas” of early numeracy in a form and language readily understood and, in time, retained by teachers;
- reflect, where possible, the structure of mathematics;
- give a sense of a possible order in which strategies are likely to emerge and be used by children;
- allow the description of the mathematical knowledge and understanding of individuals and groups;
- form the basis of planning and teaching;
- provide a basis for task construction for interviews, and the recording and coding process that would follow;
- allow the identification and description of improvement where it exists;
- enable a consideration of those students who may benefit from additional assistance;
- have sufficient “ceiling” to describe the knowledge and understanding of all children in the first three years of school; and
- build on the work of other successful, similar projects such as *Count Me in Too* (see below).

These principles informed the process of developing and refining the framework as is outlined in the next section.

The development of the framework

For 1999, the decision was taken to focus upon the strands of Number (incorporating the domains of Counting, Place Value, Addition and Subtraction Strategies, and Multiplication and Division Strategies) and Measurement (incorporating the domains of Length, Mass and Time). In 2000, the strand of Space (incorporating Properties of Shapes and Visualisation and Orientation) was added to the framework.

Within each mathematical domain, growth points were stated with brief descriptors in each case. There were typically five or six growth points in each domain. To illustrate the notion of a growth point, consider the child who is asked to add two and nineteen. Some young children will either use pen strokes or objects to make 2 ands to make 19 then put them together and count all to find the total. Other children will start at two and count on 19 more, keeping track on their fingers or by some other method. Yet others realise that it is commutative and know that the problem is the same as 19 plus 2 thus solving the problem still by counting on but in an easier way having used a strategy to simplify the problem. Another child might use a derived strategy and recognise that 2 plus nineteen is like 2 plus nine and then add the ten later. These children may all succeed in solving the problem but they are operating in this situation with different growth points (GP) varying from GP1 to GP5. The ENRP growth points for Addition and Subtraction Strategies are shown in Figure 1.

0. Not apparent
   Not yet able to combine and count two collections of objects.
1. Count all (two collections)
   Counts all to find the total of two collections.
2. Count on
   Counts on from one number to find the total of two collections.
3. Count back/count down to/count up from
   Given a subtraction situation, chooses appropriately from strategies including count back, count down to and count up from.
4. Basic strategies (doubles, commutativity, adding 10, tens facts, other known facts)
   Given an addition or subtraction problem, strategies such as doubles, commutativity, adding 10, tens facts, and other known facts are evident.
5. Derived strategies (near doubles, adding 9, build to next ten, fact families, intuitive strategies)
   Given an addition or subtraction problem, strategies such as near doubles, adding 9, build to next ten, fact families and intuitive strategies are evident.
6. Extending and applying addition and subtraction using basic, derived and intuitive strategies
   Given a range of tasks (including multi-digit numbers), can solve them mentally, using the appropriate strategies and a clear understanding of key concepts.

Figure 1. ENRP Growth Points for Addition and Subtraction Strategies.

These growth points were used in the creation of assessment items, and in the recording, scoring and subsequent analysis. They were also used in the Professional Development and became an integral part of the planning process and the on-going assessment.

Growth points, levels and stages

The growth points are clearly a key element of this framework. While they provide a guide to teachers and a clear way of thinking about the children’s learning there are some important aspects to acknowledge.

- Each student does not necessarily pass each growth point along the way. The interpretation of these growth points reflects the description by Owens and Gould (1999) in the *Count Me In Too* project: “the order is more or less the order in which strategies are likely to emerge and be used by children. . . . intuitive and incidental learning can influence these strategies in unexpected ways” (p. 4).
- A student who is capable of operating at a particular level does not necessarily use those strategies at a particular time. For example “counting back” in subtraction is at a lower level than derived strategies but many people would count back for a problem such as 1001 – 3.
- The growth points should not be regarded as necessarily discrete. The extent of the overlap is likely to vary widely across young children, and “it is insufficient to think that all children’s early arithmetical knowledge develops along a common developmental path” (Wright, 1998 p. 702).

Rather than a document specifically written for research, the framework is a document written for teachers with the intention that it is used by them to such an extent that they take ownership of it. It is a guide and not a statement of development stages in the sense of children being locked into certain growth points according to age, or in the sense of all children moving along the same developmental path.

The development of the interview

Once the early drafts of the framework were developed, assessment tasks were created to match the framework. A major feature of the project is a one-to-one interview with every child in trial schools and a random sample of around 40 children in each reference school at the beginning and end of the school year (February/March and November respectively), over a 30- to 40-minute period.

There have been many studies that have used interview techniques (For example: Doig & Hunting, 1995; Pearn, 1996; Rowland, 1999). Such techniques enable the interviewer to gain insight into children’s thinking. Teachers interviewing their own students will be better able to target their teaching. However it is difficult for teachers not to use a session with a student as a teaching opportunity. For this reason the interview is deliberately not an open interview but rather set questions with set language and strict protocols.

Children are interviewed individually, and presented with a series of carefully-graded tasks, according to carefully documented protocols. Although the full text of the interview involves around 50 tasks (with several sub-tasks in many cases), no child moves through all of these. The interview continues until they reach a point where the child is judged from the responses to be unable to proceed to higher levels. The interview is of the form “choose your own ending”, in that the interviewer makes one of three decisions after many tasks. Given success with the task, the interviewer continues with the next task in that mathematical domain as far as the child can go with success. Given difficulty with the task, the interviewer either abandons that section of the interview and moves on to the next domain or moves into a detour, designed to elaborate more clearly the difficulty a child might be having with a particular content area.

All tasks were piloted with children of ages five to eight in non-project schools, in order to gain a sense of their clarity and their capacity to reveal a wide range of levels of understanding in children. This was followed by a process of refining tasks, further piloting and refinement, and where necessary, adjusting the framework.

The growth points for which they are intended to provide evidence influence the form and wording of the tasks. The consideration of student responses to a given task led to a refining of the wording of a given growth point.

The interview provides information about those growth points achieved by a child in each of the domains. Our aim in the interview is to gather information on the most sophisticated strategies that a child accesses in a particular domain. It is important to stress that the growth points are “big mathematical ideas or concepts”, with many possible “forms of progress” between them. As a result, a child may have learned several important ideas or skills *necessary* for moving to the next growth point, but perhaps not of themselves *sufficient* to move there. Also, to achieve many of the growth points requires success on several tasks, not just one or some.

Of course, decisions on assigning particular growth points to children are based on a *single* interview on a *single* day, and a teacher’s knowledge of a child’s learning is informed by a wider range of information, including observations during everyday interactions in classrooms. However, teachers agree that the data from the interviews are revealing of student mathematical understanding and development, in a way that would not be possible without that special opportunity for one-to-one interaction. It appeared that the children also enjoy that special time having the teacher “all to themselves”. Teachers report that children appreciated the opportunity to show what they knew and could do.

Professional development

The design of professional development included four strands that were interwoven throughout the project. These strands were the knowledge of children and the way they think, the knowledge of the mathematical thinking of the particular children in the class, the knowledge of the mathematics, and the knowledge of mathematical pedagogy. The professional development was delivered through the mechanisms of statewide conferences, regional meetings, school teams and support for the individual teacher within the classroom.

For example the task based interview, administered by the teachers, could be considered as part of the professional development. It provides information about the first two knowledge strands – children's understanding of mathematics, but in some cases it also challenged knowledge of mathematics and the interview approach with its emphasis on children's verbal explanation raised some aspects of pedagogy. Teachers first met the interview at a Statewide Professional Development where they were given both an overview and a chance to see the detail of each section through observation of recorded interviews. They also had an opportunity to record an interview. This was followed by further support at a school level and with a team member working with individual teachers during their first interviews sometimes demonstrating as well as providing other support. There was also discussion at meetings within the school and direct contact with project personnel for any queries. The interview touches on most (but not all) of the mathematics curriculum areas in the first years of schooling. Volume and capacity and Chance and Data are not included, though they have been part of the professional development and knowledge about them has been accessed through other assessment activities.

The State-wide professional development days included a range of aspects of teaching as well as dealing with project specific aspects such as the framework and interview. Aspects of teaching such as the use of open questions, opening and concluding a lesson, and the use of manipulatives were discussed. Each of the domains was also looked at in detail with teaching ideas and issues as well as ongoing assessment. Some sessions were for all but each professional development day had some sessions where there was a degree of choice. Because the whole team from the school was involved in the professional development usually a school planned their involvement in the sessions so that their coverage was complete and the teachers could then share what they had learned with the other team members on their return to the school environment. The knowledge of mathematics pedagogy was the main aspect of many sessions on the State-wide days but while this may often have been the main focus, the knowledge of mathematics and the knowledge of how children develop and learn mathematics was also a part of these sessions. In the later two years teachers started to play a greater role in these days with some sessions being run by teachers from the schools involved.

The regional or cluster meetings involved all staff from 3-5 schools in a region in a two hour after school professional development session. They generally occurred once a month (except in some months where there was a Statewide meeting of all teachers), and each one was led by a member of the research team who worked closely with all schools in that region. Each cluster meeting had a theme with a specific issue to discuss and some classroom activities to try. These cluster meetings provided opportunity for teachers to share their ideas and their insights and gave real value to the input from the teachers. Often there were tasks for teachers to try back in their classrooms and opportunity to share their children's work and discuss their observations and ideas at the following cluster meeting.

The other level of professional development was one-to-one in the schools and in the classroom. Each cluster leader from the research team visited their schools on average about once a month and spent time in the classrooms working alongside the teachers. This enabled a working relationship to be built between the research team member and the teachers in the schools. Sometimes on visiting a classroom the cluster leader "borrowed" the class to try out a new activity. On other occasions a demonstration lesson was given in response to a specific request from the teacher or team leader within the school. A team teaching approach was often used with the classroom teacher and the cluster leader working together. Another common experience was the visitor working with a small group of students or just roving and working with students as the situation indicated, generally being another "hand" in the classroom. Finally there were times when the cluster leader just acted as observer. Within the school on these visits there was a lot of opportunity to talk with individual teachers both in classrooms and in the staffroom, and to directly respond to the teachers' needs.

The impact of these professional development approaches can be evidenced both through student growth and teacher change.

Student growth

Student growth in the 35 trial schools where teachers are actively involved in the project has been compared with growth in a set of 35 reference schools, matched on a range of characteristics including among language backgrounds, socio-economic areas, regions and size. Growth is based on the growth points assigned as a result of the interview. It is important to note that the assignment of Growth Points is an "on-balance judgment," not an aggregate of completed items, as on a more orthodox test. It should be said that students in all 70 schools demonstrated growth. However, looking at the data overall, children in trial schools outperformed those in the reference schools at *every grade level* and *in all of the mathematical domains* studied.

Domain	**Assessment time**	**Reference Schools Mean N=295**	**Trial Schools Mean N=942**
Counting	March 1999	0.83	1.07
	Nov 2001	3.49	4.15
Place Value System	March 1999	0.38	0.46
	Nov 2001	2.24	2.67
Addition & Subtraction Strategies	March 1999	0.19	0.30
	Nov 2001	3.03	3.69
Multiplication & Division Strategies	March 1999	0.31	0.44
	Nov 2001	2.85	3.28
		N=295	**N=884**
Time	March 1999	1.04	1.10
	Nov 2001	2.63	3.20
Length Measurement	March 1999	1.56	1.58
	Nov 2001	2.92	3.41

Table 1. Means showing growth over the three years.

An ANOVA looking at the student growth over the six domains which were assessed in both March 1999 and November 2001 shows the difference between the trial and reference schools to be significant ($p<.001$) for each domain. This was a comparison of the 295 students in reference schools and 942 for the number domains and 884 for the measurement domains in trial schools. The means for the two assessments are shown in Table 1.

The graph in Figure 2 shows the growth of students in the Addition and Subtraction Domain across the three years of the project. The triangles symbolise trial schools while the reference schools are shown by rectangles. Each pair of graphs represents a cohort of students progressing through the project. Horne and Rowley (2001) described a process of creating an interval scale from the growth points so that means and comparisons could easily be made. This interval scale is created from the growth points and is close to them so that the number 5 can be considered as approximately growth point 5.

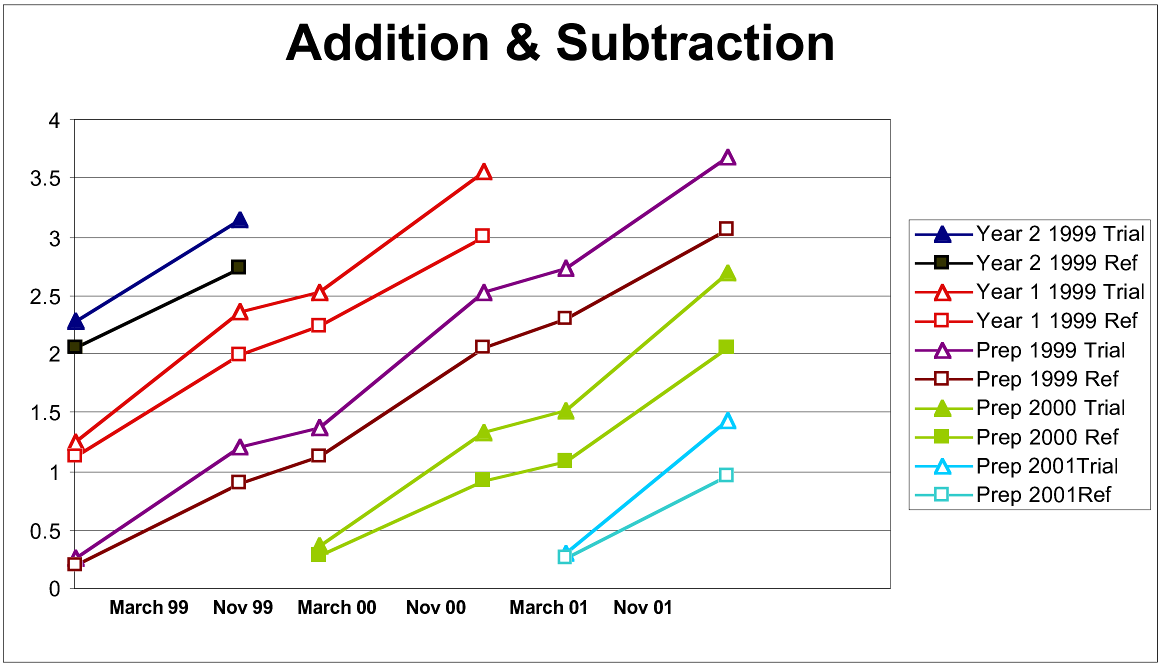

Figure 2. Growth from March 1999 to November 2001.

