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Svensk Förening för **M**atematik**D**idaktisk **F**orskning
Swedish Society for Research in Mathematics Education

Mathematics and Language

Edited by

Christer Bergsten and Barbro Grevholm

Proceedings of MADIF 4

The 4th Swedish Mathematics Education Research Seminar
Malmö, January 21-22, 2004

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Preface

This volume contains the proceedings of *MADIF 4*, the Fourth Swedish Mathematics Education Research Seminar, with an introduction by the editors. The seminar, which took place in Malmö January 21-22, 2004, was arranged by *SMDF*, The Swedish Society for Research in Mathematics Education, in co-operation with Malmö högskola. The members of the programme committee were Christer Bergsten, Barbro Grevholm, Ingvill Holden, Thomas Lingefjärd, and Marie Skedinger-Jacobsson. The local organiser was Marie Skedinger-Jacobsson at Malmö högskola.

The programme included three plenary lectures, one plenary panel, and twenty paper presentations. We want to thank the authors for their interesting contributions. The papers have been reviewed by the editors, and some minor editorial changes have been made without noticing the authors. The authors are responsible for the content of their papers.

We wish to thank the members of the programme committee for their work to create an interesting programme for the conference, and Marie Skedinger-Jacobsson for her valuable help with the preparation and administration of the seminar. We also want to express our gratitude to the organiser of *Matematikbiennalen 2004* for its valuable financial support. Finally we want to thank all the participants at *MADIF 4* for creating such an open, positive and friendly atmosphere, contributing to the success of the conference.

Christer Bergsten, Barbro Grevholm
Editors

Content

Introduction
Mathematics and Language

Christer Bergsten, Linköpings universitet
Barbro Grevholm, Høgskolen i Agder

For the fourth mathematics seminar arranged by the Swedish Society for Research in Mathematics Education, the theme was chosen to be *Mathematics and language*. In this volume we have collected the contributions to the seminar. It contains the three plenary lectures, the plenary panel presentation, and eighteen paper presentations.

The crucial role of diagrams in mathematical reasoning is the focus of Willi Dörfler's plenary presentation *Mathematical reasoning and observing transformations of diagrams*. Recognising that diagrams of many different kinds are ubiquitous in mathematics, and that reasoning often deal directly with the transformations of diagrams, building on their structural properties and often rule-governed transformations, rather than with abstract ideas involved in mathematical concepts, the character and relevance for mathematics education of Peirce's notion of diagrammatic reasoning is outlined. This points to the importance of perceptive aspects of mathematical thinking. Diagrammatic reasonning is used to solve tasks, to be distinguished from a representational use of diagrams. By offering examples from elementary arithmetic, and from linear and abstract algebra, Dörfler highlights the power and usefulness of different kinds of diagrammatic reasoning.

Terezinha Nunes argues in her keynote conribution *How mathematics teaching develops pupils' reasoning systems* that the theories of Piaget and Vygotsky are coherent and complementary, and combined will give a broader picture of children's learning of mathematics. Using the conception of thinking systems, she shows by looking at pupils' working with multiplicative tasks, that the schema of one to many correspondence is used by very young children when mediating thinking tools are available, as well as in multiplicative reasoning out of school. Multiplication as repeated addition does not support this crucial schema for developing this kind of reasoning. To build powerful reasoning systems in classrooms, important principles and tools for the systems to work must be analysed.

In her plenary address *Assessing students' knowledge – Language in mathematics tests*, Astrid Pettersson offers an insight into the Swedish old and new systems of national testing in mathematics education. By looking at the variables authenticity, feedback and documentation, she concludes that the new system opens up for a broader and more supportive assessment process, for both teachers and students, with an aim to develop and analyse rather than to judge and condemn.

In the plenary panel, moderated by Christer Bergsten, the conference theme *Mathematics and language* was discussed by Håkan Lennerstad, Norma Presmeg and Åse Streitlien. Focusing on language in mathematics education, one can study both its structure and its use. The latter will be in focus when analysing communication in the classroom – this was the direction taken by Åse Streitlien in her contribution on discourse patterns in a primary mathematics class. The aspect of meaning was then added by Håkan Lennerstad, which led into the study of language also as a structure with the question: Why don't we teach the grammar of Mathematish? To analyse both the use and the structure of language in the mathematics education context, Norma Presmeg offered examples of how the ideas of semiotic chaining had helped mathematics teachers to reflect on and develop their practice. Follow-up questions from the audience demonstrated well that language is an important and deep issue in mathematics education.

The paper presentations open up with a contribution in Swedish, *Reflekterande samtal för pedagogisk utveckling* (Reflective conversations for pedagogical development) by Ann Ahlberg, Jan-Åke Klasson and Elisabeth Nordevall. In this study the overall aim is to explore how the special pedagog and teachers work together in order to develop the teaching in mathematics. The results show that the conversations contribute to the initiation of processes, which help the teachers to make visible their standpoints and awareness of values. The conversations help the teachers to develop a reflective attitude which give them preparedness for action, and knowledge to better understand and change their own practice.

Kristín Bjarnadóttir writes about *Teachers' preparedness of 'Modern Mathematics' in Iceland*. A historical account is given of the Icelandic school system and in particular teacher education and how it was prepared for the radical alteration of modern mathematics and its language in the 1960s and onwards. She presents four measures not taken by the authorities that she claims hindered the development of mathematics teaching in Iceland. She also raises the question if other circumstances such as the Icelandic inheritance of home- and self-education did mend the situation.

In the paper *On reasoning characteristics in upper secondary school students' task solving* the authors Tomas Bergqvist, Johan Lithner and Lovisa Sumpter investigate what it is that makes students succeed or fail in a problematic situation. They ask in what ways students manage of fail to engage in plausible reasoning as a means to make progress in solving tasks. Another question is the role of use of established experiences, algorithmic reasoning and piloted reasoning. The results show that most students use algorithmic reasoning or repeated algorithmic reasoning. Sometimes students' conceptual understanding is not sufficient for plausible reasoning.

Lisa Björklund's contribution has the title *Teachers and assessment – A description of Teachers' actions connected with the mathematics national test for school year 5*. In the paper the following questions are discussed: 1) Do teachers verbalise their observations and assessment of pupils' knowledge achievement in the competency profile enclosed about the pupils' competencies? 2) How do teachers assess pupils' work in one part of the text? 3) What attitudes do teachers have towards the National Test in mathematics and towards pedagogical assessment? The results show that teachers are generally pleased with the structure and content of the test. They also feel that the recommendations regarding what pupils should be able to achieve are reasonable in relation to the goals to be attained. Most teachers do not verbalise their assessments of pupils' knowledge. The author claims that the Swedish model for assessment can be improved via professional development of teachers in the area of assessment.

On the role of problem solving and assessment in Swedish upper secondary school mathematics in Finland is the title of the contribution by Lars Burman. In the survey the teachers agree that there are important strategies in problem solving to focus on in their instruction. Two out of three teachers construct at least one new problem for a text in mathematics. The results show that most teachers (more than 90 %) agree that problem solving ability should have impact on pupils' marks in mathematics. More than every second teacher uses tests as part of the instruction and during courses as a complement to the course test. All teachers want to make students take responsibility for their own learning and almost all try to make students aware of their knowledge. The impact of matriculation examination was confirmed but not very strong.

Hamper or Helper: The role of language in learning mathematics is based on the author's, Bettina Dahl's, doctoral thesis work. She has investigated four Danish and six English students and interviewed them in order to find out how they see language in connection to learning mathematics. There are various views of language in relation to how the pupils explain how they learn a new mathematical concept. Students use language as a thinking tool or they think it has a dual nature as it both facilitates and hampers learning. Another group of students think that language hampers learning.

Elsa Foisack writes about her work for the doctoral thesis in pedagogy in the paper entitled *Deaf children's concept formation in mathematics*. The concepts of multiplication with whole numbers and length are investigated. No difference was found concerning the steps towards comprehension of the concepts for the deaf pupils compared to those of hearing pupils. As in earlier research, it was found that deaf pupils need more time to learn mathematics than hearing children normally do. Of importance for the learning of deaf children is also the structure of sign language and the lack of an established terminology in mathematics.

Mikael Holmquist's paper, *Prospective mathematics teachers' learning in geometry*, is based on a study of student teachers in Gothenburg. He has shown that the concept images of the students more rarely correspond to the mathematical concept definition. Some issues about the consequences for the prospective teachers' work in the classroom are raised. What kind of concept definitions and concept images are on the teacher's repertoire? What kind of referents and criteria are the bases for the teacher's standpoint in assessment and validation? The emerging result will form the basis for deeper studies of what these prospective teachers learned in geometry and how they will express this in their teacher practice.

KULT-projektet – Matematikundervisning i Sverige i internationell belysning, is the contribution (in Swedish) by Johan Häggström. The KULT-project stands for Swedish school culture – classroom practice from a comparative aspect and is part of a greater study, The Learner's Perspective Study. A large data material has been collected through video tape recording of classroom activities by three different cameras at the same time. Teachers and pupils comment on the tape recorded sequences immediately after the lesson. The method opens opportunities for a deeper understanding of the studied teaching and the relation between teaching and learning.

In *Limits of functions – how students solve tasks*, Kristina Juter presents a study of university students justifications of how they solve tasks on limits of functions. The study is part of a larger study published as her licentiate thesis on students' concept formation of limits of functions. The study was carried out at a Swedish university at the first level of mathematics studies. Two groups of students on consecutive semesters solved the same tasks. Students' solutions are analysed and categorised and the results are presented in some detail from five of the given problems and reveals that there are many aspects to work with in order to improve students' learning.

Sinikka Kartinen presents in her paper *Learning to communicate – Communicating to learn in mathematics classrooms* an investigation of the mathematical problem solving process in a collaborative learning situation with in service teachers. The goal is to develop an appropriate analytic tool to highlight collaborative problem solving processes in the learning of mathematics, to investigate the role of cultural tools in the collaborative learning of mathematics teachers, and to investigate the processes of teacher participation in the collaborative learning of mathematics pedagogy. The study yields useful information about teacher learning and development from both the social and mathematical point of view. It also provides educators with tools to develop curriculum as well as instructional solutions for the mathematics classroom.

The linguistic side of mathematics is discussed by Thomas Lingefjärd and illustrated with examples form an algebra course taught with a focus on language. How do students and teachers handle the language of mathematics and how do they change back and forth between common language and the language of mathematics? He presents many arguments for teaching the language of mathematics and points out that all students should be taught the importance of the language of mathematics to better understand the subject.

Matematish – a tacit knowledge of mathematics by Håkan Lennerstad and Lars Mouwitz has the purpose to highlight the symbolic notations of mathematics and present some hypotheses. The authors stress the language aspect of the symbolic notation system and call it ‘Mathematish’. Using speculative reasoning in combination with empirical underpinnings form history of mathematics and their own teacher experience, they argue that Mathematish is a complete language and hope that this will create fruitful analogies with other languages.

Kategorisering av små gruppers handlingar is the title of the paper by Stefan Njord, Gunilla Svingby and Barbro Grevholm. The aim of this study is twofold: to study how a smaller group of pupils without the presence of a teacher collaborate to solve the experiments the group decides to work with and to study how the group uses the artefact (a graphical computer program). The detailed analysis of the actions in the group creates the categories interpretation, verification, trial and error and computation with formulas. Different sequences of these categories of action are used by the groups. The strategies of cooperation and the use of the artefact are described in the paper.

A theoretical framework for analysis of teaching-learning processes in algebra is presented by Constanta Olteanu, Barbro Grevholm and Torgny Ottosson. The framework is based on the variation in ways of making sense of the object of learning and the effectiveness of the communication during the lessons. The aim of the framework is to help in understanding the classroom interaction and the influence it has on the forming of the object of learning in algebra. The framework is a tentative try to combine Sfard’s focal analysis and the reification theory with Marton’s theory of variations.

Rudolf Strässer starts from the definition by Wartofsky on artefacts in his presentation *Artefacts – Instruments – Computers.* He develops the concept further as Warfofsky does in primary, secondary and tertiary artefacts. The explanation of the concepts instrumentalisation and instrumentation is given in connection to the development of utilization schemes. This leads the author to information and communication technology, ICT. Illustrations from the use of Dynamical Geometry Software are used to shed light on the points on artefacts. In the conclusions Strässer claims that utilizations schemes found by empirical studies can enrich the picture, which the researcher has of the process of teaching and learning mathematics.

