Developing and Researching Quality in Mathematics Teaching and Learning

Edited by

Christer Bergsten and Barbro Grevholm

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Preface

This volume contains the proceedings of *MADIF 5*, the Fifth Swedish Mathematics Education Research Seminar, with a short introduction by the editors. The seminar, which took place in Malmö January 24-25, 2006, was arranged by *SMDF,* The Swedish Society for Research in Mathematics Education, in co-operation with Malmö högskola. The members of the programme committee were Christer Bergsten, Morten Blomhøj, Barbro Grevholm, Mikael Holmquist, and Kristina Juter. The local organiser was Per-Eskil Persson at Malmö högskola.

The programme included two plenary lectures (Werner Blum, Barbro Grevholm), one plenary panel (Gerd Brandell, Erkki Pehkonen, Jeppe Skott), eleven paper presentations (Christer Bergsten, Rita Borromeo Ferri, Gerd Brandell, Torbjörn Fransson, Johan Häggström, Håkan Lennerstad, Lisbeth Lindberg, Thomas Lingefjärd, Alistair McIntosh, Tine Wedege, Magnus Österholm), and two short oral presentations (Eva Riesbeck, Allan Tarp). In this volume the plenary addresses and ten of the papers are included. We want to thank the authors for their interesting contributions. In addition to the pre-conference peer-review process, the revised final papers were submitted after the conference and re-reviewed by the editors. The authors are responsible for the content of their papers.

We wish to thank the members of the programme committee for their work to create an interesting programme for the conference, Per-Eskil Persson for his valuable help with the preparation and administration of the seminar, and the special reactors to the papers for initiating stimulating discussions during the paper sessions. We also want to express our gratitude to the organiser of *Matematikbiennalen 2006* for its valuable financial support. Finally we want to thank all the participants at *MADIF 5* for creating such an open, positive and friendly atmosphere, contributing to the success of the conference.

Christer Bergsten, Barbro Grevholm
Editors

Content

Developing and Researching Quality in Mathematics Teaching and Learning

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Some notions related to characteristics of human activities are extensively used and seen as important, even critical, but are nevertheless difficult to grasp or define, and even more so, measure in clear and undisputable terms. Quality is no doubt one such notion. In our professional efforts we strive for quality mathematics teaching, expecting that it will result in quality of learning on behalf of our students. However, there are at least three problematic issues involved in this enterprise – how can quality in teaching be described, how can quality in learning be described, and how are these two dimensions related? There are two relevant dimensions of activities, that is one dimension of practice related to how teaching and learning can be developed toward a higher quality, and one dimension of inquiry related to researching the notion of quality of teaching and learning mathematics. At the research seminar reported on in this volume, both of these dimensions were discussed during the plenary and parallel sessions.

In his plenary address *Investigating quality mathematics teaching – the DISUM project*, Werner Blum discussed general criteria for quality in mathematics teaching. He outlined three strands, which are seen as critical, based on support from empirical research: *Demanding orchestration of the teaching of mathematical subject matter*, *cognitive activation of learners*, and *effective and learner-oriented classroom management*. In his and Dominik Leiss' paper, some modelling tasks used in large scale German development projects are discussed to illustrate how these criteria can contribute to the development of quality learning outcomes. Barbro Grevholm offered a general discussion of the question *What is quality in mathematics teaching and learning?* in her plenary address. Based on meanings of the terms involved and some views from the literature, she concludes that existing knowledge about what constitutes effective teaching is insufficient and that teachers lack means of successfully sharing such knowledge. She suggests relating learning quality to competence models of mathematics knowledge and finally points to many critical issues that need to be investigated in future research.

The papers discussed several different aspects of quality in the teaching and learning of mathematics. The teaching format of large group lectures was the focus of the paper *Lecture Notes – On lecturing in undergraduate mathematics* by Christer Bergsten. Based on the literature and data from a case study he identifies some critical aspects of quality. Rita Borromeo Ferri is in her paper *The teachers' way of handling modelling problems in the classroom – What we can learn from a cognitive-psychological point of view* using a cognitive perspective, and the notion of thinking styles, on teachers' way of handling mathematical modelling in the classroom. In the study by Torbjörn Fransson, *Students developing utilisation schemes for an artefact to solve problems in three dimensional analytic geometry*, some aspects of the influence of the use of concrete material on the quality of learning geometry are investigated. Not only the variation of teaching material may influence the quality of learning, but also what features of the mathematical content are possible to experience by students, as investigated by Johan Häggström in the paper *The same topic – Different opportunities to learn*. According to Håkan Lennerstad in his paper *Completing mathematics by teacher and student reflection*, to achieve quality in the learning of mathematics the formal mathematics normally taught in school settings needs to be complemented by a reflective mathematics, where the dialogue is seen as an important tool. Lisbeth Lindberg aims to research teaching and learning of mathematics in the vocational subjects in the Swedish upper secondary school, and has in her paper *To search for mathematics teaching and learning in vocational education* a focus on quality of research approaches and methods. In the step towards more advanced mathematical thinking, Thomas Lingefjärd sees in his paper *The use of Langford's problem to promote advanced mathematical thinking* generalization as a key process, and in particular generic abstraction was embodied in this problem as seen from students' protocols. At a more basic mathematical level competence in mental computation was investigated by Alistair McIntosh in *Mental computation of school-aged students: Assessment, performance levels and common errors*. In his big Australian project, aimed at improving students' quality of mental computation, a scale to measure such competence was developed. In adult mathematics education the issue of developing numeracy has been a goal, and Tine Wedege provides in *Numeracy as a tool in adult education: Success or failure?* reflections on an evaluation of the quality of a new Danish programme for adult education. In the final paper by Magnus Österholm, *A reading comprehension perspective on problem solving*, it is found from a literature review that the relationship between reading comprehension and problem solving is complex and that the reading process can affect as well as act as an integral part of the problem solving process. However, not much research has focused on this relationship.

The contributions in this volume taken together illustrate how the notion of quality enters into all levels of mathematics education, and we hope that the reading will provide some insights as well as lead to a deepened interest in pursuing further investigations into the important and complex issue of quality of teaching and learning mathematics.

Investigating Quality Mathematics Teaching – the DISUM Project

Werner Blum, Dominik Leiß
Universität Kassel, Germany

Abstract: *In this paper, we will first give our definition of "quality mathematics teaching", and then discuss the role of appropriate tasks for quality development. Afterwards, we shall introduce the project SINUS which aims at quality development as well as the project DISUM which aims at investigating research questions that have come up in the context of SINUS, especially how students and teachers actually deal with demanding mathematical modelling tasks and how this can be improved. As an example, we shall present and analyse the "Filling Up" task. Then, we shall report on how students treated this task and other tasks from DISUM. Thereafter, we shall report on how experienced SINUS teachers dealt with this task in the classroom. In both cases, strengths and difficulties are analysed. We shall close by discussing some implications of our findings on teaching and for further research.*

The aim: quality teaching

The teaching of mathematics in school is, from the beginning, aimed toward supplying students with knowledge, skills, competencies and attitudes so as to become intelligent and responsible citizens and to be able to use mathematics in a well-founded manner when solving real world or intra-mathematical problems. There is a wealth of empirical evidence from educational research indicating that the desired effects of mathematics teaching can only (at most) be achieved if the teaching obeys certain criteria for *"high-quality teaching"*. The following set of quality criteria constitutes our definition of *"Good Mathematics Teaching"* and is the basis of all our research and development activities. The ultimate yardstick is how mathematics teaching contributes towards students' mathematical achievement. It is based both on theories about learning and teaching mathematics, and on empirical findings. We distinguish between three categories, with associated criteria:

I. Demanding orchestration of the teaching of mathematical subject matter

(1) Providing multiple opportunities for learners to acquire competencies such as mathematical modelling or reasoning mathematically.

The bases for criterion (1) are (a) the normative assumption that the quality of an individual's mathematical education is demonstrated in certain *competencies* (in the sense of the Danish KOM Project, see Niss 2003); and (b) broadly documented empirical insights from studies into situated learning that the acquisition of competencies does not happen in some magical transfer from other activities such as performing algorithms, but only (at best) by means of well-aimed, direct competency-oriented activities.

(2) Creating manifold connections, within and outside mathematics.

These two criteria – and all the others to follow – may sound trivial and self-evident, but they are not at all trivial as they are violated in the world's classrooms every day.

II. Cognitive activation of learners

(3) Permanently stimulating cognitive activities of students, including meta-cognitive activities - the conscious use of strategies and reflections upon one's own activities.

The basis for criterion (3) is obviously a *constructivist* view of learning, and the effectiveness of meta-cognitive activities has been shown by numerous empirical studies.

(4) Fostering students‘ self-regulation and independence as much as possible, and, based on firm diagnoses, reacting adaptively to needs of individual students.

The criteria in category II and in particular in category I, are close to the subject material, in contrast to many other definitions of quality teaching. In addition, there are several criteria concerning general "classroom management":

III. Effective and learner-oriented classroom management

(5) Distinguishing clearly between learning and assessing, and using students' mistakes constructively as good learning opportunities.

(6) Varying teaching methods and using media flexibly, while fostering students' communication and cooperation.

(7) Structuring lessons clearly and using time effectively.

In all aspects, the *teacher* has a crucial role to play, so that we can speak (in the words of Weinert, 1997), of "learner-centred and teacher-directed" teaching.

Unfortunately, these criteria are *not sufficient* to assure students' achievement, since there are many other factors that influence achievement, such as willingness to expend effort, or the general status of education in society. These criteria are deemed *necessary,* in that disregarding them guarantees failure, as can be observed everywhere every day. Fortunately, there is sufficient empirical evidence suggesting that these criteria are *weakly sufficient*. This means that taking into account certain non-trivial combinations of these criteria will (other conditions being stable) result in better learning outcomes. Stated more simply, research suggests that efforts to realise good mathematics teaching are worthwhile!

Empirical studies show that everyday mathematics teaching in our country (Germany), is often far from good teaching in the above sense. This holds also for most countries in the world, even for some of the best-performing countries in the TIMS Study, as can be clearly seen from the TIMSS Repeat Video study. The unsatisfactory TIMSS and PISA-2000 results have led to efforts in Germany to improve mathematics teaching (we shall say more about this below), and since then, the German results have improved, so that in PISA-2003, the German performance nearly reached that of Sweden (the two means are no longer significantly different). However, there is still a lot to do to implement good mathematics teaching, and this holds also for most countries in the world.

Tasks as a vehicle for quality development

How can mathematics teaching be improved? An important vehicle is provided by mathematical *tasks* (see Christiansen & Walther, 1986, for the crucial role of tasks in mathematics teaching). By far, the most important activity for students in mathematics lessons and tests is dealing with tasks, and students' mathematical competencies are advanced by appropriate activities when solving tasks. Mathematics teachers have to select or construct appropriate tasks and to create learning environments guided by these tasks. So the key to improving mathematics teaching is through the *"New culture of tasks"* which involves treating (What?) a broad spectrum of competency-oriented tasks, in ways (How?) obeying the aforementioned quality criteria.

"Competency-oriented" means that the given task requires not only knowledge and technical skills, but also some additional competencies such as modelling or 'reasoning'. Here are four examples of tasks, suitable for 14-15-year-olds.

Example 1:

"Filling Up"

Mister Stone lives in Trier, 20 km away from the border of Luxemburg. To fill up his VW Golf he drives to Luxemburg where immediately behind the border there is a petrol station. There you have to pay 1.05 Euro for one litre of petrol whereas in Trier you have to pay 1.30 Euro.

Is it worthwhile for Mister Stone to drive to Luxemburg? Give reasons for your answer.

In order to solve this task, one has to – explicitly or implicitly – define what "worthwhile" should mean and to make some assumptions, for instance about the petrol consumption of the car.

Example 2:

"Sugarloaf"

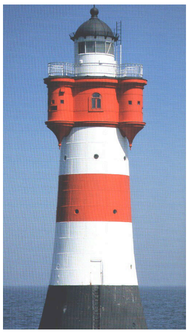
From a newspaper article:
The Sugarloaf cableway takes approximately 3 minutes for its ride from the valley station to the peek of the Sugarloaf mountain in Rio de Janeiro. It runs with a speed of 30 km/h and covers a height difference of approximately 180 m. The chief engineer, Giuseppe Pelligrini, would very much prefer to walk – as he did earlier, when he was a mountaineer, and first ran from the valley station across the vast plain to the mountain and then climbed it in 12 minutes.

How far is the distance, approximately, that Giuseppe had to run from the valley station to the foot of the mountain? Show all your work.

Most people will use the Pythagorean theorem here, but this is not necessary. Again one has to make some assumptions, perhaps tacitly. An appropriate solution could be "1.4 km (approx)".

Example 3:

"Lighthouse"



In the bay of Bremen, directly on the coast, a lighthouse called "Roter Sand" was built in 1884, measuring 30.7 m in height. Its beacon was meant to warn ships that they were approaching the coast.

How far, approximately, was a ship from the coast when it saw the lighthouse for the first time? Explain your solution.

This kind of task is rather well known. After some obvious assumptions (earth as a sphere etc.) one gets a solution with the aid of Pythagoras, something like "20 km (approx)".

Example 4:

“Giants’ shoes”

In a sports centre on the Philippines, Florentino Anonuevo Jr. polishes a pair of shoes. They are, according to the Guinness Book of Records, the world’s biggest, with a width of 2.37 m and a length of 5.29 m.

Approximately how tall would a giant be for these shoes to fit? Explain your solution.

One has to make assumptions about the ratio between men’s shoe size and height, and “the rule of three” leads to something like “30 m (approx)”.

These are all *modelling* tasks, that is, a substantial demand within these tasks is to simplify and structure the given real situation, to translate it into mathematics and to interpret mathematical results obtained (see, for example, Blum et al., 2002). We will return to modelling later.

Another common demand within these tasks is reading and understanding a given text (this is part of the *communication* competency in the sense of Niss, 2003). In addition, one has to use representations, to argue and justify what one is doing, and of course to use mathematics technically and symbolically.

The SINUS project

The afore-mentioned “New culture of tasks” was a constituting principle of the *SINUS* project. SINUS is an abbreviation of “**S**teigerung der Eff**i**zienz des mathematisch-**n**aturwissenschaftlichen **U**nterricht**s**”, that means “Increasing the efficiency of mathematics and science teaching”. SINUS was established by the German government (both federal and all 16 states) in 1998, soon after the release of the unsatisfactory German TIMSS results, with the central aim of improving mathematics teaching, and hence improving students’ mathematical achievement. SINUS started with 180 schools, and at present nearly 2000 schools are involved. The guiding principles of SINUS are:

- The “New culture of *teaching”*; that is following consistently the quality criteria in all teaching activities.
- The “New culture of *tasks”*; that is treating a broad spectrum of competency-oriented tasks in ways that obey the quality criteria, and using such tasks also in written tests and examinations.
- The “New culture of *cooperation”*; meaning that the whole staff of each school participates in these efforts to improve teaching quality - not only

individual teachers. Moreover, teachers cooperate across schools, and schools cooperate with universities or even with the school administration.

The SINUS programme is regarded by politicians as the most successful educational programme in Germany ever. Probably this is true, and the reason why SINUS has been so substantially extended and is still running. The improved German performance between PISA-2000 and PISA-2003 is certainly also due to SINUS. However, classroom observations indicate numerous *shortcomings* even in the SINUS programme. For example, there is still a big *gap* between actual teaching practices and the quality criteria. It is important to say that most of these shortcomings are not due to a lack of practical realisation of existing knowledge and expertise by the SINUS teachers. Rather than being due simply to an insufficient implementation of quality teaching in the classroom, shortfalls are identified with a *lack of knowledge* concerning both

- actual procedures and difficulties encountered by *students* when solving such cognitively demanding tasks, and
- actual and appropriate ways for *teachers* to act when treating such tasks – such as how to diagnose students' solution processes and how to intervene in case of students' difficulties.

A trivial prerequisite for both of these is knowledge of the following:

- the cognitive demand of given mathematical *tasks* (which competencies are needed on which cognitive level for solving a task).
- hard empirical data about outcomes – that is evidence of actual effects of task-driven *learning environments* on students‘ achievements and attitudes.

So, there is a lack of corresponding *research* – it is really surprising how little we still know about the micro-structure of students and teachers dealing with cognitively demanding mathematical modelling tasks.

The DISUM project

This lack of knowledge was in 2002 the starting point for the research project *DISUM* (see Blum & Leiß, 2003, and Leiß, Blum, & Messner, 2006). DISUM is an abbreviation of "**D**idaktische **I**nterventionsformen für einen **s**elbständigkeitsorientierten aufgabengesteuerten **U**nterricht am Beispiel **M**athematik", that means "Didactical intervention modes for mathematics teaching oriented towards self-regulation and guided by tasks". DISUM is an interdisciplinary project between Mathematics Education (W. Blum), Pedagogy (R. Messner) at the University of Kassel, and Educational Psychology (R. Pekrun) at the University of München. It concentrates on modelling tasks, mainly in grade 9, and investigates how students and teachers deal with such tasks and how this can be improved.

Among the DISUM activities that have been or are being carried out are the following:

1. The *construction* of appropriate modelling tasks.
2. Detailed cognitive and subject matter *analyses* of these tasks (constructing the "task space", based on the modelling cycle).
3. A detailed study and theory-guided description of actual *problem solving processes* of students in laboratory situations (involving pairs of students, sometimes with and sometimes without a teacher).
4. A detailed study and theory-guided description of actual *diagnoses* and *interventions* from teachers in these laboratory situations.
5. A detailed study of regular *lessons* with such modelling tasks, taught by experienced teachers from the SINUS project, and a theory-guided description of these lessons, emphasising our quality criteria.
6. The construction of various instruments to *measure* students' achievements and attitudes.
7. The construction of manageable and promising *tools* for the training of teachers in "well-aimed coaching" of modelling.

Among other things, we have developed a classification system for teachers' *interventions* (see Leiß & Wiegand, 2005) which is very helpful, both analytically and constructively: We distinguish between interventions related to

- students' interactions and the organisation of work
- students' affects and motivation
- the content itself
- the strategic meta-level
- mere diagnosis

What we have also achieved is a further development of existing *learning strategy models,* to increase tractability for students. Our model is comprised of five components:

- Goals ("What do I wish to achieve with this task?")
- Volition ("I do not wish to be diverted from solving this task")
- Organisation ("How do I distribute my time? Who can help me, whom can I help?")
- Strategy ("Can I imagine the situation? What is known, what is unknown? Do I know similar tasks? What are the next steps?")
- Evaluation ("What have I achieved so far? Is this reasonable?")

During 2006/07 our plans for DISUM include:

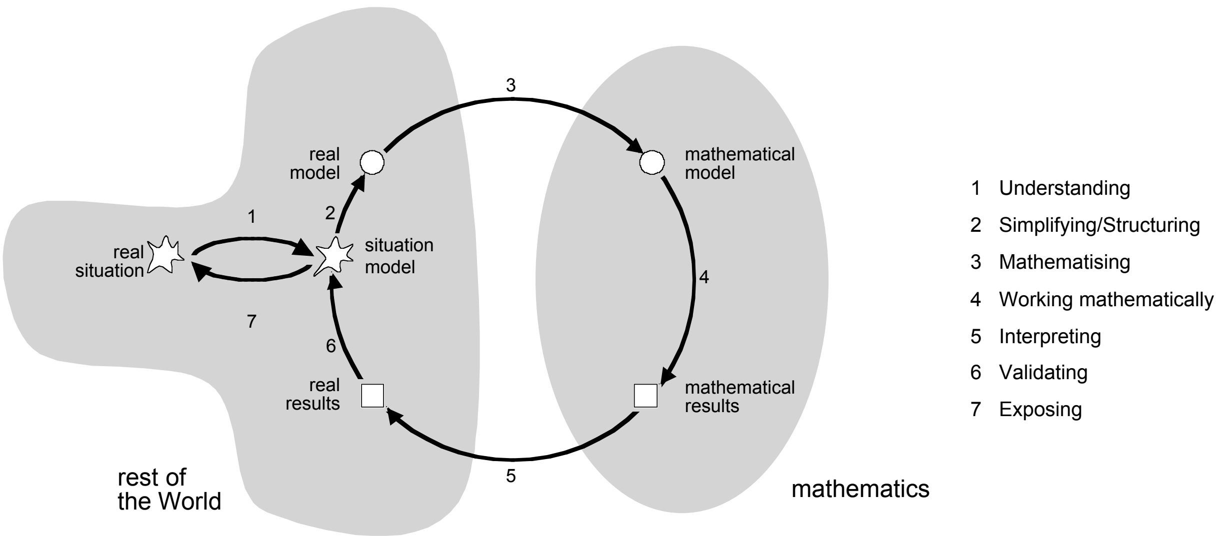
8. A detailed study into the influence of the *training of teachers* in well-aimed coaching, for purposes of enhancing mathematical achievement and

affects, especially the modelling competencies of students, in comparison with the effects of “normal” instruction or of no teaching at all (students working totally on their own).

9. The *implementation* of our instruments and findings in school classrooms and in teacher education programmes, in order to improve both everyday teaching and teachers’ expertise, in particular in the SINUS context and in the context of the implementation of our new national Educational Standards (“Bildungsstandards”) for mathematics.

An example: analysis of the “Filling Up” task

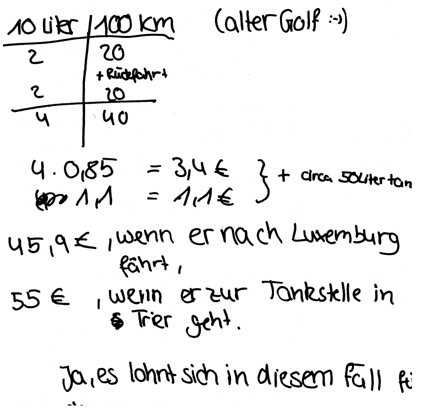
A global cognitive analysis of “Filling Up” (see above) yields the following ideal-typical solution, oriented towards the so-called modelling cycle (see Blum & Leiß, 2006):



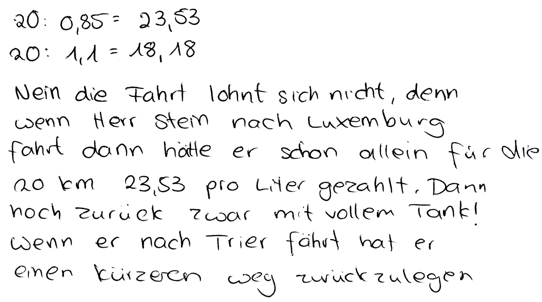
First, the text has to be read and the problem situation has to be understood by the problem solver, that is, a so-called *situation model* of the given problem situation has to be constructed. Then the situation has to be simplified, structured and made more precise, leading to a real model of the situation. In particular, the problem solver has to define what “worthwhile” should mean. In the standard model, this means only “minimising the direct costs of filling up and driving”. Mathematisation transforms the real model into a mathematical model. Working mathematically (calculating, solving equations, etc.) yields mathematical results, which are interpreted in the real world as real results. A validation of these results may show that it is necessary to go round the loop a second time, for instance in order to take into account more factors such as time, pollution or real costs for driving (e.g. obsolescence, insurance, repairs). The process ends in an exposition containing a recommendation for what Mr. Stone should do. Dependent on which factors have been considered, the recommendations might be quite different.

This version of the modelling cycle we use has been influenced by various sources, among others by the cognitive theories of Reusser (1998) and Verschaffel, Greer, and de Corte (2000). It is more oriented towards the problem solving individual than other versions. In particular, step 1, reading and understanding, is individually shaped, meaning that the resulting situation model is an idiosyncratic construction of the problem solver. This version of the modelling cycle has proved extremely helpful for our purposes. In particular, it shows potential cognitive *barriers* for students, and enables good predictions with respect to problem solving processes. Thus, it gives teachers an invaluable basis for diagnoses and interventions. Of course, actual individual problem solving processes are usually not as linear as suggested by this model, as they often go several times back and forth between the real world and mathematics. It is an interesting task to study individual modelling processes and to identify individual "modelling routes" (see Borromeo Ferri, 2006).

Here are two typical solutions of students to the "Filling Up" task:



Standard model: comparing (only) the costs of driving and filling up



Traditional "solution": extract all numbers from the text and calculate these somehow, no matter what it may mean

How do students deal with demanding modelling tasks?

As mentioned above, we have observed, videographed and interviewed 9th-graders solving modelling tasks, working in pairs. We have selected pairs of students for each of four "competence levels", from weak Hauptschule to strong Gymnasium students.

Generally speaking, all difficulties of students can be represented in the modelling cycle, and all steps in the modelling process have been observed as actual cognitive barriers, though with different emphases in different tasks. Let us take the "Sugarloaf" task as an example. Here, the first step, reading the text and understanding both situation and problem, was the most difficult part of the whole task. Here is an example (with two students from level 2, strong Hauptschule students, see excerpt 1).

Excerpt 1 (Hauptschule students)
O.: How many kil... How long that takes with the three minutes, that is when it drives 30 km/h, that thing.
P.: Yes.
O.: And is over here in three minutes – how much does it cover then? How many kilometres or how many metres? How should I know that?
P.: Wait, wait. That is …
O.: 15 km, with 15 km/h it takes half an hour.
P.: Rule of three!
O.: Rule of three!
The pupils calculate with the "rule of three" (proportions) that the distance is 1.5 km.
O.: 1.5 km.
P.: How long does it take ...?
O.: 1500 meters. That is the distance that it takes approximately.
P.: That is it takes him 1.5 km.
O.: Yeah but what – perhaps the 12 minutes mean something as well.
P.: Here they ask how *far* he has run and not how *long!*
O.: 1.5 km, 1500 m.
P.: Yes.

After successfully calculating the distance from the given velocity and time, the students think they are done with the task, which means, they switch to a common strategy: "You don't have to actually understand the situation, just use the given data in some way." In other words: The students do not construct an appropriate situation model.

A similar and very typical problem solving strategy can be seen in the next example where the "Giants' shoes" task is "solved" (by two students from level 2, strong Hauptschule students; see excerpt 2):

Excerpt 2 (Hauptschule students)

*C.***:** Well, to calculate, from these two figures, the height, the size of the man –
If the width of the shoe is 2.37 m and the length 5.29 m, then should, I believe, 2.37 m times 5.29 m – then you have the height of the man, I believe.

The students do not know what to do, and so they just calculate an area by multiplying the width and the length of the shoe. More generally speaking: They use a schema that is immediately at hand. Another couple uses the Pythagorean theorem, and since they forget to draw the square root they get a result which seems numerically reasonable.

The last mentioned mistake (forgetting the root) is not typical since a common relative *strength* exhibited by the students was – not surprisingly in Germany – the procedural part of the modelling process, that is numerical calculations, solution of equations, application of the "rule of three", use of the Pythagorean theorem and the like. On the other hand, a uniform *shortcoming* of the students was the lack of validation and substantial reflection in the end of the process. We know from research into teaching and learning how important it is to look back and to reflect on one's own problem solving process, that is – in the words of Reusser (1998) – "to extract the relevant conceptual-schematic and processual-strategic characteristics of a problem solution in an abstracting way" (*abstraction réflechissante* in the sense of Jean Piaget). More generally, no conscious use of problem solving strategies by the students was recognisable, in particular, no student seemed to have some version of the modelling cycle as a strategical guiding tool at his or her disposal. As a consequence of the absence of validation, the students did not try in the end to improve a solution they had reached but were all satisfied with any solution whatsoever.

How do teachers deal with demanding modelling tasks?

The following examples are taken from a "Best Practice Study" in the DISUM context where experienced teachers from SINUS included our modelling tasks in their regular lessons. Here are a few results of our observations.

First of all, most lessons clearly stood out positively from typical German lessons. Most lessons had a structure like the following:

1. Presentation and short discussion of the task
2. Dealing with the task individually
3. Solving the task in small groups
4. Writing solutions individually or as a group
5. Presentation of the students' solutions, in the whole class or in new groups
6. Reflection on the solutions

In these lessons,

- students *had* opportunities to model, to argue, to communicate,
- mental activities *were* stimulated,
- students, for the most part, *could* work independently,
- the atmosphere *was* tolerant towards mistakes and free of judgmental assessment,

to mention only a few of our quality criteria. Here is an example of how a teacher handled a mistake in a very productive way. It is a lesson in a grade 10 Gymnasium class with the "Lighthouse" task. The students had refined their model by including the height of the ship, and the result was that the higher the ship, the shorter the distance, that is the later the lighthouse can be seen from the ship – an obvious nonsense! The reason for that result is that the students used a wrong model, which, in effect, involves $\sqrt{H-h}$ instead of $\sqrt{H-\sqrt{h}}$ in the distance calculation (here H, h are height of lighthouse and ship – the students used only concrete figures in their calculations, no variables). The teacher lets the students finish their work and makes his own calculations in parallel, and only in the reflection phase does he point to this mistake. Thus, the teacher reveals the inherent cognitive conflict, and then asks the students to deal with this problem and find the mistake by themselves. In fact, the students discover their mistake and correct it independently.

The next example shows, in the context of the "Filling Up" task, how a student's question can lead to fruitful reflections. A student says "That's strange, if we say 8 litres [for the consumption of the car], and other groups say 6 or 7 litres, then we will all get different results later on." Later on, the teacher uses this question for functional reflections: How does the result depend on the initial data and how accurate can the result then be? This is an important question in all examples.

However, this was an exception. In most cases, there were no further reflections on the solution processes, in particular no functional reflections. However, such functional analyses are necessary for real understanding and an indispensable part of the *abstraction réflechissante* in the sense of Piaget and Reusser. Often, even the *teachers* seemed to be satisfied when students had *any* right solution. Equally, the counterpart of the afore-mentioned absence of problem solving strategies on the students' side was the absence of the stimulation of such strategies by teachers. Such stimulation seems not to be a part of teachers' everyday repertoires, even not of these "Best Practice" teachers.

Another interesting aspect observed in several lessons is also less positive: the strong influence of the teacher's own conceptualisation of the task on his or her interventions and thus on the students' solutions. Here is an example, again in the "Filling Up" context. In an interview before the lesson, the teacher had solved the task by using the standard model which takes into account only the direct costs of filling up and driving, nothing else. Right at the beginning of the lesson, a student asks what "being worthwhile" means. The teacher responds "whether it financially makes sense" and "whether it is cheaper", thus leading the students directly (if unintentionally) to the standard model. Later on in the lesson, the following dialogue arose (see excerpt 3):

Excerpt 3 (Gymnasium class)

S2: You could also ask if maybe his workplace is past the gas station in Luxemburg because then he'd have to go that way, anyway.

T: Yeah, okay, we still have to be realistic. If we take too many assumptions into account it'll get too tricky.

So, the teacher again prevents the students from extending the standard model. In the end, it was even possible in our study to identify within a mixture of students (from various classes) students from particular classes, just by the kind of solution they had achieved.

Some consequences for teaching and research

Our observations show – in accordance with other studies such as TIMSS video – that there seems to be a lot of room for improving mathematics *teaching,* for instance:

- more opportunities for students' competency-oriented activities and less algorithmic exercises,
- more connections,
- more reflections,
- a better balance between independent and teacher-directed phases.

These aspects have to be integrated systematically into *teacher education*. More than at present, teachers have to be expert in quality teaching as well as expert in selecting and using tasks; for analysing the cognitive and didactical potential of tasks, and for providing quality-oriented treatments of these tasks. How this can be adequately implemented in teacher education is not only a developmental but a *research* question. Other research questions involve:

- undertaking detailed cognitive analyses of mathematical tasks used in everyday teaching and everyday examinations, the empirical testing of these tasks, and analysis of competency-based explanations for assessing the difficulty of the tasks;
- further studies into the cognitive processes of students while solving competency-oriented tasks; in particular identifying the role of various problem solving strategies;
- further studies into the effects of teachers' interventions in task-based learning environments on students' achievements and affects.

We hope to be able to report on some of these questions at some other events in the near future.

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What is Quality in Mathematics Teaching and Learning?

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Abstract: *In this paper the meanings of the terms quality, teaching and learning are discussed, and some views from the literature and interview data on quality in the teaching and learning of mathematics from different actors in the educational enterprise are highlighted. These data all point in the same direction, indicating a state of the art as insufficient existing knowledge about what constitutes effective teaching and that teachers lack means of successfully sharing such knowledge. A number of unanswered questions on these issues still remain to be investigated.*

Introduction

The theme of this Madif5-conference is *Developing and researching quality in mathematics teaching and learning*. Thus a natural question is 'What is quality in mathematics teaching and learning?' The word quality indicates that we are dealing with good properties of a phenomenon. But 'good teaching' or 'learning in a good way' does not mean the same thing for all of us. What have researchers written about good teaching and learning? I will discuss some views from researchers on the issue. If we ask students, student teachers or teachers what they consider to be good teaching and learning, we might get different kinds of answers. Then whose conceptions of quality are we researching? Are we developing quality in teaching when we are developing teaching? Do we achieve improved learning when we have developed quality in teaching? Maybe some of the concepts we normally use here are too ill-defined or vague to enable us to reach any conclusions? I will discuss arguments from research and views from practice. Are there implications for practice or suggestions for future research to be found here?

