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Svensk Förening för **M**atematik**D**idaktisk **F**orskning
Swedish Society for Research in Mathematics Education

Research and Action In the Mathematics Classroom

Edited by

Christer Bergsten
Göte Dahland
Barbro Grevholm

Proceedings of ***MADIF 2***

The 2nd Swedish Mathematics Education Research Seminar
Göteborg, January 26-27, 2000

Skrifter från **S**vensk Förening för **M**atematik**D**idaktisk **F**orskning, Nr 1

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Preface

This volume contains the proceedings of *MADIF 2*, the Second Swedish Mathematics Education Research Seminar, with an introduction by Barbro Grevholm. The seminar, which took place in Gothenburg in January 26-27, 2000, was arranged by *SMDF,* The Swedish Society for Research in Mathematics Education, in co-operation with NCM, the National Center for Mathematics Education. The members of the programme committee were Ann Ahlberg, Christer Bergsten, Göte Dahland, Barbro Grevholm, and Ulla Runesson

The programme included four plenary lectures, two plenary panels, and ten paper presentations. The four plenary lectures and eight paper presentations are published here. We want to thank the authors for their interesting contributions. The papers have been reviewed by the editors, and some minor editorial changes have been made without noticing the authors. The authors are responsible for the content of their papers.

We wish to thank the members of the programme committee for their work to create an interesting programme for the conference, and Bengt Johansson and Birgit Eriksson from NCM for all their help with the preparation and administration of the seminar. We also want to express our gratitude to NCM for its valuable financial support. Finally we want to thank all the participants at *MADIF 2* for creating such an open, positive and friendly atmosphere, contributing to the success of the conference.

The editors
Christer Bergsten
Göte Dahland
Barbro Grevholm

MADIF 2 – Conference programme

Wednesday January 26, 2000

Time	*Programme*	*Contributors*
09.00-09.15	Opening	Barbro Grevholm, Bengt Johansson
09.15-10.10	Plenary talk 1	Christer Bergsten
10.10-10.40	Break	
10.40-11.40	Paper session 1	Thomas Lingefjärd Ann Ahlberg et al Merethe A.Nilssen & Guri Nortvedt Arne Engström Eva Taflin
11.45-12.15	Information/Discussion	
12.15-13.15	Lunch	
13.15-14.10	Plenary talk 2	Barbara Jaworski
14.15-15.15	Paper session 2	Gunnar Gjone Rolf Hedrén Allan Tarp Ingemar Holgersson Per-Olof Bentley
15.15-15.45	Break	
15.45-17.00	Plenary panel 1: *Reactions on the sessions of the first day*	Göte Dahland (chair) Ole Björkqvist, Barbro Grevholm, Ingvill Holden, Richard Noss
17.00	Happy Hour	
19.00	Conference dinner	

Thursday January 27, 2000

Time	*Programme*	*Contributors*
08.15-09.10	Plenary talk 3	Michèle Artigue
09.15-10.10	Plenary talk 4	Roger Säljö
10.10-10.30	Break	
10.30-12.00	Plenary panel 2: *Methods and results of research in mathematics education today and tomorrow*	Jeremy Kilpatrick (chair) Michèle Artigue, Celia Hoyles, Frank Lester, Roger Säljö, Inger Wistedt

Content

Research and action in the mathematics classroom

Barbro Grevholm

In her plenary lecture Michèle Artigue deals with *Didactic engineering and the complexity of learning processes in classroom situations*. Introduced in the early 1980:s didactic engineering was intended for two fundamental issues: The question of dealing with the complexity of classroom phenomena in research methodologies and practices and for the question of relationships between research and action on the educational system. She exposes the theoretical base for didactic engineering and presents some prototypical examples of it. Two examples of Brousseau's engineering are given: introducing rational numbers and the puzzle situation, and one example of differential equations on university level. Finally she discusses the potential and limits of didactic engineering as a research methodology and claims that there is a necessary complementarity between it and the more naturalistic methods.

The title of Christer Bergsten's plenary talk is *Faces of Swedish research in mathematics education.* His presentation is based on literature, previous overviews and data collected in 1997 through a questionnaire to researchers in Sweden. He describes the development in Sweden as a move from content oriented and experience based guidance to research based didactics. The term ”matematikdidaktik” (didactics of mathematics or mathematics education) became more widely used after a conference in 1986. Problem areas and research methods used are reviewed, and some examples of Swedish research studies from different time periods are presented. The dissemination of research results is a critical question. Many persons in Sweden doing research and developmental studies in mathematics education have a background as teachers and thus keep their research close to the teaching and learning practice, something which may be viewed as a Swedish tradition in mathematics education research. Finally he expresses the hope that this conference will contribute to increase awareness and interest also among practitioners for what research can offer.

Barbara Jaworski in her plenary lecture talks about co-learning partnerships in mathematics teaching and teaching development. Her title is *The Student-Teacher-Educator-Researcher in the mathematics classroom*. She explains the concept co-learning emanating from Wagner and continues with Pearson's suggestion that teaching is intended to create learning as an answer to the question what is learning and what does it mean to develop it. In her introduction she points out the complexity of the classroom setting. After elaborating some problems in mathematics teaching and its development she gives examples relating to co-learning partnerships. In these many concepts and factors are discussed such as inquiry, classroom norms, diversity, and teacher as researcher in tackling dilemmas. Partnerships on different levels are discussed. Tom Cooney's concepts of mathematical and pedagogical power are related to co-construction in the co-learning partnership. She asks how a co-learning partnership is constructed and how it operates. The discussion on development of norms for co-learning leads to questions as in what ways teachers are socialised into the norms of inquiry and reflection. Finally she presents a diagrammatic representation of participants, concepts and relationships as a starting point for further dialogue about how to develop teaching of mathematics through co-learning at all levels.

From natural language to mathematical reasoning: Word problems and the socialisation of children's thinking is the title of Roger Säljö's plenary talk. His interest is how people reason, argue and act in different communicative practices and how they learn to do so. His perspective is socio-cultural. A few examples illustrate what is going to be his point. He discusses the fact that people develop intellectual and discursive tools, physical tools and social institutions. Learning how to translate from every day language to expressions in mathematical or logical terms is a powerful socialisation of people's minds and an advanced kind of skill. He claims that we have to learn to disregard from the real world and argue in a textual reality. Textual worlds are often different from physical worlds. The discussion continues with questions about attending to the world and attending to texts about the world. He claims that it is extremely important to assist children in bridging the gap between a text about the world and the real world and this is an issue of being made aware of how models relate to physical reality. This learning must come through interaction, arguing and discussion. The skill of realising how to co-ordinate models and mathematical expressions is a discursive skill. Säljö finally claims that learning how to move around in text based realities is a complex aspect of our cognitive socialisation that requires systematic challenges and guidance.

In the paper presentations Merethe Anker-Nilssen and Guri A. Nortvedt talk about *Girls and Mathematics - Focusing on the current situation in the Norwegian upper secondary school*. The aim of the project is to increase recruitment and level out gender differences in upper secondary school. The presentation first outlines findings in the research literature and then describes some findings in a qualitative research study. The aim of the study is to investigate students' attitudes towards mathematics in order to see what makes mathematics a preferred choice of subject. Four different reasons are found: extra credits, mathematics is needed for further education or employment, interest in the subject and mathematics is helpful to understand other subjects. The next part of the study will focus on assessment format, attitudes and expectations towards testing, self evaluation, alternatives to traditional written tests and teaching aids.

Per-Olof Bentley presents *A study of students' ways of experiencing ratio and proportion.* He starts with an introduction of some concepts from literature such as extensive and intensive quantities, homogeneity and relates to studies by Lybeck, by Kaput and West, and by Lamon. Both qualitative and quantitative data are used. In-depth interviews were carried out and three categories found are called explicit, implicit and absolute proportionality. These are elaborated and exemplified. Educational implications are to devote more attention to ratio and proportion.

Arne Engström speaks about *Rationality and intersubjectivity – some preliminary starting points to understand the communicative character of mathematics.* He wants to put forward some preliminary starting points for a theoretical frame-work in a study of the communicative character of mathematics. Mathematics education is related to the ideas of intellectual education and citizenship of the reformation. He claims that the rational action is based in an intersubjective understanding and treats in his theoretical discussion common points from Habermas, Piaget, Wittgenstein, Schütz and Pierce. The conclusion is that problems in having a meaningful and relevant mathematics education should be sought in failures in establishing a functioning social interplay in the mathematics classroom rather than in pupils' lack of ability.

Using symbol-manipulating calculators (SMC) in upper secondary schools is the title of the presentation by Gunnar Gjone. The study, initiated by The National Examination Board in Norway, has the purpose of investigating conditions concerning the use of SMC for final exams in mathematics in the second year of upper secondary school. The question is if it is possible to construct examination problems and organise test situations that mirrors real challenges and possibilities and still provides a base for giving individual marks. The author discusses the use of graphs in solving problems, equality and the formation of the limit concept and illustrates with some exam tasks. In his discussion about forms of knowledge Gjone finds Sfard's notion of structural and operational conceptions helpful but asks if the definitions of these concepts should be somewhat redefined with respect to the new technology.

Rolf Hedrén presents *Alternatives to standard algorithms. A study of three pupils during three and a half years*. The three girls in the study were not taught standard algorithms during their first five years in school. They were encouraged to use their own written methods, including drawings, for all kinds of computations that they could not do mentally. They often worked in a group and were not shown the standard algorithms until their sixth year in school. The results show that the girls could find their own methods on their own and sometimes with the help of peers or teachers. Their methods were less effective and more like methods used in mental arithmetic. They acquired a good number sense and good ability in mental computation and preferred their own methods even after they were taught the standard algorithms.

How can we describe young children's arithmetic abilities? is the question Ingemar Holgersson discusses. The objectives of the study have been to investigate whether the schemes for cognitive development of numerical abilities developed in the US can be used to analyse Swedish children on an individual level and to see how these abilities develop during the time in pre-school. Tasks on sequencing, counting, abstraction and problem solving were used in a study with 16 six year olds. The results of the length of their number sequence is comparable with the results from the US. The most striking result is the great span in ability between different children, a finding also consistent with that of Fuson. A remaining question is why some children use derivations based on a growing number of relations while others resort to counting procedures even for basic number relations. Holgersson thinks that longitudinal studies, focused on important key questions, could contribute to the understanding of this phenomenon.

Thomas Lingefjärd writes about *Mathematical modelling and prospective teachers.* Three studies are discussed with the questions: How do pre-service teachers relate mathematical models to reality when using software tools to generate the models? What conceptions and misconceptions lie behind the decision to believe more in a mathematical model than in real-world phenomena? Some results from the third study are discussed with examples of the kind of tasks students worked with. Authority and responsibility are exposed. When students are forced to explain and argue for the models inaccuracies and misconceptions are revealed that may be hidden otherwise. The conclusion is that teachers on all levels need to be cautious about what students actually understand about the modelling process and how they interpret it.

Allan Tarp presents a paper on *Killer-equations, job threats and syntax errors*. His starting point is the fact that an increasing number of students turn away from mathematics in school and from math-based education programs at university level. He claims that modern research seeks explanations within human factors as students, teachers and cultures. Postmodern counter research looks for hidden possible explanations in mathematics itself. His study identifies unnoticed syntax errors within mathematics and problematic top-down practice that allows killer-equations in the classroom. The study reports successful change of practice with a bottom-up approach that is more user-friendly.

The subject in most of the papers touches upon research and action in the mathematics classroom. Many different aspects are exposed as design of the action through didactic engineering, the importance of action through communication, action in co-learning partnerships on different levels, gender questions, the relation between mathematical models and reality, the use of tools in mathematical activity and alternative pathways for learning. There are many connections between the content in the different papers and two or several authors bring up similar questions from different entrances. In all the authors in their papers have contributed in constructive ways to the development of mathematics teaching and learning through their insightful discussions. The readers should be able to get inspiration for new actions in the mathematics classroom or for new research studies.

Didactic engineering and the complexity of learning processes in classroom situations

Michèle Artigue
Université Paris 7 Denis Diderot

Introduction

In order to understand a concept, it is generally useful to look at its "history", to try to determine what kind of problems motivated its emergence, shaped its development. Without any doubt, such an historical entrance is specially helpful when dealing with didactic concepts. Most of them are still under development, strongly contextualised and dependent from the didactic culture where they emerged. More than by general definitions, they take meaning through the ways they are used and transformed by the didactic community. For such a reason, I begin this text dedicated to didactic engineering by briefly evoking the context of its emergence before presenting its theoretical basis and, in the second part, the main characteristics and phases of an engineering work. In the third part, I illustrate these theoretical and methodological considerations by some typical examples ranging from elementary to university level. In the fourth part, finally, I briefly discuss the potential and limitations of didactic engineering work and the role such a work can play now in the didactic field where a variety of ways of dealing with the complexity of teaching and learning phenomena have been developed.

Didactic engineering: From where does this concept come?

The notion of didactical engineering emerged in the didactic field in France in the early eighties. As appears in a seminal text written by Chevallard (1982), it was seen as a means to approach two fundamental issues.

1. The question of dealing with the complexity of classroom phenomena in research methodologies and practices;
2. The question of relationships between research and action on the educational system.

At that time, in France as in many countries, didactic research was in search of scientific legitimacy and was tempted to borrow scientific criteria and methods from well established domains, such as psychology. Researchers were thus led to escape classroom complexity through what we have then called "external methodologies" such as tests, questionnaires, interviews, with an overemphasis in validation on the statistical comparison between control and experimental groups as if it were the only pathway to scientificity. The concept of didactic engineering was developed in order to overcome the limitations of such an attitude which are now well known (Schoenfeld, 1994), scientifically to take into account the complexity of the systems we wanted to investigate and specifically to find methodological ways for dealing with the complex intimacy of classroom functioning. As regards the second point, relationships between research and practice, the ambition was to find ways of rationalising action on the educational system and to create, in some sense, an engineering science for didactic design. As expressed by Chevallard, in the text quoted above, engineering work relies on scientific knowledge (this is not a requirement for innovation whose essential value is newness even if this newness often only results from the poor institutional memory of educational systems), but engineers have to work with more complex objects than the refined objects of science. They have thus to manage pragmatically problems that science is unwilling or not yet able to tackle, and they have pragmatically to prove the effectiveness of their constructions. According to Chevallard, action research, which was predominantly seen at that time as the way of linking research and action, located in an intermediate position, fulfilled neither the requirements for validity of research, nor those of action.

Let us note that initially, two different terms were introduced: the term of "phenoménotechnique" borrowed from the philosopher Bachelard for the research dimension, and the term of "didactic engineering" for the development dimension. Very soon, however, the two collapsed and only the second survived. In fact, didactic engineering developed mainly as a research methodology. Entering the rational vision supported by the term of engineering for development didn't fit the educational culture. Teachers were more likely to see themselves as "artisans", artists rather than engineers or users of engineering products. Moreover, there were neither appropriate structures nor persons ready to take charge of the huge amount of development work necessary in order, for example, to transform prototypes built for research into viable and robust engineering products. Adaptation of research products to ordinary teaching remained mainly uncontrolled.

The theoretical base of didactic engineering

As mentioned above, didactic engineering emerged and developed in the French didactic culture, relying on its theoretical frames. It particularly relies on the theory of didactic situations initiated by Brousseau (1997) and its evolution has been shaped by the evolution of the theory. There is no doubt that my presentation given here is different from that which I would have given ten years ago or even five years ago.

Introducing didactic engineering I have thus to enter the theory of didactic situations, at least in order to stress some fundamental points, and hope that I will be able to avoid possible misunderstanding.

1. The theory of didactic situations, which began to develop from the late sixties, was initially inspired by constructivism and Piaget's epistemology: learning results from adaptation in some kind of biological sense to "problematic situations". In that sense, it is a constructivist theory.

2. The theory of didactic situations is not a cognitive theory. Its central object is not the cognising subject but the didactic situation: a construct, which denotes the complex set of interactions between students, teachers and mathematics at play in classroom situations. The didactic situation shapes and constrains the adaptive processes students can develop and thus the mathematical knowledge, which can be constructed. One essential assumption is that, without understanding the situation, you cannot interpret students' behaviour in cognitive terms.

3. The aim of the theory is to understand learning and teaching processes and the ways these interact, but beyond that, the theory also aims at developing rational means for controlling and optimising such didactic situations.

In fact, in the theory, classroom situations are modelled at different levels and, to keep here a reasonable level of complexity, I will distinguish two main levels: the "a-didactic" level and the didactic one.

In the a-didactic model, students are modelled as cognitive subjects, in the classical sense. The model focuses on the students/mathematics interaction. One central object is the notion of "medium" presented by Brousseau as a system that reacts to students actions, both in a collaborative and antagonist mode. This medium is defined in terms of material objects and also in terms of knowledge. Knowledge in the medium is knowledge stabilised, ensuring the required familiarity with the mathematical objects at play, giving some kind of reality to the mathematical world. Generally mathematics media are not very rich in material components but they are rich in knowledge components. When students do not work in an isolated way, possible actions and reactions from others have also to be integrated. Taking into account the characteristics of the mathematical situation proposed to students and of the specific medium which shapes their interaction with mathematics, the a-didactic model aims at giving account of possible actions (from the students), of their respective cost both cognitive and technical, of the feed back students can receive from the medium, and the means of control or self-validation these feed-back induce. Didactic variables are those which change the economy of the interaction. At this stage, initially, the teacher was not introduced into the model. This is no longer the case (Perrin-Glorian, 1999), but such integration remains limited to what can enter the model of the students/medium interaction. For instance, some teacher's actions can be modelled in terms of enrichment of the natural retroactions provided by the medium. Other can be modelled in terms of adding new pieces of knowledge to the medium. One hypothesis at the basis of such a modelling is that in order to understand students' behaviour we have to understand their economy, and also start from the principle that the most economical practices are those most likely to appear.

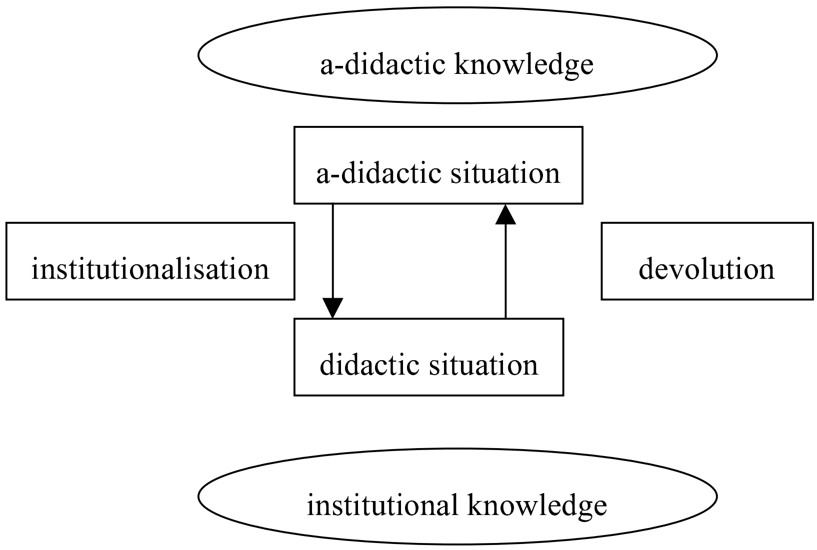
Such a kind of modelling is limited: the student is seen as a pure mathematical subject, which is far from being close to reality. But we all know that even very simplified models can be productive. This is certainly the case with the a-didactic model and I will try to illustrate this point in the third part.

The second level for modelling is the didactic level. At this level, students are also modelled as institutional subjects. Thus their adaptation is constrained not only by the characteristics of the mathematical situation and the way interaction with it has been organised, but also by their knowledge of institutional norms which fix the respective positions and roles of teachers and students. The concept of "didactic contract" which labels the expectations from the teacher with respect to the students, and conversely, as regards the mathematical content, is an essential tool for didactic analysis. Of course, at this didactic level, the teacher is an essential part of the model.

Adaptive processes developed by students are in fact a subtle mixture of a-didactic and didactic processes. Understanding that subtlety is essential in order to understand and even to control what can be learnt in a given situation. By institutional adaptation, indeed, students can learn to behave as good institutional subjects without learning the mathematics we want them to learn.

One fundamental assumption of the theory of didactic situations is that some level of a-didacticity is necessary to mathematics learning. When elaborating engineering designs or observing classroom situations, researchers referring to the theory of didactic situations are thus especially sensitive to the processes, which allow the creation and maintenance of such a-didactic phases. Taking into account the fact that both students and teacher perfectly know that they meet in a didactic institution with specific mathematical aims, the birth of an a-didactic phase requires what has been denoted by Brousseau as a “devolution process”. Through the devolution process, the teacher tries to give mathematical responsibility to students, and let them forget, at least for a moment, that the task given to them has a specific learning aim. The pressure of the didactic contract has to be made as low as possible. All through the a-didactic phase, the teacher has to maintain, through adequate decisions and mediations, which may differ from one group of students to another, this devolution of responsibility.

“A-didactic” phases produce mathematical knowledge, but a form of knowledge strongly dependent on particular actions and contexts attached to the a-didactic situation. Another essential role of the teacher is to help students relate what they have produced in the a-didactic work to more institutional forms of knowledge, those targeted by the didactic institution. Thus, the necessity of the “institutionalisation” process, which can be seen as an inverse process of the devolution process, is displayed. Devolution and institutionalisation, in a given situation, organise the relationships between the two levels we have introduced in the modelling, according to the following schemata:



According to the theory of didactic situations, understanding, a priori, the mathematical potential of a designed situation or, a posteriori, the mathematical life of an observed situation, requires understanding this complex game between the respective responsibilities of teacher and students, between the two layers in the modelling of the situation.

This is not an easy task and appearance can be misleading. This explains why validation processes for didactical engineering cannot rely on the statistical comparison between experimental and control group performance. Validation necessarily has to be of an internal nature and to refer to the internal analysis and control of the engineering design. It is thus mainly based on the confrontation between what is called the a-priori analysis of the engineering design and the a-posteriori analysis based on the data collected during the experimentation or after it.

More globally, the different phases of an engineering work reflect these theoretical bases: preliminary analysis, design of the engineering project with specification of both macro-level and micro-level choices and associate variables, a-priori analysis, experimentation and a posteriori analysis (Artigue, 1988). I will now illustrate this theoretical presentation by some prototypical examples.

Some prototypical examples

Didactical engineering at elementary level

Didactical engineering research began to develop at elementary level and two prototypical examples are provided by the long term engineering designs elaborated independently by Brousseau (1981) and Douady (1984), in order to tackle the extension of the numeric field towards decimals and rational numbers. I cannot enter here into the details of these engineering designs. I will only evoke two key situations in Brousseau's engineering: the first one deals with the introduction of rational numbers, the second one with the extension of the product operation to such numbers.

Brousseau's engineering: Introducing rational numbers.

Rational numbers can be introduced through different problems and contexts, favouring different conceptions of these. Epistemological analysis leads us to distinguish two main conceptions: division and commensurability. The most common didactic choice corresponds to the first conception (the traditional parts of tarts) and the didactic difficulties it generates are now well known. Brousseau's choice favours the second conception: commensurability.

The problem proposed to pupils is the following: how to compare the thickness of sheets of paper? Pupils are working by groups, each group is given different piles of sheets of paper of different thickness, some of these being very close. Different groups can have common and different types of papers. They are asked to find a way inside the group to compare thickness, then to write a message allowing the comparison of thickness between groups without any new manipulation.

One fundamental characteristic of the situation is that thickness is not directly measurable. Of course, some comparisons can be made just by perceptive means but the necessity to compare with other groups without more manipulation obliges the pupils to look for other strategies. In order to classify the sheets according to their thickness, pupils are thus induced to use piles of paper and a linear modelling of the relationship between number of sheets and thickness of the pile. The fact that piles are available favours this strategy and, in that case, the approximate character of the linear model does not prevent them making correct comparisons.

Through repeated experimentation, this situation has proved to be very robust and its strong a-didactic potential has been evidenced. Finally, each type of paper will be characterised by one or more couples of whole numbers, referring to the manipulations made by pupils, for instance 12mm for 25 sheets and 21mm for 30 sheets for the types below.

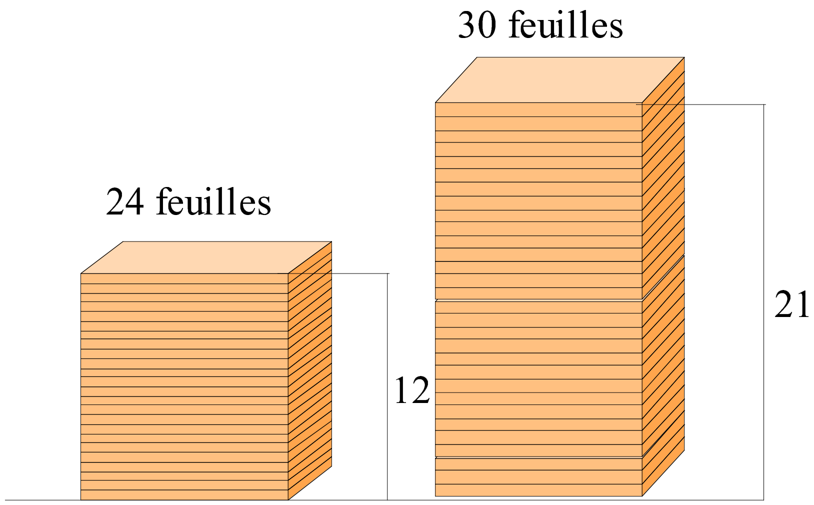


Figure 1. Two piles of paper measured by the pupils

Comparison relies on simple proportional reasoning and, of course, corresponding knowledge has to be part of the medium: for the first pile, we obtain 24 mm for 50 sheets, this paper is certainly thinner than the second one with 21 mm for 30 sheets ; pupils can also say that, for the first paper, the number of sheets is about double the thickness, which is far from being the case for the second pile. What is implicitly at play in this situation is the ordered structure of rational numbers.

Note that, even if a lot of work can be developed in this context about equality and order of rational numbers, even if pupils progressively discover and formulate a lot of rules, test them on real piles, then use piles in a more metaphoric way in order to support mental and written calculations, knowledge is there strongly attached to this context and not necessarily easily transferable to another context. Notations introduced by pupils and progressively simplified for economical reasons are not necessarily the conventional notations and it will be the responsibility of the teacher to decide when to connect the vernacular notations of the classroom to the standard fractional notations and move from the first semiotic registers to the standard one, while keeping for a time the discursive support of the paper context.

In Brousseau's engineering, the same context is then used in order to introduce the sum of rational numbers, but the extension of the product law to rational numbers is out of range in the same context. For this extension, once more, different conceptions can be mobilised. Brousseau favours the conception of product as an external operator through the situation known as "the enlargement of the puzzle".

Brousseau's engineering: The puzzle situation.

The problem proposed to pupils is the following: you have to construct an enlarged puzzle similar to the given model; the length of 4 cm in the original puzzle has to become …cm in the enlarged puzzle.:

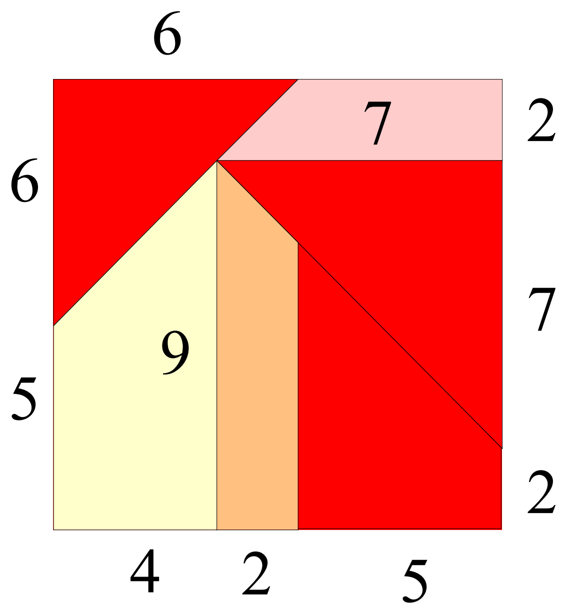


Figure 2. The original puzzle used in the problem

Pupils work in groups. Conditions for enlargement can be different from group to group. In each group, pupils share the construction of the different pieces. The material medium provides them with a validation process, which is obvious. If they can put together the different pieces and make a puzzle similar to the original one, they have succeeded. In the medium of course, there is also some perceptual knowledge about enlargement and similarity together with some geometrical knowledge necessary to reproduce rectangular triangles and trapezia with given dimensions. The fact that students work in groups prevents them from making uncontrolled adjustments and, at the same time, makes the situation compatible with time constraints.

The mathematical variables of the situation are carefully chosen, taking into account what can be expected from the pupils in terms of basic strategies. Parallel sides of the square are cut into two and three parts by the pieces. Thus, if pupils use an additive model for enlargement, they will not be able to make a square with the pieces. Correspondences between lengths defining the enlargement are also important didactic variables: for some choices, the situation will be easy, for instance if it is: 4 cm gives 8 cm. It will not lead to new mathematical knowledge. The correspondence: 4 cm gives 6cm, which is generally used, is an intermediate one in terms of complexity. Basic strategies rely on additive models. Here the additive model (adding 2 cm to each dimension) does not lead to success. But this additive vision can be improved without rejecting an additive conception, if pupils are able to see “6” as “4 plus half of 4”. Correspondences such as 4 cm gives 5 cm, or 4 cm gives 7 cm are more complex. There is little doubt that the mathematical aims of the puzzle situation cannot be achieved if pupils are not faced with this complexity, but working in the classroom with different correspondences help to approach this complexity, while preserving an a-didactic functioning.

Once more, this situation has proved its robustness and adequacy. It has become a “classic”, not only used with its initial aim, at elementary level, but also, at secondary level, for supporting mathematical work on proportionality in the geometrical setting.

In that case, it has been noticed that teachers were often tempted to present the data in a table such as the following:

initial	2	4	5	6	7	9
enlarged		7				

Table 1. Example of how teachers tend to present data

At first sight, this can appear as a simple notational choice but, using this semiotic register, one kills the mathematical problem. For middle school pupils, indeed, this semiotic register is strongly attached to proportionality in France, and introducing it is like contractually saying to pupils: you are in a situation of proportionality, you have to use a multiplicative model. One skips from an a-didactic adaptation process to a contractual one.

These two situations illustrate the kind of work which researchers make when, elaborating engineering research projects, they try to satisfy the commitments of the theory, that is to say:

- ♦ to give students a maximal responsibility in the development of their mathe matical knowledge ;
- ♦ to make, as much as possible, new mathematical objects and techniques appear as optimal tools in the solving of mathematical problems ;
- ♦ to ensure that the meaning and interest of these problems can be easily per ceived by students ;
- ♦ to ensure that expected solutions are accessible collectively to the group of stu dents, mostly in an autonomous way, through their interaction with the medium;
- ♦ to ensure that the expected behaviours, if observed, necessarily result from the targeted construction of knowledge.

In Brousseau’s engineering as in Douady’s engineering, such an ambition is reasonably fulfilled. Extension of didactic engineering towards more advanced levels sets up new problems. Such a nearly autonomous functioning is often out of range and teacher’s mediations, even in a-didactic phases, are likely to play an essential role. In the following part, we illustrate this point by referring to an engineering research carried out at university level on differential equations (Artigue, 1992, 1994).

Didactic engineering at university level: Differential equations

After presenting the research project and the main choices attached to the engineering, I will focus on two situations. The first one allows a functioning similar to what has been described up to now: students’ interactions with the medium are sufficient to produce the expected social construction of knowledge. In the second one, such a functioning is out of range and didactical modelling of the situation has to include a careful analysis of the teacher’s role and of the possible effects of the decisions (s)he takes in order to overcome the cognitive limitations of the students’ interaction with the medium.

The didactic engineering project: An overview
From an epistemological point of view, in the solving of differential equations, one can distinguish three essential settings :

- The algebraic setting in which one looks for exact solutions and their expression through implicit or explicit algebraic expressions, series development or integrals.
- The numerical setting in which one looks for approximate solutions and the control of such approximations.
- The geometrical or qualitative setting in which one tries to identify the geomet rical and topological characteristics of the phase portrait of the equation.

French undergraduate teaching was and is still focused on algebraic solving. Such a course gives students the impression that differential equations always have algebraic exact solutions, that there is a specific recipe for each type of equation and that the main aim of research in that area is to find the missing recipes. This is a stable object but an obsolete one if we consider the present development of the field where qualitative and numerical approaches are of growing importance.

Our ambition in the research project was to develop an engineering design more satisfactory from an epistemological point of view, by opening teaching to qualitative and numerical settings, to experiment and analyse their conditions of viability.

In the phase of "preliminary analysis", we tried to understand the reasons for the stability of such an obsolete object. Our first hypothesis was that, facing a stable equilibrium point of a complex dynamical system, we had to understand the constraints at the source of such a situation. That is, no change could be made viable without substantially modifying the system of constraints. Our second hypothesis was that stability resulted from constraints of a different nature: epistemological, cognitive and didactic which intertwined and reinforced mutually. If, for example, one considers the constraints, which act as an obstacle to an early introduction of qualitative setting:

- *From an epistemological point of view*, one can invoke the late and quite autonomous development of this approach initiated by Poincaré in the late nineteenth century. The difficulty of the mathematics problems which motivated its development (the three bodies problem, problems of stability of dynamical systems…) and the impossibility of using directly such problems in an elementary didactic transposition of this approach.
- *From a cognitive point of view*, one can invoke the strong flexibility between the algebraic and the graphic semiotic registers required by qualitative solving, the subtlety of qualitative proofs and their mathematical needs in terms of elementary analysis (proving that solution curves intersect or do not intersect, analysing infinite branches…).
- *From a didactic point of view*, one can invoke the easiness of traditional alge braic teaching. The fact that qualitative solving cannot be managed algorithmically, that it needs a strong graphic support which is not easy to develop and negotiate taking into account the poor institutional status of the graphic register at university level and, at that time, the limited number of books which could inspire such an integration of a qualitative approach.

Starting from this analysis, we entered the conception phase, trying to play with the identified set of constraints in order to move them sufficiently enough to ensure the viability of another equilibrium at a reasonable cost. The design was based both on global and local choices. Global or macro-didactic choices govern the whole organisation, local or micro-didactic choices govern specific phases or situations. Our main macro-didactic choices were the following:

- ♦ Deal with the cognitive and didactic constraints linked to the status of the graphic register through a specific preliminary module on functions and graphs.
- ♦ Deal with the complexity of qualitative solving by using computer software assistance for creating a progression in the complexity of the tasks proposed to students, by officially introducing a set of methods for qualitative study and specific geometrical arguments for managing qualitative proofs with limited analytic competencies.
- ♦ Deal with time constraints by a reduction of the content in algebraic solving and, more globally, a change in the status given to equations accessible to exact solving: for instance, simple cases such as linear equations or equation with separate variables were considered as simple cases which could serve as a reference or comparison tool in more complex situations.

Finally this resulted in the following seven-step structure:

1. To which needs do differential equations respond? (This part of the engineering design relied on the results of previous research on differential and integral processes in mathematics and physics (Alibert et al, 1989)).
2. An introduction to the qualitative approach assisted by computer drawings of slope fields, isoclines and phase portraits.
3. The algebraic approach: exact solutions for first and second order linear differential equations and differential equations with separate variables.
4. The complementarity of the algebraic and qualitative approaches.
5. An introduction to numerical solving, Euler's method and refined Euler's method.
6. The basic tools and methods of qualitative approach: fences, funnels…
7. The integration of the different approaches in the solving of more complex problems.

As announced above, after this global presentation, I will focus on two key situations in this engineering design, belonging respectively to phases 2 and 4.

Associating equations and phase portraits
During phase 2, students are introduced to the qualitative approach. This introduction strongly relies on computer tools. The fundamental notion is that of tangent field associated to a first order differential equation in the form $y' = f(x,y)$ and the associated notion of solution curve as a curve compatible with the tangent field at every point. Students are asked to draw some very simple tangent fields, then they are provided with tangent fields drawn with the use of computer software and they are asked to draw solution curves. The aim of such an activity is to give meaning to the notion of

solution curve by making students become physically aware of the constraints imposed on the drawing by the tangent field.

Then students are faced with a task of association between differential equations and phase portraits. They are given seven differential equations: $y'=\frac{y}{(x+1)(x-1)}$, $y'=y^2-1$, $y'=2x+y$, $y'=\sin(xy)$, $y'=\frac{\sin 3x}{1-x^2}$, $y'=\sin x\cdot\sin y$, $y'=y+1$, and height phase portraits, two of them corresponding to the same equation: $y'=\sin(xy)$ but with different windows.

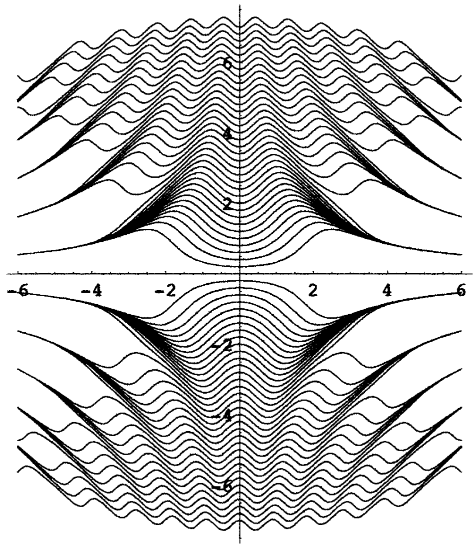


Figure 3 a.

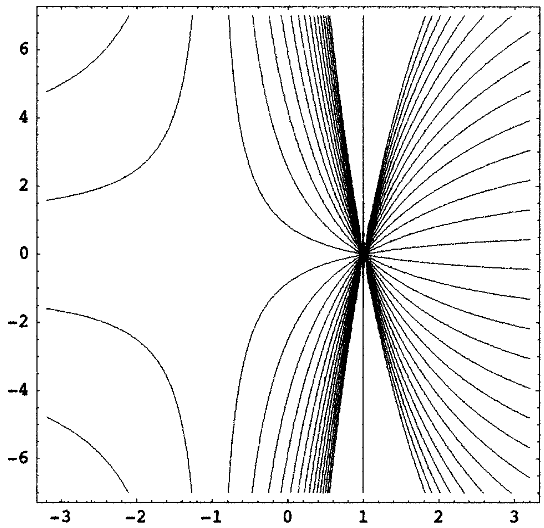


Figure 3 b.

Such a task, at first sight, looks very complex but its variables have been carefully chosen in order to make success accessible to beginners working in small groups, in a reasonable amount of time (about one hour). For each equation, four criteria at least can allow association: existence of symmetries, periodicity, particular solutions, isocline 0 easy to find, singularities, areas for increasing, decreasing solutions. In other terms, the means of validation offered by the medium are very rich and, moreover, as far as the associations progress, the task becomes less and less complex. Experimentation carried out with different kinds of students attests to the robustness of this situation and the fact that, coming after a first introduction as described above, it can be solved by students in a nearly autonomous way. Let me add that equations have also been chosen in order to prevent association by pure analogy of forms, for instance waves and trigonometric functions. With beginners in the field who have not yet developed economical methods for qualitative study, this situation results in a very rich set of criteria. These are then collectively discussed with the teacher, and a table of conversion between their algebraic and graphic form (for instance: "in the equation $y'=f(x,y)$, f doesn't depend on x" and "the set of solution curves is globally invariant through horizontal translations") is established and institutionalised. This provides them with a basic toolkit for engaging in qualitative approach.

Connecting qualitative and algebraic approaches

The second situation I would like to evoke in this engineering design introduces phase 4 dealing with the respective potential and limits of qualitative and algebraic approach and their connection. Once more, the variables of the situation are carefully controlled. As expressed in Artigue (1994):

> (a) Starting a qualitative study must be easy, as what is at stake in the situation is not located in difficulties at this level. For example, one could arrange things such as the horizontal isoclinal line is made up of straight lines, and so that certain particular solutions, which are relatively easy (e.g. isoclinal lines) allow the research to be organised by providing a division of the plane for the solution curves. (b) The algebraic solving, while it does not give rise to any particular difficulties, must not be too easy; in particular, the expressions obtained for the solutions should not be self-evident. (c) The qualitative solving, although easy at the start, allows broad categories of solutions to be determined, to foresee in what way they will vary, but must not allow all the problems set to be solved: for example, the existence of such and such a type of solution, or the nature of such an such an infinite branch. (d) At least some of these properties should, however, be accessible to algebraic solving.

For instance, the equation: $y' = x(y^2 - 1)$ fulfils these conditions.

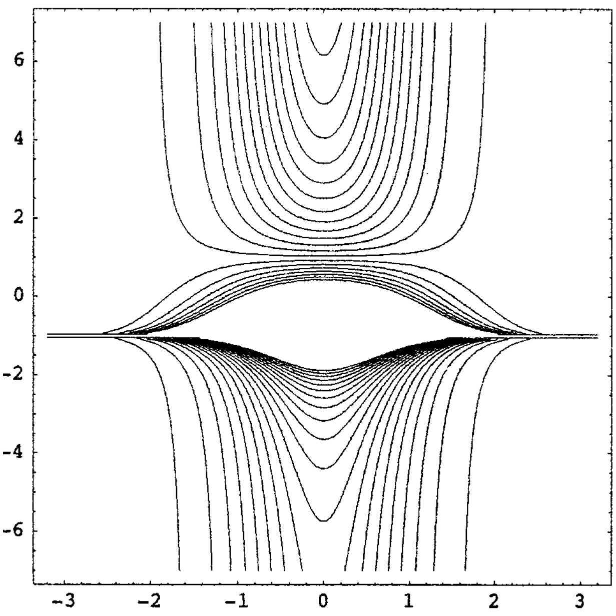


Figure 4.

The isocline of slope 0 is the reunion of three straight lines, two of them, the horizontal lines $y = 1$ and $y = -1$ being particular solutions. As f is a C^1 function on the plane, the theorem of existence and uniqueness of solutions is satisfied in the whole plane. This allows to divide the plane into three different regions: $y > 1$, $-1 < y < 1$ and $y < -1$ where solution curves necessarily stay. The sign of y' in the three regions is easy to determine. Moreover the vertical axis is a symmetry axis for the phase portrait. Solutions in the upper region are decreasing for negative *x*, necessarily cut the *y* axis, then increase, but qualitative study doesn't give in an immediate way the nature of their infinite branches. Solutions in the intermediate region are increasing then decreasing and it is easily proved that they have horizontal asymptotes when *x* tends towards infinity. This asymptote is necessarily the particular solution $y = -1$ as, if not, the derivative y' would tend towards infinity. In the lower region, the situation is a bit more complex. For negative *x*, solutions are decreasing, some of them cut the *y* axis but one cannot ensure that they all cut the *y* axis. Those cutting the *y* axis have $y = -1$ as an horizontal asymptote both on the left and on the right. Thus conditions a) and c) are satisfied.