Eva Taflin, Kerstin Hagland and Rolf Hedrén ask *What mathematical ideas do pupils and teachers use when solving a rich problem?* From videotaped lessons and interviews they draw the conclusions that teachers’ own mathematical ideas and solution methods direct their pupils, that teachers sometimes have difficulties producing feedback building on their pupils’ solutions, and seldom make generalisations out of their pupils’ solutions. In ten examples drawn form their data they analyse and argue for these results.

Allan Tarp talks about *Mathematism and the irrelevance of research industry*. He formulates an irrelevance paradox linked to the relevance paradox by Mogens Niss. He also claims that mathematics education research increases together with the problems it studies. The irrelevance paradox can be solved by using a postmodern sceptical LAB-research to weed out LIB-based mathematism coming from the library in order to reconstruct a LAB-based mathematics coming from the laboratory, as he phrases it. The reader has to find out what he means by the fact that the ‘Cinderella-difference’ is making a difference in the classroom.

In the plenary lectures, debate and papers presented in this volume, the relation between mathematics and language is viewed and discussed from many different perspectives and the authors see the importance of being aware of the language aspect of mathematics in teaching and learning. We hope that you as a reader will find several of these perspectives interesting and fruitful and that they can add something to your understanding of the character of mathematics teaching and learning. We thank all the contributors for the many and varied papers and are confident that the discussion on mathematics and language will continue.

Mathematical Reasoning and Observing Transformations of Diagrams

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Introduction

This contribution is located in the context of the philosophy of mathematics by the American philosopher and pragmatist Charles Saunders Peirce. Yet, it is readable and understandable without a detailed knowledge of the stance taken by Peirce. The interested reader might consult the papers Dörfler (2004a, 2004b) or Hoffmann (2001, 2002). This especially holds for the notion of diagram and diagrammatic reasoning which were introduced by Peirce to explain, on the one hand, the stringency of mathematical proofs and, on the other hand, the possibility of inventions and constructions in mathematics, or what he calls "surprising observations". Thus he says (in Peirce, Collected Papers 3.363):

> It has long been a puzzle how it could be that, on the one hand, mathematics is purely deductive in its nature, and draws its conclusions apodictically, while on the other hand, it presents as rich and apparently unending a series of surprising discoveries as any observational science. Various have been the attempts to solve the paradox by breaking down one or other of these assertions, but without success. The truth, however, appears to be that all deductive reasoning, even simple syllogism, involves an element of observation; namely, deduction consists in constructing an icon or diagram the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts. ... As for algebra, the very idea of the art is that it presents formulae, which can be manipulated and that by observing the effects of such manipulation we find properties not to be otherwise discerned. In such manipulation, we are guided by previous discoveries, which are embodied in general formulae. These are patterns, which we have the right to imitate in our procedure, and are the icons par excellence of algebra.

From this it already comes clear that a diagram might be of a great variety: geometric figures and algebraic expressions as well. For short, diagrams in Peirce are special (iconic) signs which have a clearly defined and recognizable structure and which can be manipulated according to (conventional) rules for transformations and compositions, cf. again the above mentioned papers. The crux of all that is that empirical and perceptive observation becomes a decisive part of mathematical reasoning, of devising and understanding proofs and mathematical arguments. Mathematical reasoning in this view is not so much the handling of abstract ideas in one's mind but the observation of the effects of one's manipulations of diagrams. The mathematical ideas rather reside in the invention of diagrams and of their fruitful manipulations, transformations, compositions. Mathematics in this sense studies the general properties and regularities of certain diagrams and of the operations with them. In accordance with the triadic sign concept of Peirce those diagrams will be interpreted by their users in many different ways and will be related to "objects" (in the sense of Peirce) also in various ways. In a way, I am analysing here only the sign aspect (representamen) of that Peircean triad "sign, object, interpretant" but apparently this for mathematics is of crucial importance.

From diagrammatic reasoning derives also the absolute reliability and security of mathematics, its so-called logical necessity. This differentiates observation of diagrams also from empirical observation in the natural sciences. Diagrammatic observation "sees" that a certain relationship will hold in all conceivable instances of the respective type of diagrams. This is enabled by the generic character of the (mathematical) diagrams: each single instance or token fully presents the respective type according to an adequate perspective on the token. Like, say, any inscription of the letter "a" presents that letter (as a type of inscriptions under a certain perspective). Finally, it should be emphasized that diagrammatic reasoning is very much different from algorithmic calculations. Though it is rule based it needs creativity and inventiveness like composing music.

Diagrams

Despite here the stance is taken that mathematical development to a great part consists in the design and intelligent manipulation of diagrams no general definition of the notion of diagram is given but rather several examples and descriptive features are presented. Generally speaking, diagrams are kind of inscriptions of some permanence in any kind of medium (paper, sand, screen, etc). Those inscriptions mostly are planar but some are 3-dimensional like the models of geometric solids or the manipulatives in school mathematics. Mathematics at all levels abounds with such inscriptions: Number line, Venn diagrams, geometric figures, cartesian graphs, point-line graphs, arrow diagrams (mappings), arrows in the Gaussian plane or as vectors, commutative diagrams (category theory); but there are inscriptions also with a less geometric flavor: arithmetic or algebraic terms, function terms, fractions, decimal fractions, algebraic formulas, polynomials, matrices, systems of linear equations, continued fractions and many more. There are common features to some of these inscriptions, which contribute to their diagrammatic quality as understood here. But I emphasize that by far not all kinds of inscriptions, which occur in mathematical reasoning, learning and teaching have a diagrammatic quality. Quite a few of what are taken as visualizations or representations of mathematical notions and ideas do not qualify as diagrams since they lack some of the essential features. Mostly this is the precise operative structure which for genuine (Peircean) diagrams permits and invites their investigation and exploration as mathematical objects. On the other hand, diagrams are of such a wide variety that a generic definition appears impossible and impracticable, as well. Accordingly, the various kinds of diagrams in a Wittgensteinean sense are connected by family resemblances and by the ways we use them. Some widely shared qualities of diagrams are proposed in the following:

– diagrammatic inscriptions have a structure consisting in a specific spatial arrangement of and spatial relationships among their parts and elements. This structure often is of a conventional character.

– based on this diagrammatic structure there are rule-governed operations on and with the inscriptions by transforming, composing, decomposing, combining them (calculations in arithmetic and algebra, constructions in geometry, derivations in formal logic). These operations and transformations could be called the internal meaning of the respective diagram.

– another type of conventionalised rules governs the application and interpretation of the diagram within and outside of mathematics, i.e. what the diagram can be taken to denote or model. These rules one could term the external or referential meaning (algebraic terms standing for calculations with numbers, a graph depicting a network or a social structure). The two meanings closely inform and depend on each other.

– diagrammatic inscriptions (can be viewed to) express relationships by their very structure from which those relationships must be inferred based on the given operation rules. Diagrams are not to be understood in a figurative but in a relational sense (like a circle expressing the relation of its peripheral points to the midpoint).

– diagrammatic inscriptions have a generic aspect which permits to construct arbitrary instances of the same type of diagram. This leads among others to consider the totality of all diagrams of a given type (like all triangles, all decimal numbers).

– there is a type-token relationship between the individual and specific material inscription and the diagram which it is an instance of (like between a written letter and the letter as such).

– operations with diagrammatic inscriptions are based on the perceptive activity of the individual (like pattern recognition) which turns mathematics as diagrammatic reasoning into a perceptive and material activity.

– diagrammatic reasoning is a rule-based but inventive and constructive manipulation of diagrams to investigate their properties and relationships.

— diagrammatic reasoning is not mechanistic or purely algorithmic, it is imaginative and creative. Analogy: the music by Bach is based on strict rules of counterpoint but yet is highly creative and variegated.

— many steps and arguments of diagrammatic reasoning have no referential meaning nor do they need any.

— in diagrammatic reasoning the focus is on the diagrammatic inscriptions irrespective of what their referential meaning might be. The objects of diagrammatic reasoning are the diagrams themselves and their already established properties.

— diagrammatic inscriptions arise from many sources and for many purposes: as models of structures and processes, by deliberate design and construction, by idealization and abstraction from experiential reality, etc. And they are used accordingly for many purposes.

— efficient and successful diagrammatic reasoning presupposes intensive and extensive experience with manipulating diagrams. A widespread "inventory" of diagrams, their properties and relationships supports and occasions the creative and inventive usage of diagrams. Analogy: an expect chess-player has command over a great supply of chess-diagrams which guide his or her strategic problem solving. Consequence: learning mathematics has to comprise diagrammatic knowledge of a great variety.

Using Diagrams

Another dimension of explicating the notion of diagram or of diagrammatic reasoning is which uses are made of them in mathematics. First, the most widespread usage is to use the admissible operations and transformations to solve a given task. This comprises calculating a numeric value, solving equations, constructing a proof in geometry, finding a derivation (in formal logic) and many others. Thereby one operates with the inscriptions by exploiting and observing their structure and its changes. Thus, this is a material and perceptive activity guided by the diagrammatic inscriptions. It is like in other material actions: to be successful one has to have acquired an intimate experience with the objects one is operating with, which here are the inscriptions. This is required and not so much abstract or conceptual knowledge. There are algorithmic operations (consider the Gauss algorithm) but much of diagrammatic reasoning is highly creative because the appropriate operations with the diagrams have to be first of all devised and deployed.

This first type of use is the only one which I want to subsume as diagrammatic reasoning. It is essential that the diagrammatic inscriptions themselves are the objects of the activity which produces knowledge about and experience with the diagrams. Second, the other usages of diagrammatic inscriptions I will call representational. The first kind of representational use is when a diagrammatic inscription is taken as a model for some other material or virtual structure from any science including mathematics itself or from any practice. This is captured by terms like application of mathematics or mathematization. It is not the place here to discuss that any further. I only remark that therein lies an important source for the design of diagrams which then within mathematics become the topic of diagrammatic reasoning. A second type of representational use is widespread in mathematics education: to use diagrams as representations for to be (by the learner) constructed abstract objects. The diagrams are taken as a means for mental or cognitive constructions and thus have little interest in themselves. They are then more kind of a methodological scaffold possibly unavoidable but to be dismissed when successful. This is diametrically opposed to diagrammatic reasoning where the focus is on the diagrams themselves as the objects of study and of operations and not on their doubtful mediation with virtual objects. In this representational view mathematics is a predominantly mental activity supported by diagrams whereas mathematics as diagrammatic reasoning essentially is a material and perceptual one. And this does not reduce mathematics to meaningless symbol manipulations since the diagrams have meaning through their structure, their operations and transformations and of course via their applications. This holds for all diagrams as considered here in a way completely analogous to how geometric figures can have meaning.

Observing Diagrams

In this section I will present some examples which hopefully offer to the reader the experience that mathematical proofs in many cases depend on the observation of structural relationships and regularities within transformations of diagrams. Other examples can be found in Dörfler (2004a). In all examples the results of previous "experiments" with diagrams are used as established formulae or "theorems".

There is the surprising result of: 11x11=121, 111x111=12321, 1111x 1111=1234321 etc. This can be "explained" by observing a diagram like the one below. One of the rules used here is the decimal multiplication algorithm which in itself does not predict the observed relationships in the above diagrams. The "understanding" of the surprising results derives from recognizing the pattern of 1's which is produced by the algorithm. The usual common interpretation of the symbols might be helpful but the essential point consists in the perceptive observation of the outcome of one's operations on the diagrams. These would hold even if there were no interpretation of the symbols as numbers. A precondition for this diagrammatic reasoning clearly will be a close familiarity with the diagrams and proficiency in their operations. This possibly sheds new light on the role of "calculations" conceived in a wider sense as intelligent and creative operations with diagrams.

1	1	1	1	1	1	1	x	1	1	1	1	1	1	1
1	1	1	1	1	1	1								
	1	1	1	1	1	1	1							
		1	1	1	1	1	1	1						
			1	1	1	1	1	1	1					
				1	1	1	1	1	1	1				
					1	1	1	1	1	1	1			
						1	1	1	1	1	1	1		
1	2	3	4	5	6	7	6	5	4	3	2	1		

Based on these first observations there is a rich space of further diagrammatic experiments and thought experiments with those diagrams. There is also the possibility of changing the diagrammatic rules, e.g. by choosing different bases for the place value system.