The growth graph shown in figure 2 represents all cohorts of students and shows the means for each testing period. Of interest too though is the spread of students across the growth points. Figure 3 follows the group of students in the trial schools who were part of the project for the whole three years and shows the percentage of students at each growth point. A vertical line at any time period is divided so that the regions are representative of the percentage of children demonstrating achievement of that growth point. One aspect this graph demonstrates is that there is a spread across the growth points at each year level and that teachers are providing learning experiences suitable for all of the children in their classes.

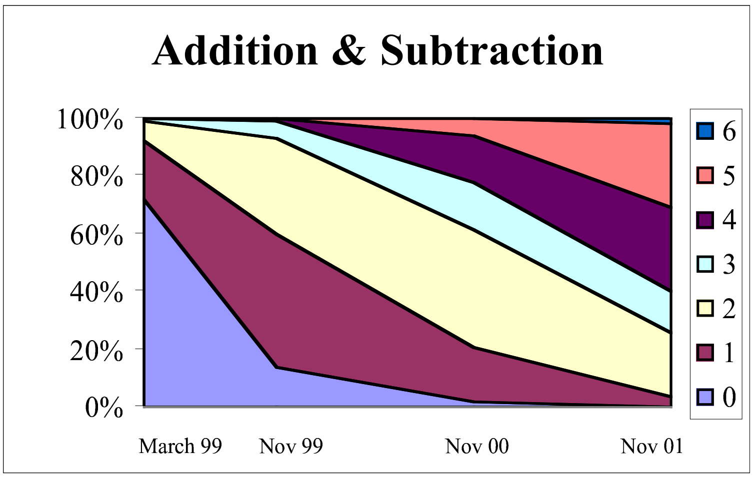

Figure 3. Spread across growth points of children present in trial schools for three years.

Teacher change

Teachers' growth and change is the other major indicator of success. Given the clearly successful efforts of trial school teachers in developing children's mathematical skills and understandings, the research project has been looking at successful teachers' practice to try to discern those aspects of "what the teacher does" that make a difference. After slightly more than one year's involvement in the project, teachers were asked to identify changes in their teaching practice (if any). There were several common themes emerged including among others: more focused teaching (in relation to growth points), greater use of open-ended questions, providing more chance for children to share strategies used in solving problems, and offering greater challenges to children, as a consequence of higher expectations.

Considering the more focused teaching, 72% of teachers specifically stated that the starting point for lesson planning is largely statements of mathematics content rather than activities. This is exemplified in the following statement from Ms H, a middle aged teacher who was asked near the end of the second year whether she thought her teaching had changed as a result of the project.

> Amazing, yes, I have. Definitely
>
> I think in the way that probably more now my teaching is more focussed, on just one - like in past times I've done what's called an "activity maths" where you might have some children working on space and some people working on measurement, and kids rotate, but I don't do that any more. What I try and do now, because of the project, is focus on one area of learning and the whole class working on that area at their own level. And I think that's what's been wonderful. You're more focussed on what you're trying to get out of the kids. And that has been a huge change. (Ms H)

The mathematics content was not only the starting point for planning but also the focus of the whole lesson and with this focus the teachers are catering for the range of students in the class. Much of this focus has come from the framework providing the basis for the mathematics content. More than 70% of trial school teachers claim to have the framework in their heads to some extent, and a further 26% understand it but don't have it in their heads. It informs teaching.

> … I feel much more informed about how they're learning, what they need to learn next and what's reasonable to expect to try and get children, most children achieving by the end of the year.

Catering for the range of students in the class has meant the use of different strategies. Many of the teachers use open questions that allow students to operate at different levels while still tackling the same task.

Another change has been the nature of classroom questions. Asking students to explain their own strategies is something that is expected in trial school teachers' classrooms. Approximately 86% of teachers ask students for explanations at least a few times a week. More than a third of teachers ask students to explain their own strategies every day.

> Until they're explaining how they got there. It's great that you've got the answer that I want to know, that you understand why that's right, and you can prove it to me. Yeah, because the children who've got it all together, that's no problem. I know that they know, but there are other children who are not really sure how they got there or whether they've fully understood why that's the right answer. 'Cause you see the doubt sometimes and they'll give an answer, in the circle somebody did it today, and it was like yes, no. It's like, can I be really confident in myself that this is right? (Ms S)

Teachers have also commented that they have become more reflective as exemplified by this statement.

> Yes, I think we all knew to a certain extent what good teaching should look like, but I don't think we always practiced good teaching, so what ENRP has also provided us with different activities, and the confidence to try things, you know, that you're not looking for worksheets, you know that you can do what you need to do with concrete materials, and you can extend children's thinking, and also it sort of helped me link children's knowledge from one area to another, I think it just changed, I think it's just a change in thinking, about how you approach, you sort of become a much more reflective teacher, I suppose that you actually think about what your doing, why you are doing it, is this the best way, what other things can I use, how could I make it more practical. Oh, yeah, I think I'm a much better teacher, and it's a lot about making connections to, like if your doing counting, then you relate that to your multiplication, you know if your doing that, you relate it to your place value, it's the linking of all knowledge, and then once you've got a bit of knowledge you can link it out elsewhere. Yeah, into the maths area and beyond. (Ms CS)

The change has not only been in individuals but in the way teachers operate within the school. The team has been an important aspect as has the sharing between schools in the local area. The teachers own words express this.

> I told you that, its also, part of the ENRP, is it's given us an opportunity to be together as a team, we've all had the same input, we've all been down there, or we've all been at cluster, we've all heard the same sorts of messages, so you've actually got a supportive team to work with. You're all hearing the same thing, there is no, someone imposing anything on you, you're working through together, and we've probably developed what they call a professional learning team in a true sense, that you actually work together, you share together, you ask for help.

> Well, I think you know, and also, going away together, you sort of actually get to know people, even better, so you actually develop strong friendships as well, and that supports the teaching and learning, like you know that you feel

> confident to say "so what are you doing in that", or "how are you doing that" or "I've got this to share", you know that your not just big noting or anything, your just saying "I know this works, have a go", or and confident to go to another classroom and say, "oh that's good, can I have that", rather than everyone owning things, its sort of…

Further data in 2001 has added to this knowledge of teacher change but is still in the process of being analysed. Making the most of every opportunity, building on student knowledge and working on improving student attitudes towards mathematics are some of the characteristics that show clearly in the case study teachers.

Conclusion

Overall the project has made a difference to both students and teachers in the schools involved. This shows in the student growth and in the teachers' comments and changes observed in their classrooms. The project was a collaborative venture and the teachers were very much co-researchers in the collaboration. The central research team feel they have been privileged to work with these teachers and feel that they too have learned much from the opportunity to work with these teachers. This research has changed the students, the teachers and the researchers.

Acknowledgements

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Numerical Skills and Arithmetic Performance

Bo Johansson
Uppsala universitet

Introduction

Whether pupils in primary mathematics education should learn to write the numerals and also use them in mathematical calculations has been the subject of an ongoing debate among Swedish mathematics teachers, educational researchers and teacher educators. Usually arguments have been voiced for a postponement of the teaching of numerals. For example, Nordlund (1910) held that a firm number concept had to be formed before the introduction of the numerals. Too early introduction of the numeral system made the children mentally represent numbers by numerals, which was a grave obstacle to the acquisition of mental calculation. Also Wigforss (1925) was against an early introduction of the numerals, which may divert attention from the solving of the problem to the drawing of the symbol. He suggested an initial use of tallies to represent the numbers and that the children must have substantial practice in writing the numerals before using them for calculations.

An early empirical investigation of children's level of skill in writing numerals (Wigforss, 1946) showed that more than 50% of the children entering school (at 7) understood the meaning of all the 10 basic numerals and that about one third could draw the numerals. These findings made Wigforss change his view and argue (Wigforss, 1946; Wigforss & Roman, 1951) in favor of numeral practices already at the beginning of the first school year. Further investigations (e.g. Malmquist, 1961) showed a still higher level of numeral skills among school starters. Despite this, experts in the area have been skeptical towards early numeral exercises (e.g., Sjöholm, 1949; Magne, 1986; Unenge, Sandahl & Wyndhamn, 1994; Alhberg & Hamberger, 1995). Arguments have been that premature introduction of the numerals results in reversals or mirror-image writing, that the children learn to write the symbols without understanding the number meaning, and that early numeral practices impede the acquisition of the number concept. However, these arguments are not supported by empirical investigations. The three experiments in this paper represent a preliminary effort to study the development of early numerical skills and the relation between numerical skills and the number concept as measured by arithmetic performance.

The views presented in the Swedish debate receive support from the results of Hughes' (1986) investigations. His argument was that there is a natural development of the symbols used by children to represent a given quantity of objects starting with idiosyncratic responses (“scribbles”) and changing into pictographic responses, or drawings of the objects to be counted. The third stage is the use of iconic responses, for example, making a tally for each object. Finally, the conventional numeral symbol system is acquired. His conclusion was that there is “a serious mismatch between the system of symbols which children are required to learn, and their own spontaneous conceptualizations” (Hughes, 1986, p.78). Following Hughes’ conclusions children should be allowed to use pictographic and iconic symbols until they have developed a readiness for the conventional symbol system.

The experimental procedure used by Hughes (1986) has been criticized by Munn (1998), who argued that his tasks did not have a clear communicative purpose and that the design was cross-sectional. Munn, using a task encouraging the child to write symbols that signaled a given number, found that the key developmental shift was not with regard to form (from concrete to abstract) but in how the symbols were used (from pre-functional to functional). The above raises three questions: 1. What is the level of numeral writing skill among children entering school today?, 2. Is skill in writing the numerals positively or negatively related to the number concept?, and 3. If numeral writing is related to the number concept, which is the mechanism? Is there a relation between numeral skill and digit memory or does numeral skill affect how the numbers are represented?

Study 1

Introduction

This study was made in 1991 and 1992. Part of Wigforss’ (1946) investigation of number skills in school-beginners was replicated to find out if level of skill in writing the numerals had changed during the period from early 1940 to early 1990. A second purpose was to investigate the correlation between the numeral writing skill and arithmetic performance. Unenge, Sandahl and Wyndhamn (1994) and Ahlberg and Hamberger (1995) argued that early acquisition of the numerals may be an obstacle to the development of the number concept. To find out if that meant a negative correlation between numeral writing skill and arithmetic performance, Wigforss’ arithmetic test was also replicated. Magne (1986) argued against numeral practices with young children, because it results in many reversals or mirror-image responses. To study Magne’s hypothesis an analysis was made of the relation between number of mirror-image numerals produced and number of arithmetic problems solved.

Method

Subjects

Two samples were investigated: 372 school beginners (7 years old) and 161 6 years old children attending pre-school. Testing was made in early autumn a few weeks after school had started. Data were collected by pre-school and comprehensive school teacher trainees. Each student tested four children, randomly selected from the class. Before testing, teachers and parents were informed in a letter asking for permission to carry out the investigation.

Tests

The children were given ten different tests, only the three tests of relevance for this study are presented here. A complete report is given in Johansson (2001).

Numeral writing

The child was given a sheet of paper and a pen with the instruction “Would you like to write the numerals you know here!” The responses were coded into four categories following Wigforss’ (1946) definitions: 1. Numerals correctly written, 2. Reversals along the vertical dimension (mirror-image reversals), 3. Omitted numerals, and 4. Unreadable numerals.

Addition and subtraction

The 7 year olds were given three addition (3+2, 2+6, 7+8) and subtraction (5-1, 7-3, 15-8) problems taken from Wigforss (1946), whereas the 6 year olds were given three slightly different addition (1+4, 3+4, 5+7) and subtraction (5-4, 8-3, 13-6) problems. The wording of the addition problems was “Imagine that you have x pears/balls and then get y more, how many do you then have altogether?” and for subtraction “Imagine that you have *x* pears/balls and then you give away *y*. How many do you have left?”

Procedure

The different tests and instructions were printed in a booklet and the students practiced giving the test and coding the responses before actual testing. The children were tested individually in a separate room in the local pre-school or school.

Results

Writing of numerals

The following results are given for each age group: number of correctly written numerals (Figure 1), number of reversals (Figure 2), number of omitted numerals (Figure 3) and number of incorrectly written numerals (Figure 4), for each of the numerals 0 to 9. For sake of comparison Wigforss’ (1946) results are also presented.