This research seminar is the fifth in order organised by the Swedish Society for Research in Mathematics Education, in addition to the one that took place in 1998 before the society was formally established. Themes of earlier research seminars have been *Research and action in the mathematics classroom* (2000), *Challenges in mathematics education* (2002), and *Mathematics and language* (2004). These seminars have been documented in the SMDF Newsletters and books of proceedings (Grevholm, 2002; Björkqvist, 2002; Brandell, 2002; Bergsten, Dahland & Grevholm, 2002; Bergsten & Grevholm, 2003, 2004). This time focus is on quality in both the process of teaching and of learning mathematics. Three crucial concepts are involved: quality, teaching and learning. What do these concepts mean to us? Let us start by trying to capture the meaning of them.

About terminology or the crucial concepts

Quality

Central in the discussion must be what we mean by quality. My largest dictionary (Websters College Dictionary, 1992) gives the following meanings, which are relevant in this context:

1. An essential characteristic, property or attribute;
2. Character or nature, as belonging to or distinguishing a thing;
3. Character with respect to grade of excellence or fineness;
4. Superiority or excellence.

To excel means to do very good and when we discuss quality in mathematics teaching and learning we primarily mean excellent, good or distinguished teaching and learning.

We are investigating the good properties of mathematics teaching and learning. Of course one can also mean inquiring into the essential character or nature of mathematics teaching and learning.

Now, in the terms 'excellent teaching' or 'essential character of teaching' there is evaluation built in. What is excellent and what is essential? This judgement must depend on the person who evaluates, on his or her personal values as based in the theoretical foundations the person uses. We cannot expect to find a unified view on these concepts. I will come back to this fact later.

Learning

Consulting the dictionary again we get the reply on learning:

1. knowledge acquired by systematic study in any field of scholarly application
2. the act or process of acquiring knowledge or skill

And it adds that in psychology it means the modification of behaviour through practice, training or experience.

The learning theories could be grouped along individualistic, collectivist or interactionist perspectives (Bauersfeld, 1994). Different metaphors have been used such as acquisition or participation (Sfard, 1998). Judging from all the different theories about learning that have been developed during the last century this phenomenon and concept is a complicated one, which is hard to capture.

We can observe that it is difficult to know when learning takes place, where it takes place, why it takes place, what it actually means to the learner, who is learning, what is actually learnt, if the learning is stable and what the learning influences in the learner's future actions. Why do we need so many different theories about learning? Why do we need behaviourism, information processing theory, constructivism (radical, social), interactionism, activity theory, socio-cultural theory, and many others not mentioned? Is it because learning is such a difficult phenomenon to model? Is none of the metaphors good enough to capture the concept? Are there so many different aspects of learning? Or do researchers feel that the theories we have used so far are not doing justice to the phenomenon?

Teaching

The dictionary again says:

1. to impart knowledge of or skill in; give instruction in
2. to impart knowledge or skill to
3. to impart knowledge or skill; give instruction, especially as one's profession or vocation. As synonyms to teach are given: instruct, train, educate

Normally a learning theory has a companion among theories about teaching. But it seems that researchers are more eager to bring forward new learning theories.

In contrast to learning it is easy to capture teaching, at least formal teaching. It is normally well known who is teaching, when the teaching takes place, where it takes place, why it takes place, what is taught (at least intended to be taught), who are taught and what the aim of the teaching is.

What teachers and researchers know much less about is the relation between the teaching offered and the learning that is supposed to be a result of that teaching.

Researchers writing about good teaching and learning

Let us start by listening to some researchers' voices about good mathematics teaching and learning. Stigler and Hiebert (1999) state that "The teaching profession does not have enough knowledge about what constitutes effective teaching, and teachers don't have means of successfully sharing such knowledge with one another " (ibid., p 12). This is s strong claim that indicates that teachers need new tools to talk about their knowledge of good (effective) teaching.

Krainer (2005) asks what good mathematics teaching is and how research can inform practice and policy. He claims that telling what is understood by good, relevant or successful teaching is going beyond describing and interpreting things, it means establishing a norm. He asks who should define these norms, for whom and with what consequences. What role can research play? According to Krainer researchers in mathematics education can choose among at least three positions concerning the question of good teaching (ibid., p. 76):

> *Refusing norms*. [...] Each school, each class, each teaching situation is unique, has its genuine context and thus needs specific norms. [...] Good

> teaching is a matter of particularization and of local knowledge generation by teachers. [...]
> *Establishing norms*. [...] it is the task of researchers to discuss and systematize the results [...] and to work out general norms. Good teaching is a matter of adapting to prescribed norms, it is a result of "generalization" by researchers. [...]
> *Negotiating norms*. [...] The negotiation needs evidence-based support and orientation. Good teaching is a matter of "particularization" and "generalization" [...] Research is about increasing our *understanding* of teaching and about making normative assumptions about good teaching explicit, and also about *further developing teaching*.

Krainer sees the future of research in mathematics education as belonging to this third position. Researchers are expected to contribute to the improvement of mathematics teaching not only by descriptions and interpretations but also by working in teacher education and informing the wider public. Suggestions by researchers might include norms for good teaching. He claims that the goal is to raise our society's expertise for good mathematics teaching. Thus researchers, teachers, students, policy makers are expected to be experts in arguing what constitutes good teaching. And the difference between experts and laymen is the art of precise observation by experts. Observation can be defined as noticing the relevant differences. Krainer concludes that noticing differences is an important means of increasing the knowledge of teachers and policy makers. A contribution to improve teaching can be made by doing, presenting and sharing research in mathematics teacher education. Thus let us share one research report on good mathematics teaching.

Quality in teaching mathematics, one example from research

In their paper, Wilson, Cooney and Stinton (2005) discuss what constitutes good mathematics teaching and how it develops. They use the perspectives of nine high school teachers. The authors' notion of good teaching is presented as the converging views of Dewey, Polya, Davis and Hersh, and the NTCM Standards. According to them Dewey would define good teaching as that which enables students to realize broader educational goals of becoming a literate citizen capable of directing one's own life through informed and reasoned choices. Polya argued that the primary aim of mathematics teaching is to teach people to think. Polya emphasized among other things that teachers should be interested in the subject, know the subject, know about ways of learning, give students "know how", focus attitudes of mind and habits of methodical work and he suggested not to force it down the students' throats. The authors see Polya's commandments emphasizing a process-oriented teaching style as consistent with Dewey's notion of education.

For Davis and Hersh the ideal teacher is the one that invites the students to 'come and let us reason together'. These scholars envision teaching as an activity that promotes thinking and problem solving rather than the accumulation of information. That is also the perspective of the standards developed by NCTM.

The authors point to the fact that what characteristics best capture good teaching seem to vary according to the philosophical underpinnings of the research. Thus they relate to both deterministic studies and to interpretative studies. These are some of the research results they classify as deterministic:

Associated with effective teaching eleven variables were found: clarity, variability, enthusiasm, task-oriented or business like behaviours, opportunity to learn, the use of student ideas and general indirectness, criticism, the use of structuring comments, types of questions, probing and the level of difficulty of instruction (Rosenshine & Furst, 1971). The first five variables are more important. The construct of clarity and variability attracted most attention from researchers (Cooney, 1980).

A model representing good teaching was developed by Good, Grouws and Ebmeier (1983). It concerns the time allocation for one class period: review and practice of mental computation (8 minutes), developmental portion of the lesson (20 minutes), seatwork (15 minutes), and assignment of homework (2 minutes).

Other studies have investigated behaviours of expert and novice teachers and quantified differences between them in time used for correction of homework (2-3 and 15 minutes, respectively) and number of problems covered per day (40 and 6-7, respectively). Expert teachers were more efficient in organizing and conducting lessons, had a better knowledge of content, had clearer lessons, and were more adapt to explain why, how and when mathematical concepts are used.

Interpretative studies, on the other hand, emphasize teachers' beliefs and action in the classroom. Good teaching must be inferred from what the researcher is studying in such cases. Thus Wilson and Goldenberg (1998) can be assumed to see good teaching as involving student-centred instructional style in which mathematics is treated conceptually. Schifter (1998) sees the notion of good teaching as that in which teachers reflect on their own understanding as well as on their students' understanding.

Wilson, Cooney and Stinton (2005) worked with nine high school teachers, who participated in a project, PRIME (Partnerships and reform in mathematics education). The teachers participated in three audio-recorded interviews, where the overarching questions were:

What constitutes good mathematics teaching? How do the skills necessary for good mathematics teaching develop? All nine teachers emphasized that good teaching requires prerequisite knowledge (about mathematics and about students), promotes mathematical understanding, engages and motivates students, and requires effective management of the classroom environment. Seven of the nine teachers made explicit reference to connecting mathematics (to enhance student understanding). Four of the nine teachers referred to helping students visualize mathematics by using computers or calculators, drawings or concrete materials. Six of the nine teachers mentioned the need to assess students' understanding. Eight of the nine teachers saw good lessons as refraining from telling.

Teachers' perspectives on developing good teaching were that it is based on experience, education, personal reading and reflection, and interaction with colleagues. The teachers spoke of learning to teach as a complex enterprise. The authors conclude by claiming that in order to improve the conversation between teachers and teacher educators and to facilitate the development of good teaching, teacher educators need to listen to how teachers are thinking about good teaching.

Quality in learning mathematics, examples from research

One way of expressing what is included in good mathematics learning is to investigate what is said about the aims for the learning. Good learning can then be seen as reaching the aims of the curriculum. There are many ways, though, to express the competencies aimed for.

Jeremy Kilpatrick (2004) refers to the US situation and says we have had mastery learning and we have had competency-based education. We have not yet had 'Proficiency' at any level and the word has not yet become quite so contaminated as other terms. What the committee (that wrote the report 'Adding it up') meant by mathematical proficiency is close to what others might mean by mastery of mathematics, or numeracy, or competence. Mathematical proficiency comprises the following strands (ibid., p. 150):

- *Conceptual understanding* – comprehension of mathematical concepts, operations, and relations
- *Procedural fluency* – skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- *Strategic competence* – ability to formulate, represent, and solve mathematical problems
- *Adaptive reasoning* – capacity for logical thought, reflection, explanation, and justification
- *Productive disposition* – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

The committee pointed to parallels in mathematics teaching to each of these five strands.

Now, this way of presenting the intended outcome of mathematical learning can be compared with the framework KOM by Mogens Niss and Thomas Højgaard Jensen. They set out to answer the question: "What does it mean to master mathematics?" They decided to use a competency based approach and gave this definition:

> Possessing mathematical competence means having knowledge of, understanding, doing and using mathematics and having a well-founded opinion about it, in a variety of situations and contexts where mathematics plays or can play a role. A mathematical competency is a distinct major constituent in mathematics competence. (Niss, 2004, p. 183)

The Danish KOM project has identified eight such competencies, forming two clusters. The first cluster is:

The ability to ask and answer questions in and with mathematics and the cluster contains

- Mathematical thinking competency – mastering mathematical modes of thought
- Problem handling competency – formulating and solving mathematical problems
- Modelling competency – being able to analyse and build mathematical models concerning other areas
- Reasoning competency – being able to reason mathematically

The second cluster is *The ability to deal with mathematical language and tools* and contains

- Representation competency – being able to handle different representations of mathematical entities
- Symbols and formalism competency – being able to handle symbolic language and formal mathematical systems
- Communication competency – being able to communicate, in, with, and about mathematics
- Tools and aids competency – being able to make use of and relate to the tools and aids of mathematics. (Niss, 2004, pp. 184-186).

This framework can, according to Niss (2004), be used in three different ways. It can be used in a normative way when we decide goals and aims of teaching and learning, design curricula, set priorities, and produce teaching material. It can be used in a descriptive way when we want to know and understand what actually happens (or does not happen) in mathematics education. And finally it can serve as a meta-cognitive support for teachers and students when they work with questions concerning the path the teaching or learning is currently taking.

As for the concept of proficiency, the competency framework also contains possible consequences for mathematics teachers. The KOM-project asked the question: "What does it take to be a good mathematics teacher?" The answer is:“A good mathematics teacher is one who can effectively foster the development of mathematical competencies with her/his students.” (ibid, p. 188)

This way of expressing what good teaching is reveals how good mathematics teaching is interpreted as that which results in good mathematics learning. The relation between teaching and learning that we assume to be there is taken as the definition of good teaching.

There are many parallels in the two frameworks of proficiency and mathematical competence. One difference is maybe that the KOM-project does not relate to the personal disposition of the students and attitudes and beliefs as much as the proficiency framework does.

David Ausubel and Joseph Novak express good learning as meaningful learning, the kind of learning where the student changes earlier knowledge structures in a lasting way. The learning can be meaningful or rote learning and those are seen as the ends of a spectrum in which learning can be placed. It is the learner himself who can decide if the learning will be meaningful (Ausubel, 1978; Novak, 1998).

So far we have dealt with views from research on quality in mathematics teaching and learning. These theoretical perspectives do not reveal much of how to implement the ideas in practice. What if we turn to mathematics education practice and ask for students’ and didacticians’ views?

Some views on quality in mathematics teaching and learning

I interviewed a student and asked: “What do you mean by quality in mathematics teaching?” The doctoral student in mathematics didactics replied:

> I imagine different quality criteria, which can be the basis for judging if mathematics teaching is qualitatively good:
>
> - In the subject instruction by the teacher, the subject knowledge must be well defined. If you teach about equations you have to define equations and to problematize this definition. If you work with functions, the functions and variables must be defined in a way that is understandable. The teaching must be directed towards the concepts, not only towards procedures and calculation techniques.
> - When the teacher teaches different mathematical contexts and solutions of different problems the teacher must during the process be able to control if the students can manage to absorb, accommodate or understand the mathematics that is presented. Thus the teacher must be able to present the subject matter in a way that others can manage to get grip of and he/she must be able to notice the students’ degree of understanding.
> - Good teaching consists of good communication. The communication goes not only from the teacher to the students in a lecture or in one to one con-

versations. There must also be good communication from the students to the teacher. The teacher has to understand what the student is mediating and be able to use this in the following teaching. It must also be good communication between the students.

- Good teaching is student active teaching. Good teaching emphasizes activities that promote mathematical thinking and reasoning, not only drill and techniques.

I asked: "What do you mean by quality in mathematics learning?" The doctoral student answered:

> I interpret this question to mean how I think the student best learn mathematics, or what conditions are important for students to learn the subject with quality.
>
> - "Never explain anything" said a teacher who won a price as good teacher. That was the secret behind his success. Students can easily come into situations where they do not get the opportunity to think by themselves because they are too quickly introduced to ideas, methods and procedures to follow. Students own the knowledge best if they had an active role when creating it. Thus to get a reason to think yourself is very important.
> - To get an overview of the subject matter is important. By investigating a problem area and work through the whole forest of many dark corners until you can see the whole area in front of you from a little hill is satisfactory. Said in another way, you get a reason to develop your inner concept map in such a way that you can see the connections between the different parts. This is best done if you yourself are allowed to manage the agenda and drive the investigation in the direction you feel is interesting for you.
> - My own experience is that I learn much through collaboration with others. When I express my ideas in oral or written form to a peer or a colleague my thought process is brought further. I can check my thinking. At the same time I get feedback from those I communicate with, which again can lead me to new reasoning. There is no doubt about the fact that the social dimension is an important quality criterion in this connection. (Author's translation)

Comparing what the student brings forward with the outcome of the study by Wilson, Cooney and Stinton (2005) we notice that the student mentions many of the characteristics they found. The student mentions that good teaching requires prerequisite knowledge (about mathematics and about students), promotes mathematical understanding, engages and motivates students, and requires effective management of the classroom environment. He also mentions the need to assess students' understanding. And he explicitly talks about good lessons as such where the teacher refrains from telling.

Here is the voice of a didactician answering the same two questions:

> You have phrased the question in such a way that I need to be very careful in my response. In English we use the word 'quality' in several senses, my dictionary offers the following:
>
> 1. that which makes a thing what it is; nature, kind, property, attribute
> 2. grade of goodness
> 3. excellence, of high grade
>
> When I use the word quality in relation to mathematics teaching and mathematics learning I am principally focusing on the third of these, that is, quality in mathematics teaching (or learning) can be interpreted as 'excellence' in mathematics teaching and learning. However, in making this statement the first meaning is also implicit. One of the problems we face in defining excellence in mathematics teaching (and learning) is that we do not have sufficient knowledge to be able to talk with precision (and consensus) about the 'essential quality (properties)' of mathematics teaching and learning.
>
> So, when I say that I aim, in my own practice, for 'quality teaching and learning mathematics' I do so in the context of my own personal and to some extent 'ideal' notion of what excellent teaching and learning might mean. I also mean that I continue to explore what the quality of mathematics teaching and learning is.
>
> When as a teacher educator I say that my aim is to see developments in classrooms that achieve 'quality in mathematics teaching and learning' then I mean that I want other teachers to share not just a sense of aiming for excellence (because their ideal of teaching and learning might be quite different from mine). I want them to combine the quest for excellence with a similar quest for understanding the quality (essence) of (excellent) teaching and learning.

Notice that this didactician mentions the lack of sufficient knowledge to talk with precision about the essential quality of mathematics teaching and learning. This standpoint coincides with the claim quoted above by Stigler and Hiebert. We also notice that the didactician sees quality in a similar way as it was described in the introduction.

Finally I want you to listen to some student teachers' descriptions of a really good teacher. During many courses I interviewed student teachers about how they want to express the characteristics of a good mathematic teacher. These are some of the answers they gave (many answers came back in one course after the other). I have grouped the answers in four categories:

The teacher's mathematical subject knowledge and knowledge about teaching
Has the knowledge that is needed
Explains well
Is interested in teaching
Is calm and gives good instruction
Can bring mathematics down to a level where all can understand

The teacher's pedagogical content knowledge
Can invent new ways of explaining when the first one did not work
Is competent, understanding and has a good pedagogical ability
Can come down to the level of students
Cares about his/her students, their joys and worries
Has patience when students do not understand

The teacher's ability to support and give motivation
Good at getting others to learn
Wants students to learn and makes them want to learn
Supports and encourages students
Understands that an inspiring lesson and an engaged teacher gives motivation

The teacher's personal qualities
Is calm, fair, understanding and good role model
Good at being a friend to the students
Has humour
Is human
Is sharp, funny and engaged

All these expectations might frighten anyone who ever thought about becoming a good teacher. Many of the expectations that student teachers express can be related to the results from the study mentioned earlier (Wilson, Cooney, & Stinton, 2005): Good mathematics teaching demands prerequisite knowledge, to promote mathematical understanding, to engage and motivate the students, to assess students' understanding and requires effective classroom management.

Is quality in teaching important for quality in learning?

In a study by Desforges and Cockburn (1987) the teachers that took part were fascinated and sometimes mystified by children's ways of learning and capacity to think. Here are some teachers' voices:

- I don't know how they learn. They certainly don't learn by copying out of books.
- I wish I knew how they learn. I haven't got a clue. When we get stuck on something you can almost see them thinking and I'd love to know what is going on in their heads.

- Children do think about things, transfer ideas, do the abstraction and arrive at a conclusion – and they are well able to share their knowledge. (ibid, p. 29)

Now if teachers do not know how students learn, how can they know how to teach in order to improve the learning? Through assessing them, teachers do know what students have not learned, but this knowledge is remarkably stable over decades. How come that it is so difficult to influence the learning outcome when we know what is lacking?

In an ongoing study at University of Agder, called Learning Communities in Mathematics (LCM), researchers and doctoral students are working with teachers in schools to build communities of learning and the concept of inquiry is fundamental to the way of working (Jaworski, Fuglestad, Bjuland, Breiteig, Goodchild, & Grevholm, 2007). Teachers and didacticians are inquiring into mathematics, into mathematics teaching and into mathematics learning. The aim is to develop teaching in order to achieve improved learning experiences. Thus in this case there is an assumption that developing teaching will lead to improved learning (Hundeland, Erfjord, Grevholm, & Breiteig, 2007). In this study I have raised the questions: What do we mean by improved learning? What do we mean by developing teaching? It is going to be interesting to see if and how these questions are answered, when the study is finished.

Implications for future research

Are Stigler and Hiebert (1999) right when they claim that the teaching profession does not have enough knowledge about what constitutes effective teaching, and teachers do not have means of successfully sharing such knowledge with one another? If so, there is a clear indication that more research is needed to find out what is good mathematics teaching and learning. Questions for such research can be specified. Both teachers and researchers can ask questions such as those presented below and try to find answers to them.

For an international conference on Innovative teaching and learning in May 2006 the following questions were suggested (Brydon, 2005):

- How do you make learning in the classroom more exciting and effective?
- Have you personally changed your teaching practice recently to make learning more effective? What motivated you to do so?
- What more can schools do to encourage and support classroom teachers to explore new ways of teaching and learning (what needs to change, and, importantly, at what level)?
- How easy is it to try new approaches to teaching and learning (what are the obstacles)?
- If you could share your most creative teaching and learning strategy with others, what would that be?

- What outdated teaching and learning strategy have you abandoned recently, and why? What did you replace it with?
- How can teaching colleagues help you to create a more innovative and exciting classroom? Perhaps tell us how this has happened at your school.
- What role do students have in creating more effective and innovative class-rooms?
- Do you have a creative and innovative idea about teaching and learning that you haven't tried yet – but would like to someday? What is it and what's holding you back?
- If you are an education academic or school leader, what advice do you give to teachers who wish to become more innovative in the classroom?
- Why is innovation in teaching and learning so important, anyway?
- How can you be sure that a particular innovative teaching and learning practice will actually improve learning outcomes?

As long as teachers and researchers do not have answers to these questions we cannot claim that we know what quality is in teaching and learning mathematics.

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Lecture Notes –
On Lecturing in Undergraduate Mathematics

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Abstract: *Lectures is a common teaching format in undergraduate mathematics. This study sets up to investigate the notion of a quality lecture, based on a literature review and a case study of one lecture in a beginning calculus course. Critical aspects discussed include information delivery, connections, rigour-intuition, algebraic-imagistic modes, gestures, socio-mathematical norms, a mathematical mind, inspiration, personalisation, and general quality criteria for mathematics teaching, all of which came into play in the lecture studied.*

Introduction

Lecturing has a long and strong history as a teaching format at universities. Depending on countries and traditions it goes along with tutoring, seminars, classes, small group work (including computer laboratories), and home assignments as examples of traditional additional teaching offered to students. Lectures still being one of the major formats used in undergraduate mathematics teaching (see e.g. Holton, 2001), while considering the problems identified in this field, it is relevant to investigate what quality in a lecture for beginning university students might be. In this paper this notion of quality in undergraduate mathematics lectures will be examined by using theoretical notions from relevant literature and empirical data from a case study, with a focus on what is actually happening in the lecture hall. In a follow-up study focus will be on the students' perspective.

A lecture will here refer to a time scheduled talk on a pre-announced topic to a larger group of people, where the speaker (mostly alone) is overlooking the "crowd" from a podium position, and the people in the "crowd" are sitting (close) together in lines of chairs facing the speaker. It is a social and situational setting, in which both general and specific frames come in play. A lecture in a course, which will be the focus in this paper, is one of a series of lectures constituting the course, along with other teaching activities.

The educational value of lectures of this kind has often been questioned, by reasons such as the following: they turn students into passive listeners instead of active learners; they are most often linearly well ordered outlines of a ready made mathematical theory, not offering a view of mathematics as a human social activity, coloured by creativity, struggles, and other emotional aspects (Alsina, 2001); they are often not understood by the students (see e.g. Rodd, 2003, p. 15).

Other critical aspects of the lecture format in university teaching are discussed in Bligh (1972), such as the lack of feedback and social interaction. Experience based advice for good lecturing is found also in Krantz (1999). Rodd (2003) makes the case that "university mathematics departments recognise the potential of lectures, not as information-delivery venues, but as a place where the 'awe and wonder' of mathematics can be experienced" (p. 20), claiming that 'acting participation' and 'identity and community' can also be experienced as a 'witness', such as in the context of experiencing in a theatre. Imagination being an essential part of the mathematical experience, effects of inspiration may be an essential outcome from a good lecture. The issue of inspiration is also emphasised by Alsina (2001), who "unmasks" a number of myths about undergraduate mathematics education, which "have a negative influence /…/ on the quality of mathematics teaching" (p. 3), such as self-made teacher, context-free universal content, deductive top-down perfect theory presentation, and non-emotional audience (pp. 3-6). According to Millet (2001), it is possible to teach large classes within a student centred paradigm, which he finds crucial for student success.

In a case study, Barnard and Morgan (1996) investigated the match/mismatch between the aims and the practice of a lecturer, analysing one lecture on "Basic pure mathematics" for first year student teachers. The lecture set up aims at a general level and a 'content-related' level. In his practice, his general aims of moving the students from a computational via a descriptive towards a deductive attitude to mathematical work, were sometimes forced aside when engaged at specific content-related levels of knowledge of facts, justification, understanding, and 'culture', putting more emphasis of the first two of these levels. These were also the main foci of the assignments and assessment tasks.

Another factor influencing the planning and performing of an undergraduate mathematics lecture is the lecturer's ideas and reflections about the aims of the lecture, in terms of beliefs about mathematics and doing and learning mathematics, of his/her students' struggles and ways of conceptualising mathematical ideas and methods, etc. Researching the thinking of undergraduate mathematics teachers, Nardi, Jaworski and Hegedus (2005) identified a *spectrum of pedagogical awareness*, including four levels labelled as *naive and dismissive*, *intuitive and questioning*, *reflective and analytic*, and *confident and articulate* (p. 293). Even the empirical data were drawn from tutoring, the authors "see teachers' awareness developing in this context as feeding into other, more widespread teaching formats" (p. 293). It seems reasonable to expect that a higher level of pedagogical awareness may contribute to the quality of a lecture from an educational point of view. In the context of limits of functions, this is indeed of relevance, due to the well researched problems students have bringing together intuitive and formal conceptions into a functional understanding (se e.g. Harel and

Trgalova, 1996, pp. 682-686) and the many different concept images they construct (Przenioslo, 2004).

In a study on different linguistic modes used in an undergraduate mathematics lecture, Wood and Smith (2004, p. 3) note that "the lecturer is working in a number of modes: oral language, written language, mathematical notations, visual diagrams and is organizing the students' attention through to each through both verbal and non verbal means. Lecturing is a mixed mode activity", and observe differences between the lecturer's language in the writing during the lecture, which is constructed dialogically while talking, and the writing in the textbook and computer help files on the same topic, where the latter is more impersonalised. In addition, "in the spoken text /…/ the lecturer makes use of a range of words like *actually, fairly, obviously* to personalize and introduce values and judgments into the presentation" (p. 7). These differences of modes and representational forms require a lot from the students, and Wood and Smith conclude that "Student answers to the examination question reveal that there is considerable difficulty in telling a coherent story incorporating the rules of grammar and the use of mathematical language and conventions" (p. 11).

Criteria have been proposed for quality in mathematics teaching, as in Blum (2004) where three strands are seen as critical, based on support in empirical research, i.e. *Demanding orchestration of the teaching of mathematical subject matter* (competence oriented, creating opportunities to acquire these, and making connections), *Cognitive activation of learners* (stimulating cognitive and metacognitive activities), and *Effective and learner-oriented classroom management* (foster self-regulation, foster communication and cooperation among students, learner-friendly environment, clear structure of lessons and effective use of time). According to Blum, "taking into account (not necessarily all but) certain non-trivial combinations of these criteria will – other conditions being stable – result in better learning outcomes" (p. 2). The criteria have been developed at school level but may be considered, in relevant aspects at least at face value level, when discussing quality of lectures in undergraduate mathematics.

A case study

In order to investigate the relevance of the theoretical terms used above for discussing quality lecturing, a case study was performed at a Swedish university, where a lecture in first year calculus in a regular education programme in engineering was observed and analysed. The lecturer was also interviewed in connection to the lecture. These data and the literature review form the basis of a discussion of the content and usefulness of the concept of quality lecture.

The protocol from one lecture of 2x45 minutes will be shown, not to evaluate it as a good or bad lecture, but to discuss ways of analysing the concept of quality in lecturing. To give the reader insight in what actually happened, and at least partly why, this protocol and the interview protocol are displayed as fully as space allows.

The lecture takes place in a first year calculus course for engineering students. There are around 140 students in the inclined lecture hall, behind the podium there are three sets of three vertically adjustable whiteboards. The lecturer, who is well experienced in this kind of activity, is a professionally trained mathematician, and has co-authored a textbook in calculus. The lecture takes place at 10.15 to 12.00, with a 15 minutes break at 11.00. The topic of the lecture is 'standard limits', the third lecture of the course after introducing the concept and basic properties of limits and of continuous functions.

The lecture protocol

The protocol is structured in three columns, to the left is a "copy" of all that was written on the whiteboard (as carefully taken down by notes of the author during the lecture observation), in the middle some of the authentic words of the lecturer that went along with it, and to the right some of the author's comments for clarification. Vertically, time is running chronologically. Some reflections on smaller sections of the protocol will be inserted.

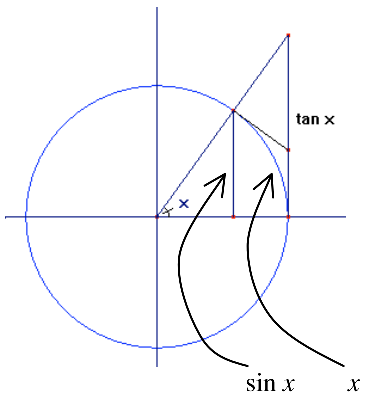
	We continue with limits.	The very first words said.
When $x \to \infty$ we have (1) $\frac{\ln x}{x^\alpha} \to 0$, with $\alpha > 0$ (α constant) (2) $\frac{x^\alpha}{a^x} \to 0$, with $a > 1$ (a constant)	*To be able to compare different kinds of functions we will need standard limits.*	Gives general comments.
$\ln x, x^\alpha, a^x$ are ordered by size (for large x)	*Speed table.*	Gives examples orally.

The lecturer begins by stating facts, telling what there is. He then goes on saying that we shall look at some different kinds of functions and standard limits. A motivation is given by saying that they will be used when doing the derivative. After listing the standard limits, thus defining which are important, the lecturer goes on to prove "some of them". In fact, he proves all of them but one, for which he gives a hint of how to tackle it. He refers to previous lectures, where limits of the "kinds" $0 \cdot \infty$, $\frac{0}{0}$ and $\frac{\infty}{\infty}$ all were identified as "difficult".

Proofs of some (1) We start by showing that $\frac{\ln x}{x} \to 0$ as $x \to \infty$	*We want to "squeeze".*	Pointing here and there in the expression.
As known we have $0 < \ln x < x - 1$, for $x > 1$ From this it follows that $0 < \ln\sqrt{x} < \sqrt{x} - 1$, for $x > 1$	*We could look at the square root of x.*	

i.e. $0 < \ln x < 2(\sqrt{x} - 1)$. Whence we have $0 < \frac{\ln x}{x} < \frac{2(\sqrt{x}-1)}{x} = \frac{2}{\sqrt{x}} - \frac{2}{x}$ $\to 0$ as $x \to \infty$ $\quad\quad$ $\to 0$ as $x \to \infty$ $\frac{\ln x}{x} \to 0 \;,\; x \to \infty$ The general case: $\frac{\ln x}{x^{\alpha}} = \frac{\frac{1}{\alpha}\ln x^{\alpha}}{x^{\alpha}} \to 0$ as $x \to \infty$ $y = x^{\alpha} \to \infty$ for $\alpha > 0, x \to \infty$	*A bad but better inequality for large x.* *Squeezed.* *This lump tends to zero.* *Not so uncommon in math that you start with a special case.*	Using log-rule for $\ln\sqrt{x} = \frac{1}{2}\ln x$ Concluding. Marks in the second fraction all but $\frac{1}{\alpha}$. Comments on the method.

The lecturer takes it for granted that the students are aware of why it is a good strategy to ”squeeze”. There is no explicit reason given why to ”look at the square root of *x*”. It is also not commented about in what aspect the obtained inequality is ”bad” but “better”; how this is understood by the students is unclear. The formal cogency is mixed up with everyday expressions like ”this lump”. A meta-comment is inserted about how it sometimes is possible to show a general case from a special case. The example just shown supports this comment.