In their famous book "Grundlagen der Mathematik" Hilbert and Bernays analyze operations with arrays of strokes (or points) the observation of which leads to much of what are taken to be properties of natural numbers. The natural numbers are interpreted as types of arrays of strokes two of which are of the same type if they can be matched one by one. Addition and multiplication appear as operations with those arrays which clearly show a diagrammatic character. Properties like evenness and oddness are observable qualities of those diagrams in the form of specific arrangements of the strokes. A good example of diagrammatic reasoning is the statement that the sum of two odd numbers (diagrams) is even. This results from observing the combining of two odd diagrams in an appropriate way. In this kind of diagrammatic reasoning that statement is a way of reporting one’s observations (and not a statement about abstract objects):

********		*********		*****************
*******	plus	**********	gives	*****************

Here the generic character of the diagrams is an important feature which provides the generality of the assertions about the diagrams. Similarly, diagrammatic reasoning by inspection of the following diagram

********		*****		*************
********	plus	****	gives	************

implies the rule “even + odd = odd”. In the same manner the corresponding rules for multiplication are obtained by diagrammatic reasoning with rectangular (product) arrangements.

Also the next example – as the others as well – is well known and only serves the purpose of orienting the attention of the reader to the role of perception, observation, pattern recognition and manipulation of concrete inscriptions as a constitutive part of mathematical thinking.

The young Gauss is reported to have found the sum of the first 100 positive integers by thinking of those numbers as being written down in the following way

1	2	3	4	…	49	50
100	99	98	97	…	52	51

and adding the two numbers in each of the 50 columns to get $50 \times 101 = 5050$ as the required sum. This is very similar to our first example: a certain recognized pattern in a diagram gives the result. Here the generic character (for even numbers) can be seen: a thought experiment with the respective diagram gives the formula $((n/2)x(n+1))$. Further experiments with those diagrams will lead to another more general diagram for arbitrary n, like the following:

1	2	3	4	5	6	7
7	6	5	4	3	2	1

I do not deny that an understanding of the involved symbols as natural numbers is helpful or even necessary for recognizing the relevant pattern. But for the latter a certain regularity, namely constant sum in the columns, is most important, and that is not inherently related to natural numbers. Thus, the diagram is added to the known properties of natural numbers and enlarges the knowledge about them. In a similar way one can analyse many other number patterns like triangular, square, rectangular numbers. In all cases besides symbolic presentations graphic ones using arrays of dots is another kind of diagrammatic reasoning based on experiments with and observation of diagrammatic structures. To that already point names like "triangular", "square" or "rectangular".

Within linear algebra there is a wealth of examples for diagrammatic reasoning. The basic diagrams there are matrices and their operations. Consider

$A=(a_{ij})$ an $(m\times n)$-matrix and $\alpha=(a_j)$ an $(n\times 1)$- matrix(vector). Then the i-th component of the product $A\alpha$ is

$$a_{i1}a_1 \quad + \quad a_{i2}a_2 \quad + \quad \cdots \quad + \quad a_{in}a_n$$

or more detailed the vector $A\alpha=(b_i)$ is given as:

$$\begin{array}{ccccccccc} b_1 & = & a_{11}a_1 & + & a_{12}a_2 & + & \cdots & + & a_{1n}a_n \\ b_2 & = & a_{21}a_1 & + & a_{22}a_2 & + & \cdots & + & a_{2n}a_n \\ & & & & & & \cdots & & \\ b_m & = & a_{m1}a_1 & + & a_{m2}a_2 & + & \cdots & + & a_{mn}a_n \end{array}$$

An empirical investigation of this diagram exhibits a column-wise regularity, which can be expressed as

$$A\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$$

where $\alpha_j=(a_{ij})$ is the j-th column-vector of A. This is a result which is a stringent consequence of the operation rules for matrices and the diagram above cannot be doubted, it is an apodictic argument though (or possibly because of) being based on "pattern recognition".

Once such a "pattern" is established as a formula or theorem, it can fruitfully be used to derive further consequences. Assuming α to be the i-th unit vector $\varepsilon_i\left(a_j=0 \text{ for } j\neq i, a_i=1\right), i=1,\ldots,n$, leads to $A\varepsilon_i=\alpha_i$, which of course can be recognized from other diagrams also. Here it becomes even more prominent that the important thing are the operational rules and not so much the (referential) meaning of the symbols manipulated. We only use our knowledge how to operate with the symbols. But still it is not a meaningless, purely formalistic game: we discover surprising and fascinating relationships for the diagrams. Thus diagrams play here manifold roles. They are, on the one hand, the objects of reasoning properties of which are detected and described (by new diagrams). On the other hand, diagrams are the means for mathematical reasoning by which relationships and regularities become observable patterns.

As another example we study one of the proofs of Cramer's rule for the solution of a regular square system of linear equations $Ax=b; A=(a_{ij})$ an nxn matrix, $x=(x_i)$ the solution vector, $b=(b_i)$ the right-side vector. Then by assumption the inverse A^{-1} (with $AA^{-1}=A^{-1}A=$ identity matrix) exists and from previous diagrammatic operations one knows that $A^{-1}=\left(A_{ji}/|A|\right)$ where $|A|$ is the determinant, and A_{ji} is the cofactor of a_{ji} in A. Then $x=A^{-1}b$ and therefore

$$x_i=(1/|A|)(A_{1i}b_1+A_{2i}b_2+\ldots+A_{ni}b_n)$$

Now $A_{1i}b_1 + A_{2i}b_2 + \ldots + A_{ni}b_n$ is observed to be the result of expanding the determinant of the following matrix A_i by the i-th column

$$A_i = \begin{pmatrix} a_{11} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,i-1} & b_2 & a_{2,i+1} & \cdots & a_{2n} \\ & \cdots & & & & & \\ a_{n1} & \cdots & a_{n,i-1} & b_n & a_{n,i+1} & \cdots & a_{nn} \end{pmatrix}$$

since A_{ji} is just the appropriate cofactor resulting from deleting the j-th line and i-th column in A_i or equivalently in A. Thus $x_i = |A_i|/|A|$. This clearly is recognised by observing invariant patterns when carrying out diagrammatic operations or experiments. For the latter, intimate experience with those diagrams and their previously observed properties is indispensable. Of course, several experimental trials with the diagrams will be necessary before a useful pattern will be discovered. In any case, it is scrutinizing the diagrams, which is at the core of "inventing" the proof. In the hindsight, this might then be presented as the "idea of the proof". We should therefore not expect our students at any level to be able to independently produce proofs without preceding intensive work on the respective diagrams. The reader might interpret this for instance in the case of (Euclidean) geometric proofs. The reader is also encouraged to have a look in a standard textbook on linear algebra and to read some of the proofs under the pretext of diagrammatic reasoning. He/she will observe again and again the importance of observing and recognizing patterns of relationships in the produced diagrams, which are constitutive for the respective proof. Instructive examples are: row rank equals column rank; matrix of a linear transformation; basis change for linear transformations. But of course already the basic properties of the matrix operations are good examples for diagrammatic reasoning.

Reading a finished (diagrammatic) proof demands first of all proficiency in recognizing patterns in diagrams. Devising a proof mostly is based on inventing new diagrams or parts of them. This becomes most clear in geometric proofs in the form of auxiliary lines and figures. Here I will refrain from studying geometric proofs because the diagrammaticity of mathematical reasoning might be more unexpected in other fields. For calculus see Dörfler (2004a).

As another example for a crucial invention I take the standard proof of the Cauchy-Schwarz-Inequality for an inner-product (α,β), i.e. $(\alpha,\beta)^2 \leq (\alpha,\alpha)$ (β,β). One invents a new diagram $(\alpha + x\beta, \alpha + x\beta)$, x any real number, and then observes the transformation: $0 \leq (\alpha + x\beta, \alpha + x\beta) = (\alpha,\alpha) + 2x(\alpha,\beta) + x^2(\beta,\beta)$, which is using the conventional properties of the inner-product. From diagrammatic reasoning with quadratic polynomials one now knows that $b^2 - 4ac \leq 0$ if $ax^2 + bx + c \geq 0$ for all x. And this gives for the above diagram

$$4(\alpha,\beta)^2 - 4(\alpha,\alpha)(\beta,\beta) \leq 0$$

which is the desired inequality.

Clearly, this kind of diagrammatic reasoning presupposes intimate acquaintence with the handling of symbols and with ascribing generality to the respective expressions. But still the diagrammatic operations and their observation adds to all this and constitutes the core of the proof, its stringency and security. Thus, mathematics cannot be reduced to diagrammatic reasoning but the latter is an essential component of its specific quality and character. Specifically, having at hand a great inventory of diagrams, diagrammatic relationships and operations is a precondition for mathematical inventiveness and productive ideas. The latter very often are rich and productive diagrams of some sort. Take as an example the Pascal triangle in combinatorics or possibly simple number relations in the context of developing number sense.

Design

In this section I will present examples for a specific type of proof. It is those proofs which consist in the purposeful design or construction of a certain kind of diagrams or in the proof of the possibility of such a construction. In a sense, those are constructive existence proofs by exhibiting diagrams with the desired property or properties. A simple example is the proof that between any two fractions m/n and p/q there is another one: Assuming $m/n < p/q$ we find $mq < np$ because of $mq/nq < np/nq$. Then for any k between $2mq$ and $2np$ the fraction $k/2nq$ will be a required fraction. Or, the Euclidean proof that for any given set of prime numbers we can find one not in this set is also of that kind.

The next example on the first glance does not give the impression that in essence it is the design of diagrams, which is at the centre of the proof of the theorem. It is the well-known theorem by Kronecker about the existence of roots for polynomials. More technically the theorem reads as follows. For any polynomial $P(x)$ over a field F (i.e. the coefficients of P are elements of F) there is an extension-field F_1 of F where P has a root (i.e. in F_1 there is an element r with $P(r)=0$ over F_1). Thereby one can assume additionally that P is irreducible over F (i.e. P is not the product of two polynomials over F each of degree 1 at least). The proof starts by considering the ring $F(x)$ of all polynomials over F, which can be considered to be a class of diagrams in the sense used here. Then the general construction of the field $F(x)/P(x)$, of $F(x)$ modulo $P(x)$, is employed which can be introduced as consisting of all equivalence classes of $F(x)$ modulo $P(x)$. Thereby $p_1 = p_2(P)$ if $p_1 - p_2$ is a multiple of P in $F(x)$. Denoting by $[p]$ the class of $p \in F(x)$ the field operations on the classes are given by $[p_1]+[p_2]=[p_1+p_2]$ and $[p_1]x[p_2]=[p_1 x p_2]$. The latter definitions have diagrammatic character but the notion of an equivalence class itself is not of a diagrammatic quality. Yet, the whole "construction" can be described easily in a diagrammatic view. In each class of $F(x)/P(x)$ there is a unique polynomial p of degree less than the degree of P. If n is the degree of P then $F(x)/P(x)$ can be viewed as the set of all polynomials $a_0 + a_1 x + \ldots a_{n-1} x^{n-1}$ over F with the usual addition and a certain multiplication. The latter results from $p_1 x p_2$ (polynomial product) by reduction modulo P, i.e. it is the remainder of the division of $p_1 x p_2$ by P. By various diagrammatic manipulations one demonstrates that those operations for those diagrams satisfy all the properties of a field. The field F clearly is contained in the new field $F_1 = F(x)/P(x)$ and thus P can be viewed as being a polynomial over F_1. Among all the diagrams of F_1 there is the special diagram x (i.e. we have $a_0 = a_2 = \ldots = a_{n-1} = 0$ and $a_1 = 1$), and for this diagram we find according to the diagrammatic rules of F_1 that $P(x) = 0$ in F_1 since $P(x) = 1 \cdot P(x) + 0$, i.e. 0 (the zero polynomial in F_1) is the remainder when dividing P by P. But this is just the same as saying that x is a root of P in F_1. To summarize: the proof can be interpreted in a diagrammatic way as the design of a class F_1 of diagrams containing the elements of F for which a sum and a product can be defined such that F_1 is an extension field of F; and in F_1 there is a diagram $r(= x)$ which is a root of P over F_1. The important property of this proof by design is that we can construct a diagram r which is a root of P (this is easy: just say that r has the property $P(r) = 0$) and which is element of an extension field (this is the hard and possibly surprising part). Ontologically, the theorem and its proof are not about abstract objects but about perceivable, observable and materially manipulable objects, viz., the diagrams $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$.