Figure 1. Correctly written numerals

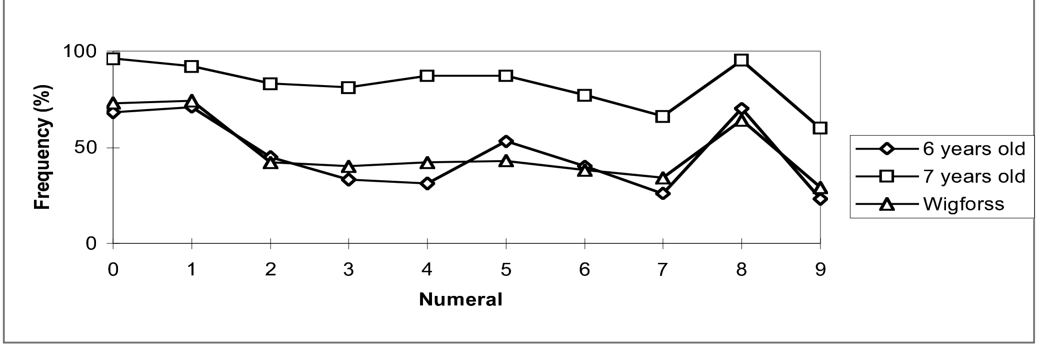


Figure 2. Reversed numerals

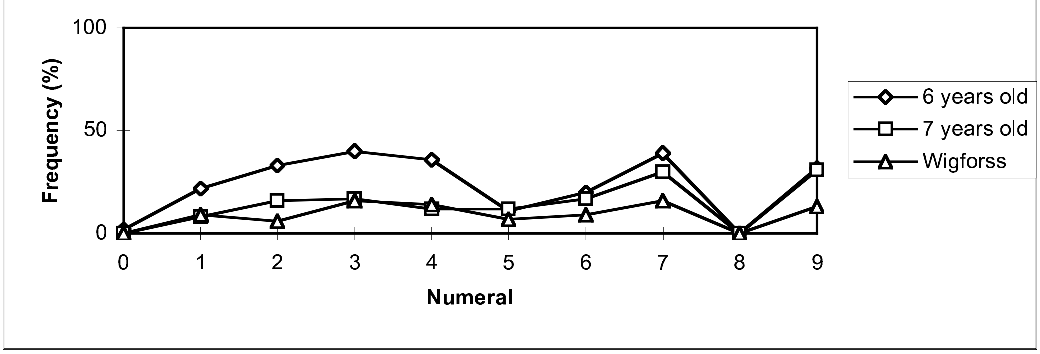


Figure 3. Omitted numerals

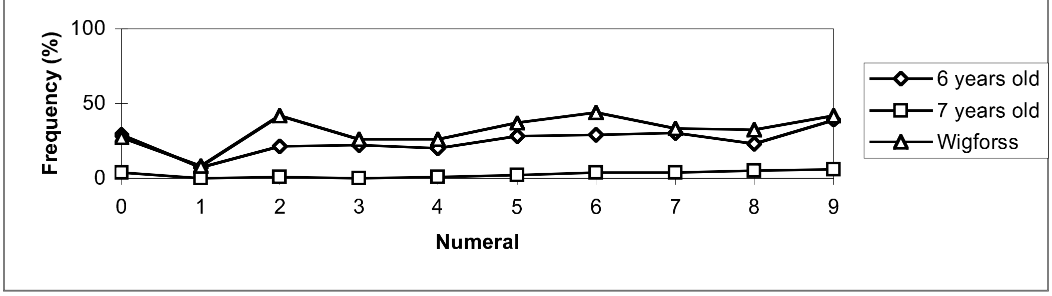
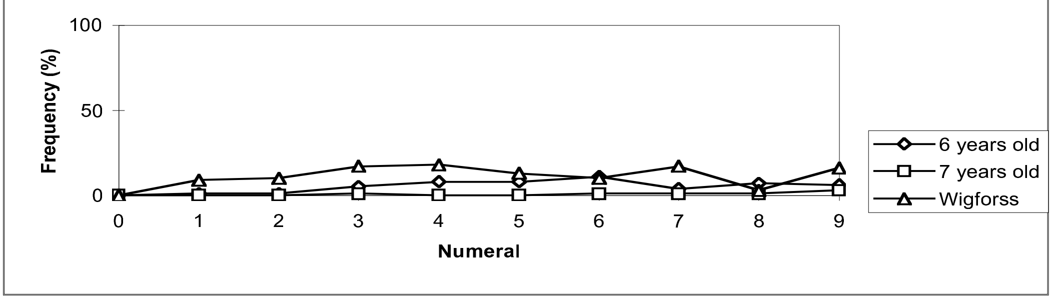


Figure 4. Incorrectly written numerals



As can be seen, the 7 year olds tested by Wigforss performed at about the same level as the 6 year olds tested in 1992, whereas performance for the 7 year olds tested 1991 was considerably higher. In other words, the school-starters of today seem to be one year ahead of the school-starters of 1942.

Figure 2, on reversals, shows that each age group produced between 1 or 2 such responses. Accepting reversals (Wigforss, 1946) as readable numerals, the results mean that the 7 year olds tested in 1991 could correctly write (albeit not perfectly) almost all the 10 basic numerals. This is a surprising contribution to the debate on numeral practice in early mathematics education: most children have learned to write the numerals before entering school.

The fact that the child can write a numeral, however, does not prove that this knowledge affects the number concept. To find out if there is such a relationship the correlations between numeral performance and arithmetic problem solving was analyzed. The categories of omitted and incorrectly written numerals were added under the heading incorrect numerals. To obtain a clearer picture of age-related development, the six year olds were divided into children younger and children older than 6,4, respectively. The seven year olds were partitioned into younger than 7,3, between 7,3 and 7,7, and older than 7,7. The partitioning criterion was to obtain groups with about the same number of children. The results are given in Table 1.

	Number of arithmetic problems solved for children aged:				
Number of numerals:	6, 4 or younger	6,5 or older	7, 3 or younger	7,3 to 7,7	7,7 or older
correctly written	.18	.24**	.28**	.12	.05
reversed	.40**	.04	-.09	.00	-.11
omitted and incorrect	-.37**	-.13	-.33**	-.24**	-.29**
Number of children (n)	66	95	144	121	107

** $p \leq 0.01$.

Table 1. Correlations between numeral and arithmetic test performance.

Positive correlations were obtained between numerals correctly written and arithmetic problems solved for the oldest half of the 6 year olds and the youngest third of the 7 year olds. For reversals a positive correlation with arithmetic performance was obtained for the youngest age group. Finally, for omitted and incorrectly written numerals the correlations were negative in four of the five age groups. Accepting arithmetic performance as a valid measure of the number concept, these results give no support to the statements that early numeral writing, or writing reversed numerals, should be an obstacle to the development of the number concept. Rather, the results indicate that not knowing to write numerals is a negative factor in the acquisition of the number concept. It should be added that strong ceiling and bottom effects may give a biased picture of the relations; a replication with younger subjects seems necessary.

Study 2

The tests in Study 1 do not measure the child's understanding of the meaning of the numerals. Unenge, Sandal and Wyndhamn (1994) argue that the fact the child can write a given numeral does not prove that the child understands the meaning of the symbol. One aim of this study was to measure children's understanding of the cardinal meaning of the numerals.

In Study 1 it was found that early mastery of the numerals may enhance the acquisition of the number concept (measured by arithmetic performance). However, the experiment gives no clues as to why this should be the case. As early as 1884 the Swedish mathematics educator Velander suggested that the numerals facilitate memorizing the numbers. His argument was that the numerals afford a small set of units easy to grasp and summarize and hence easy to use in mental calculations. This argument is in effect the opposite of Sjöholm's (1949) thesis that the numeral is "the symbol for a symbol" /my translation/ and therefore very difficult to handle. As a first test of how the numerals are used in mental calculations, the correlation between numeral writing and digit memory was analyzed.

Method

Subjects

Subjects were 84 children aged 5 to 6 years and attending pre-school. The parents and the pre-school teachers were asked for permission before testing. The children were partitioned into three age groups with 28 children in each: 1. Mean age 5,2 (range 4,8-5,4), 2. Mean age 5,8 (range 5,5-6,2), and 3. Mean age 6,5 (range 6,3-6,8).

Tests

The children were tested with a total of 12 different tests. Due to space limitations, only the 5 used in the analysis will be presented here.

Numeral writing

The children were given the same numeral writing instruction as in Study 1 and the responses were coded according to the four categories already described.

From numeral to cardinal number

The children were presented a numeral written on a card (numerals taken from Angelin, 1978) with the instruction “I would like you to draw as many tallies as the numeral says”, with the numeral 2 as practice. Two series were constructed: A. 3, 5, 7, and 11 and B. 3, 6, 8 and 14. Each child was tested with only one of the series.

From cardinal number to numeral

The child was presented a sheet with a given number of tallies. The instruction was “Draw the numeral that tells how many tallies there are”. Both correctly written numerals and reversals were scored as correct. Children tested with Series A in the former test were given Series B in this, and vice versa for Series B.

Addition tests

Six different problem series were constructed: 4+1, 3+4, 5+4, 13+4; 3+2, 2+4, 3+6, 5+12; 1+3, 5+2, 7+2, 7+8; 1+4, 4+3, 2+7, 5+12. Each child was tested with two of the series. The instruction was “Imagine you have x bolls/apples and then you get y more. How many do you then have all together?”

Digit memory

A digit memory test, modeled after WISC, was used.

Procedure

The children were tested individually in the local preschools by two trained research assistants.

Results

Table 2 presents mean performance on the different tests with the children grouped into three groups according to age.

	Child group with mean age of			
Test	5,2	5,8	6,3	R^2
Numerals to number (%)	69	73	85	.04
Number to numerals (%)	55 --------------	81	85	.15**
Addition (%)	0.7	0.9 -------------	1.5	.21***
Digit memory (%)	51	53	58	.02
Correctly written numerals (mean)	2.4--------------	4.4	4.8	.19***
Reversed numerals (mean)	1.3--------------	2.4	2.5	.09*
Errors and omissions (mean)	0.4	0.3	0.4	.01
Number of children (n)	28	28	28	

*p<.05, **p<.01 ***p<.001 _R^2 =explained variance

Table 2. Performance on the different tests for the three age groups.

The results were subjected to an analysis of variance with age group as a between subjects variable. Significant results were followed up by Tukey tests of simple main effects. A line in the table signifies a statistically reliable difference ($p<.05$). The results show that the sharpest increase in number skills took place in the age interval 5,2 – 5,8 years of age, whereas the increase in arithmetic performance was concentrated to the next age interval.

Next, correlations were computed between the measures of numeral writing and the other four test results, see Table 3.

Numeral writing and age group	Numeral to Memory number results	Number to numeral	Addition results	
Correctly written numerals				
Oldest group	.30	.62	.56	.30
Middle group	.63	.54	.61	.54
Youngest group	.36	.57	.57	.20
Reversed numerals				
Oldest group	-.11	.14	.05	.30
Middle group	.36	.39	.42	.22
Youngest group	.43	.47	.56	.35
Incorrect numerals and omissions				
Oldest group	-.16	-.07	-.23	-.37
Middle group	-.30	-.08	-.05	-.05
Youngest group	-.12	-.04	.10	.10

$r > .37$ is significant at the .05 level.

Table 3. Correlations.

The strong correlations between correctly (and in 2 of 3 cases for reversals) written numerals and performance on the number to numeral test is taken to indicate that the ability to write numerals is not only an acquisition of a certain form but also the learning of the cardinal meaning of the symbol. The correlations between correctly written and reversed numerals (2 cases of 3) on the one hand and number of addition problems solved on the other are a replication of the Study 1 results supporting the conclusion that numeral writing develops hand in hand with the number concept. The results for digit memory, however, were inconclusive, indicating that numeral writing is not effective through an increased digit memory span.

Therefore, it was decided to pursue another track to analyze why numeral skill has a positive effect on arithmetic performance, namely that numeral skill affects strategy used to solve the arithmetic problem. Consider the problem “Lets say you have 3 crowns and then you get 4 more. How many do you then have all together?” One way to solve this problem is to imagine first three coins, or look at three fingers, then four coins/fingers and finally count all items to find the sum – seven. Another strategy would be to simply imagine the numerals 3 and 4 and then compute the sum, either directly or with a decomposition into simpler number combinations , e.g., by using the doubles: “3 and 3 equals 6, 1 more makes 7”. In other words, children with a low level of numeral skill may have to use various counting-all strategies whereas children having acquired the numerals may also have access to derived strategies such as direct retrieval, doubles or ten complements. To test this hypothesis, the third study was run.

Study 3

To test the hypothesis presented above the children were presented with both numeral writing tests and arithmetic tests, followed by post-solution questioning on strategy used to solve the problem. Because the first two studies had demonstrated that many pre-school children can write the ten basic numerals, the present children were also asked to write the numerals for two- and three-digit numbers. As in the former experiments, the children were tested with many different tests; only those of relevance for the hypothesis under study will be presented.

Method

Subjects

The subjects were 6 years old children having completed their pre-school class (testing made in late May). Of a total of 60 children enrolled in the school, permissions for testing were obtained for 54.

Tests

Two numeral writing tests

were run. First, the experimenter asked the child to write the numeral she or he knew, then the child was asked to write (numbers said): 13, 18, 23, 54, 62, 106 and 2089 in numerals. To concentrate result presentation only correct responses are given.

Arithmetic tests

The subjects were given one addition and one subtraction test, each with five problems given in order of difficulty. After the child had solved a problem the experimenter asked questions about solution procedure.

Procedure

Each child was tested individually in a separate room in the school. All test sessions were video-recorded and the strategies were scored by both the experimenter and the author. Scorer agreement was .91.

Strategies

The distinctions between direct modeling, verbal counting and derived strategies (e.g. Bergeron & Herscovics, 1983) were taken as points of departure when categorizing the children's descriptions of their solution procedures. Due to space limitations only the strategies for the addition problems are presented:

Concrete Counting All (CCA): Fingers or other countable objects were counted out one by one to represent one of the addends followed by a similar process for the other addend. Then all the objects or fingers were counted to determine the sum. On a few occasions abbreviated counting strategies (see Baroody, 1987) were used, these were also categorized as CCA.

Sequence Count All (SCA): The child started with "one" and counted up to the cardinal value of the first addend, then the child continued to count a number of steps equal to the second addend.

Sequence Count On (SCO): The child started with the cardinal value of the first or largest addend, counting on from there while the second addend was enumerated.

Decomposition strategies (DE): The child decomposed the given numbers into easier number combinations, using doubles or combinations of 10.

Direct recall (DR): The child relied on facts recalled from memory, without recourse to counting. To be defined as a direct recall answer, the response had to be given within 2 sec.