(2) $\frac{x^{\alpha}}{a^{x}}$ Let $a^{x} = y$, i.e. $x = \frac{\ln y}{\ln a}, y \to \infty \; (a > 1)$ (exc) <stl(1)> (3)  For $0 < x < \frac{\pi}{2}$ $\quad$ $\sin x < x < \tan x$ Divide by $\sin x$ (> 0) to get $1 < \frac{x}{\sin x} < \frac{1}{\cos x}$, i.e. $\cos x < \frac{\sin x}{x} < 1$ $\because \frac{\sin x}{x} \to 1$ as $x \to 0+$ We also have $f(x) = \frac{\sin x}{x}$ even, i.e. $f(-x) = f(x)$, so $\lim_{x \to 0-} \frac{\sin x}{x} = 1$ $\quad$ $\because \frac{\sin x}{x} \to 1$ as $x \to 0$	*If you do this you will get back that first standard limit.* *Then I want to make a diagram. We can start with small x.* *If you would want to run along this path or that path I claim you would choose this one.*	Pointing, making gestures. Pointing in the diagram that the arc *x* and the vertical sin *x* are about the same for small *x*. The lecturer is using gestures to illustrate coming from the right or the left: his left arm is out and his right hand in front moving form right to left, and vice versa.

The previous formal cogency is here complemented by a reasoning supported by a diagram which is not clearly logically valid (the critical stance is $x < \tan x$), something that is seen also in the use of an everyday metaphor where the lecturer is choosing a path on behalf of the students. When reasoning by an even function, the students must themselves fill in the details of how this property is used here. Another intuitive trait of the presentation is gesture. To say ”tends to” and to use arrow notation is another way of displaying an intuitive conception of limits. The *lim* symbol is used once, for the first time during the lecture, but quickly abandoned. However, it appears soon again, maybe due to practical notational reasons.

(4) proved previously (5) $\lim_{x\to0}(1+x)^{1/x} = \lim_{x\to0} e^{\ln(1+x)^{1/x}} = \lim_{x\to0} e^{\frac{1}{x}\ln(1+x)} = e^1 = e$ $\to 1$, by (4)	*It is not nice when both the base and the exponent are moving. …use log-rules, taking down the exponent*	Refers to the definition for rewriting the logarithm. Marks $\frac{1}{x}\ln(1+x)$ using the words *that lump*.
(6) $\lim_{x\to0}\frac{e^x-1}{x} = /\text{let } e^x - 1 = y/ =$ exercise (7) $\lim_{x\to0} x^{\alpha}\ln x = /\text{let } y = \frac{1}{x},\ y\to\infty \text{ since } x\to 0+/$	*Those are the ones we have, previously extracting from polynomials /…/ later on Maclaurin expansion /…/ polar and some crap, avoid standard limits, but building on /…/ Now we shall look at some examples.*	He is here talking about the inverse, the logarithm. One can hear, immediately after the last sentence, the sounds of note books being opened.

The lecturer's words along with (5) give emotional and independent life to mathematics, also hiding the fact that the continuity of the exponential function is used but hidden in the argument. After finishing the proofs, some “connecting” comments are made about how this knowledge can be used. It must also be noted that many of the students are only listening and not taking notes until the lecturer announces that “we shall look at some examples”, when at once there is a sudden and strong sound of note books being opened.

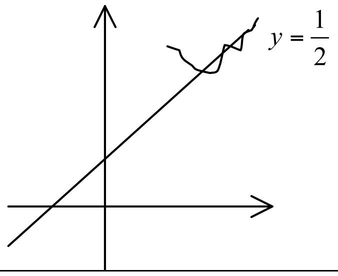
ex 1) $\lim_{x\to0}\frac{e^{\sin 7x}-1}{x} = \lim_{x\to0}\frac{e^{\sin 7x}-1}{\sin 7x}\cdot\frac{\sin 7x}{7x}\cdot 7 = 1\cdot1\cdot7 = 7$ $\frac{\sin\Delta}{\Delta}\to 1$ as $\Delta\to 0$ $\frac{e^{\Delta}-1}{\Delta}\to 1$ as $\Delta\to 0$ (it must be the same expression in Δ)	*Not true – also not true – now it is true* *This is how you should read standard limits.*	The lecturer adds on factors one by one, students laugh. The lecturer talks and points, from close and from far away, points at $1\cdot1\cdot7$.

ex 2) $\lim_{x\to 0} \frac{\arcsin x}{x} = /\text{sätt } y = \arcsin x,\ x = \sin y,\ y \to 0 \text{ as } x \to 0/$ $= \lim_{y\to 0} \frac{y}{\sin y} = \frac{1}{1} = 1$ OBS: $\frac{\arcsin x}{x} \neq \frac{\sin(\arcsin x)}{\sin x} = \frac{x}{\sin x}$ *nope* $2 = \frac{\pi}{\pi/2} = \frac{\sin \pi}{\sin \pi/2} = \frac{0}{1} = 0\ ,\ 2 = 0$	*We do one more.* *You get* $\frac{0}{0}$. *Is trying to get rid of arcsin x.* *Upside down.* *Try some angles.* *Now we will have a break.*	Short explanation. Time is exactly 11.00

The lecturer is in example 1 setting up socio-mathematical norms (Yackel and Cobb, 1996) for how one ”should” do, at the same time as he is building on an intuitive grasp of the variable concept as general-exchangeable. One could call it a *pseudo formalism* to say only ”upside down” about the line $\lim_{y\to 0} \frac{y}{\sin y} = \frac{1}{1} = 1$. We also note that the lecturer is aiming at preventing the students from doing a common mistake, in the last line above (i.e. cancelling “sin” from the numerator and denominator, as if it was a number that can be divided). The examples often somehow speak for themselves, no motivation is given to why these particular examples were chosen, or why the rewriting using log-rules is made to evaluate the example $\lim_{x\to\infty} x(\ln(1+x) - \ln x)$. The lecture turns into a kind of ritual with its *raison d’être* taken for granted, as it seems by lecturer and students alike. All the examples are lined up according to a “this is how to do it” model of presentation.

The main part of the second half of the lecture is spent on techniques, or practical thinking, and only a minor part on the theoretical superstructure or validation of the techniques, i.e. theoretical thinking. The organisation of the mathematical work is algebraic, imagistic and non-numerical (cf. Barbé et al., 2005; see also Sierpinska, 2005).

$f(x) - (kx + m) \to 0$ as $x \to \infty$ iff $f(x) - kx \to m$ and this m is found. In the example: $\frac{f(x)}{x} = \sqrt{\frac{1}{1 - 1/x}} \to 1$ as $x \to \infty$, so if $y = kx + m$ is an asymptote as $x \to \infty$ we have $k = 1$. m is then found $m = \lim_{x\to\infty} f(x) - kx$ (if the limit ∃) $f(x) - kx = x\sqrt{\frac{x}{x-1}} - x = x\left(\sqrt{\frac{x}{x-1}} - 1\right) = \frac{x\left(\frac{x}{x-1} - 1\right)}{\sqrt{\frac{x}{x-1}} + 1} =$ $= \frac{x}{(x-1)\left(\sqrt{\frac{x}{x-1}} + 1\right)} = \frac{1}{1 - \frac{1}{x}} \cdot \frac{1}{\sqrt{\frac{1}{1 - 1/x}} + 1} \to \frac{1}{2},\ x \to \infty$	*If both exist…* *One can extract an x. Maybe raise to higher terms by some conjugate thing.*	Makes a circle around $\frac{x}{x-1}$ to the left in this row.

$\because\ y = \frac{1}{2}$ is an asymptote to f as $x \to \infty$ (check the asymptote as $x \to -\infty$) 	*It could look like this but it doesn't.* *The calculation does not show how big is the difference.*	Points out that *f* can also have a vertical asymptote, moving his right arm vertically to illustrate his point. Draws a sketch. Laughter. Gives an example, laughter.

The comments of the lecturer contribute to the creation of socio-mathematical norms, e.g. that a diagram can illustrate an idea without being exact. How the students understand this is not clear, what reason could there be to draw an incorrect diagram? Gestures used along with choices of words create patterns for thinking, which might constrain conceptions to certain standard situations.

The interview

Two days before the interview with the lecturer, which lasted for about 45 minutes in an informal setting, he was given the transcribed protocol. A sheet with nine questions formed the basis of the interview, given also to the interviewee when starting up. To have a relaxed and informal discussion, there was no audio recording made during the interview but careful notes were taken by the interviewer (the author) and transcribed immediately after the interview. The exact words of the interviewee were written down as much as possible. The summary of the protocol shown below contains the main issues raised, by using quotations and summaries put into a story-like format. A short time after the interview this text was shown to the lecturer who confirmed that it gave an accurate account for what he said and meant.

Summary of protocol

When we talk about the role of the lectures within the course, he says he is *trying to "extract" the main core of the course, some parts more informally, the details can be found in the literature. To present some things carefully, show how to do rigorous proofs for those who are interested, knowing that students do not always read the textbook. Pointing at critical details, sometimes giving examples.* For the definition of a limit of a function he used in his first lecture the formal ε–δ characterisation to some extent, but put more emphasis on an intuitive image of closeness, using diagrams of graphs. The reasons for doing limits are taken from within mathematics – derivatives, integrals and asymptotic behaviour. These latter concepts are motivated by applications outside of mathematics. *I emphasise that limits is the most important concept for the whole course, all other concepts are building on it.*

Reasons given for using lectures in this course state that lectures are cheaper than other forms of teaching, and that when doing formal lectures there is less lecturing during classes. The lecture format is good, and students most often attend lectures to a greater extent than classes.

He finds it difficult to describe what kinds of (specific) goals he sets for each lecture. *I want to present, to make things seem true, the most important I think is that students believe they understand better what a concept means. To exemplify what you can handle practically, to illustrate the standard way of doing things. Lectures can look very different, some being richer with examples.* Later on in the interview, discussing quality, he repeats this argument: *Some lectures get a little colourless, building a ground, lectures differ.* When asked what the students got out from this particular lecture or why students attend lectures, he says that they normally go to the lectures, *I don't know why. It is a smooth way to get something done, they think they can use things from the lecture, collect materials, thinking the lecturer will say something that is useful for the exam.* Even he says he is not good at "feeling", during a lecture, if the students understand, it happens that he experiences a "mood" that something is difficult, and then makes some modifications of how to proceed.

We discuss what makes a lecture a good lecture, the issue of quality. He here comes back to arguments similar to what he said about goals: *I can sometimes feel it has been good, sometimes experience fears – not interesting enough, too many examples? But the students maybe see it differently, that it is good with many examples. The students should get some (beginning of an) insight, a better understanding of some concepts, a better image, make them believe that some things may hold. There is no need to include everything in detail, just do some more "popular" descriptions and let the students themselves fill in the details from the textbook. At least some of them do this.* He goes on describing how the students shall experience some kind of engagement, and get some kind of deeper understanding: *One has to make the basic standard limits one's own. I think it is easier to remember something that you once have understood why it is true. Getting the basic picture, you remember it whether you want it or not. That is how I felt when I was a student.*

He points out that there are some things that are more difficult to grasp this way, such as limits involving the logarithm. *If this lecture was of high quality I don't know, maybe somewhat dispersed. Other lectures can be more coherent.* That this course is identical for several study programmes is *making you less free with for example the order in which things are being presented.*

Students often come asking questions during the break or after the lecture. It can be about explaining things on the whiteboard, to fill in some details, but it can also be what their teacher at high school has said, something from applications or about what will happen later on in the course. It is the more able students that ask these questions.

When commenting on this particular protocol, he says: *I was not always careful with the wordings, such as the difference between a function and its graph. The protocol gives an impression that all was kind of relaxed, which is something I strive for*. Concerning the lack of numerical interpretation of limits, he gives the reason that *It is easy to get a misleading impression from the numerical behaviour of for example* $x^{\frac{1}{10}} \ln x$ as $x \to 0+$. *I don't know if the students would get an easier access to the concepts this way*.

When asked about learning to lecture he says that it is learning by experience: *I do things slower now than some years ago, when I wanted to cover all topics, now I skip some and leave it to the students*. It is not common to visit others' lectures but there are many informal discussions with colleagues.

Discussion

A first thing to note is that such questions about lectures that are in focus here, provided it is a lecture as part of a course, cannot be treated in isolation from the other formats of teaching that together with the lectures make up the course. However, there may be some quality characteristics of a general kind that apply to lectures as such. A lecture in a course being one of a series of lectures constituting the course, has a consequence that the "crowd" (the students) after a few lectures get used to the lecturer's way of lecturing. The students that attend the lecture do so, it can be assumed, to increase their ability to pass the course. This implies that the lecturer has the advantage that the "crowd" is not only willing but often even anxious to listen and take notes. The protocols support these observations. What the lecturer puts forward is considered (by the students) the core of the course, the most important issues defining the course, which can be inferred from the common tradition among students to copy and even sell lecture notes, also in cases where there is a textbook available. Another feature of this kind of lecture is that the lecturer has a fair control of the listeners' (at least formal) pre-knowledge related to the content of the lecture.

In project tasks, organised group work, and steered individual tasks, the role of the teacher is less directly visible, and the personalisation of teaching is reduced to a minimum, as well as the social and affective interplay between students and teacher. These aspects of the teaching situation are influencing the process more in a dialogical classroom management, making the teacher as a *Person* critical to a higher degree. This is even more the case in a lecturing format, especially in a lecture hall situation with an audience of a big group of students.

Referring to the literature reviewed and the observations displayed in the protocols, ten critical issues will be discussed in the protocol analyses of the case study presented here.

Information delivery: The lecture is rich in presenting mathematical information such as basic theorems, proof methods and problem solving techniques. At a face level of analysis the lecture is displayed as a demonstration of facts and procedures. In terms of quality one must ask *why* as a lecture, *what* information is chosen and *how* is it demonstrated? Interview data do not provide a clear reason why lectures are used but give some insight into the personal process of didactic transposition, by words such as *trying to "extract" the main core of the course*, *Pointing at critical details*, and *to illustrate the standard way of doing things*. Concerning the *how* issue, it will be discussed below.

Connections: In this lecture no external connections (outside mathematics) are made. From the interview we know that some applications are at least mentioned in other lectures – in fact the lecturer states that lectures can be very different. After proving the standard limits some internal connections (within mathematics) are given, as is also the case at the very end of the lecture. On the whole, however, also these are sparse and never worked out in details, apart from referring to a known inequality in the proof of standard limit (1). We can also include under this heading the level of *coherence* of the lecture, i.e. how well the different parts of the lecture are connected. This is seen as a condition for quality according to the interview. The lecture had two distinct parts, a theoretical part and an "applied" part, i.e. demonstration of *related* techniques by examples, the "know-how" connected to the theoretical tools.

Rigour-intuition: Throughout the lecture, in proofs and examples, mathematical rigour is maintained. Only on one occasion, discussing the inequality $x < \tan x$, is this approach put aside in favour of an intuitive reasoning, or rather an attempt to convince, based on a diagram. This is in some contrast to what the lecturer says in the interview, that the most important is to get the students get a feeling of understanding: *There is no need to include everything in detail*. At the same time he wants to *present some things carefully, show how to do rigorous proofs for those who are interested*. In this lecture most things he presented were done so carefully, in addition to adding metaphorical language and gestures in line with a more intuitive approach. This effect was also seen in Wood and Smith (2004). However, there are also situations where things are taken for granted which could possibly present a problem for some of the students, such as algebraic rearrangements or taking an equality or theorem as known.

Algebraic-imagistic modes: The different mathematical registers of algebra and diagrams are closely linked to the previous aspects rigour-intuition. The algebraic mode is dominating, while diagrams are presented at four separate occasions. Each of these diagrams is functional as an aid to reasoning, at one occasion as the only basis for a logical conclusion, using the path metaphor (Lakoff and Núñez, 2000) combined with a personification.

Gestures: As seen from the lecture protocol, gestures are often used by the lecturer to make ideas visible, to illustrate. In the interview the lecturer also puts an emphasis on the word *illustrate* as an overall goal for his lecturing. This can be seen as part of the game to lecture, as features of acting.

Socio-mathematical norms: Not only the written mathematical messages on the whiteboard play the role of institutionalisation, i.e. stating what officially counts in mathematics, but also oral messages telling "how to do it", such as the expression *This is how you should read standard limits*. All this taken together gives the lecture as a whole the role of establishing socio-mathematical norms, possibly by the students seen as necessary for passing the exam. The sudden activity of writing in the notebooks when examples were to come support this, as does the wordings in the interview: students may think that *the lecturer will say something that is useful for the exam*.

Mathematical mind (ways of doing/thinking, beliefs, attitudes): A big proportion of what the lecturer is saying when doing the proofs and the examples concern ways of thinking and useful ideas and techniques in mathematics. He is at the same time acting like a model mathematician, at the same time doing what he is preaching, sometimes just doing it without giving reasons or excuses (*We do one more* or *then I want to make a diagram*). As a model he is a person, using not only the formal language of mathematics but also metaphors and everyday wordings (*We want to "squeeze"*), including normative expressions such as *It is not nice when both the exponent and the base are moving*. The students can witness "live" how mathematics can be done (cf. Rodd, 2003).

Inspiration: As observed in the literature review, the issue of inspiration is by many writers seen as one of the key features and potentials of lectures. However, this is not mentioned at all in the interview, where the objectives concern student conceptual understanding and demonstrating functional mathematical tools. Thus, from the data presented here, one can only infer implicitly from the lecture protocol the potential "awe and wonder" that students might experience, and inspiration to go on doing exercises by their own or in the class. That students were in fact active listeners showed by observable reactions at different occasions, such as laughter or notes taking. The examples chosen for demonstration were to some extent on an advanced level, some possibly even out of reach for many to understand "in real time", a feature that could contribute to inspiration: is it really possible to decide on the convergence of such a number sequence, and will we even learn how to compute its limit value later on?

Personalisation: The lecturer as a person is clearly visible in the lecture notes by his use of a personal non-formal language to balance the algebraic flow on the whiteboard, his use of gestures and humour. From the interview we know that in his lectures he is striving for a relaxed atmosphere, which can be one way of expressing that he and his students can have a nice time together doing mathematics in a non-authoritarian mode. The impression from the observer was that Mathematics was the dominating "person" at the lecture but that it was communicated through a *Person* as a human activity.

General criteria for quality mathematics teaching: In relation to the teaching quality criteria by Blum (2004), the protocol and interview data show clear traits of parts of all three strands (cf. above), such as competence orientation, stimulating cognitive and meta-cognitive activities, providing a clear structure and effective use of time, while some others are less visible, partly due the very format of a lecture.

Conclusion

Is it possible to set up a set of criteria for a quality lecture? One option could be to assign an observed "level" to each of the categories discussed above. The kind of validity such a procedure could produce would probably become more well defined as well as increase by basing the evaluation on a whole series of lectures in a course. Indeed, in the interview the lecturer emphasised that lectures in the same course can be very different. However, as argued above, due to the crucial role for the lecture format of personal characteristics, the full relevance of such an approach for developing teaching quality cannot easily be identified. Instead the meaning of the kind of study presented here could lie in its potential to initiate a more focused discussion, based on well researched theoretical terms and empirical observations, on what in fact takes place in undergraduate mathematics teaching, to increase among practitioners an increased pedagogical awareness of the kind discussed by Nardi et al. (2005), which can lead to a development of quality in undergraduate mathematics teaching, not only in lecturing but as an integrated educational enterprise.

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The Teachers' Way of Handling Modelling Problems in the Classroom – What We Can Learn from a Cognitive-Psychological Point of View

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***Abstract**: In this paper I present one aspect of my current study of analysing teachers and pupils in context-bound mathematics lessons from a cognitive perspective. Using the mathematical didactical and cognitive-psychological approach of mathematical thinking styles, I will depict the teachers' way of handling modelling problems in the classroom, where the present focus lies. Looking at modelling from a cognitive perspective has largely been neglected in the current discussion regarding modelling. Results of the study presented in this paper will show what we can learn from the behaviour of teachers in context-bound lessons, when we look at them from a cognitive viewpoint.*

A short overview of cognitive aspects in mathematical modelling

Before I describe my study, I give only a short overview of cognitive processes as a much neglected aspect within the discussion on mathematical modelling.

With regard to cognitive processes one has to mention the extensive work by Richard Lesh and his team (Lesh & Doerr, 2003). In his theoretical approach, Lesh primarily refers to works by Piaget, Vygotsky, Dienes and other psychologist. Lesh's work, however, has another emphasis than the one of the study discussed in this paper, which are the following ones: Besides the fact that the individual modelling processes of pupils will be reconstructed with the cognitive-psychological approach of mathematical thinking styles on a micro-process-level, also the teachers' handling during the pupils' modelling process and their classroom discussion afterwards will be reconstructed. A design which was especially developed for this study was necessary. It will be described later in this paper.

One aspect is analyzing the individual modelling processes of pupils, which I on the basis of my analysis call *individual modelling routes* (Borromeo Ferri, 2006). However, we often forget the role of the teachers in this context. The teacher is the person who is helping the students during the modelling process and who is discussing problems later in the classrooms. Both – helping and discussion – can be very different from teacher to teacher because of their different mathematical thinking styles. Later on I will say more about this approach.

Coming back to the state of the art of cognitive aspects within the modelling discussion, the paper by Treilibs (1979) is worth mentioning (cf. Treilibs, Burkhardt, & Low, 1980) as his analyses focuses on the individual, to be precise, on the pupils, during modelling. Treilibs focuses on determining how learners build a model. Consequently he does not examine the complete modelling process, but instead concentrates on the so-called 'formulation phase' during which the model is formed.

Matos' and Carreira's research (1995, 1997) puts a special emphasis on 10th grade learners' cognitive processes. They reconstruct the representations of the pupils while solving realistic problems. They analyse the creation of conceptual models (interpretations) of a given situation and the transfer of this real situation into mathematics. In their results, they point out the numerous and diverse interpretations which learners use while modelling.

In the discussion of modelling, the works mentioned above were widely marginalized notwithstanding aspects of cognitive psychology. Looking at Blum et al. (2002), this also becomes very evident. The question of beliefs (see Maaß, 2004) has increasingly gained importance over the last years. A more intensive discussion of cognitive influences on the individual while modelling in math lessons has yet to take place. In this context, the role of the teacher will also have to be taken into consideration.

My current study provides a coherent analysis of four different aspects from a cognitive perspective: Firstly, analyzing learners and teachers in contextual mathematics lessons, secondly, analyzing micro-processes at an individual level, thirdly, analyzing groups of pupils during the process and, finally, fourthly, considering the role of the teacher at the same time.

This comprehensive analysis would therefore also yield new insights for the current discussion of modelling. Especially the linking of the approach of mathematical thinking styles to modelling, or rather the investigation of the possible influence of mathematical thinking styles on the entire modelling cycle, are new aspects which are introduced into the discussion by this study.

The mathematical didactical and cognitive-psychological approach of mathematical thinking styles

In this chapter I present the theoretical framework of mathematical thinking styles, which I developed in my PhD-thesis (Borromeo Ferri, 2004)[1]. In the current study, mathematical thinking styles are used as theoretical 'glasses' to analyze teachers and pupils in context-bound mathematics lessons and form the basis

[1] I have to add, that in my theses I described in a historical way a lot of approaches concerning different ways of thinking including Kruteskii. In this paper I have no space for that. But Kruteskii reconstructed not really thinking types, for him, these were types of abilities an individual can have.

for the research project and the data analysis. This approach is also a new aspect in the field of modelling research.

In the PhD-thesis already mentioned, the following definition of a mathematical thinking style is developed: *Mathematical thinking style* is the term I use to denote “the way in which an individual prefers to present, to understand and to think through mathematical facts and connections using certain internal imaginations and/or externalized representations. Hence, mathematical style is based on two components: 1) internal imaginations and externalized representations, 2) the holistic respectively the dissecting way of proceeding.” (cf. Borromeo Ferri, 2004, p. 50)

In my thesis, I use a laboratory design to reconstruct and analyze different mathematical thinking styles of 12 students attending 9^{th} or 10^{th} grade, i.e. I am able to describe the ‘existence’ and distinctness of three mathematical thinking styles:

- *Visual thinking style (pictorial-holistic thinking style)*: Visual thinkers show preferences for distinctive internal pictorial imaginations and externalized pictorial representations as well as preferences for the understanding of mathematical facts and connections through existing illustrative representations. The internal imaginations are mainly effected by strong associations with experienced situations.
- *Analytical thinking style (symbolic-dissecting thinking style):* Analytic thinkers show preferences for internal formal imaginations and for externalized formal representations. They are able to comprehend mathematical facts preferably through existing symbolic or verbal representations and clearly define their expressed ideas in formalisms.
- *Integrated thinking style:* They combine (in the same parts) visual and analytic ways of thinking to the same extent.

Mathematical thinking styles should not be seen as mathematical abilities but as preferences how mathematical abilities are used. This is one of the principles of a mathematical thinking style I set up with reference to Robert Sternberg, a cognitive-psychologist, and his theory of thinking styles (Sternberg, 1997). Mathematical thinking styles are set on an unconscious level of personality so that an individual does not know about his or her mathematical thinking style. Concerning what I mentioned in the chapter before (teacher as the person, who helps students), a mathematical thinking style can influence the way of teaching mathematics and also the way how the teacher discusses modelling problems, which is one of my hypotheses.

Werner Blum and his team (see DISUM-project) are focusing also on the teachers’ behavior during the modelling process of the students. In contrast to my work, they try to reconstruct and differentiate the ‘types of interventions’ of the teachers rather than the influence of mathematical thinking style on these processes, where my focus lies.

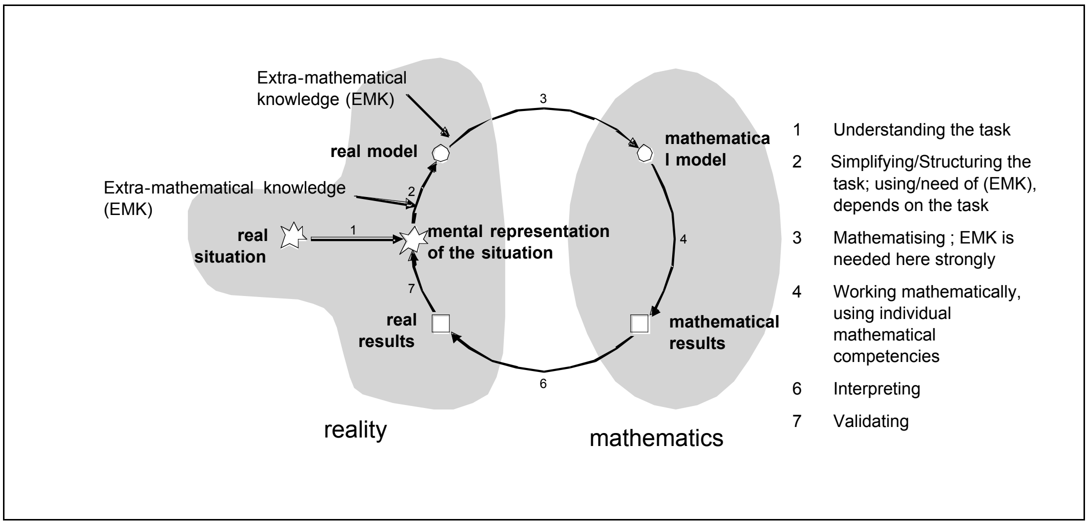
On the basis of the interviews in my thesis, I can reconstruct the phenomenon that some pupils describe that they do not understand their teachers while being taught math by them. This has nothing to do with the fact that the explanations of the teacher are bad. I set up the hypothesis that the mathematical thinking style of the teacher does not match with the mathematical thinking style of the learner. Then both, the teacher and the learner are not talking in the same 'mathematical language'.

In my current study I use mathematical thinking styles as a mathematical didactical and cognitive-psychological approach for analyzing my data. In addition, I take the theory of mathematical thinking styles out of the lab into the classroom.

The "modeling-cycle under a cognitive perspective"

In this chapter I briefly present the so-called 'cognitive modelling-cycle' I am referring to in my analysis. Within the didactic literature of modelling there exist different modelling cycles. For the purpose of my study with regard to cognitive aspects, the reconstructed handling of the teacher with modelling problems as well as the reconstructed individual modelling routes of the pupils are illustrated with the help of the modelling cycle according to Reusser (1997) and Blum and Leiss (2005). Reusser assumes that a so-called *situation model* exists in which an individual illustrates the situation depicted in the task through what can be called a mental picture. Blum and Leiss (2005) have adapted the situation model for their work on the DISUM-project. As I do not call this a situation model, I use the term *mental representation of the situation*. In my sense, this term better describes the kind of internal processes of an individual after reading the given modelling task. Concerning these phases of the modelling cycle (real situation, mental representation of the situation, real model, mathematical model, mathematical results, and real results) I want to analyze whether there are differences in the way teachers deal with modelling problems and whether mathematical thinking styles have influence on this handling processes.

Confer the following illustration for what I call the modelling cycle under a cognitive perspective:



The study

Research questions

The following questions are central to my study. Looking at each question, one can suppose that only one of these would be enough for a study in an educational setting. However, especially the connections of theses questions, which combine different levels of analyzing procedures in the classroom, were very fruitful for the purposes of my study till now. Concerning the focus of my paper, especially hypothesis gained to question one are of great interest.

1. What influences do the mathematical thinking styles of learners and teachers have on modelling processes in context-bound mathematics lessons?
2. Can the differences between situation model, real model and mathematical model (as described in didactic literature on modelling) be reconstructed from the learners' ways of proceeding? What role do they play with regard to understanding the relationship between mathematics and 'the rest of the world'?
3. Are there differences in the pupil-pupil and teacher-pupil interactions in lessons, if their mathematical thinking styles match or not?

Methodology and design of the study

The design of the study is highly complex as the research questions require different levels of data collection and data analysis. As far as the evaluation is concerned, the design has turned out to be a useful tool due to its multilayeredness. The project is carried out within the context of qualitative research. Quantitative research seems to be inappropriate given the focus of the study on the internal cognitive processes of learners and teachers.

The investigation is conducted in three 10th grade classes from different *Gymnasien* (German Grammar Schools). The sample is comprised of 65 pupils and 3 teachers (one male, two female).

Altogether three lessons are videotaped in one class. Before I started videotaping, each individual of a class had to do the questionnaire on mathematical thinking styles which I developed on the basis of my thesis. This questionnaire is evaluated independently by me and my research student by reconstructing the individual learner's mathematical thinking style. In addition, an interview is conducted with the teacher to reconstruct his or her mathematical thinking style which also includes biographical questions. Questions are also asked about his/her study of mathematics at university but also about his/her current view of mathematics or about reasons why his/her view of mathematics might have changed over the course of their teaching life.

In the first lesson, the learners work on one, possibly two not too complex modelling tasks. Pupils are divided into groups of five on the basis of the evaluation of the questionnaire and according to their mathematical thinking styles. One group is videotaped during the modelling process. The only guideline I give the teachers regarding their lesson and how the tasks should be dealt with is to work in small teams. In the second and third lesson, two further but more complex modelling tasks are worked on. Referring to the first lesson, the camera is directed at a group desk and records a view of the class, teacher, and blackboard during plenary discussions. Additionally, the teachers are equipped with a minidisc-recorder strapped to their body in order to record all their interactions with the learners. Thus, I try to record the teacher's help or suggestions during modelling as this could possibly influence the pupil's modelling process. The modelling tasks selected for the learners are of central importance as they delineate the field for the analysis. The tasks are analyzed with regard to subject matter aspects and from a cognitive viewpoint. They are taken from the DISUM-project by Blum and Messner. The evaluation of the data that has been looked through and has been analyzed by now includes the reconstruction of the individuals' modelling processes in the videotaped groups as well as plenary talks and interviews of teachers. In accordance with grounded theory (Strauss & Corbin, 1996), codes were formed and used in order to break up and reassemble data.

Results and hypothesis of the study

On the basis of the data analysis till now, the following hypotheses could be generated:

1. The teachers' mathematical thinking styles can be reconstructed and manifests itself during individual pupil-teacher conversations as well as during discussions of solutions and while imparting knowledge of mathematical facts.

1a. Teachers, who differ in their mathematical thinking styles, have the preferences of focusing on different parts of the modeling-cycle while discussing the solutions of the problems.

2. Different mathematical thinking styles of the learners result in different observable *modelling routes*.

2a. The learners' different mathematical thinking styles manifest themselves during the modelling process in such a way that the *starting-point* of the modelling route seems to occur during different phases.

I do not go into detail concerning hypotheses 2 and 2a here because I will discuss hypotheses 1 in part and 1a in detail as these constitute the focal subject of my paper.

Due to the fact that I use mathematical thinking styles as theoretical 'glasses' for my analysis, two aspects are relevant for the reconstruction of mathematical thinking styles: On the one hand, statements from the interviews with the teachers and, on the other hand, the teachers' actions and interactions during the actual lessons. Thus, statements made in the interviews and actual utterances can be compared. In the following I present the results gained from two teachers, Mrs. R. and Mr. P. On the basis of the interviews, Mrs. R. is reconstructed to be a visual thinker, while Mr. P. is reconstructed as being an analytic thinker. The following parts of the interviews can only be an illustration to make clear, which mathematical thinking style they are attributed to.

Answers from Mrs. R. and Mr. P. to two questions taken from the interview:

Interviewer: Please describe in five terms what mathematics means to you.

Mrs. R.: Oh (5s) a good question, okay (3s), an interesting subject (5s), logical thinking, ehm (3s) making connections. Ehm, tasks, yes tasks also belong to it.