The best known special case of the above of course are the complex numbers where $F = R$ (the real numbers) and $P(x) = x^2 + 1$. Thus the resulting diagrams are of the form $a + bx$, and $x = i$ is a root of $x^2 + 1$ in $F_1 = C$. The product in F_1 results from

$$(a+bx)(c+dx) = ac + (ad+bc)x + bdx^2 = (ac-bd) + (ad+bc)x + bd(x^2+1)$$

which in F_1, i.e. modulo $x^2 + 1$, is $(ac - bd) + (ad + bc)x$. The reader will recognize the usual product in C where we write $a + bi$ instead of $a + bx$. The diagrams in C can be designed more directly, of course, without the use of the polynomials. This proceeds by considering all diagrams of the form $a + bi$, by defining a sum and a product for them based on $i^2 = -1$ (a stipulated diagram again) and by demonstrating via diagrammatic manipulations that thereby results a field. Focusing on the diagrams, their design and their operations instead of looking for "numbers" which are denoted by those diagrams turns this construction into a rational and even perceivable and observable one. The complex numbers thereby loose their common imaginary and mythical quality. Thus the diagrammatic point of view contributes to demystifying mathematics. Of course, there remains the infinity of R which is beyond diagrammatic means. Yet, on the level of C this does not pose specific problems.

To make the design of a root of $P(x)$ and of a field containing it even more transparent I choose the specific case of $F = Z_5$, i.e. the field of residue classes of Z modulo 5 which we denote for the sake of simplicity of writing by 0,1,2,3,4. Consider the polynomial $P(x) = x^2 + 2$, which easily is seen to have no root in Z_5 since the squares in Z_5 are 0,1 and 4 ($x^2 + 1$ would have 2 as a root since $4 + 1 = 0$ in Z_5). The elements of $F(x)/P(x)$ are therefore the diagrams $a + bx, a$ and b in Z_5, which are 25 elements among them all of Z_5 and, for example, $2x,\ 3x,\ 4x,\ 2 + 3\mathrm{x}$, etc. For the sum, we have for example

$(2 + 3x) + (3 + x) = 0 + 4x = 4x$; and for the product

$(2 + 3x)(3 + x) = 1 + 2x + 4x + 3x^2 = 1 + x + 3x^2 = x + 3(x^2 + 2) = x$ modulo P.

The latter more easily is obtained by using $x^2 = -2 = 3$ in Z_5 or better in F_1. It is then a matter of diagrammatic reasoning to convince oneself that those newly designed diagrams with their operations of sum and product have all the properties of a field. Most of them are direct consequences of the respective properties holding in Z_5. For the multiplicative inverse one has to solve the equation $(a + bx)(c + dx) = 1$ with a, b given for $c, d \in Z_5$. If $b = 0$ then $c = 1/a$ and $d = 0$; otherwise $c = a/(a^2 + 2b^2)$ and $d = (-b)/(a^2 + 2b^2)$ (observe that $a^2 + 2b^2 \neq 0$ for all $a, b \in Z_5$ not both zero). In this (finite) case one has a complete survey of all diagrams and there is absolutely no need for abstract objects, which the diagrams possibly stand for. At least in these cases the mathematics is about the writing and manipulating of diagrams according to conventional rules, which derive from specific purposes and intentions, which can be viewed to be a possible interpretant of the diagrams (the signs) in the sense of Peirce. Possibly one has then to take the diagrams as their own objects to complete the triadic sign relationship of Peirce.

A similar analysis could be carried out for many other mathematical "constructions". I just mention some more examples: direct products of algebraic structures (design is the writing of ordered pairs); design of finite geometries; existence of (combinatorial) graphs with certain properties.

Conclusion

I hope the reader has got an idea of what is meant by diagrammatic reasoning and of its power and usefulness in mathematics. But I hasten to emphasize that mathematics cannot and should not be reduced to diagrammaticity. There are powerful ways of mathematical thinking and reasoning which appear to evade diagrammatic methods, see Dörfler (2004b). Of very great interest also for the learning of mathematics possibly is the intricate interplay of diagrammatic and other ways of presenting mathematical ideas, their relationships and differences.

References

Dörfler, W. (2004a). Diagrams as means and objects of mathematical reasoning. To appear in: Developments in mathematics education in German-speaking countries. Selected papers from the annual conference on didactics of mathematics.

Dörfler, W. (2004b). Diagrammatic thinking: Affordances and constraints. To appear in M. Hoffmann, J. Lenhard & F. Seger (Eds.), *Activity and sign – Grounding mathematics education*.

Hoffmann, M. H. G. (2001). Skizze einer semiotischen Theorie des Lernens. *Journal für Mathematik-Didaktik 22*(3/4): 231-251.

Hoffmann, M. H. G. (2002). Peirce's "diagrammatic reasoning" as a solution of the learning paradox. In *The quiet revolution: Essays on process pragmatism* (pp. 147-174). Hrsg. von Guy Debrock. Amsterdam: Rodopi Press.

Peirce, C. S. (1931-1958): *Collected papers, vol. I-VIII*. Cambridge, MA: Harvard University Press.

How Mathematics Teaching Develops Pupils' Reasoning Systems

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Recent theoretical discussions have pinned constructivism and social constructivism against each other. In this paper, I argue that Piagetian constructivism and Vygotsky's social constructivism are coherent and complementary. If we can reach a synthesis of these two theories, we will have a more encompassing approach to analysing how pupils learn mathematics and how mathematics teaching develops their minds. I suggest that the theories are consistent because they are based on the same metaphor of the mind and that they are complementary because they explain different aspects of the development of reasoning. Together they can help us understand the developments in pupils' reasoning systems that result from changes in the thinker (Piaget's contribution) and in the thinker's activity when using different thinking tools (Vygotsky's contribution). In order to develop these ideas, I will first discuss the concept of thinking systems. I will then work with a simple example in mathematics education, multiplicative reasoning. I will first consider the origin of multiplicative reasoning - i.e., the development of the thinker, and then discuss how mathematics teaching can affect pupils' reasoning systems in this domain. To conclude the discussion, I will consider a research agenda for mathematics education based on the conception of thinking systems.

Reasoning systems

Systems theory was applied to reasoning by Piaget and the Russian developmental psychologists in the first half of the twentieth century. They were attempting to solve the same problem and envisaged systems theory as the solution. The problem they were trying to solve was the mind-body problem, posed by the contrast between biological and higher mental functions.

Biological functions are typically carried out by specialised organs. For example, digestion is carried out by the digestive system; breathing by the respiratory system. Such functions involve a constant task performed by the same mechanisms leading to an invariant result. If we consider breathing as an example, the task is to bring oxygen to the cells in the body. This is accomplished by an invariant mechanism: oxygen is received by the blood cells and transported to all the cells in the body. The invariant result is that the cells receive oxygen.

In contrast, higher mental functions are not carried out by a specialised organ but through the co-ordination of different actions. They are carried out by functional systems. According to Luria's definition, in functional systems "a constant task [is] performed by variable mechanisms bringing the process to a constant result" (Luria, 1973, p.28). I will take two of Luria's examples to make this point. The first one is 'remembering'. It is easy to be misled into thinking that we have a specialised organ for remembering: the brain. But Luria points out that remembering involves functional systems rather than a single biological unit. Imagine it is your partner's birthday and you want to remember to buy some flowers before going home. Your task is to remember to buy flowers. You can accomplish this through a variety of means. You can simply repeat this to yourself many times until you think it is now impossible for you to forget. You may tie a knot around your finger: as you don't normally have a string around your finger, this will remind you to buy the flowers. You might write it down to help you remember - on your palm, where it will be very visible. Or on a yellow sticker, for example, and paste it on your wallet. Or you may type it into your electronic diary and set an alarm to go off just before you leave your office. These variable mechanisms can be used with the same end: to recover the information. No single biological unit can account for all the different mechanisms you may call upon.

A second example used by Luria is locomotion. Walking involves the same organs: it is a biological function. But locomotion can be accomplished by variable mechanisms: it involves a functional system. If your aim is to go from A to B, you may walk, swim, ride your bike, drive, or fly. The end result will be to get to B. One of the mechanisms will be chosen for practical reasons. Although it could be argued that all of these mechanisms are under the control of some brain centre that helps us make connections between different places, this is evidently not so. If I fly from London to Tokyo, I might not have the faintest idea of how London and Tokyo relate to each other. When I take a taxi from the airport to the hotel, I do not need to know anything about the spatial relations between the airport and the hotel. Locomotion is not a biological function and many of our target destinations could not be achieved if all we could do was walk.

Higher mental systems are open systems: they allow for the incorporation of tools that become an integral part of the system. When we take notes in order to remember something, writing becomes an essential part of remembering. When we fly from London to Tokyo, the aeroplane becomes an essential part of our locomotion. Vygotsky suggested that what is most human about humans is this principle of construction of functional systems that allow activities to be mediated by tools.

He termed this ‘the extra-cortical organisation of complex mental functions’ to stress that these functional systems cannot be reduced to the brain.

Even the most elementary mathematical activities are carried out by functional systems. Solving the simplest addition problem, for example, involves a functional system. Paraphrasing Luria: we have no specialised organ for addition. If asked to solve the problem ‘Mary had five sweets and her Grandmother gave her three more; how many does she have now?’, a pupil can find the answer through a variety of mechanisms. The pupil can put out five fingers, then another three, and count them all; the pupil can put out just three fingers and count on from five; the pupil can recall an addition fact, 5+3, and use no fingers. If this were a large number, the pupil might decide to use a calculator. These are variable mechanisms that bring the invariant result of finding the answer to addition problems.

For educators, one of the most significant features of higher mental functions is that they are open systems: the variable mechanisms - which are often created through the incorporation of tools - can be replaced by taking into the system something new from the environment. When a mechanism is replaced with something new, the system changes. Children first solve addition problems using their fingers to represent the objects in the problem. The principle used by the pupils’ reasoning system in this representation is one-to-one correspondence: one finger represents one sweet; one counting word is tagged to one finger; the last counting word indicates the number of sweets. When pupils replace the use of five fingers with the word ‘five’ by itself, the system changes: instead of one finger for each sweet, one word represents all five sweets at the same time. This small change has a huge impact on the reasoning system: whereas the pupil has a limited number of fingers, number words continue indefinitely on. A system with fixed limits becomes much more powerful because its limits are removed by a change in tools.

This change - from using fingers to using words - is not simple because it requires refinements of the principles that organise the reasoning system. Fingers represent sweets through one-to-one correspondence but this principle is not sufficient for pupils to understand numeration systems with a base. It has been widely documented that pupils need to understand additive composition of numbers in order to be able to use number systems efficiently. Vygotsky (1978) himself pointed out that forming complex mental systems mediated by tools involves a complex and prolonged process subject to all the basic laws of psychological evolution. Sign using activity by pupils - such as the activity of using numeration systems to quantify answers to problems - is neither simply invented by children nor passed down by adults. Children’s own activities and the signs they know are initially not connected. When they become connected, a major development is accomplished.

To sum up: The points of convergence between Piaget's theory and Vygotsky's theory reside in the use of the same metaphor of mind, the search for a solution to the same problem, the acknowledgement of variations and invariants in thinking systems, and the acknowledgement of qualitative developmental processes that precede the possibility of mediated action. On its own, each theory is incomplete. Piaget had a theory for the development of children's reasoning schemas but did not have a theory about the consequences of acquiring conventional systems of signs. Vygotsky did not have a theory for the developmental processes that precede mediated action but stressed the increased power that conventional systems of signs bring to our reasoning.

The consequences of these gaps are that Piaget's child can understand number but cannot solve numerical problems. To solve numerical problems, we need numeration systems. Vygotsky's child can count but may not know when and how to use counting to solve problems. As mathematics educators, we must bring these two together: we must understand how children organise their actions and help them incorporate new tools into their thinking systems.

The description of simple problems, like the addition problem mentioned earlier on, only gives a glimpse at how this process of co-ordinating pupils' own activities with conventional signs works. Because the problem is so simple, the example conveys the false idea that a reasoning system will inevitably change - and change for the better - when a new mechanism is incorporated into it. Unfortunately, as mathematics educators know only too well, there isn't necessarily a happy end to all stories. An analysis of multiplicative reasoning will illustrate this point.

Multiplicative reasoning

Piaget's (1965) hypothesis was that reasoning about multiplicative situations starts with pupils' use of one-to-many correspondence as an organising principle. One-to-many correspondence encapsulates the concept of ratio or a fixed relation between two variables, which are at the core of multiplicative reasoning. Starting from the Piagetian work, I will argue that mulplicative reasoning cannot be reduced to repeated addition and that successful teaching about multiplicative reasoning should attempt to promote the incorporation of systems of signs into the correspondence reasoning. This is the contribution from Piagetian theory to mathematics education: the identification of a schema of action that forms the basis of multiplicative reasoning. From this starting point, we might ask which systems of signs used in mathematics should be coordinated with this reasoning and what is the best route to accomplish this in the mathematics classroom. The discussion that follows will be based on research as far as possible. But there are many points where only hypothetical answers are possible presently: we do not have the research to answer many of the questions raised here.