Don't know, guessing and not identifiable (GU): Some children could not describe their solution procedure, others said that they had guessed the answer and some descriptions were too vague to be categorized. All these responses were brought together under the heading of guessing (GU).

Not given (NG): The problem was considered beyond the child's current performance level and was therefore not given to the child.

Table 4 shows mean performance on the various tests (not subtraction) described above. The children correctly drew 7.7 of the numerals from 0 to 9, solved a mean of 2.5 multi-digit numeral problems, and solved about 3 of the arithmetic problems.

Numerals test	Tests: Single digit numerals	Multi-digit numerals	Addition-problems
Performance			
(means)	7.7	2.5	2.9
(max)	10	6	5

Table 4. Mean performance on the two numerals tests and the addition test. Maximum performance on each test is also given.

In Table 5 are given the results from the computation of correlations between the drawing of numerals (single-digit and multi-digit numerals) and strategies used to solve the addition problems.

	Strategies						
Numerals test	Problem Not given	Guess-ing	Concrete counting all	Sequence count all	Sequence count on	Decom-position	Retrie-val
Numerals							
Single-digit	-.02	-.03	-.26	-.12	-.10	.08	.42
Multi-digit	-.41	.05	-.23	-.26	.26	.23	.34

r > .22 is significant at the .05 level.

Table 5. Correlations

The results show that children solving many numeral problems also made heavy use of the retrieval and decomposition (multi-digit numerals only) strategies, with a reverse relation for the strategies concrete counting all and sequence count all (multi-digit numerals).

Discussion

All three experiments have shown positive correlations between number of numerals correctly written and arithmetic performance, which suggest that knowing to write the numerals correctly (or reversed) concur with the development of the number concept, contrary to the opinion of many mathematics teachers, educational researchers and teacher educators. It seems that learning to draw a given numeral not only means the learning of a given form, but also the acquisition of its cardinal value. In addition, the present experiments suggest that the reason why numeral writing goes hand in hand with the development of the number concept is not through an increase of digit memory, but through the providing of an efficient system for imagining the numbers. In effect, Velander (1884) considered the numerals “fully concrete” /my translation/, just because they were easy to grasp and turn into distinct images.

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Projects as an Educational Strategy[1]

Marie Kubinova
Charles University, Prague

Introduction

By a project we mean, in line with e.g. Lehmann (1999), Ludwig (1996), Meyer (1997), Quartapelle (1999) and others, a challenge to carry out various activities (both mathematical and non-mathematical) during which pupils, guided by the teacher or on their own, during and/or outside school hours, discover mathematical concepts and regularities, acquire new knowledge and/or get to know the possibilities of using mathematical knowledge outside mathematics. These activities go on within mathematics (in a mathematical context, e.g. a project *Pythagoras' Theorem*, *Origami and symmetry*) or outside mathematics (e.g. a project *Water* in which pupils learn the concept of dependency, practise operations with decimals and fractions and use physical, geographical, scientific, etc. knowledge as well as real life knowledge).

We consider the project method to be a challenge for a teacher to become a co-ordinator of pupils' independent work (based on their activity when acquiring information, analysing it and using it) and to enable pupils to become active participants of the project who have considerable competencies and who are able to co-operate with each other and use the available information.

Our priority is to construct and realise in school such projects which are based on the ideas of constructivism and to overcome the disadvantages of the "traditional teaching" (at least in the Czech Republic) such as: memorised and one-sided cognitive learning based on the didactic transformation of subject matter which emphasises passing ready-made knowledge to pupils; isolated and fragmented school subjects and the separation of school from everyday life; low motivation; mechanisation and rigidity of school work; disregard of children's interests.

Some characteristics of projects

Projects which we prepare for pupils are based on some characteristics which in our opinion contribute most by their motivating role in the teaching of mathematics. They are: realisation of pupils' needs and interests; development of pupils' competencies and potentials; self-regulating learning, motivation; changing roles in the teaching/learning process; implicit role of a teacher; orientation towards

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the presentation of results; co-operation in teams; interdisciplinary character; social relevance; change in the conception of school.

In this contribution, we will describe in more detail three projects which we used in our own teaching at the basic school. While ‘solving’ them, pupils get an opportunity to:

- get to know mathematics from a different point of view, namely as a suitable method for solving problems from various different fields and from everyday life and as one of the ways of how to understand reality,
- learn about the potential of mathematics and other school subjects and their own potentials,
- be directed to independent work, to defending their opinion, to coping with failures and discovering their causes, to using results of their work outside mathematics, to co-operating with others, looking for solving methods outside mathematics and coping without the teacher‘s help,
- use previous knowledge and various different solving methods in problem solving and look for an optimal solution and optimal strategy.

The project *We look for a number* – diagnostic version[2]

One version of the project *We look for a number* was prepared for 14-year-old pupils from grade 8 and 15-year-old pupils from grade 9 as two worksheets, which only differed in the instruction which of the numbers *A*, *B* should be chosen and which one should be looked for.[3] It was our intention that the two worksheets were identical except for the label of the INPUT and OUTPUT because, among others, we also wanted to diagnose if pupils pay attention to the assignment of the problem they are supposed to solve and if they will distinguish the two types of projects and discover their connection.

The two types of projects were posted on the notice board and pupils had a sufficient number of worksheets[4] at their disposal. Their task was to: look for pairs of numbers *A*, *B* so that the conditions given in the worksheets were

[2] The project *We look for a number* was realised with different worksheets in the school year 1998-99 in grade 7, first as a *motivational* (for the motivation of new subject matter) and *expositional* (for the introduction of new subject matter – the ‘discovery’ of negative numbers and their properties), later as *fixing* (for the consolidation of already known subject matter – the drill of the algorithms for counting with integers), and finally as *diagnostic* (assessment of the level of understanding operations with integers).
[3] If number *A* was given, number *B* was looked for via a sequence of inverse operations to the given operations.
[4] An illustration of one worksheet is appended.

fulfilled; prepare a set of similar problems with solutions for their classmates and for younger students; present results of their work to their classmates.

Eighth graders worked independently or in groups outside the school lessons, ninth graders worked in groups in two successive mathematics lessons. Altogether 92 pupils took part in the project. Pupils became interested in the project and lively discussions among pupils and between the teacher and pupils originated. Their focus was mainly questions connected with properties of numbers and arithmetic operations. In all classes pupils agreed that they would present results of their work via (a) filled worksheets and (b) a set of worksheets with tasks for their classmates.

The observations of pupils' work, discussions with them and analyses of their written work brought about some interesting phenomena:

Interest in work which went beyond their 'normal' schoolwork – pupils worked with interest, they accepted the task as their own, kept bringing new and new solutions of the given task. Without any major outer motivation, they carried out a number of sums with numbers which in 'traditional' teaching would be unthinkable.

Minimal co-operation among students in a group – group work which we observed in mathematics lessons was organised differently in different groups. The prevailing model was where one member of the group organised work of the others. Co-operation among group members was minimal even in classes in which pupils were used to group work.

Strong influence of the used educational strategy on the use of solving strategy – groups in class 9A[5] worked with both types of problems at the same time. However, they discovered the mutual connection of them very quickly and immediately used it in their further work. For most groups from the other classes, the first key factor of the success of their work was the realisation that they were, in fact, solving two different types of problems. Not surprisingly, most of them concentrated on solving the problem in the direction from the given number *A* to the unknown number *B*. The second, intellectually and numerically more demanding type when number *A* was looked for via number *B*, was not solved at first by pupils at all. When speaking to them, we found out that the majority did not consider the difficulty of the tasks at all, they supposed that the two worksheets were identical. They chose the solution 'from *A* to *B*' because they considered it to be natural. Four groups and three individual pupils even refused to accept both types of worksheets because 'they only need it once'. One group and one pupil came later to collect the second type because 'they need it after all'. Only three groups from classes other than 9A discovered the connection between the two types of tasks and used it in their work.

[5] Project teaching was used very often in this class.

Neglecting the input on the worksheet – this was the most frequent mistake in pupils' work in 'We look for number *A*' which could be caused by the stereotype 'the problem is solved from the front', as one pupil explained.

Choice of a correct inverse operation – if pupils discovered the direction in which the task should be solved, they had no problems with the choice of the correct inverse operation.

Different attitude to problems with no solution or with several solutions – class 9A was unusual in that the pupils also handed in problems with no solution or with several solutions. In other classes, it was very rare that pupils handed in worksheets in which the input number was chosen so that the problem had several solutions or no solution. They justified it by saying 'we cannot hand in unsuccessful problems with several solutions' or 'an unsolved problem'.

Feedback – in class 9A pupils emphasised the need to check if their proposed solutions were correct, in other classes there was no feedback.

Minimal innovation in the creation of new tasks – worksheets which pupils prepared for their classmates more or less copied those given by the teacher. Tasks prepared by younger pupils mostly concerned operations with natural numbers.

The project *We look for a number* can be easily modified according to the situation in the class. The above comments suggest that work on projects of this type can help pupils overcome stereotypes in grasping problems and in their solutions. This is exemplified by the different approaches to the solution of the project by class 9A where project teaching had been used quite often compared with other classes in which different educational strategies had been used.

The project *We look for a number* – motivational version

A modified version of the project 'We look for a number' was used in grade 6 as a motivational project. We wanted to:

- create a suitable climate for group work because the class was newly established (it consisted of pupils from different classes) and pupils were at first not able to co-operate at all,
- support pupils' inner motivation to the work beyond the normal scope of school work,
- lead pupils in such a way that they independently get into the expositional phase and discover the 'basis' of a decimal number.

The sixth graders were excited about the project. Groups formed spontaneously without the teacher's intervention. However, during the first lesson when the worksheets were prepared in such a way that all the prescribed operations were realised within the domain of natural numbers, the groups did not function properly. Individual members of groups did not co-operate, there was no control of work inside groups.

During the second lesson problems were given which could not be solved within natural numbers and the work became so time consuming and technically and intellectually demanding that it required pupils to co-operate. This probably was the beginning of the creation of a suitable climate, which will enable us to use project teaching in the future.

Pupils were fascinated by the possibility of creating ‘their own’ numbers. During their first lesson they solved all the prepared stock of worksheets. Some pupils prepared their own worksheets at home, others wrote new solutions into already filled worksheets.

During the second lesson, in several groups at once, the ‘discovery’ of a decimal number was made as an object which:

- can solve the situation in which we look for a quotient of two natural numbers and can show that it is not a natural number,
- is not given in advance but originates during the process of division of one natural number by another.

It transpired that pupils could write decimal numbers, some of them could even add them. However, their knowledge was mostly formal and connected with the separate model of a decimal number which expresses price of goods. The next lessons confirmed that pupils accepted the concept of decimals into their knowledge structure as recognised notation via the agreed system of signs (digits, decimal number) without any connection to the process in which the decimal system originated.

The work on the project was the stimulus for several pupils to discover other properties of decimal numbers through the algorithm of division of one natural number by another. Other pupils elaborated at home an extensive set of examples of decimals with the help of which they showed that there were two ‘types’ of decimal numbers: ‘decimal numbers which will stop’ (decimals of the type 0.25) and ‘decimal numbers which repeat’ (periodical decimal numbers even though some of the numbers they gave were not periodical). When the pupils presented results of their work to other pupils in a mathematics lesson, a rich discussion originated in which pupils: made the first (incomplete) classification of divisors according to the ‘type’ of decimal; discovered the possible lengths of the period and ‘that in some numbers there is also something before the period’; described the rules for the order of decimals.

It was very important for us that the pupils’ spontaneous interest in the results of work of their two classmates brought about a working climate which meant a possibility to use, in a natural way, the constructivist approach to teaching even though it meant that the teacher had to react to the situation by the change of the content of the lesson. For the first time in five weeks of common work, we succeeded in creating working climate in the class which addressed most of the pupils. Pupils showed their excitement from the 'discoveries' they made and required new problems to solve.

The project *Energy for all*

The project 'Energy for all' has been in a simple form assigned since 1994 as a fixing project when topics Percent and later Statistics were being covered in class. The pupils' task was to carry out a statistical survey on the consumption of electricity in their households, to present their results in a suitable way and possibly to suggest ways to save energy. In all but one case the parents supported the project and co-operated. In some cases they even accepted the results of the project and observed the proposed suggestions! So results of school work were used outside school as well. It had a motivational effect on some of our pupils. This was manifested, for instance, in that they offered topics for other projects.[6]

At the beginning of the school year 1999-2000, we got into an altogether different situation. A discussion about various sources of energy took place in one physics lesson in grade 9 when the revision of the subject matter *Work and energy* took place. During that day a group of pupils brought their suggestions for projects on energy which could be solved by others. It was a very extensive list of topics that the pupils called *Energy for all* and which consisted of the following areas[7]:

(a) *Can I manage my own energy?* Intake and distribution of energy (tables of energy values of food, ways of distribution of energy, balance between the intake and distribution of energy, ...). Healthy lifestyle (my opinions of the possibilities of living in a healthy way every day, what other people think about it, ...).

(b) *How much do we pay for the electric appliances in our households?* How much energy do we use (statistical survey on the consumption of electric energy, heat and gas), proposals for saving energy, experiences with them. How much electricity does my cassette recorder (computer, television, video,…) use? Can we buy an electric appliance which saves energy?

(c) *Ecology-friendly energy sources.* Solar, thermal, water, wind, nuclear, ... power stations. Energy of biomass. Power stations – pros and cons.

(d) *Advantages and disadvantages of various types of transport* (car, air, train, boat, ...).

[6] For instance, the projects *What is missing in our neighbourhood*, *The place where I live,* and *Smog on a highway*.
[7] The areas are given in the form which the pupils used.

The project *Energy for all* was organised as the after-school activity. In a month, pupils gathered an enormous amount of materials, including statistical surveys and summaries, graphs and diagrams. They also organised and analysed two questionnaires. The results of the project were presented in several school subjects.