As regards the algebraic solving, it doesn't require a lot of technical knowledge, just the decomposition of a rational fraction into its simple elements. But the expression we obtain at the end is: $y = \frac{1 + Ae^{x^2}}{1 - Ae^{x^2}}$, *A* being an arbitrary real constant, which is not obvious to interpret. Moreover, students can be trapped by the illusion that one value of A corresponds to one solution which is only the case for negative *A* or *A* more than 1. In fact, the connection with the previous qualitative work is here very helpful. Solutions defined on *R* obtained for negative *A* correspond to the intermediate region. Solutions defined on *R* in the lower region correspond to $A > 1$. And to values of *A* between 0 and 1 correspond three different solutions, two in the lower region, one in the upper region, with vertical asymptotes. Conditions b) and d) are thus satisfied.

Experiments carried out with first year students show that here, interactions with the medium are not enough to make accessible all the required mathematical work in an autonomous way and in a reasonable amount of time. An efficient management of the session requires important mediations from the teacher, specially when students have to point out what is left open by the qualitative study and what are the possible conjectures, and then what has to be connected between the results of the qualitative and the algebraic study. For some of them, additional help is necessary, for instance in order to get exact solutions under the form $y = g(x)$. Trying to provide the maximum of responsibility to students through the piloting of these necessary mediations, in order to maintain a certain level of a-didacticity, is not evident at all. Most often, for such situations, what is left to the responsibility of students is the execution of some technical parts of the mathematical work, the teacher taking charge of the reflective and more conceptual part, through "ostension" techniques or techniques which, as the "Socratic maïeutic" give the impression that students are associated with the production of knowledge, although this is not really the case.

The modelling of such situations requires the integration to the a-didactic / didactic model of the teacher as a full actor of the situation, whose role cannot be reduced to the management of devolution and institutionalisation processes. By her or his mediations, (s)he regularly modifies the medium, the way students interact with it and the possible cognitive effects of this interaction. Understanding these changes is a necessity if one wants to understand and even control the mathematical world open to the students. It is far from being easy.

Potential and limits of didactic engineering as a research methodology

Didactic engineering, as a research methodology, entered the didactic scene with the ambition better to take into account the complexity of classroom phenomena, the complexity of relationships between mathematical learning and the social and institutional characteristics of the environments where it takes place. Certainly, the ambition was not only to understand but, beyond that, to find the ways of controlling some of the resulting didactic phenomena, through the determination of key didactic variables and their control. There is little doubt that this approach has been productive for research, both from fundamental and applied points of view. From an applied point of view, it was the source of a great diversity of engineering products from elementary school to university, covering a lot of mathematical topics, supported by a strong and coherent theoretical background and benefiting from the development of the theory.

From a fundamental point of view, didactic engineering has also been productive. For years, it has been a privileged means for testing the validity of the theoretical assumptions upon which it relied: those of the theory of didactic situations. For instance:

- difficulties met at finding what Brousseau calls "fundamental situations", that is to say situations characteristic of some domain of knowledge, when working at more advanced mathematical level;
- unexpected and resistant discrepancies between a priori and a posteriori analysis;
- difficulties met in the diffusion of engineering products, once validated by research;

have certainly played an essential role in the evolution of the theory of didactic situations. Even if, as stressed above, the central construct of the theory was, from the beginning, the "situation" seen as a complex of social relationships between various actors (students, groups, teachers) and mathematical knowledge, these different actors were not considered in the same way. The complexity of the cognitive and affective economy of students' behaviour was highly recognised, but the teacher was not really considered as a "problematic subject" who deserved the same attention. Seen as a partner of the researcher, many often involved in the design of engineering projects, (s)he remained in some sense an unquestioned, transparent subject. The complexity of her or his role, of her or his personal and institutional determinations was not really taken into account. The above mentioned difficulties contributed to change this simplistic vision of the teacher and, as a consequence, the theoretical basis, the conception and a priori analysis of engineering designs.

During the last ten years, research about teachers had an exponential increase, addressing their conceptions and beliefs first, then the way institutional and situational constraints shape the decisions they take, analysing more globally their professional gestures and their professional development (Margolinas and Perrin-Glorian, 1997). Most of this research work was not carried out in the framework of didactic engineering but through more naturalistic methodologies, better adapted to the exploration of such a new field of investigation. Indeed, and we touch here on one of the evident limitations of didactic engineering as a research methodology, this doesn't give us access to the "natural life" of didactic systems. It works with constrained systems. Thus, if it allows us to test our didactic constructs through the ability we demonstrate to produce or reproduce didactic phenomena, it is less adapted to empirical or semi-empirical work in emergent field.

Didactic engineering can certainly be considered as an efficient means to approach the complexity of didactic processes, both for theoretical purposes and teaching design aims. But, as any methodology, it has both potential and limits and there is a necessary complementarity between it and the more naturalistic methods which have taken increasing influence in didactic research these last ten years (Sierpinska and Kilpatrick, 1998). Such complementarity is also at play today in French didactics. Naturalistic methods are used in order to refine for instance our teacher's models, and such refined models are then integrated into didactic engineering. Conversely, didactic engineering is used to test theoretical constructs which may emerge from anthropological perspectives relying on more naturalistic methodologies.

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Faces of Swedish research in mathematics education

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Introduction

The recent appearance of international research reviews (Grouws, 1992; Bishop et al, 1996) as well as national (e g Arzarello & Bartolini Bussi, 1998; Blum et al, 1992; Douady & Mercier, 1992; Kieran & Dawson, 1992) in the field of mathematics education is a significant mark of the state of the art, showing that the field has become so vast and complex that it has become difficult to overlook. This is also underlined by the fact that its state as a scientific discipline has been the focus of a recent ICMI study (Sierpinska & Kilpatrick, 1998). Swedish research studies were reviewed already in the general overview by Werdelin (1973). More recently, information about the work of present Swedish researchers, based on answers to questionnaires, was given in Johansson (1991, 1994). A survey of recent Swedish research on mathematical disabilities is made in Sahlin (1997). See also Engström (1999).

This presentation is based on looking at Swedish research publications from the past, on the answers to questionnaires that were sent to Swedish researchers in the end of 1997 (reply from 30 persons) and to departments of teacher training, education, and mathematics in 1999 (reply from 10 departments), and comments from colleagues. By showing different faces of Swedish research in mathematics education the product set of their characteristics will display the varieties and potentials of the research efforts. After a look at history, a short review of research problems and methods, examples of research results, research communities, and future perspectives will be presented.

Beginning with a general outlook on the object of the research interest, i e the teaching and learning of mathematics in Sweden, it has been said that before the time of the ”new math” there had been very little change in the way mathematics was taught at schools (Magne, 1986). The teacher showed the students, using explanations and visual tools (object lessons), how a mathematical technique worked (for example two digit multiplication), and the students worked, under the guidance and help from the teacher, through a number of tasks to practice this technique from a book of exercises, which was the only ”textbook”. This was called teaching by rules, and was considered the most efficient method for mathematics teaching in the upper grades of compulsory school. In the lower grades the heuristic method of learning[1] was considered to be more adapted to the level of thinking of young children (Bergsten, 1939). Both these methods

[1] The student should first get so simple tasks that he is able to solve them by himself, with his own methods, then by systematic training with more and more difficult tasks he will develop mathematical skills. The key word was learning by practice. (Bergsten, 1939)

were rooted in the strong Swedish tradition of elementary mathematics teaching to keep the learning activities close to the reality of the young child, using concrete materials and visual tools to make children develop their mathematical skills through their own activities. (See e g Wigforss, 1952) The ideas behind the new math were therefore strongly criticised (Lindström, 1968) already before its curriculum came into play in 1969, but nevertheless it put a stop to a promising development of mathematics teaching in the country. Now, it did not take long before the poor results of the new programme, mainly on computation skills, were highlighted in the PUMP project (Kilborn, 1979), and the low Swedish results (in international comparison) from the second IEA study 1980 (Skolöverstyrelsen, 1983), started intense activities on a national level to continuing education of mathematics teachers in compulsory school. The emergence of the electronic calculator during this time ”complicated” the situation, and the time since then in mathematics education in Sweden has been, as I see it, ”a search for identity”. The need for research in mathematics education has therefore only been increasing since the new math changed the scene, even the curriculum from 1980 (”back to basics”), and from 1994 (again with more emphasis on mathematical thinking and understanding) have tried to put mathematics education in Sweden on a positive development.[2] The Swedish results in the international TIMSS study (Skolverket, 1996) indicated that the efforts made had got a positive pay off.

Face 1: *History*

The development of Swedish research and developmental studies in mathematics education could be termed as a move from content oriented and experience based ”guidance” to research based ”didactics” (*didaktik*). The title of the last book from one of the nationally most well known mathematics educators in Sweden during the 19th century, Karl Petter Nordlund, is ”Vägledning vid den första undervisningen i räkning” (”A guidance to early arithmetic teaching”, my translation; Nordlund, 1910).

On the international level this move or shift parallels the paradigm shift in mathematics education research around 1980, when positivistic models of hypothesis testing gave way to methods more apt to the practitioner’s perspective. To quote Kilpatrick (1992, p. 31), ”research in mathematics education was moving out of the library and laboratory and into the classroom and school”. This also explains the shift: When scientific research put itself outside the (strong) guidance tradition it did not have any effect on it, when moving inside, it did. It was also at this period when research journals in mathematics education appeared[3] as well as research institutes[4].

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[2] See Wyndhamn (1997) for an analysis of the mathematics curriculum development in Sweden; see Håstad (1978) for a description of the development of mathematics education in Sweden 1950-1980.
[3] Journal for Research in Mathematics Education, Educational Studies in Mathematics, Zentralblatt für Didaktik der Mathematik
[4] Shell Centre, Institut für Didaktik der Mathematik, Instituts de Recherche de l’Ensignement des Mathématiques.

The book by Nordlund (1910) is a very detailed description of what lessons in elementary mathematics topics should look like, with extensive use of concrete material and student activity. It is interesting to note that the work by for example Bergsten (1939) and much later the textbook by Anderberg (1992) still belong to the same tradition. At the turn of the century (i.e. 1900) there was a strong movement to make mathematics teaching more ”åskådlig” (it is hard to find an appropriate English word for this Swedish word, which means making mathematics more clear or lucid, for example by using visual means for displaying mathematical meaning), by mathematics educators like Ehlin, Kruse and Setterberg, influenced by the Perry-movement in England (Wistedt et al, 1992). Nordlund had already used that principle for 40 years by then (his first mathematics textbook dates from 1867), possibly carrying on the tradition from Comenius' work in Sweden 200 years earlier[5].

Wigforss extended this tradition of ”åskådlighet” (*lucidity*) by the development of diagnostic testing materials of high quality that became widely used (e.g. Wigforss, 1946). The development of standardized tests on a national scale, by Wigforss, to support the marking system, was another early significant contribution (see Kilpatrick & Johansson, 1994). Other research studies during the period before the new math were few and individual products rather than long term results from mathematical education research groups or milieus, for example the ingenious early interview study by Jonsson (1919) and the powerful factor analytical studies by Werdelin (1958, 1961).

The shift in mathematics teaching with *the new math* initiated, by the problems it evoked, an increased interest in the nature of mathematical skills and knowledge, and also in teacher training. This is shown by the increased number of projects and studies in the field that appeared (e.g. Kilborn, 1979). Also in the US, there was an explosion in the number of articles that appeared in the field (Kilpatrick, 1992). Subject matter based *didaktik* was the focus of a conference in Marstrand (Marton, 1986), after which the term *matematikdidaktik*[6] was beginning to be widely used. The first course at university level in Sweden by the name *matematikdidaktik* was organized by Wyndhamn and Unenge in Linköping in 1985 (10 credit points). During the 15 years that have followed more than 15 PhD works have been presented, the largest number within the phenomenographic approach, along with many other studies. Textbooks for teacher training in this new research based *matematikdidaktik* like Unenge et al (1994) or Bergsten et al (1997) look very different from those mentioned above in the ”guidance” tradition.

In some of the national research reviews that have been presented a ”national tradition” or ”trend” has been able to identify. In Germany, for example, there is the *Stoffdidaktik* tradition since the 1950:s (vom Hoffe, 1998), in Italy the two trends of

[5] Nordlund's introduction of the heuristic method for teaching, as an alternative to the mechanistic teaching methods ”by rules” that were common at that time, was inspired by his teacher Kjelldahl in Uppsala (Bergsten, 1939).

[6] English translation *didactics of mathematics* or *mathematics education*.

concept-based didactics and *innovation in the classroom* were identified (Arzarello & Bartolini Bussi), in the Netherlands there is the *realistic approach* (DeCorte & Vershafel, 1986), in France there is the well known conception of *didactic engineering* (see the paper by Artigue in this volume). And so forth. Is it possible to find a typical ”Swedish” tradition in mathematics education research, an identifiable trend that dominates, or has dominated, the national scene?

Face 2: *Research problems*

The range of problems that are studied in the field of mathematics education research can be listed along several dimensions – focus on different actors (students, teachers...), organisation of teaching (groupings, individualisation...), mental processes (reasoning, visualisation...), focus on topic areas (geometry, algebra...), and many more. Within each dimension the focus can be on a general level (how does understanding in geometry develop?) or on a more specific level (how do students in grade 9 understand the concept of similarity?). A look at 23 Swedish PhD works during the period 1919-1999 (of which 15 are from the last ten years) that could be classified as belonging to the field of mathematics education, showed an almost non-overlapping distribution of research problem areas: problem solving (Wyndhamn, 1993), arithmetic with school beginners (Neuman, 1987; Ahlberg, 1992; Ekeblad, 1996), structure of mathematical knowledge (Werdelin, 1958; Bergsten, 1990), computers in mathematics education (Hedrén, 1990; Dahland, 1998), long term development of mathematical ability (Pettersson, 1990), students ways of solving arithmetic tasks (Jonsson, 1919), the organisation of learning (Ekman, 1968; Dunkels, 1996), fractions and reflective thinking (Engström, 1997), effects of curriculum change (Håstad, 1978; Kristiansson, 1979; Hellström, 1985), individualisation (Larsson, 1973), mathematical modelling (Wikström, 1997), understanding graphs (Åberg-Bengtsson, 1998), teachers' and students' conceptions of mathematics/ teaching and learning mathematics (Löthman, 1994, Sandahl, 1997), teachers' different ways of handling content (Runesson, 1999), influence of social factors on mathematics achievement (Chen, 1996). Other studies focus, additional to the above mentioned areas, on gender, communication between students and between students and tasks, informal (”everyday”) knowledge, quality of children's mathematical thinking, teacher students, mathematical disabilities, and undergraduate mathematics education.

From a quantitative point of view, a big proportion of the Swedish research has been made within a number of projects, sometimes on a large scale, often funded by the National Agency for Education. Examples of such projects are (in alphabetical order) ADM, ALM, ARK (including DIM and RIMM), BIM, DIS, DOS, GEM, GUMA, HÖJMA, MYT, PUMP, and ”Matematik i en skola för alla” (Mathematics in a school for all), ”Problemlösning som metafor och praktik” (Problem solving as metaphor and practice), and ”Vardagskunskaper och skolmatematik” (Every day knowledge and school mathematics).[7]

[7] For explanations of acronyms and references please contact present author.

Problem areas in mathematics education that recently have attracted most interest and research attention in Sweden are (according to the questionnaire from 1997 as mentioned above) young children's conceptions of mathematics and early number learning, teachers' and teacher students' conceptions of teaching and learning, assessment and evaluation of knowledge, technology in mathematics education, problem solving and communication in the classroom, mathematical disabilities, gender issues, and research approaches such as phenomenography and constructivism. Learning issues in undergraduate mathematics have recently come into focus in some studies. Topic oriented studies are very few, as well as theoretical studies of epistemological character, and there is no *stoffdidaktik* tradition in the German sense (exceptions are for example the work by Kilborn, 1979, and by Bergsten, 1990). Qualitative methods are dominating, in particular in combination with the phenomenographic approach, and more or less well structured methods of triangulation are often used to increase the validity of the studies.

Face 3: *Research methods and results*

Methods of research in mathematics education in Sweden have followed the international trend, i e from an early dominance of quantitative studies towards an increasing number of qualitative studies. Experimental designs, with experimental and control groups, using pre- and post-test techniques, have been used by for example in Werdelin (1968), Hedrén (1990) and Ahlberg (1992). Examples of studies using psychometric methods are Werdelin (1958, 1961), Bergsten (1990), Pettersson (1990), and Chen (1996). A number of national survey studies have been produced by the National School Board, for example on grades 5 and 9 in 1992 (Skolverket 1993a, 1993b). Some interesting longitudinal studies have been conducted, for example on early number conceptions (Neuman, 1987) and alternatives to standard algorithms (Hedrén, 2000). Today interview techniques are dominating the scene, but Jonsson's early study (Jonsson, 1919) proves there is a long tradition in Sweden for qualitative methods. Modern techniques such as video recordings have been used in Löthman (1994) and Dunkels (1996).

Phenomenography (Marton, 1981) has a strong position in mathematics education in Sweden, as in the studies by Neuman (1987), Ahlberg (1992, 1997), Ekeblad (1996), and others. Piagetian constructivism has fewer exponents (e g Engström, 1997), and so has the Vygotskyan perspective (Wyndhamn, 1993; the paper by Säljö in this volume). Studies on a more subject oriented theoretical level are also less frequent (e g Kilborn, 1979; Bergsten, 1990).

It is often said, by practitioners, that research is very interesting but not of so much use in the daily work in classroom reality. Now, research results are always of a theoretical nature, and even if they sometimes are on a very general level, they can nevertheless form a basis for designing curricula, as well as teaching or learning situations. For example, knowledge at different levels of the education system of the five examples of major findings of research in mathematics education, given by Niss (1998), would be a safe basis to avoid many mistakes in the design of teaching and learning activities.

It is not possible, in this limited format, to list the ”Swedish results”. Instead some sharp results that seem to have obvious teaching implications will be shown, ordered chronologically from 1919 to 2000.

Mental calculation methods

In a study by Jonsson (1919) a series of interviews on mental calculation methods was conducted. His results are still relevant for the discussion today when the teaching of formal calculation methods (‘standard’ algorithms for addition, multiplication and so on) in elementary school is questioned, in favour of building on children’s own methods. Jonsson found that despite the extensive training on only one formal method for doing additions fourteen out of fifteen students interviewed in grade 2 used other methods when they were free to choose. He also found that students chose the methods, in each calculation, that needed as little thinking effort and time as possible.

Rules first or discovery learning?

The effects on learning of different organisations of learning situations have been much studied in pedagogical research. Since mathematics is, among other things, a rule-using discipline, it has been common to use two different methods of instruction, i.e. rule first (given by teacher or book) – then practice, or let the students themselves discover the rule from the material (discovery or heuristic learning). Studies on this problem were of interest for example for the design of programmed teaching in the sixties. This was studied for example in the BIM project. In an experimental study by Werdelin (1968) with 211 grade 6 students, one group (A) got the principle (law of distribution) before the examples, group B first some examples, then the principle, and more examples, and finally group C only examples. To measure the effects of the different treatments one test was given immediately after the experiment, one test two weeks later, and one test to measure transfer effects. There was a significant difference at the immediate test to the advantage of group A, a difference that however disappeared after two weeks. There was no difference on the transfer test. The main conclusion (from this and other studies) was that the advantages of the heuristic method become visible in a long term perspective, when combined with other important more general aspects of learning, such as drawing your own conclusions and make a synthesis of what you experience. One could comment that this research provides scientific support to the ideas of Nordlund in the 19^{th} century (cf above).

Doing mathematics without using understanding

In a study on the different steps involved in solving a complex mathematical task, Ekenstam and Nilsson (1977) showed how students strongly depend on their familiarity with certain standard patterns when solving mathematical tasks, rather than trying to think what the problem was about, or consider the meaning of the mathematical expressions as starting point for their reasoning. To construct their tests, Ekenstam and

Nilsson chose a 'top task', e g an equation like $\frac{3x-2}{2}=\frac{x}{3}$ and broke it down to the small steps that were used in solving the task. This produced a series of tasks like $3(3x-2)=2x$, $9x-6=2x$, $7x-6=0$ and $7x=6$. To find out at which step the students had problems, each of these five tasks were included in the test items, along with some similar tasks that changed e.g. the x to a t, the - to a +, or the particular numbers involved. From 10 top tasks 130 items were thus constructed, distributed over 10 tests of mixed items. The tests were distributed to a sample of 2000 students beginning upper secondary school so that each item was solved by approximately 200 students. It was observed that the solution frequencies strongly depended on minor changes, from a mathematical point of view, of the task characteristics. As an example, by rotating the same right angled triangle, the solution frequency to an area calculation task changed from 39% to 64%, or (in a simplification task) inverting the expression $\frac{(mv)^2}{mv^2}$ caused a change from 41% to 17%. Another remarkable observation was the big difference in difficulty between the simplification tasks $\frac{a^2}{a}$ and $\frac{a}{a^2}$, a difference that disappeared when visible coefficients were used: $\frac{3a^2}{6a}$ and $\frac{6a}{3a^2}$. The main conclusion from the study is that the mathematical skills of many students are based on the application of trained patterns rather than on understanding of what they are doing. A similar conclusion is made in recent interview studies by Lithner (1998, 1999) on undergraduate mathematics problem solving.

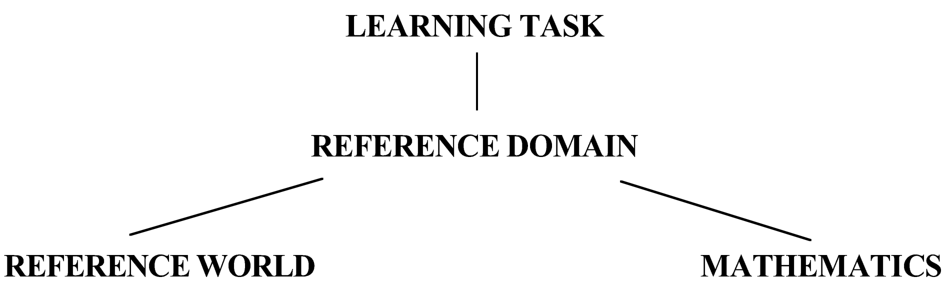
The effect of using electronic calculators

In Sweden there were some early studies on a broad basis investigating the effects of the use of the electronic calculator in elementary mathematics teaching. In a study during the years 1977-1983, called the RIMM project (Hedrén & Köhlin, 1983), 7 classes were studied during their years in the Swedish ”middle grades” (grades 4-6). Using an experimental design with experiment and control groups, it was clearly shown that the experimental groups, where the calculator was consequently used, showed the same ability as the control groups to do arithmetic calculations, mentally as well as with pencil and paper, but showed significantly better results on number estimation, choosing correct operation, and to use relevant information when solving word problems. The explanation given for the results was the increased amount of training on problem solving that was possible when the calculator reduced the time needed for performing the necessary calculations. This idea also formed the basis of the longitudinal project ADM, investigating the use of computers and later graphic calculators in secondary school (Björk & Brolin, 1995).

Connecting everyday knowledge to mathematics teaching

There has been a long tradition in Sweden to relate children's mathematical work in school to their every day world outside school, something that is also visible in Swedish curriculum texts. However, this has caused a tension in school mathematics between every day mathematics (since ICME5 known as ethno-mathematics) and ”academic math”, i.e. mathematics as a scientific discipline. In a three years project ”Everyday knowledge and school mathematics”, the aspect of using everyday knowledge to learn mathematics was highlighted (Wistedt et al, 1992). The project was a co-work between a teacher, a pedagogue, and two mathematicians, the main material videotaped and audio-taped classroom activities, including group work, and discussions among teachers.

Using everyday knowledge in school mathematics means for students that their *reference world* (everyday experience) in some way must connect to the *world of (school) mathematics*. To do this connection the student has to create a *reference domain* that picks out those aspects from the reference world that come into play in the mathematics ”game” in school. This reference domain is a kind of model world (the terms are from Schoenfeld, 1986). To use everyday knowledge for learning mathematics the reference domain is the crucial link between the existing intuitive knowledge of the student (his/her reference world) and the new knowledge (of mathematics) he/she is trying to construct. The different modes of thinking and usage of words in these worlds create conflicts in the learning process, and one main conclusion is that instead of being a process of induction from the reference world via the reference domain to the world of mathematics, learning mathematics by connecting everyday knowledge means that the student is using knowledge and experiences from two worlds – the everyday world and mathematics – when solving the problems. The results show that students use knowledge from both worlds, the reference world and mathematics, when constructing their reference domain.



This means that the learning paradox is coming into play, since it seems as the student needs an existing conception of mathematical abstractions to move from reference world to reference domain, a domain which is supposed to be the link to understanding the mathematical abstractions. To use some kinds of manipulatives for learning, for example, presupposes that the manipulatives be interpreted in a way that presupposes knowledge of the mathematics they are supposed to learn by the material. To come out of this paradox, and to learn mathematics by connecting to everyday knowledge, a dialectic view of learning in a cultural perspective is needed. Connecting everyday knowledge can work as an instrument in a mutual communication between a personal world of experience and a cultural tradition such as mathematical thinking. The main result from the study is the opening of a way to bridge the gap between everyday knowledge and mathematics, by showing the way students construct reference domains that build both from contexts of practice and of theory.

Young children's mathematical thinking (pre-school and first school years)

This problem area has attracted several Swedish researchers, with the most extensive studies by Ann Ahlberg and Dagmar Neuman. An increased interest in pre-school mathematical experience can be noticed the last years, with a number of recent Swedish publications.

As an example, Ahlberg's study "Children's ways of handling and experiencing numbers" (1997) is showing the complexity of early number experience. The study is part of the project "Numerosity and the development of arithmetic skills among visual impaired children, hearing impaired children and children without these impairments". It is an interview study within the phenomenographic framework, where the interview is treated as a *conversation with a structure and a purpose*. 38 children from 3 different pre-schools (average age 6.7 years) were interviewed on 3 different kinds of tasks, *every day problems*, *decomposition problems* (cf Neuman, 1987)*, and contextual problems* (cf Ekeblad, 1996). Children were not allowed to use any manipulatives. Interview outcomes were classified into a number of main categories, in line with the phenomenographic approach, and were analysed under the main headings **Ways of handling numbers** and **Ways of experiencing numbers**. One of the main results is that there was not a one-to-one correspondence between the way children handle numbers and the way they experience them, as shown in the matrix on next page.

Five categories of handling numbers were identified, of which **Saying numbers** and **Counting** were the most frequent. The four identified categories of experiencing numbers all come into play, more or less, when children handle numbers by counting, structuring, and using known facts. This pairing of different dimensions deepens the picture of the complexity of early numerosity development, and Ahlberg concludes:

> When trying to grasp numerosity children handle numbers in a vast array of ways, and thereby experience different aspects of numbers. However, in spite of using different ways of handling numbers, the numbers may appear in the same way to them and they may experience the same meaning. Consequently, there is not only one pathway, but many pathways to numbers."
>
> /…/
>
> Understanding numbers and learning arithmetic skills is not only a question of the quantification of objects or fingers. Neither is it a matter of learning how to count on the number sequence or developing logical thinking. It is instead, a question of being able to explore and discern different aspects and possible qualities of the numbers - of experiencing numbers in the sense of sensuously and simultaneously perceiving different aspects of numbers. (Ahlberg 1997, p. 109)

	WAYS OF EXPERIENCING NUMBERS			
Ways of Handling Numbers	**Number Words**	**Extents**	**Positions in Sequence**	**Composite Units**
SAYING NUMBERS				
Random Number Words	•			
Equal Numbers	•			
Successive Numbers	•			
ESTIMATING	•	•		
COUNTING				
Double Counting	•		•	
Counting and Tapping	•	•	•	•
Counting and Looking	•	•	•	•
Counting and Listening	•	•	•	•
Finger Counting				
Using Fingers; Counting All	•	•	•	•
Using Fingers; Touching	•	•	•	•
Using Fingers; Looking	•	•	•	•
STRUCTURING				
Seeing	•	•	•	•
Using Derived Facts	•	•	•	•
USING KNOWN FACTS	•	•	•	•

Table from Ahlberg, 1997, p 85

The results have clear implications for teaching.

Problem solving

Jan Wyndhamn, together with Roger Säljö, has done extensive work within the socio-cultural and situated cognition framework. Wyndhamn, with Riesbeck and Schoultz, has recently finished a project called ”Problem solving as metaphor and practice”, where problem solving activities in classrooms and teachers' views on problem solving were scrutinized using qualitative data techniques (Wyndhamn, Riesbeck & Schoultz, 2000). Some of the results indicate that problem solving in practice most often reduces to solving word problems in class, and that group work activities become just another version of ordinary work with mathematical tasks. No transfer effects were found between everyday mathematics and academic math (cf Wistedt et al, 1992). Problem solving in school mathematics seems to reduce itself to a metaphor *for* a practice, related more to the organization of teaching than to mathematical content.

One critical question is the dissemination of research results. It is not always easy to pick up a ”result” from a study and ask practitioners (e.g. teachers) to use it. It depends, among other things, strongly on the origin of the research question and the level of generality of the result. Bishop (1998) argues that researchers should become more aware of the fact that practitioners are the only actors for change:

> ”The research site should be the practitioners' work situation, and the language, epistemologies, and theories of practitioners should help to shape the research questions, goals and approaches.” (p. 43).

Many of the persons in Sweden doing research and developmental studies in mathematics education have a background as teachers, keeping their research close to the teaching and learning practice. Maybe this is the ”Swedish tradition” in mathematics education research. The question of information and a common discourse still remains, however. The only Nordic research journal in mathematics education, *Nomad* (Nordic Studies in Mathematics Education) is still young and has not yet succeeded to reach a broader audience among practitioners. The journal *Nämnaren* for teachers of mathematics has been the most important source in this respect in Sweden for the last 25 years (though not a research journal), as well as the mathematics teacher congress *Matematikbiennalen* (every second year since 1980) and its regional follow-up meetings, and meetings arranged by mathematics teacher associations. In some recent publications Swedish research has been made available for a broader audience, primarily for use in teacher training (e.g. Ahlberg, 1995; Gran, 1998; Neuman, 1989). Hopefully, the present conference will contribute to an increased awareness and interest also among practitioners what research can offer.

Face 4: *Research communities*

Before the ”shift” after the new math most Swedish research in mathematics education took place at departments of education. At the end of the seventies some PhD programmes also involved departments of mathematics, but it is not until recently that departments of mathematics have begun to create research milieus in *matematik-didaktik*, as in Umeå. The first PhD thesis with a mathematics education content at a department of mathematics in Sweden was Dunkels (1996) in Luleå. Other sites in Sweden for research activities in the field of mathematics education are the national testing institutions in Stockholm (PRIM) and in Umeå (Nationella provgruppen). These institutions do test constructions and research on assessment in mathematics, providing long term descriptions of mathematical skills and attitudes of Swedish school children. In fact, results of such measurements often produce the strongest direct influence on practice. As an example, the studies by Lindblad (1978), followed up by Ljung (1987), caused a change of the prerequisites for entering teacher training colleges, and the second IEA study started huge efforts on a national level to educate Swedish teachers of mathematics (see Utbildningsdepartementet, 1986)

There are also networks and organisations outside universities that play an important role for research in mathematics education. The network *Women and mathematics* under the leadership of Barbro Grevholm has organised a number of international conferences in Sweden, with proceedings of research papers, one of which was the ICMI study conference on gender issues in mathematics education (Grevholm & Hanna, 1995; Hanna, 1996). The present conference was organised by the new and independent *Swedish society for research in mathematics education* (SMDF[8]).

Face 5: *Looking ahead*

With this history behind, what might the future of research in math education look like in a small country like Sweden? In fact, some opposite trends can be identified at present. On the national level teacher education seems to move towards establishing a more generalized educator profile, with less emphasis on teaching subject matter towards a teacher as an administrator and supervisor of learning. On the local level, at departments of mathematics and didactics, new research milieus for mathematics education are being established, and teacher training programmes include courses of a scientifically oriented *matematikdidaktik*. Again, on the government level, resources have been given for researching and educating teachers of mathematics in subject matter and *didaktik*.

The increased interest among teachers as researchers (as mentioned above), and the increased emphasis of a research based teacher training, are important backups for the future development of Swedish *matematikdidaktik*. For this we need people that can inspire the way Andrejs Dunkels did, and Gudrun Malmer is doing. I also believe that the expansion of research in *matematikdidaktik* into the departments of mathematics will be a necessary and important factor for a promising development of the range and quality, and in the search for an identity, of Swedish research in mathematics education. Today, measured in number of publications, Swedish research in mathematics education is hardly visible on the international scene in the increasing stream of articles and books. No doubt a change is on the way. Using a metaphor one could say that research in mathematics education in Sweden is a kettle of water heated up so that it, maybe, soon will start boiling.

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[8] See the web page of SMDF at www.mai.liu.se/SMDF/

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The Student-Teacher-Educator-Researcher in the mathematics Classroom

Co-learning partnerships in mathematics teaching and teaching development

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Co-learning

A co-learning partnership derives from the idea of co-learning as presented in an article by Jon Wagner (1997) analysing relationships between researchers and practitioners in educational settings. One style of such relationship is called a ‘co-learning agreement’. According to Wagner,

In a co-learning agreement, researchers and practitioners are both participants in processes of education and systems of schooling. Both are engaged in action and reflection. By working together, each might learn something about the world of the other. Of equal importance, however, each may learn something more about his or her own world and its connections to institutions and schooling. (Wagner, 1997, p. 16)

In this paper, I extend this notion of co-learning agreement to relationships between educators and teachers, and between teachers and students. Thus ‘researchers and practitioners’, from Wagner’s definition, become researchers *and* educators *and* teachers *and* students. An important aspect of co-learning is the responsibility of the learner as an agent of inquiry. Thus all participants might be considered researchers and this will be a major theme of the paper.

The complexity of mathematics teaching and its development

What is mathematics teaching and what does it mean for mathematics teaching to develop? Pearson (1989) suggests that teaching is intended to create learning, and this seems a reasonable starting position. Mathematics teaching is intended to create the learning of mathematics. In a similar vein we might say that mathematics teacher education is intended to create the learning of mathematics teaching. It sounds easy and straightforward. But, can we judge the success or effectiveness of teaching by assessing the learning, which takes place?

In the UK in recent years the notion of judging the effectiveness of teaching through assessing learning has become a dogma. We now have national statutory tests at every stage of education from early primary to 16+. The most recent call for further tests means that children will be tested formally every two years. The pressure on teachers to teach to these tests is overwhelming.

Research has shown us the importance of studying children's learning; what the individual child thinks, does, can achieve and so on. An impressive array of scholars has contributed to our body of knowledge relating to children's learning of mathematics.

Much of the research has had a psychological focus, largely deriving from the work of Piaget, and caricatured by Bruner (1985) in his words "a lone child struggles single-handed to strike some equilibrium between assimilating the world to himself or himself to the world (p. 25)". More recently, scholars have gone back to Vygotsky in an attempt to chart the social threads of human learning and increasingly to conceptualise mathematical learning in a sociological frame. For example the constructivist movement of the 1980s shifted into a socio-cultural domain in the 1990s and is still shifting in recognition of the socio-political forces affecting learning. A simultaneous philosophical revolution has challenged perceptions of mathematics itself, and provided alternative visions of mathematical thought and its development – from a hard science to a sociological phenomenon.

In mathematics education terms like social constructivism, inter-subjectivity, and communities of practice have tripped liberally from the pens of theorists and researchers. As a background to conceptualising mathematics teaching, there is recognition of a counterpoint between the learner as an individual cogniser, and the learner as member of a society encompassing diverse communities of practice. The mathematics classroom can be seen as an intersection of social and cultural groupings and creeds, driven by political forces and societal demands, and striving to create a mathematical discourse that enables all students, whatever their personal and social trajectories, to learn mathematics. In all of this, how does a teacher start to conceptualise and realise the learning of mathematics? How can mathematics educators work with teachers to support an effective conceptualisation and realisation?

Why is it that after 100 years, say (Piaget and Vygotsky were both born in 1896), of research and theory into learning, and more recently into mathematical learning that mathematics educators have not been able to convince politicians that mathematical learning is not a simple matter ensured by frequent testing? And moreover, that teaching is even more complex. Perhaps a significant difference between politicians and educators is that politicians want the answers tomorrow and that educators revel in delving deeply into complexity – trying to make sense, to analyse, to characterise, but rarely offering answers. And teachers are somewhere in between, trying, sincerely in the main, to enable children's learning of mathematics, coping with multitudes of pressures from external and internal sources.

This introduction has tried briefly to paint the complexity of the classroom setting, the knowledge brought by research and theory to this setting, and the problematic reality for a teacher in constructing mathematics teaching. This paper proposes that no simplification of the complexity (e.g., according to political dogma), or segmentation (e.g., according to theoretical creed) is going to help conceptualisation of teaching and its practical interpretation. The idea of co-learning partnerships is proposed as a means of acknowledging and dealing with diversity to achieve a mathematical learning community.

Some problems in mathematics teaching and its development

In mathematics lessons, an environment needs to be created through which all students can have the opportunity to gain access to mathematics, learn mathematical skills, develop an ability to apply mathematics in everyday circumstances and experience joy in being able to do and understand mathematics. A question for this paper is what creating such an environment entails, and what are the problems.

Walter Doyle (1986) suggested that teachers and students collude unconsciously to reduce cognitive demand in classrooms – students behave better when tasks are easy and straightforward, leading teachers to set easy and straightforward tasks. Thus the setting of easy and straightforward tasks becomes a covert part of the didactic contract of the classroom.

Celia Hoyles (1988) commented on "a transmission model of teaching and learning [mathematics] where knowledge and expertise is assumed to reside with the teacher" (p. 156). She asks, “When, for example, we observe that there is little negotiation of mathematical meaning between teacher and pupils, we must question whether this is due to lack of initiative or confidence on the part of the pupils, to lack of diagnostic skill on the part of the teacher, or to constraints built into the classroom situation” (p. 156). She reflected on classroom constraints that obviate effective teaching:

> We know that teachers and pupils tend not to search together in a genuine and open way to uncover mathematical meaning. We know, for example, that pupils want teachers to 'make it easy' or 'tell them the way' and we have to recognise the powerful influences on teacher practice which almost compel an algorithmic approach. We need to find a significantly different mode of education and practice in our classrooms, new roles for teachers which they value and which they see as significant for the mathematics learning of their pupils. (p. 162)

Desforges and Cockburn (1987) reported, as a result of working with teachers extensively over ten years, that they had seen no evidence of classrooms where what they call *higher order skills* are seen to be operational consistently over substantial time periods. According to their research, even ‘good’ teachers are so bound by the pressures, constraints and demands on a teacher’s time and energy that they cannot sustain enquiry methods, draw on the spontaneous skills and interests of children, and have the capacity to monitor each individual child, seeing when to intervene and when to leave alone (p. 142). Their conclusion includes the following statement:

> We set out on this investigation with the suspicion that the teacher’s job is more complex than that assumed by those who advise them on how to teach mathematics. Put bluntly we have found what teachers already know: teaching mathematics is very difficult. But we feel we have done more than that. We have shown that the job is more difficult than even the teachers realize. We have demonstrated in detail how several constraining classroom forces operate in concert and how teachers’ necessary management strategies exacerbate the problems of developing children’s thinking. (p. 155)

They claimed that the teachers concerned, although espousing belief in aspects of good practice and striving to achieve the development of higher order skills in students, nevertheless were unable to succeed within the current system.

David Reynolds (1997) made reference to the Third International Mathematics and Science Study, in which countries of the Pacific Rim were more successful than those in the UK. He pointed out those school practices that would prepare employees for collaborative work, understanding and creativity are seen in the Pacific Rim to include 'the power of collaborative group work to deliver vastly improved traditional outcomes'. However, he sees little in the Pacific rim countries to compare with the progressive methods employed in British primary schools:

> The Pacific Rim will choose to pull the lever of the group, a marked contrast to our British inability to conceptualise and implement group-based learning, which remains simply a progressive sound-bite.
> ... they have an agreed technology of practice as well as in the precise nature of what that technology is. In Taiwan it would be inconceivable that their groups of students, who are among their brightest in terms of achievement, would be encouraged to discover their own home-made technologies of teaching. - [i.e.] to view themselves as philosophers engaged in the constant debate and discussion on the nature of the goals of education.

Reynolds suggests that a solution to the problems of teaching mathematics effectively might result from adopting methods of *interactive whole class teaching*, currently practiced in Pacific Rim States. Yet anyone familiar with teaching will recognise that any phrase such as 'interactive whole class teaching' will have a multiplicity of interpretations in classrooms. Some might be effective, some not, but the rhetoric itself will not ensure effectiveness.

These references point to some of the problems of conceptualising an effective approach to teaching mathematics. Variously authors point to problems in lack of cognitive demand, transmission teaching, making tasks 'too easy' for pupils, undirected group work, and so on. The 'pressures, constraints and demands' on teachers become worse rather than better. Mathematics might be seen to be a cognitively demanding subject – hard to teach and to learn (Cockcroft Report, 1982). How are students to be encouraged to engage with the cognitive demand? What approaches by teachers will enable such engagement? And will such approaches provide equitable access for all students? Peter Gates (1999) writes:

> ... unfairness, injustice and prejudice are not abstract concepts of macro-social analysis of an internecine class struggle. They are felt through the disappointment, hopelessness and frustrations of ordinary people as they get through their everyday lives....
>
> Mathematics Education plays its part in keeping the powerless in their place and the strong in positions of power. ... It does this through the authoritarian and divisive character of mathematics teaching. (Gates, 1999, pp. 46-47)

It is relatively easy to point towards the problems, but what are the solutions – what is that 'significantly different mode of education and practice in our classrooms' that will overcome the problems and result in more widely effective mathematical learning?

I want to suggest, not answers or solutions to these and other problems, but an approach to teaching that regards all participants in the teaching process, and its development, as learners, seeking together to overcome the problems. I will address the idea of such an approach through a number of examples from research involving students, teachers, educators and researchers.

Examples relating to co-learning partnerships

Inquiry into mathematics and its learning and teaching

I shall present first a two-fold example of classroom and teacher enquiry into mathematics, mathematics learning and mathematics teaching.

Episode 1: A group of girls have been set a task by their teacher, George. It involves fitting together four squares of the same size in various configurations (all squares touching at an edge or a corner, but not overlapping) and finding out what perimeters are possible in the resulting figures. After doing this with four squares, the girls are asked to extend their thinking to other numbers of squares, 2, 3, 5, 6, ... , and to try to generalise. They conclude that perimeters will all be even numbers, and they justify this conclusion. George then asks whether they can find an *odd* perimeter. In tackling this challenge they arrange squares so that the edge of one square touches just half the edge of the adjacent one. In the case of four squares, the perimeter is 9, an odd number.

Hence odd perimeters are possible when squares are arranged in this formation. George now asks whether a non-integer perimeter is possible. The girls seem to think not. But what if they could overlap the squares ... ? No, says George - no overlapping is allowed.

Episode 2: A video-recorded sequence from this lesson is watched by the teachers in the same mathematics department as George. They discuss what they have seen and raise issues related to teaching and learning. They recognise, for example, the quality of the girls' thinking, including elements of generalisation and testing of conjectures, and ask how such thinking develops. One issue concerns George's intentions for the lesson – had he planned that the girls would seek odd perimeters, use the half-squares method, or look for a non-integer perimeter? One of his colleagues suggested that it seemed as it the girls were "actually teaching you something". George's response to these questions was as follows:

> "The thing is, I was ad-libbing a lot of the time, so things were coming out that I hadn't thought about before, which was good, because it was extending me as well, and extending themselves, you know." (Open University, 1989)

Here we have two episodes of interaction. The first involves interaction between a teacher and his students in the classroom where mathematical thinking and inquiry is taking place.