Mr. P.: In five terms describing mathematics, yes, playing with numbers, playing with variables, logical thinking (3s), building logical connections, yes and there is also a connection to reality. For me, mathematics is the language of physics.

Interviewer: Which view of mathematics do you think you give the pupils while teaching?

Mrs. R.: That they know that mathematics will be good if they keep the overall view. Often I tell them that I like mathematics. I am not a formalist. When I get a task, the first thing I do is drawing a sketch. For me it is not so important that they do everything formally in a correct way but that they understand that mathematics can help them in their way of thinking.

> *Mr. P.:* That they mainly learn to recognize structures and, yes, I give them the connection to reality mostly through physics because I am also a physics teacher. But in mathematics I believe that they have to learn to think in structures and that they are able to 'move' within these structures so that they are able to see and to build formulae.

In the following, I illustrate the reactions of the two teachers to a scene from the classroom discussion to make hypothesis 1a more clear and to show *how the teacher handles with this problem*.[2] Pupils in both classes work on the 'lighthouse task', which I took from the DISUM-project:

In the bay of the city of Bremen, a lighthouse measuring 30.7 m called "Red Sand" was built directly on the coast in 1884. It was meant to warn ships which were approaching the coast with its beacon. How far was a ship still away from the coast when the lighthouse could be seen for the first time? (Round up to full kilometres.)[3]

After the learners wrote the solutions on the board, Mrs. R. and Mr. P. reacted as follows:

> *Reaction of Mr P.:* That was really good. You all did a very good job to solve this problem. But what I am missing as a maths teacher is that you can use more terms, more abstract terms and that you write down a formula and not only numbers. This way corresponds more to the way of thinking physicians and mathematicians prefer, when you use and transform terms and get a formula afterwards […]" [Mr. P. developed a formula with the pupils after this statement.]

> *Reaction of Mrs. R.:* So we have different solutions.[4] But what I recognized and what I missed in our discussion till now is the fact that you are not thinking of what is happening in the reality! When you want to illustrate yourself the lighthouse and the distance to a ship, then think for example of the *Dom*.[5] I can see the *Dom* from my balcony. Or ehm, whatever, think of taking off with a plane in the evening and so on. Two kilometres. Is that much? Is that less?

The analysis of their lessons shows that their mathematical thinking styles became evident in the discussion of reality-based tasks in the plenary as well as during one-on-one talks with the learners. Due to the limited space of this paper, I cannot give examples for the latter. What is more, in relation to modelling, the

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[2] In this paper I have no space to show also a scene of an interaction between teacher and pupil; the reactions and especially the way they handle with the problem in the classroom discussion should be made clear here are the focus of this paper.
[3] The solution is 20 kilometres; you can solve this with Pythagoras' theorem or Cosinus.
[4] One solution presented was 2 kilometres for the distance from the lighthouse to a ship.
[5] This is the name of a famous fair in Hamburg.

following interesting connection (of which only the crucial point is mentioned here) can be made:

- Mr P. as the more analytical thinker focussed less on interpretation and validation. For him, the subsequent formalisation of tasks in the form of abstract equations is important. Accordingly, the real situation becomes less important.
- Mrs. R. as a visual thinker interprets and, above all, validates the modelling processes with the learners. This becomes evident in her very vivid, reality-based descriptions she uses and provides for the learners.

Summary and discussion

As already mentioned in the first chapter, the study has a very complex design because the research questions address many different levels of contextual mathematics lessons. Although the analysis of the pupils' performance has not been described, it is still sufficient to deduce implications for the teaching and learning of modelling. What can be reconstructed with respect to the pupils is that individual modelling routes of learners differ because of the influence of mathematical thinking styles.

In the analysis it can likewise be shown that advice from the teacher and the discussion of reality-based tasks in the plenary serve to emphasise or even avoid certain phases of the modelling process. It is a fact that most teachers and pupils are most of the time unaware of their preference for a certain mathematical thinking style. Besides this aspect one has to bear in mind that pupils are supposed to see the *point* of mathematics, as is often demanded, with the help of reality-based tasks or lessons. The latter shall make the pupils aware of the connection between mathematics and reality. But can this work if the teacher formalises to a great extent and does not validate much? Or what, on the other hand, happens to mathematics, if the focus is put too strongly on reality? And what about teachers and students who prefer different mathematical thinking styles? Does this also interfere with the construction of meaning during the modelling process?

The hypotheses generated here give rise to new questions which have to be addressed in the near future. It also becomes evident that looking on teachers and pupils in context-bound mathematics lessons from a cognitive-psychological angle, several (modelling-)processes, which could be addressed as 'normal' before, are a little bit more opened up now and give us more space for understanding what happens on a micro-level.

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Students Developing Utilisation Schemes for an Artefact to Solve Problems in Three Dimensional Analytic Geometry

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***Abstract**: When working with concrete materials in mathematics education, utilisation schemes about how to use the materials must be constructed. In this study, eighteen upper secondary students work on three-dimensional analytic geometry with an artefact available, and the development of their utilisation schemes is investigated by variation in task design. How these schemes develop and influence the solution process depend more on the students' pre-knowledge than on how they are guided through the tasks.*

Concrete materials in mathematics education

Artefacts such as concrete materials or computers are used in school during mathematics classes. Often the aim is to help pupils visualise mathematical objects and processes. Studies of using concrete materials in mathematics education show that mathematical achievement, as well as students' attitudes towards mathematics, can be improved through a long-term use of concrete instructional materials, provided that teachers are knowledgeable about their use (Sowell, 1989). Several studies indicate that, when there is an option, students often show a preference to solve tasks by using an algebraic approach rather than a visual approach (e.g. Eisenberg & Dreyfus, 1991; Presmeg & Bergsten, 1995). However, Hart (1993) writes that children "do not automatically realise how to use a diagram or what its intended message is. Its special features need to be taught just as other aspects of mathematics need to bee taught" (p. 57).

Wartofskys' (1979) general definition of an artefact includes a cultural creation such as language. In this paper we are only interested in artefacts such as concrete materials designed and/or used for educational use. Bergsten and Fransson (2006) divided those artefacts in three categories, *static*, *responsive* and *dynamic* artefacts. A static artefact can be manipulated but does not change its form, and no response is given back to the user, who has to make an interpretation from the actions performed. When dealing with a static artefact it is the way you interact with it that defines the learning outcome. To get something out from it, the students need utilisation schemes (Strässer, 2004), which change the artefact to become an instrument for learning (Rabardel & Samurcay, 2001). Students' interaction with an artefact for solving a mathematical problem is a complex process. To be able to understand this communication between students, between students mediated by the artefact, and students communicating with him/herself with or without the artefact must be investigated. Communication here includes talk, gestures, writing, etc., i.e. all sorts of mediating (Sfard, 2001).

In an earlier study, Bergsten and Fransson (2006) investigated to what extent and how students working with a static artefact interact with the model, and finally how the interaction influenced the solution process. The result showed that the static artefact used played an important role in building up a sense of understanding of the problem situation. There was also evidence showing that the artefact served as a vehicle for communication and that it supported students to validate steps in the solution process. The students' pre-knowledge played an important role for what interaction took place when dealing with planes in three dimensions, a topic the students had never met in their mathematics courses before – there was a low interactivity.

Purpose

To be able to make use of an artefact, students need some kind of utilisation scheme. Such a scheme can be constructed by the teacher and taught to the students. By working with the artefact and using their pre-knowledge, students can also construct the utilisation scheme by themselves (Bergsten & Fransson, 2006). In this case, the utilisation scheme will be based on the students' own interaction with the model, which may lead to a functional way to solve the problem at hand. In the study presented here, we had half of the students work with tasks designed to guide them to a specific utilisation scheme, while the other students work with tasks not guided in this respect. All of the students have the same final task (the target task), the purpose being to investigate the utilisation schemes used in this task, in relation to the work during the previous tasks. Another aim was to study how the utilisation schemes influence the solution processes.

Method

Design of the model and the tasks

To study utilisation schemes, we designed two different sets of tasks, dealing with analytic geometry in three dimensions. To their help the students had an artefact, from now called the *model*, specially designed for this and the previous study (Bergsten & Fransson, 2006). This model, made of four sides a steel mesh as an open rectangular cylinder, is 16 squares wide, 20 squares deep and 27 squares high, each square being approximately 4 cm^2. One of the sets of tasks (T_2) has tasks that guide the use of the model. In these tasks the students have to mark some points on the model. For the other set of tasks (T_1), the students themselves have to decide how to use the model. See Appendix for details of the tasks and a figure of the model. In the introductory task, common to T_1 and T_2, two intersecting orthogonal straight lines, each parallel to two of the opposite sides of the model, were marked with strings in the model. The students were asked to describe the location of the point of intersection. The purpose of this task was that the students should feel a need for a coordinate system and to begin developing utilisation schemes. When this first task was completed, the tutor defined a coordinate system on the model with wooden stitches. After this the groups got different kinds of tasks. Three of the groups got a set of open tasks (T_1). The other three groups got a set of tasks (T_2), where they were asked to mark points on the model and where "small steps" tasks were designed to guide the students to a specific kind of utilisation scheme. Finally, both groups got a common target task:

> *The line L_1 is passing through the points (7,0,12) and (15,16,20) and L_2 through the points (0,4,11) and (20,8,19). Investigate if the lines L_1 and L_2 intersect or not; if they do, determine the point of intersection.*

For the target task the purpose was to see if the groups stuck to their own developed utilisation scheme or not, and to investigate any differences between the groups that had the guided set of tasks and the groups that had the open tasks. The author was tutoring the group work, though groups worked autonomously and most of the time without the tutor being present. All groups were videotaped working on the tasks, and the analyses of the tapes were supported by a transcription method, an elaborated variation of an *interactivity flowchart* (see Kieran, 2001), aimed at highlighting the flow and linking of arguments as well as the level and kinds of interactions between the different students, and between the students and the artefact. As an example, some lines, quoted in the result section below, are shown from students working on task 4 in set T_2.

Line	***Alex***	***Börje***	***Clara***	***Comments***
1. Shall we do it like we did?			Initiate	
2. Mmm, let us start with the *x* and *y* directions.	Follow up			*Turning the model so the xy plane is towards them*
3. It is exactly the same.		Confirm		
4. Yes, we don't have to look, we have the numbers here.	Explicate			*i.e. at the model* *i.e. on the task sheet*
5. But it can be helpful.		Response		
6. Yes, it can.	Confirm			
7. I have to see it.			Question	

In line 1, Clara takes an initiative which Alex follows up in line 2. In the comment interactivity with the model is described. Clara's initiative is confirmed by Börje in line 3, and the discussion goes on. In this short paper there is not space to discuss the possible advantages or disadvantages of using this specific transcription method, or how it was useful for the purpose of this study, the dynamics of the groups not being the focus.

A-priori analysis

After the introductory task a positive orthogonal coordinate system was defined on the model and the origin was located in one of the lower corners by the tutor. The groups working on the T_2 tasks were then given two points (0,0,12) and (20,0,16), and were asked to determine some points on the line through the given points. After that they were again given the point (0,0,12) and a new point (20,16,16) and the same way asked to determine some points on this line. These groups were dealing with straight lines in two dimensions first, as the lines were located on the faces of the model, and after that straight lines in three dimensions (passing through the interior of the model). The purpose was to guide them into a utilisation scheme where they used the model to project any given line to the sides of the model. The model supports this visualisation, and with their pre-knowledge of straight lines in two dimensions the students should be able to solve these tasks. They were also asked to mark the given points in all the tasks, to guide them to actively use the model. All groups were then given the points (7,0,12) and (15,16,20), the groups working with T_2 asked to mark the points, and asked to determine some points on the line through these points. In all tasks they should determine at least one point located outside the model. In a second task, the groups were asked to decide if three given points were located on the line passing through the same points. Finally, the groups got the target task.

To analyse the tasks, dealing with lines in three dimensions, consider a straight line L and a point on this line. To move from this point to another point on the line involves a movement in all three directions, as described by the formula $L:\ (x,y,z) = (x_0,y_0,z_0) + t(\Delta x,\Delta y,\Delta z)$, where (x_0,y_0,z_0) is a point on the line, Δx, Δy and Δz are the differences between the coordinates of two points on the line, thus making up a direction vector, and t is a real number.

We may interpret the movement as a move in one direction at a time, for example Δx steps in the x-direction followed by Δy and Δz steps in the y- and z-directions, respectively. The model supports this interpretation, as the students are able to look at the line through the xz-plane and the yz-plane, and by this also see the projection of the line on these planes. The model also supports visualisation of the projected line on these planes. However, since the students at this level normally have not been working in school with straight lines in dimensions higher than two, we can not expect them to use the symbolic representation of the line given above. Considering their background knowledge, they may try to calculate a slope. Here they have to realise the fact that a three-dimensional line has different slopes in different directions. The model may support the students to calculate two slopes, one that they can visualise in the *xz*-plane and one in *yz*-plane, k_x and k_y, respectively.

The model also supports a direct three-dimensional interpretation of representing the movement from one of the two points to the other in terms of a vector (Δx, Δy, Δz). Just by counting squares they can determine $\Delta x = 8$, $\Delta y = 16$ and $\Delta z = 8$ for the given points. Further, using proportionality, the may scale (8,16,8) down to (1,2,1), which they may relate to the model, and may further be able to combine several vectors (1,2,1) to reach new points.

In the final task, they were asked to determine the point of intersection of two given lines. If you have equations for the lines, you easily find the intersection point using a system of equations. To find an intersection point between two lines with no given equations given is harder. The model supports a visual representation of the lines, and by looking through it you may perceive visually an intersection point and by counting squares you can determine, for example, the *x*-coordinate for this point. Now, using the knowledge of the straight line in two dimensions it should be possible to determine that you have found the intersection point. In this case it is (10,6,15).

The students

Eighteen upper secondary school students from two different Swedish schools (nine students from school S_1 and nine from S_2) voluntarily accepted to join the experiment. The students were divided in six groups with three students in each group, three groups from each school. In school S_1 there were three students available one day and six students another day, thus deciding the groups. In school S_2 students were randomly assigned to the small groups. In all six groups there was a mix in gender. Three students (forming group S_{11}) followed the technology programme and the others were enrolled in the science programme. All of the students had taken course D in mathematics in the Swedish secondary school system. Some of them had also taken an optional course, including some coordinate geometry in higher dimensions than two. Some of the students also had done some programming in Cad. They all had good or very good grades from their mathematics courses. An overview of the sample is shown in the table below, where the * indicates that these students had taken the optional course.

Group	S_{11}	S_{12}	S_{13}	S_{21}	S_{22}	S_{23}
Tasks	T_1	T_2	T_2	T_2	T_1	T_1
Programme	Technology	Science	Science	Science*	Science*	Science

Results

For the first task, all six groups defined a coordinate system on the model to solve the task. The location of origin was different between the groups, but most of the groups defined it in one of the upper corners of the model. One of the groups (S_{23}) defined the origin to be located in the middle of the interior of the model. Due to space limitations, only the work of four groups work will be described in some detail. We first report on the three groups working on T_2, and then the groups working on T_1.

Group S_{12}

Two boys (Anders and Bengt) and one girl (Cilla) make up this group (all names given are fictitious), working on the set of tasks T_2. In the second task, two of the students count squares to locate the two given points. They mark this with sticky tack. While looking at the model, Anders suggests that they should determine the slope for the straight line. They calculate the slope for the line by taking the differences of the coordinates, writing $k = \frac{\Delta z}{\Delta x} = \frac{4}{20} = 0.2$ and decide, by looking at the coordinates, that $m = 12$[1]. They write down $y = 0.2x + 12$. The points they are asked to determine they get by putting $x = -5,\ 5,\ 10,\ \text{and}\ 15$ in their equation. Anders and Cilla start counting squares, when working with the third task, to locate the given points and mark them. After that, Anders suggests: *"Can't we use a formula like $z = kx + ly + m$"*. They write down the equation $z = 0.2x + 0.25y + 12$. With this equation the try to solve the task in the same way that they solved the second task. They put $x = 5$ and $y = 4$ in the equation and get $z = 14$ (writing (5,4,14) on the paper). Anders wants to use the model to verify if they have found a point on the line, and says: *"Should we use a string anyway?"* Cilla agrees: *"Just to see."* Now, Anders and Cilla mark the line and Anders, bending down looking at the model, says: *"It doesn't fit."* After that, Anders suggests that they could write down three two-dimensional equations and see if they can help them. By looking at the coordinates Anders writes down $y = 0.8x$ and $z = 0.25y + 12$. After a long silence they try for $x = 5$ and get (5,4,13). They verify that the point (5,4,13) is on the line by counting squares in the model. After this they determine four points on the line, for $x = -5,\ 5,\ 10,\ 25$. Now they say they are finished with the third task.

Working on the fourth task, Anders and Cilla begin counting squares to locate the given points [(7,0,12) and (15,16,20)] they also mark by putting a string between the two points. Anders suggests that they should solve this task in the same way that they solved the previous and Bengt agrees. By looking at the coordinates for the given lines, Anders writes down $y = 2x - 14$ and $z = 0.5y + 12$. From these equations the group determines that (0, –14,5), (8,2,13) and (10,6,15)

[1] In class students have used to the equation $y = kx + m$ for a straight line.

are points on the line. For each point they determine they verify that it is on the line by looking at the model.

They solve the fifth task by checking if the x coordinate, for the given points, gives correct y- and z-values by their equations. Finally, in the sixth task Anders and Cilla mark the given points and the new line in the model. They look at the model and Anders says: *"It looks as if they intersect."* They determine, as shown above, two equations, $y = 0.2x + 4$ and $z = 2y + 3$, for the line L_2. On a paper they write, L_1; $y = 2x - 14$ and $z = 0.5y - 12$ and L_2; $y = 0.2x + 4$ and $z = 2y + 3$. By using these equations they get, by setting the y-coordinates equal, that the point of intersection is for $x = 10$. This x-value gives that L_1: (10,6,15) and L_2: (10,6,15). They don't verify, by looking at the model, that it is the right point they have determined.

Group S_{13}

Due to space limitations, the work of this T_1 group will be presented only shortly. When solving the tasks, the students mark the given points and put a string between them, and they also turn the model a lot. They draw coordinate systems on their paper. Solving the second task, they determine an equation ($z = kx + m$) for the two-dimensional line, and by choosing some values for x they determine the z-coordinates. When working with the third task they try to project the line into two two-dimensional lines, representing this by the three equations $y = \frac{4}{5}x, z = \frac{1}{5}x + 12$, and $y = 4z - 48$. Choosing values for x and using two of the three equations solve the problem for them. They verify that it is the right points by putting a string into the model to see if it seems to be a reasonable solution. They start the work with the fourth task by marking the given points and they also mark the line by using a string. They realise that using the same method as in task three they will solve this task too and that they do not have to use the model at all, but all three students say that they have to visualise it. Using the already mentioned method to solve the task, they do not verify the points by using the string as in task three. The fifth task is also solved by the same method, i.e. checking, with their equations for the line, if the x-coordinate for the given points produces those points. For the sixth task, they determine two equations for the line L_2. By solving the equation system, with two of the equations for L_1 and the equations for L_2, they determine that the lines intersect in (10,6,15). They also verify this by looking into the model.

Group S_{21}

This group, working with T_2, is composed of one girl (Anja) and two boys (Ben and Carl). In task 2, Carl suggests that they should use a string to mark the line. After doing that Anja says: *"You just have to see the k-value"*, and shows with her hand to the model. Ben follows up her idea and suggests: *"We can determine* *the k-value by taking Δ there"* (showing Δz in the model) *"and by taking Δ there"* (showing Δx in the model). After Ben has determined the slope to be $\frac{1}{5}$, Anja states: *"So, if we move five squares there so…"* They solve the task by adding 5 and 1 to the x- and z-value, respectively, and determine that (5,0,13) and (25,0,17) are on the line. They do not verify by looking at the model that the points are on the line. After having marked the given points in task 3, Anja makes a suggestion: *"I suggest that we determine a k-value for this"* (showing $\frac{\Delta z}{\Delta x}$) *"and one k-value for another direction"*. They determine that $\Delta y = 16$, $\Delta x = 20$ and $\Delta z = 4$, and from this they get, $k_y = \frac{\Delta y}{\Delta x} = \frac{4}{5}$ and $k_z = \frac{\Delta z}{\Delta x} = \frac{1}{5}$. Now Anja explains: *"If we move five steps in this"* (pointing in z-direction), *"we shall at the same time move one step in x and four steps in y"*. Carl follows up her idea: *"Five steps here"* (pointing at x) *"gives one step"* (pointing at z-direction) *"and four for y"*. From the points (0,0,12) and (20,16,12) they add $\Delta x = 5$, $\Delta y = 4$ and $\Delta z = 1$ respectively. Doing this they write down $+(5,4,1)\cdot n$ on a paper. They determine that (5,4,13) and (25,20,17) are points on the line. They do not verify, by looking at the model, that the point is on the line. In the fourth task they work in the same way, and determine that $\Delta x = 1$, $\Delta y = 2$ and $\Delta z = 1$, and Carl says: *"We can create that kind of thing you know"*, and writes $+(1,2,1)\cdot n$. Now they do as in the last task, i.e. add (1,2,1) to (7,0,12) and (15,16,20) respectively, and determine that (8,2,13) and (16,18,21) are points on the line. They do not verify, by looking at the model, that the point is on the line.

For solving task 5, they take one of the points they should examine and subtract the given coordinates (7,0,12), which they named *Ö* and see if the difference is equal to $n\cdot(1,2,1)$. Here they write: given point – *Ö*=n(1,2,1).

In the target task, while marking the new given line, Carl says: *"I don't think that we have to mark all these lines… looks like they intersect, but we can't be sure. We have to calculate"*. Ben suggests: *"We can see if we see where they intersect"*. After marking the lines Carl suggests: *"We have to create such a thing"* (pointing at (1,2,1) on the paper) *"for the lines"*. Anja follows up Bens' idea: *"Can't we see if we found one point"* (by looking at the model) *"and see if it is right"*. Ben continues: *"yes … Here somewhere, for this x-value it looks like they intersect",* and writes down $x = 10$, $y = 6$ and $z = 15$. By calculating as before they find that L_2: $(0,4,11)+(5,1,2)n$, and Ben suggests: *"I believe n=2, check up what happens if n=2"*. Doing that they find that (10,6,15) is the point of intersection and verify that the point is correct by doing the calculation $(10,6,15) - \ddot{O} = (3,6,3) = n(1,2,1)$. Carl is not content and suggests: *"Maybe we should try to find a method to do this without having to see and check"*. Ben follows up: *"Yes, maybe … but this was a nice method I think. What kind of method should that be?"* Carl writes down the following equation:

$(0,4,11)+(5,1,2)n_1 = (7,0,12)+(1,2,1)n_2$. Solving this equation they find that $n_1 = 2$ and $n_2 = 3$, which gives that the point of intersection is (10,6,15).

Group S_{22}

There were two girls (Aina and Bea) and one boy (Christer) in this group (T_1). In the beginning of solving task 2, Bea suggests: *…move this strings so we are able to see where it is.* After doing that the group is silent. Watching the model with the line marked, Christer suggests: *We can do it easy, why not just double everything? Start in that point [pointing at (15,16,20)] and move to that point [pointing at (7,0,12)]. Then moving with the same change in the coordinates we find a new point, very far from the model but still on the line. … We can also half the change to find a point inside the model.* After the group have written $\Delta x = 8$, $\Delta y = 16$ and $\Delta z = 8$ on a paper, Bea says: *We can either move a…move…choose a negative vector, isn't that in the opposite direction…see it as a vector. …if we want to move a third vector we just multiply the vector with a third.* With this method, i.e. (7,0,12) – (8,16,8) and (7,0,12) + (4,8,4), the group found out that (–1, –16,4) and (11,8,16) are two points on the line.

When deciding if the points in task 3 are on the line or not Bea suggests: *… if we start with x … because, if Δx is one we can decide Δy … how big Δy and Δz are in proportion to that … and then multiply by for example twenty three.* When Aina asks: *What are you saying, multiply by that?* Bea explains her idea: *Multiply everything by twenty-three and then we get the coordinates for y and z and then see if they correspond [meaning if they correspond to the coordinates of the given points].* After this the group is silent for a while when Bea is calculating. She writes $\Delta x = 1$, $\Delta y = 2$ and $\Delta z = 1$ on the paper and then saying: *The question is if I have to … if I have to start in that [pointing on the paper] and add that I got when multiplying with twentythree to the difference here [pointing on the paper] then … don't you think so?* After doing some calculations and finding some mistake in it, she finds it hard work and decides to start from the beginning again. Christer suggests: *Couldn't we see it as two different functions and see it in two different planes… we don't get everything at once but …* Bea agrees: *Maybe we have to do it in two steps…* Aina does not agree and suggests: *But, can't we just compare … no... But yes, if you compare this two or … compare that and that [pointing on the paper] and see if it is the same … the same direction … compare that and that, if we get the same numbers wouldn't it be on the line then?* After that, they find that (11,8,16) – (7,0,12) = (4,8,4), which gives that the change is the same as between (15,16,20) and (7,0,12), and therefore the point (11,8,16) is on the line. They do the same calculations for the other points.

After this they start with the final task, and Christer asks: *The question is if we can solve this with the same method.* He continues: *I still think it would be smooth if we could describe it with two equations. Then we could compare two of them … when comparing y similar to something x in both directions and get it to* *the same point. If we don't get the same point, then they don't intersect.* Bea suggests that they should mark the new line: *If we think that it would be two strings in the model … if they intersect in a point they should be in the same spot in both the xy-direction and the x … both in the xy-direction and the y-direction.* Aina marks the line L_2 in the model with a string. Bea and Christer want to split the line in two equations, and Bea says: *Now I have two equations, x- and y-direction [writes down y=2x-14 and y=0,2x+4] … but … how can we decide if they intersect?* After letting the y coordinates in the equations be equal they find that x = 10. This gives them that y = 6, by using one of the equations for L_1. At the same time, Aina, looking real hard at the model, says: *That looks like it is correct.* Now Bea determines two more equations, $y = 2z - 24$ and $y = 0.5z - 1.5$. After that they let these y-values be equal and find that z = 15. All three look into the model and agree that it looks right.

Groups S_{11} and S_{23}

Due to space limitations, the work of two groups working with T_1 will be only shortly described. The group S_{11} uses the model only after intervention by the tutor. When dealing with tasks about a line in three dimensions they represent the line with two equations and by using these equations in a fruitful way they solve the tasks. In the target task, they get the wrong intersection point in their first two attempts. After the second time the tutor marks the lines in the model and asks them if they have found the correct point, and they find that their intersection point was located far away from the model. After their third attempt they verify, by using the model, that they have found the correct point.

When the group S_{23} students work with lines in three dimensions in task 2, to find some point on the given line, they mark the line in the model and try to visualise the solution. They note that the line has different slopes in different directions and determine one slope in the yz-plane and one in the xz-plane. They find the slopes by counting the difference of coordinates, as presented in the task sheet, and use that the coordinates change proportionally, for example when Δz = 1 then Δy = 2 and Δx = 1. They use the vector approach for task 3, without looking at the model. To solve the fourth task, they mark the lines with strings and decide visually that they should intersect. To be certain, they represent them with two equations respectively, after looking at the projected lines in the model. By setting some coordinates equal for the intersection point they get the equation $x + 5 = \frac{8x}{20} + 5$, from which they find the solution.

Discussion and conclusions

Even though the model invites to a visual approach all groups solved the target task algebraically, which according to results from previous research often is the preferred method. However, most groups related their solution processes to the model, with more or less elaborated utilisation schemes. The students seem to find it easier to get ideas to start solving the task by using a visual approach but they do not find this sufficient to be certain that they have found the correct solution. As one student from group S_{22} says, *I like the idea to have a real coordinate system to look at. Well, I suggest that solving it using equations can be done, but having something to look at then makes me start thinking of different solutions.*

Two of the groups doing tasks T_2 consistently used a projective solution process for the target task, describing the lines with two equations from the projected lines. This method was in line with the utilisation scheme promoted by the design of the tasks. The third group (S_{21}), however, used a vector model, which of course also works on the tasks with lines in the planes on the faces of the model. The members of this group had all taken the optional course and were thus familiar with the idea of a vector, though not explicitly using the word *vector*. Consequently, they solved also the target task by adding a multiple of the vector between the given points on the lines. They developed a utilisation scheme where they looked at the model to find a suitable multiple of the vector to add, and then tried it out algebraically on the two lines. They expressed a dissatisfaction of having to be reliant on the model, and developed a purely algebraic method to solve the problem, setting up the vector equation $(0,4,11)+(5,1,2)n_1=(7,0,12)+(1,2,1)n_2$. We conclude that all three groups working on the guided tasks thus stuck to their utilisation schemes developed during the first tasks. These schemes clearly related to their different pre-knowledge.

Of the groups doing the open tasks T_1 only the work of group S_{22} was reported here in some detail. This group was familiar with the vector concept and used this consistently on tasks 2 and 3, using the model only to visualise the lines. In task 3 there was some discussion on looking at different planes, probably referring to the faces of the model, and solving the task in two steps. It is Christer, who during task 2 wanted to make things easy by multiplying vectors, now apparently found some difficulties when the lines were passing through the interior of the model. Bea followed his idea and they solved the target task by separating variables, having two equations each of two variables for each line, i.e. the same method that was used by the students in the T_2 groups S_{12} and S_{13}. Aina tried to remember some method for this kind pf problem, but it seems as the solving process is influenced by having the model available to visualise the planes with the projected two-dimensional lines. Aina was here concentrated on the model, and after finding a solution they all validated it by looking at the model. The T_1 group S_{11} did not use the model at all, while all the other groups used it in all tasks. The S_{11} students followed the technology programme and were there used to handle coordinates in three dimensions. They said that they had not calculated with coordinates in three dimensions but they brought a method (or understanding) of how to handle the coordinates, and obviously felt safe feeling no need to verify their solutions, thus failing to observe their mistake. However, after the intervention by the tutor of how to use the model, they later turned back to the model to validate their solution by visualisation.

To summarise our tentative conclusions, students stay in the target task with their utilisation scheme as developed during the previous tasks as long as they find it functional. However, the way they take advantage of the model depends more strongly on their pre-knowledge, and if the problem situation is familiar the artefact does not come into play for the solution process. One main role of the artefact for the students, in this and our previous study, is to validate the solutions as well as the methods used. When students are not guided into a specific utilisation scheme, they may not benefit from the potential advantages that the artefact offers. On the other hand, with appropriate pre-knowledge, non-guided tasks in this respect may open up for a more varied use of the artefact, using utilisation schemes that support alternate ways to solve the problem at hand.

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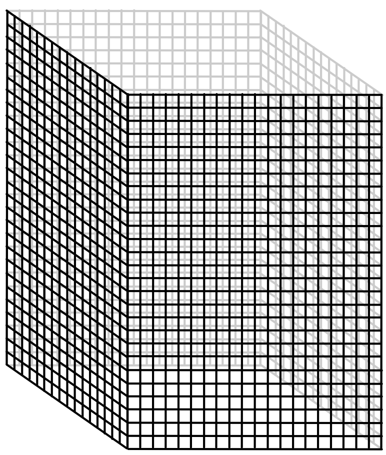
Appendix

T_1: 1. In the model there are two straight lines marked. These lines intersect in one point and your task is to describe the location of this point.
2. Through the points (7,0,12) and (15,16,20) there is a straight line. Your task is to determine some more points on this line. At least one of the points you determine must be located outside of the model.
3. Are the points (23,32,28), (11,8,16) and (10,9,15) on the line (from task 2)? Motivate how you know if they are on or not.
4. Examine if the lines L1 and l2 (see below) intersect, and if they do, locate the intersection point.
L1: through the points (7,0,12) and (15,16,20)
L2: through the points (0,2,11) and (20,8,19)

T_2: 1. In the model there are two straight lines marked. These lines intersect in one point and your task is to describe the location of this point.
2. Mark the points (0,0,12) and (20,0,16) at the model. Through these points there is a line. Your task is to determine some more points on this line. At least one of the points you determine must be located outside of the model.
3. Mark the points (0,0,12) and (20,16,16) at the model. Through these points there is a line. Your task is to determine some more points on this line. At least one of the points you determine must be located outside of the model.
4. Mark the points (7,0,12) and (15,16,20) at the model. Through these points there is a line. Your task is to determine some more points on this line. At least one of the points you determine must be located outside of the model.
5. Are the points (23,32,28), (11,8,16) and (10,9,15) on the line (from task 4)? Motivate how you know if they are on or not.
6. Examine if the lines L1 and l2 (see below) intersect, and if they do, locate the intersection point.
L1: through the points (7,0,12) and (15,16,20)
L2: through the points (0,2,11) and (20,8,19)

The model



The Same Topic – Different Opportunities to Learn

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Abstract: *The teaching and learning of mathematics involve intricate processes and many different factors may have impact on learning outcomes. The study reported in this paper, however, has a quite narrow focus on how the mathematical content is treated. It is part of a larger study and is based on the assumption that what students learn or do not learn, in respect to a certain content, is dependent on what features of the content have been possible to experience by the students. Three lessons where the same mathematics is taught are compared and substantial differences are found.*

Comparative studies

Most large international comparative studies in mathematics education involve comparing learning outcomes or student achievement. In the series of IEA-studies (FIMS, SIMS and TIMSS) more and more variables – mathematics curriculum, size of mathematics classes, grouping and streaming of students, amount of mathematics lessons and homework, teacher education, attitudes of students and teachers, etc. – have been included in the studies in order to understand and explain differences in achievement between countries (IEA, 2005). Subsequently, in the TIMSS-studies of 1995 and 1999 the ”teaching process” was included through the video-recording of 8th grade mathematics lessons (Hiebert et al., 2003). One aim was to identify what is typical of teaching in high-achieving countries. An interesting result was that, despite a huge effort, it turned out to be hard to find any factors that could explain the differences in achievement. After having coded and examined more than 60 aspects of mathematics teaching in between 50 and 140 taped lessons from the seven countries the research group alleged, ”... we had difficulty finding lesson features that correlate with differences in achievement” (Givven, 2004, p. 208). A further analysis of how the mathematical content was dealt with, from a student perspective, was needed.