It was very important that the project was initiated by pupils themselves. They accepted the project as ‘theirs’. The work was organised by two boys and a girl who guided their classmates – they distributed individual tasks among them and organised the presentation of results.

Conclusions

Our long-term experience with using projects in the teaching of mathematics shows that well chosen and well used projects can be one of the means of creating a positive climate for a quality change in a pupil‘s personality. In such a project there is enough space for the *development of one‘s own learning strategies* and at the same time enough time and space for solving many different problems and reaching some results. It motivates pupils towards an active approach to their learning. Therefore, we can understand a project as a *specific educational strategy based on partnership.*

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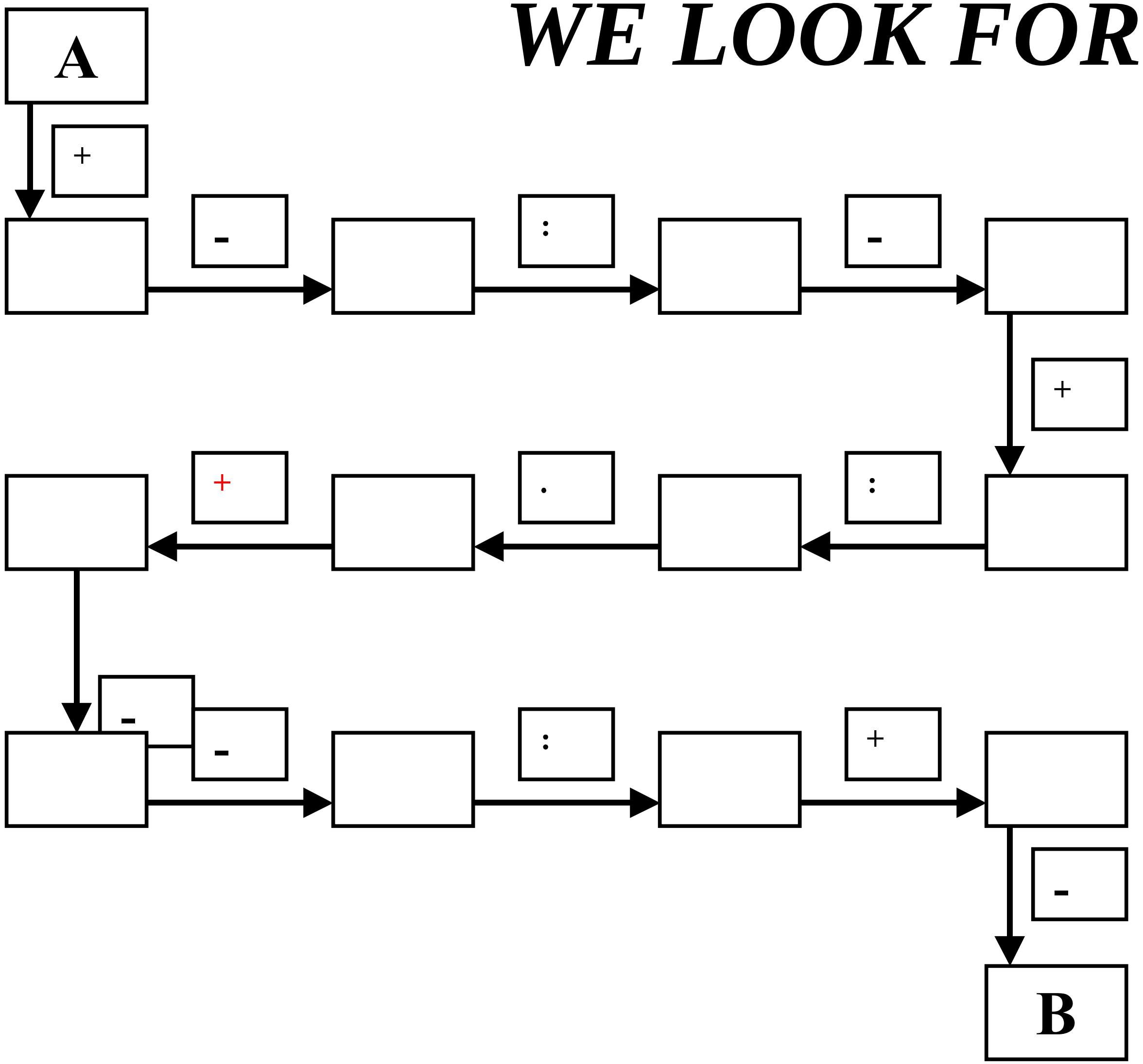
WE LOOK FOR A NUMBER



1. ***Choose number B***
2. ***Fill in the numbers in boxes so that after carrying out the suggested operations number A is:***
    - ***three more than B,***
    - ***three times smaller than B,***
    - ***equal to B,***
    - ***integer,***
    - ***negative,***
    - ***divisible by eight,***
    - ***equal to the square of a natural number,***
    - ***prime,***
    - ***equal to the square of B,***
    - ***equal to the square root of B,***
    - ***periodical.***
3. ***If there are several possible solutions, find all of them.***
4. ***If there is no solution, justify.***

Learning Velocity Graphs – The Case of Laura and Fiona

Variation as a critical feature for learning

Ulla Runesson
Göteborgs universitet

Comprehending graphical representations of time versus velocity (a $v(t)$ curve) indicating negative velocity seems to be problematic to students (see e.g. Goldberg & Andersson, 1989; Nemirovsky, 1994). In this paper I demonstrate how two students in grade eight learn to interpret this kind of graphical representations during a mathematics lesson.

Theoretical framework

The point of departure taken is that for a certain learning to happen there must be certain critical features present in the learning environment. The assumption is based on a theoretical framework, developed in detail by Marton and Booth (1997) and Bowden and Marton (1998). They argue that variation plays an important role for learning, that certain learning presupposes an experienced variation of the object of learning. We learn to experience and act in the world by discerning critical features or aspects of objects and situations by focusing on them simultaneously. The way in which the learner experiences an object is due to which aspects of the object that is in her focal awareness at the same time. Certain aspects of the object or the situations are discerned, they are in the fore of her attention, while others remain in the background. To understand something in a certain way implies that certain aspects must be discerned and held in our awareness in a certain way. Further, an aspect is discerned only if it is experienced as a dimension of variation, thus that it could be something else.

Description of the study

The aim of the study was to analyse and describe that which was critical for learning in terms of the aspects and dimensions of variation related to the object of learning that was present in the learning environment. The base for analysis was a section of a more extensive data set of videotaped records of fifty-five secondary maths and science lessons. This data set was made available to a multidisciplinary research team comprising expertise in different areas, for instance mathematics and science education, sociology, developmental psychology (Clarke, 2001). The particular extracts of data chosen for this study, are one video recording of a whole class discussion and a sustained conversation between two students during a mathematics lesson.

The lesson, which was narrowly examined in this study, was a maths lesson in grade eight. After an introduction in a whole class setting, the students worked in pairs. The task was to match five different situations with a moving object changing velocity as a rate of time, to eight different graphs (figure 1). Two of the situations implied a bi-directional movement (a bouncing ball and a ball thrown into the air). That is, these did not only include a change of velocity, but a change of direction of the object, as well. Opposite to the unidirectional motion of a falling stone for instance, a bouncing ball is bi-directional (i.e. moves in different directions). A bouncing ball changes velocity *and* direction. In that sense this situation – and the graphical representation – is more complex than the others are. To two of the students – Laura and Fiona – studied particularly in this study, these situations were the most problematic ones.

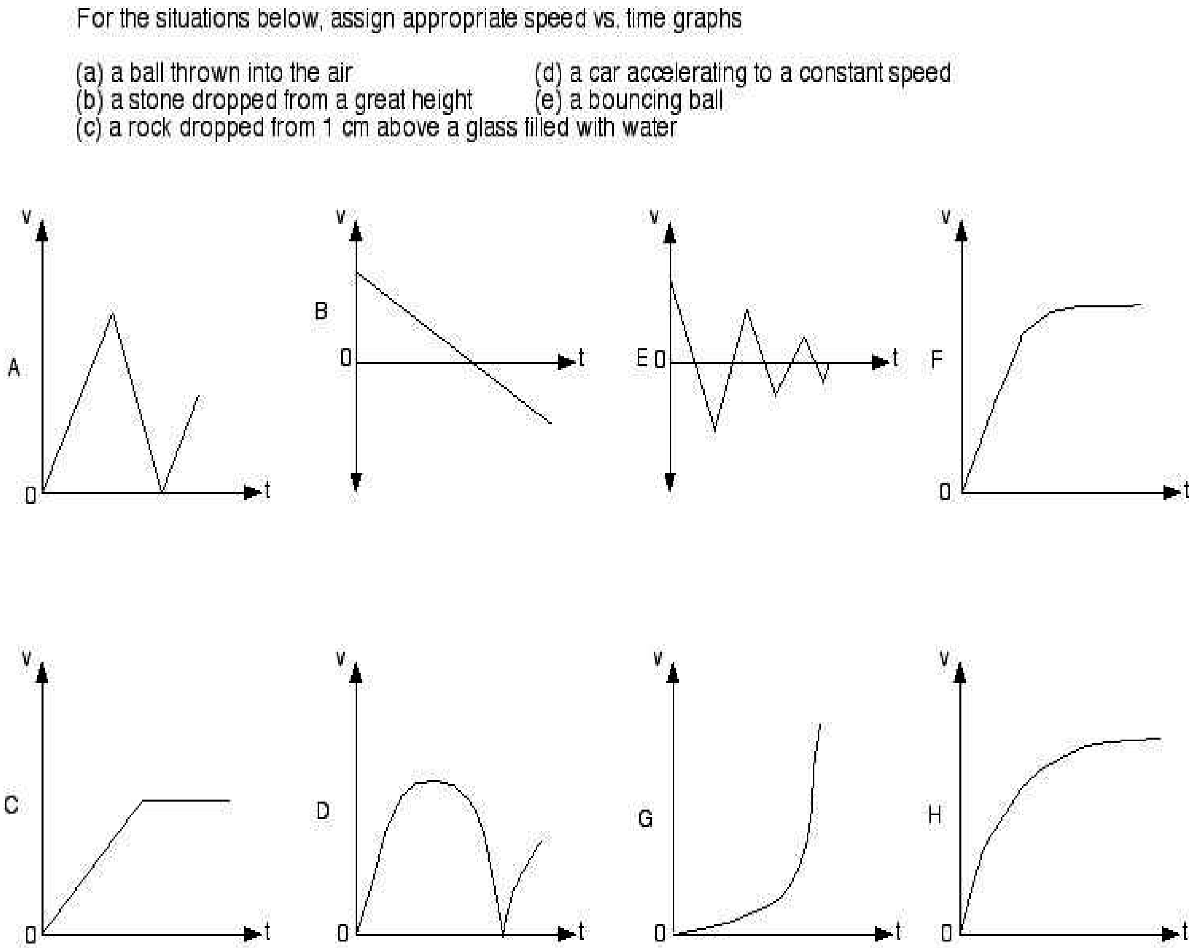


Figure 1. V(t) graphs presented to the students

The focus of my analysis was the object of learning (i.e. that which was possible to learn) that was jointly constituted in the interaction between the teacher and the students in the whole class setting on the one hand, and between the students working together on the other. Similar analysis of the classroom interaction has been presented before (see e.g. Runesson, 1999; Rovio-Johansson 1999).

Results

It was found that the students' ways of handling the task in the peer interaction very much reflected how this was handled in the whole class setting. The presence as well as the absence of variation of certain aspects of the object of learning, constituted in the whole class setting, was found to be critical for their learning.

The lesson started with a whole class session for about 15 minutes, followed by peer work.

In the whole class instruction graph B (see figure 1) was presented to the students. However it was not chosen as representing a ball thrown into the air, as was the case in the text book, but a car slowing down, stopping and then reversing, moving in opposite direction with an increasing speed. This situation was supposed to illustrate how a graph intersecting the x-axsis indicate a change of direction. In different ways the teacher opened for variation. For instance, she *shifted focus* (e.g. between the situation and the graph) she made *comparisons* (e.g. of the speed in different points of time) and pointed to *similarities and differences.* In this way, she brought out a pattern of variation.

When working in pairs, the students should chose the appropriate graph and make a quick sketch of the graph. The focussed students, Laura and Fiona, chose graph D as representing "a ball thrown into the air". For "a bouncing ball", they also chose graph D, that is the same graph representing both the situations. When Laura and Fiona were discussing the problem with the bouncing ball (e), it was suggested that it could be either graph D or E. Laura, who suggested D, argued:

> L: Wouldn't be like E, it's not, a ball's not gonna like –
> F: (filled in) Go backwards. It's going to go woo-woo (moving hand back and forth once fast two and half times horizontally). It's not going to do that.

In my interpretation, the girls must have experienced that alternative E, with the graph intersecting the x-axis, represented a change in direction. Thus, they have discerned one critical feature of that kind of graph. However, they took the change of direction for-granted – that this kind of graphs represent a horizontal change of the direction only. The girls explicitly stated that the change of the movement of the ball is vertical. It does not "go backwards", and therefore alternative E, must be rejected. However, Fiona who probably kept to the idea that a graph intersecting the x-axes represents a change in direction, suggested B. Laura rather impertinently put off the suggestion with:

L: Like B? It’s not like B.
F: D. (Fiona immediately corrected)
L: Yeah that’s what I reckon. It’ll go bo-ing (moves pencil in hand as a bouncing ball) except it’ll a little lower each time. I reckon. So I reckon it’ll be like D.

It seems as it did not appear to the girls that the graph could represent a horizontal and a vertical movement as well. They did not open up for this variation; that the change of speed could imply a vertical *or* a horizontal movement. The fact that they omitted the possibility that a graph intersecting the *x*-axis could indicate a vertical as well as a horizontal change of direction, was critical for their solving of the problem, and thus for their learning.