The second involves interaction between teachers, reflecting on a classroom episode, and inquiring into issues concerning the teaching and learning of mathematics. In the first, the teacher creates a situation in which students inquire mathematically. Their inquiry is guided partially by the teacher's questions and partially by their own directions of thinking. It is the girls who suggest, with justification, that all perimeters will be an even number, leading the teacher to ask whether an odd perimeter is possible. It is the girls who try out the half-square shift, to discover an odd perimeter.

The teacher acknowledges that he too is learning from the students' direction of inquiry. The teacher's wider mathematical experience influences the questioning and guides the constraints. For example, when one girl suggests that the squares might overlap, the teacher rules out this possibility. On the other hand some discussion of what they would then be finding if they had overlapping squares might have been fruitful for the girls' wider conceptualisation. Here is a teaching issue.

When is it appropriate for the teacher to constrain the situation, and where might it be appropriate to let students follow up and question their own ideas? Such issues are made explicit when the teachers talk together as a group. The value of this seems to be in drawing attention to a range of possibilities that might not be obvious in the pressures of classroom interaction. Once highlighted, they can then be discussed separate from the particular classroom incident from which they arose, leading to a developing epistemology of classroom interaction.

In these two episodes we see mathematical knowledge growing for both students and teacher, and we see knowledge of teaching growing for the teachers. Thus both of these situations involve co-learning, although the notion of a co-learning partnership was not explicit in either of them. The co-learning situation embodies individual as well as common knowledge. Differential power relationships between participants are interpreted through varying roles and responsibilities. Characteristics of a fruitful co-learning situation might be seen to include elements of inquiry, reflection and critical questioning. A co-learning *partnership* implies an *explicit* arrangement agreed between participants.

Creating classrooms norms

I worked, as a researcher, with a mathematics teacher, Ben, for nine months studying his teaching and talking with him about his thinking and decision making in constructing teaching (Jaworski, 1994, Chapter 9). One day he referred to a lesson that I had not observed:

Episode 3: Did I tell you about the interesting incident which I had there? One was explaining to the other about trig – it was Rachel to Pat, and I was sort of talking with them and I went away, and then suddenly realised what I'd been saying. I was not talking about trig – I wasn't even talking about that. I was talking about the role of the teacher and the learner, and their responsibility. And that's a really peculiar position for a maths teacher to get into in some ways isn't it? You know, I've left my subject, in effect, for other people to teach, and I'm there teaching how to take on different roles. It's a funny situation. I didn't talk about any maths at all. Pat was saying, "I don’t understand", and Rachel was getting really annoyed about this. I said to Pat – "As a learner you've got to think about what she's saying and say: "Stop – this is where I don't understand." – that's your responsibility, and if you can't do that, Rachel can't help you. And I said to Rachel, "She’s having problems with what you're saying – can you say it in a different way?" Then I walked away. I didn't talk about the real problem with the maths. (Jaworski, 1994, p. 177)

This example illustrates co-learning between two students, their teacher and a university researcher. The two students were working on problems in trigonometry. One was trying to explain to the other, but the interaction was not proving successful. The teacher's intervention did not involve sorting out the mathematical problems, but tried to help the girls see how they might work together more effectively to sort out the problems themselves. Thus the teacher might be seen as creating norms for effective interaction leading to mutual learning between students.

We might regard this episode as indicative of a higher-level of cognitive demand, manifested, not just in encouraging students to think through a problem for themselves, but also in challenging them to make decisions about their *way* of working on the problem, and in many cases about the problem on which to work. The result of such challenge in Ben's classroom, and others that I studied, was that students were deeply engaged in and reflective of their own thinking and learning. This contributed to their active construction of mathematical concepts within a supportive social context whose interactions ensured that constraints arose and were resolved.

The teacher's articulation of the episode for the researcher led to the teacher's recognition of his teaching role and overt perception of its value in his teaching repertoire. For the researcher, it provided further evidence of the nature of co-learning partnerships. Interaction between teacher and researcher encouraged the articulation and growth of knowledge on the part of the teacher, and the teacher's communication allowed the researcher to contribute to a wider knowledge base in teaching.

Conceptualising the terrain to deal with diversity

In a research project overtly conceptualised as a co-learning partnership between two teachers and two university researchers (Jaworski & Potari, 1998), one of the teachers, Jeanette, articulated her vision of working with student diversity to enable progress in mathematical learning:

Episode 4: In an investigative lesson, I provide the stimulus for the initial problem and then give them some time to explore and so I now see a bit on a hill side, bit rocky, fairly open, maybe a bit bleak. Some of them will stay very close to me not physically but close to me metaphorically, not close to the problem or their friends. Others will start perhaps to go round the problem, trying things, maybe coming back, making sure they are doing alright and then go off again. One or two will skidaddle down the path and find something very interesting or get nowhere and come back again.

And so my role will be making, I will make sure that the ones who want to stay with me are walking with me round, maybe round or round about, encouraging them to go off. Sometimes I feel I push them and they wont go, and perhaps I will leave them and come back and they haven't gone anywhere. And then I will have to take them on with me a little bit further and then try and push them off again and they will go. And other times I will come back and ask something, that boy over there haven't seen him for a while, let's go and see what's happening. And could be they are finding something quite nice. It's when they've gone off on their own track and, and they don't think they've got anywhere and they ask for help to come back. That's fine. It's when they've gone off on their own track and I'm not sure whether they are getting anywhere or not.

It could be like with Simon (a student in her top level Year 10 class), it's simply we are just, are on different levels. He knows what he is doing and when he can eventually explain it to me then it's absolutely fine, so I can just leave him to do it. But others sort of seem to be going off in a sort of way but actually aren't getting anywhere. When do you give them a rope and pull them back?" (Potari and Jaworski, in preparation)

The metaphor of the hillside was used by this teacher to speak about student diversity and her approach to working with her students, varying her interactions according to their needs, and recognising questions this raised for her. It allowed the team of four to work on issues of classroom interaction related to lessons that had been studied.

Episode 5: In one such lesson, Jeanette had set a problem to groups in her class to design a box, using a minimum of material, to hold 48 cubes, each of side 2cm. Her aim here was to address concepts of volume and surface area from the mathematics curriculum. The lesson had shown Jeanette's interactions with various groups in the class to be related to the particular thinking of the students concerned. However, as time progressed Jeanette became more aware of the particular mathematical goals of the task and achieving these goals with the whole class. One such goal was that students should perceive the properties of a box with minimum surface area. However, only some of the groups had come close to appreciating these properties. Jeanette was therefore faced with shifting into direct instruction mode and explaining the concept to the class, or of leaving many students in their current lack of resolution of the problem, neither of these particularly satisfactory. Jeanette's approach for tackling diversity had come up against time constraints in addressing the curriculum, indicating the need for more time or alternative approaches. (For further details of this lesson, see Jaworski & Potari, 1998)

My purpose here is not to offer solutions for the identified dilemma. A long experience of teaching and working with teachers makes clear to me that such dilemmas occur every day in a teacher's life. As these issues came to light among the four participants of the research, the inherent teaching dilemmas were discussed.

The nature of a co-learning agreement is that the dilemmas can be recognised and tackled. However, in this case, students were not central to the recognition and tackling of the dilemma, but rather passive recipients of its consequences. It might be that the students themselves could have contributed to its resolution.

Teacher as researcher in tackling dilemmas

The Mathematics Teacher Enquiry (MTE) Project studied the contribution of teacher-research to the development of mathematics teaching (Jaworski, 1998). One teacher, Sam's research focus was on students who were 'productive' or 'resistant' to his teaching. He wanted to find out what cause these reactions and how he might overcome resistance to enable more effective learning.

Episode 6: This episode is taken from a lesson in which activity was based around exploring the effects of placing operators (+, -, x, ÷) and pairs of brackets between the three numbers 6,3,2 and inspecting the outcomes (e.g., (6+3)+2 or 6+(3÷2)). An object of this lesson was perception by students of generality in the use of brackets in algebraic expressions (Jaworski, 1998).

Sam discovered that some students readily moved from their activity in trying special cases, to a generalisation of the process, which was what Sam had hoped for. This group he termed 'productive'.

However, one group became very cross with Sam, and resisted when he tried to push them beyond their specific cases. They felt some success in discovering that there were 32 distinct cases of placing brackets and operators, which had been one of the questions of the lesson. It seemed, to them, that they had achieved what they had been asked. To be asked, subsequently, to do more, seemed to devalue what they had achieved so far. Thus, not only did this resistant group not perceive the need for generalisation, they felt unhappy with the teacher's apparent lack of appreciation of their effort and achievement.

For Sam this was extremely salutary. His personal focus as a result of the event was how to adapt his teaching so that it would be more sensitive to the needs of these students. The mathematical issues were somewhat implicit in this focus.

Sam, the mathematician, had in mind the generality behind specific examples of concepts – in this case, algebraic representations and the need for brackets. Mathematically, it seems essential for students, to appreciate the generalities involved. For those who remain at the level of the particular, their mathematical development is limited. However, pushing students too rapidly towards such generality may result in their losing interest or confidence both in their mathematics and in the teacher's teaching.

Thus a question arises: what tasks, questions, or classroom activities will enable most students to move less problematically to mathematical generality and abstraction? How is this recognisable by a teacher? For Sam, how does it link to his perceptions of students being resistant to his teaching? Such questions might be a part of Sam's further research.

Sam's engagement in research, his identification of research questions regarding productivity and resistance, led to his recognition of characteristic of learners resistant to his approach. He subsequently interviewed students in the resistant group to try to understand better their perspectives and learning needs. His interactions, in the MTE project, with other researchers provided a forum for airing and discussing the issues he recognised.

Such discussion in the MTE group led to wider awareness of such issues, of research approaches to learn more of students' perspectives, and of the value of talking with colleagues about teaching issues. The project charted a growth of knowledge through such interactions identifying them as a kind of co-learning activity.

Co-learning agreements or partnerships. Partnerships between teachers and learners.

A co-learning partnership in mathematics teaching or in the development of mathematics teaching may be expressed by an adaptation of the quotation from Wagner earlier. The italicised words are the ones I have changed.

In a co-learning *partnership, teachers* and *learners* are both participants in processes of education and systems of schooling. Both are engaged in action and reflection. By working together, each might learn something about the world of the other. Of equal importance, however, each may learn something more about his or her own world and its connections to institutions and schooling. (adaptation of Wagner, 1997, p 16)

The focus on education, institutions and schooling is significant because mathematics education is taking place within these systems. The focus on the 'worlds' of the participants is significant because it is these worlds from which they gain their experiences, which form their historically derived knowledge, and according to which they make decisions and judgements (Scribner, 1985). According to socio-cultural theorists, Jean Lave and Etienne Wenger, learning takes place through participation within communities of practice, these communities having their own socio-cultural norms into which newcomers have to be socialised. Initially the newcomer is (legitimately) peripheral to the practice, but is drawn into the practice through participation. Thus we might see students and teachers being socialised into the norms of schools and classrooms, into mathematics classrooms and the operative practices of mathematics teaching and learning.

Jill Adler (1996) has questioned just what the practice is, in the mathematics classroom, into which its participants are socialised. It is clearly not the practice of being a mathematician. It is a practice of learning and teaching mathematics within the particular worlds impinging on school and classroom, governed by cultural, societal, economic and political forces. All of these influence the ways in which mathematics is perceived and communicated. For example, whether the focus is on basic number, on utilitarian factors, or on rules and algorithms; whether motivation is in the tests or examinations driving the curriculum. If the practice has anything to do with acting or thinking mathematically in the style of a mathematician, then the norms for such a practice have to be created relative to these powerful influences.

Thus we might see a co-learning partnership being a community of practice with its own norms. It is important to recognise that Wagner's definition was talking about a *research* relationship. So, although this is not explicitly stated, it must be recognised in reinterpreting his words. Wagner's definition talks about *action* and *reflection* and *working together*. An interpretation for the mathematics classroom would put the word mathematics in front of each of these: i.e., *mathematical action*, *mathematical reflection* and *mathematical working together*.

We saw one manifestation of these norms in George's classroom in which his students acted, reflected and worked together on mathematics. Where George and his colleagues were concerned, the focus was mathematics teaching: thus George engaged in the action of mathematics teaching, reflected on his mathematics teaching and worked together with his colleagues on issues related to mathematics teaching. George and his *students* were co-learners of *mathematics*, as George acknowledged overtly. George and his *colleagues* were co-learners of *mathematics teaching*. In both cases, as the participants worked together knowledge grew and the community developed. A question to be asked here, is to what extent may these participants be seen to engage in *research*?

Didactic contract

Guy Brousseau (1984) has talked about a *didactic contract* between a teacher and students in a classroom from which the activity of the classroom derives, and which in its turn is strengthened by this activity. In George's classroom, for example, questioning and enquiry seemed to play an important role. In the brief episode described, the teacher questioned and students engaged in inquiry.

However, students' own questioning could have resulted from the prevalence of questions. In Jaworski (1994, p. 113), I described a classroom where the teacher stopped himself from giving an instruction and asked the question "what am I going to ask you to do?", one student spoke for many in the response "ask questions". In this classroom *asking questions* was one of the norms of mathematical activity. Terry Wood and Tammy Turner-Vorbeck (in press) have written about *argumentation* being an explicit classroom norm – the expectation that students would challenge each other to explain and justify their solutions to problems.

Thus the didactic contract is a recognition of the norms in the community. These norms are developed and strengthened as activity proceeds. In one classroom, described in Jaworski (1994, p. 152/3) a boy tells his group he has a prediction for the 5 by 5 case he has just been working on. One of his peers says to him, "don't you mean a conjecture in maths it's a conjecture". Later the same boy said, "are you sure 6 by 6 is 37, because that was my conjecture". In this classroom the norm of 'conjecturing' was being developed and reinforced through this dialogue, as it entered into and became a part of classroom discourse. We might say that expectations of conjecturing were a part of the didactic contract.

The didactic contract is an expression of the (agreed?) expectations between students and their teacher. It might be overt, or implicit. In the examples just mentioned, questioning, argumentation and conjecturing were overt. Although, how they became overt is significant and not obvious. Sometimes norms are covert. Walter Doyle (1986), as I said earlier, suggested that teachers and students collude *unconsciously* to reduce cognitive demand. Thus reduction of cognitive demand becomes a covert part of the didactic contract of the classroom.

Mathematical and pedagogical power

Whether norms are developed overtly or covertly, the didactic contract is an evolving co-construction by all participants in a classroom. Bauersfeld (1994) suggests that "Teacher and students interactively constitute the classroom culture". Thus, its participants constitute the classroom culture through co-construction. In a co-learning partnership, this co-construction needs to support or engender co-learning – i.e., it needs to includes elements of inquiry, reflection and working together. These elements might be seen to encourage what Tom Cooney has called *Mathematical Power*. For the learner of mathematics, this is "the ability to draw on whatever (mathematical) knowledge is needed to solve problems".

Mathematical power is "the essence of intelligent problem solving within the context of teaching mathematics" (Cooney, 1994, p. 15). In George's classroom we might see the students (and George himself) as developing mathematical power, both in the mathematics they are learning, and in their appreciation of the processes through which they are learning it. Simultaneously, through George's participation in the classroom activity and in raising issues with colleagues, George might be seen to develop what Cooney has called *Pedagogical Power,* – the ability to draw on whatever *pedagogical* knowledge is needed to solve problems.

Cooney writes "pedagogical problem solving has to do with recognizing the conditions and constraints of the pedagogical problems being faced" (1994, p. 15). I would add to this the aims, possibilities and opportunities provided by such problem solving. Pedagogical power is vested in such pedagogical problem solving through processes of reflection and analysis. George gains pedagogical power through engagement in classroom activity, reflection on this activity, and analysis of the situations in which he engages together with both his students and his colleagues. Development of mathematical and pedagogical power can be seen as an outcome of the co-learning partnership in George's classroom.

George's learning is recognisable from his own words in the episodes described. This might be regarded as a social construction: i.e., through inter-subjectivity with students and colleagues, George constructs personally his knowledge of mathematics and pedagogy (Jaworski, 1994). However, this knowledge does not exist in isolation from knowledge deriving from all the other communities in which George participates. Lave and Wenger (ibid) see learning as a process of enculturation where learners as “peripheral participants” in the community grow into “old stagers”, those who represent the community of practice. They write, “... newcomers legitimate peripherality ... involves participation as a way of learning – of both absorbing and being absorbed in – the “culture of practice” ... mastery resides not in the master, but in the organization of the community of practice” (Lave and Wenger, 1991, p 95).

Thus, knowing, or cognition, is *situated* in the practice. Teachers might be seen as growing into the practices of the community where their teaching is situated – those of schools and classrooms. These classrooms are situated within a wider socio-political community with a variety of cultural influences. The development of knowledge of teaching can be seen as a fundamental part of *participating* in teaching within this social setting.

So, George, as an old-stager in the school and classroom environment might be seen as thoroughly socialised into its practices. However, he can also be seen overtly to influence his own growth relative to these wider practices. To what extent these various influences are harmonious or in conflict in a teacher's growth of knowledge deserves further study. Central to this paper is the influence of the teacher's complexity of knowledge on the classroom environment in which he operates and its implications for students' mathematical development. How is a co-learning partnership constituted and how does it operate? What are the roles and responsibilities of its participants?

Developing norms for co-learning

In order for the co-learning partnership to operate effectively for mathematical learning, students have to participate as co-learners of mathematics, engaging together in inquiry and reflection. If they are to be socialised into such participation, how are the norms created? It seems that creation of norms must be first of all the responsibility of the teacher, but that students need to be active in the longer term construction of norms. For example, for questioning to become a norm, someone first has to ask questions, or to ask for questions to be asked. A teacher might initially lead by asking the questions herself, then encouraging questions from students. Or she might begin by seeking overtly to engender a questioning culture.

Julie Ann Edwards (2000) contrasted two situations, one in which particular norms developed over time through example and encouragement by the teacher as part of mathematical activity, and another in which the requisite norms were 'taught' overtly before mathematical activity could begin. She found that the overt teaching of the norms did not show more success that their covert encouragement. Sometimes, overt teaching leads to what Mason, adapting Brousseau, has called the *didactic tension*; the problem that,

The more explicit I am about the behaviour I wish my pupils to display, the more likely it is that they will display the behaviour without recourse to the understanding which the behaviour is meant to indicate that is the more they will take the *form* for the substance (Mason, 1988, p. 33).

For example, the teacher wanting his students to conjecture might have them conjecturing in given situations but never appreciating the value of conjecturing as part of mathematical justification and proof. The didactic tension is just one problem that teachers face in striving to establish the norms that seem essential to the activity they would promote. Teachers can become aware of such problems through their own inquiry into teaching. This might involve some form of research into their own practice, as in the case of Edwards mentioned above, or it might emerge through collaborative reflection as seen in the case of George and his colleagues.

In the case of Ben, quoted above, it was reflection alongside an external researcher that resulted in his recognition of the norms he wanted to promote, and his approach to 'teaching' them. Sam, in the 6+3+2 lesson, was overtly researching his teaching to discover the source of students' resistance to his ways of teaching. Through this explicit inquiry, he was able to learn something about his own teaching approach as a result of finding out the students' perspective. Jeanette, in her metaphor of the hillside, was able to articulate aspects of her role in working with the needs of different students.

We see here teachers here taking responsibility for creation of norms and also inquiring into students' interaction with norms. Students in Ben's class were divided on whether he knew the answers to the questions he asked them (Jaworski, 1994, p. 153). Some felt he did, others not. However, their tackling of the questions went beyond a ritual response: there was substance as well as form (Mason, 1988). George's students were socialised into engaging in an inquiry approach and providing justification for their conjectures. It seems clear that the way students respond is crucial to the development of norms. The extent, to which students appreciate their own role in the process, and its contribution to co-learning, seems worthy of further study.

Roles, relationships and responsibilities within the co-learning process are complex. It seems that a position of greater power would carry more responsibility. Thus a teacher would have responsibility for drawing students into questioning and decision-making, and a teacher-educator would have responsibility for drawing teachers into the processes involved.

Teachers' engagement in inquiry and reflection at a pedagogic level is central to the development of a co-learning partnership. It is a question for teacher-educators how such activity by teachers originates. In what ways are teachers 'socialised' into the norms of inquiry and reflection? In most of the examples discussed, this 'socialisation' became evident through interactions with a researcher from outside the school environment. Discussion with the researcher encouraged reflection and stimulated inquiry.

The researcher's questions often led teachers to deep searching of reasons for decisions and judgements which then led to further inquiry. As the researcher in these situations, I recognised the potential for teaching development of this asking of questions, as well as the role of the researcher in encouraging a teacher to sustain inquiry and reflection. The examples quoted from Sam and Jeanette arose as part of projects that were collaborative between teachers and researchers.

Researcher as educator

Wagner talks about the 'unavoidable intervention' of the researcher in educational practice. Thus a researcher is not a neutral outsider but a full participant influencing the directions of social practice in education – even when no overt intervention is planned.

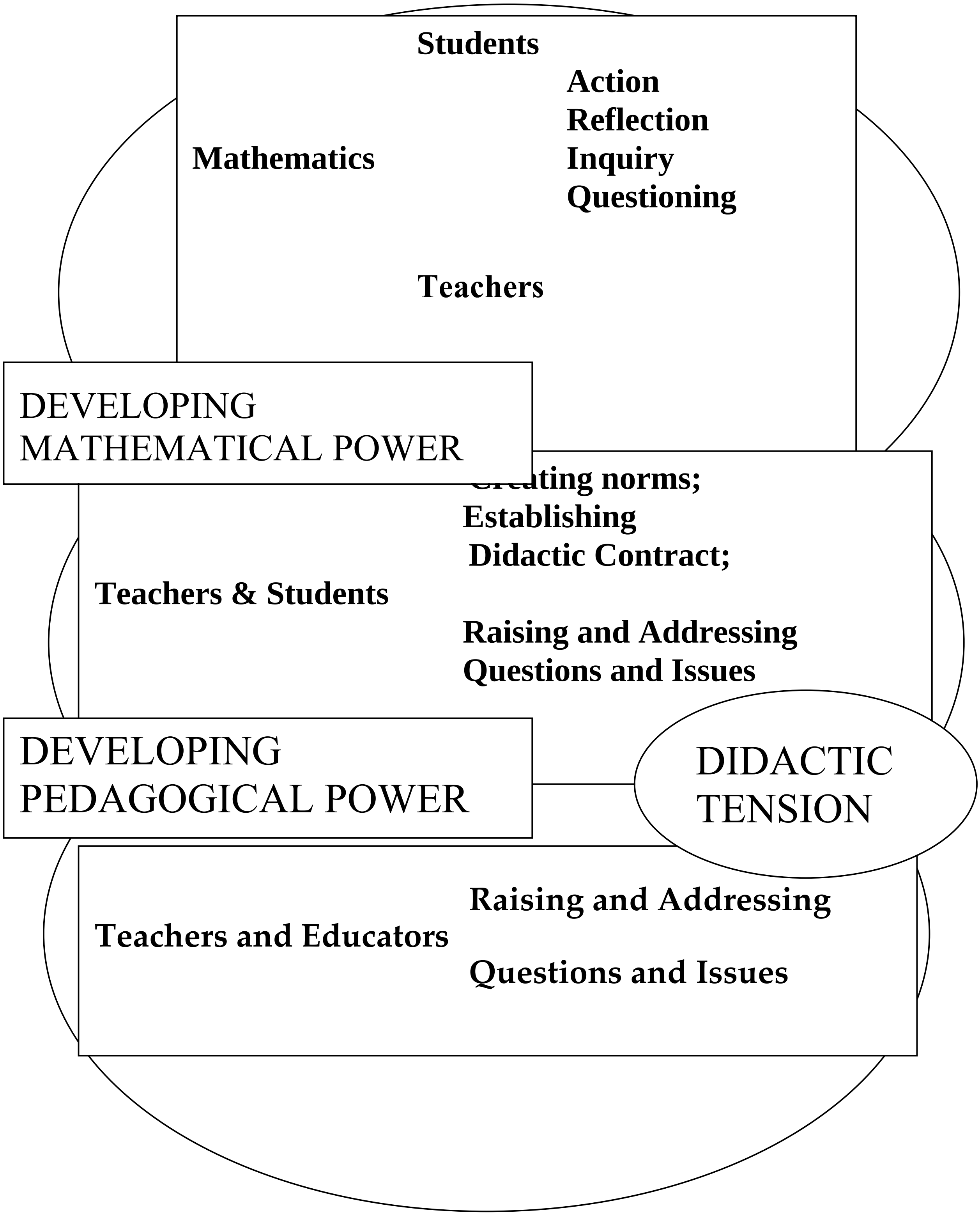
My own involvement as a researcher working with teachers in various research projects has indicated ways in which the researcher influences teaching through the research process. In fact, it has often seemed that more effective engagement by teachers in educational issues of deep significance has resulted from a researcher's naive questions, than is achieved through specifically planned courses for teachers.

By 'naïve' questions I means those that seek genuine information rather than those that are designed to elicit particular answers. It is the struggling with these naïve questions that leads to real engagement with issues. These might be mathematical issues as learners seek to engage with mathematics and to learn mathematics; they may be pedagogical issues as learners struggle with the teaching-learning process and ways in which teaching might support mathematical learning.

A start to conceptualising co-learning

I have tried to make sense of the many factors and concepts discussed above to start to get a grip on this notion of co-learning at so many levels. Inevitably there are still more questions than answers, but this is a characteristic of the epistemology of co-learning agreements. The following diagrammatic representation of participants, concepts and relationships is a starting point for further dialogue between those interested in developing teaching of mathematics through co-learning at all levels.

A co-learning community in developing mathematical learning and teaching



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From natural language to mathematical reasoning: Word problems and the socialisation of children's thinking[1]

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Introduction

The background of my presentation is an interest in how people reason, argue and act in different communicative practices in various corners of our complex society, and how they learn to do so. The ability to learn is one of the most significant features of human beings. Ten thousand years ago, our ancestors living in this part of the world dwelled in caves, had to endure very hard life conditions, and had an average life expectancy of about 25 years. Biologically, genetically, in terms of brain capacity and in most other respects, we are identical to these people. However, at a socio-cultural level almost everything is different. And all of these differences in how we live, work, travel and communicate are rooted in our ability to learn at a collective and at an individual level.

Starting from a socio-cultural perspective (Vygotsky, 1987; Wertsch, 1991; Säljö, 2000) on human development, which is my background, reasoning and arguing imply using linguistic and physical tools to analyse and make claims about events in the world - be they real or imagined. Thus, I am not taking the perspective of considering learning, for instance, mathematics as a question of acquiring an essentially ready-made body of knowledge that can be applied - as the metaphor goes - to various problems in 'real life' or in the 'real world' (which are two other dubious metaphors in this context). Instead, my main interest here is to offer some reflections on how people reason and argue, and how they quantify, maybe even mathematise, in order to make a claim or to prove a point in a conversation or when they solve a problem inside or outside the formal school setting. And, in addition, I want to say something about how we understand a particular kind of difficulty that people might have, that of moving between everyday discourse and analytical languages.

To give a more digestible introduction to my presentation, and to get me to the point more quickly, I will give a straightforward illustration of what I mean by learning as an issue of learning to reason. In 1997, the Swiss scholars Reusser and Stebler published a study of mathematics achievements amongst ten to twelve year old students in which the following items were included:

A boat sails at a speed of 45 km/h. How long does it take this boat to sail 180 km?
John's best time to run 100 m is 17 seconds. How long will it take him to run 1 km?

[1] The research reported here has been financed by the Swedish Council for Research in the Humanities and Social Sciences (HSFR).

This problem pair, and many similar ones, has been used in several studies (cf., e.g., Verschaffel, De Corte & Lasure, 1994) over the past two decades. What is interesting about them is the dramatic difference in which the children manage to handle these problems successfully. In Reusser's and Stebler's study, and in the one by Verschaffel, De Corte & Lasure (1994), the first problem was correctly solved by between 85 and 90 per cent of the participants. Viewed as an instance of mathematical reasoning, and disregarding for the moment the referential problem of what is said in the text, the second problem was solved by an even higher percentage of students. The only catch was that their response was a simple multiplication of 10 by 17. Only a tiny fraction of the students in both these studies, 4,5 and 2,5 per cent respectively, indicated in some manner that there was something problematic about the second item.

In parenthesis, even the first - and 'standard' problem - in this pair is a bit strange to someone coming from the West Coast of Sweden. A sailing boat that sails at a speed of 45 kilometres an hour is a rare and quite expensive piece of equipment! But let me leave the realism of this aside.

Together with my mathematician colleague Jan Wyndhamn I made a study in the mid-1980's (Säljö & Wyndhamn, 1988) in which we were after the same problem of how students interpret word problems. We asked the following pair of questions to the same age groups:

A cow produces 18 litres of milk per day. How much milk does the cow produce during one week?
Lisa goes to school and she has 6 lessons per day on the average. How many les sons does she have per week?

Again the first item is essentially unproblematic. About 90 per cent of the students performed the multiplication 7 by 18. In the second case, the situation turned out very differently. Among low achievers in mathematics, for instance, about 40 per cent managed this problem in the sense that they realised that the reference of the concept of week in this case (in Sweden) is five.

The problem I want to address in what follows is: *What does this tell us about how children learn to reason?* Why do these items result in such differences in performance? I think that if we analyse such performance differences in some detail, we might get somewhat deeper in our understanding than establishing that children do not know how to solve word problems, or, alternatively, that they cannot read properly, which is the explanation preferred by some mathematics teachers. Such explanations are not particularly useful, since they give us no clues as how to guide people when they attempt to master complicated forms of reasoning. So let us stay open, at least initially, and not resort to the usual argumentation that these differences can be explained by referring to some abilities and knowledge that people do not have. Instead, what items of this kind in all their simplicity illustrate, is something very fundamental about how difficult it sometimes is to move between describing the world in our natural everyday language and using other models of that reality (since everyday accounts of the world are also models). Underlying these differences in performance on tasks that are identical in terms of how to quantify, there is a story about an educational discourse in schools and classrooms that has become increasingly abstract.

The situated nature of human reasoning

Over the past ten years or so, there has been much discussion about the situated nature of knowledge. The general line of argument has been that the assumptions of general cognitive skills, transferable knowledge and/or nicely bounded developmental stages of the Piagetian kind have been overrated. Context and situational specifics determine how people reason to a much larger extent than the dominant cognitivist paradigms have been willing to accept. Thus, and to return to the problems above, it is inconceivable, or at least highly unlikely, that children watching a 100 metres race in an athletics competition would assume that the runner could keep the same speed for 1 kilometre, which is the 'real world' interpretation of the mistakes they make on the first kind of problem about John running. This kind of mistake they would only make in school. The world, and all its various social practices, are much more complicated than the notions of a general cognitive processing device or a developmental stage would imply. Thus, during the 1980'ies and 1990'ies we have seen work that illustrate that what children can do in terms of counting when acting as street vendors, they seem unable to do when given a test in a school-like setting of the same kind of problem (Carraher, Carraher & Schliemann, 1985).

To many, these findings about the situatedness of knowledge and understanding seem to undermine the very possibility of making claims about regularities across domains and/or situations. Indeed, it was seen as a threat to the ambitions of most established educational approaches. How can learning be meaningful and powerful, if skill and knowledge are local and dependent on circumstance? Is not the very definition of knowledge and, as we sometimes call them 'facts', that they are true and verifiable independently of where they occur? What is the point of teaching people skills and methods, if they are not transferable? These were the kinds of worries raised by some. However, the realisation that human knowledge is relative to circumstance and premises is, of course, no threat to our possibilities to learn about human learning and development. It just forces us to leave simplistic mechanistic notions about what human thinking and acting are all about, and to accept the fact that we transfer knowledge and skill in a somewhat more subtle manner than was assumed by the behaviourists, the cognitivists and Piaget. Human beings, like no other creature on this planet, are able to learn both individually and collectively, and they <u>do</u> take knowledge and skills from one situation to the next. If they did not, we would not meet here today, as the systematic creation of knowledge that we deal with as researchers and teachers would not be possible.

The tool-producing and tool-using animal

In a sociocultural theory of learning and development, individual intellectual growth as well as the increase in mastery of physical skills occur mainly in three different dimensions. Thus, people develop:

1. Intellectual/discursive tools (concepts, theories, systems of measurement, ideologies, discourses about nature and society etc.)
2. Physical tools (technologies and socio-technologies)

3. Social institutions (i.e., enduring collective practices where knowledge and skills are developed and reproduced)

Perhaps the most impressive element in our intellectual history is the manner in which we have been able to transform intellectual, or as I prefer to call it: discursive, knowledge into physical tools. Watches, compasses, computers, mini-calculators and thousands of other artefacts are repositories of human knowledge and insight. We have managed to convert our number system and a set of mathematical operations into a physical device that we know as a mini-calculator. By using this device, tasks that were very difficult and time-consuming to do in your head, such as multiplying or dividing four digit numbers, immediately became simple and easy to deal with.

However, my ambition here is not to go into this fascinating story of how people are able to convert their knowledge into physical artefacts. Instead, I want to dwell on the manner in which we appropriate certain intellectual and/or discursive tools, and how we move between everyday discourse and mathematical modes of reasoning.

The issue of how people learn to use analytical tools of the kind offered by mathematics and logic are interesting from a socio-cultural perspective. First, mathematics and logic are analytical also in the sense that truth conditions of expressions are not dependent on references to real world events. In mathematics and logic, meaning is established internally within the system itself. Mathematics is in itself a universe of meaning in which concepts are defined by relations to other concepts and operations. However, when you put such resources to use for making claims about real world events, as in the problems above, you run into the problem of referentiality in an empirical sense. Or, expressed differently, when mathematical notations and expressions are co-ordinated with an outside world - be it physical or imagined - and when you begin to count money, distances or whatever, the problem of reference to an outside world appears. Learning in this setting thus implies being able translate between expressions in mathematical or logical terms, and expressions that are made in everyday language (or in various kinds of institutional languages). This is precisely what so-called word problems are all about. Learning how to do this is a powerful socialisation of people's minds, and if you look at it in terms of our cognitive history, it is a very advanced kind of skill.

The second point I want to mention here is that learning to mathematise or to do logical reasoning is very clearly an issue of mastering a set of communicative or discursive rules. One must learn how to make claims and argue systematically within one or more discourses, and realise what is a valid claim to make in a particular situation and what is not. This is a fascinating learning process, and it is a matter of learning what Wittgenstein calls 'language games'. But before continuing on this line, let me just give two brief illustrations of what I mean by learning a particular form of discourse.

The first example is meant to illustrate the general idea of what it means to learn a particular form of discourse, and how you can use it to make claims about something you have seen, heard or read. Although, in this case, the outcome of the mastery of a particular form of discourse was perhaps not a very big success. It is about the economist, the Total Quality Management Consultant, of a company that was offered a cultural experience by his boss.

Tom goes to the symphony

A company president who had been given tickets for the performance of Schubert's Unfinished Symphony couldn't attend, so he passed them to his Total Quality Management Consultant. The next morning, when the president asked the consultant if he had enjoyed the concert, he was handed the following memorandum.

For considerable periods of time, the four oboe players had nothing to do. The number should be reduced and their work spread over the whole orchestra, thus eliminating peaks of activity. All of the 12 violins were playing identical notes. This seemed unnecessary duplication, and the staff of this section should be cut drastically. No useful purpose is served by repeating with horns the passage that had already been played by the strings. If all such redundant passages were eliminated, the concert could be reduced from two hours to twenty minutes. If Schubert had attended to these matters, he would probably have been able to finish his symphony after all.

What makes this story funny, and the unknown author very perceptive, is how clearly it illustrates how the discursive tool – i.e., the ways of reasoning and thinking of economics – do not co-ordinate very well with the event that is being described. If this is what our Total Quality Management Consultant assumed that going to a concert was all about, he lives a rather strange life. But the general point is clear, to learn is to master systems of discourse as intellectual and practical tools, and to be able to apply them to what you encounter.

My second example concerns the learning of what we call logical reasoning. In a very famous study in the history of psychology, the Russian psychologist Luria (1976) in the early 1930's travelled to the southern republics of what was then the Soviet Union. He was interested in what he called illiterate peasants, i.e., people who did not have any formal schooling and who could not read and write. What he observed, to make a long story very short, was among other things the problems the indigenous farmers had in handling so-called syllogisms:

In the Far North, where there is snow, all bears are white. Novaya Zemlya is in the North and there is always snow there. What colours are the bears there? (p. 108)

In response to such exercises, participants would often respond in the following manner:

"There are different sorts of bears." [Interviewer repeats syllogism]
"I don't know; I've seen a black bear, I've never seen any others. Each locality has its own animals: if it's white, they will be white; if it's yellow, they will be yellow."
Interviewer: But what kind of bears are there in Novaya Zemlya?
"We always speak only of what we see; we don't talk about what we haven't seen."

If we look at the nature of the communicative problems in this situation in detail, and the difficulties that our cautious farmer has, they are quite interesting. What the respondent is doing is in some sense not unreasonable. He is arguing that he does not know what the bears are like in this strange northern place that the interviewer is talking about, since he has not been at that particular place. He prefers to speak about what he has seen and what he knows of through personal experience. In many respects this is a commendable strategy for how you should argue; do not claim to know how things are in a place where you have not been. However, what our cautious friend does not realise is that in this particular kind of language game, the rules are different. In this discursive tradition, invented by the Greeks, you are supposed to see if the conclusion follows - as we put it - logically from the premises.

The problem for the respondent is that he is arguing about the world, while the test assumes that you limit your attention to what is in the text. Thus, this exercise should be read with the attitude: "if we assume that all bears in the North are white, and if Novaya Zemlya is in the North, what does then follow about the bears in Novaya Zemlya?" In other words, to answer this question you have to temporarily disregard the issue of how the world is organised and whether all bears up north really are white or not. The states of fact in the physical world are temporarily immaterial, and you should not appeal to your knowledge or lack of knowledge about the world in order to establish if the reasoning is logical or not. Expressed differently, what we have to learn to do in this kind of reasoning is to disregard the real world, and to argue in a textual reality. And text worlds are often very different from physical worlds.

Attending to the world and attending to texts about the world

From a psychological point of view, and as an instance of problems of learning, the manners in which we use texts and concepts to refer to the physical reality are extremely interesting. To learn how to co-ordinate an expression or a claim with an outside world, one has to be aware of the nature of the discourse that is expected, and what claims that are being made in a specific situation. It is a matter of discerning what is figure and what is ground, what is assumed and what is claimed. This is, I will argue, a typical skill in identifying and moving within and between different kinds of discourses. It is not about what we know about the world in any abstract sense. All the children in the little study that Jan Wyndhamn and I did know that they go to school five days a week and not seven. The problem is one of realising this at the very moment in which you are doing word problems.

What learning implies in this case, or in many instances of logical reasoning, can be described as a particular kind of alienation from our normal, everyday attitudes to language and symbols. In a literate knowledge tradition, such as ours, one must learn when to appeal to the real world for confirmation of what is true and not true, and when to attend to the inner logic of a claim or a statement. This is an extremely complex learning process in a modern society, especially when we consider all the new manners in which we are able to model and mathematise the world by means of simulations using information technologies. In what sense is what happens on the screen real or true?

If we return to the so-called word problems, and the difficulties children have when being cognitively socialised, we can reflect on some of these complexities. The most frequent explanation of why children fail on the kind of difficulties that the example with John running and Lisa going to school above present them with, is that they fail to make what is called realistic considerations (Reusser & Stebler, 1997; Verschaffel, De Corte & Lasure, 1994), i.e., they fail to keep in mind what the world is like. However, this is an all too simple explanation, and it actually begs the question as it assumes that there is a transparent and indivisible world to refer to. Even if we all believe, I hope, that there is a real physical world out there, the manner in which people refer to this world when talking and writing is very complex (Greer, 1997). In another famous example in the literature on word problems (the example is drawn from the Third National Assessment of Educational Progress on thirteen year-olds in the US), we find that children often respond to the following item without considering what it means as a statement about the real world:

> *1128 children are going on a trip in buses. Each bus can carry 36 children. How many buses are needed?*

In the original test, less than a quarter of the students who correctly performed the division 1128 divided by 36 gave the correct answer 32 (cf. Greer, 1997). The majority of the rest gave answers that would imply that they operated with fractions of buses, an interesting physical entity in this setting. However, even though the numerically correct expression 31 and a third of a bus is a strange utterance about the world in this case, one cannot argue that it is absurd. There is no problem in finding equivalent statements that make perfectly good sense. For instance, today there is a big political concern in this country about the fact that currently the birth rate (measured in terms of children per woman) has dropped to a low of 1.6 children per woman from 2.1 which was the figure we had some years ago and which allegedly is healthier for a social system such as ours. Again, one could argue that 1.6 or 2.1 children is a strange entity. However, in a world of national statistics this is a perfectly natural mode of reasoning and quantifying. National statistics is one kind of reality, living children of flesh and blood is another kind. None of these two realities is per se more real than the other, they are simply different.

What is interesting about these word problems from my point of view is that they reveal some of the extreme complexities of learning in a literate culture that has at its disposal many powerful discourses and ways of referring to reality. It is therefore important to realise and to classify correctly the problems that children have when dealing with these abstract exercises. From a psychological point of view they are at best learning to establish what is assumed and what is claimed, i.e. to single out what the assumptions in a problem are and how they relate to some specific conclusion. Often mathematics teachers seem consciously to avoid the kind of ill-formed problem that I have pointed to. They intuitively realise that these are much more difficult. By avoiding such problems, they unknowingly give precedence to one kind of mathematical reasoning and one kind of theory of learning; the kind where the semantics and extra-mathematical references should not pose a problem (Lave, 1992). In the standardised version it is basically the internal logic of the mathematical expressions and operations that counts as mathematics. However, it is extremely important to assist children in bridging this gap between a text about the world on the one hand, and the real world on the other. This is not a problem about learning how to apply mathematical reasoning, as the metaphor goes, it is an issue of being made aware of precisely how models relate to physical reality, and this is crucial to all learning in our scientific culture.

To a large extent this is something which has to be learned through interaction, arguing and discussion. The communicative mode in which children have to be made aware of such difficulties is in discussions. Only later can they be assumed to master this in the text version. Thus, there is not very much to practise in the traditional sense. To manage these kinds of difficulties you urgently need dialogue and interaction, a knowledgeable person who can draw your attention to what is assumed and what is claimed in a particular kind of textual version of the world. Thus, the problems that children have in this context can be described not as the lack of knowledge, but as a reflection of the fact that they are somewhat lost in a communicative sense.