One to many correspondence and multiplicative reasoning

Piaget's initial investigations on correspondence can be illustrated quite simply. The interviews with children started with the well-known method of asking the children to take one red flower for each vase. The red flowers are then put aside in a bundle and the child is asked to take one yellow flower for each vase. The vases are then taken away and the child is asked whether there are as many red and yellow flowers on the table. Piaget suggested that this problem is not too difficult for children at about age 5. They can understand that if the number of red flowers is the same as the number of vases, and the number of vases is the same as the number of yellow flowers, then there are as many red as yellow flowers. This is an example of the famous transitive inferences studied by Piaget: if A = B and B = C, then A = C.

Piaget continued this interview by putting all the vases back on the table and asking the children how many flowers would be in each vase if all the flowers were distributed evenly in the vases. Children who succeeded in the preceding question also knew that there would be two flowers per vase. Piaget's final test of children's understanding of correspondence was then to put the flowers away, leaving only the vases on the table, and ask the children to pick up the correct number of drinking straws so that each flower would be placed in one straw. The children could see the vases but not the flowers. In order to solve this problem, the children would have to think: there are two flowers per vase. To take the same number of tubes, I need to take two tubes per vase. Piaget suggested that children at ages 5 and 6 show a good degree of success in these problems.

In the last few years our research team investigated pupils' use of one-to-many correspondence reasoning to solve a variety of multiplicative reasoning problems. Kornilaki and I examined pupils' solutions of multiplicative reasoning problems in action; with Bryant, Watanabe and van den Heuvel-Panhuizen, I investigated solutions to written problems, and with Park I investigated the teaching of multiplication to young children.

Kornilaki (1999) asked young pupils in English schools to solve the following problem: In each of three hutches there are four rabbits; all the rabbits will eat together in a big house; the child's task was to place on the big house the exact number of food pellets so that each rabbit had one pellet. In front of the child was a row of three hutches but no rabbits. Kornilaki observed that 67% of the 5-year-olds and all of the 6- and 7-year olds were able to pick up the exact number of pellets needed to feed the rabbits. The 5- and 6-year-olds had two ways of solving the problems. One route to solution was by establishing a correspondence between the pellets and the hutches, placing four pellets in correspondence with each hutch. The second solution involved counting: the children first determined the number of rabbits by pointing four times to each hutch as they counted, and then took that number of pellets. The 7-year-olds could either use these correspondence solutions or solve the problem through arithmetic, because they had learned multiplication tables. It is significant that the lack of knowledge of multiplication tables did not disadvantage the 6-year-olds in comparison with the 7-year-olds: all the children in both groups were successful. Thus the principle of correspondence was used by the younger children to solve multiplication problems before they learned about multiplication in school; the older children could use a new mechanism, multiplication tables.

In another study, Kornilaki (1999) gave children a slightly more difficult problem: ‘I bought three boxes of chocolate; in each box, there are four chocolates. How many chocolates do I have?’ What makes this problem more difficult is not that it requires a different reasoning schema nor that the problem is about boxes of chocolate: it is that there was no starting point for the children. In the previous problem, Kornilaki gave the children a starting point: she set out in front of the children a row of cut-out paper hutches. The representation of one of the variables facilitates the use of the correspondence schema because the child only needs to create a representation for the second variable. In this problem, the children had no such initial representation of one variable. The problem involves the same numbers, so there is no extra difficulty in terms of counting. But the children have to come up with a representation for both variables on their own. This representational difficulty significantly changes levels of success: 37% of 5-year-olds, 70% of 6 -year-olds, 87% of 7-year-olds and all of the 8-year-olds succeed in this more difficult problem. Because the schema of correspondence is still the solution chosen by the majority of 5- to 7- year olds, the comparison between the results of this experiment and the previous one shows that representing both variables is a considerable step in children's progress. It also suggests a course of action for teachers. It is quite likely that young children can profit from solving problems presented along with the representation for one variable and that they will, in time, come to co-ordinate this activity with the representation of both variables on their own. This would help them progress in their ability to solve multiplication problems as a result from new co-ordinations between reasoning and representations.

Young pupils can also solve such problems when they are presented through drawings because drawings facilitate the use of correspondence. We have presented the following problem to approximately 1000 children in England: ‘In each house in this street (the drawing shows 4 houses) live 3 dogs. Write down the number of dogs that live in this street.’ The proportion of correct responses for 6-year-olds in this problem is approximately 60%. Although it is not as high as the level of success when the children have cut-out paper hutches in front of them, it is remarkable that 6-year-olds, who did not receive instruction on multiplication, can show such high level of success. Figure 1 shows one of the children's productions in a problem where they are asked to draw the number of carrot biscuits necessary to feed all the rabbits inside all the houses: in these productions, the children left no doubt about the mechanisms they used to solve the problem.

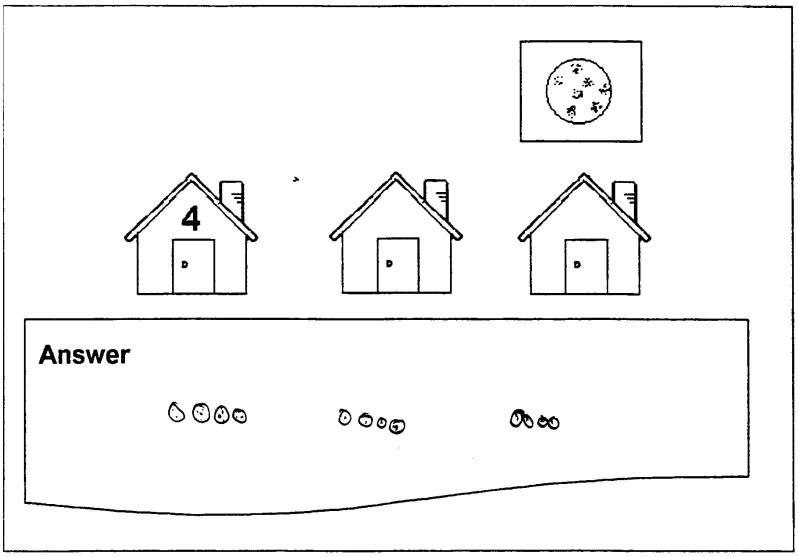


Figure 1. One drawing of the number of carrot biscuits necessary to feed each rabbit in the huts.

Reasoning by correspondences is not restricted to solving multiplication problems. It is also used to solve multiplication problems where information on a factor is missing. There are two ways in which the information about the factors could be missing; we will consider each one in turn.

The first missing-factor multiplication problem would be: ‘I had a party; each child that came brought me three flowers; I got 12 flowers; how many children came?’ In this case, the children know the correspondence - 1 to 3; if they repeat this until the total of flowers is 12, they will know how many children came to the party. This problem, known in the mathematics education literature as ‘quotitive’ or ‘measure’ division problem, is in my view actually an inverse multiplication problem. This is not simply a matter of terminology: if we think about it as an inverse multiplication problem, we have a theory about how the child will solve the problem. In the same way that missing-addend problems are initially solved by children through addition strategies - the child figures out how many to add to one set to arrive at the total - quotitive division problems should be solved by children initially through correspondence. This analysis also allows for some predictions regarding children’s success. Children should show a higher level of success in direct multiplication problems than in the inverse ones. Note that this is not a prediction about division problems in general because sharing division problems are as easy for children as the direct multiplication ones.

A second possibility would be to say: ‘I had a party. Three children came. Each child brought me the same number of flowers. I got 15 flowers. How many flowers did each child bring to the party?’ This problem has the same structure as the problems described earlier on in the sense that it involves a fixed ratio. However, because the ratio is not described, the children will find it very difficult to use the correspondence schema. Thus this is an inverse multiplication problem where we expect much less success.

Kornilaki did find that direct problems were easier than both types of inverse problems and that inverse problems of the type traditionally described as quotitive division were significantly easier than this second type of inverse multiplication question. The rates of success in quotitive division problems were 30%, 50%, 80% and 83%, respectively, for 5-, 6-, 7-, and 8 year-olds whereas in the second type of inverse multiplication problem they were 10%, 30%, 56%, and 80% for the same age levels.

An analysis of children's strategies showed that in the first type of inverse multiplication problem, where the children knew the ratio, the vast majority of the children who solved the problem correctly (83%) did so through correspondence reasoning, either creating an explicit representation of both variables or creating an explicit representation of the groups of flowers while counting the children in correspondence with each group. In the second type of division problem, where the ratio was not known, a correspondence solution could only be implemented by trial-and-error. Only 21% of the children successfully used the correspondence solution whereas about 50% seemed to be able to understand the inverse relation between multiplication and division and actually shared out the total number of flowers into three groups.

Reasoning by correspondences can also be documented amongst young children solving simple proportion problems before they have been taught about proportions in school. The example in Figure 2 shows a problem adapted from van den Heuvel-Panhuizen. It was given to approximately 1,000 pupils in England. Because these were given to whole classes of pupils, it is not possible to describe their strategy.

Only some of the children make marks on the booklets, giving us a clue to their solution process. However, the similarity between the percentage of children solving the missing-factor multiplication problems and these simple proportion problems is suggestive of the use of similar approaches to the solution of the two problems.

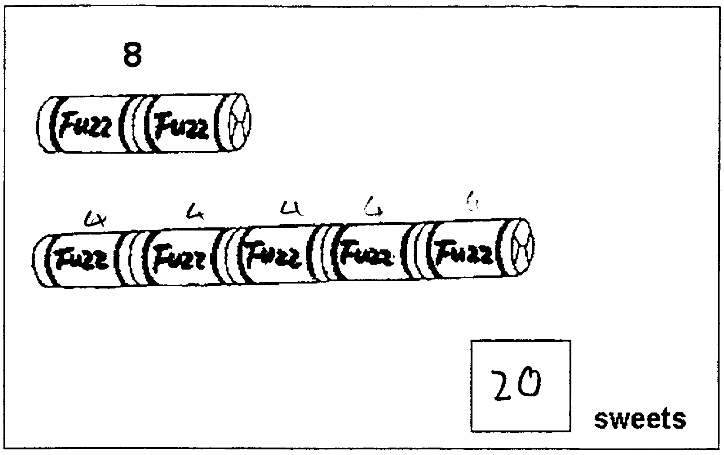


Figure 2. A simple problem involving proportions: the top roll has 8 sweets altogether; how many sweets in the bottom roll?

These analyses of children's solutions to multiplicative reasoning problems suggest that mathematics teaching can develop pupils' reasoning system by helping them co-ordinate their correspondence reasoning with counting. As indicated by Vygotsky, a very significant moment in children's development is that when children co-ordinate their actions with systems of signs. The hypothesis about the development of the concept of multiplication is made more specific by Piaget's theory, which suggests that the schema of correspondence is the crucial action in the case of multiplication.

The hypothesis is by no means trivial: the teaching of multiplication in many countries is based on repeated addition, not on correspondence reasoning. It is noteworthy that the mathematics education literature contains arguments in favour and against the use of repeated addition as the basis for multiplicative reasoning. Whereas Yanomashita and Matsushita (1996) argued that repeated addition is only a means to solve multiplication problems but does not represent its meaning, Fishbein et al. (1985) and Steffe (1994) appear to suggest the opposite. Our hypothesis is based on the analysis of reasoning systems developed here: if the child's multiplicative reasoning is based on correspondences, not on addition, the best way to develop the child's reasoning system is to promote its co-ordination with new systems of signs. It leads to a very specific prediction: that children taught about multiplication through correspondence reasoning will make more progress in solving multiplicative reasoning problems than children taught about multiplication through repeated addition.