> What, then, do the higher-achieving countries have in common? The answer does not lie in the organisation of classrooms, the kinds of technologies used, or even the types of problems presented to students, but in the way which teachers and students work on problems as the lesson unfolds. (Stigler & Hiebert, 2004, p.14)

The research reported here is also comparative, but on a smaller scale. I view the comparison not only as an opportunity to learn from other cultures. A comparison is often necessary to make the familiar, and all that is taken for granted, visible. To contrast the familiar with something less familiar or even better, with something quite different, can make important features that are too well-known and established, possible to discern.

Focus on the mathematical content

Studies of factors that may have an impact on learning outcomes (see e.g. Pong & Morris, 2002; Gustafsson & Myrberg, 2002) suggest that factors, on a general ”political level”, can merely provide opportunities for good teaching and learning, not guarantee that it will take place.

> One possible conclusion that can be drawn from across these meta-analysis […] is that to improve school learning, we should focus on those variables that impact directly on the learning experiences of students such as teaching and feedback. (Pong & Morris, 2002, p. 11)

It seems that residential area, school, parents’ level of education and other ”distal factors” (Pong & Morris, 2002) cannot explain differences in results. If you want to understand why some students learn better than others, the teaching process cannot be ignored and treated as a ”black box”.

But even inside the mathematics classroom, some of the factors that appear to be ”close” to the teaching and learning may have little impact on student learning. As mentioned earlier, one lesson from the TIMSS video study is that aspects of a mathematics lesson that are fairly easy to observe, e.g. mood of instruction, the number of students in the class, how students are grouped etc., seem to affect student achievement much less than *how* the mathematics content is handled by the teacher and the students.

> A focus on teaching must avoid the temptation to consider only the superficial aspects of teaching: the organisation, tools, curriculum content, and textbooks. The cultural activity of teaching – the ways in which the teacher and students interact about the subject – can be more powerful than the curriculum materials that teachers use. (Stigler & Hiebert, 2004, p. 15)

This leads to one of my assumptions for the analysis. *What* is possible for students to learn is to a high degree related to *how* the mathematical content is handled in the classroom. Different ways of handling the content makes it possible to learn different things. The objective of this study is to capture the interaction about the mathematical content rather than looking at more easily obtainable features. By comparing lessons I will try to answer the question: How is the same mathematical content dealt with in three different school settings and what are the implications for students’ opportunities to learn?

Theory of variation

In the theory of variation (Marton & Booth, 1997; Runesson & Marton, 2002; Marton, Runesson & Tsui, 2004) learning always has an object, called the *object of learning*. The object of learning can be seen or understood in qualitatively different ways. The different ways of understanding are distinguished by what aspects of the object of learning an individual can discern and keep in focal awareness at the same time. *Learning* in this theory means learning to be able to see the object of learning in a new way, which indicates that new aspects have been discerned. An experience of *variation* is required in order to facilitate discernment of new aspects. The aspects of the object of learning that are varied (e.g. in a mathematics lesson) are more likely to be experienced. Aspects that are kept invariant are not possible to experience, unless the student creates the variation by her/himself by using previous experience.

As an example let us consider students that are given the opportunity to experience a variation in the number of solutions by solving a series of system of equations like the following (with one, infinitely and no solutions). They may probably learn that the number of solutions is not to be taken for granted.

$$\begin{cases} x+y=3 \\ 2x-y=5 \end{cases} \qquad \begin{cases} x+y=3 \\ 2x+2y=6 \end{cases} \qquad \begin{cases} x-2y=3 \\ x-2y=1 \end{cases}$$

Students, on the other hand, that only solve systems of linear equations with singular solutions are not provided with the opportunity to learn that these systems have anything but one unique solution. Without an experienced variation in the aspect of ”number of solutions” the possibility that students would discern this feature is very small. In the series of systems of equations above other vital aspects may at the same time be kept invariant, e.g. the letters used for unknowns (x and y). There is no variation in this aspect.

By identifying what features of a mathematical concept are kept invariant and what aspects are varied during teaching and learning situations a ”pattern of variation” can be formed. Differences between lessons can then be described in a qualitative way by means of the different patterns of variation. It has been shown that this approach works well in describing subtle, yet important, differences in how the object of learning is handled (see e.g. Runesson, 1999; Runesson & Mok, 2005). Pang (2002) demonstrates in his PhD-thesis that what students learn, or don’t learn, can be explained by means of the patterns of variation they have been offered in teaching. Of course there is no ”absoluteness” in the theory. Some students in a class may not, for different reasons, discern important features of the content, even though these features are made possible to experience.

The data

The data have been collected within the Learners’ Perspective Study (LPS study, 2005; Clark, 2000). It is an international study where sequences of competently taught mathematics lessons are documented. In all participating countries at least 10 (often 15-18) consecutive mathematics lessons in three 8^{th} grade classes are videotaped with three cameras. The teacher camera follows and records the teacher. The student camera is directed towards 2-4 students sitting next to each other. The whole class camera captures most of the classroom. After each lesson video-stimulated interviews are conducted with two of the focused students. Data from each sequence of lessons consist of classroom videotapes, transcripts translated to English, teacher questionnaires, student and teacher interviews, copies of students' notebooks and textbooks etc. The Swedish data was gathered in ”KULT-projektet” (KULT-projektet, 2005; Häggström, 2004) during 2002 and 2003. In this paper only a minor part of my research is reported. The present comparison is made between three specific instances in lessons in China (Hong Kong and Shanghai) and Sweden based on videotapes and transcripts.

Three lessons

The lessons are chosen because the *same topic* is taught. The analysis is restricted to the parts of the lessons where the concept of *system of linear equations in two unknowns* is introduced to the students for the first time. In this section, I will describe how the mathematical content is handled as these parts of the lessons unfold, followed by an analysis in the next.

Hong Kong

After a short revision of *linear equations with one unknown*, the teacher presents a problem. It is written on the blackboard and the students are asked to find the number of rabbits and chickens.

> A farmer has some rabbits and some chickens. He does not know the exact number of rabbits and chickens, but in total there are ten heads, and there are twenty-six legs.

In the following there are many shifts between teacher-led discussion and student-work, individually and in pairs. The problem is handled in three ways. The first method used is ”guess and check”. The teacher leads the way to the solution by posing questions; ”Can all of them be chickens?”, ”Can all be rabbits?”, ”Can there be five each?” etc. Secondly, the problem is represented by the formulation of *one* equation, $2x + 4(10 - x) = 26$. This is done with similar firm guidance from the teacher. When asked by the teacher, only five students say they could have managed to do this by themselves. The equation is not solved, probably because the answer is already known. Then the teacher introduces system of equations.

> Teacher: *I am now going to teach you an easier method. It is also about using equations, but it is simultaneous equations in two unknowns.*

After a few minutes of teacher-led discussion the teacher has written two equations on the blackboard.

$$\begin{cases} x+y=10 \\ 2x+4y=26 \end{cases}$$

This example is used to ”define” the concept of simultaneous linear equations in two unknowns. No method for solving the system of equations is demonstrated.

Teacher: *Well, okay, well, we call it simultaneous equations. Simultaneous means the equations will be listed out together. A moment ago ... Nancy has asked me why it is called linear. This is because we can draw a straight line from this kind of equation.* [...] *... this is called two 'yuan', that means how many unknowns are there?*

Student: *Two.*

Teacher: *Two. It is simultaneous that means the equations are put together.*

After this introduction a worksheet is distributed to the students. In the first task the students are asked to find the corresponding y-values, for $x = 0, 1, 2, 3$, for the two equations separately and to list them in tables as a step in finding a common solution. How to find the first two y-values are explained and shown by the teacher. The students start to work to fill in the rest of the two tables.

Shanghai

This lesson begins with a swift revision (4 minutes) on the topic of *one linear equation in two unknowns* and its' solutions. The teacher then announces the topic of today's lesson and shows a slide with three questions about the concept of a *system of linear equations in two unknowns*.

Q1. What is a ”system of equations”?

Q2. How can you tell whether a system of equations is a system of linear equations in two unknowns?

Q3. Identify whether the given is a system of linear equations in two unknowns.

1) $\begin{cases} x+y=3 \\ x-y=1 \end{cases}$ 2) $\begin{cases} (x+y)^2=1 \\ x-y=0 \end{cases}$ 3) $\begin{cases} x=1 \\ y=1 \end{cases}$

4) $\begin{cases} x/2+y/2=0 \\ x=y \end{cases}$ 5) $\begin{cases} xy=2 \\ x=1 \end{cases}$ 6) $\begin{cases} x+1/y=1 \\ y=2 \end{cases}$

7) $u=v=0$ 8) $\begin{cases} x+y=4 \\ x-m=1 \end{cases}$

The teacher tells the students to read a section in the textbook and to discuss the three questions in pairs. After a couple of minutes the teacher calls for attention. The students' answers to the first two questions are obviously the same as in the textbook.

Student: *A system of equations is formed by a number of equations.*

[…]

Student: *There are two unknowns in the equations and the indices of the unknowns are one. This is called system of linear equations in two unknowns.*

Teacher: […] *He has just mentioned the definition of system of linear equations in two unknowns.*

These important points are then repeated a number of times in the following whole-class conversation before they move on to the third question.

Teacher: *Okay, these two points, oh then, let us take a look at the following questions with these two points.*

All eight items in question 3 are discussed in whole-class, one at a time. Reasons for or against them being a system of equations in two unknowns or not, are given by students in each case. Four of the examples meet the requirement and four do not. The lesson then continues with a focus on the meaning of a *solution* to a system of linear equations in two unknowns and what is required of a pair of numbers (x, y) to be a solution.

Sweden

The teacher begins the lesson by returning to a problem that previously has been handled by an equation in *one* unknown. Some students had tried to use *two* unknowns and the teacher now shows that this approach also is possible.

After this the teacher and the students formulate a new problem together. One student is asked to "think of a number" (x) and the teacher "thinks of another number" (y). The student tells the teacher what number he is thinking of and based on this the teacher writes the equation, $x + y = 60$, on the whiteboard. After examination the conclusion that there is not enough information to determine the two numbers is reached in the whole-class discussion. There are many possible solutions to this equation, so the teacher adds another condition and gets the following on the board.

$$\begin{cases} x + y = 60 & (1) \\ x = 14 \cdot y & (2) \end{cases}$$

[*Note.* · is used for multiplication]

Teacher: *Now I have two conditions and two unknowns. Now we can easily calculate the whole ... So, let's do it.*

The teacher uses the method of substitution and finds the value of *y*. The whiteboard now shows the following.

$$\begin{cases} x + y = 60 & (1) \\ x = 14 \cdot y & (2) \end{cases}$$

$$\text{(1) and (2) gives:} \quad \begin{aligned} 14y + y &= 60 \\ 15y &= 60 \\ y &= 4 \end{aligned}$$

The problem is not solved completely since the student's number (*x*) was already revealed by mistake. The students' attention is then directed back to the two equations when the teacher points to them. A ”definition” of a *system of equations* is made from this example.

Teacher: *What is this then?*

[T points to the equations (1) and (2) on the whiteboard]

Teacher: [...] *Well, it's two equations. A system of equations, it is called.*

[T writes ”ekvationssystem” (system of equations) beside the equations]

After this introduction the teacher writes a similar system of equations from the textbook on the board for the students to solve. A few minutes later the teacher writes the solution on the board, in silence. The students continue to work with similar tasks in their textbook, while the teacher walks between the desks and talks to individual students.

The analysis

The intention of the analysis is to first describe how the mathematical content is handled in terms of what aspects are varied and what aspects are kept invariant (pattern of variation), and secondly to discuss the students' opportunities to learn in the light of the patterns of variation observed.

Hong Kong

During the introduction the rabbit-and-chicken problem is kept invariant and dealt with in three different ways. The method for representing and solving the problem is thus varied. The use of one equation is contrasted to the use of a system of equations. This contrast offers a possibility to discern features such as *two* unknowns and *two simultaneous* equations. The characterisation of the equations as *linear* is mentioned only as a reply to a student's question but not further discussed.

The meaning of *a system of linear equations in two unknowns* is only given by positive examples. There are no counterexamples that may point to important features of what is *not* included in the concept.

Shanghai

Already from the revision in the beginning of the lesson it is clear that many of the features involved in the new concept seem to be familiar to these students. That includes the concept of one linear equation in two unknowns, the number of solutions to a linear equation in two unknowns and how to determine whether a proposed solution is correct or not.

The concept of *a system of linear equations in two unknowns* is introduced by verbal description of some characteristics – two equations, two unknowns and degree one. During the seven and a half minutes when question 3 is discussed in a teacher-led mood of activity, a certain pattern of variation appears. Eight proposed systems of linear equations in two unknowns are considered. The important points from the first two questions – *two equations*, *two unknowns* with *indices of one* – are kept invariant. By an apparently very deliberate choice of items in question 3 the teacher generates a specific pattern of variation. The contrasts that are formed between the eight examples makes it possible for the students to discern some of the features that might be critical when it comes to understanding the concept of system of linear equations in two unknowns. Some of these features are:

The variables must be of degree one. Even if you ”know” this it may, in some cases, be difficult to interpret the mathematical symbols correctly. Some of these instances are,

- $(x+y)^2$ is not of the first degree even though $x+y$ is.
- xy is not of the first degree even though x and y are.
- $y/2$ is of degree one but $1/y$ is not.

There must be two, but not three, variables. In the last example *x, y* and *m* are used. The teacher and the students seem to interpret this as three variables. However, the letter *m* is often used to denote a constant, not a variable or unknown. The use of letters with a more similar ”status” (e.g. $x - y - z$ or $r - s - t$) would have made the interpretation more straightforward.

Other letters beside x and y can be used. The letters *u* and *v* are used in one example.

Two equations can be merged together to what might look like one equation. It is probably not evident to students that an algebraic expression like $u = v = 0$ can be interpreted as two equations merged together.

Sweden

In the first short sequence the teacher keeps a previous problem invariant while the way to represent it is varied. The contrast to the earlier use of an equation in *one* unknown when trying to solve the problem might make the feature of *two* unknowns discernable.

In the next sequence the two numbers ”thought of” are kept invariant and the number of conditions are varied. At the same time as the number of conditions are increased from *one* to *two,* the number of solutions changes from *many* to *one*. The need for *two* equations when you have *two* unknowns is highlighted.

During the same episode the meaning or interpretation of the letters x and y are elaborated on in a sophisticated manner. They are simultaneously kept invariant – the student and the teacher keep thinking of the *same* two numbers – and varied when only the first condition, $x + y = 60$, is examined – *different* number pairs are suggested as solutions. The letters x and y are at the same time considered to denote two distinct but yet unknown numbers and two variable numbers.

As in the Hong Kong lesson the meaning of *a system of linear equations in two unknowns* is only given by positive examples. There are no counterexamples that may point to some of the features that are not included in the concept. Apart from there having to be *two* equations, no real discussion of the specification or requirement for the concept is done. The focus is almost immediately turned to the procedure of solving, where the meaning of *a solution to a system of equations* also is given by positive examples only, without any counterexamples. The general picture is that, after the introduction, the method of solving is kept invariant for the rest of the lesson and is used to solve different examples of systems of equations, which is what varies.

Different opportunities to learn

How is the mathematical content dealt with and what are the implications for students’ opportunities to learn? I have observed several differences in respect to what features of the concept of system of linear equations in two unknowns are elaborated on during the introduction in the three lessons. One such aspect is the system of equations as *a method for problem solving*. This aspect is taken for granted in the Shanghai lesson, quite differently from the other two lessons, where the concept is introduced as one way, among others, to represent and solve a particular problem. According to the theory of variation it is not likely that the Shanghai students would experience this aspect during the sequence analysed.

I find the most striking difference, however, to be the deliberate use of non-examples and contrasts in the third question from the Shanghai lesson. In the Swedish and Hong Kong lessons the attention is quite rapidly turned from the concept of system of equations to the solving of the same. In the Shanghai lesson *what is* and *what is not* a system of linear equations in two unknowns is further elaborated on. The pattern of variation that is generated while the Shanghai class and their teacher discuss the eight items in question three gives the students the opportunity to experience aspects like, *two equations can be written as one expression*, *three unknowns are not allowed*, *first degree expressions and excep-*

tions, and *different letters may be used for unknowns*. None of these aspects are varied during the introduction in the Swedish and Hong Kong lessons and these students are thus not provided with the opportunity to discern these features. In a similar way, nor are these students given the opportunity to experience and learn that there are *other* systems of equations than the systems of linear equations in two unknowns, as this is taken for granted.

My claim that the differences I have observed, and described by means of what pattern of variation is offered, will affect what is possible for students to learn does not go beyond the analysed sequences. From the data analysed here it is not possible to say anything about the students' opportunities to learn outside of these quite short sequences. As a consequence I will include more lessons, both from the three classrooms analysed in this paper, as well as lessons from other documented mathematics classrooms in the Learners' Perspective Study, into my further research. Perhaps it is possible to observe significant differences also when longer sequences of mathematics lessons are included in the analysis.

I am not yet prepared to draw any conclusions regarding cultural differences from these data. The recorded mathematics lessons are not necessary typical of the school cultures in China and Sweden. They merely provide examples of what is considered good teaching from the different countries. The obviously deliberate use of variation and contrast found in the Shanghai lesson, however, seems to be in accordance with a Chinese tradition of teaching described elsewhere (Gu et al., 2004), and possibly not an isolated example. This I intend to keep in mind for my further research.

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Completing Mathematics by Teacher and Student Reflection

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Abstract: *The value of mathematical dialogues between students and teachers is not restricted to the educational value for students. They are also fundamental for the teachers' ongoing professional development. Furthermore, student activity, mediated by teachers, may be used to complete "official" mathematics. The goal of this paper is to bring together dialogue, student thinking, teacher development, and the dialogue seminar reflection tools aiming towards an extended version of mathematics – extended in conceptual direction. Reflection on student-teacher practice may have a natural output as "reflective mathematics", completing formal mathematics. The formulation of reflective mathematics is intended as a remedy of the common and reasonable complaint by students that mathematics is meaningless and fragmented.*

Introduction

A teacher gradually accumulates teaching experience during years of teaching. However, a large volume of teaching is not equivalent to high professional skill. Development of skill is strongly dependent on the reflective attitude towards the ongoing practice. Can anything interesting be said about the basis of such reflection?

This question has an extremely wide scope. It is here focused on attitudes towards students as not only learners, but also as authentic producers of mathematical thinking with potential value for teachers' view of mathematics, and as representatives of the present student culture. Attitudes towards mathematics itself are also in focus, for example whether it may be reinterpreted by student thinking and student activity. We start by discussing reflective methods, particularly action research and the dialogue seminar.

Action research focuses teachers' development by evaluating their own practice. It is described in Miller and Pine (1990) as

> An ongoing process of systematic study in which teachers examine their own teaching and students' learning through descriptive reporting, purposeful conversation, collegial sharing, and critical reflection for the purpose of improving classroom practice. (p. 57)

Action research is often carried out as collaboration between researchers at a university and teachers at an elementary school. The terms "researcher" and "teacher" have a different flavour in action research in that teachers here are considered as the primary researchers. Researchers and teachers together document the school activity in different ways, and successively make evaluations and adjustments. As we shall see later, the dialogue seminar has more the form of teacher-to-teacher meetings. Action research has often reported good improvements both in students' learning and teacher skill. It seems however as if action research not so often formulates students' views of the subject – how the general image of the subject, as mathematics, can be improved. Action research appears to be a pedagogic method primarily aimed at the improvement of the classroom practice and not so much at developing the way the subject is described. However, the method of action research can certainly also be used in this direction.

Student-teacher dialogues are two-sided, but the attention on dialogues has been largely one-sided – focusing on student learning but not much on teacher learning. It would be valuable if teachers' long term professional development also is an articulated purpose. The one-sidedness makes students into pure consumers of education, and not producers. Students' sense of responsibility in their own education is important. However, the intention here is not to increase the formal demands on students. The intention is that they are seen as thinking persons, whose mathematical work can be important for others, not only for themselves. What students are asked for, or invited to, is to try to formulate sincerely their mathematical attempts and thoughts, and engage in dialogue with teachers and/or other students. Older students may be involved in more regulated cooperation with teachers.

Then, what are teachers asked for? For teachers it is very important to find ways to discuss and formulate their teaching practices with other teachers and with other expertise. Action research is one way. We focus in this paper on the dialogue seminar, which is a tool for reflective practice. Here organized dialogues among teachers are seen as the main tool for formulating and extracting knowledge. The dialogue seminar is discussed later.

The importance of dialogues in learning is well established. Johnsen Høines (2004, p. 101) describes very clearly that the *differences* in view between persons engaged in a dialogue is the energy driving the ongoing mutual discovery which is typical for a dialogue, here by citing Dysthe (1999):

> Without the differences the interaction would not have any function. The understanding would not develop. *Different voices are not enough to create meaning; the tension and struggle between them create understanding*.

In the dialogue seminar there is also an explicit recognition of the authentic differences, "fruitful disagreements", that may exist and may appear in a dialogue (Berg, 2005, p. 101). The goal of a dialogue is not to reach a common conclusion, so such an expectation is inappropriate.

The lack of recognition of student-teacher dialogues for teachers' development reflects teacher educators' view of student teachers, and mathematicians' view on mathematics student teachers. Relations in classrooms propagate from teacher education to school. If teacher educators do not recognize their learning when teaching student teachers, one cannot expect student teachers to recognize their learning as teachers when they meet students after teacher education. Increased contact areas are called for between at least three cultures of mathematicians, teacher educators and students. The dialogue seminar has often been used for culture-bridging purposes.

However, increased contact areas between cultures are not without risk. When two cultures meet, both cultures are to some extent jeopardized. This requires a mutual respect, not only between individuals. In particular, the mathematics culture can be seen as deviant and fragile, and may lose important characteristics if these are not clearly recognized. One way this can happen is that a mathematics teacher, when facing the depths of some students' difficulties, may dismiss important mathematical ideas. There is a risk of inventing less general versions of concepts that solve immediate problems, but result in more problems in the future. This is not to say that the teacher should not negotiate with students about teaching methods.

The articulation of mathematics from the standpoint of the stories of mathematical activity is in this paper called "reflective mathematics". The term stands for everything that gives meaning and insight to formulas and their manipulation, and explanations that contribute to making calculations predictable. Both need to be valuable for more than a few persons. Without reflective mathematics, the subject is meaningless and unpredictable formula manipulation. Students can contribute significantly to the construction of reflective mathematics, mediated by teachers. But reflective mathematics is not only effective metaphors found by students. It may involve fundamental different views of mathematics that makes the subject more available, formulated by teachers, but perhaps originating in teacher practice. An example of this is formulated later.

By frequently expressed student difficulties one may say that reflective mathematics today has a weak position in mathematics. This is related to aspects of its linguistic character, which we next turn to. To illustrate this, let us compare the activity of a mathematics teacher to that of a chemistry teacher. A chemistry teacher uses words and argumentation to explain properties and reactions of chemical compounds (subject matter). A mathematics teacher uses words and argumentation to explain mathematical argumentation. Note that in mathematics, explanations and the subject matter, what is to be understood, are both words. Now, if the mathematics teacher becomes very familiar and articulated in the mathematical argumentation, and the student feedback is weak, there is not much difference for the teacher between the two types of argumentation. Mathematical argumentation may become enough. The situation may be summarized in the following short dialogue that may follow a long uninterrupted teacher presentation.

Student: *Ok... and can you now explain it?*
Teacher: *That is what I just did!*

Sentences that for the student appear as subject matter, and not explanations of it, appeared for the teacher as explanations. The mathematical language may have become the natural language for the teacher, but not (yet) for the student. The situation corresponds in a chemistry context to the teacher demonstrating chemical reactions without a word of comment.

Mathematics without reflective mathematics can be seen as a mute mathematics, although it is not silent. Mathematics is almost entirely a linguistic practice.

The event that mathematics argumentation replaces "argumentation from the outside" as described above, relies on basic properties of languages. Language users are normally not conscious of the language used since we usually focus the content we talk about, and not the language itself. M. S. Smith (1994, p. 10) writes:

> In most normal everyday language use, we are not especially aware that we are following rules. We even select many of the words unthinkingly. When saying "he was kissed" we do not consciously refer to a passive rule for constructing the passive sequence. We are more concerned with expressing our thoughts and understanding what people are saying.

However, the language can be made visible. M. S. Smith continues:

> It is possible, however, to shift our attention to the sounds, letters, words and constructions we are using. If, for example, someone suddenly asks a question such as:
> 'What is the word for an animal you keep at house?'
> 'What words did she actually use when she refused?'
> 'What is another way of saying "I don't mind if I do"?'
> then the listeners' conscious attention is directed suddenly to the language itself, and not just to meaning and messages. We could call this going into the meta mode.

The 'meta mode' is equally important for becoming conscious of the linguistics of the symbolic language of mathematics. Bakhtin (1981) underlines the need for different languages to be able to see a language: "Languages throw light on each other: one language can after all see itself only in the light of another language."

The term *Mathematish* (Lennerstad & Mouwitz, 2004) denotes the symbolic language of mathematics seen as a language, in comparison with other languages.

To summarize: partly due to the properties of languages, mathematical activities easily become pure linguistic practices – argumentation "from the inside".

Then authentic views on mathematics from students and teachers disappear, which is the disappearance of reflective mathematics.

A university project for student influence

The project "Student influence of textbooks" is a starting point of this paper (see Lennerstad, 2005a, and Lennerstad, Erman and Samuelsson, 2006). It was funded by the Swedish Council for Higher Education. Students on a calculus course at undergraduate level were able to post their mathematical questions and comments on a web page. Teachers and graduate students answered the questions. The aim was twofold – the obvious one of helping the students, and the less obvious one: to use the communications to improve the textbook used, which was Lennerstad (2002).

The questions were stated in relation to the textbook. The author studied the questions afterwards, and made several changes as a result of this. The changes in the book were not vast, but noticeable. The book is now printed in a new version, Lennerstad (2005b), including the student-inspired revisions.

Initially, formulating mathematical questions was by students looked upon as a strange task. By habit, the very restricted task to *answer* a specific mathematical question was preferred, not the unrestricted task to find questions. But this seemed to be only an initial problem. Students also have reported learning from other students' communications.

When references were investigated for the project, it proved to be virtually impossible to find previous projects where the course material was intended to be modified as a result on the student feedback. Three projects were found, Frith, Jaftha, and Prince (2004), Larson (1999) and Porter (1995). In none of them a textbook was under change – all referred to web material.

It was equally difficult to find such projects for elementary school or high school. Of course, teachers learn from dialogues with students, and textbook authors attempt to reach real students. However, in the absence of systematic ways of doing this, the image of students' mathematical problems may mainly be formed by those students that teachers talk to, while other students have different unformulated problems. An author often writes the text to fit an ideal student. How well does this ideal student correspond to real students? The answer to this question is of course of basic significance for the value of the textbook.

A main aim of the project was to make the image of the ideal mathematics student more realistic, both in requiring feedback from all students, and by letting the students make the formulations by themselves – not directed by teachers' questions. One inference of the lack of similar projects is that the teacher culture does not value the importance of systematic student feedback for long term improvement of education – other than the natural feedback that takes place in mathematics classrooms.

In the next section we argue that the formulation of a systematic image of students' view of mathematics, available for teachers, may be of fundamental importance for the quality of teaching.

Reflective mathematics – defragmentizing mathematics

The purpose to discuss the nature of mathematics is here to *reach more reasonable teacher expectations* towards mathematics and mathematical activity, particularly in any kind of mathematics education. The purpose is to avoid teaching practices that fail, and where the reasons for the failure become clear years later. It is to be more prepared for events in the mathematics classroom, and to be able to design successful didactical projects. This is of course a fundamental purpose of mathematics education in general.

Avoiding failure requires many kinds of insights and competencies, but we here focus knowledge in "reflective mathematics", which concerns meanings of mathematical concepts and calculations in formulations accessible for students. It is possible to do very good mathematics without ever being aware of this mathematical knowledge – without the need to formulate it. This is a common circumstance in linguistics in the sense that we constantly may improve in our native language without the need of being grammar-conscious. This is relevant for mathematics in view of the dominant symbolic language. An underlying assumption here is that symbolic mathematics poses the main problems for the student collective, and reflective mathematics is intended as a bridge to symbolic mathematics. With this purpose, the two need to be tightly connected.

Aiming at the concepts of mathematics "underlying" formulas, we start by discussing the meaning of "conceptual" in mathematics. In the context of the meaning of "understanding", Anna Sierpinska describes that

> The distinction between "seeing" and "seeing as" is important for mathematics whose very nature does not allow for "seeing" its objects, but always to "see them as". (Sierpinska, 1994, p. 10)

Thus, conceptual descriptions in mathematics are in principle always metaphoric. About "conceptual representation" and "conception", Sierpinska writes (ibid.):

> While a conceptual representation is defined as expressible totally in words, a "conception" may be very intuitive, partly visual and not necessarily logically consistent or complete. A person who has a "conception" of, for example, the mathematical concept of a limit, "has some notion" of it, has "some understanding" of it not necessarily of the most elaborate level.

For Sierpinska, a "conception" does not have to be expressible in words. In both cases it concerns mathematical understanding that is not restricted to symbolic representation. The relation of symbolic versus non-symbolic representation of a concept will sooner or later be important. However, it is important to be able to communicate and elaborate concepts before the "symbolic state", which thus need to be made in native language, images and other ways of expression. Reflective mathematics cannot restrict itself to symbolic representation, but should be related to it.

As described above, school work is a major source for reflective mathematics. All cultures that are involved with mathematics may contribute. The term "mathematics" has very different meanings for mathematicians, elementary school mathematics teachers, journalists, technicians, students, parents. A conceptual discussion is needed in order to start to understand these different views. We will later discuss the dialogue seminar, which is a natural tool for such cross-cultural interchange.

Conceptualities of mathematics are often discovered by teachers during their practice. Here teachers are forced into discovery by the pressure of students in need and engagement by teachers. Repeated such discoveries and explanations are extremely valuable for textbook teachers and the mathematics culture. Such discoveries may require reformulation of fundamental mathematical issues. They are important since their source is students' work.

We give next an example to further describe the notion of reflective mathematics. It is a result from the author's dialogues with students.

An example of reflective mathematics

As an example of reflective mathematics we describe the two major mathematical generalizations that children encounter in elementary school. The first goes from sets of identical objects to numbers – which represent the cardinality of a set – the number of objects of the set. The second goes from numbers to letters – which represent numbers. The first connects reality to mathematics, while the second generalization is inside mathematics, since both numbers and letters belong to the mathematics realm. Both represent conceptual difficulties for children, which cannot be expected to be overcome by calculation practice only.

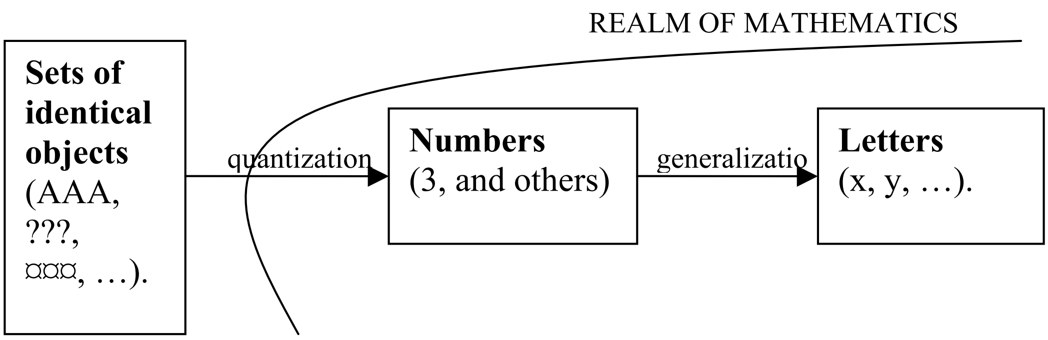

Figure 1. Two mathematics generalizations in elementary school: from sets to numbers, and from numbers to letters.