Let me add two final observations. This skill of realising how to co-ordinate models and mathematical expressions is a discursive skill, it is not a skill that resides in our brain as a biological entity. You have to be socialised through communication in order to master these kinds of difficulties as is evidenced by the problems that the farmers without schooling had in Luria's study (cf. Wistedt, 1994a, b). The acquired nature of this skill is also what the Greeks realised by forcing their students to practise on syllogisms. But it is not a skill that emerges solely through drill and practice. It is an analytical insight that has to do with discovering what you can do in texts and what you can do in the physical world, and what you have to think of when moving between. The second observation is also quite interesting in my opinion. Normally international comparisons, such as IEA and TIMSS, reveal that children at the ages that I have been using as illustrations in Eastern countries (Japan, Hong Kong and other countries) outperform children in Western Europe and the US in mathematics achievement. However, it is interesting to note that in a study by Yoshida, Verschaffel and De Corte (1997), it is shown that this does not apply to these kinds of problems. When the semantics are problematic, and when the reasoning implies that children have to connect statements with an outside reality, the Japanese children were no better than their Belgian colleagues. This also testifies to the very basic psychological difficulties that are involved when learning to master these kinds of problems. Learning how to move around in text based realities is an important aspect of our cognitive socialisation, but it is a very complex one that requires systematic challenges as well as guidance.

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Girls and Mathematics –
Focusing on the current situation
in the Norwegian upper secondary school

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Abstract
The research project ”Girls and mathematics” will run for the academic year 1999/2000. The aim of the project is to suggest actions to increase recruitment and level out gender differences in mathematics at upper secondary level. This article consists of two parts, first it will briefly outline findings in research literature and second it will describe some findings in a qualitative classroom study. Four different groups of students and their teachers have been chosen for observation and interviews. What reasons do students give for choosing mathematics when it becomes optional? Other issues such as seating and teacher commitment will be commented on shortly as well.

Introduction

The research project "Girls and Mathematics" was initiated by the Norwegian Ministry of Education based on findings in the TIMSS-survey. The project period is the academic year 1999/2000. The aims of the project are:

- to investigate attitudes towards mathematics among students, especially girls, who have chosen to take mathematics
- to suggest actions to be made to make more students, especially girls, choose mathematics when it becomes optional
- to suggest actions to be made to level out differences in achievement

The results of the TIMSS-survey pointed to some important results but it also raised questions. A qualitative research approach is suitable to investigate these questions more deeply and to work towards the aims of this project. The main sources of information for the project are:

- research literature
- classroom observation
- small surveys among students
- the TIMSS-results
- interviews with teachers and students

Mathematics in the Norwegian school system

At present Norwegian children start school at the age of six (since 1997). They do seven years of primary school and three years of lower secondary school before entering upper secondary school. Students attending upper secondary school are of age 16+ - 18+.

An academic stream is offered as well as a vocational. Mathematics is compulsory in the first year for all students. For students in the academic stream this means five lessons of mathematics each week throughout the year. This course is divided into two parts. The curriculum in the first part of the course is the same for all students, while the students choose between two different curricula depending on whether they want to continue to take mathematics or not for the last part of the course.

Mathematics is optional the next two years, and students can choose between two streams[1]:

- MX – for students preparing for studies in mathematics, the natural sciences a.o.
- MY – a more social sciences oriented course

The TIMSS results report that while an equal amount of boys and girls choose to follow the offered course 3MY, only 35% of the students in 3MX are girls. The TIMSS-results also revealed significant gender differences in achievement among the students in the 3MX course. The differences found in the TIMSS-material are not mirrored in the students' grades, but still the Norwegian Ministry of Educational Affairs considers these findings alarming.

Research literature

One of the goals of the project is to give a review of international research literature on gender and mathematics. The focuses of the project have been to get a clearer picture of explanations to why girls tend to have more negative attitudes towards mathematics and why girls in some countries and/or under certain circumstances achieve less in mathematics than boys do. The following is a short review of such explanations.

These explanations can be placed on different levels. Because of the consequences of where one chooses to focus the explanations it is important to be aware of this. If for instance, one chooses to emphasise biological explanations, this will imply that there is nothing "we" can do to change the situation. If on the other hand, one chooses to focus on the curriculum it implies that there is a potential for change.

The different levels of explanations can be summarised as follows:

- biology
- with the girls themselves
- parents
- society/ culture
- curriculum
- books and teaching material
- mathematics as a subject
- the classroom
- assessment format

[1] Both streams qualify for studies at colleges and universities to the same extent.

Biology

Media has given biological differences between boys and girls and how this influences achievement and attitudes in mathematics a lot of attention lately. The project has chosen not to focus on these explanations for several reasons. These are:

- The differences between girls' and boys' attitudes and achievement have changed over time, which indicates that biological differences can not have a larger impact on the differences than the environmental factors
- The differences between girls' and boys' attitudes and achievement vary between countries, which indicates that society and culture affect these results to a greater extent than biological factors
- If biological factors would show to have an influence on these matters then these factors are hard if not impossible to change

With the girls themselves

One kind of explanation that falls into this category is that girls have their own cognitive way of learning in mathematics, which is different from the way in which boys learn. Many of these kinds of explanations argue along with Belenky et al (1986). In "Women's way of knowing" Belenky et al argue that the research on the brain and cognitive development has mainly been conducted on the male brain. This implies a belief in that girls' experiences influence their brain in such a way that they develop different cognitive abilities.

Others argue that girls learn in a better way when they experience knowledge in a connected way while boys have a more separate way of knowing. This research often argues with Gilligan's (1982) "In a different voice" from which the concepts of "connected" and "separate" ways of knowing are taken.

A substantial amount of the research in the field of "gender and mathematics" has focused on girls' lack of confidence in their ability to do mathematics[2]. This lack of confidence makes girls not take mathematics when it becomes optional. Some research results also point to the fact that the lack of confidence can have a negative impact on future learning and the development of attitudes towards mathematics.

Meyer and Koehler (1990) in Revak (1995) focus on what they call "attribution system". When girls do well in mathematics they tend to say that it was because the test was easy etc instead of like boys have a tendency to do, to say that it's because they are good at mathematics. On the other hand when girls don't do well they tend to blame themselves while boys tend to blame external factors like the test, having a bad day or the teacher.

Some researchers have focused on girls' so-called learned helplessness (see for instance Beyer, 1995). These explanations point to that girls are taught to be helpless. This implies that when girls work with mathematics they need a lot of support from their peers and teacher. Boys on the other hand are taught to be more self-reliable and autonomous in their learning.

[2] See for instance Willis (1996), Hanna (1995), Fennema (1995), Finne (1996), Niederdrenk-Felgner (1996).

Parents

Parents influence their children when it comes to choosing a career. Parents may have different expectations regarding daughters and sons that again may influence the daughters' and sons' choice to take mathematics (Adda, 1995). Parents may also contribute to what girls and boys consider as appropriate gendered behaviour and this might not include girls doing mathematics (Elwood & Murphy, 1998).

The society

There are some widespread myths about gender and mathematics in the society today and these myths have a negative impact on girls' attitudes towards mathematics. One such myth is that girls and boys have equal opportunities when it comes to choosing/not choosing mathematics when it becomes optional when they in reality don't (Willis 1996). As mentioned earlier many factors influence boys and girls in different directions in their education.

Another such myth is that girls are born without an interest in mathematics and that they are not as good at doing mathematics as boys are (Kreinberg & Lewis 1996). This again can make girls feel deviant if they like mathematics.

In our society there also exist certain ideas about what is considered "feminine" and "masculine". Mathematics is often associated with masculinity, which may influence girls in another direction because in taking mathematics they will feel less feminine[3]. As Niederdrenk-Felgner (1996) writes:

> Still decisive are traditional ideas in their environment about roles and gender-specific attributions: mathematics, science, and technology subjects tend to be linked with the masculine domain,…

These myths and expectations from society around them influence girls in such a way that they do not take mathematics when it becomes optional. Girls tend not to see the relevance of mathematics to their future life to the same extent as boys do (Beyer, 1995; Reilly et al, 1995; Revak, 1995).

The teacher may also have a certain influence on students' choices. Some research indicates that teachers tend to expect higher grades from girls than from boys before they recommend students to take mathematics (Adda, 1995; Smart, 1996).

The curriculum

Another level to look for explanations is in the curriculum. Traditionally it has been written by men and some would say *for* men. The argument would then be that since men have written it, it influences both content and language (Willis, 1996; Forbes 1996). One consequence might be that teaching according to this curriculum could alienate girls.

Books and teaching material

Textbooks have traditionally been written by men and the examples used are often taken from boys' world of experience. This makes it hard for girls to identify themselves with "the mathematical world" found in books and other teaching material (Solar, 1995; Niederdrenk-Felgner, 1996; Smart, 1996).

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[3] See for instance Willis (1996), Hanna (1995), Niederdrenk-Felgner (1996).

Mathematics as a subject

Some researchers criticise mathematics as a science. They focus on the fact that mathematics has been constructed by white middle-class men and that this has influenced what mathematics has become and also the school subject and the teaching (see for instance Fox Keller, 1985). This has affected mathematics in such a way that girls don't feel that mathematics is relevant to them.

The classroom

The classroom is a complex arena for finding explanations when it comes to girls and mathematics. Some research has been done on the classroom processes but it has proved difficult to find clear relationships between what happens within four walls and consequences in achievement and attitudes. Some researchers have for instance documented that girls receive less attention than boys do in the classroom (Leder, 1996; Fennema, 1995; Solar, 1995; Niederdrenk-Felgner, 1996). Intuitively this would imply that girls don't get the same opportunities for learning and that they would develop more negative attitudes towards mathematics. This has so far not been documented.

Jungwirth (1995) has analysed the communication between teacher and students in the classroom and found that there are certain patterns of communication typical for the mathematics classroom. The communication and hence the patterns are dominated by the teacher. It seems that boys are better at following these patterns than girls and that may give the teacher the impression that the girls are not as good at mathematics as the boys are. Thompson (in Murphy & Elwood, 1998, p 164) suggests that when "*communicative style does not reflect ability, observer bias seems very likely to occur*."

Some research results also indicate that girls seem to be more comfortable with questions that are taken directly from what they have learned in class rather than questions that demand more thorough analysis (Reilly et al, 1995).

In connection with this some research results indicate that encouragement from their teacher is more important for girls than for boys (Grevholm, 2000). This can be seen as an effect of girls' lower confidence in their ability to do mathematics than boys, and the "learned helplessness" factor referred to earlier.

In connection with research results that indicate that girls may have their own cognitive way of thinking and/or learning, separate girl classes in mathematics have been suggested and tried out. So far there are no clear results pointing to that this is an advantage for the girls when it comes to learning outcomes and attitudes (Harding in Saif, 1995).

Assessment format

Some research has been conducted on gender and assessment but not necessarily in connection with mathematics and mathematics teaching. Some research indicates that girls are more comfortable with coursework assessment (Clark, 1996; Smart, 1996; Leder, 1999). This is in sharp contrast to the strong tradition of final written exams in mathematics.

Other research focuses on the context of tasks. When the context of the task is taken from boys' world of experience the girls' may feel alienated because the context is unfamiliar (Elwood & Murphy, 1998).

Elwood & Murphy (1998) also mention communication in connection with assessment:

> Students' learned styles of communication and ways of working combined with their preferred choice of reading material exert a powerful influence on their solutions and form of responses they consider appropriate. (p. 178)

Summing up

What will be most interesting for this project is the research being done on gender and assessment besides research on boys' and girls' view on mathematics as a subject. Also results from research on the classroom processes will be taken into account. As stated earlier in this article the explanations that give potential for change are of main interest. In the next project period project members will have two main considerations when choosing literature to be reviewed. The first main consideration will be practical ones as to what can be changed in Norwegian classrooms. The second consideration will be to search for literature that can give additional information to observations in the qualitative study.

Qualitative study

The mandate of the project group is to suggest actions to increase recruitment, among girls especially, and to level out gender differences in achievement. In the qualitative study the aim is to investigate students' attitudes towards mathematics in order to try to isolate what makes mathematics a preferred choice when choosing subjects for your studies. In order to understand what makes mathematics a favourable subject, it is important to gain insight into what students already choosing this subject think of it.

The project group follows four different groups of students during the school year 1999/2000:

- The Blue school : 2MX: Male teacher (B1)
  2MY: Female teacher (B2)
- The Green school: 3MX: Female teacher (G1)
  3MY: Male teacher (G1)

The Green school is situated in the countryside while the Blue school is situated in a rural area. In choosing the school in the rural area schools where the applying students had extremely high or low grade averages were ruled out. Schools with certain cultural profiles were ruled out as well.

The four groups will be observed approximately ten school lessons each, and in addition some students and all teachers will be interviewed at three different occasions during the school year. All students (about 80) will be given simple questionnaires on occasions such as starting school, mid term exams, deciding what to do next year, final exams and others. The observations and the questionnaires will form the basis for choosing students to be interviewed as well as determining themes for the interviews.

The four groups are of variable group size, the smallest group consisting of only six students and the largest of 29. The number of girls varies between one[4] and eight. The students are familiar with the project aims, those of them that are 18+ have given

[4] Another girl dropped out of the smallest group just a few days before the presentation. Her reason for doing so was that she had been elected president of the class. In addition she was already attending more classes than what is asked for according to norms for secondary schooling.

their own consent to participate in the research project. For students under the age of 18, the parents have given their consents. The climate of all groups is friendly, and we perceive them as including. Some students will sit in pairs or small groups in all classrooms, but in their teaching the teachers use this to a different extent. Mainly the students themselves choose whether to sit on their own or in pairs. If seated in a pair they are usually offered the choice of whom to co-operate with as well.

In the autumn term of 1999 we have been focusing on the following:

- Reasons for choosing mathematics
- Teaching/the organisation of the classroom
- Assessment/evaluation
- Plans for further education

This article will mainly focus on the first point giving a summary of students' reasons for choosing mathematics and presenting some students' voices. The importance of the teacher to the students will be briefly touched upon and some issues about the organisation of the classroom will be presented.

Reasons for choosing mathematics

All students have answered a questionnaire as to why they have chosen mathematics. Mainly four different reasons are given:

- Extra credits
- Mathematics is needed or beneficial for further education or employment
- Students are interested in the subject
- Mathematics is helpful in order to understand other subjects, for instance physics

Students give these reasons as single arguments or in combination with one or more of the others. No gender differences are found in the students' reasons. The two first reasons are the most frequent ones, but even to claim to have a special interest in the subject is common among the students who have opted for mathematics. Only a few students report mathematics as a supporting subject to their other choices. The following quote from a questionnaire illustrates how students give multiple reasons for choosing mathematics.

> In the second grade mathematics was all right, with a good class climate and a teacher we felt at ease with. The most important reason however, was the extra credits I will get by doing Mathematics. They will come in hand when I try to become a Physician (Medical Doctor). Rose 3MY

Extra credits

For the past few years students completing optional mathematics courses have been given extra credits. These come in hand when applying for popular programs at universities or colleges. Especially students wanting to go into medical school or to study media report this as a reason for choosing mathematics.

> To do Mathematics for two years gives four extra credits. So it pays well. I could have chosen German as well, now, but I believe Mathematics is more interesting than German.
> Lilly 3MX

Usefulness

Other students report that mathematics is required (one or two years of mathematics in addition to the compulsory curriculum depending on what you want to qualify for) or a benefit for further education or employment. Students wanting to qualify for specific studies report that mathematics is required or asked for. Other students report that they have a feeling of it being helpful.

> I want to be a Physician (Medical doctor), and [so I] must do Physics, Maths and Chemistry — Violet 2MX

> Because I have a feeling of Mathematics being asked for at a later stage. A lot of companies prefer that you know some Mathematics. — Onslow 2MY

Interest

Both boys and girls in both streams report that they have an interest in the subject or that they enjoy mathematics more than other subjects.

> Because I am quite good at mathematics. — Emmett 2MX

MX or MY?

As reported earlier, while comparable numbers of boys and girls go into the MY program fewer girls than boys choose to take MX courses. Why do students choose to follow one stream or the other? Are these choices strategic ones or are they based on knowledge of the content of the curricula or demands in future studies[5] or employment?

Among the students in the MX program, there are students who claim that they did not consider this question when choosing. Others claim that they have a special interest in mathematics as a discipline or that doing this course is a way for self-fulfilment. Lilly is a good example of a student demonstrating this view. She is not planning to go on studying mathematics; she wants to study social sciences at the university. She offers this explanation in an interview. I had not yet questioned her upon this issue, but she apparently found it important to share this information with me.

> You might wonder why I have chosen MX, not MY? I did MX in the second grade, and then I was going to have this subject for two years, because I wanted to prove to myself that I could do it as well. Because I had heard that it was soooooo difficult.
>
> …. And it is difficult in the third year, it is particularly difficult, but I wanted to prove to myself that I was smart enough. In a way I could do it for self-fulfilment.
>
> Lilly 3MX

Some of the students in the MY program indicate in their reasoning that they conceive their curriculum as

- easier
- offering more time to work on different parts
- being for the students that are interested in mathematics without conceiving themselves as specialists

[5] These demands are not formal ones, but as to what will be helpful in order to understand better.

Also choosing this stream is considered a tactical choice among some students. Rose clearly reports tactical reasons as well as other more personal considerations. Rose switched from 2MX to 2MY just before the end of 1st term last year.

> … I started out doing 2MX.
> ... Well, T is very energetic when she is teaching and all that. And it is, with her, well it became an even more stressing subject than it really is meant to be, so I went there for three months, then, but I found out that I rather wanted to switch over if it was possible. Well, it was to get a better grade as well.
> Interv: Yes, you thought that MY would be easier?
> E: Yes, it was too.… I am glad I switched over. Rose 3MY

What do students report as positive?

The students observed in the classroom study report to be relatively positive towards mathematics. In interviews they offer opinions on mathematics learning and teaching in general as well as their own experiences as mathematics learners. What do students report as positive in regard to the teaching? What do they report positive attitudes towards? All the issues raised by the students are about their teacher or the organising of the teaching.

For the students in the third year (3MX and 3MY) keeping their second year mathematics teacher is reported as positive. They are familiar with this teacher, and they have clear expectations as to how their school year will be. For instance they know something about what they can expect of help, about what tests will look like, to what extent their teacher will be supportive and connected matters.

This last point is also of great importance to the students that are in their second year. They report that their teacher is important for their learning. Some of them go into such issues as the teachers' competence as a mathematician. The students in all four groups describe their teachers as being very able. A couple of the students even characterise their teacher as a "genius".

In their descriptions of the teachers the students describe them as caring persons. Caring in this context could mean both about students' socially well being and about students' learning. In the students' descriptions the teachers employ these qualities to a different extent, but they sincerely see their teachers as caring.

This is probably the reason why students' quotes tell that they (the students) conceive their teachers as important for them in order to feel good about school. All students interviewed have reported that they feel comfortable about being in the mathematics classroom, and that the opportunity to ask the teacher questions is one very important part of this. All teachers allow their students to ask questions during blackboard demonstrations. What we observe is that while in some classrooms questions are raised frequently, in other classrooms students rarely ask why and how[6].

Some students tell that they do not feel comfortable asking the teacher for further explanations in front of the whole group. Instead they raise questions during individual or group work. All teachers dedicate at least half their lessons to work on items or problems. During this time they move about in the classroom approaching students at student requests or to check on students. The two female teachers and one of the male

[6] One explanation to this might lie in the nature of the answers provided by the teacher, but this material is not sufficiently analysed to draw such a conclusion at this stage.

teachers will regularly check on all their students during a period of independent work. The last male teacher teaches a group consisting of only six students. In this classroom the teacher and the students appear as a learning community together. Except during blackboard demonstrations the teacher sits in the middle of the room and participates in a group discussion with the students when they work on exercises. Students will ask both fellow students and the teacher for help when needed.

In all groups students are free to approach other students with questions, or to work in pairs or groups of three. For some students this is comforting, especially when given the opportunity to choose themselves among their classmates exactly who they want to work with. All teachers are flexible on the organisation of students' seating, even though one of the teachers have chosen to split up a group that was more off task than on.

Students feel secure about asking their peers. Some report that they feel more comfortable asking a peer for an explanation since he or she is likely to have the same kind of questions and because there will probably be similarities among the students in regard to reasoning about the mathematical content. Several students claim that their teacher knows too much mathematics to recognise the problems the students face in trying to make sense. When working with peers they are given the opportunity to discuss and reason on equivalent levels.

Among the students working in pairs and the reasons they give for choosing partners, the most frequent argument for their specific choice is to work with a friend. Again students make choices that make them feel secure.

A few students report that the grouping is accidental, an effect of where they found a free seat on their first day in the autumn term. What is crucial for going on working together is that they feel they are on a comparable level, making it possible to discuss troublesome items.

Only one pair, two girls, report that they have actively chosen each other for pedagogical reasons. These students also find time to work together after school to work on exercises or to prepare for tests, exams or just the next period. These girls organise their work as a discussion oriented co-work with equal responsibilities to offer explanations. They also feel free to pose questions. The line of work after the common reasoning may take two paths: Sometimes they work out solutions on exercises on their own before coming together again, other times they work out the solutions together. These girls also report that they through their work discover alternative algorithms to what the teacher and the teaching offers.

For several of the other students working in pairs working together means working individually until you face something you can not solve on your own. Then you ask your peer. Other students prefer to work individually all the time and direct their questioning to the teacher when they need help or just a hint. What we suggest is that the opportunity to choose for yourself how to work satisfies the needs of a large quantity of the students and help provide a positive class climate.

Girls and mathematics – where do we go next?

In order to work towards the aims of the project, further information on some aspects of mathematics teaching needs to be collected. Deeper insight into both student and teacher attitudes towards testing is required. Some interviews have been conducted where students were asked to reflect upon different aspects of testing and alternatives to written tests and exams. Still findings in transcripts provide more questions than answers.

Future interviews will focus on

- assessment format
- attitudes and expectations towards testing
- self evaluation
- alternatives to traditional written tests
- teaching aides

In order to collect information about some of these issues we will conduct student and teacher interviews on the "Red" school as well as the Blue and the Green. The Red school is participating in the extension of an OECD project titled "Assessment as a link between instruction and learning in mathematics"[7]. As a part of this project all students write their own "book of rules" where they are free to write examples, descriptions, make drawings, explanations, rules etc of their own choice. Students are encouraged to bring this book to tests and exams and use it as an active aid. Will these students' attitudes towards assessment (written tests) differ from the attitudes described by the students in the Blue and Green schools? Do the students in the Red school give other reasons for choosing mathematics in the first place?

At the Blue and the Green school we would like to question students and teachers as to what they consider important in order to increase recruitment in mathematics. An interesting question in this context is as to while the girls already doing mathematics mainly feel good about their choice, why are they so few?

Some of these answers might be found in the explanations of students dropping out of mathematics. So far two girls and one boy have dropped out of the groups we are observing. These students will be interviewed. Also students in their second year not continuing with mathematics in their third year will be questioned on this matter before school ends in June.

It is also in our intentions to interview school counsellors at the Blue and Green schools to see what they do to recruit students. Are actions schools take in order to recruit students the same actions as teachers and students suggest? What information do counsellors report that students ask for? What information can counsellors provide?

So far the observation and interviews cumulate more questions and a few answers to the original research questions. Clearly in addition to more observation and interviews more searches through research literature will be helpful to suggest actions to made to increase student recruitment and level out gender differences in the Norwegian upper secondary school. A further review of research literature and a linkage between literature and observations are also part of future plans. A full project report will be written in July 2000.

[7] Readers can find a description of the OECD project in "Changing the subject" edited by Black and Myron Atkin (1996, p 211).

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A study of students' ways of experiencing ratio and proportion

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Introduction

At the University of Gothenburg there was a one-year preparatory course for students who planned to attend the teacher-training programme in mathematics and natural science. These students joined a seven-week course in mathematics.

Proportion is considered to be one of the most important concepts in mathematics and is useful also in several other subjects. Due to the high applicability it is probably especially important for teachers to understand this concept. Ratio is linked to proportion and therefore inevitable to avoid in this approach. It became natural to examine the students' apprehension of both ratio and proportion.

Aim

The aim of the study is to try to understand ways of experiencing ratio and proportion for a certain group of students. The focus is on how this group of students apprehend the concepts of ratio and proportion. It is important to point out that my ambition has not been to look for every way of apprehending these concepts. Marton and Booth (1997) have shown that even small groups of subjects present a variation of apprehension.

Background

Extensive and Intensive Quantities

In modern research within the topic of problem solving it is common to distinguish between extensive and intensive quantities. (Harel & Confrey, 1994)
An extensive quantity consists of a number and a referent. The referent is the measure in some unit or element in a discrete set of objects. The number is multiplied with the referent or "unit".

The ratio of two extensive quantities can make an intensive quantity. An intensive quantity can be a ratio in different forms, for instance "2 parts of concentrated lemonade for 5 parts of water". But also scale and unit conversion factors are examples of intensive quantities. A lot of things can be described by the expression "x per y".

Homogeneity

The concept of homogeneity refers to the character of the data. If you for instance have a series of data of how the price of a specific thing depends on the weight and when every ratio of the price and the quantity is constant, homogeneity is prevailing.

Price	Amount	Ratio
$2.50	2 Kg	1.25
$3.75	3 Kg	1.25
$5.00	4 Kg	1.25
$10.00	8 Kg	1.25

It is therefore possible to conclude that 12 Kg cost $15.00. The price is not constant, however, because when you buy a bigger amount of something the price per unit often goes down.

Lybeck's Study

Lybeck (1980) studied students' ways of experiencing proportion. He used the volume of water in one experiment and weight (or power) and length in another.

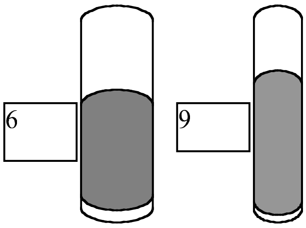


Figure 1

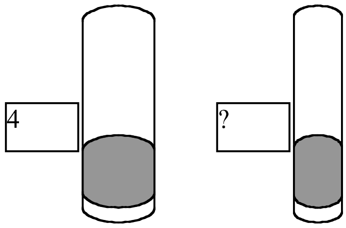
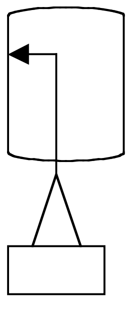


Figure 2

The first experimental situation consisted of two different cylinders of water. There was a scale on each cylinder. The diameter of them was different. In the first case there was water in the first cylinder up to 6 on the scale. It was then poured into the second cylinder and rose to 9. The water was then poured out and new water put in. This time it was up to 4 on the scale in the first cylinder. If we now pour the water from the first cylinder into the second cylinder, how high would it be on the scale?

In the second experiment missing value problem was also used. An object was weighed on a spring-balance. You could check on a scale the weight of the object.



The following table was presented to the subjects:

Length	Mass
4 mm	20 g
10 mm	50 g
20 mm	100 g
50 mm	250 g

Every weight was converted to a corresponding length on the spring balance. What would the weight of an object with the length of four on the scale be?

Both problems led to proportional reasoning. Several different ways of reasoning were found. The outcome space was two-dimensional with two main categories:

- *A-form*, a so-called *within-measure* comparison, and
- *B-form*, an *across-measure* comparison.

Each of the main categories had four subcategories, which referred to more or less implicit forms of reasoning.

In the first subcategory a way of more direct proportional reasoning was found. It could be expressed as a ratio or as a constant multiplied by one of the variables:

$y = 1{,}5 \cdot x$ or $\frac{y}{x} = \frac{3}{2}$

The second subcategory was a proportional increase or a decrease expressed as an addition or subtraction of a ratio or a factor like

$y = x + \frac{1}{2} \cdot x$ or $y - x = \frac{1}{2} \cdot x$

In the third subcategory the subjects realise that the increase or decrease could not be absolute but they are, however, unable to quantify it.

In the fourth subcategory the increase or the decrease is seen as absolute:

$y = x + 3$ or $y - x = 3$

In all the subcategories the expressions could be seen in an inverted way.

The concept of homogeneity is implicitly hidden in the experimental situation and therefore not dealt with at all.

Lybeck's study was a further development of the research done by Karplus et al (1975).

Kaput and West's Study

Kaput and West found four different, but not completely different, patterns of reasoning, namely the *build-up strategy*, the *abbreviated build-up strategy*, the *unit-factor approach* and the *formal equation-based approach*.

The approach of the study seems to be procedural, as procedure is focused and not apprehension.

Build-up strategy

The build-up strategy is a type of co-ordinated increment of quantities by the following problem from Kaput and West:

> *A restaurant sets tables by putting seven pieces of silverware and four pieces of china on each placemat. If it used thirty-five pieces of silver-ware in its table settings last night, how many pieces of china did it use?*

Here we have two quantities that is to say "number of silver" and "number of china", which increment by different numbers, seven and four respectively. The solution of the problem is built up gradually by incrementing the two different quantities.

> *For seven silver, there is four china;*
> *For fourteen silver there is eight china;*
> *For twenty-one silver there is twelve china;*
> *For twenty-eight silver there is sixteen china;*
> *For thirty-five silver, there is twenty china.* (Kaput & West, 1994)

Abbreviated build-up strategy

In the abbreviated build-up process a structure of division and multiplication is prevailing.

In the above-mentioned problem you first examine how many placemats that were laid:

$$\frac{35\ silverware}{7\ silverware\ /\ placemat} = 5\,placemats$$

This is of cause the division structure. It will be followed by a multiplicative structure.

$$5\,placemats \bullet 4\ china\ /\ placemat = 20\ china$$

This two-step approach, division and multiplication, is a more efficient way of reasoning. It is therefore termed an abbreviated build-up strategy. (Kaput & West Maxwell,1994)

The unit-factor approach

This approach is usually used in problems with continuous variables. Kaput and West therefore use another problem called the "Italian dressing problem":

"To make Italian dressing you need four parts of vinegar for nine parts of oil. How much vinegar do you need for 16 decilitres of oil?"

First it is necessary to find the amount of vinegar per parts of oil.

$$\frac{4\ vinegar}{9\ oil} = 0{,}44\ vinegar\,/\,oil$$

Then how much vinegar do you need for 16 decilitres of oil? For one decilitre of oil you need 0,44 decilitre of vinegar, for 16 decilitres you need

$$16\ dl \bullet 0{,}44\ dl\ vinegar\,/\,dl\ oil = 7{,}04\ dl$$

By using the approach to find the amount per unit you make it possible to scale up to the requested amount. It is therefore termed the unit-factor approach. (Kaput & West Maxwell, 1994)

The formal equation-based approach

In this approach an ordinary equation is used. This means according to Kaput and West that the subject does not fully need to understand the actual problem and the different concepts used. They actually claim that it is easier to solve the problem by this approach.

$$\frac{7\ silverware}{35\ silverware} = \frac{4\ china}{X\ china}$$

The approach is of course termed due to the use of equations. (Kaput & West Maxwell, 1994)

Lamon's Study

Lamon (1994) studied twenty-four sixth-grade children's pre-instructional thinking. She used missing value problems and found that students have a lot of pre-instructional thinking, but they were not using the formal symbols in an equally developed way. In the balloons-problem the students were asked the following:

Ellen, Jim and Steve bought 3 helium filled balloons and paid \$2.00 for all three. They decided to go back to the store and get enough balloons for everyone in their class. How much did they have to pay for 24 balloons?

Lamon found four different ways in which the students solved the problem.

- The subjects grouped the balloons in groups or sets of three and then stated that you need eight such groups in order to get all twenty-four. Eight times two make sixteen.
- They used the build-up strategy.
- The subjects first calculated the price per unit. Then they multiplied by the total number. This is the unit-factor approach.
- Three balloons were \$2.00 and 2 divided by 3 is $^{2}/_{3}$. How many 24ths is $^{2}/_{3}$? It is $^{16}/_{24}$ and therefore the answer is 16.

The last approach is a within-strategy. The scaling factor in the first measure space was calculated and then it was applied in the second measure space. The majority of the children used the first strategy.

The subscription problem:

$$y = x + 3 \quad \text{or} \quad y - x = 3$$

The student was shown a card in a magazine that offers three plans for subscribing the magazine. (1) You may subscribe for a 6-month period, during which time you will receive three bills each for the amount of \$4.00. (2) You may subscribe for a 9-month period, during which time you will receive three bills each for the amount of \$6.00, each. (3) You may subscribe for a 12-month period, during which time you will receive three bills each for the amount of \$8.00, each. Do you get a cheaper rate if you buy the magazine for a longer period of time?

Only thirteen students of twenty-four were successful. Six different strategies were found. The two predominant were:

- Compare the price for 12 months with the cost for two six-month periods and then the 9-month period is in the middle.
- Group the months' periods in sets of 3 months. For a 6-month period the price per 3 months is the same as for each 3-month period in the 9- and 12-month periods.

Fifteen out of twenty-four students solved the next problem.

The apartment problem:

In a certain town, the demand for rental units was analysed and it was determined that to meet the communities' needs, builders would be required to build units in the following way: every time they build three single units, they should build four two-bedroom units and one three-bed unit. Suppose a builder is planning to build a large apartment complex containing between thirty and forty units. How many units should be built to meet this regulation? Suppose one built 32/40 (choose one). How many one-bedroom units, two-bedroom units and three-bedroom units would the apartment building contain?

Eleven students used a unitising approach.

The unit was:
1-bedroom apartments
2-bedroom apartments
3-bedroom apartments

The units of units were:
Three units of 1-bedroom apartments
Four units of 2-bedroom apartments
One unit of 3-bedroom apartments

The units of units of units were:
Those units of apartments together form one eight unit of apartments.
The units of units of units of units were:
How many 8 units are needed?
4 or 5 of that 8 unit.

Lamon concludes that unitising and norming processes play an important roll in developing more advanced strategies for proportional reasoning. (Lamon, 1994)

Method

Approach

The basic assumption in this study is mainly post-positivistic. The ontological assumption is that the world is real and possible to observe, however not fully objectively. Yet it is necessary to strive for as high a degree of objectivity as possible.

Epistemologically knowledge is seen as non-falsified hypotheses. These hypotheses could be considered as facts or probable laws. (Guba & Lincoln, 1994) In this paradigm the focus is not on experimental arrangements, but rather on studying reality as it is, in this case how students solve maths problems. Apprehension is not possible to study directly, since it is not possible to study what is going on inside their heads.

Both quantitative and qualitative data have been used. Quantitative data were the result of an inquiry test and qualitative data were transcribed interviews.

Pilot Study

First a pilot study was conducted. In order to examine the students' knowledge or ability in ratio and proportion they were given a diagnostic test. There were indications from the test that some problems were severe. The problems used were standard "missing value proportional reasoning word problems". This is a particular case of "reasoning in a system of two variables between which exist a linear functional relationship". (Karplus, Pulos & Stage, 1983, p. 219)

The Interview Study

In-depth interviews were then carried out with ten students and the interviews were tape-recorded. Having summarised the recordings, a qualitative analysis was applied to the data. In "The Discovery of Grounded Theory" Glaser and Strauss (1967) describe how a theory can be generated directly from the data. The theory in this case is not generated by the testing of hypothesis but is successively growing from the data. This growth takes place step by step according to a certain ritual. The first step is to conceptualise the data, a process called "open coding". Different ways of reasoning are attributed to a concept.

In order to sharpen the limits of every concept a comparative analysis and a dimensional analysis are used.

The concepts are then arranged into categories due to their character. The categories are then compared in order to further define the category-limits. By use of the comparative and the dimensional analysis relations between the categories can be established.

The theory, which in this way is developed out of the data, is called *grounded theory.*

Theoretical sampling means that the theory that is successively developed out of the data is also the basis for the sample of subjects or for new questions that should be put. In this case the interviews were made in intervals. In the meantime analyses were carried out which led to changes of the sample. Special attention was also paid to the way the subjects reasoned about the concept of "parts".

Theoretical saturation means that after collecting a certain amount of data, new data will not develop the theory further, since saturation is reached.

Problems

The tasks given the students to solve were the following two "missing value problems".

The lemonade problem:

> *You are about to make lemonade. On the bottle you read: 2 parts of concentrated lemonade and 5 parts of water. You wish to get 8 litres of mixed lemonade. What amount of concentrated lemonade should you add?*

The vinegar problem:

> *We are going to mix vinegar, 2 parts of vinegar and 5 parts of water. We have 8 litres of water. How much vinegar shall you add?*

Results

Three qualitative different categories were found namely "Explicit Proportionality", "Implicit Proportionality" and "Absolute Proportionality".

Explicit Proportionality

The concept of "parts" was expressed as a ratio for *all* quantities. This ratio was also possible to apply to different quantities and not only to a specific quantity. One of the students, Peter said when trying to solve the lemonade problem:

That $\frac{2}{7}$ *of 8 litres is the concentrated lemonade*

Another student, Anna said:

There are 2 parts of lemonade in 8 litres.
Two sevenths should be concentrated lemonade of 8 litres.
Then it makes

$$\frac{2}{7}\cdot 8=\frac{16}{7}=2\frac{2}{7}\ \text{litres}$$

Sven claimed that: *Two sevenths of eight litres* $\frac{2}{7}\cdot 8=\frac{16}{56}$

You can also take a proportional percentage multiplied by 8 litres.

This way of experiencing ratio or proportion is usually considered to be the most developed category. Lybeck (1980) termed it A-forms. This is also known as “within approach”. Ratio is composed of quantities within the same dimension or “unit”.

B-forms or the “between approach” are also represented in this category. In this case the ration is composed of quantities of different dimensions or “units”. When Kate solved the lemonade problem, she said:

8 divided by 7.

Concentrated lemonade $\frac{2 \cdot 8}{7}$

Water $\frac{5 \cdot 8}{7}$

Notice that eight has the unit “litres” and seven “parts”. Kate used the same approach for the vinegar problem. She answered immediately:

$\frac{2 \cdot 4}{5}$ = 1.6 litres.

Implicit Proportionality

The students representing this apprehension did not operate with ratio at all.
When attacking the lemonade problem Per said:

2 dl and 5 dl make only 7 dl

He argued in the following way:

2 parts make 5 litres
3 parts make 6 litres
1 part makes….

Per transformed “parts” into litres and then operated in this dimension not scaling it properly to reach other quantities. Per seemed unable to deal with quantities of more than one dimension.

At first Lena stated that she ought to get a relation for 1 litre and then multiply by 8.

2 dl concentrated lemonade and 5 dl water makes 7 dl mixed lemonade.

Then she tried to multiply with some number in order to get as close to 8 as possible. She found 11 and got 77dl that was the closest she got to 8 litres. She concluded.

This means 22 dl concentrated lemonade.

Absolute Proportion

In this category the concept of proportion was used as if the “parts” did not express anything about other quantities than one litre.

The students changed the “parts” in order to describe proportions for other quantities. This inability not being able to transfer the information in the “parts” to other quantities into other dimensions is characteristic to this category. This is termed “absolute proportion”(my definition) due to the fact that the students operated only in “parts”, not in any other dimension.

Kim concluded the following:

That the lemonade is 16 parts and the water is 40 parts.

Roger on the lemonade problem:

1 litre to be divided into 7 parts

but he did not seem to complete this way of reasoning. Instead he said:

$2 \cdot 8 = 16$ *parts*

On the Vinegar Problem Roger said:

$2 \cdot 4 = 8$ *parts*

But then he switched approach and continued:

1 litre with 7 parts makes a certain number (x).

$2 \cdot x + 5 \cdot x$ *make* 4 *litres*

This is the only equation-based approach in the group. (Kaput & West Maxwell, 1994)

Discussion

The aim was to try to find out the students' ways of experiencing ratio and proportion.

This aim was fulfilled. Three different categories were found and several subcategories. The first category contained the sub-categories, a "within" subcategory and a "between" subcategory. It seemed to me that due to the experimental situation or the character of the problems, the two other categories, "implicit proportionality" and "absolute proportionality", emerged.

A majority of approaches in these categories would be classified as "wrong" or not successful. These categories seemed new but could, however, depend on the interpretation of the data. My analysis is of course only an attempt to interpret these data.

Lybeck's main categories A-form and B-form were found in the category of "explicit proportionality". The character of his experiments seems to hide the other two categories. Probably due to the concepts of "parts" and the relativity of that concept, the two other categories, "implicit proportionality" and "absolute proportionality" are exposed. It seems natural to conclude that these categories are more or less linked to the concepts of "parts".

Of Kaput and West's four approaches, "the build-up strategy", "the abbreviated build-up strategy", "the unit-factor approach" and "the formal equation-based approach", only three were found.

It seems that Kaput and West could have had a different epistemological approach than I have had in this paper. They focus on approaches. They seem to have a more procedural focus. With a focus on approaches it is possible to conclude that three of their approaches were found.

The build-up approach could be identified in one of the examples in the category of "implicit proportionality". Per argued in the following way:

2 parts make 5 litres, 3 parts make 6 litres, 1 part makes….

This is clearly the "build-up approach".

The "abbreviated build-up strategy" was also found. When Kate solved the lemonade problem, she said:

8 divided by 7.

Concentrated lemonade $\frac{2 \cdot 8}{7}$

Water $\frac{5 \cdot 8}{7}$

The "unit-factor approach" was not found.

The "formal equation-based approach" was represented by one student, Roger, who first started with another approach and then switched strategy and said:

1 litre with 7 parts makes a certain number (x).

$2 \cdot x + 5 \cdot x$ *make* 4 *litres*

The use of an equation is evident.

The result from Lamon's investigation shows that unitising and norming processes play an important roll in developing more advanced strategies for proportional reasoning. (Lamon, 1994). This is also found in my investigation but not to a large extent. In the "explicit proportionality" category several students create a ratio like "two sevenths". For instance several students also try to find out how many litres there are per part. This seems to be congruent with as the "abbreviated build-up strategy".

Part of the result seems to be similar to that found earlier. What could be new in this paper is the category of "absolute proportionality" in which the subjects operate with "parts" all the time. It is also possible that in the category "implicit proportionality" the conversion from parts to litres in order to avoid operating in two different quantities is a new strategy.

In future research it would be interesting to focus on students who have not successfully solved the problems. This is of course not a new approach but if one focuses the aspects of relativity of the concept of "parts" it would still be interesting.

Educational Implications

Ratio and proportion should be devoted bigger attention in class. It seems to be necessary to apply proportion to several different quantities more or less simultaneously. It is also necessary to use exercises with quantities of at least two different dimensions. The key issue seems to be an inability to relate quantities of different dimensions to each other.

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Rationalitet och intersubjektivitet – några preliminära utgångspunkter i ett försök att förstå matematikundervisningens kommunikativa karaktär

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Inledning

> Mathematik ist keine Menge von Wissen.
> Mathematik ist eine Tätigkeit, eine Verhaltensweise,
> eine Geistesverfassung.
> Hans Freudenthal, 1982

I det inledande avsnittet ges utifrån en bestämning av matematiken en bakgrund till de frågor om *rationalitet* och *intersubjektivitet* i matematikklassrummet som denna uppsats avser att diskutera. Avsikten är att utarbeta en teoretisk begreppsram för empiriska studier av klassrumsinteraktion. Jag kommer att relatera matematikundervisningen till upplysningens idéer om bildning och medborgarskap. Det är en vidareutveckling av vissa tankar som redan fanns i mitt avhandlingsarbete (Engström, 1997).

Matematik

Matematik spelar en viktig roll i vårt samhälle. Den kan användas för att studera och analysera samband i omvärlden. Den är också en av våra äldsta vetenskaper. I modern naturvetenskaplig forskning är matematik en viktig hjälpvetenskap. Förståelsen av avancerade modeller som ofta används inom den naturvetenskapliga forskningen förutsätter mycket goda kunskaper i matematik. Detta nära samband mellan matematik och naturvetenskaperna får dock inte förleda oss att tro att matematik är naturvetenskap.