We (Park & Nunes, 2001) tested this prediction in an experimental study with 42 children attending two schools in London. The children's mean age was 6 years and 7 months and they had received no instruction on multiplication in school, according to their teachers. The children were randomly assigned to one of two instruction groups: repeated addition or correspondence. The children were pre- and post-tested on a set of mixed additive and multiplicative reasoning problems. We use the phrases ‘additive’ and ‘multiplicative’ reasoning problems, rather than addition and multiplication, because the problems included both direct and inverse (i.e., missing addend and missing factor) problems. During the teaching phase, the children in both groups solved a total of 16 problems, which could be represented by the same arithmetic sentences. For the repeated addition group, the problem was phrased as a sum of two identical sets; for the correspondence group, the problem was phrased as a question where two variables were in a fixed ratio to each other. For example, for the arithmetic sentence 2 x 3, the repeated addition group solved the question: ‘Tom has three toy cars. Ann has three dolls. How many toys do they have altogether?’ The same arithmetic sentence was exemplified in the correspondence group by the question: ‘Amy’s Mum is making 2 pots of tomato soup. She wants to put 3 tomatoes in each pot of soup. How many tomatoes does she need altogether?’

Consistently with our theoretical framework, we expected the children to make different levels of progress in the multiplicative reasoning problems from pre- to post-test: we expected the children in the correspondence group to make significantly more progress in multiplicative reasoning problems than the repeated addition group. This prediction was supported by our results. Although the groups did not differ at pre-test, their performance was significantly different at post-test, with the correspondence group performing better than the repeated addition group in multiplicative reasoning problems. This difference could not be explained by a similarity in the grammatical structure of the problems because the verbal description of problems in the pre- and post-test was varied and did not simply follow the description of a ratio situation. For example, in three problems much of the information was visually presented (see one example in Figure 3).

You can't see all the windows in the front of the building. How many windows are there altogether?

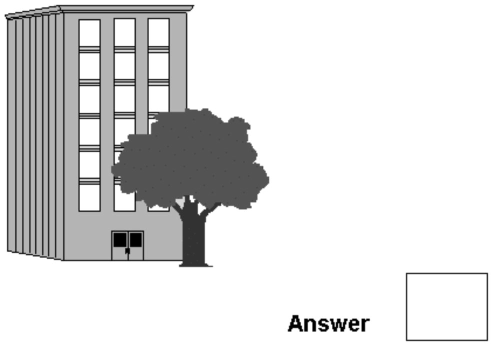


Figure 3. Example of a multiplication problem from the pre- and post-test which does not refer to ratio.

In conclusion, evidence seems to support the idea that children's reasoning schema for multiplicative situations is based on setting correspondences between variables. In order to develop their reasoning systems, mathematics teaching should lead the children to use a variety of mathematical tools in connections with this reasoning schema.

Reasoning by correspondences can create powerful systems for solving multiplicative reasoning problems. Relevant evidence comes from a variety of situations where people solve problems outside school: children selling products in the streets, foremen working out the size of walls from scale drawings, fishermen calculating the amount of processed sea-food from the amount fished (Nunes, Schliemann, & Carraher, 1993), and peasants calculating volume (Soto Cornejo, 1992), all reason mainly by correspondences. They all have replaced the overt actions of a correspondence schema with a new system of signs: instead of setting objects in correspondence, they use number words followed by the quantities indicated. Although they may occasionally make computational errors, they hardly ever make errors in their reasoning.

Three examples are presented here as illustration. The first one comes from the work on street mathematics (Nunes et al., 1993). A girl was selling lemons, which cost 5 cruzeiros (the Brazilian currency at the time) each. Posing as customers, we asked for 12 lemons. She calculated the price by separating out 2 lemons at a time, as she said: 10, 20, 30, 40, 50, 60. She replicated the ratio 2-10 until she reached 12 lemons. Note the similarity in the activity of correspondence, which was carried out in a more powerful way because the 1-5 correspondence was changed into 2-10.

Another illustrative example comes from our work with fishermen. The fishermen we interviewed in Brazil sold the fresh fish they caught to middlemen. The middlemen salted and dried the fish to be sold far from the ocean. In order to know more about their own commercial activities, the fishermen must understand that the quantity they sell to the middleman is not the same quantity sold by the middleman to the customer. When the fish is salted and dried in the sun, there is a loss of weight. The quantity of processed fish is proportional to the quantity of unprocessed fish. The same is true, for example, for the connection between quantity of unprocessed and shelled fish.

A fisherman was told that there is elsewhere a kind of shrimp that yields 3 kilos of shelled shrimp for every 18 kilos that you catch; if a customer wants to buy 2 kilos of shelled shrimp, how much do you have to fish for him? The fisherman calculates: One and a half kilos [processed] would be nine [unprocessed], it has to be nine because half of eighteen is nine and half of three is one and a half. And a half-kilo [processed] is three kilos [unprocessed]. Then it'd be nine plus three is twelve [unprocessed]; the twelve kilos would give you two kilos [processed] (Nunes et al., 1993, p.112). His use of correspondence is quite clear: for each quantity of unprocessed food, he names the corresponding amount of processed food. This is accomplished by performing the same operation on each variable: if one is halved, the other is also halved. This type of solution is known in mathematics education as 'scalar' reasoning, in contrast to 'functional' solutions. In functional solutions, an operator is identified, which can be applied to one quantity to calculate the value of the other one. Functional solutions have not been reported in unschooled adults and are less common than one would expect even in British adolescents (see Nunes & Bryant, 1996, for an analysis).

The third example I take from Soto Cornejo (1992), who interviewed rural workers in the North of Chile. The workers sold wood for processing into vegetable coal by volume. It has been documented often that students have difficulty with the concept of volume and make many mistakes in calculating volume, particularly if there is a decimal point involved in the calculation. Soto Cornejo drew a lorry showing the dimensions 5 meters by 2 meters by one and a half meters, and asked the worker to calculate the volume of its trailer. An illiterate worker reasoned like this: 'First I make a layer one meter high and always five meters long. That will give you five cubic meters. (Note that the layer is an imagined object that he sets into correspondence with the measures of volume) And that two times (the width of the trailer is two meters), that makes ten cubic meters. Now I've got 5 (*sic*) centimetres two times. We will take 5 centimetres (*sic*), this makes five times five, twenty five, that is two and a half cubic meters. The total is ten plus five, fifteen cubic meters.'

Once again we see the use of correspondence reasoning: to each layer corresponds a volume, and the layers are simple (lxlx5) so that the correspondence is easily established. This is a most imaginative way to solve a problem that students find difficult after 7 or 8 years of school.

Multiplicative reasoning out of school is abundant with examples of scalar reasoning; functional reasoning is almost completely absent. My conclusion is that outside school people form powerful reasoning systems for the solution of multiplicative problems by using new symbolic tools from arithmetic instead of manipulatives. But they do not appear to refine the reasoning principles involved in the system to generate functional solutions easily.

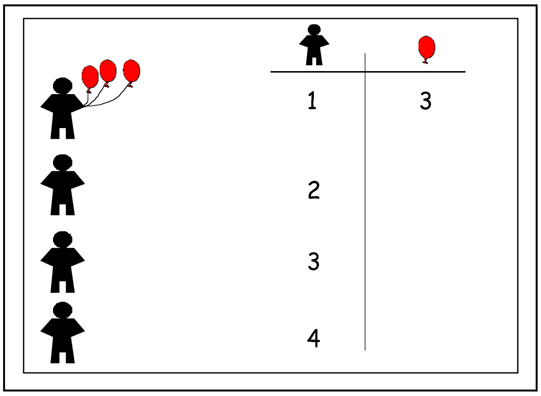
What happens in school? Can mathematics teaching develop pupils' reasoning system?

Piaget considered Vygotsky's position optimistic with respect to didactic intervention: 'One must guard against an excessive bio-social optimism into which Vygotsky sometimes seems to fall' (Piaget, 1962). According to Piaget, teaching will have a positive influence if it is coherent with the pupils' reasoning; otherwise, teaching might actually be ineffective or even lead the pupils astray.

The teaching of multiplication in many countries may not take pupils' multiplicative reasoning into account. In many countries pupils are taught that multiplication is the same thing as repeated addition. And in many countries pupils seem to develop misconceptions about multiplication and have difficulty with proportional reasoning (e.g. Hart, 1988). This analysis of multiplicative reasoning using systems theory offers hypotheses about what could be changed in the teaching of multiplication in order to promote the development of pupils' multiplicative reasoning. Many of these ideas will already be used by teachers. They are not necessarily new ideas in this sense. What is a new outcome from this analysis is a framework that can provide coherence and help choose - and test - effective ideas.

The first one is related to the representation of multiplicative reasoning problems in the classroom. Earlier on I suggested that it is appropriate for teachers of young children to promote the co-ordination of the correspondence schema of action with counting. But what next? How can this schema be translated into paper and pencil representation?

I would hypothesise that the first translation might be into tables that show the correspondences, rather than into arithmetic operations. Figure 4 shows an example of a problem used in the classroom to support primary school pupils aged 7-8 years in developing their multiplicative reasoning. A mixture of figurative and numerical representations was used to strengthen the connection between the schema of action and paper and pencil representation.



Each child has 3 balloons.
Can you draw the rest of the balloons?
Can you write how many balloons there are altogether in the table?

Figure 4. One example of the use of tables in the teaching of multiplication.

A second co-ordination with symbolic systems may be to represent the problems through graphs. With the same group of primary school pupils, graphs were used after they had worked with tables for a few lessons. The teacher was initially sceptical about the possibility of working with graphs with such young children. After the sessions, he was enthusiastic. A small scale pilot study in the school showed that the children participating from this programme made significantly more progress than the control children from pre- to post-test, although the intervention only lasted a few weeks. A more detailed study is still needed.

The consequences of introducing the idea of graphs in close connection to the solution of multiplication problems can only be speculated about. The difficulties that pupils have in establishing connections between graphs and functions may be much less important if they learn about graphs in the manner suggested here. Algebraic representation of graphs and tables could be the third type of representation used in this context, which could be introduced perhaps through the explicit representation of the constant ratio in graphs and tables. But our explorations have not gone that far yet.

Conclusion

I conclude with a research agenda, rather than a solution. The use of systems theory helps us understand how reasoning systems become more powerful through mathematics teaching. But we know that progress is not an automatic result. In order to be certain that we are building more powerful reasoning systems in the classroom, we must investigate which principles are essential for the system to work well, the variable mechanisms or tools that teachers can insert into the reasoning system, and what type of changes we expect to accomplish. I suggest that in many cases we should not be satisfied with the simplest changes, where the same principles are used without the refinements that can result from the incorporation of new tools. If we are able to understand the principles used by pupils in the organisation of their reasoning, we should also be able to examine which refinements they need, and investigate ways of promoting these refinements in the classroom.

References

Hart, K. (1988). Ratio and proportion. In J. Hiebert & M Behr (Eds), *Number concepts and operations in middle grades*. New York: Macmillan.

Hoyles, C., Noss, R., & Sutherland, R. (1991). The ratio and proportion microworld. Final report of the Microworlds Project. Institute of Education, University of London.

Kornilaki, K. (1999). Young children's understanding of multiplication: A psychological approach. PhD thesis, Institute of Education, University of London.

Luria, A. (1973). *The working brain*. Wandsworth (UK): Penguin.

Nunes, T. & Bryant, P. (1996). *Children doing mathematics*. Oxford: Blackwell.

Nunes, T., Schliemann, A. & Carraher, D. (1993) *Street mathematics, school mathematics*. New York: Cambridge University Press.

Park, J. & Nunes, T. (2001). The development of the concept of multiplication. *Cognitive Development, 13,* 1-11.

Piaget, J, (1965). *The child's conception of number*. New York: Norton.

Piaget, J. (1962). Comments on Vygotsky's critical remarks concerning *The Language and Thought of the Child and Judgement and Reasoning in the Child*. In L. Vygostky, *Thought and language*. Cambridge MA: MIT Press.

Soto Cornejo, I. (1992). Mathématiques dans la vie quotidienne des paysans chiliens. Faculté de Psychologie et des Sciences de l'Education, Louvain-la-Neuve, Belgium.

Vygotsky, L. S. (1978), Mind in society. In M. Cole, V. John-Steiner, S. Scribner & E. Souberman (Eds.). *The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Watanabe, A., Nunes, T., Bryant, P., & van den Heuvel-Panhuizen, M. (2000). Assessing young children's multiplicative reasoning. *Newsletter of the Developmental Psychology Section the British Psychological Society, 55,* 7-15.

Yamonoshita, T. & Matsushita, K. (1996). Classroom models for young children's mathematical ideas. In H. M. Mansfield, N. A. Pateman, & N. Descamps-Bednarz (Eds.), *Mathematics for tomorrow's young children: International Perspectives on curriculum*. Dordrecht: KIuwer Academic Press.