The students’ way of handling the task in the peer interaction very much reflected what had happened in the whole class interaction. For instance, graph E was rejected by the girls as representing the bouncing ball due to it did not go "backwards". Exactly the same word was used in the whole class interaction, but about another object and another kind of motion (a car going forward and reversing). In that episode, the teacher opened for variation of change of a horizontal direction. This dimension of variation was identified in the peer interaction as well.

By comparing the patterns of variation that was constituted in the whole class and the peer interaction respectively, it was possible to identify some features in the learning environment that were critical for the students' learning. According to the theoretical framework taken and described previously, the pattern of variation - that is what is varying, is invariant, what is left out, or is taken for granted - is a defining condition for learning. I will point to some features of the pattern of variation that was constituted in the whole class interaction that I would argue affected the students' possibility to learn.

First, the way the teacher changed one of the examples, from a bouncing ball to a car, seemed to be critical for the students’ learning. The very idea of mathematical representations is that they are general, for instance a graph can represent many situations. From the point of view of mathematics, one could believe that it does not matter if the example is about a ball or a car. However, from the point of view of the students, these situations are quite different. Firstly, the change of direction is different. Secondly, in terms of the learners' everyday experience of the force that affects the velocity and the motion of the object, the two situations are quite different. From their everyday experience they "know" that the change of the velocity and direction of the car is caused by a human being (pushing the accelerator and reversing the car). So, in this case a "visible force" accomplishes the change. For the ball, however, the only force that is visible is the force caused by the throwing hand. So, that it was never pointed out that the graph represented a change of direction vertically as well as horizontally

- that this aspect did not make up a dimension of variation, but was taken-for-granted - seemed to be critical for the girls' learning.

Discussion

It is reasonable to assume that the teacher changed the situation represented by the graph (i.e. from representing a ball thrown into the air as in the text book, to represent a car first moving forward and then reversing with the velocity changing) was done out of good will. It is usual that teachers in the introduction of the lesson take examples that in some respect are somewhat different from those ones the students will encounter when they solve problems on their own. This could be a way to demonstrate the generality of mathematical representations or models. However, in this case, this change was critical for learning, particularly since an important general aspect of the $v(t)$ graph (i.e. how the graph represents vertical as well as horizontal direction of velocity) was not indicated.

From the theoretical standpoint taken, that which is varied is likely to be discerned. It is reasonable to assume that the understanding of the generality of graphs representing negative velocity (i.e. horizontal and vertical change) takes the experience of a certain pattern of variation. Obviously, the girls, when working on their own, did not experience that the same graph indicating negative velocity could represent a variation of situations (in this case changes in different directions). So, one could ask; would another pattern of variation in the whole class interaction have made this learning possible? I would argue that there would have been other possibilities to discern this critical aspect, and thus the generality of the graph, if the teacher instead had illustrated the same graph with two different situations (i.e. keeping the graph constant while varying situations).

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Process Oriented Research and its Reflection in Pre-Service Mathematics Teachers Education – A Case of Diploma Theses

Nada Stehlíková, **Darina Jirotková**
Charles University, Prague

Our conception of the didactics of mathematics

Hejny (Hejny and Stehlíková, 1999) distinguishes two polarised streams in the research of the didactics of mathematics. The *content oriented stream* focuses on the curriculum, textbooks and teaching aids and looks for ways of opening the world of mathematics to as many people as possible through the change of the content and teaching methods. The *process oriented stream* concentrates on what is going on in a student's or teacher's head when they are doing mathematics, solving problems, discussing mathematics and explaining it to another person; this includes the study of interactions, classroom learning, etc. – i.e. cognitive and communicative processes in mathematics education. Whilst the first type of research is traditional in Czech mathematics education research, the latter type has only gained more attention in the last ten years.

Advocates of process oriented research believe that teachers play the key role in the process of improving the teaching of mathematics. This assumption has been acknowledged by both national and international research which shows that neither a change of curriculum nor a change of textbooks and teaching aids can bring about the desired shift from instructivist towards constructivist approaches on their own.

One of the ways of influencing teachers' strategies of teaching is through action research.

Action research and student teachers as researchers

Much research has been devoted to 'the-teacher-as-the-researcher' movement (action research) in the didactics of mathematics and it has been widely acknowledged that a teacher who carries out experimental and research work changes his/her pedagogical consciousness and attitude in a positive direction (see e.g. Boero and Szendrei, 1998; Cooney and Krainer, 1996; Crawford and Adler, 1996; Edwards and Hensien, 1999; Hatch and Shiu, 1998, and others). Moreover, action research contributes to the difficult task of implementing research results in practice because "on the one hand teachers must know about research issues in mathematics education, and on the other hand researchers must know the challenges that teachers have in their practice, and the solutions of these challenges should be one of the main issues to study" (Carrillo, Coriat & Oliveira, 1999).

Mason's (1998) idea that the most significant products of research in mathematics education are "the transformations in the being of the researchers" because it is "*their* questions that change, *their* sensitivities that develop, *their* attention that is restructured, *their* awarenesses that are educated, *their* perspectives that alter", supports the above claim.

Action research has been reflected in the pre-service education of teachers, too. More and more researchers claim that student-teachers can also take part in action research (e.g. Hatch, Shiu, 1998 – UK example, Peter-Koop, 2001 – German example). Tutors/researchers from our Mathematics and Mathematical Education Department have started to involve student teachers in action research and, similar to Germany (Peter-Koop, 2001), the student teachers' research results form the basis for their *diploma thesis*, which they have to write in order to get a teaching degree in mathematics.[1] In this contribution, we will focus on diploma theses of future elementary teachers and future mathematics teachers trained at the authors' university as one of the manifestations of the process oriented research in the preparation of future teachers. 'Diploma student' is the student teacher who writes a diploma thesis, 'supervisor' is a person who supervises him/her.

Traditional diploma theses

Traditionally, diploma theses reflecting the content oriented didactics of mathematics have prevailed at our faculty. They mostly come under quantitative research paradigm and concentrate on the content of mathematics, textbooks, teaching aids, curriculum, standards, possibly different ways of teaching a topic. Teaching experiments are carried out solely to try proposed teaching methods in practise. The following are examples of such theses: 'Congruences modulo and their use in arithmetic', 'Graphic solution of equations', 'Mathematical test as a means of evaluation', 'Comparing mathematical textbooks', etc.

Alongside these types of diploma theses, the new types[2] (hereinafter NT) reflecting the process oriented research started to appear. In the text below, we will attempt to determine their common characteristics and illustrate our considerations through two examples.

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[1] Some of them continue with their research project within the PhD study.

[2] 'New' refers to being new within the education of teachers in the Czech Republic, we are aware that similar work has been done in other countries for some time now (e.g. Peter-Koop, 2001).

New types of diploma theses – two illustrations

Illustration 1: ***Didactical elaboration of a non-standard arithmetic structure***

The diploma thesis 'Didactical elaboration of a non-standard arithmetic structure' was written by a female student, let us call her Molly. She was a future mathematics teacher. Molly started working with the first of the authors (NS) when she was in her first year of a five-year study. She took part in a series of semi-structured interviews, which were conducted within the research on a student's ability to structure mathematical knowledge. She was very excited about the mathematical topic she was working on – so called restricted arithmetic RA[3]. She studied it at home and often came back to the experimenter (i.e. NS) with new suggestions. Very soon, the previously formal interviews were transformed into a qualitatively different setting in which the research purposes grew less and less important, while the teaching-learning purposes became prominent. Molly continued investigating RA with growing autonomy, formulated hypotheses, definitions, tentative theorems, meeting regularly with the experimenter to discuss her work. Her 'co-operation' with the experimenter spanned all five years of her study and culminated in her two-hundred-page diploma thesis on the topic.

Her diploma thesis consists of two parts. The first, a *mathematical* section comprises (1) the mathematical description of restricted arithmetic, and (2) some results of investigating magic squares, Pythagorean triples and pyramids within restricted arithmetic. In both cases, these are Molly's own mathematical results which have not been elaborated anywhere in literature. The second part is *didactical* and consists of (3) a teaching experiment, which Molly prepared, carried out and analysed (including an overview of terms she was using – constructivist approaches to teaching, clinical interview, how to make protocols, concept mapping in the didactics of mathematics), and (4) a proposal for a mathematical project for primary and secondary students on restricted arithmetic. The appendix includes protocols from the teaching experiment.

Illustration 2: ***Concept creation process in geometry of the elementary school***

The diploma thesis 'Concept creation process in geometry of the elementary school' was written by a female student – Dana. Her supervisor was the second author of this contribution (DJ). Dana was a future elementary teacher. Her diploma thesis involved a qualitative analysis and its research tool was the well-known YES-NO game. The rules of this game are simple. It is played by two players A and B (these can be either individuals or groups). A set of objects represented by models, icons, or written form is given. Player A chooses one element from the set mentally. The task of player B is to guess using yes-no questions only which element has been chosen. The game has been developed so

[3] For its description see Stehlíková and Jirotková, 2001.

that it can be used with elementary pupils and the set of objects for this age-group consists of models of solids.

Dana prepared the game and played it with elementary pupils. She recorded all interactions and interviews she had made with the pupils. She transcribed them and complemented the transcriptions by indicating where necessary the pupils' non-verbal expressions. These protocols were then analysed from the point of view of the concept creation process. Dana was very surprised to see how much information she could extract about the pupils' images of not only solids, but also of various geometric relations, what experiences pupils have, what the level of their communication abilities is, etc. She gradually developed her ability to analyse the protocols and gained sensitivity to important cognitive and communication phenomena. The relation with her supervisor developed from 'student – researcher' to 'researcher – expert' and finally to a 'partner' relationship.

Some characteristics of NT diploma theses

Long-term elaboration: It is not unusual for the diploma students to work on their diploma theses for three or more years. Very often they volunteer to take part in some research as subjects and then they get interested in the topic and continue with their own part of the research.

Student's own experiment: We believe that clinical interviews can help student teachers to learn both about how pupils think and how they themselves think. Therefore, interviews often become part of the research diploma that students carry out. Recording, transcribing and analysing a teaching experiment is a valuable source of information for student teachers from the point of view of both mathematics and pedagogy. When analysing their experiment, the students reflect and critically evaluate their own behaviour in the roles they played during the interview – the role of teacher and researcher. They realise the deficiencies in their mathematical formulations, their teaching methods, communication techniques, etc. They develop a sensitivity to use, or avoid where possible, such situations.

Student's introspection: Introspection is one of the characteristics of the methodology built into the Prague Seminars and has become a part of diploma theses as well. The students are asked to reflect on (a) themselves being problem solvers, and (b) themselves being teachers/researchers. In agreement with Duffin and Simpson (2000), we take into account that introspection should be complemented by other techniques in order for us to get more creditable results, and thus this technique is accompanied by co-spection (Duffin and Simpson characterise it as "sharing of our own personal reactions to experience"; in our case it is the sharing between the diploma student and the supervisor) and research 'as if from inside' with the student trying to take into account how the learner might reason whilst he/she is observing the experiment.

Mutual influence between a diploma student and supervisor: The long-term relationship between a diploma student and a supervisor influences both parties. During their work together they develop mutual trust which is very important for the 'success' of their co-operation. While the supervisor first plays the role of a teacher, he/she later becomes a *discussion partner*. The student learns how to do research in mathematics education, gets to know his/her learning and teaching styles, etc. On the other hand, the supervisor gets new and fresh ideas from the student who might have different opinions and offer different perspectives, which the supervisor can immediately use in his/her teaching practice. He/she also improves his/her sensitivity to students' reactions, which has a positive impact on his/her teaching, too.

Supervisor's involvement in the topic: It is also distinctive to the research at the Faculty of Education that researchers (and teachers) go through the similar situations (usually at a higher level) as their students to get a deeper insight into a student's thinking processes (see also Littler and Koman, 1998). For instance, in illustration 1, the supervisor herself first investigated RA (which had not been previously elaborated anywhere) before offering it to Molly. In illustration 2, the supervisor carried out many experiments with the YES-No game first. Our experience has shown that this fact contributes profoundly to the positive and motivational climate between the diploma student and the supervisor.

Change of roles: The change of roles is the most important characteristics of NT diploma theses. By the change of roles we mean that the participants, i.e. a diploma student and his/her supervisor, assume certain roles during the elaboration of the diploma thesis: the role of *pupil*, *teacher*, *researcher* and possibly *expert*.

Roles of pupil, teacher, researcher, and expert

In illustration 1, we can distinguish several roles of the supervisor and several roles of the diploma student (roles are given in brackets). The supervisor [researcher] carried out a series of semi-structured interviews within the research on structuring mathematical knowledge. The (future) diploma student took part in them as a subject [pupil] and was asked to investigate RA. Later, as the didactic intentions of the supervisor grew more dominant, she [teacher] guided Molly [pupil] in Molly's own discovery of RA in a constructivist way. While doing so, Molly prepared [researcher/teacher], carried out [researcher/teacher] and analysed [researcher] a teaching experiment based on RA. The supervisor [expert/researcher] discussed with Molly the phases of her teaching experiment.

In illustration 2, the diploma student [pupil] first experienced the game in her classes with the supervisor [teacher]. Then the supervisor [researcher/teacher] offered the diploma student [pupil] a set of problems which concerned the game above and which had both a mathematical and a didactical character. They proceeded to prepare together [expert – researcher] the experiment, which Dana [researcher] later analysed. The supervisor assumed the role of [expert]. Dana [researcher/teacher] realised her experiments during her teaching practice.

From the examples it is clear that the roles are not easily separated and in fact, one person can play several roles in a particular situation.