The first generalization provides children with appropriate meaning to the number symbols 1, 2, 3, …, hopefully. However, teachers working with children with special difficulties rapidly recognize the depths of abstraction that are embedded in these extremely common symbols, irrelevant and unseen for others. Most of us learn to calculate and use calculations in our everyday life, which does not necessarily mean that we "understand" (nor need to "understand") these symbols (Sierpinska, 1994; Smith 1994). This is well in accordance with the linguistics of mathematics. We may well learn and do well at a surface level, without even being aware of the existence of deeper levels. This is of course not always true, as for example the second generalization indicates.

This is underlined by a work by Skemp (1982), who identified two levels where students may work: surface/syntactic level and deep/semantic level. Some students may try to master the "symbol practice" itself, while some may try to understand and work with the underlying meanings of the symbols. Goodchild (1997) found from empirical material that almost all students follow either one of these ways of work. They were reluctant to switch level, either to make sense of syntactic operation, or to facilitate complex tasks by using an efficient formalism.

Before the second generalization, children among other things learn operations from addition to division, calculation methods, the place value system, the decimal point, and more. Despite the quantity of number manipulation in school, numbers are among children not often considered as objects that may be combined and manipulated. Again from Sierpinska (1994 p. 7), who discusses Greeno (1991):

> It is a very poor understanding, Greeno says, if a person, asked to calculate mentally "25·48" represents to himself or herself the paper and pencil algorithm and tries to do it in his or her head. A better understanding occurs if the person treats 25 and 48 as objects that can be "combined" and "decomposed": 48 is 40 and 8…

In Andersson and Bengtsson (2001) two teacher students describe how they compared the mathematics knowledge in two fifth grade classes, where one class had little mathematical dialogue, and one had much dialogue. Of course, many other factors varied, which may influence the findings. The first class was slightly better in calculations than the second. However, when given the question "In which ways can you write 5?", children in the first class typically did not answer at all, while children in the other class filled the paper with calculations as "$1+4$, $2+3$, $6-1$, $\frac{25}{5}$, …". For them, the number 5 was obviously decomposable. The students in the second class had certainly seen this type of question before, but the point here is that students may have very different views of the flexibility and decomposability of numbers.

To summarize this discussion, one could say that it would be a large conceptual gain in mathematics understanding if all children regard *numbers as objects that evidently can be combined and decomposed in many different ways*. Children may see a similarity between numbers and construction toys such as Lego, for example.

This reflective mathematical observation can be extended slightly. Sometimes the converse question appears, for example whether $\frac{1}{2}$ and 0.5 is the same number. A strongly related statement is that all numbers can be written in many ways. Does the question whether $\frac{1}{2} = 0.5$ or not arise from a misinterpreted uniqueness of mathematics saying that "symbols which are different have different meanings"? Such a notion could be counteracted by establishing the obvious existence of synonyms in the formal language of mathematics, as well as in Swedish, English and other natural languages. The metaphor of "number line" for numbers can also help, in that ½ and 0.5 are represented by the same point on this line.

The fact that *any number can be written in many different ways* is fundamental for mathematics, since the main part of most mathematical proofs consists of rewriting the same expression in such a way that is more suitable for the goal of the proof. Without this synonymic property of mathematics, one can therefore question the possibility of mathematical proof.

Note that these two views of numbers, as combinable and decomposable on one hand, and as naturally having many synonyms on the other, is mathematical knowledge that is not often well established in textbooks. Furthermore, these statements cannot be written in symbolic language. Such observations do not appear by themselves from hours and hours of calculation. Some kind of appropriate mathematical reflection is needed. These statements are examples of reflective mathematics.

We also shortly comment the second generalization during elementary school, from numbers to letters. It is easy for teachers, but may appear very strange for students. Teachers often say that one can do the same thing with letters as with numbers, and we think of the fact that letters may be replaced by numbers, so the same rules are valid. But in many other obvious respects, which students may have in mind, this is not true. For example, it is not possible or meaningful to transform the number "x" into decimal form, as can be made with $\frac{1}{4}$. Furthermore, the goals of calculation are entirely different. It is possible to calculate $2+3$ and end up with 5, or calculate $345 \cdot 73$ and use a certain way of structuring the multiplication. Nothing of this is relevant when numbers are replaced with letters. We may consider that $x + y = y + x$, but do not calculate anything. We contemplate, summarize and discuss rules of calculation. Students get tasks such as to simplify $(1/x + 1/y)/(1/x - 1/y)$, although it may not at all be clear when this goal is reached. Actually, such goals cannot be strictly specified. One strict way would be to count the number of symbols in the answer, but this does not always give the "mostly simplified" answer according to the mathematics culture. This difficulty is related to the famous assertion by Wittgenstein that there are no rules for how to use rules.

A teacher who claims that one can do the same thing with letters as with numbers refers to formal truth, but not to activities and goals. The exceptional focus on truth itself can also be observed in research reports in mathematics, where the main question is the truth of results and why they are true (proofs) while the meaning of the result, and its significance and relevance usually receive minor attention. Reflective mathematics tries to formulate this second aspect, which obviously is essential for students' learning of mathematics.

The subject of mathematics drastically changes its character at this generalization, which a teacher focusing formal truth may not notice. Different mental capabilities of the students become important. This change is known often to cause problems, which should be taken into careful consideration in textbooks and by teachers.

Thus, "reflective mathematics" aims at being a general and metaphorical description of formal mathematics, providing more meaning and overview to formulas and concepts, essentially making formal mathematics more accessible. However, the development of reflective mathematics relies on the courage of any mathematics active person to try to formulate significant mathematical problems, questions and considerations from one's own authentic personal viewpoint. Reflective mathematics can grow from mathematical dialogue concerning real questions, including those that occur between different mathematics cultures. We have no stronger tool for thinking than our native language. We cannot do without this tool if we want to formulate central conceptual facts in mathematics, regardless of the shadows cast by its formidably powerful symbolic language. This language is powerful but can only express a part of the essential mathematical knowledge.

Formulation of reflective mathematics requires a fundamental change in attitude to mathematics, towards an attitude that is more akin to that in the humanities. Mathematical errors are not only disturbances to be corrected, but potential sources of discovery of reflective mathematics, and opportunities for respectful dialogue. Each person's pronounced view of mathematics is important by itself, and may be important for the formation of reflective mathematics. The concepts of mathematics can be defined as the meaning of its formulas, and they are both abstract and not easily described in writing. Typically, they need dialogue to come alive.

The dialogue seminar

The scientific development in mathematics since the birth of the symbolic notation has been very fast. It has often been developed formally only, with meaning sometimes arriving later. Both the speed and the formality can be related to the power of this language. Certainly, an underlying idea of the formalist approach by Hilbert and the mathematics logic project of Frege was that formalism is self-sufficient. As described above, conceptual observations in mathematics are preferably made in dialogue, also between different mathematics cultures (teachers, students, mathematicians), which represent different mathematical experiences.

The dialogue seminar is partly designed as a framework for cross-cultural dialogues. We do not here give a full description of the method, but in Göranzon and Hammarén (2003, p. 9), major goals are described as follows:

> The dialogue seminar method is a method of working that aims to (i) create a practice of reflection (ii) formulate problems from the dilemma (iii) work up common language (iv) train the ability to listen.

Furthermore,

> As a method, the dialogue seminar expands the perspective of the concept of knowledge by extending its field to encompass the nature of practical knowledge.

The dialogue seminar does not only aim at knowledge that can be written. Also practical knowledge and skill are central. This is well in accordance with the teaching profession that clearly does not rely on knowledge only – there is also a large component of unformulated skill.

All these four goals are important for the development of reflective mathematics: (i) create reflection on mathematical activity, (ii) viewing difficulties ("dilemmas") as opportunities of better understanding of mathematics and how it naturally is understood, (iii) to create a nuanced language about mathematics that complement the formal language, and (iv) to train different mathematical cultures, mainly students, teachers, mathematicians and teacher educators, to listen seriously to each other.

Dick Tahta (1984, p. 46) formulated a classical dilemma in that has been discussed above. He stated that one of the two most obstinate longstanding problems is *Why is traditional algebra so difficult for a large majority of students?*

The dialogue seminar requires collective work to continue over time. It works with examples, both from the involved individuals' experiences, and from the literature. The participants are "coordinated" by studying one common text. Each participant actively prepares herself/himself before the seminar by writing a reaction on that text, possibly from experience. During the meeting each participant reads the text for the others, after which comments are allowed, while criticism is not.

Furthermore from Göranzon and Hammarén (2003, p. 6)

> In Plato's writing on Socrates' dialogue, dialogue is an instrument of understanding. But the understanding is of a special type, and is never a synthesis. It is based on a concept of truth that can never be captured or made permanent.

In this view, textbooks do not contain knowledge of this type. Textbooks contain mere images or shadows of knowledge, from which true knowledge may emerge under benevolent circumstances. This poses two tasks to textbooks:

1. containing a selection of the most appropriate "shadows of knowledge",
2. to communicate that this knowledge is only "shadow knowledge", and to suggest developments.

This observation also indicates that in Plato's view, practical knowledge is a knowledge that is essentially too complex to be written, but can be made visible in a group of listening, engaged and experienced persons. Plato's note also indicates the dangers in languages. It is tempting to see linguistic expressions themselves as knowledge.

In Sfard (2005, p. 406), the author states that

> The teacher could hardly be blamed for being a captive of her own discursive ways. While in the midst of intensive interaction with a group of children she could not allow herself the luxury of multiple interpretations.

Sfard claims that reflection on practice is difficult from the inside, it needs an outside view. She furthermore describes the possible power of educational research:

> The power of educational research lies in its being the art of multiple interpretations. By making clear that there are many narratives to be told about any given instance of educational practice, this research loosens the oppressive grip of old discursive habits and sets us free to consider new options.

This view of educational research is very much in parallel to the goals of the dialogue seminar.

In Järfälla outside Stockholm, Sweden, the project "Höja nivån" ("raising the level"), led by Pi Högdahl, has significantly decreased the number of students that leave elementary school without a grade in mathematics – see Högdahl (2005). This result has been achieved by providing mathematics teachers time and opportunity to meet and discuss mathematics and educational problems from the practice in their mathematics classes. Continuing this, a dialogue seminar has recently started in these schools, supported by the Swedish National Agency for School Improvement. It is led by Pi Högdahl, Håkan Lennerstad and Martin Gode, and has as central theme translations between mathematical formulae

(Mathematish) and Swedish. This has the purpose of charging strange mathematical formulae with concrete or dramatic meanings for students, demonstrating the rules of formulae in detail, and encourage natural language in the mathematics class. It attempts to shed light upon the linguistic difficulties in mathematics that appear in practice. Teachers meet and reflect about such translations and their value in practice.

In Ericsson and Söderström (2006) the outcomes of the dialogue seminars during the fall of 2005 are documented. Teachers were in general very content with this form of professional development, allowing rich opportunities to express and listen to teacher experiences. Teachers developed also understanding of linguistic properties of the symbolic language of mathematics – Mathematish. For example, Mathematish synonyms were often talked about.

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To Search for Mathematics in the Vocational Teaching and Learning – an Overview of Theories and Methods

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Abstract: *In 1994 a new curriculum was introduced in Sweden. All students at upper secondary school have to take the introductory mathematics course A, including those students enrolled in vocational programmes. Data show that many pupils fail on this course. I am interested in looking for signs of the teaching and learning of mathematics in the vocational subjects in the Swedish upper secondary school. During the last decades there has been research in the use of mathematics in different vocations. As this field is close to mine I am looking at methods used by these researchers to answer their research questions to help me find an adequate method for my research. This article also gives an overview of the content of mathematics in the compulsory school and course A and from documents talking about linking mathematics to vocational subjects.*

Introduction

In 1994 a new curriculum was introduced in Sweden ('Lpf94'; Skolverket, 2005), with a course based structure for the programmes in the upper secondary school. In the written document of Lpf94 it is indicated that there could be many good opportunities to study the relevance of mathematics and the connections between mathematics and work.

> Upper secondary school mathematics should thus be linked to the study orientation chosen in such a way that it enriches both the subject of mathematics and subjects specific to a course. Knowledge of mathematics is a prerequisite for achieving many of the goals of the programme specific subjects (Skolverket, 2005, English version).

The need for a broad competence of the vocational teacher is emphasised by the National Board of Education:

> The committee will also stress the importance that the vocational teachers will get a broader competence. It is important that the vocational teachers have better knowledge than their learners both in the vocational subjects and in e.g. mathematics…/…/ For many learners in the vocational programmes it is of utmost importance that the vocational teachers support the education in the core subjects, thereby giving legitimacy to the whole study programme of the learner. (SOU, 1996; My translation)

There is thus expressed a wish for better knowledge and broader competence for teachers to support learning. That means, among other things, to know more about other subjects that could be linked to one's own in order to create cooperation among teachers in planning for good teaching and learning. As a mathematics teacher educator I have visited many mathematics classrooms and observed mathematical activities, though still not many classrooms and workshops of vocational subjects to observe the way those teachers use mathematics. My interest is to investigate if and how the vocational teacher is supporting learning in mathematics. In the study I will visit the vocational programmes to search for mathematics when the vocational teacher is teaching in the vocational subject.

My overall research question concerns what kind of mathematics and how this mathematics is used and/or taught/shown/exposed by the vocational teachers (who are not mathematics teachers) in the vocational subject. To be able to find out about that I need to develop methods that are efficient when visiting the vocational workshop or classroom to find evidence of mathematics teaching and learning.

Aim and method of this paper

The aim of this paper is to present some of my findings when trying to answer the following questions:

- What theories and methods have been in use in earlier studies to search for mathematics in vocational education?
- What will be the most appropriate methods for my study?

In searching for methods I had to extend the field of studies for this paper to also include searching for mathematics in workplaces and vocations. I will discuss my reasons for this later.

In the literature review the focus will be to inquire into different approaches of research using different theories and methods when focusing on what kind of mathematics and how it is used in vocational classes and workshops, workplaces, and vocations – mathematics that can be transparent or hidden.

In order to be able to answer the questions above I have used an explorative approach. I have investigated different sources to collect data for this purpose. These are the database MATHDI, the NCM database, the proceedings from the PME and ALM (1994-2005) conferences[1], and theses from the Nordic countries (2000-2005). I have used the database MATHDI as this base has publications within the didactical field of mathematics education. At PME and ALM confe-

[1] MATHDI is Mathematics Education Database, NCM is Nationellt Centrum för Matematikutbildning, i.e. National Centre for Mathematics Education in Sweden, PME is The International Group for the Psychology of Mathematics Education, ALM is Adults Learning Mathematics – a Research Forum.

rences there have been working groups addressing work based related issues so there might be reports in the proceedings from those conferences. The key words I have used are mathematics, vocational education, workplace, vocation, research and methodology.

Results of the literature review

Invisible mathematics

A crucial factor for the work I intend to do is how easy or difficult it is to see mathematics in the vocational courses for a researcher. As there is not much work done about the teaching and learning in vocational courses, I also look at research into the use of mathematics in the workplace.

The most important result of related research is that, in most cases, mathematics in the workplace is hidden, contained in a black box, or present in the workplace only as "frozen mathematics" (Gerdes, 1986). It is also named invisible mathematics (Coben, 2000) as it is not seen as mathematics. Chevallard (1989) is using the notion implicit mathematics. Wedege (2005) relates it to "Mathematics – that´s what I can´t do" - Thus it is not seen as mathematics even if it is present in the (workplace) situation.

Consequently, Strässer and Bromme (1992) argue that researchers asking workers directly in a questionnaire about the use of mathematics in the workplace often will get a denial, i.e. the answer that there is no mathematics. Later Strässer (2000) argues that the growing use of technology adds on to "this process of hiding mathematics from societal perception" (p. 241). Most of the mathematical knowledge used in workplaces remains implicit, sometimes unconscious.

These findings have three consequences for my work:

1. It is a delicate matter to reveal the mathematics which is used in the workplace.
2. The researcher must spend time in the workplace to be able to "see". This implies a methodology, which cannot rely on simple surveys or questionnaires to find out about the use of mathematics in the workplace.
3. It is a difficult aim to "see" and this needs thorough planning to handle mathematics to be used and/or taught especially in the classroom.

Testing right/wrong

The report *Mathematics counts* (Cockroft, 1982) gave evidence that a questionnaire can only give answer to what does not work, but not evidence to why it does not work. Much research in those days focused on students' shortcomings. In the late 1980's there was a shift from just investigating errors that students made in performing mathematics by using paper and pencil tests to other testing methods. Even so some research done by Necher and Tecibal already in 1975 pointed out the possibility to get different answers from students using written or oral tests with the same mathematical content (Noss & Hoyles, 1996, p. 32).

My interest is to find any kind of mathematics in the vocational education that is taught/used by the vocational teacher. I am not going to evaluate the knowledge. This means that testing is not a valuable method in this study.

Comparative studies, transfer

Scribner´s and Fahrmeier´s study from 1984 (de Corte, 1987, p. 643) is a comparative study where they looked at the reasoning of dairy workers versus high-school students in a series of tasks involving calculations for milk crate packing. Pettito in 1985 (de Corte, 1987, p. 644) argues that inappropriate transfer from school arithmetic is revealed. The dairy workers were highly flexible in the arithmetic strategies they used, whereas the high-school students were very inflexible; when new practical arithmetic problems demanded revision of calculation strategies for optimizing, students inflexibly continued to apply their school-learned procedural rules. The method used was paper and pencil test. The result of this research is that no transfer happened from school mathematics to a work situation (de Corte, 1987, p. 643).

Evans (1999, 2000) is bringing up the concept of transfer and he proposes a reformulation, where he is talking about describing the practice in a transfer relationship and to analyse similarities and differences between discourses, for example college versus everyday or vocational mathematics. Evans uses statistics methods but also interviews of work and in relation to situated material. His theoretical framework is mainly within the postmodernist area.

It could be of interest to analyze if the teacher has any idea of transfer from the subject mathematics to the vocational subject.

Activity / Critical incidents

A vocational classroom is a multifaceted place. The teaching and learning takes place in an activity. According to Vergnaud, "The most theoretical challenge, for researchers that try to analyse professional practices, is to trace and identify conceptual components of activity, right or wrong" (in Bessot & Ridgway, 2000, p. xxiii). The object is to reveal and to see mathematics.

Eisenhart (1988, p. 102) argues that "all human activity is fundamentally a social and meaning-making experience", and that mathematics is seen as a human activity in the ethnographic research tradition. The research of Lave and Wenger (1991) shows that mathematics in practice is more or less linked to routine activities. This means that mathematics is used in well-known problems with well known strategies specific to each of these problems. Mathematics is visible in activities associated with routine activities for at least the experienced staff. In this case the expert was compared to the novice. Lave and Wenger say that to be aware of mathematics you will have to observe situations when the situation becomes a mess. All research using ethnography involves learning the language of the workplace. It is also time consuming as it takes time to learn the specific terminology of the vocational subject. As a researcher you have to spend time to understand the culture that is at hand.

Hoyles, Noss and Pozzi (1999) are talking about breakdowns. This could be when the person is in a new situation where the old procedures do not work and the worker feels insecure and starts to argue or question the old way to do it. Mathematics becomes visible. Hoyles, Noss and Pozzi were using a variety of different methods. They collected documents from three different professions, nursing, banking and flying (pilots). These documents were analysed regarding mathematical content. They interviewed senior staff members and have pre-interviews with those they observed by using ethnographic methods. They used simulated interviews, questionnaires, and teaching experiments.

From the analyses of the documents there arose what those researchers call visible mathematics. This means that it includes mathematical symbols and representations but also strategies and methods used in mathematics classrooms. The researchers are claiming that this first construction of visible mathematics was an incomplete map showing sometimes less and sometimes more. The map is differed as in banking there was little context when specific mathematics should be used. In nursing and aviation there were stronger indications of the implementation of particular activities. In the nursing field they also found pre-requisite mathematics material such as basic mathematics.

The visible mathematics was revealed mostly in two kinds of activities (Noss & Hoyles, 1996, 2004). In those they had to find solutions by using well known procedures or algorithms or to carry out routine data, measuring and plotting vital sign data. The message from their research is that practitioners *do* use mathematics in their work, but what they use and how they use it may not be predictable from considerations of general mathematical methods. Moreover, strategies depend on the nature of the activity, whether it is routine or breakdown, and the resources available (ibid).

My analysis of the paragraph above is that in the breakdown situation the vocational teacher will expose the mathematics used or not used and even in unexpected situations. This is happening in an activity and then the researcher has to be present. That means that one must use an ethnographic approach.

Researchers and teachers in collaboration

Hogan and Morony carried out research run by the Australian Association of Mathematics Teaching Inc (AAMT) in Australia (Bessot & Ridgway, 2000). In this research schoolteachers were co-researchers. The aim was to deepen the understanding of what kind of mathematics that is used in different workplaces, in order for the teachers to be able to discuss this with their students. Workers from a sample of workplaces were shadowed half a day by a teacher-researcher, who was just observer and interviewer. Workers were interviewed to explore issues that had arisen and to give more information. The teacher-researcher could also return to ask more questions. These researchers pointed out the value for teachers to belong to a research community and to know more about research out of experiences. The outcomes of the research were workplace stories.

In 2000 Wedege presented her thesis (Wedege, 2000), where she used the same method that was developed and carried out by Hogan (1997). Later she used nearly the same approach in a study where 25 adult teacher educators visited one of eleven workplaces to observe competent workers (Wedege, 2004). They were using a visit form designed by Wedege and presented in her thesis to document the findings from half a day's visit. They did not interview the workers during or after the visit. They used an approach with tree types of data collection – observation, interview and collection of artefacts.

Wedege (2000, 2004) also describes another research where she is using the method to shadow a key worker to describe the action that takes place. She also photographed for documentation and presented the observations as a descriptive story with what she calls episodes, which are particularly interesting incidents. If my study could be done in cooperation with other researchers I would try out the method described by Hogan and used by Wedege. I will however analyse the protocol that Wedege developed and see if that is useful for my study.

Swedish developmental work taking place in vocational education

The main purpose of a the project DUGA (Kilborn, 1996), which was run in vocational programmes, was to analyse the mathematical content in vocational textbooks and materials for different vocational courses, and to compare this to the mathematics course the students had to take. The teachers were interviewed about their vocational teaching focusing on the students' mathematical knowledge.

The developmental project KAM (Grevholm, Lindberg & Maerker, 2002), funded by the Swedish Board of Education, had as it focus to see if there could be ways for more collaboration between the mathematics teacher and the vocational teacher. The purpose for this was to enhance the students' understanding of mathematics by integrating parts of it in the vocational subject. In that project many ways to collect data were used – to collect textbooks, manuals, use tape recorders, take field notes, and to interview the teacher.

These two projects are as close as I can come to compare to my own study. The questions for these studies were different, but the setting is similar. They took place in the vocational classrooms and workshops. There was a group of researchers involved in these studies. I will be the only researcher. The data collection was varied. This might give good possibilities to investigate more aspects of mathematics used/taught/exposed by the vocational teachers.

In the Australian project *Education for Mathematics in the Workplace* (Bessot & Ridgway, 2000) all the researchers come to present mathematics as an activity. This view of mathematics in a workplace is important when choosing a research approach. My conclusion so far is that this is the case in the vocational classroom /workshop as well.

Discussion and conclusions

In trying to answer the first question for this paper, *What theories and methods have been in use in earlier studies to look for mathematics in vocational education?*, the literature review has been worked out to find what has been done recently in order to develop a research approach for my study which will take place in the vocational programmes in Swedish Upper Secondary Schools. First of all the content for this study is mathematics. This means that the researcher must know the mathematical content and the level of mathematics and its implementations in the field of the study. As mathematics can be seen as an activity, the ethnographic research tradition can be applicable.

I have here presented my literature findings from not only the vocational educational field, from which there have not been so many documentations from research or developmental work. That is why I looked at studies from workplaces and vocations. I believe this is relevant for my own research.

Over the years there has been a trend towards a multimodal approach to collect data with more technical devices. The devices are smaller and easier to handle. There is a mix of quantitative and qualitative data to be analysed.

A vocational classroom is a multifaceted place. The teaching and learning takes place in an activity. The researcher has to be in the activity and use a lot of tools to find any signs or evidence of mathematics that is transparent or hidden. In the literature I have found good examples of different research methods. The researchers' theories have however not been as explicit.

My answer to the second question, *What will be the most appropriate methods for my research?*, based on the discussions above, is the following. As I am the only researcher in my study I will have to use different tools when visiting the vocational classroom or workshop to find what I am looking for, that is signs and evidence of mathematics used by the vocational teacher in the vocational classroom or workshop. Examples of such tools are tape and video recorders, field notes and working materials such as textbooks and manuals.

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The Use of Langford's Problem to Promote Advanced Mathematical Thinking

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Abstract: *Starting with a general discussion of the issue of understanding in mathematics, this study sets focus to generalization as a key process to be promoted in advanced mathematical thinking. Interview data from students working on Langford's problem showed a great variety in how to treat the problem, where expansive, reconstructive as well as disjunctive generalizations came into play. It is found that this problem embodied key principles needed to generate generic abstraction leading into more formal mathematical thinking.*

Introduction

College students' deficiencies in mathematical understanding and ways of thinking reach well back into the F-12 mathematics curriculum and "students' mathematics education is in full swing by the time they enter college" (Steen, 1998). Unfortunately, many college courses also fail to address advanced mathematical thinking (Carlson, 1998). Consequently, it is important for college and university mathematics instructors to be aware of the nature of students' deficiencies, the cognitive reasons for such deficiencies, and so be in a position to adapt introductory college mathematics instruction accordingly. The design of courses that facilitate transition to advanced mathematical thinking will require "a simultaneous focus on issues of pedagogy and learning alongside the challenging matters of content order and course organization" (Ferrini-Mundy, 1998). This paper deals with the question if it is possible to give students an assignment promoting advanced mathematical thinking and at the same time give rich opportunities for active investigation, analysis, and reflection. Is it possible to observe and validate such a transition towards a more advanced mathematical thinking?

There is a visible trend in modern education that shows an increasing emphasis put on learning through problem-oriented or problem-based educational methods. The underlying idea is to improve the quality of students' learning about complex problems or phenomena in the world through assignments that give rich opportunities for active investigation, analysis, and reflection. Such methods entail an increased use of a wide variety of different information sources. When studying mathematics at the tertiary level, many students use tools like graphing and symbol-manipulating calculators and a variety of sophisticated computer programs. Students also use assorted textbooks and other reference books, and many of them are likely to turn to their family members, friends, colleagues, or maybe neighbors as a reference group.

One could argue that if students do seek information in a variety of ways, then the way these students study is close to the way people ordinarily work in many different professions. People are often valued for the everyday jobs or projects they do, their ability to work effectively with others, their responses to problem situations, and their capacity to find tools or information that will help them complete an assignment. In occupations as well as in modern studies, it is important and most likely beneficial for the individual to be open and flexible in approach. It is desirable and would be natural if examinations in mathematics could mirror that fact.

Theoretical framework

Mathematical understanding

Since ancient times, people have been concerned about understanding (and lack of understanding) in connection with mathematics. Henri Poincaré underlined the ambiguity of the meaning of the verb:

> What is understanding? Has the word the same meaning for everybody? Does understanding the demonstration of a theorem consist in examining each of the syllogisms of which it is composed in succession, and being convinced that it is correct and conforms to the rules of the game? In the same way, does understanding a definition consist simply in recognizing that the meaning of all the terms employed are already known, and being convinced that it involves no contradiction? (Quoted in Sierpinska, 1994, p. 72)

Sierpinska (1994) claims that researchers in mathematics education have different objectives when discussing the question of understanding mathematics. Some objectives are more pragmatic (to improve understanding), others are more diagnostic (to describe how students understand), and still others are more explicitly theoretical or methodological. What unites researchers is that they all have a theory of what understanding is, explicitly expressed or not. According to Sierpinska, there are at least four different theories or models of understanding in mathematics. To begin with, we have theories that are centered on hierarchies of levels of understanding. One such example is the van Hiele (1986) levels, but there are others.

Second, we have models that describe understanding as a growing ”mental model,” ”conceptual model,” ”cognitive structure,” or something similar. The term cognitive structure comes from Piaget (see for example Piaget, 1978) and several authors refer to Piaget when constructing their model for the understanding of mathematics.

Third, Sierpinska (1994) mentions models that look at the process of understanding as a dialectical game or interplay between two ways to apprehend understanding. The dialectical dualism may be illustrated by a concept considered as a tool in a problem-solving process and at the same time viewed as an object to study, analyze, and develop in a theoretical way. One well-known example is Skemp's (1978) discrimination between instrumental and relational understanding. According to Skemp, instrumental understanding is what it takes to reach the right answer, while relational understanding means that you understand both what to do and why. Another way to describe this is as operational versus structural understanding (Sfard & Linchevski, 1994).

The fourth type of understanding is a historical-empirical perspective in which the epistemological obstacles are united by today's students (Sierpinska, 1994). Robert and Schwarzenberger (1991) claim that from a psychological perspective, it is meaningful to focus on tertiary students' growing ability to reflect on their own learning of mathematics. They argue that advanced mathematical thinking includes the ability to separate knowledge of mathematics from meta-knowledge of mathematics, which includes, for instance, how correct, relevant, or elegant a solution is. They further advocate that students at this advanced level should have a great amount of mathematical knowledge, experience of mathematical strategies, and well-functioning methods together with aptitude for communicating those skills at least on a basic level. According to Robert and Schwarzenberger, research shows that students vary greatly in this respect.

Vinner (1997) extends the distinction between rote and meaningful learning with what he calls conceptual and pseudo-conceptual behavior. Conceptual behavior is characterized by the consideration of concepts, "as well as relations between concepts, ideas in which the concepts are involved, logical connections, and so on" (p. 100). In contrast, pseudo-conceptual behavior is based on rote learning, lack of reflection upon appropriateness of answers or reasons for errors, lack of underlying meanings assigned to symbols and words employed, use of superficial similarities, and "the belief that a certain act will lead to an answer that will be accepted" by an external authority (p. 115). Thompson and Sfard (1994) describe a similar distinction when they define "grasping the meaning" as "having the ability to think about the objects hiding behind the words" (p. 22).

Learning through assignments

Working on an assignment is an active learning process. Students are more likely to understand and retain material that they have used in an assignment to solve a problem, and it can serve very well as a bridge between concretization and abstraction. Mathematics is characterized by its abstract nature, and for the initiated, moving about within this abstraction is characterized by Devlin:

> When mathematicians define some abstract mathematical object or system as a "set of objects" satisfying certain properties, it usually doesn't matter what the members of the set are; rather, what counts are the operations that can be performed on those members. In fact, even that is not quite right. The real interest is in the properties of those operations. (Devlin, 1997, p. 57)

For the mathematics student or his or her instructor, whether in elementary school or college coursework, coming to grips with this abstraction is an educational challenge that continues to confront all those involved. But what happens to a student's learning when a group of students get the same assignment in mathematics? Is it self evident that the same learning process, the same evolution of their generalization and abstraction knowledge, the same active learning process undeniably occurs?

Generalization and abstraction

Although related, the processes of generalization and abstraction have important distinctions. Both require an individual to look for commonalities, to isolate properties, and to stress certain features while ignoring others (Mason, 1996). Generalization involves an extension of an existing set of familiar objects or processes, while abstraction requires a shift of attention from the objects or processes themselves to the structure and/or relations among the entities. Focusing on the structure requires a mental re-construction to create a new object, itself subject to operations and having a set of properties (Dreyfus, 1991; Tall, 1991).

Generalizing

Generalizing is an integral part of classroom practice, where seeing the general in the particular lies behind most instructors' examples and exercise sets. However, to be a successful instructional tool, both teacher and student must focus on the same aspects of the learning experience. Mason (1996) makes the distinction between "looking through" and "looking at" an object, such as "working through a sequence of exercises and working on these exercises as a whole" (p. 65). He cautions that while the instructor may see the general in a particular example, his or her students may only see the particular in what is offered as a general example. Mason illustrates how, for some, an example is an example of something, while, for the others, it is simply a totality in itself.

> The sum of the angles in a triangle is 180 degrees. What is the most important word, mathematically in that assertion? I suggest that it is … a. That tiny indefinite article signals the adjectival pronoun any, which in turn signals the adjective every, which refers to the scope of variability being countenanced. … Yet in many classrooms, it is the 180 that is stressed, presumably so that students will remember it. But failure to use this fact is rarely due to forgetting whether it is the 180 or some other number, but rather, due to lack of appreciation of the generality, the invariance, being expressed. (p. 67)

Harel and Tall (1989) distinguish three different kinds of generalization which depend on the individual's mental construction:

1. Expansive generalization occurs when the subject expands the applicability range of an existing schema without reconstructing it.
2. Reconstructive generalization occurs when the subject reconstructs an existing schema in order to widen its applicability range.
3. Disjunctive generalization occurs when, on moving from a familiar context to a new one, the subject constructs a new, disjoint, schema to deal with the new context and adds it to the array of schemas available. (p. 39)

Abstraction

An abstraction process occurs when students focus their attention on specific properties of a given object and then considers these properties in isolation from the original. This might be done, for example, to understand the essence of a certain phenomenon, perhaps later to be able to apply the same theory in other cases to which it applies. Such application of an abstract theory would be a case of reconstructive generalization – because the abstracted properties are reconstructions of the original properties, now applied to a broader domain. However, note that once the reconstructive generalization has occurred, it may then be possible to extend the range of examples to which the arguments apply through the simpler process of expansive generalization.