Assessing Students' Knowledge – Language in Mathematics Tests

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The theme of this conference is mathematics and language. In my talk about the national tests in mathematics for the compulsory school with a language perspective, I will focus on three important concepts relevant to language and assessment – authenticity, feedback and documentation. I will relate the concepts to the old and to the new test systems for different school years.

Authenticity	Old system	New system	Year 9	
Feedback	Old system	New system	Year 9	
Documentation	New system	Students	Teachers	Test instructions

Authenticity – Are the tasks more authentic now, in comparison with the tasks in the old system? What can be said about problem contexts and the formulation of questions?

Feedback – What kind of feedback to the students and to the teachers is there in the old system and in the new one? And in what way do we express the feedback from the results of the national tests?

Documentation – How do the teachers document their students´ knowledge? In this aspect – documentation – I give examples only from the new system. What kind of words do teachers and constructors of tests use when they describe different qualities in the students´ solutions?

So this is the agenda for this lecture. But let me first introduce the two different test systems.

The old system

From the middle of the 1940s until the middle of the 1990s, the grading system in Sweden was norm-referenced. At the beginning this was the case only for school years 2, 4 and 6 in the compulsory school. Later there were compulsory achievement tests in mathematics only in school year 9 and in year 3 for two of the course programmes in the upper secondary school. The aim of the tests was to get maximum uniformity in teachers' grading, throughout the country. The teacher should use the results of the test to obtain information about the average level of the class as well as the distribution of individual grades in the class, all in relation to classes across the country. The test gave information as to how many students in a class should have the different grades, but the test did not inform the teacher which student should have a special grade. These achievement tests were used for a group-referenced grading system, i.e. the knowledge and achievement of all the students in the country who belonged to a particular category and studied the same course were compared with each other. In this system it was very important that the tests were objective in the sense that all teachers could assess the students' solutions in exactly the same way. The test items had to have a high discrimination index and had to provide variation in difficulty. The result of the test was the guiding factor for the teacher in the grading of the students.

The new system

A new national curriculum for compulsory school and for the upper secondary school came into effect in the autumn of 1994. It defines the underlying values, basic objectives and guidelines of the school system. In addition, there is a nationally defined syllabus for each individual subject. The compulsory school syllabi indicate the purpose, content and objectives for teaching in each individual subject. These are of two kinds: those that must be achieved, and those for which it is the duty of schools to give all students a reasonable chance of achieving. Some examples of goals are the following:

The school should strive to ensure that all pupils

- develop a sense of curiosity and a desire to learn, develop their own individual way of learning
- develop confidence in their own ability
- learn to listen, discuss, reason and use their knowledge as a tool to
  - formulate and test assumptions as well as solve problems
  - reflect on experience and
  - critically examine and evaluate statements and relationships
- take personal responsibility for their studies …
- develop the ability to assess their results themselves and to place their own assessment and that of others in relation to their own achievements and circumstances

Examples of goals to attain in the compulsory school:

The school is responsible for ensuring that all pupils completing compulsory school

- have mastered basic mathematical principles and can use these in everyday life
- can use information technology as a tool in their search for knowledge.

To coincide with the introduction of the new curriculum and syllabi, a new grading system has come into effect. Under this system, grades are awarded on a three-grade scale from the eighth year of schooling onwards. The grades are

Pass, Pass with Distinction and Pass with Special Distinction. In the upper secondary school the grade Failure is added and the students in upper secondary school are graded after every course. The grading is goal-related; i.e. the grades relate students' knowledge and achievement to the goals set out in the syllabus. Only the teacher awards grades.

At the end of school year 9 national tests are held in the three subjects Swedish, English and mathematics in order to assess students' level of achievement. The tests provide support for teachers in awarding grades. The testing of mathematics for grade 9 consists of both traditional tasks and more open ones. In one part of the test, calculators are not allowed. Depending on the nature of the task some tasks are more atomistic and some are more holistic. The holistic ones are assessed with the help of assessment matrices. Often we also have an oral test for grade 9.

There are tests in these same subjects at the end of school year 5, but it is not compulsory for the municipality to use them. The main purpose of the subject test for school year 5 is not only to check that the students have achieved the requirements of the curriculum and syllabus. They also have a diagnostic purpose. In the test material there is also a scheme for self-assessment. The teacher is advised to integrate the subject test within the ordinary teaching, the intention being that both the ways in which the student has worked with the problem as well as the answer will be taken into consideration. There are tasks for both individual work and for group work. To help teachers to describe the mathematical knowledge of the student they may use a proposed Competence Profile. The teacher should then consider both their assessment of the student's work on the subject test as well as their overall assessment of the student's mathematical knowledge. Our hope is that the teachers can, with the help of the profile, gain a more balanced picture of the student's knowledge in mathematics. For the test in school year 9 there is a similar profile, but only referring to the test - a test profile.

The diagnostic materials in mathematics consist of two parts, one part for use in pre-school and up to grade 6 and one part from grade 6 to grade 9. Each part consists of a scheme for analysis and diagnostic tasks. The purpose of the materials for analysis is to help teachers analyse and document the students' knowledge in mathematics. The same scheme is to be used for students of different ages. When using the scheme it will show the student's development over several years. Students are allowed to express their knowledge in different ways: action, pictures, words and symbols. Three different areas are focussed upon in the first scheme, Measuring and spatial sense, Sorting, Tables and diagrams and Number sense. In the second scheme for grade 6-9 we focus on Measuring, Spatial sense and geometrical relations, Statistics and probability, Number sense and Patterns and relations.

The following overview shows the different test materials for the compulsory school in Sweden.

Diagnostic materials Not compulsory	Pre-school and up to grade 6	A booklet for analysis A booklet with tasks
	From grade 6 up to grade 9	A booklet for analysis A booklet with tasks
Subject test Not compulsory	Grade 5	4-5 different parts + group-tasks and self-assessment
Subject test Compulsory	Grade 9	2-3 different parts + group-tasks and/or oral test

The National Test is not meant to steer teachers in their grading but rather to help them to assess whether and how well the individual student has reached the goals for the subject. The starting point for the construction of a test is the view of knowledge expressed in the curriculum and the view of the subject in the syllabus as well as the criteria for the different grades. With the new test the teacher cannot determine the level for his class as a whole compared with other classes in the country. It is important to have tasks involving different content areas of the subject. The problems in the material should be designed in such a way that the student has the opportunity to show different competencies in mathematics. The new national test works in a goal and knowledge context. It has in part other demands than that of a norm-referenced test. The new national test must consist of more varying tasks and the students must have opportunities to show their competence in different ways. Mathematical competence is so much more than merely knowing certain mathematical content and skills. It is also essential to communicate knowledge and to present mathematics in written, oral, visual and symbolic forms. In addition it is important to use mathematical strategies, models and methods within one’s present knowledge and skills to create new skills and methods utilising a range of facts, concepts and processes.

Authentic tasks – the old and the new system, year 9

In the old system we had a lot of short tasks but also different themes in one part of the test. The themes could be

- Sri Lanka
- Traffic and Environment
- Sports Booth

In the new system we also have a lot of short tasks and different themes but also more extensive tasks and tasks for pairs/groups, such as

- Bus Frequency
- Inheritance
- Perimeter

My conclusion is that the contexts of the tasks are not more authentic now compared with the tasks in the old system, but the way of posing questions is more authentic in some tasks in the new system.

In the old system we asked things like

- How many …
- Compute…
- What is (the price)…
- How high….
- How much …
- When….
- Estimate…

In the new system we also have questions like

- Why
- Explain
- Investigate
- Describe

Some examples of tasks from the new system (school year 9):

Example 1: There are 11 people working in a company. Their monthly salaries are:

15 000	13 000	47 000	15 000
13 000	55 000	15 000	13 000
16 000	16 000	13 000	

Work out the average and the median monthly salary. Which measure – average or median – best reflects the salaries of the group? State the reasons for your choice and explain why you believe the other measurement is not as good.

Example 2: Perimeter

In this task you will be working with four different geometric figures. All figures must have a perimeter of 12 cm.

You should work with the following geometric figures:

- a rectangle, where the length is twice as great as the width
- a square
- an equilateral triangle
- a circle.

You should study and compare the areas of the figures. What conclusions can be made?

> When assessing your work, the teacher will take into account the following
> - how clearly and correctly you have drawn the figures
> - whether you have made the correct calculations
> - how well you explain your workings and methods
> - how well you have stated the reasons for your conclusions.

Example 3: Hassan says, ”An increase from 40 to 80 is a 100 % increase”. Amir says, ”Then a decrease from 80 to 40 is a 100 % decrease”. Who is right and who is wrong? Explain in both cases why it is right or wrong.

Feedback of test results – the old and the new systems, year 9

We can illustrate the comparison of working in the old and the new systems in the following way:

The old system	*The new system*
1. Construction of the test	1. Construction of the test
2. The students work with the test	2. The limits for the three grades are established
3. The teachers assess and send results back	3. The students work with the test for standardizing
4. The interval for grade 3 is established and sent to schools/teachers	4. The teachers assess the students’ work.

In the new system the teachers can, if they so wish, give the students more feedback about their knowledge than in the old system.

The main difference between the two systems is that the teachers receive the grades together with the test, rather than after the students have worked with the test. In the old system the teachers got feedback with a total sum and an interval for school year 3, a mean and a standard deviation. In the new system they get a sum of different “Pass-points” and “Pass with distinctions-points” but also an assessment matrix. With the assessment matrix the teacher can discuss with the students how he/she has solved the tasks, the strengths and weaknesses in his/her solution. (See Appendix)

Feedback in the new system – year 5

Even for year 5 the teachers have the opportunity of giving students feedback by means of a competence profile, where the teacher together with the student can make notes.

Another way to get feedback for the teacher and the students is to let the students assess their knowledge by themselves. Here are some questions from the self-assessment scheme

How do you feel when doing the following?	**Very sure**	**Pretty sure**	**Unsure**	**Very unsure**
Estimating approximately how long a bus is.				
Looking in a newspaper to see how long a TV programme is.				
Working out 8 – ___ = 3				
Deciding which number is greater – 3.8 or 3.14.				
Working on tasks other than those you are used to.				
Explaining to a classmate how you solved a task.				
Working with someone else.				
Working on your own.				

Another form of feedback is for the students to answer questions about mathematics:

1. What do you think you are good at in mathematics?
2. What do you think you need to practise more in mathematics?
3. Give examples of one or several tasks in the test, which you think were good. Explain why.
4. When do you think you learn mathematics best?
5. Write more about yourself and mathematics.

Documentation in the old and the new system

As we have seen, teachers could document students' results on the national tests in the old system as a sum of points. In the new system, they can document the results as different kinds of points but also by notes in an assessment matrix. In the new system we also have diagnostic material. I would like to leave the tests for a while and talk about the diagnostic materials. This material consists of two different parts – an analysis scheme and a part with problem tasks. The two parts focus on different areas. Let's look at the analysis scheme for measurement and spatial sense.

Some examples of rubrics in the scheme:

Shows self-confidence in own ability
Shows pleasure, interest etc.
Takes responsibility for his/her learning
Deals with and solves problems
Uses ”Measurement and spatial sense”
Words in common usage, understands words like longer, heavy, greatest etc
Basic spatial sense
Maps and drawings
Geometrical objects
Patterns
Symmetry
Length
Volume
Mass (weight)
Area
Angles
Time

What kinds of words do teachers use in describing students' knowledge? About 50 teachers from pre-school up to grade 8 in a municipality in Sweden have used the scheme. Here is a summary of their documentation.

Shows self-confidence in own ability
Here the teachers often write adjectives, like “positive, interested-uninterested, happy-unhappy, curious, engaged, immature, certain-uncertain, aktive-inactive”. Other teachers use verbs such as “takes responsibility for his own work and for group tasks, leads the group work, tries willingly, takes responsibility for planning, wants to be noticed and recognized, wants to have a lot of help, does homework, needs a lot of support”

In the other boxes we can have three different categories, the teachers who describe “Know”, “Know and what”, “Know and where”:

”Know”: No problems, Yes, OK, recognise, have understanding, participate, explain, compare, tell, describe, understand concepts, and understand hour and half an hour
”Know and what”: gives correct names to triangle, rectangle, size buttons, uses ruler
”Know and where”: making bread, doing woodwork and handicrafts

Some of the teachers also write what the students do not know: have difficulties with, mixed different concepts etc.

An example of a teacher’s documentation:

> The student (9 years old) saw a pattern in the carpet. He calculates how many squares the carpet consists of. He first uses his fingers to estimate the squares and uses this measure to calculate how many squares the carpet consists of.