Naturvetenskaperna syftar till att beskriva och studera företeelser i vår fysiska omvärld. Naturvetenskaperna uppställer teorier vars sanning och förklaringsvärde visas genom experiment. Naturvetenskapliga teorier bevisas inte genom detta, utan de görs mer eller mindre troliga. Det finns dock alltid en möjlighet att modellen eller teorin inte gäller, att man via experiment får resultat som strider mot modellens förutsägelser.

Matematik däremot är en mänsklig konstruktion, där man utifrån ett litet antal givna förutsättningar, s.k. axiom, kan *bevisa* att olika utsagor är sanna. Till skillnad från många andra vetenskaper, t.ex. samhällsvetenskaperna, som ofta är flerparadigmatiska, går det inom matematiken att avgöra om ett resonemang är riktigt eller felaktigt utifrån givna villkor. Däremot kan man vara oense om huruvida en viss metod är att föredra framför en annan, eller hur ett visst problem ska formuleras och om hur ett visst resultat ska tolkas.

Det är, som Damerow (1996) betonar, en väsentlig skillnad mellan att undervisa i matematik och i t.ex. biologi. När en lärare i biologi demonstrerar någonting på en växt så försöker han/hon lära ut något om växter. En matematiklärare däremot som demonstrerar ett begrepp på verkliga objekt avser inte att arbeta med verkliga objekt utan försöker kommunicera en mental konstruktion som inte har någon motsvarighet i den verkliga världen. Ett träd på klassrumstavlan är en ikonisk modell av ett verkligt träd, medan en triangel ritad på tavlan är en modell av en *abstrakt idé*. Lärandet i matematik bygger på både *abstraherande* och *generaliserande* processer.

Matematik är ytterst ett medel för den enskilde att beskriva och analysera omvärlden och sin egen situation. Förmågan att använda matematik är av samma karaktär som förmågan att använda det egna språket (Högskoleverket, 1998). Uppfattad på detta vis ges matematikutbildningen en humanistisk dimension där medborgarskap och bildning ses som viktiga aspekter av matematiken.

Matematik brukar uppfattas som universell och logiskt nödvändig. Den amerikanske matematikern Benjamin Peirce[1] definierade matematik som ”science of drawing necessary conclusions” (citerat i Steen, 1999, s. 270). Jag kontrasterar min egen uppfattning mot dels platonism[2], dvs. att denna universalism skulle äga en motsvarighet i en idévärld, dels mot social idealism, att matematiken är situerad i den meningen att exempelvis vad vi uppfattar som logik är avhängigt sociala omständligheter. Samhället spelar en roll för hur vi kommer att uppfatta vad logik och förnuft är, men inte hur som helst, för som Piaget (1965) säger, det vore att förväxla förnuftet med ”statens förnuft”.[3] Låt mig tillbakavisa den sociala idealismen med den amerikanske filosofen Mark Bickhards ord:

> If all knowledge is just whatever society says it is, why don’t we agitate or persuade society to simplify mathematics? Wouldn’t our rhetoric be better directed in that way? Wouldn’t our world be much simpler if pi (π) simply equalated the integer 3? (Bickhard, 1995, s. 257).

Piaget betonar inledningsvis i sin essä *Sociologiska förklaringar* (Piaget, 1965/1974) att den mänskliga kunskapen väsentligen är kollektiv och att det sociala livet utgör en av de viktigaste faktorerna i de förvetenskapliga och vetenskapliga kunskapernas uppkomst och tillväxt. Men vilka är sambanden mellan logik och det sociala livet? Jag kommer att diskutera detta nedan.

Låt oss jämföra *nio* i följande satser:

a) Runt vår sol rör sig *nio* planeter.
b) Tre gånger tre är *nio*.

Till skillnad från i sats a är nio en nödvändighet i sats b. Vad är det som gör denna nödvändighet? Beror det på konformitet eller är det bara konventioner? Det är ingendera. Matematiska sanningar är logiskt sanna och barn utvecklar en förståelse för sådana sanningar. Men på vilka grunder?

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[1] Benjamin Peirce var professor i matematik och astronomi vid Harvard och far till Charles S. Peirce.

[2] Det finns enligt platonismen en unikt korrekt matematik. Så uppfattades Euklides geometri länge som själva grundvalen för allt vetande om rumsliga förhållanden. Under 1800-talet uppkom ett antal s.k. icke-euklidiska geometrier. Vid sekelskiftet upptäcktes ett antal paradoxer inom mängdläran, vilken hade utvecklats av Cantor i slutet av 1800-talet. Tillsammans kom de att i grunden förändra synen på matematik. Platonismen är inkorrekt och oförenlig med matematikens senare landvinningar att det existerar skilda, men lika giltiga former av talteori, algebra, topologi – beroende på vilket mängdbegrepp som används. Eftersom varje sådant mängdbegrepp är matematiskt lika giltigt så existerar ingen unikt korrekt matematik. Platonismen är därför falsk.

[3] Il est clair, en effet, que n’importe quelle action de «la société» sur l’individu n’est pas source de raison, sans quoi celle-ci se confondrait trop souvent avec la «raison d’Etat» (Piaget, 1965 s. 146).

> The growth of knowledge is always something undertaken by a subject where the function of that knowledge is to build up a viable set of presentations, including re-presentations over time. The child who believes that 7 + 2 = 9 but that 2 + 7 = 5 has the task <u>not</u> of checking out one of these beliefs against an otherwise mysterious reality but rather of constructing a self-consistent system of knowledge (Smith, 1993, s. 509).

Nödvändig kunskap är universalia. Den kunskap som en individ har utvecklat är inte en personlig egendom. Alla kan utveckla och därmed ”äga” kunskapen 7 + 5 = 12. Nödvändig kunskap är självidentisk, dvs. det är samma kunskap som förvärvas av alla dem som förvärvar den. Nödvändig kunskap är sann i alla möjliga världar (se fotnot 13). Man kan också, som Wittgenstein hävdar, säga att logiken avspeglar vårt sätt att tänka.

> 131. Die logischen Gesetze sind allerdings der Ausdruck von ›Denkgewohn-heiten‹, aber auch von der Gewohnheit zu denken. D. h., man kan sagen, sie zeigten: wie Menschen denken und auch, was Menschen »denken« nennen (Wittgenstein, 1956/1994, s. 89).
>
> …
>
> 133. Die Sätze der Logik sind ›Denkgesetze‹, ›Weil sie das Wesens des menschlichen Denkens zum Ausdruck bringen‹ – richtiger aber: weil sie das Wesen, die Technik des Denkens zum Ausdruck bringen, oder zeigen. Sie zeigen, was das Denken ist, und auch Arten des Denkens (Wittgenstein, 1956/1994, s 90).

Under senare år har *reasoning*, ett lite svåröversatt ord i detta sammanhang, fått en allt större uppmärksamhet i den internationella matematikdidaktiska diskussionen.[4] Det handlar om det för matematiken så centrala – att resonera, att argumentera, göra slutledningar och leda något i bevis – med andra ord om att kommunicera det som vi har möjlighet att veta något om. Det är intressant att notera att det amerikanska matematiklärarsällskapet, NCTM, i sitt läroplansarbete, *Standards 2000*, har fört in *reasoning* som ett viktigt moment i matematikutbildningen från förskolan och uppåt. NCTMs senaste årsbok (Stiff, 1999) har just frågan om hur man utvecklar *reasoning* i matematikundervisningen som tema. *Reasoning* ges här en betydligt vidare betydelse än den traditionellt har för matematiker (deduktion och formell bevisföring). Steen (1999) varnar för att okritiskt anamma *reasoning* som slagordet för 2000-talets matematikutbildning och gör paralleller till tidigare satsningar på problemlösning, back-to-basics etc. Jag menar dock att det finns goda skäl att uppmärksamma *reasoning* i den vidare innebörden. Dess tydliga kommunikativa karaktär reser frågor om förhållandet mellan *rationalitet* och *intersubjektivitet*. Maher (1998) ger ett illustrativt exempel på hur man kan arbeta med att resonera och argumentera i de lägre årskurserna i grundskolan som är taget från hennes egen klassrumsforskning.

Piaget diskuterar i essän *Les opérations logiques et la vie sociale* (Piaget, 1965/1995) logiken utifrån de skilda klassiska sociologiska utgångspunkterna hos Tardes sociologiska individualism och Durkheims sociologiska holism. Han pläderar för ett *tertium quid*:

[4] Se t.ex. Drouhard et al (1999), Krummheuer (1999) och Wood (1999).

> If logical progress goes hand in hand with progress in socialization, is it because the child becomes capable of rational operations due to the fact that social development makes him capable of cooperation; or, on the contrary, is it because his individual logical acquisitions permit him to understand other people and thus lead to cooperation? Since the two sorts of progress go completely hand in hand, the question seems to have no solution except to say that they constitute two indissociable aspects of a single reality that is at once social and individual (Piaget, 1965/1995, s 145).

Hur matematiken uppfattas har viktiga implikationer för undervisningspraktiken. Även om denna aspekt inte är i fokus för detta paper så låt mig ändå få beröra detta med en hänvisning till den tyske matematikdidaktikern Heinrich Bauersfeld som menar att *fundamentalt olika undervisningspraktiker* uppkommer om matematiken uppfattas som en objektiv sanning, en samhällelig skatt, något existerande, eller som *en praktik av gemensam matematisering*, som styrs av de regler och överenskommelser som uppkommer ur denna praktik.

> The first conviction will lead teachers to "introduce" children, to use "embodiments" and "representations", which are structurally as "near to the structure mathematically meant" and as little misleading or distracting as possible. Children's errors will find corrections toward the expected correct answer and so forth. Objectively existing structures and properties also give space for "discovery" activities, given that the expected findings are in reach of the present cognitive aptitudes (e.g., "zone of proximal development").
>
> The latter conviction will lead teachers to organize their activities as part of a practice of mathematizing, that is, as a challenging and supportive "subculture" specific to <u>this</u> teacher and <u>these</u> children in <u>this</u> classroom, which functions toward developing the students' "constructive abilities", their related self-concept, and self-organization, rather than as a management through product control and permanent external assessments. The diversity of subjective constructions of meaning and the necessity to arrive at viable adaptations – "taken-as-shared meanings" and "taken-as-shared regulations" – requires optimal chances for discussions based on intensive experiences and aiming at the negotiation of meanings. (Bauersfeld, 1993, s 140)

Därmed blir matematikens relation till upplysningens idéer om medborgarskap och bildning synliggjorda.[5] Jag vill knyta matematikutbildningen till upplysningens klassiska tankar om bildning som myndiggörande och medborgarskap i en demokrati. Jag ställer detta i bjärt kontrast till de målrationella och individualistiska synsätt som idag breder ut sig i det allmänna skolväsendet.

Bildning

Borde vi återupprätta ordet bildning, frågade sig Gunnar Bergendal (1985), då rektor vid Lärarhögskolan i Malmö, för 15 år sedan – frågan känns lika aktuell idag – och fortsatte:

[5] I de nordiska länderna är det främst Stieg Mellin-Olsen (1987) och Ole Skovsmose (1994) som anlagt ett sociologiskt perspektiv på matematikutbildningen. I det senmoderna samhället med dess konvulsioner (migration, globalisering, etc) för nationalstaten vill jag hävda att det inte längre är möjligt att undandra matematikutbildningen dessa aspekter. Frågor som rör demokrati, medborgarskap och bildning är centrala för matematikutbildningen i det senmoderna samhället.

> Gå tillbaka till dess ursprungs källor – att bildas, formas. Då blir bildning det skeende i vilket min kunskap formas. Från livets början till dess slut. Kunskapen bildas i skärningen mellan min livsvandring och de traditioner jag möter, går in i, föds in i – därav kunskapens på en gång individuella och kollektiva karaktär. Och då är det så tydligt att god kunskap bildas bara i levande traditioner, med personligt ansvarstagande och personlig tolkning, och inom den helhet som människors villkor – politiska, ekonomiska, kulturella, ... – utgör (Bergendal, 1985, s 82).
> …
> För bildningen – i motsats till utbildningen – finns inte ett på förhand uppsatt mål, bildning är sökande, prövande, värderande. Bildningen tillhör livet självt, den berikas av erfarenheter, mot- och framgångar i familjen, skolan, arbetet, kort sagt i mångfalden av mänskliga gemenskaper (Bergendal, 1985, ss 82–83).

Situationen för skolmatematiken är bekymmersam. Allt för ofta leder matematikundervisningen i våra skolor till leda, ångest och utslagning bland eleverna. Det är nog få elever som upplever matematiken som frigörande. Kanske ligger en av orsakerna däri att vi försöker undervisa i ämnet som om det fanns en matematik fri från alla de kulturella och sociala sammanhang vi människor lever i? Bergendal (1985) pekar på att matematiken, alltsedan den antika grekiska kulturen, har burit på en spänning mellan den rena matematiken som en logisk struktur och den verklighet som vi mäter och påverkar. Man kan säga att med den icke-euklidiska geometrins framväxt i början av förra seklet lyfte matematiken från verkligheten och utvecklades till en rad abstrakta system som kunde fyllas med olika verklighetsinnehåll. I det växelspel mellan denna matematik och de exakta naturvetenskaperna har den utveckling skett som fört den mänskliga kunskapen till materiens innersta och universums yttersta. Men, inflikar Bergendal,

> … för kunskapen om människan och människans samhälle är vetenskapen matematik utomordentligt problematisk. Ty denna kunskap beror på mångtydigheten i språket och andra uttrycksformer och på att entydiga gränser mellan subjekt och objekt inte kan dras, medan matematiken bygger på sina begreppsbildningars entydighet (Bergendal, 1985, s 64).

I konflikten mellan formallogikens universella anspråk och människornas framhåller Bergendal att

> … en skolmatematik som står på människornas sida handlar om verkligheten genom människornas egna erfarenheter och är inbäddad i vardagsspråket och andra mänskliga uttrycksformer. Den har andra värderingar och meningsskapande sammanhang än den matematiska vetenskapen, bestämda av det vartill kunskapen skall användas (Bergendal, 1985, s 64).

Vid sidan av en formell skolmatematik har det naturligtvis alltid funnits en informell, en folkets, matematik. Folkliga räkne- och problemlösningsmetoder har funnits i de flesta yrken. Längs våra kuster har det byggts båtar och skepp allt sedan vikingatiden. Katedraler och sockenkyrkor har byggts runt om i landet sedan 1100-talet. Sådd och skörd, smide, snickeri och fiske – alla dessa verksamheter har inbegripit en folkets matematik som utvecklats utan formell skolgång. Det har varit en matematik som baserat sig på på en ingående kunskap om och handhavande av de hantverksprocesser, råmaterial och verktyg, som använts. Skolmatematiken har i stor utsträckning kommit att fjärma sig från denna informella och kontextbundna matematik.

Matematikklassrummets fenomenologi[6]

Som lärare slits vi ofta mellan elevens egen, ofta idiosynkratiska, förståelse av matematiken och vår egen uppfattning av ämnet baserad på en ”korrekt” förståelse av vad matematik är. Det kan vara svårt som lärare att skapa sig en mening eller innebörd i den matematik som eleverna ger uttryck för. Jag ska här diskutera hur en individ konfronteras och arbetar med matematiska idéer. I detta sammanhang ska matematiska idéer inte uppfattas som universella objekt utan de får sin mening på det sätt som en elev uppmärksammas på dem. Det innebär ett avvisande av den cartesianska dualismen mellan subjekt och objekt, mellan eleven och det matematiska objektet.

Fokus i min diskussion ligger på socio-kognitiva aspekter av lärandet. Avsikten är tudelad. Dels gäller det att försöka förstå hur en individ skapar sig en förståelse av den värld som framträder för henne/honom i klassrummet. Det är individens erfarenhet av världen, av matematikklassrummet, av den sociala interaktionen, som vägleder hennes/hans handlingar, snarare än yttre bestämda begrepp av matematiken i sig. Dels gäller det att förstå villkoren för hur lärandet i matematik, uppfattad som en kommunikativ rationalitet, kan befrämjas.

Det kan vara viktigt att här göra några avgränsningar. Jag avser inte att diskutera t.ex. Piagets centrala begrepp reflekterande abstraktion och konstruktiv generalisering, begreppsbildning i matematik, matematikens essens eller liknande frågor då dessa faller utanför ramen för denna uppsats.

Även om jag själv utgår från ett radikalkonstruktivistiskt perspektiv är det sannolikt så att de teoretiska utgångspunkter jag har, för att förstå rationalitet och intersubjektivitet, kan delas eller omfattas i olika utsträckning av andra ansatser. För en hänvisning till andra forskare som utifrån liknande utgångspunkter diskuterat förhållandet mellan radikalkonstruktivismen och interaktionismen se t.ex. Cobb & Bauersfeld (1995) och Bauersfeld (1998).

Grundläggande utgångspunkter

Här ska mina grundläggande utgångspunkter för en förståelse av rationalitet och intersubjektivitet i klassrummet diskuteras utifrån Jürgen Habermas teori om det kommunikativa handlandet, Jean Piagets sociologiska arbeten[7], Alfred Schütz sociala fenomenologiska teori om livsvärld, Ludwig Wittgensteins språkspelsteori och Charles S. Peirce teckenteori (semiotik). En sådan diskussion öppnar naturligtvis för kritiska anmärkningar om eklekticism. Men en sådan kritik vore att missförstå min avsikt. Jag

[6] Den ursprungliga idén till denna uppsats fick jag av läsningen av Tony Browns (1996) artikel *The phenomenology of the mathematics classroom* samt hans bok *Mathematics education and language* (Brown, 1997). Brown diskuterar där utifrån Alfred Schütz' sociala fenomenologi hur en individ erfar och skapar sig en matematisk förståelse utifrån ett livsvärldsperspektiv. Min framställning nedan av Schütz arbeten bygger i huvudsak på Browns båda arbeten. Även om jag i flera avseenden delar Browns perspektiv så har jag genom arbetets gång, bl.a. genom att ta upp centrala idéer hos framför allt Jürgen Habermas och Jean Piaget kommit att göra en något annorlunda läsning av fenomenologin än den Brown gör.

[7] Piagets sociologiska arbeten är av någon anledning inte särskilt uppmärksammade, främst kanske för att de flesta arbeten inte funnits tillgängliga annat än på franska. Piaget innehade under ett antal år bl.a. en stol i sociologi vid universitetet i Genève. Den tredje delen av hans *chef d'œuvre* (Piaget, 1950) ägnades bl.a. åt sociologin. *Études Sociologiques*, som är en samling essäer, utkom 1965 och i engelsk översättning först 1995. En svensk översättning av den inledande essän kom redan 1977! Det finns ett flertal mycket initierade genomgångar av Piagets sociologiska arbete, bl.a. Mays (1982), Chapman (1986, 1988), Kitchener (1991/1996) och DeVries (1997).

avser att i min diskussion behandla ett antal gemensamma beröringspunkter som jag funnit hos dessa författare. Det vidare arbetet kan komma att visa på oförenliga ansatser eller antaganden bakom dessa vilket jag då får ta ställning till.

Kommunikativ handling – Habermas och Piaget

Människors åtgärder och handlingar får sin mening och giltighet genom det tanke- och åsiktsutbyte som sker. För ett sådant samspel krävs en kommunikativ kompetens – en förmåga att granska tankegångar kritiskt utan öppen eller dold styrning av de maktintressen som samhällslivet genomsyras av. ”Mit dieser *kommunikativen Praxis* vergewissern sie sich zugleich ihres gemeinsamen Lebenszusammenhangs, der intersubjektiv geteilten *Lebenswelt* (Habermas, 1981/1997, s 32).

Min diskussion om de gemensamma utgångspunkterna hos Habermas och Piaget tar avstamp i Piagets essä *Sociologiska förklaringar*:

> Varje socialt samspel [*interaction sociale*, min anm.] framträder sålunda som om det manifesterade sig i form av regler, värden och tecken. Själva samhället utgör å andra sidan ett system av interaktioner som börjar med relationerna mellan individerna två och två och utsträckes ända till interaktionerna mellan var och en av dem och samtliga andra. Det utsträckes också till den påverkan som alla föregående individers handlingar, dvs handlingarna i alla historiska interaktioner, utövar på de som utförs av de aktuella individerna (Piaget, 1965/1974, s. 38).

Piagets inflytande på Habermas märks på flera sätt. Dels i direkta hänvisningar till och diskussioner av Piagets arbeten som i Habermas (1976, 1981a/1997a). Dels i de klara paralleller som finns mellan Habermas och Piagets syn på t.ex. kommunikation och deras distinktioner av sociala kriser (se t.ex. Chapman, 1986).[8]

När man talar om handlingars rationalitet avses vanligtvis en målrationell innebörd. Men detta är bara en aspekt utifrån vilket handlingar kan göras rationella, dvs. utföras mer eller mindre rationellt, menar Habermas. När man använder sig av termen rationell förutsätter vi ett nära samband mellan rationalitet och vetande, men [...] Rationalität hat weniger mit dem Haben von Erkenntniss als damit zu tun, wie sprach- und handlungsfähige Subjekte *Wissen erwerben und verwenden* (Habermas, 1981a/1997a, s 25).

Begreppet *kommunikativ rationalitet* går tillbaka till

> … die zentrale Erfahrung der zwanglos einigenden, konsensstiftenden Kraft argumenta-tiver Rede, in der verschiedene Teilnehmer ihre zunächst nur subjektiven Auffassungen überwinden und sich dank der Gemeinsamkeit vernünftig motivierter Überzeugungen gleichzeitig der Einheit der objektiven Welt und der Intersubjektivität ihres Lebenszu-sammenhangs vergewissern (Habermas, 1981a/1997a, s. 28).

[8] Habermas (1981/1997a) diskuterar, med hänvisning till Austin, kommunikationens dubbla struktur i språkbruket – dels i det *lokutionären*, dvs. det propositionella, sakinnehållet, i språkakten och dels en *illokutionären* komponent, där talaren fullföljer en handling genom det han säger. Det är genom detta senare som *Verständigung* kommer tillstånd, en process av ömsesidig förståelse. Här ska framför allt en jämförelse göras med Piagets essä *Les opérations logiques et la vie sociale*. Även om Habermas diskussion är betydligt mer utvecklad och utförligare är ändå parallellen slående. Detsamma gäller Habermas distinktion mellan ekonomiska kriser och legitimitetskriser. Se framför allt Piagets essä *Essai sur la théorie des valeurs qualitatives en sociologie statique («Synchronique»).*

Habermas hänvisar till Piagets begrepp *decentrering* och skriver att kognitiv utveckling allmänt innebär en decentrering av en egocentriskt präglad världsförståelse. Han för in begreppet livsvärld som ett korrelat till förståelseprocessen. ”Kommunikativ handelnde Subjekte verständigen sich stets im Horizont einer Lebenswelt” (Habermas, 1981a/1997a, s. 107).

> Wenn wir in dieser Weise Piagets Begriff der Dezentrierung als Leitfaden benützen, um den internen Zusammenhang zwischen den Strukturen eines Weltbildes, der Lebenswelt als dem Kontext von Verständigungsprozessen und den Möglichkeiten rationaler Lebensführung aufzuklären, stoßen wir wiederum auf den Begriff kommunikativer Rationalität. Dieser bezieht das dezentrierte Weltverständnis auf die Möglichkeit der diskursiven Einlösung kritisierbarer Geltungsansprüche (Habermas, 1981a/1997a, s. 110).

Habermas knyter vidare explicit an till bl.a. Wittgensteins språkspelsbegrepp i sin kommunikativa handlingsmodell.

> Der Begriff des kommunikativen Handelns setzt Sprache als Medium einer Art von Verständigungsprozessen voraus, in deren Verlauf die Teilnehmer, indem sie sich auf eine Welt beziehen, gegenseitig Geltungsansprüche erheben, die akzeptiert und bestritten werden können (Habermas, 1981a/1997a, s.148).

Med ordet *Verständigung* (förståelse) avses inte ett tillstånd utan en process av gradvis *ökande uppskattning av varandras utgångspunkter* (se Israel, 1999). När skiljaktiga meningar klaras ut kan man nå samförstånd (*Einverständnis*). Kommunikation och handling ska inte likställas Språket är ett kommunikationsmedel som tjänar förståelsen, medan aktörerna genom att göra sig förstådda (*verständigen*) samordnar sina handlingar för att uppnå bestämda mål. Kommunikativa handlingar är komplexa därigenom att

> … die gleichzeitig einen propositionalen Gehalt, das Angebot einer interpersonalen Beziehung und eine Sprecherintention ausdrücken” (Habermas, 1981a/1997a, s. 143).

Samverkan (*coopération*), skriver Piaget,

> … is the source of three sorts of transformation in indvidual thinking, all three of which are of nature to permit individuals to have greater consciousness of reason immanent in all intellectual activity.
>
> In the first place, cooperation is a source of reflection and of self-consciousness. On this point, it marks an inversion of meaning, not only in relation to specifically individual sensory-motor intelligence, but also in relation to social authority, which engenders coercive belief and not true deliberation.
>
> In the second place, cooperation dissociates the subjective from the objective. It is thus a source of objectivity, and rectifies immediate experience into scientific experience, whereas constraint is limited to consolidating the former by simply promoting egocentrism to the rank of sociomorphism.
>
> In the third place, cooperation is a source of regulation. Over and above simple regularity perceived by the individual and heteronomous rule imposed by constraint in the areas of both knowledge and morality, it installs autonomous rule, or the rule of pure reciprocity, a factor in logical thought and the principle behind notional systems and signs (Piaget, 1965/1997, s. 239).

Wittgensteins språkspelsteori

I *Filosofiska undersökningar* (1953/1998) gör Wittgenstein upp med sin tidigare uppfattning, framförd i *Tractatus,* där han såg språket som ett avbildande system av refererande namn. I stället hävdar han nu att språket primärt utgör ett samspel, ett *språkspel*.

Meningen hos ett språkligt uttryck bestäms av dess användning, dvs. de regler som utvecklats i denna språkliga praxis. Härav följer då naturligtvis frågan hur vi kan förstå eller mena något med ett uttryck när meningen av det är bestämd av användningen, vilket i sin helhet inte kan vara närvarande vid användningen eller menandet. Wittgenstein för här ett resonemang om regelföljande. Själva möjligheten att följa en regel och att använda ett språkligt uttryck förutsätter en redan etablerad offentlig praxis. Att tala om meningen av en sats är att tala om den roll den spelar i ett bestämt språkspel. Att förstå denna mening (i en sats) handlar om att kunna delta i det aktuella språkspelet i de aktuella situationerna (se Svensson, 1992).

> 23. Men hur många arter av satser finns det? Kanske påstående, fråga och befallning? – Det finns *otaliga* sådana arter: otaliga sätt att använda allt det som vi kallar ”tecken”, ”ord”, ”satser”. Och denna mångfald är inte något fast avgränsat, något en gång för alla givet; utan nya typer av språk, nya språkspel, kan man säga, uppstår och andra föråldras och blir bortglömda. (*En ungefärlig bild* av detta kan matematikens förvandlingar ge oss.)
> Ordet ”språkspel” är här avsett att framhäva att *talandet* av språket är en del av en aktivitet eller av en livsform (Wittgenstein, 1953/1998, s. 21).

Härav skulle man kunna dra slutsatsen att språkspelsteorin handlar om *vardagsspråket,* men det avvisas av Svensson (1992). Denne hävdar att Wittgenstein främst var *logiker*, och inte ägnade sig åt empirisk språkvetenskap. De kommentarer Wittgenstein ger till den faktiska språkanvändningen är endast avsedda att belysa aprioriska, logiska, förhållanden. Det är vårt tänkande och våra tankeformer Wittgenstein undersöker, inte vårt språk och våra språkliga uttryckssätt.

För vårt vidkommande är det av stort intresse att Wittgenstein även karakteriserade matematiken som ett språkspel. Hans arbeten innehåller också många matematikfilosofiska avgöranden, så avvisar han t.ex. de dittillsvarande matematikfilosofiska försöken att ge matematiken en säker grundval.[9]

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[9] Inom matematikfilosofin har därvid olika riktningar framträtt:

- Logicismen, som betecknar matematiken som en komplex struktur av rent logiska samband, en gren av den symboliska logiken. De företräddes av framförallt Frege och formulerades klarast av Russell och Whitehead i deras arbete *Principia Mathematica*, 1910-1913.
- Intuitionismen/konstruktivismen, som avvisar logiska slutledningars regler och föredrar att bygga upp matematiken på intuitiv grund genom konstruktioner. Den främste företrädaren var holländaren Brouwer. Intuitionismen kan ses som en renodlad version av Kants syn på matematik som apriorisk begreppskonstruktion. För ordningens skull bör man kanske påpeka att det finns viktiga skillnader mellan Brouwers och Kants konstruktionsbegrepp.
- Formalismen, som ser matematiken som ett system av formaliserade axiomatiska teorier. Matematiska resultat ses som logiska konsekvenser ur givna axiom. Den främste företrädaren vad David Hilbert. Ett avgörande grundskott mot formalismen gjordes av Kurt Gödel 1931 då han publicerade sitt *Ofullständighetsteorem* som lade Hilberts försök att upprätta ett sammanhängande, motsägelsefritt och uttömmande system i ruiner. Teoremet la fast gränserna för vår kunskap. Numera anses försöken att upprätta en säker grundval för matematiken som missriktade. Inom den matematik-filosofiska diskussionen har också en annan riktning framträtt med en fallibilistisk och

> 16. Wozu braucht die Mathematik eine Grundlegung?! Sie braucht sie, glaube ich, ebenso wenig, wie die Sätze, die von physikalischen Gegenständen – oder die, welche von Sinneseindrücken handeln, eine *Analyse*. Wohl aber bedürfen die mathematischen, sowie jene andern Sätze, eine Klarlegung ihrer Grammatik.
> Die *mathematischen* Probleme der sogenannten Grundlagen liegen für uns der Mathematik so wenig zu Grunde, wie der gemalte Fels die gemalte Burg trägt.
> ›Aber wurde die Fregesche Logik durch den Widerspruch zur Grundlegung der Arithmetik nicht untauglich?‹ Doch! Aber wer sagte denn auch, daß sie diesem Zweck tauglich sein müsse?! (Wittgenstein, 1956/1994, s. 378).

Wittgenstein inte bara tillbakavisar absolutism och apriorism, utan hävdar också att matematiken är syntetisk apriorisk:

> 43. Man könnte vielleicht sagen, daß der synthetische Charakter der mathematischen Sätze sich am augenfälligsten im unvorhersehbaren Auftreten der Primalzahlen zeigt.
>
> Aber weil sie synthetisch sind (in diesem Sinne), sind sie drum nicht weniger *a priori*. Man könnte sagen, will ich sagen, daß sie nicht aus ihren Begriffen durch eine Art von Analyse erhalten werden können, wohl aber einen Begriff durch Synthese bestimmen, etwa wie man durch die Durchdringung von Prismen einen Körper bestimmen kann.
>
> Die Verteilung der Primzahlen wäre ein ideales Beispiel für das was man synthetisch *a priori* nennen könnte, denn man kann sagen, daß sie jedenfalls durch eine Analyse des Begriffs der Primzahl nicht zu finden ist (Wittgenstein 1956/1998, s. 246).

Ovanstående följer egentligen utifrån uppfattningen att matematiken inte är en upptäckt utan en uppfinning, ”Der Mathematiker aber ist kein Entdecker, sondern ein Erfinder” (Wittgenstein, 1956/1998, s. 111).

Vari ligger det specifika i en *matematisk* diskussion till skillnad från t. ex. en politisk? Drouhard et al (1999) pekar på två avgörande aspekter – *motsägelser* och *motstånd*. I en matematisk diskussion erfar individerna *motsägelser*, dvs. att andra kan vara säkra på en motsatt uppfattning och inte låta sig övertygas med auktoritära argument eller maktanspråk. Den andra aspekten är att det karaktäristiska i nödvändigheten i en matematisk utsaga är att den bjuder *motstånd*. En vägg bjuder motstånd om du försöker gå igenom den. Du kan emellertid forcera väggen med en yxa. För en matematisk utsaga finns ingen yxa, eller annat verktyg, som kan göra att du uppfattar en falsk utsaga som sann. Wittgenstein för ett liknande resonemang:

> ”Wenn du diese Regel annimmst, *mußt* du das tun”. – Das kann heißen: die Regel läßt dir hier nicht zwei Wege offen. (Ein mathematischer Satz.) Ich meine aber: die Regel führt dich wie ein Gang mit festen Mauern. Aber dagegen kann man doch einwenden, die Regel ließe auf alle mögliche Weise deuten. – Die Regel steht hier wie ein Befehl; und *wirkt* auch wie ein Befehl (Wittgenstein, 1956/1994, s 406).

Det finns invändningar att resa mot Wittgensteins betoning på regelföljandet för att karakterisera matematiken. Möjligtvis gör Wittgenstein en felaktig analogi mellan

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kvasi-empirisk syn på matematiken (se t.ex. Tymoczko, 1986; Ernest, 1991, 1998 och Hersh, 1998). En relativt lättillgänglig och mycket initierad introduktion till matematikfilosofin ges i Sandmel (1995a, 1995b). Sandmel redovisar dock bara kort de nyare kvasi-empiriska strömningarna, vilka han bestämt avvisar.

språk och matematik. Som Hersh (1998) understryker så har både språk och matematik regler, men till skillnad från språk så är matematikens regler inte godtyckliga, utan har en *inre nödvändighet.*

> It's true, school and society tell us,
>
> 3 + 5 = 8.
>
> But in politics, in music, or in sexual orientation, some people reject the dictate of school and society. Some people dare question the Holy trinity, the American flag, whether God should save Our Gracious Queen, and so on. But *nobody* questions elementary arithmetic. A few poor souls trisect angles by compass and straight-edge, despite a famous proof that it can't be done. But in that problem the discoverer easily confuses himself. We *never* get letters claiming that
>
> 3 + 5 = 9
>
> If arithmetic can be whatever you like, why has no one in recorded history written
>
> 3 + 5 = 9?
>
> Anthropologists in the Sepik Valley of New Guinea finds surprising practices and beliefs about medicine, about rain, about gods and devils. Not about arithmetic. (Hersh, 1998, ss 206–207).

Alfred Schütz' sociala fenomenologi

Inom samhällsvetenskaperna har intresset för att utforska människors vardagsliv vuxit sig starkt framför allt under det senaste decenniet. Därigenom har det fenomenologiska begreppet *livsvärld* kommit i fokus. En av de mer framträdande forskarna, vars studier av vardagslivet, dess strukturer och samhällsvetenskapernas relation till denna verklighet, är den tyskfödde, sedemera amerikanske, sociologen Alfred Schütz. Som grundare av fenomenologin brukar annars den tyske filosofen Edmund Husserl räknas.[10] Inom svensk matematikdidaktisk forskning har den fenomenografiska ansatsen en stark ställning och inom den har producerats ett antal avhandlingar (se Engström, 1999); den senaste är Runesson (1999).

Habermas använder sig som vi visat ovan av livsvärlden som en grund för sin teori om det kommunikativa handlandet. Han är dock kritisk till en del aspekter av Schütz' livsvärldsbegrepp (se framför allt kapitel sex i Habermas, 1981b/ 1997b).

Schütz skiljer sig på en del väsentliga punkter från Husserl fr.a. i vad gällde frågan om livsvärldsforskningen kan återföras på ett transcendentalt medvetande vilket Husserl ansåg eller om den faktiska livsvärlden är yttersta grund (Bengtsson, 1999). Schütz försökte jämka samman de, som han uppfattade, vetenskapliga kraven på objektivitet med de subjektiva dragen i den fenomenologiska metoden.

Hos Schütz utgörs livsvärlden av de världsliga händelser och sociala konventioner som konstitueras och rekonstitueras genom människors vardagliga och oreflekterade handlande. Schütz kallar detta för *kunskap av första ordningen*. Den är bestämd av livsvärlden och är ordnad i för oss kända idealtypiska mönster, eller *typifieringar. Kunskap av den andra ordningen* utgörs däremot av den vetenskapliga expertförståelsen med vars hjälp samhällsvetaren tolkar och förstår livsvärldens vardagliga mönster.

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[10] Utmärkta introduktioner till fenomenologin, och Alfred Schütz' arbeten finns i Bengtsson (1998, 1999). Fenomenografin som vuxit fram vid Göteborgs universitet har ett speciellt förhållande till fenomenologin, vilket inte diskuteras här, utan jag hänvisar till dess företrädares egna arbeten fr.a. Marton och Booth (1997).

När Schütz tar avstånd från Husserls sätt att söka sanningen i de oreflekterade fenomenen utgår han från människors förmåga att kommunicera och förstå varandra. Livsvärldens vardagskunskap bygger på en intersubjektiv förståelse baserad på två principiella antaganden om människors upplevelse av omvärlden: För det första perspektivens *ömsesidighet* och för det andra perspektivens *meningskongruens* (Bäck-Wiklund, 1998).

Perspektivens ömsesidighet är ett antagande om att för att ett samtal skall bli meningsfullt mellan två människor så måste deras perspektiv vara utbytbara. De skall kunna inta varandras ståndpunkter och positioner och fortfarande vara i stånd att förstå varandra. Som medlemmar av ett samhälle, en kulturkrets, utgår vi från föreställningen att olika skeenden, föremål, handlingar och personer har samma innebörd för andra som för oss själva.

Antagandet om perspektiven meningskongruens innebär att två parter i ett samtal antar att de båda tolkat situationen lika. I stället för att behandla frågan om intersubjektiviteten som ett filosofiskt problem behandlade Schütz det som ett praktiskt problem som handlar om *möjligheterna till mellanmänsklig förståelse* (Bäck-Wiklund, 1998).

Tecken och teckensystem

Klassrummet kan ses som bestående av olika människor vilka var och en handlar i enlighet med hur världen framträder för henne eller honom. Matematiska fenomen förstås genom tecken snarare än som fakta, vilket innebär att matematiska idéer som utvecklas inte har några stabila förkroppsliganden (*embodiments*) – yttre framträdelser (laborativ materiel, skrivna symboler, eller de ramar inom vilket de används) kan tolkas på olika sätt. Klassrummet är den värld som är inom räckhåll (*Welt im reichweite*) vilket förstås genom de tecken som används. Tecken i denna mening hänförs inte till de Saussure's begrepp tecken utan snarare till Peirce. De olika tecknen ingår i ett system av meningssammanhang (jämför med Peirce teckenteori nedan i avsnitt 2.4).

Tankescheman

Med hjälp av typifieringar känner vi igen olika situationer i vardagsvärlden och vi utvecklar olika tankescheman för detta. Schütz uppfattar tankescheman av fyra ordningar:

Tankeschema av första ordningen: *apperceptuellt schema.*

Detta schema omfattar världen av yttre framträdelser. Objekt ses som saker i sig själva utan någon referens. Ett barn kan t.ex uppfatta $x^2 + y^2 = 1$ som en blandning av bokstäver och siffror utan någon betydelse. En mer erfaren elev ser det som ekvationen för en cirkel.

Tankeschema av andra ordningen: *appresentativt schema.*

Världen uppfattas som en värld av tecken. Skolklockan uppfattas inte som en sak i sig utan som en signal att det är dags att gå hem. Uttrycket $x^2 + y^2 = 1$ uppfattas som en representation av en cirkel.

Tankeschema av tredje ordningen: *refererande schema.*

Det här är den värld där jag ser mig själv handla, den värld som jag föreställer mig att jag arbetar i, givet min läsning av yttre framträdelser. Om jag använder mig av en algebraisk notation, så föreställer jag mig en geometrisk figur. Det refererande schemat omfattar området av mentala bilder där mina tankar fungerar. $x^2 + y^2 = 1$ *ses* som en cirkel

Tankeschema av fjärde ordningen: *tolkande schema.*

Det här är den relation jag antar mellan världen av yttre framträdelser och världen jag föreställer mig existerar. Oerfarna och erfarna matematiker kan ha olika sätt att framkalla mentala bilder till algebraiska symboler. Det tolkande schemat består av området av personliga sätt att kombinera mentala bilder med algebraiska symboler.

Charles Sanders Peirce semiotik

Jag har ovan diskuterat hur socialt samspel framträder som regler, värden och tecken. Det som jag kommer att intressera mig för här är inte hur dessa tecken överförs utan *meningsskapandet av dessa tecken* i Habermas' och Piagets efterföljd på kommunikation som skapande av mening. Jag kommer härvid att diskutera Peirce teckenteori (semiotik). Charles Sanders Peirce (1839–1914) var grundare av pragmatismen och anses vara en av den nordamerikanska kontinentens största filosofer.

Abduktion

Piaget och Peirce har båda sina rötter i den Kantska filosofin. Det finns en intressant parallell i hur de båda försöker ersätta de aristoteliska begreppen om abstraktion och generalisering[11] i sina respektive diskussioner om en matematikens epistemologi.[12] Det finns en paradox i matematikfilosofin som avser hur man kan avvisa empirismen som grund och samtidigt förklara matematikens stora tillämpbarhet på verkligheten. Kants väg ut ur detta dilemma mellan rationalism och empirism var konstruktivism.[13]

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[11] Med abstraktion avses i denna tradition att bortse från egenskaper som är tillfälliga och individuellt säregna och se till det *väsentliga*, det som är gemensamt för en klass. Genom att minska innehållet eller antalet kännetecken kan man öka omfånget och därmed stiga till en högre abstraktionsnivå. Generalisering innebär att från enskilda iakttagelser sluta sig till partikulära eller universella satser, dvs man sluter sig från iakttagna fall till icke-iakttagna fall, ev. till samtliga fall.

[12] Se framför allt de arbeten som utvecklats vid IDM i Bielefeld av Hoffmann (1996), Otte (1998).

[13] Kant gör i sina kunskapsteoretiska arbeten en distinktion mellan två typer av kunskap, *a posteriorisk* kunskap som grundas på erfarenheten, dvs. empirisk kunskap, och *a priorisk* kunskap, som är oberoende av erfarenheten. Han för också in distinktionen mellan *analytiska* och *syntetiska* omdömen. De analytiska omdömena kan avgöras genom en logisk analys av det omdömet säger. "En kropp har en utsträckning" är ett analytisk omdöme då det innefattas i begreppet kropp att det har en utsträckning. Det vore självmotsägande att tala om kroppar som inte har en utsträckning. Ett sådant omdöme är ett nödvändigt sant omdöme, ett analytiskt a priori omdöme.

Syntetiska omdömen utsäger något som inte ligger i själva begreppet. De är i viss mening sammansatta. I mitt inledande exempel om nio planeter så ligger det ingen nödvändighet i att det är nio planeter som rör sig kring vår sol. Det ligger inte i själva begreppet sol att det finns nio planeter. Ett sådant omdöme är syntetiskt aposte-rioriskt.