Role	**Cognitive competencies**	**Actions**
pupil	is able to solve presented mathematical problems, to play a game, etc., to react to teacher‘s challenges	solves problems, plays a game, etc., responds to teacher’s challenges
teacher[4]	is able to create suitable problems, questions, etc. with the given educational goals, knows the class, knows his/her pupils, can use his/her pedagogical experi-ence, has communicative skills	realise didactic intentions, possibly co-operates with the researcher, reacts positively to the pupils social environment to create problems which are relevant in pupils’ experience
researcher	is able to determine the goal of research, to prepare, realise and analyse an experiment, to formulate results of the research, to propose research implemen-tation, formulate questions for new research	(in co-operation with an expert and/or teacher) prepares a scenario of the experiment, realises it, keeps evidence, makes protocols and analyses them
expert[5]	is able to place research into a wider context, to make hierarchies of goals and research results, to offer a methodology for research	discusses with the researcher, or teacher, goals and conception of research, its results, the techniques of analysis

In the table above, we attempted to describe the above roles in a more general way. Each of them is characterised by two parameters – cognitive competencies and actions. Only some characteristics of the individual roles have been chosen for their description.

The change of roles between teacher and researcher is particularly interesting and the above mentioned literature on action research pays considerable attention

[4] Goffree et al (1999) describe the role of teacher as follows: the teacher creates conditions for student activity, creates a favourable disposition towards mathematical tasks, promotes investigative processes, sustains students’ activity, promotes communication and thinking and stimulates the development of concepts and procedures.

[5] Goffree et al (1999) characterise the expert as a person who helps the student in that he/she “points out relevant events and interesting phenomena, asks questions and discusses the answers, gives explanations and makes theoretical reflections”.

to it. We can say that the continual change between the role of teacher and the role of researcher is characterised by *distancing*. If the person distances him/herself from the event, then he/she behaves as a researcher rather than a teacher. When the didactical intentions are more prominent, than he/she assumes the role of teacher.

Note: A teacher finds it more difficult than a researcher to distance him/herself towards a pupil who is having problems with the research experiment.

Conclusions

What do we consider as having contributed to the success of our work with diploma students? We believe that is the interplay of several factors. First, we were able to offer the students tasks early in the course, which they found interesting. Next, we involved them in our research, which stimulated them into doing their own research in the area. Doing their own research project helped them see things they had not seen previously. Putting them into different roles during their work also widened their perspective profoundly. It is our belief that these factors contributed greatly to the students' willingness and eagerness to learn something new and not just write a diploma thesis. We are aware that this approach will not work with all students. Some of them will feel more secure by writing the ‘old type’ of diploma theses.

We believe that the above described way of ‘supervising’ diploma students will influence a long-term change of the students’ beliefs and their teaching styles and that they will be more disposed to the constructivist approaches to teaching and influence other practising teachers in this direction. Their long-term development as practising teachers and/or researchers will be followed.

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Vad lär lärare och elever i år 7 - 9 via rika problem?

Eva Taflin, Kerstin Hagland, Rolf Hedrén
Högskolan Dalarna

Inledning

I den senaste svenska kursplanen i matematik för grundskolan står bland annat:

> Problemlösning har alltid haft en central plats i matematikämnet. Många problem kan lösas i direkt anslutning till konkreta situationer utan att man behöver använda matematikens uttrycksformer. Andra problem behöver lyftas ut från sitt sammanhang, ges en matematisk tolkning och lösas med hjälp av matematiska begrepp och metoder. … Problem kan också vara relaterade till matematik som saknar direkt samband med den konkreta verkligheten. För att framgångsrikt kunna utöva matematik krävs en balans mellan kreativa, problemlösande aktiviteter och kunskaper om matematikens begrepp, metoder och uttrycksformer. Detta gäller alla elever, såväl de som är i behov av särskilt stöd som elever i behov av särskilda utmaningar.
> (Skolverket, 2001, s 2)

Anledningen till att problemlösning har denna centrala plats i matematikundervisningen diskuteras dock inte så ofta. Enligt vår uppfattning kan detta leda till att lärarna betraktar problemlösning som ett moment vid sidan av den ordinarie undervisningen, något som man tar till som stimulerande inslag och/ eller för att sysselsätta snabbräknande elever.

Inom både den internationella och nationella matematikdidaktiska forskningen har elevers strategier vid problemlösning i matematik och deras förmåga att klara av matematiska problem länge varit ett rikt forskningsfält. Det som enligt vår uppfattning är mindre rikligt representerat är forskning kring hur lärare lägger upp och motiverar sin undervisning i samband med problemlösning, hur eleverna uppfattar denna undervisning och vilka tillfällen till lärande som därvid erbjuds eleverna.

I vårt projekt om rika problem, RIMA, som vi behandlar i denna rapport, vill vi speciellt uppmärksamma dessa frågor. Enligt Lesh et al (1983) kommer problemlösning och tillämpningar i matematik inte att användas i skolan om inte lärare och andra praktiker blir övertygade om att dessa spelar en viktig roll när det gäller att eleverna förvärvar grundläggande matematiska idéer. Dessa tankegångar är en utgångspunkt för vårt projekt, och vi använder en typ av problem, som vi uppfattar vara särskilt ägnade för att skapa tillfällen till lärande. Sådana problem kallar vi rika problem, en term som med olika definitioner återfinns i den matematikdidaktiska litteraturen (t ex Björkqvist, 1999).

Innan vi går in på vår egen definition av rika problem vill vi kortfattat beröra vad andra forskare skrivit om problemlösning. Lester (1983) ger en definition av ’problem’, som vi uppfattar vara tämligen allmänt accepterad. Ett problem är en uppgift för vilken:

1. individen eller gruppen som möter uppgiften vill eller måste finna en lösning,
2. det inte finns en enkel tillgänglig procedur som garanterar eller komplett avgör lösningen,
3. individen eller gruppen måste göra en ansträngning för att finna lösningen.

Det är framför allt det andra och tredje villkoret som skiljer ett problem från en s k rutinuppgift, där lösningsproceduren är given, och som oftast är en ren tillämpning av ett matematikmoment, som eleverna nyligen undervisats om.

Olika forskare har i sina publikationer betonat olika aspekter av problemlösning. Polyas fyra problemlösningsfaser (Polya, 1945) är väl bekanta: Att förstå problemet, att göra upp en plan, att genomföra planen och att se tillbaka för att kontrollera lösningen. Polya betonar också vikten av heuristik, metoder och regler för upptäckt och uppfinning. Problemen ska lösas genom en dialog, läraren ska ställa de rätta frågorna för att hos eleverna skapa tillfällen till lärande.

Schoenfeldt (1983) ser en kvalitativ skillnad i om eleven fattar taktiska eller strategiska beslut vid problemlösning, en skillnad som är avgörande för framgång i verksamheten. Ett taktiskt val innefattar för Schoenfeldt standardprocedurer, algoritmiska val men också val av metod, t ex att rita ett diagram. Ett strategiskt val är av mer övergripande natur, dit räknar han en övergripande handlingsplan, som närmast hör samman med metakognition.

Jaworski (1998) tar upp problemlösning med utgångspunkt från konstruktivismen. Hon pekar på att eleverna individuellt utvecklar sin egen förståelse, men att denna samtidigt är starkt beroende av klassrumsdiskursen. Lärarens roll kan uppfattas vara att dela med sig av sitt sätt att se på problemets matematiska innehåll som en av rösterna i den sociala diskursen.

Många av de problem, som de ovan nämnda forskarna ger exempel på vid problemlösning, leder till att eleverna får tillfälle att arbeta med olika matematiska satser och samband. Ingen av forskarna använder dock termen "rika problem". Björkquist (1999) skriver däremot om rika matematikuppgifter, som är värdefulla på grund av sitt matematiska innehåll. Sådana uppgifter ska användas för att koppla olika tillvägagångssätt till varandra och utgöra utgångspunkter för att utveckla olika teman inom matematik. De kan ses som viktiga hjälpmedel för att bygga upp kognitiva scheman och kan då även fungera som nyckeluppgifter för förståelse, för minnet och för generalisering.

Vi vill slutligen peka på Wistedts studier kring vardagsnära problem (Wistedt, 1992). När man ger elever problem från en vardaglig kontext kan det, som författaren visar med exempel, hända att verklighetens realiteter tränger sig på och hindrar eleverna från att utnyttja sina matematiska kunskaper för att lösa problemen.

Med utgångspunkt i litteraturstudier och även i våra egna erfarenheter av lärares, lärarstuderandes och elevers arbete med olika typer av problem, har vi ställt upp följande kriterier för ett rikt problem:

1. Eleven ska utveckla sin matematiska kunskap genom att arbeta med problemet.
2. Problemet ska vara lätt att förstå, och alla ska ha en möjlighet att arbeta med det.
3. Problemet ska inte ha en för lösaren given lösningsstrategi.
4. Problemet ska upplevas som en utmaning, kräva ansträngning och ta tid.
5. Problemet ska kunna lösas med flera olika representationer.

Vi kan se dessa kriterier som en utvidgning av de villkor, som Lester (1983) ställer på problem. Vi vill även peka på att eleven ska tillägna sig ny matematisk kunskap eller fördjupa den hon äger. Vi vill också betona att det ska finnas flera olika sätt att lösa problemet på genom att det ska kunna leda fram till flera representationer, och att alla elever i den grupp, som arbetar med det, ska känna att de kan nå något resultat vid problemlösningsprocessen.

Syfte och frågeställningar

Syftet med det här diskuterade projektet är, som berörts i inledningen, att skaffa fördjupade insikter i hur lärare lägger upp arbete kring problem, som har en potential att vara rika enligt de kriterier, som vi angett ovan. Vi vill också studera hur elevernas eget arbete kring lösande av problemen kan leda till att tillfällen för lärande uppkommer samt hur eleverna själva uppfattar arbetet med lösande av rika problem.

Våra frågeställningar är:

1. På vilka grunder planerar lärare lektioner kring rika problem?
2. Vilka tillfällen till lärande uppfattar läraren att hon/han skapar i sin undervisning kring rika problem?
3. Vilka tillfällen till lärande uppstår, när eleverna arbetar med rika problem, och hur utnyttjar de dessa tillfällen?
4. Hur uppfattar eleverna undervisning kring rika problem?

När vi enligt punkt 1 studerar grunderna för lärarnas planering, skiljer vi på matematiska grunder å ena sidan och sociala grunder å den andra.

Genom att vi tidigare i stor utsträckning har studerat lösande av rika problem bland grundskoleelever, lärarstuderande och även bland färdiga lärare och att vi studerat relevant forskningslitteratur, anser vi oss ha möjlighet att ställa upp en hypotes. Vi tror att lärarens agerande har stor betydelse för om ett problem ska bli rikt eller inte enligt de ovan angivna kriterierna. Detta gäller både när hon presenterar problemet och när hon leder den gemensamma klassdiskussionen i samband med att eleverna redovisar sina lösningar.

Metod

Projektet startade höstterminen 2001 och är longitudinellt. Vi kommer att följa fyra olika klasser, som tillhör två olika skolor och har fyra olika matematiklärare, under skolåren 7 t o m 9. Under åren 7 och 9 kommer eleverna att få tillfälle att arbeta med tre problem, som vi anser vara rika, under år 8 med fyra sådana problem. Problemen kommer i stor utsträckning att väljas från en problembank i samråd med de deltagande lärarna. Vi söker därvid att så långt möjligt välja problem, som anknyter till den övriga matematikundervisningen. Vi kommer att med lärarna diskutera de erfarenheter av arbete med problemen, som vi själva skaffat oss, innan lärarna låter sina elever arbeta med dem. I övrigt ges lärarna full frihet att lägga upp sin undervisning kring dem som de själva anser vara bäst. De kan t ex välja att låta eleverna arbeta enskilt med problemen, låta dem arbeta i grupper eller först låta dem arbeta enskilt för att sedan samlas till gruppvisa diskussioner.

Vi samlar samtliga i projektet deltagande lärare till träffar ungefär tre gånger per termin. De därvid förda diskussionerna spelas in på ljudband.

Följande utvärderingsmetoder används:

- Videoinspelningar av lärare och några elever under problemlösningstillfällen.
- I två grupper av elever används även inspelning med ljudband, eftersom videokamernas ljud inte är tillräckligt för att fånga diskussionerna i grupperna.
- Ljudbandinspelade intervjuer med lärare före och efter lektioner. Den sistnämnda sker med hjälp av den metodik som på engelska brukar kallas *stimulated recall*.
- Läraranalys av lektioner.
- Lärarloggböcker.
- Kliniska intervjuer med elever före och efter lektioner. Dessa spelas in på ljudband.
- Intervjuer med elever efter lektion med hjälp av *stimulated recall*. Intervjuerna spelas in på ljudband.
- Test och enkäter som bjuds tre gånger under projektets gång.

Inspelningar med ljudband transkriberas för att användas i kommande analyser. Videoinspelningarna bildar underlag för *stimulated recall*. I de grupper, där diskussionerna spelats in med ljudbandspelare används de också för att vi ska kunna avgöra vem som säger vad och för att kunna följa elevernas gester och mimik.

Resultat

Vi tar här endast upp resultaten från det första problemlösningstillfället, det enda som vi i någon mån har hunnit analysera hittills. Det problem, som eleverna arbetade med vid detta tillfälle löd:

> *32 Ahlgrens bilar kostar 10 kronor. Hur många bilar får du för 25 kronor?*

Problemet kom att utvidgas på olika sätt i de fyra klasserna. I denna rapport koncentrerar vi oss på det som händer i två av klasserna och huvudsakligen på de matematiska målen. Vi tar upp frågeställningarna i ordning.

På vilka grunder planerar lärare lektioner kring rika problem?

Lärarna, som vi kan kalla Sven och Sara, uttalar sig på följande sätt om de matematiska målen för problemlösningstillfällena. Vi börjar med ett kort utdrag ur intervjun med Sven före problemlösningstillfället. ET är intervjuare.

ET Är det något annat du tycker är viktigt att lyfta fram här?

Sven Det viktigaste målet som jag har är att dom ska få öva på problemlösning att dom ska se att det finns olika sätt att lösa samma problem. Sekundärt mål med lektionen är att dom ska få lite hum om proportionalitet och så där men huvudsaken är att öva problemlösning och att öva att arbeta i grupp.