The study

In a course on *discrete mathematics*, I used the following assignment as one part of the course exam for a group of prospective teachers of mathematics. The students were asked to present their solution in a written report, which clearly would be individual and not a copy of any classmate's report. They were also permitted to use any kind of sources and help, as long as they referred to these sources in the report. The allotted time for the assignment was three weeks. All the 23 students produced written reports on the solution of the problem.

Assignment

Imagine the digits 1, 2, 3 in a sequence where each digit is used twice and arranged in the following way:

312132

Notice that there is **one** other digit between the ones (1), **two** other digits between the twos (2), and **three** other digits between the threes (3).

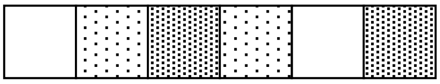
a) Is it possible to arrange the digits 1, 2, 3 (each used twice) in other ways, so that these rules still hold?

b) What will happen if you add two 4s to get a sequence of 8 digits? How many solutions will there be then?

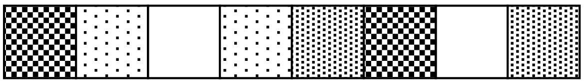
c) Investigate the problem for sequences of digits from 1 to 6, 1 to 7, 1 to 8, 1 to 9, and 1 to 10.

Argue for your conjectures and conclusions!

The problem is known under the name Langford's problem. The problem is named after the Scottish mathematician C. Dudley Langford who once observed his son playing with colored blocks. Langford noticed that the child had arranged three pairs of colored blocks such that there was one block between the red pair, two between the blue pair, and three between the yellow pair, like so:



This solution is unique. Reversing the order is not significant, because all you have to do is walk around to the other side of a given arrangement and view it from that side. Langford added a green pair and came up with:



Generalizing from colors to numbers, the above became 312132 and 41312432. The problem has attracted mathematicians, computer scientists, and others, and is only solvable with the aid of computer programming for large *n*. The connection to Langford's problem was not mentioned to the students in the study.

Student responses

The 23 students in this study were all in their second semester of mathematics in the teacher program for secondary mathematics at a Swedish university. They had previously taken courses in algebra, geometry, probability and statistics in the program, and had also learnt how to use the computer software MS-Excel to solve various problems within discrete mathematics. The assignment was their third take-home exam in the course, and the previous two assignments had been modeled by the aid of MS-Excel. The students were informed about my purpose to extend their mathematical thinking from arithmetical thinking into advanced mathematical thinking and also about my intention to observe that transition.

A small group of student's engaged MS-Excel also to solve Langford's problem. Since Excel had proved to be helpful in solving other problems in the course, they tried hard to make the problem fit within the possibilities of Excel. The fact that the problem is not really suitable for Excel the way it was proposed was neglected by these students, since they selected a tool before analyzing the problem throughout. These students can be characterized as using expansive generalization.

Student 1: *I entered the figures in Excel and tried to make up a formula for the reorganization of numbers. When I didn't find any formula that worked, I just put in the numbers for every possible combination in Excel. This worked fine for 4 digits, but was very tiresome and difficult for 5, 6, 7, and so on… digits.*

Student 2: *I know it must be a combinatorial problem, but I can't find the right formula in Excel. It drives me almost nuts, I have spent many hours on this problem by now, without getting anywhere.*

Another, much larger, group understood the connection to existing procedures and concepts within the course, such as calculations of sums of arithmetic series. This group of students managed to reach some valuable conclusions, but did not really solve the problem in full or manage to do enough generalization. Although there naturally were differences between the students, they all reconstructed ways of calculating arithmetic sums but only broadened and widened its applicability range to very small magnitude. These students can be characterized as using reconstructive generalization.

Student 3: *I have understood the problem like this. Number 1 is in position P(1) and P(1) + 2. Number 2 is in position P(2) and P(2) + 3. Number N is in position P(N) and P(N) + N+1. This yields an arithmetic sum and shows me that it is only possible to find solutions for 7 and 8, but not for any other numbers.*

Student 4: *In order to get a complete number series, the formula* $3N^2 - N$ *must be a multiple of 4. This happens when N = 4k and when N = 4k + 3, so for 7 and 8 digits there is at least one solution.*

Fortunately, 8 students actually managed to develop an impressing theoretical solution to Langford's problem. Not only did they make the necessary connection to different concepts and procedures within the course, they also constructed a new general way to look at the problem, far beyond what was asked of them in the assignment. These students may very well be characterized as using disjunctive generalization.

Student 5: *I found that there is no solutions what so ever for 6, 9, and 10, but there are solutions for 7 and 8, and larger numbers. An interesting fact is that it works when the digit sum for the combination of digits is divisible with 4, and only then. Example: 2(1 + 2 + 3) = 12 and 2(1 + 2 + 3 + 4) = 20.*

Student 6: *I have shown that there are at least two digits with the use of 7 and 8 digits which implies that there should be solutions also for 11 and 12 digits. When I continue, I find that there are solutions for 3, 4, 7, 8, 11, 12, 15, 16, 19, 20 digits, and etcetera.*

Finally, 4 students actually entered the digits 312132 into a search engine for Internet and got information about Langford's problem that way. Nevertheless, the information they found turned out to be difficult to interpret and understand for someone who had not been working with it before, and these students had to ask for help in order to better understand what they had found on the web.

Conclusions

To me it was somewhat of a surprise that the students would handle the Langford's problem so differently. It was also unexpected that the solutions would fit so nicely within the Harel and Tall's generalization scheme. The problem obviously involves generalization (because it is not possible to solve otherwise), and gives the students possibilities to observe and identify one or more specific examples of behavior of Langford's digits as typical of a wider range of examples embodying an abstract concept, something Harel and Tall label generic abstraction. There are three different principles connected to generic abstraction.

The *entification principle* states that, for a student to be able to abstract a mathematical structure from a given model of that structure, the elements of that model must be conceptual entities in the student's eyes; that is to say, the student has procedures that can take these objects as inputs.

The *necessity principle* states that the subject matter has to be presented in a way to which learners can see its necessity. For if students do not see the rationale for an idea, the idea would seem to them as being evoked arbitrarily; it does not become a concept of the students.

The *parallel principle*: When instruction is concerned with a "concrete" model, that is, a model which satisfies the entification principle, the instructional activities within this model should be designed to parallel the processes that will later apply within the abstract structure. This will mean that the instruction potentially involves only an expansive generalization, in which the concrete model is manipulated in a generic way. But it is designed to lay the seeds for a much easier reconstructive generalization at a later stage when the abstraction of the formal concept occurs in a corresponding abstract manner.

I argue that Langford's problem is such a problem that the parallel principle applies. Obviously all the students were able to use their own procedures to take the objects of the problem as inputs. Since the students are all in the mathematics teacher program, the necessity principle holds. I found that the problem provided the students with excellent possibilities to expand their generalization in a totally unrestrictive way. Since the parallel principle encourages a generalization of the procedure to be passed from the examples to the abstract concept by a process more parallel to an expansive generalization, the properties must be reconstructed in the abstract context. Consequently the passage from generic abstraction to formal abstraction remains one requiring reconstruction, but a reconstruction with potentially less cognitive strain (Harel & Tall, p. 41). I still do not know if I should be happy that 8 out of 35 students managed to reach that abstract level, or sad that not all of the students got there.

The fact that the vast majority of the students preferred to put a lot of effort into finding specific solutions on their own is nevertheless very encouraging. It proves that one very well can use alternatives to more traditional ways of examining mathematical performance and understanding, and thereby promoting active investigation, analysis, and reflection.

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Mental Computation of School-Aged Students: Assessment, Performance Levels, and Common Errors

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Abstract: *From 2001 to 2004 the author directed a project in Tasmania and the Australian Capital Territory aimed at assessing and improving the mental computation ability of Australian students. This paper reports on the research aspects of the project, which involved (a) developing a scale of competence in mental computation, (b) developing assessment processes, and (c) analysing common errors in mental computation.*

Introduction

Thirty years ago in primary schools in many countries around the world, mental computation meant 'mental arithmetic'. This typically consisted of ten or twenty questions given at the beginning of a lesson. The questions were mainly confined to basic facts, that is, to the mental calculations considered necessary so that one could perform written calculations: speed and accuracy were emphasised.

Research by Biggs (1967) revealed that this was of limited value. He noted that in 69 classrooms in English primary schools the average number of minutes per day devoted to mental arithmetic varied from nil to 11 minutes. The approach categorised as traditional had as one of its characteristics' a great deal of mental arithmetic in which speed of response is encouraged'. Among the conclusions of the study are:

'[Number anxiety] tends to increase slightly with more time devoted to mental arithmetic.'

'Allocation of time to mental arithmetic bore no relation to attainment.'

In other words, these daily speed and accuracy tests did not make the children noticeably more competent, but it did make them slightly more neurotic about numbers.

More recently mental computation has been seen to have much wider relevance in education: in particular, it has been shown (Wandt & Brown, 1957; Northcote & McIntosh, 1999) that adults in their everyday lives use mental computation for over three quarters of all their calculations, whereas written calculation and calculator use are each involved in less than fifteen percent of all their calculations. This raises the question as to why so much school time should be spent on written calculation, which has such limited use in adult life, and why so little time is spent on mental computation, which is the calculation method most commonly needed.

The relative educational values of written and mental calculation have also been questioned. A number of researchers (Plunkett 1977; Kamii & Dominick, 1997; Markovitz & Sowder, 1994; Reys, 1984) have shown that mental computation is much more efficacious in both developing and indicating number sense.

One consequence of the research into adult uses of mental computation was the realisation that competence in mental computation involves much more than knowledge of basic facts. Mental computations commonly used by adults include two- and even three-digit computations, as well as simple calculations involving fractions, decimals, ratio and percentage. A further major use of mental computation, not covered by the present study, is its use in estimation and approximation.

As a result an increasing number of authorities at both national and local level have given much greater prominence to mental computation in their curriculum guidelines and documents.

In the United States *The Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989), and in the United Kingdom *Mathematics in the National Curriculum* (Department of Education and Science, 1991) both stressed the equal importance of ability with the calculator and with mental and written computation.

In Australia mental computation is emphasised in current state and federal curriculum documents both as a critical component of functional numeracy, and as an effective means of developing number sense in students (see for example Australian Education Council, 1991).

In Norway, the draft 2006 mathematics curriculum states that after Year 2 the students should be able to 'develop and use a variety of calculation strategies for addition and subtraction of two-digit numbers'. After Year 7 they should be able to 'develop and use methods for mental calculation, estimation and written calculation'.

The Swedish national guidelines include, as one of the goals that pupils should have attained by the end of the fifth year in school, that they should 'be able to calculate in natural numbers – in their head, and by using written calculator methods and pocket calculators'. By the end of the ninth year in school they should: 'have good skills in and be able to make estimates and calculations of natural numbers, numbers in decimal form, as well as percentages and proportions in their head, with the help of written calculation methods and technical aids'.

While it appears there is much more official encouragement for a serious school programme in mental computation, there is little sign yet that official assessment programmes take account of testing mental computation. One significant exception is the state of Queensland in Australia, which since 1999 has successfully included mental calculations in its compulsory annual testing programme at years 3, 5 and 7.

Although considerable research interest has centred more recently on the strategies students use when they calculate mentally (Beishuizen, 1997; Carpenter & Moser, 1984; McIntosh, De Nardi, & Swan, 1984; Reys, Reys, Nohda and Emori, 1995), very little research has been done into levels of ability in mental computation, or into the errors made by students in computing mentally. For example, there is nothing in the mental computation literature that parallels Ashlock's *Error Patterns in Computation* (1994), which confines itself exclusively to errors in written computation. "This entire book is designed to help you learn as much as possible from the written work of children" (Ashlock, 1994, p. 13). Since children's focus of thought, and consequently their patterns of thinking, are often markedly different when they are engaged in mental computation from those they employ when calculating with pencil-and-paper, it is to be expected that the kinds of errors they make, and the reasons for these errors, may also sometimes differ.

The project

Research and development aims

Improving and Assessing the Mental Computation of School Aged Students is a three-year project (2001 – 2003) funded by DEST (the Australian federal Department of Education, Science and Training), the Education Departments of Tasmania and the Australian Capital Territory (ACT) and the Catholic Education Office of Tasmania. The project had both research and curriculum development aims. The research aims of the project were to develop a developmental framework for mental computation competence, to explore ways of assessing students' competence in mental computation, and to analyse students' errors in mental computation. The curriculum development aim of the project was to develop and trial in six schools (three primary schools and three high schools) sequential teaching material for mental computation covering Grades 3 to 10, including whole numbers, fractions, decimals and percentages. This paper deals only with the research aims and outcomes of the project.

Methodology

In 2001 - 2003, approximately three thousand students across grades 3 to 10 in twelve Tasmanian and ACT schools participated in one of a set of mental computation tests. Items were of two types: 'Short items' (S) had three seconds in which to answer and 'Long items' (L) had fifteen seconds. These differences were intended to separate items which students might be expected to know instantly from those that they could work out given time. The items were drawn from a bank of two hundred and forty-four items that included only computations with a single step (that is, no items contained more than two terms, e.g. 17 – 8). All items were recorded onto a compact disc and supplied to the schools on audiotape. Students in grades 3 and 4 had fifty items presented to them, while those in the other grades attempted sixty-five. Sixteen different test forms were used, four at each of two adjacent grade levels: Grades 3/4, Grades 5/6, Grades 7/8 and Grade 9/10, and these were linked by the use of common items both within grades and across grade levels. Nine 'link items' (lk) were also included, in which five seconds was provided for students to respond, in order to provide a basis for linking to an earlier study that had identified a developmental scale of mental computation (Callingham & McIntosh, 2001).

The items were analysed using Rasch modeling techniques. This approach to analysis allowed students' performances and all item difficulties to be estimated using the same measurement scale, so that they are directly comparable. This placed all students and all items in an ordered display from least proficient or, in the case of items, least difficult, to most proficient, or most difficult. The underlying variable was then interpreted in terms of the mental computation skills required by each item, which provided a 'profile' of students' mental computation competence. By determining the points at which there was a qualitative change in the demands of the items, eight levels of mental computation competence were identified (Callingham & McIntosh, 2002).

The testing program was repeated in 2002 with the same set of tests, and again in 2003 with some new items, particularly for fractions, decimals, and percentages. Sufficient items were maintained for the different tests to be linked together, so that all items could be placed on the same scale. The longitudinal testing confirmed the scale identified initially.

Similar items within the levels were clustered and described. Items were also separated out into the following sub-domains for closer analysis:

- Whole number single digit addition and subtraction
- Whole number single digit multiplication and division
- Two-digit addition and subtraction
- Two-digit multiplication and division
- Decimals addition and subtraction
- Decimals multiplication and division
- Fractions addition and subtraction
- Fractions multiplication and division
- Percentages.

For each item, at each grade level, all responses were categorised as Correct, Incorrect or No answer. All incorrect answers were recorded, and the occurrence of the most common errors was expressed as a percentage of the number of incorrect answers (thus excluding all 'No answer' responses). These were then analysed for clusterings of error types.

Results

Levels of performance

Eight levels of competence were identified from Level 1 (least) to Level 8 (most). As the intention was to test only levels of competence expected of most adults, the tests did not include items of particular technical difficulty. Thus higher levels of competence than level 8 could be hypothesised as within the competence of many of the students.

Table 1 shows the percentage of students in each grade who were at each of the levels. Cells containing 20% or more are shaded. It can be seen that, whereas over 10 % of Grade 3 students are at Level 5 or above, almost 10 % of Grade 10 students are below these levels. The relative lack of improvement in performance between Grades 6 and 7, and between Grades 8, 9 and 10, is also worth noting, adding weight to the belief in a performance plateau in these age groups, and perhaps to the lack of attention given to mental computation at secondary level.

Table 1: Percentage of students in each grade at each level of competence

Grade	Level 1	Level 2	Level 3	Level 4	Level 5	Level 6	Level 7	Level 8
3	16	14	35	21	9	3	1	0
4	5	4	18	29	24	18	2	1
5	1	2	4	20	34	27	8	3
6	1	1	5	13	24	30	14	14
7	0	0	4	9	24	42	14	7
8	0	1	0	8	19	32	20	22
9	0	2	1	5	13	30	25	24
10	1	1	1	5	15	31	24	23
Overall	4	3	10	14	20	26	12	11

To illustrate the developmental scale, Table 2 shows for one sub-domain (Whole Number Single-Digit Multiplication and Division), the skills that describe students at levels 1 to 6, and specific items at these levels from the tests. No item in this sub-domain was at a higher level than 6. In contrast, although only technically simple items were included in the Fractions Addition and Subtraction sub-domain, requiring conceptual understanding rather than computational skill, these calculations proved more difficult. Table 3 shows that all items fell in

Levels 4 to 8. Indeed, no items relating to fractions, decimals or percentage appear below level 4.

Table 2: Levels for sub-domain: Whole number one-digit multiplication and division

Level	Skills	Items from tests at this level
1	Can quickly double a single digit	6 x 2 (S), 2 x 10 (L)
	Can quickly multiply single digit By 10	4 x 10 (S), 8 x 10 (S), 3 x 10 (S), 5 x 10 (S)
2	Knows multiples of 2 and knows or can quickly calculate some multiples of 3, 4 and 5	7 x 2 (S), 9 x 2 (S), 7 x 10 (L), 4 x 3 (lk), 4 x 3 (L), 5 x 4 (S), 5 x 4 (lk), 5 x 4 (L), 6 x 5 (lk), 6 x 5 (L),
3	Knows/can quickly calculate multiples of 3, 4, 5	6 x 5 (S), 4 x 3 (S), 3 x 6 (lk), 7 x 3 (lk), 3 x 6 (S), 7 x 3 (S), 7 x 3 (L), 7 x 4 (L)
	Can halve even numbers to 20	Half 18 (L)
4	Can calculate product of single digit numbers	6 x 9 (L), 8 x 4 (L)
	Knows or can calculate inverse of first ten multiples of 3, 4 and 5	12 ÷ 3 (L), 21 ÷ 3 (S), 12 ÷ 4 (S), 20 ÷ 4 (L), 30 ÷ 5 (S), 30 ÷ 5 (L),
5	Knows most table facts and can calculate the others	7 x 6 (S), 7 x 6 (lk), 8 x 7 (L), 9 x 8 (S), 9 x 8 (L), 6 x 9 (S),
	Knows or can calculate inverse of most table facts	72 ÷ 8 (L), 70 ÷ 5 (L), 54 ÷ 9 (L), 56 ÷ 7 (L)
6	Knows all table facts and their inverses	8 x 7 (S), 54 ÷ 9 (S), 72 ÷ 8 (S), 56 ÷ 7 (S)

Table 3: Levels for sub-domain: Fractions, addition and subtraction

Level	Skills	Items from tests at this level
4	Knows/ can calculate $\frac{1}{2} + \frac{1}{2}$	$\frac{1}{2} + \frac{1}{2}$ (S)
5	Can add/subtract halves and quarters less than one	$\frac{1}{2} + \frac{1}{4}$ (S), $\frac{1}{2} + \frac{1}{4}$ (L), $\frac{3}{4} - \frac{1}{2}$ (L),
	Can add fractions with common denominators (totals <1)	$\frac{2}{7} + \frac{3}{7}$ (L)
6	Can add and subtract halves (and equivalents) and quarters beyond one	$\frac{1}{2} + \frac{3}{4}$ (L), $\frac{1}{2} + \frac{4}{8}$ (L), $\frac{1}{2} + \frac{5}{10}$ L), $1\frac{1}{4} - \frac{1}{2}$ (L),
	Can quickly subtract a simple unit faction from one.	$1 - \frac{1}{3}$ (S), $1 - \frac{1}{3}$ (L)
7	Can add one half and one third	$\frac{1}{2} + \frac{1}{3}$ (L)

Common errors: whole numbers

By far the most common incorrect response to items involving addition and subtraction of whole numbers was an answer that differed by one from the correct answer. Table 4 gives the number of times an error of one was made for basic addition/subtraction fact items and for addition/subtraction of larger numbers. The table excludes all cases where no answer was given to a calculation. Table 4 shows that this error type accounted for over a third of all basic fact errors in Grades 3/4 and over a quarter of all errors in Grades 5 to 8.

Table 4: Errors of 1 for all whole number addition/subtraction items by grade and type of calculation

	Basic Fact Items				Larger Numbers			
	Grade 3/4	Grade 5/6	Grade 7/8	Grade 9/10	Grade 3/4	Grade 5/6	Grade 7/8	Grade 9/10
(a) Errors of 1	394	23	14	-	367	224	74	51
(b) All errors	1082	87	53	-	1506	1784	724	361
(a) as % of (b)	36.4	26.4	26.4	-	24.4	12.6	10.2	14.1

For larger whole number addition/ subtraction items, the second most common error was an answer that was incorrect by 10. This error persisted through to Grade 9/10 students: for example, for the 15-second item 58 + 34, 10 out of 28 incorrect answers given by Grade 9/10 students were either 82 or 102.

The most common error for multiplication and division basic facts was an answer that was wrong by one multiple: for example, for the item 21 ÷ 3, 21 out of 49 incorrect answers given by Grade 5/6 students were either 6 or 8.

Common errors: fractions and decimals

Table 5 shows results from four of the fraction and decimal items from the tests. Columns three to five show the percentage of students at each grade answering correctly. There were no fraction or decimal items in the tests for Grades 3/4. The second column indicates the number of seconds given for each item. The most common incorrect answer across the grades is also given.

Table 5: Percentage of students answering correctly selected fraction and decimal items by grade, and most common incorrect answers

Item	Time (seconds)	Grade 5/6	Grade 7/8	Grade 9/10	Most Common Incorrect Answer(s)
$1 - \frac{1}{3}$	15	30	51	52	$\frac{1}{4}$ or $\frac{3}{4}$
$\frac{1}{2} \div \frac{1}{4}$	15	-	42	47	$\frac{1}{4}$
3 x 0.6	15	-	32	39	0.18
0.3 + 0.7	15	42	52	64	0.1

For the item 0.3 + 0.7, the answer 0.1 constituted 221 of 296 incorrect answers.

Discussion

Consideration of the errors associated with the present study suggests that the fundamental distinction that needs to be made for errors in mental computation is that between conceptual and procedural errors. A conceptual error is one made because the student does not understand sufficiently the nature of the numbers or the operation involved. A procedural error is one in which the student, although having an overall strategic understanding of what to do, makes either a careless error or other error in carrying out the strategy. For example, 0.1 x 0.1 = 0.1 and 3 ÷ ½ = 1 1/2 are likely to be conceptual errors whereas 58 + 34 = 82 and 3 x 5 = 18 are likely to be examples of procedural errors. While procedural errors are associated with both written and mental computation, the procedures themselves, and therefore the types of errors, are often quite different. As an example, for the item 74 - 30, a quite common error at Grades 3/4, 5/6, and 7/8 was the answer 36. It is likely that students making this error had a correct overall procedure or strategy of taking the 4 off the 74, subtracting 30 from 70, and then replacing the 4; but a lack of control over the procedure led them to subtract rather than add the 4.

As observed in this study, the errors made by students with whole number calculations tended to be procedural, whereas those involving fractions, decimals, and percentages were predominantly conceptual. For example, items such as 1 – 1/3, 0.3 + 0.7, and 30% of 80, which are typical of three types of items set, were very frequently answered incorrectly, and yet each depends on very simple arithmetical ability coupled with conceptual understanding of the type of number involved. It is hard not to draw the conclusion that, in spite of teaching over several years, very many students know very little about fractions, decimals and percentages.

Where an addition or subtraction was incorrect by one (and it is worth noting that there were also frequent cases with larger numbers where the answer was wrong by two), it appears very probable that in many cases the children's strategy was to count up or down by ones; this is reinforced by the fact that this error occurred more often in basic fact calculations when the addend was larger. A similar reason can be hypothesised for the number of multiplication/division errors that are wrong by one multiple. In both cases there appears to be an error of counting, whether by ones or by multiples of 2 to 10.

The most frequent errors associated with calculations involving fractions, decimals and percentages appear to have a mainly conceptual basis.

Decimal computation errors appear to be predominantly associated with the common misunderstanding noted by Hart (1981) and Stacey and Steinle (1998), namely "thinking that the figures after the [decimal] point represented a 'different' number which also had tens, units etc" (Hart, 1981, pp. 51-52). Examples of this error included 0.5 + 0.75 = 0.8, and 3 x 0.6 = 0.18.

Implications

This study has confirmed that mental computation can be simply and effectively assessed by a written class test. Since mental computation is now a major curriculum requirement, the major implication for assessment is that the assessment of mental computation deserves serious consideration at all levels, from classroom and school level to system-wide and national testing programmes.

The major implication for teaching of mental computation of whole numbers is that teachers need to centre on the development of efficient strategies other than counting on and back, whereas for fractions, decimals, and percentages the issue appears to be that of developing conceptual understanding.

For whole numbers, three issues should be addressed. First, when children are using counting strategies for computations, their teachers need to observe how they count and keep track of their counting, and need constantly to ask children how they arrived at their answers. Errors may be caused by inefficient use of their fingers or other procedural error, for example when adding 6 and 3, counting 6, 7, 8. If these incorrect procedures are not discovered and discussed early they can become ingrained and persist throughout primary school. Second, children need to be weaned off the increasingly inefficient strategy of counting on and back by ones, to more sophisticated, neater and simpler strategies: using doubles and near doubles, bridging ten, adding tens, using compatible numbers, using related known facts. Third, the scope of mental computation needs to be widened to include strategies for double-digit computations.

For fractions, decimals, and percentages, the main remedy appears to lie in much greater emphasis on conceptual understanding, and much less on algorithmic processes. Mental computation, unlike written computation, is very difficult to teach or learn algorithmically. It depends much more on conceptual understanding of numbers and operations and on a holistic approach to numbers. At least the algorithmic teaching of procedures should be delayed until children have a conceptual understanding of the types of number and the operations involved.

Finally, if mental computation is to take the place in schools that both society and the pronouncements of curriculum developers at system level encourage, then it needs to be given attention in terms of teaching time, exploration of efficient teaching strategies, and resources equivalent to those that hitherto have been given to the teaching of the formal written algorithms.

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Numeracy as a Tool in Adult Education: Success or Failure?

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Abstract. *In 2001, a new programme in adult mathematics education was introduced in Denmark. The aim was that the students further developed their numeracy. The development of education and teacher education was research based and an operational model of Numeracy was the pivotal point in this work – and as such a kind of quality control in the process. Three years later, in 2004, the Danish Evaluation Institute evaluated this educational programme, Preparatory Adult Education (PAE) as it was named. In the light of her experience from development and evaluation of this mathematics programme, the author questions if the concept of Numeracy is implemented in the teaching and learning practices of PAE-mathematics. Thus, the paper reports reflections provoked by the author's observations during an evaluation process – not a study designed to answer this question.*

At the beginning of the 21st century, adult and lifelong mathematics education form two sides of the same coin in Denmark. The adult educational system is built up in parallel with the mainstream education system. Mathematics is offered to adults at lower and higher secondary level. From 2001, basic mathematics education is offered to adults in the new programme of Preparatory Adult Education (PAE). The aim is that students develop numeracy, which is defined as functional mathematical skills and understanding that in principle all people in society need to have. In English speaking countries, "numeracy" is a key word in basic adult education but this was the first time that numeracy was mentioned in a Danish Act. Five years before, Lindenskov and I had imported the term from the English speaking countries, translated it into Danish (numeralitet) and reconstructed it as a concept (Lindenskov & Wedege, 1997).

At the Third Nordic Conference on Mathematics Education (NORMA01) in 2001, I presented this concept of numeracy and some of the basic ideas of PAE-mathematics (Wedege, 2005). After my lecture, Bill Barton from New Zealand asked if it was really necessary to introduce numeracy in Denmark. My answer was that the concept of numeracy and our operational model of Numeracy formed the research base for developing the new mathematics curriculum. I also regarded a new term as important to avoid a usual teacher reaction when presented for a new curriculum: "This is what we have always done".

As stated in the Act of PAE, the educational programme was evaluated in 2004. Numeracy is the aim of the education and the pivotal point of the teacher education. Nevertheless, neither teachers nor school leaders mentioned the word "numeracy" during the whole evaluation process. As a person involved in the development of the curriculum in 2000-2001 and in the evaluation 2004-2005, this observation provoked me and this paper reports my reflections on whether the model of Numeracy is implemented in the teaching and learning practices of PAE-mathematics or not.

Numeracy as a term, a concept and a tool

In English speaking countries, the term "numeracy" is used for certain basic skills and understandings in mathematics, which people need in various situations in their daily life. As mentioned above, numeracy is a key word in basic adult mathematics education. As a concept however numeracy is deeply contested in politics, education and research. Nevertheless, as an analytical concept, adult numeracy may be considered as mathematical activity in its cultural and historical context. (For a review of research and related literature on adult numeracy, see Coben et al., 2003.) In policy reports and in international surveys, the term "numeracy" is often used as a parallel to literacy. "Quantitative literacy" and "mathematical literacy" are two other terms dealing with people's mathematical competencies in relation to societal requirements (see OECD, 1995; 1999). In the Second International Handbook of Mathematics Education, Jablonka (2003) gives a critical overview of different constructions of mathematical literacy. She argues that any conception of mathematical literacy – implicitly or explicitly – promotes a particular social practice.

In the mid 1990s, the Danish language did not have a single expression corresponding to the term numeracy[1]. Nevertheless, Lindenskov and I chose to use the term *numeralitet*, which was later adopted by the Ministry of Education. In our definition, adult *numeracy* describes a mathematics containing everyday competence that everyone, in principle, needs in any given society at any given time:

- Numeracy consists of functional mathematical skills and understanding that in principle all people need to have.
- Numeracy changes in time and space along with social change and technological development. (Lindenskov & Wedege, 2001, p. 5)

It is this expression "in principle" that makes possible a general assessment of adult numeracy (as in the big international surveys) and the developing of general courses in numeracy. All adults who participate in a numeracy course will, in fact, have their own perspectives (why am I here?), their own backgrounds and

[1] Like with numeracy we do not have translation of the term "mathematical literacy", in the Nordic countries.

needs (what am I going to learn?) and their own strategies (how am I learning?). It is this definition of numeracy which is adopted as the aim in PAE-mathematics.

During research and developmental work in the Adult Vocational Training system, Lindenskov and I developed an operative model for the study of adult numeracy. It has four interrelated dimensions, which are

- *Media* (a) written information and communication (b) oral information and communication, c) concrete materials, d) time, and e) processes.
- *Context* - in the meaning of situation context - (a) working life, (b) family life, (c) educational context, (d) social life, and (e) leisure.
- *Personal intention* (a) to inform/be informed, (b) to construe, (c) to evaluate, (d) to understand, (e) to practice, etc.
- *Skills & Understanding* - Dealing with and sense of (a) quantity and numbers, (b) dimension and form, (c) patterns and relations, (d) data and chance, (e) change, (f) models.

Our construction of the operative model of Numeracy was based on paradigmatic socio-cultural studies such as those of Scribner (1984), Lave (1988), and Nunes, Schliemann and Carraher (1993), on conceptions of functional literacy such as those of OECD (1995), and on the six "big mathematical ideas" presented and discussed by Steen (1990). This model has been used and further developed as an analytical tool in adult mathematics and research. During the development of PAE-mathematics, we found inspiration in Bishop's (1988) cross-cultural studies of mathematical components in everyday activity and added mathematical activities such as counting, measuring, locating to the fourth dimension (skills and understanding) of the Numeracy model (see figure 1).

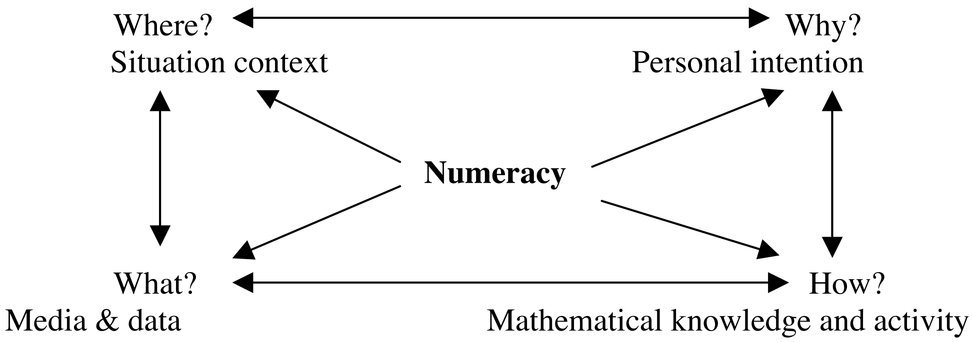

Figure 1. Four analytical dimensions of Numeracy (Wedege, 2004, p.113).