Finns det syntetiskt aprioriska omdömen? Ett sådant omdöme skulle inte vara självmotsägande att förneka, likväl skulle vi kunna avgöra apriori om det är sant eller falskt. Vi måste, enligt Kant, företa en kopernikansk vändning, dvs. vända uppmärksamheten mot de erfarna tingen och mot erfarenheten själv. Våra erfarenheter formas och struktureras utifrån åskådningsformer som tid och rum och kategorier som ting och orsak-verkan. Alla medvetandeformer uppvisar en tidslig och rumslig ordning. Matematiken, närmare bestämt aritmetiken och (den euklidiska) geometrin är, enligt Kant, exempel på syntetisk aprioriska omdömen då de kan föras tillbaka till dessa åskådningsformer. Genom detta är det inte möjligt att nå erfarenhet om tinget i sig (das Ding an sich) utan alla de olika formerna och kategorierna kommer att finnas i varje erfarenhet eftersom det är något som det erfarande subjektet

Piaget överskrider Kant genom att visa att de Kantska åskådningsformerna och kategorierna inte är aprioriska utan konstrueras genom *reflekterande abstraktion* och *konstruktiv generalisering* (se Piaget, 1975/1985, 1977). Piaget gör en stark distinktion mellan empirisk och reflekterande abstraktion och förklarar matematikens tillämpaningar i att den matematiska kunskapen har en grund i konkreta handlingar som att ordna, gruppera, föra samman etc, vilka sedan utgör utgångspunkt för vidare abstraktioner och generaliseringar.

Peirce diskuterar ett tredje alternativ till slutledning vid sidan av deduktion, att följa en ”regel” på ett enskilt fall för att uppnå ett ”resultat” och induktion, en omvänd slutledning från ett enskilt fall och resultat till en regel. Induktionen är visserligen användbar i många sammanhang, men är ingen logiskt giltig slutledningsform. Peirce kallar sin tredje slutledningsform för *abduktion*. Enligt Peirce är abduktionen vardagens slutledningsform som leder oss till den mest sannolika förklaringen till ett fenomen som väcker vår förvåning. Peirce kallar sitt begrepp *hypostaserande abstraktion*. Abduktionen innebär hypotesbildning och sanningsantagande av denna hypotes.

Det finns en annan intressant parallell mellan Peirce och Piaget. Deras respektive teorier om abduktion respektive reflekterande abstraktion är ett svar på frågan om hur ny kunskap utvecklas. Platon hävdade att idén om ny kunskap var en paradox i sin dialog Menon. I stället handlar det om en återerinring. Genom den speciella samtalsteknik, frågor och prövningar av svaren på frågor, låter Platon Sokrates utveckla sin majevtiska metod, dvs. förlösa vetandet som redan finns.[14]

Habermas diskuterar i sin *Erkenntnis und Interesse* (1968) Peirce abduktionsbegrepp som en vetenskaplig utvecklingsprincip. Habermas uppfattar abduktion som en i grunden kollektiv process, som inte endast gäller vetenskapen utan även vardagslivets kunskapsinhämtande. Dess giltighet tryggas genom dialog och samstämmighet inom kollektivets gränser.

Semiotik

För Peirce råder det ett triangulärt förhållande mellan tecken, objekt och interpretant. Ett tecken är något som för någon står för något i viss bemärkelse eller kapacitet. Det är riktat till någon det vill säga skapar ett motsvarande tecken i personens medvetande,

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själv bibringar sinneserfarenheterna. Häri ligger Kants distinktion mellan tingen i sig, das Ding an sich, (noumenon) och fenomenen, das Ding als Erscheinung, (phaenomenon).

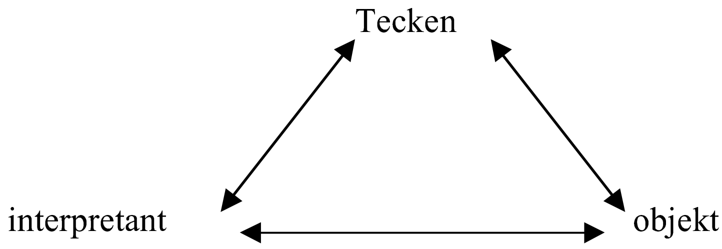
Den amerikanske filosofen Saul Kripke har i sin Naming and Necessity (1980) utvecklat en formell semantik för modallogiken (s.k. möjlig-världs-semantik) där han starkt kritiserat uppfattningen att distinktionerna nödvändig-kontingent och apriori-aposteriori sammanfaller. Begreppen nödvändig-kontingent, apriorisk-aposteriorisk (empirisk) och analytisk-syntetisk bildar motsatspar som är ömsesidigt uteslutande och uttömmande. Begreppen nödvändig, apriorisk och analytisk är inte synonyma, dvs. de har inte samma extension. Nödvändig betyder sann i alla möjliga världar, medan kontingent betyder sann i någon, men inte alla världar, dvs. möjlig, men inte nödvän-dig.

För Piaget är utvecklingen av modal (dvs. nödvändig) kunskap det centrala i hans genetiska epistemologi (se Smith, 1993, för en genomgång).

[14] En nutida förespråkare för nativismen och kritiker av konstruktivismen är den amerikanske filosofen Jerry Fodor. Begrepp är medfödda och därför oberoende av erfarenheten. Bereiter (1985) har diskuterat det som kommit att kallas "inlärningsparadoxen". Hur kan en struktur generera en annan struktur mer komplex än sig själv? Konstruktivismen är därför omöjlig.

Det finns många argument mot nativismen. Den innebär ju t.ex. att Newtons fysik skulle ligga latent redan hos Aristoteles. Det finns kunskap som inte beror på arv, perception eller erfarenhet, nämligen nödvändig kunskap. Nativismen ger i detta fall ingen förklaring alls. Se min avhandling (Engström, 1997) för en genomgång av Piagets teori.

eller möjligen ett mer utvecklat tecken. Tecknet som det skapar kallar jag interpretanten av det första tecknet. Tecknet står för något dess objekt (Peirce, citerat i Fiske, 1990, s. 63). Följande figur kan illustrera detta förhållande:



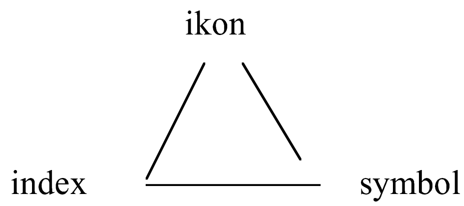
Figur 1. Förhållandet mellan tecken, objekt och interpretant hos Peirce (Fiske, 1990, s 63)

Att pilarna är dubbelriktade syftar på att varje term måste förstås i relation till de andra. Interpretanten är inte tecknets användare utan syftar på ”den egentliga betecknande effekten” (Peirce, citerat i Fiske, 1990, s. 63).

> Interpretanten av tecknet är resultatet av användarens erfarenhet av tecknet och de sammanhang där detta ingår. Gränserna sätts av sociala konventioner. Variationerna inom dessa är en fråga om sociala och psykologiska skillnader mellan användarna (Fiske, 1990).

Peirce behandlade också tre *teckenkategorier*, dvs. olika förhållanden mellan tecknet och objektet det hänvisar till. En *ikon* liknar objektet på något sätt; i ett *index* finns ett direkt samband i verkligheten mellan tecknet och objektet; i en *symbol* saknas både likhet och samband. Ett fotografi är en ikon, rök ett index som är kopplat till (indikerar) eld och ett ord är en symbol (Fiske, 1990).

Även teckentyperna kan visas i en triangulär modell



Figur 2. Förhållandet mellan ikon, index och symbol enligt Peirce (Fiske, 1990, s 70).

Vid IDM i Bielefeld arbetar en forskargrupp med att tillämpa Peirce semiotik inom matematikdidaktiken, se t.ex. Hoffman (1996) som redogör för hur läroprocesser kan modelleras semiotiskt enligt Peirce teori.

Avslutande reflektioner

Matematik är en social konstruktion. Den utövas och praktiseras av människor som lever och verkar i bestämda historiska skeenden. Matematik präglas av de sociala och kulturella villkor under vilka den har växt fram. Trots detta är matematikens *resultat* oberoende av tid och plats, av genus och etnicitet, av social klass och kultur. Matematiken uppvisar en *inre nödvändighet* vilket skiljer den från andra mänskliga uppfinningar och kunskapsområden.

Mitt intresse är att förstå de villkor under vilka en elev utvecklar och konstruerar kunskap som hon/han uppfattar som nödvändig. Detta i en social interaktion i klassrummet, vilket ger matematikundervisningen en kommunikativ karaktär. Härigenom reses frågan om förhållandet mellan rationalitet och intersubjektivitet. Jag relaterar skolans uppgifter till upplysningens klassiska tankar om bildning som myndiggörande och medborgarskap i en demokrati.

Matematikundervisningens kommunikativa karaktär

I min diskussion om matematikundervisningens kommunikativa karaktär har jag tagit en utgångspunkt i några gemensamma beröringspunkter hos ett antal forskare som Habermas, Piaget, Wittgenstein, Schütz och Peirce, vilka alla behandlat frågor om rationalitet och intersubjektivitet.

Rationellt handlande förutsätter en ömsesidig förståelse mellan människor. Rationella överväganden, argumentation och tänkande, utvecklas genom ett socialt samspel. Socialt samspel framträder som värden, regler och tecken i vid mening. Det sociala samspelet uppfattas här som en *process* mot ökad förståelse (*Verständigung*). Språket är det medel varmed denna förståelse utvecklas och etableras. Genom social samverkan sker en decentrering av det individuella tänkandet till en *förhandlad verklighet* baserad på ömsesidighet och meningskongruens. Matematiska fenomen företer inte några stabila förkroppsliganden. Yttre framträdelser och ramar kan tolkas på olika sätt. Vad vi uppfattar som rationellt och förnuftigt får sin mening i det sociala samspelet.

Vill vi studera hur förnuft och logik i vid mening – att resonera, att argumentera, att göra överväganden och dra slutsatser – kan stimuleras och utvecklas hos eleverna bör vi också studera förutsättningar för att etablera detta sociala samspel och dess utveckling. Det är meningsskapandet av dessa värden, regler och tecken vilka framträder genom det sociala samspelet som bör sättas i fokus. Problem med att få matematikundervisningen meningsfull och relevant för eleverna bör kanske i större utsträckning sökas i misslyckanden att upprätta ett fungerande socialt samspel i matematikklassrummet än i bristande förmågor hos eleverna.

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Using symbol-manipulating calculators (SMC) in upper secondary schools

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Introduction

In the school year 1998-1999 the National Examination Board (Eksamenssekretariatet) in Norway initiated a study with symbol-manipulating calculators (SMC) in mathematics. The main purpose was to investigate the conditions concerning the use of this type of calculators for final exams in mathematics in the second year of upper secondary school. [15]

The Project

The project was announced by the National Examination Board in a letter to schools. Here we find the rationale for the study:

The purpose with this experiment is to find out if it is possible to construct examination problems and organize test situations that mirrors real challenges and possibilities, and at the same time provide a basis for giving individual marks.

Schools (teachers) could then register to participate. For the study the schools themselves had to get or borrow the calculators from companies importing the SMCs. The National Examination Board had some limited funds available for supporting the teachers, and had more of a coordinating function. The study was part of a larger study on the use of information technology in different school subjects (eg. Norwegian, English).[16]

Five classes from five different schools in the southern part of Norway took part in this study. The teachers had volunteered to participate – hence there were teachers interested in technology participating. There were four male and one female teacher.

The calculators

Two types of calculators were used: The TI-89 and the TI-92. Four classes used TI-89 and one class used the TI-92 as tools through the school year and for their final exam. It should be noted that the students did not get their calculators until mid-October. These calculators have built in versions of the computer algebra system (CAS) *Derive*. The program is a somewhat restricted version of the computer (PC) version, but sufficient for the symbolic computations found in the course.

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[15] In the upper secondary school in Norway there are two streams in mathematics – *MX* and *MY*. The *MX* stream is for mathematics and natural science studies, whereas the *MY* stream is more for social science studies. The courses in the second year are denoted *2MX* and *2MY*.

[16] The study was evaluated by Nils Voje Johansen, Dept. of Mathematics, and Gunnar Gjone, Dept. of Teacher training and School Development at the University of Oslo.

The exam

Much of the project was focused on problem construction and the exam. This type of exam in Norway is usually a 5 hour written exam. For upper secondary mathematics all students have graphic calculators. The students had the regular exam in mathematics, except that one of the problems was replaced by a problem especially designed for the use of this type of calculator.

The exam papers were graded at the same time as for the other students. Three persons graded the exam papers, so that each individual paper was graded by two persons. This is the usual procedure for Norwegian exams. As graders were the two persons evaluating the study, as well as one of the participating teachers (who of course did not grade his own students). The graders were only grading exam papers from the project. This turned out to be a shortcoming of the study.

Data

The main data for the study were the students' exam papers. They were marked with school name, and also the sex of the students. We, nor the graders, had not access to the name or sex of the students at the time of the grading. Analyzing differences relating to these factors, however, has not been performed for this presentation.

We also provided a questionnaire for teachers. The answers gave us information on how the teachers reacted. The teachers were in general positive (which they had been from the start). We asked specifically about problem types suitable for these calculators. Here not many concrete suggestions were presented.

Going through the exam papers of the students, we found several interesting relations on how the students used their calculators. Some of these findings relate to the SMCs especially, whereas others relate to graphing calculators in general. By grading the exams we found that actually the regular problems (for the use of ordinary graphic calculators) provided the most interesting material. The special problem for SMCs did not give as many interesting results. Moreover, the special problem was not good for differentiating among students.

Findings

In this presentation I will concentrate on three aspects of using SMCs and graphic calculators, relating to some of the regular exam problems.

Solving graphically

We found that the students used graphic representations of functions to a large extent in solving mathematical problems. We were struck how students in several problems used a more complicated (graphic) approach, instead of using a simpler and quicker analytic approach.

Equality

Use of the symbolic-manipulating calculators also presented us with some surprises: The students were very "inventive" in using the calculator to show equality between expressions.

Concept formation – limits

We observed that several students had difficulties with the limit concept trying to use the calculator directly. The study of these three elements is based on the analysis of three problems in the set.

Solving problems graphically

We studied the solution of the following problem:

PROBLEM 1

d) The depth of water in a harbour varies with the tide. In a certain 24-hour period the depth is given by the function *f* where

$$f(x) = 8.50 + 0.70 \cdot \cos\frac{4\pi}{25}x$$

where *x* is measured in hours and *f*(*x*) is measured in meters.

1) What is the difference in depth between high and low tide?
2) How long time is it from one high tide to the next?

Figure 1. Problem 1d) from 2MX exam, spring 1999. The text translated by the author.

Solution of this problem is straightforward without using a graphic calcula-tor. The answer to the first question could be obtained by noting that the cosine function varies between –1 and +1, hence the answer is 0.70 m + 0.70 m = 1.40 m.

What the majority of the students did was to graph the function, then using either the TRACE function or MAX/MIN of functions to find the difference.

To present the correct answer to this problem the students would draw or sketch the graph indicating which calculator functions they had used. The following instructions are stated on the second page of the exam:

Graphs – use of graphing calculator

State which calculator functions you have used. It is not necessary to give detailed use of keys.

Remember to write the scale and units on the axes if you *draw* graphs as part of your answers. You do not need to include a table of the values you have calculated for the function unless specially asked to do so.

If you use the calculator to construct graphs, it is sufficient to *sketch* the shape of the curve in your answer. The sketch must show clearly how you arrived at the answer.

Figure 2. Official translation of instructions to students.

Some of the solutions contained very elaborate graphs. For some, writing up the solution to the problem took more than a full page of written text. Many also included quite extensive description on how they used the TRACE-function or the MAX/MIN – functions on the calculator.

Since this was one of the first problems on the test, it was considered an easy task and most students in our sample presented a correct solution. The graders would not differentiate between different types of solution.

Equality

Derive has the possibility for testing equality of expressions. The following problem was on the test:

> **PROBLEM 4**
>
> In the triangle *ABC we have* $\angle A = x$, $\angle B = 2x$ og $AB = 5$.
>
> a) Sketch the triangle *ABC* for different values of *x*. Explain why
>
> $x \in \langle 0, \pi/3 \rangle$
>
> b) Use $\sin(2x + x) = \sin(3x)$ to show that $\sin(3x) = 3\sin x - 4\sin^3 x$

Figure 3. Problems 4a) & 4b) from 2MX exam, spring 1999.
The text translated by the author.

We will here especially consider problem b). The straightforward ”traditional” solution of the problem would be something like the following:

sin 3*x* = sin (2*x* + *x*) = [using the formula for *sin* to a sum, and then simplifying to obtain the desired result]

We give here some of the students' answers (in translation)

Since sin 3x = sin (2x + x), we can write on our calculator
*tCollect(sin (2x + x)) = tCollect (3*sin(x) – 4*(sin(x))^3)*
➔ *true. This means that the expressions are equal for all values of x.*

A similar but simpler (?) way of writing the solution was found in some students' “solutions”:

I use TI-89 and write
*(sin (3x) = sin (2x + x)) = (sin (3x)= (3*sin(x) – 4*(sin(x))^3))*
Pressing ENTER I get <u>*true*</u>*. This shows that the expression in the first large parenthesis is equal to the expression in the second large parenthesis. TI-89 simplifies the expression to (sin (3x)= (3*sin(x) – 4*(sin(x))^3))*

Some of the students used the traditional approach, but the number of students were attempting a solution similar to the ones presented above. For this problem it was quite clear that the committee making up this examproblem had not taken into account the use of SMCs.

Another type of solution that was found used the graphic capabilities of the calculator. They were drawing two graphs: One graph of the function *sin 3x* or of *sin (2x + x)* and then another graph of $3\sin x - 4\sin^3 x$. Observing that the graphs coincided they concluded that the functions were equal.

The solutions raises several important problems.

The last ”method” of solving the problem, by observing coinciding graphs, is easy to argue against. The solution is depending on the resolution of the screen. On the other hand it is a ”solution” that can easily be obtained on a graphing calculator.

Second, is the solution by having the calculator collecting terms ”correct”? How is the calculator arriving at the answer? Could we in general trust the test for equality for functions? This leads to the principal question how the calculator-algorithm is constructed.

The third problem that we looked into is the last part of problem 4.

Concept formation – limits

The concept of limit is important in calculus. *Derive* on the TI-89 and TI-92 is able to compute values of limits.

PROBLEM 4

d) Show that the area of the triangle can be written as

$$F(x)=\frac{12.5\sin(2x)}{3-4\sin^2 x} \qquad x\varepsilon\left\langle 0,\frac{\pi}{3}\right\rangle$$

e) Investigate if the area *F* has a maximum value. What happens with the area when

$$x \to \frac{\pi}{3}$$

Figure 4. Problems 4d) & 4e) from 2MX exam, spring 1999. Translation by the author.

We will concentrate our discussion on problem 4e). The function in d) is given so that students can answer question e) without having arrived at the correct expression in d). We first note that the solution is straightforward knowing something about the properties of the *sin* function. As *x* approaches $\frac{\pi}{3}$ the denominator approaches zero, whereas the numerator approaches a number different from zero. Several students observed this fact. We also observe that the result can be obtained from a geometrical reasoning about the triangle (*ABC*). However, a number of students keyed in the expression directly, knowing that the SMC was able to calculate limits.

When I was looking at the graph I adjusted the (view)window as follows: xmin = 0 xmax = $\frac{\pi}{3}$ *since x∈ ⟨0,* $\frac{\pi}{3}$*⟩. Then I went to the graph again, on F2 zoom and A(?) zoom Fit. Then an "undelicat" graph appeared. I went to F5 maximum og pressed enter. Then I went to the left and pressed* *enter* *and then to the right and pressed* *enter*. *Then I got* maximum value of F when x = 1,0471976 and y= 1,08· 10^{12}

There is a certain rationale to approach the value from left and from right. However, the important element here is the dependency on the calculator, not trying to reason without it. This was observed for several of the students. Several would arrive at the answer "undef(ined)" and then just say that the answer is undefined. One conclusion would be that they were very limited by the SMC, and not trying to reason without it.

These examples raise question about what kind of knowledge of mathematics we found in the students using the calculators. We will discuss this question with reference to the two forms of knowledge – *structural* and *operational* – as discussed by Anna Sfard (1991).

Forms of knowledge

Sfard argues that abstract mathematical notions can be conceived in two different ways, structurally as objects or operationally as processes. For most people the operational comes before the structural, and the transition from a "process" conception to an "object" conception is difficult.

> Seeing a mathematical entity as an object means being capable of referring to it as if it was a real thing – a static structure, existing somewhere in space and time. It also means being able to recognize the idea "at a glance" and to manipulate it as a whole, without going into details. (Sfard, 1991, p 4)

In several papers Sfard has written about the operational – structural duality of mathematical conceptions, e.g. Sfard (1991, 1992). Looking at the historical development of mathematical concepts she introduces the notion of *reification.*

> …the idea of turning a process into an autonomous entity should emerge, and finally to see this new entity as an integrated, object-like whole must be acquired. (Sfard, 1991, p 18)

She states that reification is difficult to attain, but also needed to attain relational understanding (Sfard, 1992, p. 4). We will now consider the use of SMC and graphing calculators for working with functions in the light of the duality between operational and structural knowledge.

Graphing functions the traditional way (on paper) is basically a computational process (operational). Working with functions the structural conception of a functions develops gradually through secondary education.

Working with functions on a graphing calculator (or SMC) introduces new elements in this development. The graph of a function is automatic from the function expression. Entering

$$f(x) = 8.50 + 0.70 \cdot \cos\frac{4\pi}{25}x$$

gives the graph immediately. However, I will still argue that a graphing calculator is basically an operational device. The graph of a function is ”drawn”. On some SMCs one can see the graph appearing in steps on the screen.

On the other hand the graph could also be more easily viewed as an object (structural) with a graphing calculator. It can to some extent be manipulated, e.g. moved or transformed (as an object).

In the example with the cosine function (Problem 1d) I will argue that the solutions based on reading off the graph is based on an operational knowledge of functions. They do not see the cosine function as an object with some properties, it is looked upon mainly as a process. This raises a number of questions. In (Sfard, 1992) we find two priciples for teaching:

(I) *New concepts should not be introduced by help of their structural description*
(II) *Structural conception should not be required as long as the student can do without it.* (Sfard, 1992, p 6)

Based on the second principle we can ask the question if students with a graphing calculator can do without a structural conception? Where one earlier relied on seeing mathematical entities as objects, the graphing calculator – for these students and for this problem – has eliminated the need for a structural conception of a function.

Many will argue that technology has the possibility of helping students form structural conceptions, but they might be different than the ”traditional” ones. In the case of functions perhaps a more graphical structural conception of a function is developed.[17] It should be noted that the problem in question (Problem 1d) was a standard type of problem, solved most efficiently with a traditional structural conception of function. In the second problem on equality (Problem 4b) some students also used a graphical approach – drawing both graphs and observing ”they were the same”. It is however, unclear if they considered the graph of a function as a process or an object. Looking for differences in the graphs suggests that they considered the graph as some kind of object.

The use of the test for equality on the SMCs shows that the student can do without the ”traditional” conceptions.

This is even more obvious in the problem with limits. Even if many of the students just keyed in the expression, the way they tried to handle ”undefined” suggests that they looked upon limit as an operational conceptions. This is probably sufficient at this stage, but one can ask if there will be a need for the ”traditional” structural conception of limits when the students will work with an SMC. If we want the students to build a structural conception of limit, what should it be like?

Perspectives

I find the notions of structural and operational conceptions helpful in discussing knowledge relating to the use of technology in mathematics education. It is necessary to ask if ”structural” and ”operational” should be somewhat ”redefined” with respect to the new technology.

[17] We can see this aspect clearly in how some calculators handle the graphs of functions. The graphs are objects that can be manipulated. Some graphing calculators have a kind of “variable graph” function that will vary constants in the equation of the graph.

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Alternatives to standard algorithms – A Study of three pupils during three and a half years

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Introduction

In many countries the role of the standard algorithms is discussed today. Is it really necessary to teach pupils in elementary school to compute 503 - 287 with the following algorithm or something like it?

```
 1010
  503
- 287
  216
```

Perhaps it is better to ask them to find their own ways to do the computing. Some of them may do it in this way: 287 plus 13 is 300, 300 plus 200 is 500, and 500 plus 3 is 503. The answer is thus 13 plus 200 plus 3, which is 216. Will this method and other similar ones be sufficient in the pupils' future lives?

I carried out a research project in one class to try to get an answer to the above question. The pupils of this class were not introduced to the standard algorithms during their first five school years, i. e. when they were up to 12 years old. Instead, they were encouraged to find their own computational methods, sometimes on their own and sometimes working together in small groups. Only whole numbers were involved in the computing.

One of the reasons for this research is, of course, that many computations are carried out with the help of calculators and computers today. This reduces the need for paper and pencil procedures, but, on the other hand, it might increase the demand for man's ability to do computational estimation and, thus, mental arithmetic. The technical aids do computations very fast, but at the same time, they also make it possible to make miscalculations just as fast. The calculator or the computer will not check our calculations; we will have to do that ourselves.

To be able to do mental arithmetic and/or computational estimation our pupils will need number sense, an ability to operate with numbers in a skilful way. We might ask if our pupils acquire better number sense when they are allowed to find their own ways of computing than when they are taught the standard algorithms.

It should be mentioned from the very beginning that by "mental arithmetic" or "mental computation" I mean computing that is done entirely in a person's head so that only the original exercise and the answer are recorded. When a person uses some kind of drawing or records intermediate results (support notes), I will use the term "written computation" (or "alternative written computation" to distinguish it from standard algorithms). Although people tend to use the same kind of strategies in alternative written computation as in mental computation, I find the above distinction essential.

Many researchers and many teachers, as well, see constructivism, especially social constructivism, as the guiding rule of their work. I see this theory of learning as one of the reasons for my research as well. I would like to summarise it in the following three principles:

1. The learner actively builds up her/his own knowledge.
2. The learner's previous experience plays a vital role during that construction.
3. The learner's interaction and dialogue with others is crucial for her knowledge construction.

Presupposing these principles, it is difficult to see how mere teaching of standard algorithms can help our pupils to construct their own meaning of numbers and operations with numbers. It seems more appropriate to allow pupils to find their own computational methods, alone or in cooperation with others.

It should also be mentioned that I made social constructivism my guiding rule not only when designing the project but also when analysing and discussing its results.

Previous Discussion and Research

In the introductory article of the proceedings of a working group: "A Curriculum from Scratch (Zero-Based)" in the eighth international conference in mathematics education, Anthony Ralston, the organiser of the group, writes

> So, if we were inventing primary school arithmetic today, should there be any p-p-a (paper-and-pencil arithmetic) at all? The answer to this question could be, no, only if each of the following questions has an affirmative answer:
>
> 1. Can you teach children all they need to *understand* about arithmetic without p-p-a?
> 2. From the standpoint of efficiency only, are all arithmetic calculations better or, anyhow, as well performed either mentally or by calculator than with p-p-a? (Ralston, 1997, p 4)

By p-p-a the author means computing carried out with traditional standard algorithms. However, he also writes that paper and pencil (or ballpoint pen) are indispensable tools for recording intermediate results or for drawing pictures. As he sees mental arithmetic and use of the calculator as the only alternatives to p-p-a, his definition of mental arithmetic also includes the possibility to make support notes and drawings.

According to the author, a curriculum from scratch should contain other essential components than arithmetic, although arithmetic should remain the most important portion of it. "But is understanding of the operations of arithmetic … not facility with arithmetic computation which is crucial to the further study of mathematics." (Ibid, p 6).

In this connection I cannot avoid citing what Plunkett (1979) wrote on standard algorithms as far back as the end of the 1970:s.

> * (The algorithms) are *analytic*. They require the numbers to be broken up, into tens and units digits, and the digits dealt with separately.
> * They are not *easily internalised.* They do not correspond to the ways in which people tend to think about numbers.
> * They encourage *cognitive passivity* or suspended understanding. One is unlikely to exercise any choice over method and while the calculation is being carried out, one does not think much about why one does it in that way. (Ibid p. 3)

Besides, the author goes on, they are used very little even by children. They are also often applied unthinkingly to computations like 1000 - 995 or 100·26. He points out that, in these cases, it would be better to look upon the numbers holistically, for instance 995 is very near to 1000.

A lot of research projects have been carried out, where pupils were taught the standard algorithms. I will here only mention one of them, the CAN-project (Calculator Aware Number) in Britain (Duffin, 1996). In this project the children, besides using their own methods for written computation, always had a calculator available, which they could use whenever they liked. Exploration and investigation of "how numbers work" was always encouraged, and the importance of mental arithmetic stressed.

One of the reported advantages of the CAN-project was that the teachers' style became less interventionist. The teachers began "to see the need to listen to and observe children's behaviour in order to understand the ways in which they learn". (Shuard et al, 1991, p 56) The teachers also recognised that the calculator "was a resource for generating mathematics; it could be used to introduce and develop mathematical ideas and processes". (Ibid, p 57).

It ought to be stated, however, that there are also researchers with a quite different opinion. Jeromy Kilpatrick writes in an editorial of one of the issues of *Journal of Research in Mathematics Education*:

> A neglected yet critical item both in implementing the NCTM standards and in gaining a better grasp of the role skill development plays in learning mathematics concerns the folk wisdom in today's school practice. Why is it that so many intelligent, well-trained, well-intentioned teachers put such a premium on developing students' skill in the routines of arithmetic and algebra despite decades of advice to the contrary from so-called experts? What is it the teachers know that the others do not? (Kilpatrick, 1988)

Bauer (1998) is also critical of letting the pupils use their own computational methods, which are often called "halbschriftliches Rechnen" (half-written computing) in German. He points out that these methods might, in everyday school practice, often fall into the decay of normalisation and automatisation, i e the development of new algorithms. These algorithms will, however, be much less effective than the traditional ones, and therefore there will be no reason to abandon them.

I will also say a few words about number sense. In my country there recently appeared a lot of articles on this subject in our journal on mathematics education, Nämnaren (Emanuelsson & Emanuelsson, 1997; Reys & Reys, 1995; Reys, Reys & Emanuelsson, 1995; Reys et al, 1995a, 1995b).

Various authors have emphasised different aspects of number sense, and I will here restrict myself to four of them, which I believe have been especially important in my research. A pupil with good number sense

1. understands the meanings and magnitudes of numbers;
2. understands that numbers can be represented in different ways;
3. knows the divisibility of numbers;
4. knows how to use the properties of arithmetic operations.

I will discuss some applications of each of these aspects that are appropriate in connection with alternative written computation and mental arithmetic.

1. The understanding of place value is a part of this aspect, both in whole numbers and in decimal numbers. For instance, a pupil should understand that 998 is very near to 1000, and that 0.05 is greater than 0.0375. She should also acquire a good conception of very big and very small numbers, one thousand, one million, one thousandth, one millionth.
2. Above all, the connection between whole numbers, decimal numbers, and fractions belongs to this aspect. A pupil knows that 12.0 is mathematically the same as to 12, 9/3 as 3, and 2/5 as 0.4. The ability to partition numbers in different ways belongs to this aspect as well. It is sometimes practical to realise that
$8 = 6 + 2 = 5 + 3 = 4 + 4$; $36 = 2 \cdot 18 = 3 \cdot 12 = 4 \cdot 9$; $316 = 320 - 4$.
3. This aspect stresses the advantage of knowing, for instance, that 25 is a divisor of 175, that 4 is a divisor of 16, and that $4 \cdot 25 = 100$. In such a case I can easily compute $16 \cdot 175$ as $4 \cdot 7 \cdot 4 \cdot 25 = 28 \cdot 100$. (In this case I also had to use some of the properties of arithmetic operations).
4. This aspect contains the ability to transform arithmetic expression with the help of among others the commutative, associative, or distributive properties of arithmetic operations. A pupil can look upon $27 + 8$ as $27 + (3 + 5) = (27 + 3) + 5$. And $6 \cdot 83$ can be computed, either mentally or with support notes, as $6 \cdot 80 + 6 \cdot 3$. (The first and the second aspect have been used as well.) The computing of $25 \cdot 7 \cdot 4$ will be easier, if the order of the factors 7 and 4 is reversed.

Purpose and Questions

As I have mentioned earlier there were mainly three reasons for starting the project:

1. The existence of calculators and computers to make computing faster, simpler, and more reliable.
2. An increasing demand for a citizen's number sense and skill in estimation, partly due to what is mentioned in 1.
3. Social constructivism as a theory of learning.

Research has shown that the pupils' own methods for computing in the four arithmetic operations are more like effective methods for mental computation and estimation than standard algorithms are. It has also been shown that pupils will acquire better number sense by inventing their own methods for computing than by following given rules.

I therefore wanted to investigate, what effect teaching might have in a Swedish classroom, where the pupils were not taught the traditional standard algorithms during their first five years at school (ages 7 - 11), might have in a Swedish classroom. As I have already mentioned, a lot of research in this area has already been carried out. I wanted, however, to follow the process in one class very thoroughly and for a long period of time. My research questions were:

1. How is the pupils' number sense affected?
2. How is the pupils' ability to do mental computation and estimation affected?
3. How is the pupils' motivation for mathematics affected?
4. Is there a difference between girls' and boys' number sense and ability in mental computation and estimation?
5. Is there a difference between girls' and boys' motivation for mathematics?

Method

I followed one class from their spring semester in year 2 up to and including their spring semester in year 5. The reason that I started in year 2 was that in Sweden we generally start teaching the standard algorithms during that year at school. Due to some pressure from parents, the pupils who wanted were given the opportunity to learn standard algorithms during year 6. I therefore met a few of the pupils at the end of this year for clinical interviews and to ask some other questions.

To give a better picture of the experimental design, I summarise it in the following way:

1. The pupils were encouraged to use their own written methods (including the use of drawings) for all kinds of computations that they could not do mentally.
2. Mental computation and estimation were encouraged and practised.
3. The pupils had calculators in their desks. They were used for number experiments, for control of computations and for more complicated computations.
4. In all other respects, the pupils were taught in, what I would call a traditional way.

I want to add that the calculators played a minor role in this experiment. However, I chose to let the pupils have them in their desks, because they were one of the reasons for the realisation of the project.

The research and evaluation methods were mainly qualitative:

- Clinical interviews
- Observations
- Copies of the pupils' computations during observations
- Ordinary interviews with pupils
- Interviews with teachers.

I supplemented these methods with tests and questionnaires.

Three girls and three boys were picked out for clinical interviews. They were chosen to represent different levels of achievement in mathematics. The three girls formed a special group during the observations, just as the three boys did, but in order to be fair to the other pupils and also to get as much information as possible, I also made observations in groups formed by the other pupils in the class.

Clinical interviews, interviews with pupils, tests and questionnaires were undertaken at the beginning of the spring semester in year 2 and in the middle of the same semester in years 3, 4, and 5. In the clinical interviews I spoke to one pupil at a time.

In this paper I will concentrate on some activities, procedures, and achievements of the three girls mentioned.

Results and Comments

The three girls are here called Ann, Britta, and Cecilia. Especially in the beginning most of the computing was done mentally, but I always asked the girls to explain their reasoning. In this paper, everything that was said, has been translated from Swedish into English as literally as possible. In Sweden, the school year starts in August and goes on until June in the following calendar year.

As space is limited, I will only discuss multiplication. However, I want to add that the pupils came across special difficulties in all the four arithmetic operations, except perhaps addition. In subtraction, many of the pupils thought that this operation is commutative. 25 - 18 was thus solved as 20 minus 10 plus 8 minus 5, giving the result 13. For some pupils it took very long time to overcome this difficulty. In division, the main problem was that the pupils could generally not partition the numerator into hundreds, tens and units to do the computing in a suitable way. They had to find other and better ways to partition the numerator, which was sometimes very difficult for them. Therefore, they often tried a more primitive trial and error strategy.

Multiplication did not appear until year 4. The exercises were sometimes solved with the help of the distributive property but as often with repeated addition or a mixture of both.

I gave the exercise 3·28 in the clinical interviews in April in year 4. Ann and Cecilia used a mixture of methods, reasoning 20 + 20 + 20 = 60; 8 + 8 + 8 = 24; 60 + + 24 = 84. Britta's strategy was a little different: 20 + 20 + 20 = 60; 8 + 8 = = 16; 76; 76 + 4 = 80; 80 + 4 = 84.

In the same clinical interviews, I wanted to see if the girls could multiply with 10, 100, and 5 in a fast and effective way. I therefore gave them the exercises 10·12, 100·12 and 5·12. All three girls computed the first one as 10·10 + 2·10. Ann wrote 100 twelve times in the second one. She had to count these hundreds in pairs, 200, 400, 600 etc., to get the product. Britta and Cecilia wrote 10·100 = 1000; 2·100 = 200 and gave the correct answer. None of the girls could see that the answer of the third computation should be half that of the first. Instead they started once again to compute the product in different ways.

In May in year 4, to solve the exercise 2·212 both Ann and Britta wrote: 2·200 = =400; 2·10 = 20; 2·2 = 4; giving the answer 424. Cecilia wrote: 2·2 = = 4 = 400; 2·1 = 2 = 20; 2·2 = 4 and arrived at the same answer.

On the same occasion, however, they, resorted to addition when the multiplication facts got a little more complicated. To solve 6·27, Britta wrote 20 + 20 + 20 + 20 + 20+ 20 = 120; 7 + 7 = 14; 7 + 7 = 14; 7 + 7 = 14. In the same exercise Cecilia wrote 27 + 27 = 54; 54 + 54 = 108; 108 + 54 = 162. Cecilia told me that she knew the multiplication fact 6 times 7 is 42, but she declared that she was not certain enough to be able to use it. Ann solved the exercise in a manner similar to Britta's.

In September in year 5, I could again see a mixture of the two methods. Ann and Britta solved 5·44 writing: 5·40 = 200; 5·4 = 20; 200 + 20 = 220, but they reasoned 80 plus 80 is 160; 160 plus 40 is 200. Cecilia, on the other hand, could do the multiplication involved directly. Britta told me that she knew that it was possible to do the computation in the same way as Cecilia but that she found it too complicated.

Cecilia showed another mixed method in November in the same year. She was asked to compute 7·24 and wrote 3·24 = 20 + 20 + 20 = 60; 60 + 12 = 72; 72 + 72 = =144 + 24 = 168. She told me that 3 plus 3 is 6 plus 1 is 7.

When I gave the exercise 10·35 on the same occasion, none of the girls found the expected shortcut. Ann and Cecilia wrote 10·30 = 300; 10·5 = 50; 300 + 50 = 350. Ann explained that 10·30 = 300, because 10·10 is 100. Britta also computed 10·30 and 10·5, but she even had to write 30 + … + 30 (10 times) and to add these numbers in pairs and handle the other multiplication in about the same way. When I asked her about the other girls' way of doing the multiplications, she said that it was a good way.

I asked her which way she thought was the fastest one, but she only answered: "Yes, but I *do* like this".

In the clinical interviews in March/April in the same year at school, Ann and Britta wrote the solution of 7·39 as 60 + 60 + 60 + 30 and 18 + 18 + 18 + 9. Cecilia, on the other hand, used the distributive law in the following way: 6·30 = 180 + 30 = 210 + 6·9 = 54 + 9 = 63 + 210 = 273. She declared that it was easier to multiply by 6 than by 7. I want to add that, in this case, I thought it was more important for the pupil to be allowed to work out her own method at ease than it was for me to interrupt her and tell her that she was misusing the equal sign.

When asked to compute 7·199 in May in year 5, none of the girls tried the shortcut 7(200 - 1). Cecilia noted down 7·100 = 700; 7·90 = 630; 7·9 = 63. She could give the answer of 7·9 directly but not that of 7·90. The other two girls tried to use a mixture of repeated addition and the distributive law with mixed success.

The same behaviour was repeated in the last observation in April in year 6. I will cite the discussion between Cecilia and me. Obs. is observer and C. Cecilia, … means a short pause.

C. It was 8 times 298, sure, and then I took … To make it a little simpler I divided 8 so that it became 4 times 200, and then I double it, as it is half of … Yes, so it was 1600, and then I took 90 times fo … or two (inaudible) four so it … I don't know why, but I took 2, then I took 18 and then 18 plus 18 is 36 and then 36 plus 36, it is 72, and then it became, as there was a zero there behind, it became 72 … or 720, and then I took 8 (inaudible) 8, it is 64 like that (inaudible) only, then it became 2384.
Obs. Yes, only a little question there.
C. Mmm.
Obs. You took 8 times 90.
C. Mmm.
Obs. What is 8 times 9.
C. 8 times 9? 72.
Obs. Yes. Why didn't you … Why didn't you take 8 times 90 is 720 at once then?
C. No, 'cause I think it was simpler, 'cause it is the multiplication table itself, sure, and then … Then it became … It was simpler so, and then I only put a zero after.

It was probably too complicated for her to use the multiplication table when computing 8 times 90. Ann wrote 4·200 = 800 twice; 1600 + 720 + 56 = 2300 + 70 + 6 = 2376. She found her mistake and could give the correct answer while Cecilia was discussing her solution.

Comments

I cannot say that the girls became very skilful at doing multiplication, at least not in finding a fast and accurate way, for instance using the distributive property throughout. There were a lot of good attempts to do it, but as soon as the numbers involved and/or the multiplication facts to be used became more complicated, the girls resorted to repeated addition. We can see this very clearly in the observation of May in year 4, when the girls solved 2·212 by using the distributive property but stuck to repeated addition in one way or the other when solving 6·27. In the dialogue between Cecilia and me at the end of the section, she clearly declared that she could multiply 8 times 9, but that it was too complicated for her to multiply 8 times 90 in a corresponding way.

We can also see that the girls, especially Cecilia, preferred to use what they saw as simpler multiplication facts to more complicated ones. In March in year 5, Cecilia thus multiplied 6 times 30 and added 30 instead of multiplying 7 times 30 directly.

During these observations, they did not try any shortcuts that might come naturally to people more used to computing. For instance, they could not compute 10 times 35 directly, nor could they use this product to find the result of 5 times 35. Another example is the computation of $7 \cdot 199$. None of the girls took advantage of the fact that 199 is very near to 200.

Thus, we might ask if the girls learnt anything at all in multiplication. I believe that they did, although they would have needed much more time and practice to be able to understand and master the use of the distributive property. It might also be true that the class teacher and I should have given them more hints and examples to encourage them to use this property and above all to be able to find shortcuts like the ones just discussed. As I have interpreted social constructivism, such behaviour would not be in opposition to the theory, if only we could make sure that the pupils had a chance to understand our suggestions and make them fit in their own previous knowledge schemes.

Discussion

In this paper I have chosen to let the readers follow a part of my project, the performance of three girls, who were generally working together during the observations, in multiplication. This means, of course, that a lot of results of the project have been omitted.

We can see that the girls' solutions could sometimes be very primitive and complicated, e.g. Britta's solution of the multiplication exercise $6 \cdot 27$ in May of year 4. However, I could also see, at least in addition and subtraction, that awkward solutions sooner or later developed into smart methods that could be used for almost all problems of a similar kind.

That did not mean, however, that the girls always used the same method for all exercises in a given arithmetic operation. They often looked at the numbers involved and tried to adjust their methods to them. In the interview of year 4, Cecilia very clearly declared that she had a supply of different methods and that it depended on the exercise which one she chose.

Sometimes the girls were hesitant about changing methods, although they were shown more effective ones. Britta made this very clear when she was asked to multiply 35 by 10 in November in year 5. She stuck to her method of adding 30 ten times, although the other girls directly computed 10 times 30 is 300. I see this as a demonstration of the ideas of social constructivism. Britta felt a need to construct *her* method. Although she could understand and presumably even appreciate her peers' method, she was not ready to make it her own.

Although, as stated above, the girls' methods were sometimes rather primitive, we could see many examples of how they practised and developed their number sense.

Finally I will try to sum up the most important points that I think I saw when following this group of three girls.