ET Om du kommer på det sekundära målet som du sa handlar om proportionalitet vad mera inom matematik är det som du tycker att dom tränar på?

Sven Förutom proportionalitet, det beror på hur dom löser problemet lite grann. Om dom väljer att ställa upp som bråk eller om dom räknar ut det med procent eller om dom gör en konkret lösning med att rita bilder av kronor eller konkreta bilar. Jag vet inte vad dom kommer att välja för strategi. (Intervju med Sven 01-11-14)

För Sven är det ett mål i sig att öva att arbeta i grupp. Vad gäller de matematiska målen går han ut mycket öppet och överlåter i stort sett åt eleverna att välja den matematik, som de vill föra in i sitt arbete med problemet. Att öva på problemlösning är också ett mål för Sven.

Sedan följer motsvarande intervju med Sara.

ET Har du funderat redan nu om du har någon särskild indelning av lektionen, hur lång den är och hur du använder tiden?

Sara Ja först tänkte jag att dom skulle få arbeta individuellt för att försöka få fram en uppställning eller om dom resonerar sig muntligt eller om dom ritar på en gång eller om dom sätter upp ett uttryck, hoppas att det blir mångfald där då. Och där har dom olika strategier och sen ska dom få gå fram till tavlan och berätta för kamraterna då och sen diskuterar vi dom olika om vi har kommit fram till samma sak. Och sen hade jag tänkt att efter den genomgången får dom jobba ihop i liten grupp för att dels göra det här diagrammet. Och sen att dom funderar över räta linjens ekvation.

ET Är det något annat särskilt?

Sara Jag vet inte, jag tror att det är en rätt så hög nivå som jag har lagt det på.

ET Så att det kan bli svårt för alla att komma dit?

Sara Ja det tror jag, diagrammet tror jag att alla klarar då, men sen att förstå sambandet kan, dom förstår nog men dom kan inte uttrycka det.
(Intervju med Sara 01-11-21)

Sara talar om hur hon tänker organisera arbetet men nämner inte att kunna arbeta i grupp som ett socialt mål. Hon är också inriktad mot specifika matematiska mål, som eleverna ska söka uppnå, att rita diagram och fundera över räta linjens ekvation. Samtidigt är hon medveten om att målen är högt satta och att alla inte kommer att kunna nå dem.

Vilka tillfällen till lärande uppfattar läraren att hon/han skapar i sin undervisning kring rika problem?

Vid den intervju, som görs med lärarna i samband med *stimulated recall*, har de möjlighet att reflektera över sin undervisning och vad den kan tänkas ha gett för eleverna. Sven uttrycker det på följande sätt:

ET När du ser dom redovisa framför klassen vad ser du för effekter av det, plus och minus?

Sven Jag tror att det är nyttigt att få öva det framför grupp.

ET Hur blir dom matematiska fördelarna?

Sven Dom som redovisar måste tänka till hur man själv har tänkt. Sen kanske lite fördelar för dom som står där dom får visa vad dom kan. Dom är lite stolta när dom redovisar.

ET Dom har räknat ut 30 kronor 96 bilar och då ska dom betala för 24 bilar till och det där hade dom ju en ganska häftig tanke det var den du hjälpte dom 4/4 och sen kom dom på att dom där 24 bilarna motsvarades av 3/4 krona och sen räkna dom ut det.

Sven Här tror jag att jag går in för mycket och tar över deras redovisning, jag vet inte. Eller så skulle dom haft mer tid.

…

ET Jag tror att det är viktigt att du går in och förtydligar vad eleverna gör. Det är inte så enkelt att ha ord och uttryck. För du visar att 24 av 32 var 3 av 4 medans dom pratar om 3 fjärdedelar av 10 kronor.

Sven Fast egentligen så visar jag det dåligt vi har inte hållit på med bråk. Ska man visa det så skulle man kanske gå vidare och ta ett stort exempel ha en tårta som kostar 32 men vi har bara 24 hur stor del av tårtan får vi handla. Men då blir det en lång genomgång på det också.
(Intervju med Sven 01-11-20)

Efter intervjuarens påpekande ser Sven att det dök upp ett matematiskt intressant proportionalitetsproblem vid redovisningen. För Sven är det oklart om han här borde ha tagit tillfället i akt och diskuterat räkning med bråk eller inte. Han ger ändå ett exempel på hur det skulle kunna ha gått till.

I Saras klass inträffade en oväntad händelse. Medan eleverna löste det givna problemet individuellt, gick Sara runt och försökte hitta så många olika lösningar som möjligt. Bland annat uppmärksammade hon en pojke med en avvikande lösning och ett avvikande svar och tog sedan chansen att släppa fram den pojken till tavlan. Intervjuaren och Sara diskuterar motsvarande avsnitt på videon

ET	Här sitter en kille kommer du ihåg vad ni pratade om?
Sara	Jag såg att han kommit snett med hur han skulle lösa det. Jag fick förklaringen idag. Man kan ju inte dela en påse Ahlgrens bilar! Man köper bara hela. Man kan inte få ut för 25 kronor och 80 bilar.
ET	Det var han som skrev 12,50 + 12,50 på tavlan?
Sara	Nej, det var pojken bredvid det sörru.
ET	Så han var låst i verkligheten? Vad sa du då?
Sara	Du har alldeles rätt sa ja. Han var den enda i hela klassen som inte hade kommit fram till 80 bilar.
ET	Han var i verkligheten, man kan inte dela och köpa.
Sara	Jag är glad att jag fick en förklaring.
ET	Det var bra att du följde upp det idag.
Sara	Det hade jag inte kommit på själv. (Intervju med Sara 01-11-22)

Vilka tillfällen till lärande uppstår, när eleverna arbetar med rika problem, och hur utnyttjar de dessa tillfällen?

I Svens klass utvidgades problemet på följande sätt: *Vad kostar 120 bilar*? Det förutsattes stillatigande att kostnaden var proportionell mot antalet bilar. I denna klass följde vi en grupp av elever. Dessa elever såg inte möjligheten att bygga vidare på svaret från det ursprungliga problemet, att 80 bilar kostar 25 kr, vilket snabbt hade kunnat hjälpa dem att besvara den nya frågan. Efter att gruppen med negativt resultat hade prövat olika metoder att lösa problemet, uppstod följande samtal mellan två av gruppens deltagare. Eleverna hade tillgång till en miniräknare. Namnen är fingerade.

Björn	… Så här blir det: Man tar 32 gånger 3, just det här på den här uppgiften (ohörbart) eller.
Bertil.	Jaa. Vi får ta 32 gånger (ohörbart) då.
Björn	32 gånger 3 är 96.
Bertil	Mm.
Björn	Och sen tar vi 32 delat i 4 är lika med 8.
Bertil	Mm.

Björn Då tar man 96 plus, 8 och 8 är 16, nä 8 gånger 3, det är 24.
Bertil 3 gånger.
Björn Ja, 8, 8 gånger 3 chansar vi på, och det blir 24. Tar vi 24.
Bertil Det blir 120.
Björn Ett, två, noll. Alltså skulle det kosta. Nu har vi gjort fel, nu har vi bara fått fram ett svar, tre det är 30 kronor, och så tre fjärdedelar av ti, av 10 kronor, 2 och 50, 7 och 50 plus det. Det är ett hund, nä, 37 och 50.
(Observation med Bertil och Björn, 01-11-14)

Eleverna ser först att 3 gånger 32 bilar är 96 bilar. Det fattas då 24 bilar till 120 bilar. Sedan upptäcker de att 32 delat med 4 är 8 och 3 gånger 8 är 24. 96 bilar kostar 30 kronor. Sedan ska de lägga till 3 fjärdedelar av 10 kr, d v s 7,50 kr. Den här lösningen redovisades inte på tavlan, men den motsvarar den, som ET och Sven talar om ovan.

Hur uppfattar eleverna undervisning kring rika problem?

Eleverna i Saras klass kommenterade den lösning, som ET och Sara diskuterar ovan. De uppfattade att den kan vara riktig, om man utgår från verkligheten. Även här tittar eleverna på filmen, medan de ger sina kommentarer. Namnen är fingerade.

Carl Här kommer Jakob, Jakob hade väl på sätt och vis fel men ändå rätt liksom.
Birger Han skrev ju att man bara kunde få sextiofyra bilar för man kan ju inte dela på en förpackning.
(Intervju med Anders, Birger och Carl 01-11-26)

Eleverna kommenterade också lärarens förehavande vid redovisningen. Här är det KH som intervjuar.

KH Vad gör hon nu?
Carl Där visar hon för närvarande hur alla tänkte, eller jag tror det i alla fall.
KH Jaha.
Anders Eller nej, det var där hon visade hur mycket olika bilar skulle kosta.
Carl Ja, ett sådant papper fick ju vi.
Anders Det där diagrammet .
Carl Mm.
KH Varför gjorde hon det, tror ni? Vad vill hon att ni skulle förstå där då?
Carl Hur man räknar ut det.
Anders Hur man kom fram till hur priset blev ju mer eller mindre bilar det var att det bara ökade ju mindre och att det minskade när det blir mindre och att priset berodde på hur många bilar det var.
(Intervju med Anders, Birger och Carl 01-11-26)

Eleverna har en god uppfattning om vad läraren ville visa med det diagram, där antalet bilar och kostnaden för dem jämförs.

Diskussion

Vi kan se, i de korta avsnitt av intervjuerna före problemlösningstillfällena som redovisats här, att Sven och Sara har ganska olika utgångspunkter för elevernas lärande i dessa sammanhang. Sven går ut på ett öppet sätt och låter eleverna styra över den matematik, som ska komma ut ur arbetet. Sara är mer målmedveten. Hon vill att lektionen ska peka hän mot ritande av diagram och en förberedande diskussion om räta linjens ekvation, även om hon inser att detta stoff kommer att ligga på ett väl högt plan för hennes elever.

Tendensen håller i sig även i den intervju, som företas efter det att eleverna arbetat med problemet. Det verkar inte som om Sven är riktigt medveten om den intressanta matematik, som de grupper, som blev tvungna att räkna ut priset för 24 bilar, var inne på. I varje fall var han inte beredd att gå in på detta moment vid själva redovisningen, även om han hade helt klart för sig hur han skulle ha kunnat gå till väga. Kanske borde han inte vara så beroende av att klassen har diskuterat bråk förut just i skolår 7. Eleverna bör ha stött på begreppet tidigare. Den grupp, som observerades, visade ju också att de kunde hantera problemställningen, även om de använde sitt eget sätt att uttrycka sig på.

Sara är noga med att följa upp problemet också genom att i förväg försöka hitta olika elevlösningar, ja, hon vågar till och med plocka fram en, som hon tror leder till ett felaktigt svar. Det visade sig senare vara ett klokt val. Här kommer både hon och hennes elever till insikt om att verkligheten faktiskt ibland kan vara ett hinder för en mer teoretisk matematik, som Wistedt (1992) så tydligt har belyst. Carl och Birger visar även i intervjun med KH att de uppfattat att olika förutsättningar kan leda till olika svar. Sara följer också upp sin idé om att leda in problemlösningen på diagram och i viss mån räta linjens ekvation. Carl och Anders talar också i intervjun med KH om att de uppfattat vad deras lärare ville visa.

Vi vill, i varje fall i det preliminära skede som projektet befinner sig i, inte uttala oss om vilket av de här diskuterade tillvägagångssätten som är rätt eller bäst. Det enda vi kan konstatera är att ett och samma problem av lärarna kan hanteras på vitt skilda sätt, och vi tror oss också våga påstå att vad eleverna lär sig, när de arbetar med problemlösning, hänger nära samman med hur läraren presenterar problemen och följer upp elevernas varierande lösningar.

Vi anknyter till vad Lesh, Landau & Hamilton (1983) säger, nämligen att problemlösning och tillämpningar i matematik inte kommer att användas i skolan om inte lärare blir övertygade om att dessa spelar en viktig roll för elevernas kunskapstillägnande i matematik. Är detta riktigt, och vår egen erfarenhet pekar starkt i den riktningen, visar det att studier kring samband mellan uppläggning av problemlösningslektioner och elevers inhämtande av kunskap i matematik, är mycket angelägna.

Vi har i denna rapport bara kunnat ta med några korta episoder, som utspelade sig, och några tankar, som de inblandade aktörerna gav uttryck för. Vi tror dock att det som här kommit fram kan ge en antydan om något av det som vi bör gå närmare in på i den fortsatta forskningen.

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APPENDIX: E-mail addresses to the contributors

Christer Bergsten	chber@mai.liu.se
Ole Björkqvist	objorkqv@abo.fi
Morten Blomhøj	morten@mmf.ruc.dk
Doug Clarke	d.clarke@patrick.acu.edu.au
Jan de Lange	jan@fi.uu.nl
Arne Engström	arne.engstrom@pi.oru.se
Jeff Evans	j.evans@mdx.ac.uk
Barbro Grevholm	barbro.grevholm@mna.hkr.se
Kerstin Hagland	kha@du.se
Rolf Hedrén	roh@du.se
Milan Hejny	milan.hejny@quick.cz
Marj Horne	m.horne@patrick.acu.edu.au
Darina Jirotková	darina.jirotkova@pedf.cuni.cz
Bo Johansson	bo.johansson@ilu.uu.se
Marie Kubinova	marie.kubinova@pedf.cuni.cz
Thomas Lingefjärd	thomas.tingefjard@ped.gu.se
Graham Littler	g.h.littler@derby.ac.uk
John Mason	j.h.mason@open.ac.uk
Ulla Runesson	ulla.runesson@ped.gu.se
Anna Sfard	sfard@netvision.net.il
Nada Stehlíková	nada.stehlikova@pedf.cuni.cz
Rudolf Strässer	rudolf@sm.luth.se
Eva Taflin	evat@du.se
Anne Watson	anne.watson@educational-studies.oxford.ac.uk

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