In what follows, I will use the term Numeracy (with a capital N) to refer to the concept of numeracy as defined in this operational model and numeracy (with a small n) to refer to the underlying conception of numeracy as defined above.

Preparatory adult education in mathematics

As mentioned above, Preparatory Adult Education is a vital element of the Danish lifelong education model. According to Rubenson (2001), one may find three generations of the idea of lifelong learning in the period from the late 1960s until now. The first generation – lifelong learning as a *utopian-humanistic* guiding principle for restructuring education – was introduced by UNESCO. The concept disappeared from the policy debate but reappeared in the late 1980s as the second generation driven by a different interest based on an *economistic* worldview emphasising the importance of highly developed human capital, and science and technology. From the late 1990s, it seems that a third generation (*economistic-social cohesion)* with active citizenship and employability as two equally important aims for lifelong learning – at least on the rhetoric level – is taking over. Preparatory Adult Education illustrates these new tendencies. During the political debate and the educational planning process of PAE Mathematics "active citizenship", "employability" and "personal needs" were used as equivalent arguments (Johansen, 2002).

An obvious danger of lifelong learning as a political project is that learning for active citizenship and democracy is reduced to an individual project. From this perspective, it was important to notice that the following statement was formulated by the Danish government in the Bill of Preparatory Adult Education (Forberedende VoksenUndervisning), in 2000:

> Further development and maintaining of the individual's skills are not *only* an individual and private affair and responsibility. It is *also* a common societal responsibility. PAE encompasses both a democratic aspect to maintain and promote the development of active citizenship and an economic perspective linked to the demands and needs of the labour market.

In January 2000, the Ministry of Education invited Lena Lindenskov and me to develop the national mathematics curriculum and teacher training in the considered adult education programme, which also contains a literacy curriculum.

In the national curriculum of Preparatory Adult Education in Mathematics, the purpose is formulated as to ensure students the possibility of clarifying, improving and supplementing their functional arithmetic and mathematical skills. The intention of the education is to increase the students' possibilities of coping with, processing and producing mathematics containing information and materials.

A specific terminology is used and defined in the curriculum. The aim is reformulated as the adult students' further development of their *numeracy*, as described above. The content is described as a dynamic interplay between a series of *mathematical activities*, various types of *data and media*, as well as selected *mathematical concepts and operations* (see figure 2). As mentioned above, we found the inspiration to these activities (counting, localising, measuring, designing, playing, explaining) in Bishop's work.

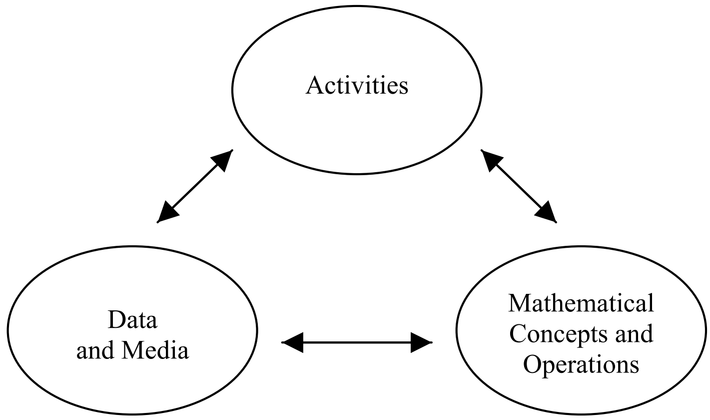

Figure 2. The content of the curriculum presented as a dynamic interplay

PAE mathematics has two levels: (level 1) *figures and quantity* and (level 2) *patterns and relations*, which in addition include the area of *form and dimension*, as well as *data and chance*.

According to the curriculum mathematical awareness is cultivated and trained in the students. The education aims to make it possible to clarify and formulate, and maybe change, students' beliefs and attitudes in relation to mathematics. Students should work with several different kinds of contexts. In addition to the mathematical context, they should work with everyday and societal contexts. The class decides upon the choice of context for class activities. With regard to individual activities, the individual students choose their contexts on the basis of what they need to learn.

The organisation, concrete aims and content should be arranged so that the background and foreground of the students take a central place. Dialogue is used to clarify and make use of the students' background and perspective. The relevance of the content is made clear by concrete connections to activities outside education. The way the problems are posed and formulated as well as the problem solving methods should be authentic in relation to the chosen context (Lindenskov & Wedege, 2001, pp. 20-22).

The principles for organisation and content of PAE-Mathematics are presented and carefully discussed in the Teaching Guidance published by the ministry of education on the web, in 2002 (Undervisningsministeriet, 2002). In this publication, one may also find the four basic assumptions concerning adults' relationship with mathematics on which the education is built:

- Adult's numeracy has great influence on their participation in education, working life and societal life and their personal organisation in everyday life. However, many adults are not aware of this.
- Adults learn better when the mathematics education is meaningful i.e. the content and the methods used are authentic and relevant.
- Many adults' school experience with mathematics is bad. This might cause blocks when they return to learn mathematics. Adults' resistance towards learning is also a well known phenomenon.
- Adults learn in different ways. Thus they profit from different learning activities and materials.

In the Teaching Guidance, the operational model of Numeracy is presented and discussed in relation to the teaching practice, and the term numeracy is used through the whole publication. In the teacher training programme, which is compulsory, Numeracy is the key concept and adult numeracy is the main thread of the education.

The teachers' views of mathematics

Before the start of Preparatory Adult Education in 2001, the Ministry of Education held three conferences to inform the mathematics teachers in basic adult education about PAE-mathematics. With the cooperation of 212 of the participating mathematics teachers (more than 90%), Henningsen and I made a survey on the teachers' beliefs and attitudes towards mathematics (Wedege & Henningsen, 2002). As representatives from the adult educational institutions that were supposed to offer PAE, these teachers were all potential future teachers in PAE mathematics.

The questionnaire comprised both open and closed questions and we analysed the material using a combination of quantitative and qualitative methods. We elicited three kinds of answers from the teachers. Mathematics in their own words (the essay), biographical information (the teacher profile) and finally the teachers were asked whether or not they associated mathematics with 18 value items constructed on the basis of Bishop's (1988) six categories (the value chart).

Ticking off the items in the value chart, the teachers generally agreed on the value items as words associated with mathematics. For example 95% of the teachers associated *rules, logic* and *order* with mathematics. However, in their essays, the teachers used different expressions to describe mathematics. No value item was found in more than 9% of the essays, and two out of three teachers (143 of 212) did not use any of the value items in their essay.

The descriptions of mathematics in the teachers' essays centred on three different types of answers, which we tentatively denoted: *everyday mathematics*, *curriculum mathematics* and *mathematics in the world*. In accordance with the rhetoric of Danish basic adult education in mathematics, the majority of the teachers' essays were in the first category of everyday mathematics.

Evaluation of PAE

In the actual Act, it was stated that the Preparatory Adult Education programme should be evaluated in 2004 and this task was given to the Danish Evaluation Institute (EVA[2]) in 2003. I was appointed by EVA as one of the five members in the evaluation group (se below).[3]

Mertens (2005) refers to arguments of what distinguishes evaluation from other forms of social inquiry. It is its political inherency; that is, in evaluation, politics and science are inherently intertwined:

> Evaluations are conducted on the merit and worth of programs in the public domain, which are themselves responses to prioritized individual and community needs that resulted from political decisions. Program evaluation "is thus intertwined with political power and decision making about societal priorities and directions". (Mertens, 2005, pp. 49-50)

Here *merit* refers to the excellence of an object as assessed by its intrinsic qualities or performance; and *worth* refers to the value of an object in relation to a purpose. So merit might be assessed by asking: How well does your programme perform? And worth might be assessed by asking: Is what your programme does important?

The evaluation of PAE was to be a *summative* in the sense that it is an evaluation used to make decisions about the continuation, revision, elimination, or merger of a programme. At the same time the evaluation was to be *formative* in the sense that the educational institutions involved were to improve the implementation of the programme locally (see Mertens, 2005). The first perspective was required by the ministry and the second was a consequence of the evaluation method employed by EVA.

The method

EVA employs different methods according to the requirements of the specific evaluation task. However, there are a number of fixed elements in each evaluation: project team, preliminary study, terms of reference, evaluation group, self evaluation, supplementary survey, site visit, report and follow-up (see web site www.EVA.dk). For the purpose of this paper, I will go into more details about five of these elements as they were realised in the evaluation of PAE:

[2] The primary task of this institute is to initiate and conduct evaluations of education - from primary school and youth education to higher education and adult and post-graduate education.

[3] EVA needed a Danish researcher with expertise within the area of adult mathematics education. My involvement in the development of PAE-mathematics and non-involvement in the implementation of the curriculum were assessed and they concluded that I was able.

The *project team* was responsible for the practical work and the methods for the evaluation, including also the responsibility for writing the final report. The members of the team were employed in EVA. The *terms of reference* was laid down by EVA describing the objective and framework for the evaluation. The *evaluation group*, where I was a member, was established and composed of people with special academic expertise in the area that is evaluated (a researcher in literacy, a researcher in numeracy, an experienced teacher in literacy and numeracy, a school leader and a Norwegian adult educational planner). The evaluation group was responsible for the academic contents of the evaluation and for the assessments and recommendations of the report. The evaluation group was appointed by EVA's board, and the members were independent of the educational programmes evaluated and also of EVA.

The *self evaluation*, which is an integral part of any evaluation, had a dual purpose: on the one hand, it should be used as documentation for the final report and its recommendations and, on the other, it should be seen as an inspiration for the evaluated educational programme or institutions for quality improvement. In the self evaluation, 10 adult educational institutions described and assessed their own strengths and weaknesses. Normally, the participants will be the teachers and also the students or the pupils and management. However, in the case of PAE, the project team had decided not to involve the students. The self evaluation was based on guidelines prepared by the project team. Some of the headings in the self evaluation report were students (profiles, learning needs), teachers (qualification), enrolment, framework of the education (aim). The self evaluation reports together with the supplementary surveys and the site visits formed the basis for the recommendations of the evaluation report.

During the *site visit* the evaluation group and the project team visited the 10 self evaluating institutions. During the visit, the evaluation group had the opportunity to interview the teachers, the students, the management team and representatives from local workplaces. The purpose of the visit was to obtain further documentation for our report. Prior to the visit, the project team prepared a checklist (questionnaire) for the evaluation group based on the self evaluation reports. This procedure was to ensure that any obscurities in the self evaluation reports are identified. However, only few of the questions prepared by the team concerned the education content, and the evaluation group added a series of questions e.g. tell about your classroom practices; do you find that there is enough time compared with the aim of the education; how do you adjust the teaching to the students' needs. The project team prepared minutes of the meeting after each visit. The minutes of the meeting are only for EVA's own use. During the site visit, I made my own notes and afterwards I used EVA's minutes to check my own. The quotations from teachers or students below stem from either my personal notes or the official evaluation report (EVA, 2005).

Worth: Is what PAE does important?

The aim of EVA's (2005) evaluation was to evaluate strong and weak points of preparatory adult education (PAE) and to assess whether the implementation of the Danish act on preparatory adult education is living up to its purpose, i.e. mainly to evaluate the worth of the programme. In the evaluation report, one finds the official results and recommendations from the evaluation group. In relation to the reflections in this paper, I find the following results relevant:

It appears that in spite of a great increase in the activity, the targets formulated on the adoption of the PAE Act in 2000 have not been reached. This is especially true of mathematics. I think that one of the reasons is that the workers' unions and the big enterprises have focussed on the literacy problem for the last 10 years. Only little attention has been paid to numeracy. Another reason might be that mathematics in people's working life is invisible or not experienced as mathematics (cf. Wedege, 2001, 2004). However, many students benefit greatly from the education, both personally and socially, and a number of them use PAE as a springboard for further education. A general example was adults now being able to help their children with their homework, and happy people telling that, after all, they were able to learn mathematics. Although the word numeracy was not mentioned by the teachers, I am sure that many students further developed their numeracy as a result of PAE-mathematics.

The first statement of the famous Math Anxiety Bill of Rights is this: "I have the right to learn at my own pace and not feel put down or stupid if I'm slower than someone else" (Tobias, 1993, p. 226). A common remark from the students was that they felt at the eye level with the teachers. They were actually treated as competent adults – another of the 14 rights.

According to the report, PAE is characterised by flexibility (EVA, 2005). This is apparent from the large number and the many different types of providers of adult education as well as from the way teaching activities are organised. Moreover, this flexibility is demonstrated by the recourse to relocation of teaching activities, in the sense that teaching may take place in business enterprises, organisations etc. However, when it comes to interpretation of the curriculum, flexibility is not only a positive term. Some of the mathematics teaching that we met during the visit didn't have anything to do with PAE. We saw for example ordinary mathematics education compensating for young students' poor mathematical skills in vocational education. In this school – like in others – we saw and heard of the use of standard mathematics textbooks. Like in an adult education school where a male student said: "He gave us a book and then we worked individually. We didn't use the material from the workplace although we asked if we could do so. The teacher gave us a textbook from grade 2 to 3. When we told him that we wanted to learn to calculate area, he said that we would meet this problem later in the book." The last comments lead us to the other dimension of the evaluation.

Merit: How well does PAE perform?

The main purpose of the evaluation was to assess the worth of PAE, however, the report also contains results concerning the merit. For example it is observed that, in general, PAE is characterised by dedicated teachers and managers capable of creating a successful environment for adult education in which the students feel safe and with teaching based on the students' needs and qualifications (EVA, 2005). In the evaluation group's opinion, the teachers in general are well qualified and capable of carrying out their teaching on the basis of the qualifications and needs of the individual student. However, among the mathematics teachers, there are many examples of this not being the case. Thus, the evaluation group recommended the individual provider to assess the teachers' qualifications in the light of the new requirements in order to ensure that all teachers include the experience of the students in their teaching and implement the teaching model from the curriculum (figure 2).

Here are a few examples from the visit: In the curriculum it is required that concrete material (e.g. juice, rice, wood, fabric) should be one of the medias used in combination with activities and mathematical operations or concepts. In a locker marked with the words "concrete material" in a well equipped classroom, I found only gadgets in plastic normally used in the children's mathematics classroom.

When students spoke in general about mathematics they often used the terms "equation", "x" and "y", which were not in the curriculum. In the following statement, I found an example of that the students' views of mathematics and their self-conceptions in relation to mathematics were not changed. A woman who was fired after many years in the same job said: "For the last 32 years I have only worked in LEGO's design department. I cannot do any mathematics." It seems that to her mathematics is still "what I cannot do" (see Wedege, 2005).

Conclusion and discussion

The purpose of the evaluation was not to investigate if the educational planners' ideas, concepts and design as manifested in the curriculum were implemented in the teaching and learning practices of PAE-mathematics. However, being an educational planner and researcher not a politician, one of my personal interests was to assess the intrinsic qualities or performance of the education (merit), particularly the implementation of the operational model, Numeracy.

In her paper "Balancing the unbalanceable", Sfard (2003) goes through the NCTM Standards in the light of theories of learning mathematics. As a part of the reform movement, she sees the Standards as a result of a serious and comprehensive attempt to teach "mathematics with a human face": "Success of educational ideas, however, is never a simple function of the ideas themselves. There is no direct route from general curricular principles to successful instruction." (p. 354) The conclusion of my reflections in this paper is in keeping with Sfard's statement.

The same goes for Skott (2006) who suggests to moderate the idea that new theoretical constructions even empirically grounded could have decisive and direct influence on institutionalised teaching and learning of mathematics. In the case of PAE-mathematics we do not deal with education already institutionalised. This is a new educational programme which replaces another programme in basic mathematics. However, the schools and the mathematics teachers are the same.

If we look at the teachers' views of mathematics before the start of PAE-mathematics, we may find a reason why nobody talked about numeracy during the whole evaluation procedure (self evaluation and interviews). As mentioned above mathematics is associated with everyday mathematics in most teachers' conceptions. Thus they do not find it necessary to use the term "numeracy" in stead of "mathematics". At one of the ministry's three information meetings on PAE-mathematics, a teacher whispered to the person next to him: "We go to this meeting; we listen and we go home doing what we are used to do." The new rhetoric was interpreted in ways that fitted with the current practices in adult mathematics education and resulted in little change to teaching.

With focus on the worth of the new adult mathematics education, "success" may be the answer to the initial question concerning Numeracy as a tool in adult mathematics education. What PAE-mathematics does is important: many students benefit greatly from this education. But looking at the merit of the new programme the answer may be "failure". PAE-mathematics does not perform according to the curriculum with a dynamic interplay between activities, data and media, and mathematical concepts and operations.

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A Reading Comprehension Perspective on Problem Solving

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Abstract: *The purpose of this paper is to discuss the bi-directional relationship between reading comprehension and problem solving, i.e. how reading comprehension can affect and become an integral part of problem solving, and how it can be affected by the mathematical text content or by the mathematical situation when the text is read. Based on theories of reading comprehension and a literature review it is found that the relationship under study is complex and that the reading process can affect as well as act as an integral part of the problem solving process but also that not much research has focused on this relationship.*

Introduction

All over the world, textbooks seem to play an important role in mathematics education at all levels (Foxman, 1999), and thereby students' reading activity and reading comprehension also play important roles. However, the reading activity can be of different kinds, for example, reading an expository text that tries to explain something to the reader or reading a problem text in order to attempt to solve the given problem. This paper focuses on the second type of reading situation; reading comprehension when trying to solve a given problem. It seems like most research in mathematics education about reading comprehension has been done in a manner that reduces reading to a potential obstacle for learning (Borasi & Siegel, 1990), for example, by focusing on how limitations in reading ability affect learning in mathematics or on readers' misunderstandings of a written task and how this can influence the solving of the task. This paper is an attempt to start from a more nuanced view of reading comprehension, and analyze problem solving from this perspective.

Purpose

The purpose of this paper is not merely to add a detailed view of reading comprehension as an important component of a problem-solving situation, but to discuss the bi-directional relationship between reading comprehension and problem solving. Therefore, the purpose of this paper consists of two parts:

A. To theorize about how reading comprehension can *affect* problem solving and how reading comprehension can be added as an *integral part of* problem solving.

B. To analyze how reading comprehension can be affected by the mathematical text content or by the mathematical situation when the text is read. In this paper, the problem-solving situation is the particular type of mathematical situation that will be studied.

Method and structure of the paper

Theories of reading comprehension will be the starting point in this paper (in section 2), that is, these theories will be used as theoretical tools when discussing the relationships between reading comprehension and problem solving. However, the two parts of the purpose will be handled differently, where the theorizing in part A will be done in a more unrestricted and exploratory way (in section 3), while the analysis in part B will be done with the help of a literature survey (in section 4). The main purpose of this survey is not to find and describe all possibly relevant references, but to find some references (of course as many as possible) that are highly relevant for part B of the purpose of this paper, in order to discuss these references from a reading comprehension perspective. A description of the literature survey follows.

The search for literature was made in two different ways. First, databases were used to search for references containing (part of) words such as ‘problem solving’, ‘reading’, ‘semantics’, and ‘linguistic’ in relevant combinations. This search was made in order to find literature that specifically deals with a relationship between problem solving and reading comprehension. Second, a less specific search was performed, for studies dealing with word problems. This second type of search for literature was made in order to find such references that potentially could be relevant for the purpose of this paper, without explicitly dealing with a relationship between reading comprehension and problem solving. There are two reasons for focusing on word problems. Firstly, a more general search for problem solving, and not specifically word problems, was presumed to yield too many irrelevant references. Secondly, a literature survey by Österholm (2004, section 3.1) found that word problems were often discussed in literature that focuses on texts and reading in mathematics.

A general search in the MathDi database[1] (using ‘basic index’) for references dealing with word problems gave 1424 results[2]. This number had to be reduced in order to be able to complete this task in a reasonable time. Therefore, it was chosen to limit the search to the titles of references (to get references that more directly focus on word problems) and to only include those published in journals (to ensure generally high quality). This resulted in 199 references, and all these

[1] http://www.emis.de/MATH/DI/

[2] All mentioned searches in MathDi were performed on 6 September 2005 and were limited to references in English. The search words used were ‘word probl*’, which resulted in references including ‘word problem’ or ‘word problems’ that were published in the years 1976-2005.

were included when the abstracts were analyzed. More details of this analysis and the results from it can be found in section 4 of this paper.

Theories of reading comprehension

Mental representations

When reading a text, a mental representation of the text is created by the reader, which describes how the reader understands the text. Many studies about reading comprehension show, or support the conclusion, that “multiple levels of representation are involved in making meaning” (van Oostendorp & Goldman, 1998, p. viii). In particular, the work of Walter Kintsch (e.g., see Kintsch, 1992, 1998) seems to have had a great influence on research on reading comprehension (Weaver, Mannes, & Fletcher, 1995). Kintsch (1998) distinguishes between three different levels, or components, of the mental representation created when reading a text: the surface component, the textbase, and the situation model.

When the words and phrases themselves are encoded in the mental representation (possibly together with linguistic relations between them), and not the *meaning* of the words and phrases, one can talk about a surface component of the mental representation.

The textbase represents the meaning of the text, that is, the semantic structure of the text, and it “consists of those elements and relations that are directly derived from the text itself [...] without adding anything that is not explicitly specified in the text” (Kintsch, 1998, p. 103). Since the textbase consists of the meaning of the text and the same meaning can be expressed in different ways, a textbase can be created without any memory of the exact words or phrases from the text.

A pure textbase can be “impoverished and often even incoherent” (Kintsch, 1998, p. 103), and to make more sense of the text, the reader uses prior knowledge to create a more complete and coherent mental representation. A construction that integrates the textbase and relevant aspects of the reader’s knowledge is called the situation model. Of course, some prior knowledge is also needed to create a textbase, but this knowledge is of a more general kind that is needed to “decode” texts in general, while the prior knowledge referred to in the creation of a situation model is more specific with respect to the content of the text.

Content literacy

As defined by McKenna and Robinson (1990), content literacy refers to the ability to read, understand and learn from texts from a specific subject area. They also distinguish between three components of content literacy: general literacy skills, content-specific literacy skills, and prior knowledge of content. Both the general and the content-specific literacy skills can be assumed to refer to some more general type of knowledge that is not dependent on the detailed content of a specific text. This type of knowledge is primarily used to create a textbase in the mental representation. The third component, prior knowledge of content, refers to knowledge that is connected to the content of a specific text, and is thus primarily used to create a situation model in the mental representation.

It is not clear to what extent mathematics in itself creates a need for content-specific literacy skills and how much of reading comprehension in mathematics depends on more general literacy skills and prior knowledge. However, the symbolic language used in mathematics seems to be a potential cause for the need of content-specific literacy skills. Also, in a study by Österholm (in press), comprehension of one mathematical text not using mathematical symbols (i.e., written in a natural language) mainly depended on the use of more general literacy skills.

Cognitive processes

Thinking about one's own reading process it seems clear that a skilled reader usually does not need to actively think very much to create a mental representation when reading. The use of syntactic and semantic rules together with the activation of more specific prior knowledge thus happens quite automatically, on a more unconscious level. In general, different cognitive processes can be more or less conscious. Perception can refer to highly automatic and unconscious processes, for example when you see a dog and directly recognize it as a dog; you are aware of the result of the process (that you see *a dog*) but no active and conscious thought processes were needed for this recognition. Problem solving on the other hand can be said to deal with active thinking, a more resource demanding process, for example when trying to remember the name of a person you meet and recognize. Thus, when reading a text without experiencing any difficulties in understanding what you read, the process has more in common with perception than with problem solving, in that the process of understanding is mainly unconscious. This is a situation representative for Kintsch's (1998) concept of *comprehension*, which "is located somewhere along that continuum between perception and problem solving" (Kintsch, 1992, p. 144).

Problem solving and reading comprehension

Problems that need to be solved can arise in different ways, but here focus is on given problems with a specific question, in particular mathematics problems given in writing. Specific theories about the problem-solving process sometimes include the reading of the problem text as an important part (e.g., see Pólya, 1990), which seems natural since one surely needs to start by reading the given problem text in order to try to *understand the problem*. Thereby, a mental representation of the text is created, that is, a mental representation of *the problem* is created. But in order not to limit the description of the result of this reading process to that the reader either has understood the text or not, and what kind of (negative) effects this might have on the solution process, a more integrated view is suggested of (1) reading the problem, (2) understanding the problem, and (3) solving the problem.

It seems quite obvious that deficiencies in literacy skills, general or content specific, can affect the attempt to solve a given problem, since direct reading errors (i.e., problems in creating a textbase) increase the risk that the mental representation contradicts the text. However, the mental representation created in the reading process does not only serve as *background* to solving the problem, but the solving process has already started, since also prior knowledge is activated in the reading process, including more specific types of prior knowledge that can be suitable for solving the problem, that is, the comprehension of the problem need not only consist of a pure textbase in the mental representation but also a situation model can be created. It could even be the case that the problem in principle has been solved through the reading process (or at least the problem *is believed* to be solved). In such a case, the problem is solved using mainly unconscious cognitive processes, that is, the problem is solved through pure *comprehension* (Kintsch, 1998) of the problem/situation. Davis (1984, p. 207) gives an empirical example of this type of solution by comprehension, where an existing mental representation of a similar problem was activated, and the person "had done this unconsciously, but had been able to reconstruct some of the process by determined introspection afterwards." Thus, this is not only a theoretical possibility, and it has also been shown that these types of unconscious comprehension processes can be used to explain behavior in such situations as action planning (Mannes & Kintsch, 1991) and decision making (Kitajima & Polson, 1995). Perhaps some observed student behavior when solving problems also can be explained by assuming that the student is relying mostly on these types of comprehension processes when trying to solve the problem, for example, when Lithner (2000, p. 165) reports that "focusing on what is familiar and remembered at a superficial level is dominant over reasoning based on mathematical properties of the components involved."

To generate the answer to the posed question in a given problem can be seen as a natural *goal* of the situation, and in order to reach that goal one needs to regulate one's behavior, that is, self-regulating processes are active. The given question can thus play a very important role also in the creation of the mental representation in the reading process since it can influence what kind of prior knowledge is activated, that is, the self-regulation seems to start already in the reading process. It has also been shown that self-regulating processes (which usually are considered as metacognitive processes) can operate at an unconscious level (Fitzsimons & Bargh, 2004). Therefore, it could be of particular interest to examine how variations of *questions* in problem texts can influence the comprehension and solution of a problem.

The literature survey

First, the titles and abstracts from the 199 references about word problems are analyzed, where the references are categorized with respect to type of research (empirical or theoretical, and what is being studied/discussed). Thereafter, in the second section, reading comprehension and problem solving will be discussed, using results from relevant references (from both types of literature searches, see above).

Word problems

Table 1 presents the number of some different types of research studies found about word problems. To only study titles and abstracts does have its limitations, and for some references it has also been difficult to decide exactly what type of research is being discussed in the full article. Some duplicates do also exist in the database, and it cannot be guaranteed that all have been found. The conclusion from these remarks is to not to take the numbers too exactly, but to see the overall distribution. Also, and more importantly, the purpose of the different categories is to find relevant literature (the named categories) for the purpose of this paper, and not to make a complete categorization and description of all references.

Overall, not many studies exist that in a direct manner examines the relation between reading and problem solving among the 199 references about word problems. However, studies that vary the wording of a text and examine the effect on the solution or solution process can also be of interest in order to see how the comprehension of the text is related to the solving.

Table 1. Hierarchy of categories of references among 199 articles about word problems, with the number of references in each category given. Named categories include references that have been studied in more detail in section 4.2. All subcategories are not necessarily disjoint.

Empirical studies	**115**
Variation of variables	75
Effect on performance (i.e., right or wrong solution)	*52*
Effect of text formulation [Category EP1]	19
Effect of reading ability [Category EP2]	4
Effect on the solving process	*15*
Effect of text formulation [Category ES]	4
Case (no structured variation of variables)	40
Discussions or theoretical studies	**77**
Types/properties of problems (including how solving can be affected) [Category D1]	12
Types of factors affecting solving problems [Category D2]	7

Reading comprehension and problem solving

This discussion will focus on the *results* from different studies, where different types of students and problems have been used, which of course can affect the results in different ways. However, this discussion will settle with the conclusion that there *exist* problems/students for which these types of results emerge, which then will be interpreted from a reading comprehension perspective. Therefore, no full meta-analysis of purposes, methods and results of studies will be performed. Several studies show that the performance in solving problems can be negatively affected by a higher complexity of the language used in the problem text (Category EP1 in table 1, e.g., Abedi & Lord, 2001) as well as by a relatively lower reading ability among students (Category EP2, e.g., Jordan & Hanich, 2000). Although these results can be seen as quite obvious, theoretically they can be interpreted as showing the need for more general literacy skills also when reading and solving problems in mathematics.

Both task context and situation context (Wedege, 1999) have been studied in the analyzed references. For word problems, the task context has been varied in different ways in empirical studies, for example, by trying to make the text more personal or interesting for the reader (Bates & Wiest, 2004). These types of studies focus on the effect the context may have on the performance among students (Category EP1), and different types of effects have been found, but it is not clear how these results should be interpreted. Does an increase in interest cause an increase in the effort that the student puts into trying to solve the problem, or does it activate more relevant prior knowledge that can be helpful when solving the problem? These types of questions have not been answered in the reviewed literature. Also, as the study by Renninger, Ewen and Lasher (2002) shows, there are many different types of interests that come into play at the same time in a rather complex way, such as interest for reading, for mathematics, and for the context described in the problem text. More generally, these results seem to depend on a rather complex, and seemingly not thoroughly investigated, interplay between properties of the text, the reader, and the situation.

Several studies show that students often seem to ignore realistic considerations when solving mathematical problems (e.g., Yoshida et al., 1997). However, other studies have altered the physical and social situation when solving a problem (i.e., the situation context), which resulted in more realistic answers among the students (Roth, 1996; Wyndhamn & Säljö, 1997). Thereby, how the student experiences the situation will affect the problem-solving process, that is, the comprehension of the situation is a relevant factor when solving (word) problems. Others (e.g., De Corte, Verschaffel, & De Win., 1985) describe a word problem as a quite peculiar type of text that can include ambiguous statements, which in the given situation need to be interpreted in a particular way (e.g., a statement that a person has $5 could in general be interpreted as either that the person has *exactly* $5 or that the person has *at least* $5). Thus, one needs specific types of prior knowledge about how statements in this situation should be interpreted, that is, one needs a type of content-specific literacy skill in this type of situation. However, as a side effect, as these type of skills evolve they seem to cause students to produce unrealistic answers in certain situations.

Another content-specific skill that seems to evolve among some students is to focus on numbers and keywords in the problem text (Hegarty, Mayer, & Monk, 1995). This surely seems to be a reading strategy specific to mathematics, since Bilsky, Blachman, Chi, Mui and Winter (1986) show that students' reading strategies can be influenced by making them read a text either as a mathematics problem or as a telling of a story. When read as a problem, the text was read with a focus on quantitative aspects and as a story it was read with a focus on more qualitative and temporal aspects.

Studies that in a more direct manner examine both the mental representation (often by letting students retell the text) and the solving of the problem consistently show a strong connection between these two aspects (Category ES, e.g., Cummins, Kintsch, Reusser, & Weimer, 1988), that is, the students solve the problem as they have understood it. Another possibility would be that one creates an elaborate mental representation of the text but bases the solution on something else (e.g., parts of the text itself and not the mental representation of the text). More detailed studies of the relationship between the mental representation and the solution show that better problem solvers mostly remember the semantic structure of the text while worse problem solvers mostly remember details in the text (Hegarty et al., 1995), and that the retelling of a problem text sometimes is made in another order than what was presented in the given text, an order that more closely resembles the calculation that is used when solving the problem (Hershkovitz & Nesher, 2001). This last result appeared both when the retelling was performed before the solving of the task (i.e., directly after reading the text) and when it was performed after the problem had been solved. Thus, the solving of the problem seems to have already begun in the reading process, a possibility discussed earlier in this paper. The existence of a specific question in the text as an important aspect was also discussed earlier. Therefore, in order to more clearly see a possible more direct effect of the *mathematical situation* (and not the existence of a question) it would be interesting to examine the mental representation before a question is given. However, no such studies have been found in this literature survey.

Conclusions

From the discussions in this paper it becomes evident that the relationship between reading comprehension and problem solving is complex. First, the reading process can affect the problem solving process, but can also act as an integral part of the solving process. However, not much research seems to have been done involving the latter relationship. Second, the literature survey has given examples of how the problem-solving situation seems to affect the reading process. However, not much research seems to have been done that directly focuses on this relationship, but the results discussed in the literature survey seem to be able to explain by assuming that the situation affects the reading process in certain ways, for example, that the reader uses specific strategies (or literacy skills) in this type of situation.

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