- Although the girls sometimes had difficulties with some of the arithmetic operations, after a longer or shorter period of time, they could overcome these difficulties and find methods, which they understood and which could help them solve the exercises.
- The methods that the girls used, were mostly less effective than the standard algorithms. On the other hand, they were more like those used for effective mental arithmetic and computational estimation. Thus, the pupils could avoid thinking in one way when doing written computing and in quite another way in mental arithmetic and estimation.
- From the methods the girls used, it could be seen that they acquired and developed good number sense.
- Even after they had been taught the standard algorithms, the girls preferred their own methods.

Anyhow, I think we have to consider whether, in the age of calculators and computers, it would not be wiser to take the chance to let our pupils develop their number sense and their skill in mental computation, even if it might cause a deficit in the use of the most effective written computational methods.

However, I think it is very important that our pupils feel that the methods which they use are really their own. To teach our pupils alternative methods instead of the standard algorithms, would, in my opinion, be totally wrong. In such a case, the algorithms would only be replaced by less effective methods, which would most probably be as unfamilar and difficult to understand to our pupils. We would land up in the situation that Bauer (1998) fears, mentioned in the paragraph "Previous Discussion and Research".

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How can we describe young children's arithmetic abilities?

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Background

My interest in how children develop arithmetic abilities began about 15 years ago when I became engaged in teacher training. Students often asked for a developmental scheme for the early arithmetic development like the one they had become acquainted with for reading abilities. My answer always was one of disappointment, that nobody had formulated such a scheme that I knew of, at least not in Swedish. This was the time when Dagmar Neuman presented her dissertation called "The Origin of Arithmetic Skills" (Neuman, 1987). Being a phenomenographic investigation it exposed different categories of uses or meanings of numbers that children show, and although they were logically ordered according to their "complexity", they were never intended to describe an individual child's development of arithmetic knowledge. What I wanted was a description of the cognitive development involved in learning arithmetic founded on empirical data on that development and not on logic. So when I in 1994 got the opportunity to do an investigation my search for a developmental scheme went abroad.

Critique of Piaget

Piaget's influence on the common view of children's development of numerical capacities has been immense stressing the ability of number conservation as a sign of having the necessary logical maturity for understanding number. One difficulty with Piaget however is that he founded his analysis of number competencies on the definition of number formulated by Russell and Frege at the end of the 19th century. This definition was set forth as a part of their ambition to formulate mathematics in terms of logic. It is noteworthy that although humans have been able to count and speak, write and think about numbers for many centuries, it was not until hundred years ago that someone found a logical consistent (although the class concept leads to paradoxes) definition for number. A number, e.g. four, is seen as the equivalence class of all sets containing four elements. So the definition is built on set theory, formulated mainly to be informative about infinite sets where one-to-one-correspondence (and not counting) plays the vital role for establishing equivalence relations. And so Piaget's investigations about children's conceptions of number are dominated by tasks that interrogate children's ability for one-to-one-correspondence and for ordering different things including number.

In the second half of the 70's researchers began to question the findings of Piaget, or rather the interpretations of his findings. There were psychological research stressing the ability to count, research directed to the cognitive abilities that the development of numerical skills demand, and research into children's problem solving strategies for simple addition and subtraction word problems (Geary, 1994).

Much of these research efforts have been summarised in Fuson (1988), Fuson (1992b) and Geary (1994). Fuson also formulated developmental schemes for different number competencies. These schemes have been the starting point for my research.

The formulation of an analysing scheme

Karen Fuson (1988, 1992b) summarised much of the research on cognitive development in connection with number. This research on children's thinking

> … unequivocably reveals children to be constructors of their own knowledge who see a given situation according to their own conceptual structures for that situation. (Fuson, 1992a, p 54)

When you analyse children's arithmetic abilities you find that there are several abilities involved (Fuson, 1988). There are abilities connected to

- **Number sequence** - such as the ability to count forwards and backwards, to step count and to start from whatever number you wish.
- **Counting** - such as being able to point at one object at a time synchronised with uttering the number words and being able to keep track of which objects you have counted and which you have not.
- **Cardinal numbers** - such as the ability to break up a number into smaller numbers.
- **Solution procedures** – such as the ability to count on from first or largest, to count back or to count up and to use known "number facts" for solving different kinds of problems.

In table 1 different developmental levels are summarised. In sequence structure abilities develop from string (where words are not separated), to separate words, to being able to start anywhere, to using number words as items for your counting, and to seeing the words as both cardinal numbers and as parts of an ordered sequence that you can mentally move on forwards or backwards at your own will. When you count, the objects or items that you are able to count change from concrete perceptual items like counters or fingers to more abstract items like the number words themselves or mental representations of perceptual items and mental objects like numbers. The cardinal structures presented in column three can be seen as a pictorial representation of the solution procedures in column four. Objects with filled contour represent those that are first or simultaneously attended to.

As can be seen these abilities are developed in parallel and are dependent on each other. This interdependence is rather intricate and not always straightforward. This makes the analysis somewhat complicated. Another difficulty is that children do not always use their most sophisticated procedures for solving a task, but depending on the situation choose to use a less sophisticated procedure (Fuson, 1988). The strategies children use are also strongly dependent on the numbers involved (Fuson, 1988; Geary, 1994). With small numbers they generally perform on a "higher" cognitive level than with larger numbers.

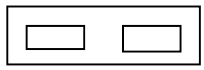
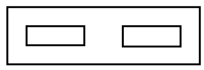
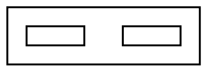
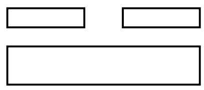
Sequence structure	**Counting and conceptual units**	**Cardinal conceptual structures**	**Solution procedures**
String **No units**			
Unbreakable list Separate words; Start from one	Perceptual unit items Single representation addend *or* sum		
Unbreakable list Separate words; **Start from one**	Perceptual unit items Single representation addend *or* sum		Level I Counting All
Breakable chain Separate words; Start anywhere	Perceptual unit items Simultaneous representation addend *within* sum		Level II Counting on, first addend abbreviated
Numerable chain Sequence unit items	Sequence unit items Simultaneous representation addend *within* sum		Level III Counting on, with first or last addend abbreviated using a general keeping-track-method
Bidirectional chain	Cardinal numbers Can be decomposed into ideal unit items		Level IV Derivations or known facts

Table 1. Developmental levels (adopted and reworked from Fuson (1988) and Fuson (1992a).

Method and objectives

My objectives with this work have been to investigate whether the schemes for cognitive development of numerical abilities developed in the US can be used for analysing Swedish children on an individual level, and to see how these abilities develop for 6 year olds during their time in pre-school.

Data were collected in the form of two video-taped interviews with 16 children born in 1988, the first in October-November in 1994, and the second with the same 16 children and one more child in May 1995. All the children participated 3 hours a day in what was called "6-årsverksamhet" (activities for 6 year olds). They co-operated 3 days a week with a Form 1 school class (7 year olds) in activities including some practical work in mathematics. Interview questions included tasks examining their abilities in number sequence, counting, number abstraction and problem solving. Data have then been analysed using different kinds of protocols to record abilities according to the analysing schemes presented above.

In order to analyse a child's abilities we have used tasks on:

- Sequence
  (forwards, backwards, before, after)
- Counting
  (linear, circle, unordered sets)
- Abstraction
  (including the Guessing Game introduced by Neuman (1987))
- Problem solving
  (join - result unknown, separate - result unknown, join - change unknown, join – beginning unknown)

Preliminary results

The tasks were created to reveal the level of development for each child. In a general sense our investigation shows that this is possible, but the analysis is not always an easy one. The solution procedure a child uses varies depending on the numbers used in the task. Another feature complicating the analysis is the child's ability to keep track for small numbers without having to use a general keeping track procedure, i.e. their ability to subitize small numbers.

Comparing the outcomes for the two interviews it is noticeable that there is only a small difference in individual children. Our expectation was that the development would be more easily detectable. The clearest change is in the length of their number sequences, where all but 3 children show an increase between the two interviews. Overall the length of their number sequence is comparable with data from the US (Fuson, 1988), with results for the autumn interviews fitting the Kindergarten data and results for the spring interviews fitting the Grade1 data.

What is striking however is the great span in ability between different children. This is consistent with Fusons findings:

> We have been struck in all of our number work by how wide the age span is for correct performance on any task. There is frequently a span of as much as 1 $\frac{1}{2}$ or 2 years in the age at which children respond correctly to some number task. (Fuson, 1988, p 416)

There is also consistency between the abilities. Generally speaking, if a child performs on a high level on e.g. solution procedure it also performs on comparable high levels on sequence, counting and cardinal number. However it seems as if the type of problem a child is able to solve and the solution procedures it uses are more consistently informative on the developmental level than the other abilities investigated, including the Guessing Game used by Neuman (1987).

Discussion

As children grow older they develop numerical abilities. This is a development that takes time. When we speak of development it is easy to impose some sort of one-dimensional image of the process. Starting from one stage, going through others, and reaching a mature, grown-up understanding. If you have a constructivist view you know that this is something you as a researcher impose on reality. This is your construction, your pattern. The interview data recorded on video are rich, but once you start to classify or categorise the answers these classifications or categorisations themselves start to become your data imposing a more definite image of the phenomenon you are investigating than is really there. Thus we have to be somewhat humble interpreting our results, cf. Mason (1998).

Although the schemes presented by Fuson are an important contribution to our understanding of children's development of number concepts, they are not always easy to apply. One of the reasons for this is children's ability to subitize small numbers that get in the way for the use of what Fuson sees as more cognitively demanding procedures. A consequence of this is that

> … many children clearly function at multiple levels that vary by the size of numbers in addition and subtraction situations. (Fuson, 1992a, p 97)

Children do use different methods for solving different tasks depending on a number of reasons. Among these is their cognitive ability. With small numbers they can use known facts or a short counting on procedure, because it is effortless. With medium numbers they may use a counting procedure with a perceptual keeping-track-method, and with big numbers they perhaps do not have a method.

However there appears to be a crucial step in children's development of arithmetic abilities that have not been exposed well enough yet. The problem is why some children spontaneously come to use known facts in derivations of other number relations, while others do not. Instead they resort to counting procedures, where the keeping-track-methods become cumbersome. Gray and Tall (1994) give evidence for this phenomenon. They have also introduced the concept of a procept to stress the double roles played by mathematical symbols. E.g. 3/7 can denote either a rational number or the division of two natural numbers. In the same way they claim that 5 can stand for the number 5 or for the result of the process of counting 1, 2, 3, 4, 5. In this way mathematical symbols often stand for shorthand notations for some process that then becomes objectified into a concept that we can speak about in its own right. Symbols that have this double role of denoting both a process and a concept they call procepts. Gray (1997) argued that a child's ability to see a number not only as the result of a process, but also as an object that can be decomposed into smaller parts, is vital for its possibility to find mathematics a manageable task.

It is this possibility of compression of processes into new objects that can mentally be operated on, that lies at the heart of mathematical abstraction. As can be seen in our study, those who are "gifted" spontaneously seek shortcuts or patterns that they can use in solving a task. Here the ability to subitize is vital, but which role it plays and how it is used needs to be investigated further. Fuson's (1988, 1992b) schemes for the development of solution strategies presupposes the development of a general keeping-track-method as a prerequisite for the use of derivations. We propose that there might be parallel processes going on: a child is engaged in establishing number relations for small numbers using processes where subitizing plays an important role and at the same time use counting while dealing with situations involving bigger numbers. The operations with smaller numbers seem to play an important role in the child's development of understanding different properties of numbers such as the commutativity of addition. It is to our mind not clear what role the development of a general keeping-track-method plays for the development of what Gray (1997) calls proceptual understanding of number.

Fennema et al (1998), investigating grade 1 to grade 3 children in the US, have found that there is a qualitative difference in the solution procedures that boys and girls use in problem solving. Boys do more derivations, while girls use more counting, generally speaking. These results are rather puzzling and bring in another dimension as well into why different children seem to do “different kinds of mathematics” in the primary school, to use an expression from Gray and Tall (1994).

Conclusions

Developmental schemes are useful for analysing children’s arithmetic skills. But this analysis is often not an easy task. The development is not straightforward and one-dimensional, but rather consists of parallel processes of expansions and integrations, resulting in what can be seen as “inconsistent” behaviour. In this process counting and perceptual aids are important. Abstraction is dependent on “well-digested” experiences with well-known objects. For the small child counting experiences provide such experiences and the use of fingers can serve as a good concrete aid. But later on fingers also can get in the way for thinking (Gray, 1997), keeping your awareness on the counting procedure and keeping-track-method you use, instead of allowing it to operate on and notice relationships between numbers.

An important issue is to learn more about why some children spontaneously use derivations based on a growing number of number relations, while others resort to counting procedures for even basic number relations. In order to do so we need to follow the growth of arithmetic knowledge in individuals and how this growth varies between individuals. We think an important contribution to the understanding of this phenomenon can be gained by longitudinal studies focused on the important key questions.

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Mathematical modeling and prospective teachers

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Introduction

The study reported here arose from my experience in teaching mathematics to prospective mathematics teachers and from the ongoing evolution of technology. During the past three decades, personal computational technology has evolved from four-function calculators in the 1970s through scientific calculators in the 1980s to graphing and symbolic calculators in the 1990s. Today, most students who study high school or college mathematics also have easy access to computers equipped with a variety of mathematical tool systems. The evolution in technology has affected the content of some of the courses in mathematics for prospective teachers and many times also the way they are taught.

During the past 4 or 5 years, there has been a distinct change in some of the courses in the program for prospective mathematics teachers at the University of Gothenburg. At the beginning of this period, the technology was introduced as an isolated part of the course, often through a visit to the computer lab. Today, the program includes courses in which the technology is an integral part of the syllabus, including the assessment. In my case, the course in mathematical modeling I teach every semester has changed dramatically during that time. As a consequence, I started to look more closely at the students' conceptions of mathematical modeling.

Thanks to the technology, students can model more complicated situations today, but as a consequence they seem to encounter more and more problems with interpreting and understanding the results provided by that technology. I have observed students mistaking model for reality, trusting the computer more than themselves, shifting the authority for mathematics to the computer, and being unwilling to take full responsibility for their own learning and performance. These observations led me to search for answers as to why students seem to forget reality when using sophisticated software to model problems and to lose faith in their own mathematical knowledge, thereby trusting in obvious distortions of the relation between mathematical model and experiential reality.

Three Studies

Two research questions led me to conduct three different studies:

- How do preservice teachers relate mathematical models to reality when using software tools to generate the models?
- What conceptions and misconceptions lie behind the decision to believe more in a mathematical model than in real-world phenomena?

The first study, in 1997, explored how students modeling with technology relate their models to reality and included 71 students, of whom 5 formed a special study group. The second study, in the spring of 1998, investigated the question of students' conceptions and misconceptions when modeling with technology. Thirty students participated, of whom 8 formed a special lab group. Finally, 70 students participated in the third study, in fall 1998, which dealt with the change of authority when modeling with technology. In this study, 5 students formed a special study group.

The data collected in the three studies came from multiple sources:

- Questionnaires
- Videotaped interviews
- Observations of the lecture sessions and the labs
- Students' lab reports
- Students' written assignments and their final exams.

The questionnaires revealed some of the students' knowledge of mathematical models when they entered the course. For example, the 250 students who responded to the following problem gave a variety of functions:

> You drop a chocolate bar from the top of the 330-meter-high Eiffel Tower in Paris. The distance of the bar above the ground depends on the number of seconds that have elapsed since you dropped it.

Exponential curve:	8%
Right half of an inverted parabola:	47%
Straight line:	42%
Don't know:	3%

Some Results from Study 3

I will briefly discuss some of the results from the third study with a study group of five students: Nina, Olga, Patricia, Robert, and Sarah (pseudonyms). The results are related to the students' responses to one of the final exam problems. Students attempting this kind of problem in an examination should, as much as possible, focus more on qualitative reasoning and less on reproduction of facts and basic routines. The fact that the students were allowed to use graphing calculators and mathematical software in their examination further stressed the importance of selecting problems that were relevant in the presence of this aid. Just as the problem should remain non-trivial in the presence of the technological tools, so the use of the technology should not be essential and the only success-creating component in the performance of the student. A relevant problem should encourage the student to make different assumptions and use different strategies where technology can serve as an aid, never as a goal. Based on these assumptions and perspectives, the following problem was selected as part of the examination. The problem concerns the heating of houses and the effect of insulation

Part A
Table 1 gives the weekly gas consumption (m^3) and average outside temperature (°C) for a particular house before the installation of cavity wall insulation.

Table 1

Temperature (°C)	-1	0	2	4	5	7	10
Gas (m^3)	206.6	195.6	173.2	149.4	115.7	116.0	82.4

Construct the simplest possible model to describe the correlation between weekly gas consumption and outside temperature.

Step 1: y = *193.2 – 11.6*x

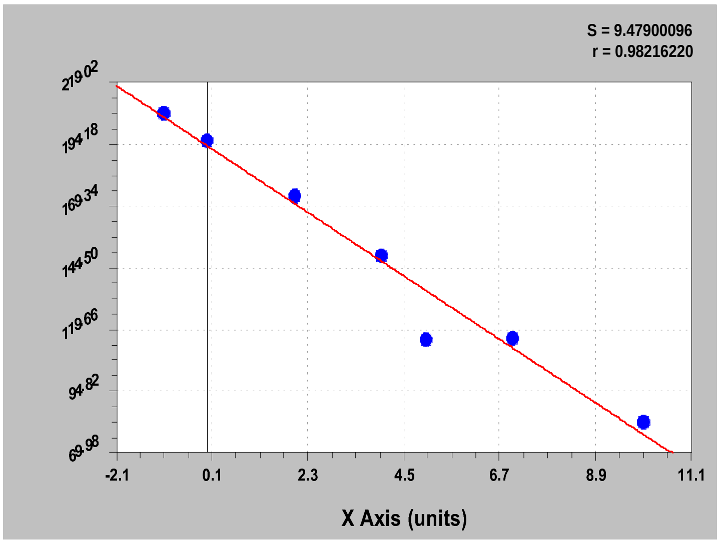

Figure 1: A model for Part A

Part B
Table 2 gives similar data for the same house after insulation.

Table 2

Temperature (°C)	-1	0	1	3	6	8	10
Gas (m^3)	134.4	127.6	120.6	110.1	89.4	72.7	59.4

Construct the simplest possible model to describe the correlation between weekly gas consumption and outside temperature after insulation.

Step 2: y = *128.2 – 6.8*x

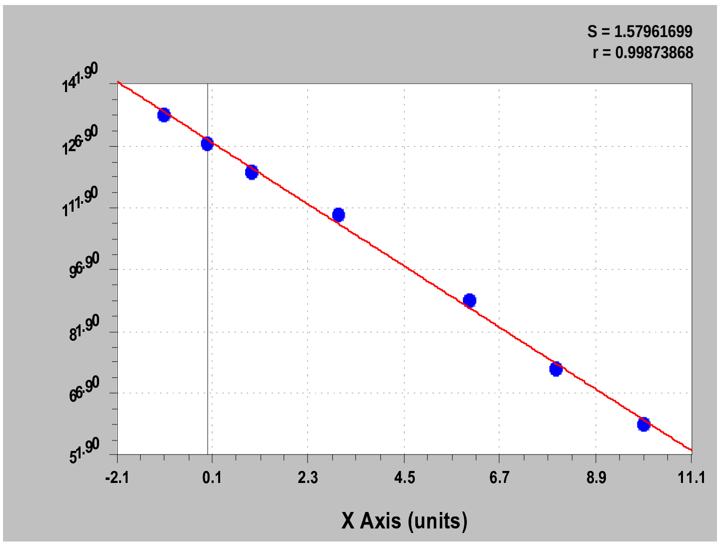

Figure 2. A model for Part B

Part C
Table 3 gives monthly averages of the outside temperature at the location of this house from October to May.

Table 3

Month	O	N	D	J	F	M	A	M
°C	10.3	6.7	4.4	3.4	3.8	5.7	8.7	11.5

Find an appropriate model to describe the annual variation of the average temperature over the year.

Part D
Write an expression for the amount of gas saved in one year by having insulation, and calculate a numerical answer for the amount of gas saved.

Authority and Responsibility

At the end of the course and when the final take-home exam was given, the students were relatively well trained in using the technology, with no visible hesitation when shifting between different software programs to draw graphs or identify models. Only one student of the five in the study group maintained a sceptical attitude throughout the course, namely Sarah, who argued that the more complicated the models became, the more dangerous it was to use computers. She said:

> It's as though we become seduced by the fancy graphs and the quickly generated results with all the decimal places. And if the model is complicated, you really don't have any chance to follow the calculations.

Sarah's observation – that the more complicated the situation, the harder it is to see through the modeling process supported by computer software – was reinforced in the final exam problem above. The gas problem began as an easy problem, but then it created unexpected difficulties for all the students in the study group, as well as most of the rest of the class.

After finding the two linear models in Parts A and B, the students ran into difficulty in Parts C and D because they needed a periodic model from October to May to illustrate the temperature changes. As a consequence, a majority of the class (and all students in the study group except Sarah) indicated that in every part of the problem, the model should be chosen exclusively by the technology. The selection principle they used relied most heavily on ranking the values of the correlation coefficients. Nina wrote:

> In order to find a model, I used the software CurveExpert. I picked a continuous and periodic function since that would also provide me with temperature values for the summer months. I decided that the function $y = a + b \cdot \sin(ct + d)$ was most suitable. After further investigation, I found that this function is not quite periodic.

Sarah, on the other hand, decided rather early in the modeling process to chose or to construct her own model, not to select one of those offered. She wrote on her exam:

> I enter those values into CurveExpert and apply curve fitting of a model like $y = a + b\cos(cx + d)$, in which I define $c = 2\pi/12$. That will force a periodicity equal to 12 months. Then I get the following model:

$y = 9 + 5.8\cos(\pi/6x + 0.86)$

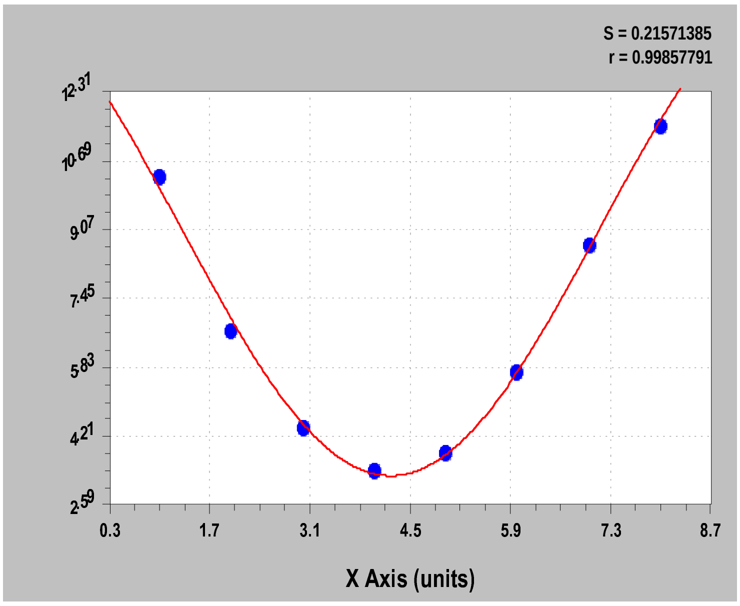

Figure 3. A model for Part C

Three of the other students in the study group, Nina, Olga, and Patricia, selected a model based on the calendar year, which meant that they arranged the monthly averages of the outside temperature from January to December, thereby yielding the following illustrative figure:

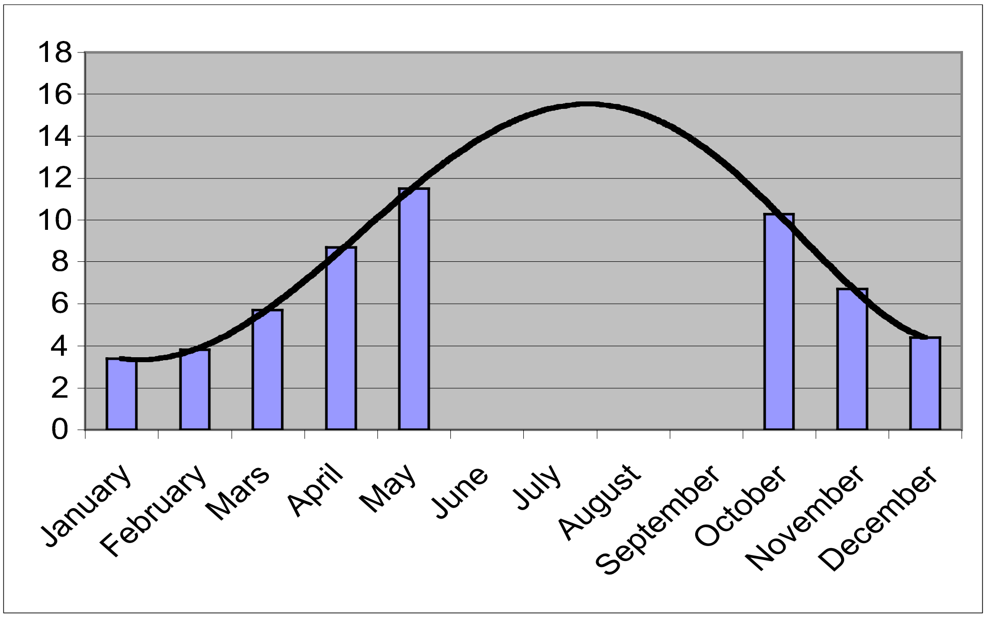

Figure 4 A different model for Part C

Both Nina and Olga became so confused by this modelling process that they subsequently employed the computer again, apparently without really needing it. Having obtained a pretty good model in Excel, they took all the points used to draw the sine curve there and generated the same model in CurveExpert. Then they integrated over the year. Presumably because they did not have the Excel figure in front of them, they then forgot to exclude the "summer gap" and got at least twice as much gas saved, as they should have. Patricia, in contrast, used the possibility of making interpretations of the graph. She wrote:

> When I see the graph and that the curve is "empty" during the warmest months of the summer, then I realize that it would be very stupid to use gas for heating when it is warmer outside than inside. And I just exclude this interval from my calculation.

In fact, the mean temperature in Table 3 over the 8 months can be roughly estimated as 7 °C. Tables 1 and 2 indicate that the amount of gas saved for a temperature of 7 °C is about 35 m^3 per week. A very rough estimate would be that the amount of gas saved over a year will probably not exceed 1200 m^3. On the other hand, Table 3 indicates that 5 months have average temperatures below 7 °C and therefore that the amount of saved gas will not be below 700 m^3.

The only student in the study group who did anything along the lines of a simple arithmetic calculation and stayed with it was Sarah. That approach helped her during her modeling process to make important and influential decisions. On her exam, Sarah wrote:

> Using an estimate based on the values from my diagram [from the fitted model], one can see that the area is about 1200 m^3. This in turn suggests that the calculated value of 1159.38 m^3 is a reasonable value for the amount of gas saved during a year.

Robert got very upset after receiving back his take-home final exam. He had neglected to take into account that the gas consumption was expressed in weeks whereas the average outside temperature for the geographical location was in months. His model thus became a mixture of two different units of time, and his result was about a tenth of what it should have been. He expressed his dismay in the final interview:

> *Robert:* I know this, I know this stuff. I know that I know this. I'm good with integrals. I can't believe that I made this error and that I didn't do a rough check before. I've spent hours and hours making nice graphs and formatting the mathematical text. And then everything is wrong!
>
> *I:* What happened to your strategy of always doing a mental and paper-and-pencil check first?
>
> *R:* I don't know. I guess I got carried away, and for some reason I thought that I didn't need the common-sense check. I just became obsessed with the problem!
>
> *I:* Do you think the problem was too complicated?
>
> *R:* No, the problem was great. It's just that I'm upset with myself. The problem sort of rips your clothing off and shows how much of the mathematics you have understood. I think it is healthy to face problems like this and to be forced to write about them, but at the same time it is almost too revealing!
>
> *I:* So what is your opinion about the trust you now put in computer-generated results?
>
> *R:* I don't know. I think that I'm as sceptical as before; at least I still know that I need to be in control. At the same time, when you learn to use computers, it is hard not to use them all the time. I have heard about people who have problems writing letters by hand after using computers for a long time, and now I can believe that.
>
> It is almost the same with me. Now I've learned to make nice graphs in Excel and CurveExpert, to cut and paste into Word documents, I hesitate to do mental or paper-

> and-pencil estimates first. It is like I'm drawn to the computer first instead, and then it is hard to stop or look back.

When students are forced to explain and argue for their models, they disclose inaccuracies and misconceptions in a way that may very well be hidden otherwise. Olga was as open about her feelings as Robert was. She and Nina had arrived at the amount of 4000 m^3. Olga had major difficulties understanding the link between the models in Parts A, B and C and the model she needed to construct for Part D. On her exam, she had written the following:

> Also in this problem I employed CurveExpert to do the heavy work and just entered the given values.… Then I took the integral of $f(x)$ from 0 to 52 to get the amount of gas saved during one year.

Like some of the other students, Olga neglected to consider that no heating was during the summer months. In the final interview, she expressed her feelings about not just the problem but the whole course:

> *Olga:* It's so typical of you guys in this course. It's always complicated and hard, never easy. It is just so typical.
>
> *I:* Do you mean the integral?
>
> *O:* I mean the whole thing. How do you think we are supposed to use computers and the results we get from them when you teachers criticise us this way? I worked a lot on this problem, and I used both CurveExpert and Excel to generate models. So how could it be wrong?

For a detailed discussion of this study and the two others, see my dissertation, "Mathematical Modelling by Prospective Teachers," completed at the University of Georgia in May 2000.

Conclusion

In short, it could be said that teachers at all levels need to be cautious about what students actually understand about the modelling process and how they interpret it. A clear focus on the validation part of mathematical modelling is undoubtedly more essential in the presence of technology than ever before.

Killer-equations, job threats and syntax errors

A postmodern search for hidden contingency in mathematics

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The difference between Modern research and Postmodern counter research

Modern research and post-modern counter research are both working in the borderland between nature and culture, between what is given and what could be different, between necessity and contingency. Out of the breakdown of pre-modern order, modernity saw the emergence of contingency. Scared by the idea of a contingent world modernity desperately began to reinstate order (Bauman, 1992). Modern research sees contingency as hidden necessity, and tries to discover the nature of this necessity wanting to produce new convincing knowledge claims "A is B". On the other side post-modern counter research tries to uncover hidden contingency in necessity wanting to produce new inspiring knowledge suggestions "A could also be B".

Although some post-modern thinking might see both culture and nature as social constructions this paper recognises a borderline between nature and culture to be drawn between numbering nature and wording culture. Nature can speak through number-meters, rulers, but since no word-meter exists, the world cannot word itself, hence all phrasings are contingent, except this meta phrasing. Phrasing is freezing, and re-phrasing is de-freezing or freeing. It is a post-modern point that a phrasing constructs what it describes and that humans are clientified by ruling phrasings and discourses (Foucault, 1972). Our convictions might be not universal truths but local truths depending on the ruling phrasing, and they might change through a rephrasing. An example of a post-modern rephrasing is seen in the following case.

Killer-equations in paradise

Once I was invited for a two-month stay at a new four years Secondary Teacher Education College in South Africa created to solve the local 1% success problem in mathematics: 90% of the students did not enter the final exam in mathematics and 90% failed. The mathematics curriculum at the college and at the high schools followed a tradition of a Platonic Top-Down mathematics describing concepts as examples of more abstract concepts all originating from the mother concept "Set". In the science education classes at the college the educational theory-tradition was that of curriculum 2005, Outcome Based Education (OBE) and Vygotskian constructivism.

After the first month I followed some students in their teaching practice at a high school in a local village called Paradise. The student-teachers received a textbook and a number of pages they were supposed to cover. In a grade 10 class two equations were written on the board by the student-teacher and solved by students in the following way:

Equations:	$\frac{M}{5}-\frac{M}{2}=3$	$\frac{y+2}{4}-\frac{y-6}{3}=\frac{1}{2}$
Solutions:	$\frac{M}{10}=3$ $M=-10(3)$ $M=-30$	$\frac{6+24}{12}=2$ y = 12·2 y = 14

Figure 1. Equations and solutions

After the period the student-teacher complained: “You ask them if they understand it and they say yes, but next day they have forgotten it all. They don’t study at home, they have too much free time and no parent support. Their friends say mathematics is not interesting. 30 minutes lessons are too short, in private schools they have 60 minutes. The ministers take their children abroad. The new curriculum 2005 also asks us to teach these equations. Something has to be done.”

Other student teachers and teachers had similar complaints: Mathematics is difficult and can only be learned through hard work, but today’s students don’t like hard work. First year high school students lack fundamental mathematical knowledge from the primary school. The teaching material is outdated and in low supplies. Many secondary school teachers are not trained in mathematics. The teachers need to be workshopped in OBE. The classrooms are too crowded to practise OBE and constructivism. The instruction has to be in English, which is not the mother language.

Designing an alternative: Rephrasing equations

In these explanations the blame for the “bad play” is placed with external factors outside the teacher’s influence: “the manager, the director and the actors”. Inspired by a postmodern view looking for alternative silenced explanations I suggested looking at “the script” by rephrasing equations into two groups: Top-Down “killer-equations” and Bottom-Up “calculation stories”.

Killer-equations are equations you never meet outside the classroom and which only serve one purpose, to kill off the interest of the students. Killer-equations are examples of Top-Down equations being examples of the general equation “A = B”, where A and B are examples of arbitrary expressions. Calculation stories or practice-equations are questions arising from social practices: the social practice of shopping e.g. contains questions like “3 kg @ ? R/kg total 14 R including a 2 R fee” leading to the calculation story or equation “x·3+2 = 14”.

Also “solving an equation by doing the same to both sides” can be rephrased as “reversing a calculation”. The multiple calculation x·3+2 is reduced to a single calculation by means of a “hidden parenthesis”: x·3+2 = (x·3)+2. This calculation consists of two steps: First the R/kg-number x is multiplied by the kg-number 3 to produce the R-number x·3. Then the fee 2 is added to produce the Total x·3+2, which is 14. Reversing the calculation consist of the two opposite steps: First the fee 2 is subtracted from the Total 14 to produce the R-number 12. Then the R-number 12 is divided by the kg-number 3 to produce the R/kg-number 4.

The reverse calculation method is identical to the old “Move & Reverse”method: a number can be moved across the equal sign from the forward side to the backward side of an equation and vice versa by reversing its calculation sign.

Calculation direction:	*Forward*		*Backward*
Total	(x·3)+2	=	14
		+2 ↑ ↓ –2	
R	x·3	=	14–2 = 12
		·3 ↑ ↓ /3	
R/kg	x	=	12/3 = 4

The "Walk & Reverse" method

Forward		*Backward*
(x·3)+2	=	14
x·3	=	14–2 = 12
x	=	12/3 = 4

The "Move & Reverse" method

Figure 2. Two traditional methods

Practising the alternative

After having discussed this rephrasing of equations with the student-teachers one of them asked me to try it out in the classroom. I accepted to take over a standard 30 minutes lesson in a grade 10 class with 50-60 students. Following the design I started to present three Bottom-Up questions:

"3 kg @ 5 R/kg total ? R"	leading to the equation $T = 5\cdot 3$
"3 kg @ 5 R/kg total ? R including a 2 R fee"	leading to the equation $T = (5\cdot 3)+2$
"3 kg @ ? R/kg total 14 R including a 2 R fee"	leading to the equation $14 = (x\cdot 3)+2$

Then I introduced the reverse calculation method mentioned above. The class did a similar problem with other numbers. I then took the class to the schoolyard and asked them to line up facing me: "We start with an R-number 5 each. Now we walk forwards to steps, a "·3 step" and a "+2 step" calculating the new R-number each time". This produced the final R-number 17. "If the final number had been 14 R what did we begin with? We can guess, or we can calculate by walking backwards reversing the calculation steps." After a "–2 step" and a "/3 step" had produced 4 R we went back to the classroom and saw the resemblance between the "Walk&Reverse" method and the reverse calculation method on the board.

By erasing the arrows the reverse calculation method became the "Move & Reverse" method. Some homework problems were given for the next period, where the student-teacher took over again after the students had written down their solution of the equation $4+3\cdot x=19$ on the back side of a questionnaire.

Evaluating the alternative

The questionnaire contained a traditional quantitative opinion question and two open questions allowing for the self-phrasing of the students:

Dear Learner. I have had the pleasure of showing you a Bottom-Up understanding of an equation $2+3x=14$ seeing an equation as a story telling about the total and how it is calculated.

1. What do you think about the idea of introducing the Bottom-Up understanding of an equation in the classroom of South African secondary schools. Draw a circle around your answer (–2: Very Bad, –1: Bad, 0: Neutral, 1: Good, 2: Very Good).
2. If you have other comments to the bottom-Up understanding of an equation you can write them here.

3. You have been living with mathematics for many years now. I would be glad if you could tell me a little about your learning life with mathematics. Just write whatever falls into your mind.

I collected 50 answers. The correctness of the method and the result were graded on a (-2, -1, 0, 1, 2) scale giving the distributions (0, 3, 7, 19, 21) and (2, 9, 1, 5, 33). The answers to question 1 were (-2, -1, 0, 1, 2): (0, 0, 2, 6, 40). As to question 2, 12 answers praised the method for being easy, 25 for being understandable and 3 for being short. As to question 3 I was amazed to find among the answers 24 occurrences of a "No math - No job" myth.

So one way of motivating equations is by job threats. Another is to keep killer-equations out of the classroom only allowing practice-equations to come in.

Why might Bottom-up mathematics be user-friendlier?

As other forms of life humans need to be connected to nature's flow of matter and energy (food) and information. In premodern agriculture humans add a cultural flow of food to nature's flow. In the modern industrial culture electrons are used to carry energy, and in the postmodern information culture electrons are used to carry information. The introduction of global TV into local cultures has uncovered the contingency of local traditions creating a post-traditional globalised society (Giddens in Beck et al, 1994). With the loss of external traditions to echo, identity becomes self-identity, a reflexive project, where the individuals have to create their own bio-graphical narrative or self-story looking for authenticity and shunning meaninglessness (Giddens, 1991).

By referring upwards a Top-Down sentence ("a function is an example of a relation") can give only one answer thus creating "echo-teaching" and "echo-reluctance". Top-Down sentences become "unknown-unknown" relations that cannot be anchored to the students' existing learning narrative. They become meaningless and only accessible as "echo-learning" (Tarp, 2000).

By referring downwards a Bottom-Up sentence as e.g. "a function is a name for calculations with variable quantities" (Euler, 1748) can give many examples thus becoming an "unknown-known" relation that can meaningfully be anchored to the students' existing learning narrative, thus extending this. Inspired by Ausubel (Ausubel, 1968) we could say that Bottom-Up learning takes place when students get a meaningful answer to their learning-question: "Tell me something I don't know about something I know".

Why might Bottom-up mathematics be unrecognised? – Rephrasing mathematics

Mathematics education is about education in mathematics - or is it? Can mathematics be rephrased and can education be rephrased? Are the actors (students and teachers) and the system clientified, caught and frozen in a "mathematics" discourse forcing them to subscribe to a Top-Down "mathematics before mathematics application" conviction?

Humans communicate about the world in two languages. A word-language assigning words to things and practices by means of sentences: "This table is high". And a number-language assigning numbers to things and practices by means of number- or calculation-sentences called equations: "The height is forty five centimetres (h=45·cm)", "3 kg @ 4R/kg total 3·4·R (T=3·4·R)". And humans communicate about the languages in two meta-languages, the grammar describing the word-language, and mathematics describing the number-language. And humans communicate about the meta-languages in two meta-meta-languages, meta-grammar describing grammar, and meta-mathematics describing mathematics.

Meta-meta-language	Meta-grammar	Chomsky	Set Relation Function	Meta-mathematics
Meta-language	Grammar *of the word-language*	Subject Verb Object	Number Operation Calculation	Mathematics *Grammar of the number-language*
Language	Word-language *Applications of grammar*	Word stories Sentences	Number stories Equations	Number-language *Applications of mathematics*
WORLD		THINGS & PRACTICES		

Figure 3. Mathematics as part of a language-house

The phrasing "Mathematics and applications of mathematics" creates a Top-Down conviction "Of course mathematics must be learned before it can be applied". A rephrasing to "Grammar of the number-language and number-language" creates the opposite Bottom-Up conviction "Of course language must be learned before its grammar". So in this case the truth is dependent upon the ruling phrasing.

Frozen by the "Mathematics and applications of mathematics" phrasing modern mathematics implements a "grammar before language" practice (or even "meta-grammar before language"), which would create global illiteracy if spread from the number-language to the word-language, thus preventing a number-language from becoming a human right. Most humans are fluent in their mother language but unable to make explicit the grammatical rules they apply, grammatical competence is mostly tacit.

So mathematics education can be about education in mathematics, but it could also be about securing the human right for a number-language respecting the tacity of grammatical competence. Forcing an explication of a definite unrelatable mathematics might be blocking for this human right.

Mixing different abstraction levels creates syntax errors

The word-language is able to differentiate between the three language levels through the three words "language, grammar and meta-grammar". Unwilling to use the two words "number-language" and "meta-mathematics" mathematics is unable to differentiate between the three language levels. It thus creates syntax errors violating Russell's type-theory saying that mixing concepts from different abstraction levels creates nonsense. We can meaningfully ask "Where in France is Paris?" but not "where in Paris is France?" And self-referring sentences like "This statement is false" are meaningless. Gödel makes the same point: mathematics can prove statements, but not itself. Non the less mathematics keeps on making syntax errors by mixing different abstraction levels. Humans might accept syntax errors through "echo-learning" but computers refuse to accept syntax errors: computer programs like MathCad thus have to operate with several different equal signs.

"2+3" is a calculation and "5" is a number. A number can be counted, read and measured. A calculation can be calculated respecting priority and sometimes in reverse order. Exchanging the words "number" and "calculation" creates meaningless sentences, hence the two words are of different type. The syntax of writing "2+3 = 5" is "<calculation><identical-to><number>", i.e. a syntax error. One way of avoiding this syntax error is to write "(2+3) = 5" meaning the result of the calculation 2+3 is identical to 5 according to the calculation "2+(3+4)" where "(3+4)" means the result of the calculation "3+4". Another way is to write "2+3 → 5" meaning "2 and 3 gives 5".

As with "2+3 = 5" also "x+3 = 5" is a syntax error. Writing "x+3 = 5–x" is a normal error since "x+3" and "5–x" are not identical calculations. Writing "(x+3) = = (5–x)" is meaningful asking when the results of the two calculations x+3 and 5–x are identical.

Writing "f(x): x+2" meaning "let f(x) be a label for the calculation "x+2" having x as a variable number" is meaningful, but writing "f(x) = x+2" is a syntax error since x+2 is a calculation and f(x) is a label. Writing f(3) = 5 is a double error saying that 5 is a calculation with 3 as a variable number. Writing f(2x) is a syntax error since "2x" is a calculation and not a variable number. Writing 2·f(x) is a syntax error since f(x) is a label and not a number. Writing y = f(x) is a syntax error and should be written e.g. y = (x+2), or y = (<f(x)>) where <f(x)> = x+2.

Talking about "the value of a function" is as meaningless as talking about "the mood of a verb". Talking about mathematics describing the world is as meaningless as talking about grammar describing the world. Mathematics and grammar describe languages, and languages describe the world. To "mathematize" the world is as meaningless as to "grammatize" the world. Mathematical models of the world are as meaningless as grammatical models of the world. The world is described by qualitative or quantitative or graphical models.

Many proofs in mathematics are based upon the power-set, the set of all subsets in a given set. A subset is meaningful, but a set of subsets cannot be a set. A set is defined by a property shared by its elements. Since no or one element cannot share anything, it is problematic to talk about an empty set and a single element set. Hence set theory and the proofs using it need a revision.

Abstraction errors

We can say that an abstraction is true if it is true whenever you meet instances of it. An abstraction is false if there are instances where it is not true.

2 meters 3 times is always 6 meters, and 2 something 3 times is always 6 something. Hence "3·2 = 6" is a true abstraction.

Although 2 meters and 3 meters are 5 meters, 2 meters and 3 centimetres are 203 centimetres, 2 days and 3 weeks are 23 days etc. Hence "2+3 = 5" is a false abstraction. Still it is taught in school as a universal truth.

In the world we always meet numbers situated in contexts carrying units, and these units have to be alike before adding. Three apples mean an apple three times: 3·apple.

It is not the number "3" but the operator "3·" that is abstracted from below. Addition only has meaning if the two operators operate on the same unit, i.e. addition only has meaning within a parenthesis:

$$T = 2\cdot 3 + 5\cdot 3 = (2+5)\cdot 3 = 7\cdot 3$$

$$T = 2\cdot 3+4\cdot 5 = 2\cdot 3\cdot 1+4\cdot 5\cdot 1 =6\cdot 1+20\cdot 1 = (6+20)\cdot 1 = 26\cdot 1$$

Adding fractions suffers from the same problem as adding numbers without units. According to the principle of a common denominator $2/3+4/5 = 22/15$. Adding numerators and denominators $2/3+4/5 = 6/8$ is considered a meaningless mistake.

However 2 cokes out of 3 cans and 4 cokes out of 5 cans total 6 cokes out of 8 cans, and not 22 cokes out of 15 cans. Now the meaningless becomes meaningful and vice versa.

Again the point is that the units should be the same before adding. 2/3 of 3 cans and 4/5 of 5 cans total 2 cans + 4 cans, i.e. 6 cans out of 8 cans, i.e. 6/8 of 8 cans.

T = 2/3 of 3 cans and 4/5 of 5 cans = 2/3·3·can+4/5·5·can = 2·can+4·can = 6·can = 6/8·8·can

In the word-language we always use full sentences to evaluate the truth of a sentence. Instead of "green" we say e.g. "This table is green". For the same reason also the number-language should use full sentences from day one, saying "T = 3·5" instead of just "3·5" thus specifying both what is being calculated and the calculation.

Standard formulations from first year mathematics as "3+5" is a third order abstraction being abstracted from reality, from the units and from the equation. Such abstractions construct mathematics as encapsulated and create serious problems to the students when they later meet wor(l)d problems.

Equations can also be solved by reverse calculations

A Top-Down approach will phrase "2+3·x=14" as an equation only solvable after equation theory has been introduced thus showing the relevance and applicability of modern abstract algebra.

2+3·x	= 14	
(2+(3·x))-2	= 14-2	+2 has the inverse element -2
((3·x)+2)-2	= 12	+ is commutative
(3·x)+(2-2)	= 12	+ is associative
(3·x)+0	= 12	0 is the neutral element under +
3·x	= 12	by definition of the neutral element
(3·x)·1/3	= 12·1/3	3 has the inverse element 1/3
(x·3)·1/3	= 4	is commutative
x·(3·1/3)	= 4	is associative
x·1	= 4	1 is the neutral element under ·
x	= 4	by definition of the neutral element
L = {xŒR | 2+3·x = 14} = {4}		

Figure 4. Solutions steps in a top-down approach.

Alternatively a Bottom-Up approach will phrase "2+(3·x) = 14" as a calculation story reporting both a calculation process (2+3·x) and a calculation product (14), thus accessible together with calculations and solvable by reversing or walking the calculations as shown above.

Bottom-up mathematics education through the social practices that created mathematics

A Platonic Top-Down understanding sees mathematics as being created by and being examples of eternal universal ideas. Alternatively a nominalistic Bottom-Up understanding sees mathematics as being created by and abstracted from social practices. According to Giddens the competence or practical consciousness developed through exposure and participance in social practices is mainly tacit (Giddens, 1984). A rephrasing of "mathematics education" could be "number-language competence" coming from bringing into the classroom the social practices of bundling, totalling and earth measuring that raises the questions creating the number language, algebra and geometry. And respecting mathematics as partly tacit knowledge. This way allows the gradual growth of tacit competencies through gradual participance in social practices (Lave and Wenger, 1991) in which the students are allowed to sense an authentic being or "Dasein" (Heidegger, 1926). This "sociological social constructivism" is different from Vygotskian "psychological social constructivism". The former accepts the meta-language to be tacit, the latter believes in a Platonic scientific meta-language to be made discursive. Another option is to give the stories of these social practices the form of fairy tales, in which case we might experience automatic assessment-free learning, suggested by the long survival of fairy tales in the non writing culture of pre-pre-modernity.

The social practice of bundling and stacking

By totalling different bundling and stacking practices are used. Thus in the case of eight apples different Total stories can be told: A 2-bundling leads to the Total story T = 4·2·apple or T=1·stack @ 4·rows per stack @ 2·apple per row. A 9-bundling leads to T = (8/9)·9·apple, a 3-bundling gives T=2·3·apple+2·apple or T= (2 2/3)·3·apple. These stories emerge from *doing* a rebundling or from *calculating* using the "rebundling-equation" T = (T/a)·a. Standardising 10-bundles leads to the decimal numbers being "Grand Totals" in disguise: T = 234 = 2·100+3·10+4·1. In Top-Down mathematics natural, integer, rational and reel numbers are existing Platonic entities. In Bottom-Up mathematics the attributes of matter, space and time might be Platonic ideas, but numbers are bundling stories abbreviated as decimal numbers able to describe these attributes with any accuracy.

> *In this way multiplication comes before addition and fractions before two digit numbers. Hence a Bottom-Up curriculum is different from a Top-Down curriculum from day one* (Tarp, 1998).

The social practices of measuring earth and uniting totals

Geometry means earth-measuring in Greek. The earth is where we live and what we live from. We divide the earth between us, and geometry grows out of questions like "How do we divide and measure earth and space?"

Algebra means reunite in Arabic. If we buy five items in a store we don't have to pay all the single prices, we can ask for them to be united into a total. If the total is 17 $ we are allowed to pay e.g. 20 $. This new total is then split into the price and the change. To check we can reunite these numbers. So living in a money based culture means being constantly engaged in a "social practice of totalling" consisting of uniting and splitting totals, and algebra grows out of the question "How much in total?" This question can be answered in four different ways:

Totals unite/*split into*	**variable**	**constant**
unit-numbers $, m, s, ...	T = a+n *T–n = a*	T = a·n **Error!** = *a*
per-numbers $/m, m/100m=%, ...	DT = Ú f dx **Error!** = *f*	T = a^n $\sqrt{n;T}$ = *a* $log_a T$ = *n*

Figure 5. One question, four different answers

The operations "+" and "·" unite variable and constant unit-numbers; "∫" and "^" unite variable and constant per-numbers. The reverse operations "–" and "/" split a total into variable and constant unit-numbers; "d/dx" and "√ and log" split a total into variable and constant per-numbers

"5 $ and 3 $ total ? $"	T = 5+3	or T = a+n
"5 days @ 3 $/day total ? $"	T = 5·3	or T = a·n
"5 days @ 3 %/day total ? %"	$1+T = 1.03^5$	or $1+T = a^n$
"n times @ (3 %/n)/time total ? %"	$1+T = (1+0.03/n)^n$ $= (1+t)^{0.03/t}$ $= \sqrt{t;(1+t)}^{0.03} \approx e^{0.03}$	or $1+T = \sqrt{t;(1+t)}^r \approx e^r$ where $e^t = 1+t$ for t small e.g. e^t is locally linear
"5 sec. @ 3 m/sec increasing to 4 m/sec total ? m"	DT = **Error!**	or DT = **Error!**

Figure 6. Practice based questions lead to calculation stories or equations

When Will the log_x Button be Included on Calculators?
A central question as " 5%/year in ? years total 50%" leads to the equation 1.05^x = 1.50 with the solution x = $log_{1.05}$ (1.50) = 8.3. This however cannot be calculated directly on a calculator. Why not?

Figure 7. An "unsolvable" problem.

The social practice of building and evaluating models

The word-language and the number-language are used to describe or model the world. Word-stories are differentiated into different genres as fact, fiction and fiddle. Fact/fiction are stories about factual/fictional things and practices. Fiddle is nonsense containing syntax errors as e.g. "this sentence is false". In the Top-Down tradition number-stories are called mathematical models or applications of mathematics. As mentioned above this phrasing is a syntax error since mathematics describes the number-language, not the world. A Bottom-Up approach can avoid this error by phrasing "number-language description" as "quantifying and calculating model" and reuse the genre distinction from the word-language by talking about fact, fiction and fiddle models (Tarp, 1999).

A fact model could also be called a "since-hence" model or a "room" model. Fact models quantify and calculate deterministic quantities: "What is the area of the walls in this room?" In this case the calculated answer of the model is what is observed. Hence calculated numbers from fact models can be trusted.

A fiction model could also be called an "if-then" model or a "rate" model. A fiction model contains contingent equations that could look otherwise. Fiction models quantify and calculate non-deterministic quantities: "My debt will soon be paid off at this rate!" Fiction models are based upon contingent assumptions and produces contingent numbers that should be supplemented with calculations based upon alternative assumptions, i.e. supplemented with parallel scenarios.

A fiddle model could also be called a "risk" model. Fiddle models quantify and calculate qualities that cannot be quantified: "Is the risk of this road high enough to cost a bridge?" The basic risk model says "Risk = Consequence · Probability". In evaluating the risk of a road statistics can provide the probabilities of the different casualties, but casualties cannot be quantified. Still in some cases they are quantified by the cost to public institutions as hospitals etc. This is problematic since it is much cheaper to stay in a cemetery than in a hospital. So risk-models might be fiddle models. Fiddle models should be rejected asking for a word description instead of a number description.

Rephrasing mathematical concepts

In the Top-Down tradition the names of mathematical concepts come from above. A Bottom-Up approach could respect these names but supplement them with other names coming from below. "Algebra" could also be called "reuniting totals". "Geometry" could also be called "earth measuring". "Velocity, density etc." could also be called "per-numbers" as the opposite of "unit-numbers". "Stochastic variables" could also be called "unpredictable numbers" as the opposite of "predictable numbers". "Linear and exponential functions" could also be called "change by adding and multiplying". "Differentiable" could also be called "locally linear". "Continuos" could also be called "locally constant" as the opposite of "interval constant" resulting from interchanging the ε and δ in the ε–δ definition. "Differential equations" could also be called "change equations".

Top down names containing syntax errors should be avoided by saying "quantify and calculate" instead of "mathematize" and "mathematical modelling", and by saying "the value of a variable" instead of "the value of a function".

Has mathematics become the God of late modernity?

Premodernity institutionalised the worship of God, the metaphysical creator, in the premodern story house, the church, and the rhetoric of this worship can still be heard preached in today's churches. When Newton discovered that the nature of forces was physical and not metaphysical, and that their effects could be quantified, calculated and predicted, the basis for the industrial culture of modernity was created. This made the quantifying and calculating number-language as important as the word-language in early modernity under names as "regning" in Danish, "Rechnung" in German etc.

The metaphysical counter reformation of the mid 1900 fuelled by the technology shocks of the risk society (Beck, 1986) and by the cognitive turn with constructivism (Piaget, 1969; Vygotsky, 1934) reintroduced a metaphysical creator in mathematics, Set, to be worshipped and teached in the story house of modernity, the school. The rhetoric of late modern Mathematics is close to that of late feudal God, e.g. "No Math-No job" and "No God-No salvation", "Mathematics is present everywhere" and "God is present everywhere". It is numbers and calculations that are used everywhere, not meta-stories about them. And such statements will marginalise all those who cannot see it. Dehumanised mathematics dehumanises humans. It is one of the challenges of postmodernity to revive the enlightenment dream of human empowerment: Humans become educated not by meeting metaphysical creators but by meeting the social practices that provide the daily *bread.*

Conclusion

Mathematics holds on to its dream of being precise and consistent in spite of its inability to fulfil it. This could be one of the hidden reasons behind today's exodus away from mathematics and math-based educations. This paper suggests the border between necessity and contingency within mathematics is moved quite considerably leaving only decimal numbers and multiplication as necessities. Inspired by Rorty we could ask: Maybe its hidden contingency should make mathematics a little self ironic and change its solidarity from the world above to the world below, from orthodoxy to human rights (Rorty, 1989). Maybe a rehumanised, Bottom-Up, meaningful, syntax error free, user-friendly mathematics will make many of today's learning problems disappear by themselves.

Fiction: "A New Curriculum for a New Millennium" – A curriculum architect contest

Last year a school in Farawaystan decided to arrange a "curriculum architect contest" in mathematics: "A new curriculum for a new millennium". Below is a fictitious response to this contest.

Organic Bottom-up mathematics: A three level bundling and totalling curriculum

The holes in the head provide humans with food for the body and knowledge for the brains: tacit knowledge for the reptile brain and discursive knowledge for the human brain. This proposal sees a school as an institutionalised knowledge house providing humans with routines and stories by making them participants in social practices and narratives, and by respecting conceptual liberty.

The chaotic learning of tacit routine knowledge can be guided by attractors (Doll, 1993), in this case by social practices providing authenticity. In the case of mathematics the social practices will be those of bundling and totalling according to the Arabic meaning of the word Algebra: reunite.

In today's post-traditional society (Giddens in Beck et al, 1994) humans can no longer obtain identity by echoing traditions, they have to create their self-identity by building biographical self-stories looking for meaning and authenticity (Giddens, 1991). Each individual student has his own learning story, a network of concept-relations, sentences. Resembling a widespread organic carbon structure a learning story steadily grows by adding new sentences to existing words: Tell me something I don't know about something I know (Ausubel, 1968). Stories can tell about the metaphysical world above and about the physical world below. Top-Down stories from above connecting metaphysical concepts cannot be anchored to the existing learning story, they become encapsulated rote learning. Bottom-Up stories from below can, i.e. stories about the social practices providing the daily bread. The three Bottom-Up mother stories are the stories about nature, culture and humans.

First the strong gravity force crunched its universe in a big bang, liberating the medium nuclear force trying to crunch the atoms of a star in small bangs liberating light. In the end the strong force crunches the star in a medium bang filling space with matter and planets and liberating the weak electromagnetic force neutralising the strong force by distant electrons. Light makes motion flow through our planet's nature creating random micro-motion and cyclic macro-motion. Molecules transfer motion through collisions and are recycled when carbon-hydrogen structures have oxygen added and removed. The weak light helps the green cells to split the weak carbon-oxygen link. The strong light, lightening, splits the strong nitrogen-nitrogen link in the air adding strength to the extended carbon-nitrogen structures from which life is build. The three life forms are black, green and grey cells. The black cells survive in oxygen free places in stomachs and on the bottom of lakes only able to take oxygen in small amounts from organic carbon-structures thus producing gas. The green cells use the weak light to remove the oxygen from the inorganic carbon dioxide structure thus producing both organic matter storing motion and the oxygen needed by the grey cells to release the motion again. Green cells form cell communities, plants, unable to move for the food and the light.

Grey cells form animals able to move for the food in form of green cells or other grey cells thus needing to collect and process information by senses and brains to decide which way to move. Animals come in three kinds. The reptiles have a reptile brain for routines. The mammals having live offspring in need of initial care have developed an additional mammal brain for feelings. Humans have developed human fingers to grasp the food, and a human brain to grasp the world in words and sentences. Thus humans can share and store not only food but also stories, e.g. stories about how to increase productivity by transforming nature to culture.

The agriculture transforms the human hand to an artificial hand, a tool, enabling humans to transform the wood to a field for growing crops. The industrial culture transforms the human muscle to an artificial muscle, a motor, integrating tools and motors to machines enabling humans to transform nature raw material to material goods. The information culture transforms the human reptile brain to an artificial brain, a computer, integrating the artificial hand, muscle and brain to an artificial human, a robot, freeing humans from routine work.

Human production and exchange of goods has developed a number-language besides the word-language to quantify the world and calculate totals. Agriculture totals crops and herds by adding. Trade totals stocks and costs by multiplying. Rich traders able to lend out money as bankers total interest percentages by raising to power. And finally industrial culture calculates the total change-effect of forces through integrating: by adding a certain amount of momentum per second and energy per meter a force changes the meter-per-second-number, which again changes the meter-number.

A three level bundling, stacking and totalling curriculum

This proposal presents an organic bottom-up mathematics growing out of the social practices of bundling, stacking and totalling. It is organised in three levels, level 1: 6-10 years, level 2: 10-14 years and level 3: 14-18 years. It is activity and question driven limiting the amount of written material. It is learner centred limiting the amount of in-service teacher training.

> *The curriculum metaphor is a tree with a trunk consisting of five fundamental social practices: bundling, stacking, totalling, coding and reporting fed by a root of basic activities. From the trunk two branches grow out, a "totals in space" branch and a "totals in time" branch reintegrating into a "totals in space and time" at three levels.*

The basic activities are carried out with different piles of pellets or beads arranged and rearranged in sand or plastic boxes or frames always followed by the question "How many in total?" The pellets are bundled in different ways, illustrated graphically, reported as a Total-story, controlled on a calculator and finally coded.

One pellet only leads to one Total-story: T = 1. Two pellets bring the names "bundle", "times" and "stack". Two pellets can be bundled as a 2-bundle one time or as a 1-bundle two times. And a 2-bundle can be stacked. This produces two Total-stories:

••	• •	:
T = 1·2	T = 2·1	T = 1·2

Figure 8. Three pellets bring the names "add" and "minus" and lead to four Total-stories:

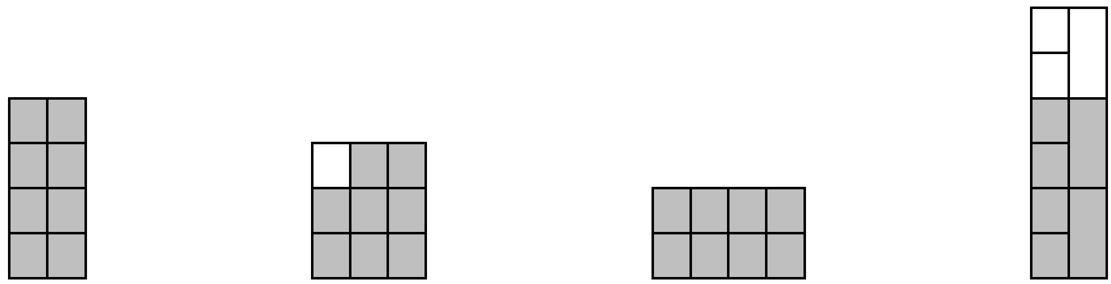
•••	•• •	• ••	• • •	••0
T = 1·3	T = 1·2+1·1	T = 1·1+1·2	T = 3·1	T = 3·1–1·1
T = 3	T = 2+1	T = 1+2	T = 3	T = 3–1

(in some cases the "·1" and "1·" can be left out)

Figure 9. Four pellets bring the names "square", "per" and "@" when the two 2-bundles are stacked

‥ ‥	∷
T = 2·2	T = 2·2 = 1·stack @ 2·rows/stack @ 2·1/row

Figure 10. Eight tiles can lead to fractions. Some fractions can be reduced througha rebundling:



T = 4·2 *T=2·3+(2/3)·3=(2 2/3)·3* *T = 2·4* *T = 4/6 = 2/3*

Figure 11. Four patterns.

Ten is used as the maximum bundle size called “X” in the beginning, T = 3·X+4·1. Later it is abbreviated to T=34 using the sign “0” for “none”. Likewise the Roman tradition can be reused by calling hundred “C” and thousand “M”: T = 3·M + 4·C + + 5·X + 6·1 = 3456.

A Total-story can be coded to hide the numbers so others will have to guess:

T = 2·5 + 1 thus becomes T = 2·a + 1

Coded total-stories are later called equations or functions. They can be analysed in tables and illustrated in figures on squared paper, where the ruler is introduced as a “counting stick” (fig. 1). The numbers of the table is calculated by walking on the floor or by “finger walking” on the table:

a = 3, T = ? a = 3 —*(·2)*→ 6 —*(+1)*→ 7 = T

Walking backwards reversing the calculation signs checks the result:

a = ?, T = 7 a = 3 ←*(/2)*— 6 ←*(–1)*— 7 = T

T = 2·a+1	**3**	**5**	**7**	**9**
a	*1*	*2*	*3*	*4*
T = 3·a–2	**1**	**4**	**7**	**10**
a	*1*	*2*	*3*	*4*

Figure 12. Two codings are needed to find the two numbers a and T (see attachment fig a)

Bundling in b-bundles and d-bundles gives the Total-story the form T = a·b+c·d. Also double coding like T = 2·a + 2·b + 1 can be analysed in tables and illustrated in space using centicubes or blocks made out of paper (see attachment fig. c). Squares with the same number can be coloured alike.

	T = 2·a + 2·b + 1:	*3*	**9**	**11**	**13**
		2	**7**	**9**	**11**
		1	**5**	**7**	**9**
		b / a	*1*	*2*	*3*

Figure 13. An analyse table for T = 2·a + 2·b + 1

Totals in space

This has three branches: Rebundling totals, adding totals and totalling forms and figures, geometry.

Rebundling totals, Level 1

Rebundling or restacking questions as "T = 2·3 = ?·5" come from e.g. sharing questions. The answer can be found by a physical rebundling using pellets or beads: 2·3 = 6·1 = 1·5+1 or by a mental rebundling using a suitable calculator as e.g. Texas Instruments Math Explorer.

From such activities a general "rebundle story" grows: 6 = (6/2)·2, 6 = (6/5)·5, 6 = (6/9)·9 or T = (T/a)·a. A rebundling into 2-bundles give birth to the names "even" and "odd".

Rebundling totals, Level 2

On this level, pellets become units, numbers become decimals, countable and measurable things become quantities and stories become equations.

Three apples become an apple three times T = 3·apple, and the counting stick now becomes a ruler counting centimetres, which can be bundled in decimetres and which has millimetres as sub-bundles: 1·dm = 10·cm and 1·cm = 10·mm. A rebundling thus can always produce a whole number giving meaning to multiplication of decimals: T = 4.3·cm = 4.3·10·mm = 43·mm.

If one of the quantities in the Total equation is a variable so is the Total: T = a·b+c·d = a·x+e. This variation can be illustrated by tables and graphs now using points instead of tiles (see attachment fig. b and d).

Calculation stories now are equations solved by reversing the calculation, i.e. moving numbers to the other side of the equal sign and reversing its calculation sign according to the rebundle story.

6 = ?·5	6 = ? + 5
6 = x·5	6 = x + 5
6/5 = x	6–5 = x

Figure 14. A short rebundling story

Now rebundling takes place between units thus changing e.g. kilograms to $ by a rebundling to known quantities.

T = 6·kg = 4·$

T = 9·kg = ?·$ T = 9·kg = (9/6)·6·kg = (9/6)·4·$ = 6·$	T = 10·$ = ?·kg T = 10·$ = (10/4)·4·$ = (10/4)·6·kg = 15·kg

Figure 15. Rebundling to known quantities

T = 100·cm = 1·m

T = 32·cm = ?·m T = 32·cm = (32/100)·100·cm = 0.32·m	T = 4.1·m = ?·cm T=4.1·m=(4.1/1)·1·m=4.1·100·cm=410·cm

Figure 16. An example of rebundling between meters and centimetres:

T = 100·% = 40·$

T = 20·% = ?·$ T=20·%=(20/100)·100·%=(20/100)·40·$= 8·$	T = 10·$ = ?·% T=10·$=(10/40)·40·$=(10/40)·100·%=25·%

Figure 17. An example of rebundling between percent % and $:

An alternative would be to use equation tables telling both what quantities to be calculated, what equation to use, what numbers to use in the calculation and how the calculation is done.

$=?	$=($/kg)·kg
$/kg=4/6 kg=9	$=4/6·9 $=6

m=?	m=(m/cm)·cm
m/cm=1/100 cm=32	m=1/100·32 m=0.32

$=?	$=($/%)·%
$/%=40/100 %=20	$=40/100·20 $=8

Figure 18. The quantity asked for, the equation to use, numbers and how it is done.

Also adding percentages can be considered an example of a rebundling, e.g. adding 5% to 40·$ two times:

To = 100·% = 40·$
T1 = 105·% = (105/100)·100·% = 1.05·40·$ which now becomes 100·%
T2 = 105·% = (105/100)·100·% = 1.05·1.05·40·$ = 1.05^2·40·$ etc. until
Tn = To·(1+r)^n

Figure 19. Adding 5% to 40·$ two times:

Another but slower way is to rebundle the 40·$ to 100·$ and then add 5·$ per 100·$:

40·$ = (40/100)·100·$, so we add 5·$ 40/100 times i.e. 2·$ totalling T1 = 40+2 = 42·$ 42·$ = (42/100)·100·$, so we add 5·$ 42/100 times i.e. 2.1·$ totalling T1 = 42+2.1 = 44.1·$

Figure 20. How to rebundle the 40·$ to 100·$ and then add 5·$ per 100·$:

Rebundling totals, Level 3

On this level power calculations are reversed as logarithm and root:

$6 = ?^5$	$6 = 5^?$
$6 = x^5$	$6 = 5^x$
$\sqrt{5;6} = x$	$\log_5 6 = x$

Figure 21. Logaritm and root

The quantities in the Total equation can themselves be Totals:

T = a·b+c·d = a·T2+T3·T4 = a·(mx+ny) + (px+qy)·(rx+sy), or
T = a·b+c·d = (kx+l)·(mx+n) + (px+q)·(rx+s) = A·x^2 + B·x +C

Figure 22. The next level of tools

In such cases the Total is called a "polynomial" to be illustrated in a two or three dimensional co-ordinate system (see attachment fig. e). A polynomial can be considered a mix of quantities controlling the appearance of a curve: The constant controls the initial level, the x the later direction, the x^2 the still later curvature, the x^3 the still later curvature or counter curvature etc. (see attachment fig. f).

ΔT = (ΔT/Δx)· Δx	in the case of macro changes, and
dT = (dT/dx)·dx = T'·dx	in the case of micro changes

Figure 23. The change of T, ΔT can be rebundled into a change of x, Δx

ΔT = Δa·b + a·Δb + Δa·Δb	or as per-numbers:
ΔT/T = Δa/a + Δb/b + Δa/a·Δb/b	in the case of macro changes, and
dT/T = da/a + db/b	in the case of micro changes

Figure 24. Considering T = a·b a stack we can find the change ΔT

DT/T = n·dx/x	or
$dT/dx = n \cdot T/x = n \cdot x^{(n-1)}$	i.e. $\mathbf{d/dx\ (x^n) = n \cdot x^{(n-1)}}$

Figure 25. The result in the case of $T = x^n$

If $T = e^x$, where the Euler number e is locally linear: $e^t = 1+t$ for t a micro number, then

dT = e^(x+dx)–e^x = e^x·e^dx–e^x = e^x·(e^dx–1) = e^x·(1+dx–1) = e^x·dx	or
dT/dx = e^x	i.e. **d/dx (e^x) = e^x**

Figure 26. The result if $T = e^x$, and e is locally linear

In the case of more variables we have e.g.

p·V = n·R·T	
dp/p + dV/V = dn/n + dT/T	since R is a constant

Figure 27. The result in the case of more variables

Adding totals, Level 1

Totals at different locations can be added remembering that only like bundles can be stacked

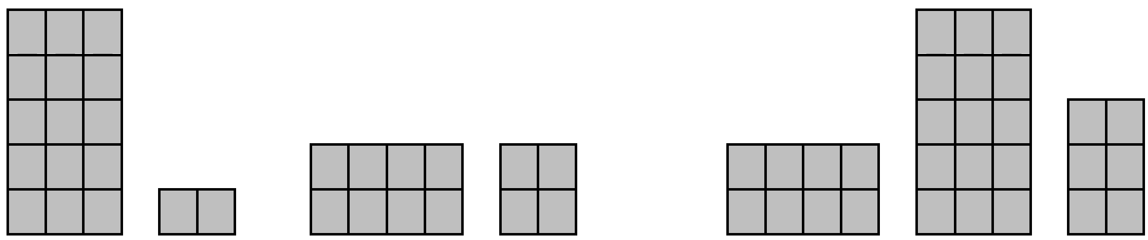


Figure 28. To add different locations

T1 = 5·3 + 1·2 T2 = 2·4 + 2·2 ΣT = 2·4 + 5·3 + (1+2)·2

T1 = T2 =	5·3 + 1·2 2·4 + 2·2	= 1·10 + 7·1 = 1·10 + 2·1
T = ΣT =	2·4 + 5·3 + (1+2)·2	= (1+1)·10 + (7+2)·1

Figure 29. The next step

Adding totals, Level 2

Totals coming from different shops can be added remembering that per-numbers never add only unit-numbers do.

T1: T2:	6 kg @ 4 \$/kg total 4 kg @ 7 \$/kg total	24 \$ 28 \$
T = ΣT =	10 kg @ x \$/kg total x \$/kg is	52 \$ 52 \$/10 kg = 5.2 \$/kg

Figure 30: Examples from different shops

Adding totals, Level 3

Totals coming from different time intervals can be added remembering that the m/s numbers are only locally constant. In this case the question is: “5 sec at 4 m/s increasing to 6 m/s total ?m”.

dT1: dT2: dT3:	dt sec @ v1 m/sec total dt sec @ v2 m/sec total etc.	v1·dt v2·dt
ΔT = ΣdT=		$\int_0^5 v\cdot dt,\ v = 4 + \frac{6-4}{5}\cdot t$ e.g.

Figure 31. Examples from different time intervals

Geometry, Level 1

Geometry means “earth-measuring“ in Greek. So geometry grows out of questions and activities related to dividing and measuring the earth we live on and from. A squared paper can be thought of as an island to be divided between two or more persons. Each person places a dot at a random location or starts a 6-step walk from a corner determined in some way by a dice. Then the paper has to be divided so they have equal distances to the border. Finally the question “How much did I get?” is posed. From this activity grows names as points, lines, midpoints, midlines or normals, triangles, “fourangles”, rectangles, size etc. All figures can be divided into triangles, and all triangles can be wrapped into a rectangle being a stack of squares and having the double size of the triangle. A ruler becomes a square counter bundling squares into 2-bundles. Different forms as cubes and cylinders or bottles are covered with paper counting surface size. Water is poured from cubes to cubes, from cylinders to cylinders and between cubes and cylinders discussing how to count the content size of water.

Geometry, Level 2

Different figures and forms get different names. Surface and content size now becoming area and volume can be calculated by equations. Rebundling stacks become reshaping areas leading to the construction and calculation of the mean and fourth proportionals. A rectangle can be divided by the diagonal producing a right-angled triangle with an outside bundled in meters and an inside bundled in diagonals c (a = sinA·c and b = cosA·c) or in sides (a = tanA·b, b = tanB·a).

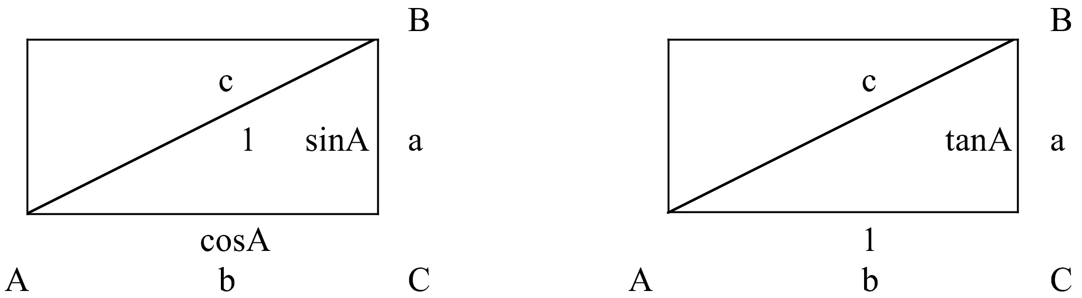


Figure 32. Triangle geometry

Design tasks lead to the golden section. Technical drawings can be made from front-, top- and side view and on isometric paper. All geometrical jobs are performed both on paper and in space.

Geometry, Level 3

Geometrical questions are translated to equations and vice versa by means of the co-ordinate system. Conic sections are put into equations. Technical drawing can now be made in perspective. Vectors are used to move and rotate figures in two and three dimensions.

Totals in time, Level 1

A total T may change in time by being added a change-number ΔT. This leads to two stories, a change-story about ΔT and a Total-story about T. Counting by 1's, 2's, 3's are examples of change stories: ΔT = 1, 2, 3 etc. Other examples are as follows in the figure.

Constant Walk, e.g. a "+2" walk	ΔT = +2	T = 6+2+2+2+....
Walking backwards provides a "–2" walk	ΔT = –2	T = 14–2–2–....
Constant Percent Walk, e.g. a "·2" walk	ΔT = +100%	T = 6·2·2·2·....
Walking backwards provides a "/2" walk	ΔT = –50%	T = 32/2/2/....
Decreasing Walk, e.g. "a to –a" walk	ΔT = +3,..., –3	T = 10+3+2+1+0–1–2–3
Swinging Walk, e.g. "a to –a to a" walk	ΔT = +3,..., –3,..., +3	T = 10+3+2+1+0–1–2–3–2 –1–0+1+2+3
Random Walk, e.g. by adding the green even dice-number and subtracting the red odd dice-numbers	ΔT = random	T = 10+4–5–1+2+...

Figure 33. Examples of how to walk

A variation to the random walk could be "dice-number six means report-time" i.e. time for a graphical report to be made both physically with beads or pellets together with the question "rearrange so the sticks have the same length ", giving birth to the word "mean".

Another variation could be "dice-number six means tax-time" where you receive or pay 1 per 3 of your fortune depending on the next dice-number is even or odd.

Alternatively a bank could be included to receive or pay out money. If both players and bank report money transferrals the names "debit" and "credit" are introduced together with the observation that debit and credit entries always go together, thus introducing accounting at an early level.

Totals in Time, Level 2

On this level five change equations appear:

$\Delta n = 1$, $\Delta T = +a$ \$	leading to linear change	$T = b+a\cdot n$
$\Delta n = 1$, $\Delta T = +r$ %	leading to exponential change	$T = b\cdot a^{n}$, $a = 1+r$
$\Delta n = 1$ %, $\Delta T = +r$ %	leading to potential change	$T = b\cdot n^{r}$
$\Delta n = 1$, $\Delta T = +r$ % $+a$ \$	leading to annuities	$T = a/r\cdot R$, $1+R = (1+r)^{n}$
ΔX = random	leading to statistics	$X \approx Xmean \pm 2\cdot Xdev$

Figure 34. The five change equations on level 2.

The first three total equations give linear graphs on "++paper", "+·paper" and "··paper" , where the "+" means a "+scale" (0,1,2,3,...) and the "·" means a "·scale" (1,2,4,8,...).

An unpredictable number X is called a stochastic variable. A variable which is not "pre-dictable" might be "post-dictable", i.e. its previous behaviour might be described in a table from which its mean and deviance can be calculated. Based upon these numbers the variable then can be interval-predicted as a confidence interval $X \approx Xmean \pm 2\cdot Xdev$. The cumulated values of a stochastic variable might give a linear graph on a normal distribution paper.

Totals in time, Level 3

On this level the change ΔT is not constant but predictable, e.g. $\Delta T/\Delta x = x\hat{}2$ or $dT/dx = x\hat{}2$. Such change equations are called difference and differential equations. They can all be solved by constantly adding the change: final number = initial number + change or $Tf = Ti + \Delta T$. In the case of micro changes this means an enormous number of addings unable for a human to perform. A computer however can do it easily in no time.

Totals in space and time: the quantitative literature

Humans communicate about the world in languages. A word language with sentences assigning words to things and actions. And a number language with equations assigning numbers or calculations to things and actions. "Word-stories" are differentiated into the genres fact, fiction and fiddle. Fact/fiction are stories about factual/ fictional things and actions. Fiddle is nonsense like "This sentence is false". "Number-stories" are often called mathematical models. Also these can be differentiated into the genres: fact, fiction and fiddle. Fact models quantify and calculate predictable quantities. Fiction models quantify and calculate non-predictable quantities. Fiddle models quantify qualities that cannot be quantified. As with word-stories also different number-stories should be treated different: Facts should be trusted, fiction should be doubted and fiddle should be rejected.

Level 1: Rebundling practices reported as Total-stories and illustrated on squared paper are examples of number- and calculation stories. Other examples are dice games of different kinds, e.g. the dice-tax-game mentioned above.

Level 2: Micro science and microeconomics. In both areas a typical question is that of rebundling one type of numbers to another kind. In physics meters are rebundled to seconds, seconds to joules, joules to degrees, volts to amperes etc. In chemistry moles are rebundled to kgs, kgs are rebundled to litres, moles to joules etc. In economics dollars are rebundled to kgs or to litres, dollars to pounds, dollars to percent etc. Statistical yearbooks are filled with tables showing quantities distributed in space and varying in time.

Level 3: Macro science and macroeconomics. In both areas the dynamics and interaction between subsystems are described and analysed, both ecological systems and economical system. Examples are Limits to Growth, Fishing Models and National Fiscal Policy Models (Tarp, 1999).

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Attachment 1: Illustrations on next page

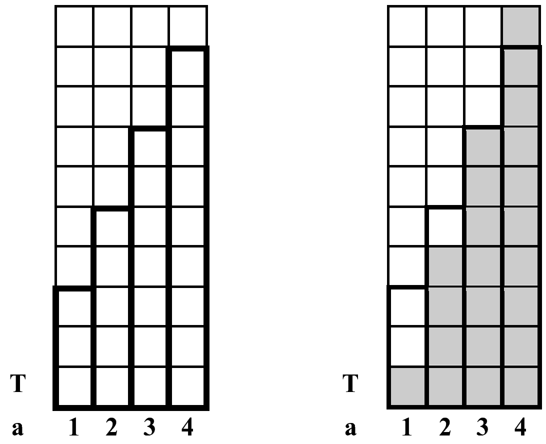

Figure a

The coded Total-stories T=2·a+1 and T=3·a–2 illustrated on squared paper

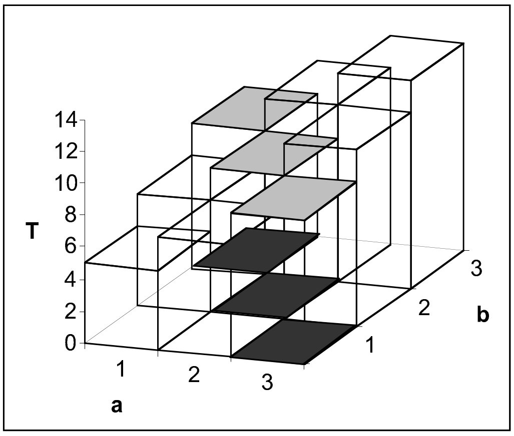

Figure c

The coded Total-story T = 2·a+2·b+1 build on squared paper. The level-9 tiles are coloured

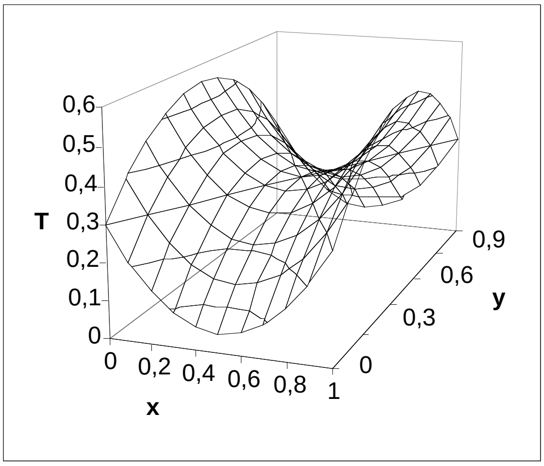

Figure e

The equation $T = x^2 – y^2 – x + y + 0.3$ illustrated in a co-ordinate system

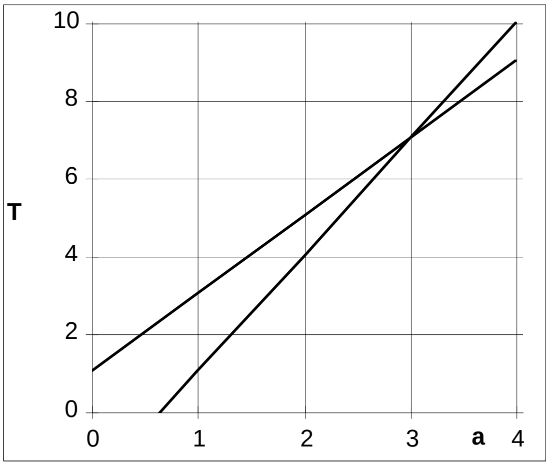

Figure b

The equations T=2·a+1 and T=3·a–2 illustrated in a co-ordinate system

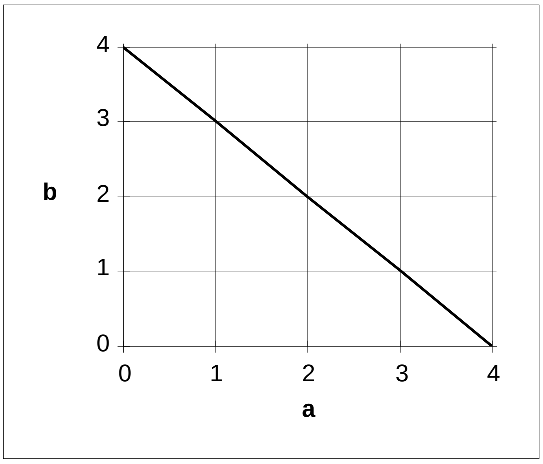

Figure d

The level-9 line of the equation T = 2·a+2·b+1 illustrated in a co-ordinate system

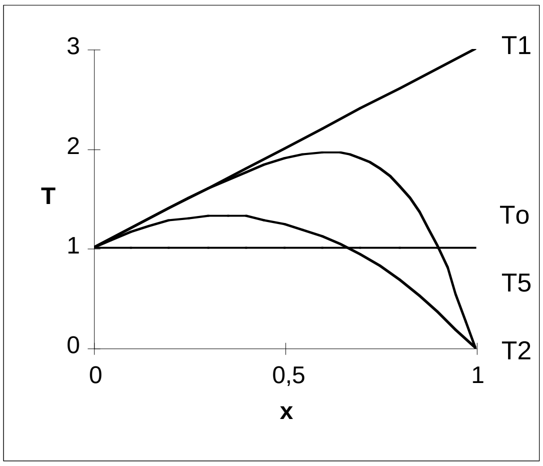

Figure f

The equation To = 1, T1 = 1+2·x, T2 = 1+2·x –3·x^2 and T5 = 1+2·x–3·x^5 illustrated in a co-ordinate system

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