Conclusion

Language is important for assessment. It is important that we use the same words for the same things, so we can understand each other. It is important that we study what words and concepts the students use when they show their knowledge and what words and concepts the teachers use when they assess the knowledge. It is also important for how the student experiences the assessment.

Assessment can be stimulating and supportive for learning. Assessment is not only a ”receipt” for knowledge displayed, but also influences an individual’s learning, his/her self-esteem and confidence in his/her knowledge. Assessment, if relevantly used, can provide a great potential for learning. But what does assessment mean for the individual? The consequences of assessment can be illustrated by the following figure:

An assessment that supports and stimulates learning means that a student’s knowledge is analysed, evaluated and expressed in such a way that the student progresses in his/her learning and feels self-confidence in his/her own ability (I can, want to, dare to), instead of an assessment that leads to a judgement and perhaps a condemnation (I cannot, do not want to, dare not).

Appendix

Assessment matrix

Problem solving capability
Comprehension and method
The assessment concerns: To what degree the student shows an understanding of the problem. What strategy/method the student chooses to solve the problem? To what extent the student reflects upon, and analyses the chosen strategy and the result. The quality of the student's conclusions. What concepts and generalisations does the student use?

Accomplishment
The assessment concerns: How complete and how well the student carries out the chosen method, makes necessary calculations and explains and defends the reasoning in the solutions.

Communication capability
Mathematical language and/or representation
The assessment concerns: How well the student uses mathematical language and representation (symbolic language, graphs, illustrations, tables and diagrams).

Clarity of presentation
The assessment concerns: How clear, distinct and complete the work of the student is. To what extent the solution is possible to follow.

	Qualitative levels		
Comprehension and method	Shows some understanding of the problem, chooses a strategy, which functions only partially.	Understands the problem almost completely, chooses a strategy which functions and shows some reflective thinking.	Understands the problem, chooses if a general strategy possible and analyses his/her own solution.
Accomplishment	Works through only parts of the problem or shows weaknesses in procedures and methods.	Shows knowledge about methods but may make minor mistakes.	Uses relevant methods correctly.
Mathematical language and/or representation	Poor and occasionally wrong.	Acceptable but with some deficiencies.	Correct and appropriate.
Clarity of presentation	Possible to follow in parts or includes only parts of the problem.	Mostly clear and distinct but might be meagre.	Well structured, complete and clear.

Panel Discussion on

Mathematics and Language

Christer Bergsten, Håkan Lennerstad, Norma Presmeg, Åse Streitlien

Introduction

Christer Bergsten, Linköpings universitet

When the theme *Mathematics and Language* was decided for this seminar it was to recognise that there are many important, and some far from obvious ways that language and mathematics go together but also to open up for new aspects of using linguistic tools to study mathematics and mathematical thinking and learning. The first critical issue is the one on definition - what do we mean when we use the word *language* in connection to mathematics? When using the expression *the language of mathematics* it is often the algebraic (or symbolic) language that is being referred to. To pinpoint this language aspect of mathematics the term *Mathematish* was used at one of the paper presentations at this seminar: Lennerstad & Mouwitz (this volume) claim that mathematical texts are bilingual in its mixed use of natural language (such as English) and Mathematish. Being imprisoned by language when analysing the use of language in mathematics, we risk to get caught in the self-reference trap, like at the huge efforts at the beginning of the last century to use logic to study the logical structure of mathematics. In relating language to mathematics education, one path to follow is to study language as a structure, another is to study the use of language. The latter will be in focus when analysing communication in the classroom – this was the direction taken by Åse Streitlien in her opening presentation at the plenary panel on discourse patterns in a primary mathematics class. To this the aspect of meaning was added by Håkan Lennerstad, which led into the study of language also as a structure – why don't we teach the grammar of Mathematish? In order to analyse both the use and the structure of language in mathematics education, Norma Presmeg gave some examples of how the ideas of semiotic chaining had helped mathematics teachers to reflect on and develop their practice. The presentations were followed up by questions from the audience, where only time gave the limit, showing that language is an important and deep issue in mathematics education. In the following the written versions of the introductory contributions of the three panel members are presented.

Remarks on Mathematics and Languages

Håkan Lennerstad, Blekinge tekniska högskola

Introduction

For students (and pupils), use of language in mathematics may have at least four meanings:

1. Use of personal vocabulary, every day experiences, images and associations to metaphorically handle mathematics problem solving and expressing mathematics activity and meaning, including mathematical dialogues.
2. Use of proper and logical natural language when writing solutions to mathematical problems.
3. Use of the mathematical formal language (numbers, equations, formulas…), a language that we may call Mathematish (see the chapter “Mathematish – a tacit knowledge of mathematics”, this volume).
4. Communication in mathematics with pupils with a different mother tongue.

In these remarks, 2 and 4 are not commented. The connection in freedom and confidence between 1 and 3 is focused.

In the aim of inviting students’ authentic thinking about mathematics, which has its reflection in teachers displaying mathematical culture, we will be lead to a fifth sense of language in mathematics. This one concerns teachers more than pupils:

5. Teacher’s use of own personal vocabulary, every day experiences, images and associations to metaphorically handle mathematical problem solving and expressing mathematical activity and meaning, including mathematical dialogues, in communication with students.

Is students’ authentic questioning present in mathematics classes?

One may ask: in which way are students present in a mathematics class? Perhaps the most common types of presence are listening to the teacher, calculation according to the textbook, some mathematical dialogue, and various non-mathematical dreaming or chatting.

Students’ authentic questioning about pieces of mathematical calculation is probably relatively absent. How calculations can be done, why they are efficient, what they mean, and comparisons to experiences and instances in other subjects where they may be useful. This may be regarded as independent thinking from personal perspectives. It can also be described as a philosophical attitude, that may produce meaning and context, partly because of pieces of mathematical culture that the teacher then may find relevant. Independent use of mathematical formal language is also uncommon by students, verbally or in writing, suggesting a status as a foreign language that is not much internalised.

Some teachers manage to inspire such independent thinking. It appears to be particularly absent in the subject of mathematics, being considered by many students as an “alien subject”.

Mathematics as anyone’s toolbox

If we, conversely, regard mathematics as a way of thinking that is natural for humans, mathematical formulas belong to everyone, and contain meanings that also belong to everyone. This is inconsistent with the common view that “mathematics is a subject you meet in school”. Contrarily, everyone meets mathematics during the first years in life, and develop individual intuitions. In school, most children for the first time meet the systematic and official language of mathematics.

If students consider mathematical formulas and meanings as belonging to everyone, it is natural to experiment and to question statements in the language of mathematics. This contradicts a view of an “alien subject”. Such experimenting is of course naturally done using the mother tongue.

What is mathematics for mathematicians?

Mathematical formal language, Mathematish, has been developed and constructed during the last few hundred years. This language has been a very powerful tool in the development, and has come to be essential in many natural sciences and technical areas. Mathematicians do not need much different descriptions of mathematical ideas than in this specialized language.

However, this strong dominance of Mathematish is perhaps not effective for everybody. Since the subject to a large extent still is described by its architects, i.e. the mathematicians, we may presently live with a description of mathematics that is not appropriate for many students.

However, like students, mathematicians struggle with mathematics. This struggle produces images and drama, which sometimes take the form of a mathematical culture. On the other hand, in the absence of mathematical culture, students often regard the teacher’s perfection at the black board to be the appropriate mathematical attitude. For students with that view, any struggle with mathematics may be strongly disappointing.

Mathematics should be actively constructed by students and teachers

Thus, in terms of mathematical meaning, there is a common ground between student’s authentic questioning, philosophy and mathematical culture. Note that mathematicians have different opinions about meanings of formulas, they agree only about the question of true or false. Students and the teacher in a class should not feel dominated by the official mathematical description, but should consider this as a resource and feel that they are allowed to construct appropriate meanings of mathematics and its activity, while doing it. Today's official description of mathematics is not enough.

The excluding Mathematish

It seems like a large part of this cultural problem lies in the extremely efficient but sometimes excluding invention of mathematics: its language Mathematish. If you feel unsafe in the language, much courage is needed to make any statement whatsoever. This is a powerful and invisible barrier for dialogues with students. Unfortunately, it is a natural invisibility in the sense that languages are, in a certain sense, naturally invisible. We are good at using our mother tongue, but not at describing its rules. For languages learned later in life, typically the converse is valid. Perhaps many students in mathematics feel that they meet a foreign language they are supposed to understand but don't know how to handle. In the absence of authentic dialogues, many students may see no alternative but to imitate formal activities during the years in school.

Mathematical knowledge: calculation, ways of calculation, and meaning

One may describe mathematical knowledge as being of three kinds: to perform correct calculations with numbers and formulas when ways of calculations are given, to see possible ways of calculation in more open problems (essential for mathematicians), and applications and interpretations of formulas and concepts. Only the first of these can be efficiently programmed in computers. The last two kinds cannot be formalized, but, being essential, should anyway be described extensively. They are essential for the success and meaning of the Mathematish practice (calculation).

The present dominating way of working mathematics education, to which today's teachers naturally belong, is very Mathematish-dominated, and neglects the last two types, for which (at least for ways of calculation) there are mainly poetic, personal and metaphorical ways of expression. Today's teachers have developed ways of calculation and ideas of meaning, which we may not need to express in order to see meaning, since the symbols themselves for teaches carry this meaning. The present educational situation can be expressed by a common saying: "mathematics is easy to teach but difficult to learn". This may be seen as expressing the profound difficulty for teachers in reaching students' mathematical thinking in true dialogues – at least in a strongly Mathematish-dominated practice.

Force of change: teacher's courage in personal mathematical reflection

Central for students' authentic questioning is teachers' reflective practice, partially breaking the present practice. Of course, children during life naturally develop abilities to reflect and make choices. In the absence of teacher's reflection, children naturally conceive school not as a place to reflect, but to learn. Therefore, teachers' use of personal vocabulary, every day experiences, images and associations to handle mathematical problem solving and express mathematical activity and meaning, may inspire and allow dialogue as well as similar activity among students in mathematics classes.

Semiotics as a theoretical framework for mathematics and language

Norma Presmeg, Illinois State University

> "The reasoning of mathematicians will be found to turn chiefly upon the use of likenesses, which are the very hinges of the gates of their science. The utility of likenesses to mathematicians consists in their suggesting, in a very precise way, new aspects of supposed states of things" (Peirce, 1998, p. 6).

Some of the originators of theories of semiotics were linguists. Ferdinand de Saussure's (1959) book, *Course in General Linguistics*, is a seminal work in this area. And Charles Sanders Peirce, himself fluent in Latin, Greek, and several other languages, makes it abundantly apparent in his writings (e.g., 1998, Vol. 2) that semiotics under girds and illuminates the study of languages and their structure. Why, then, is semiotics, defined as the study of semiosis (activity with signs), useful to mathematics educators? A hint of an answer to this question is given in the initial quotation from Peirce, and in this paper I illustrate semiotic aspects of metonymy and, in particular, metaphor, showing the relevance of "the use of likenesses" in the learning of mathematics. In a triadic model of nested signs based on the formulation of Peirce, the categorization of signs as iconic, indexical, or symbolic relates to the uses of metaphor and metonymy in semiosis. I have found these constructs to be powerful lenses in my research, both on ways of connecting home activities of students with formal mathematical concepts in school and college (Presmeg 2002), and in understanding the ways that signs support learning of mathematics at all levels. After an initial description of this triadic nested model of signs and their uses, I illustrate how metaphor and metonymy are implicated in the model, and its use in linking mathematics in and out of school.

Pierce's triadic model of semiosis, in the United States (Peirce, 1992), had its counterpart in the Swiss structural approach of Saussure, who defined the sign as a combination of a "signified" together with its "signifier" (Saussure, 1959; Whitson, 1994, 1997). Lacan inverted Saussure's model, which gave priority to the *signified* over the signifier, to stress the *signifier* over the signified, and thus to recognize "far ranging autonomy for a dynamic and continuously productive play of signifiers that was not so easily recognized when it was assumed tacitly that a signifier was somehow constrained under domination by the signified" (Whitson, 1994, p. 40). This formulation allows for a chaining process in which a signifier in a previous sign combination becomes the signified in a new sign combination, and so on. An example from Hall's (2000) dissertation research –

which built on my initial work in this area – illustrates these processes. (For further examples, see Cobb et al., 1997, 2000; Presmeg, 2002.)

An example of use of semiotic chaining in mathematics education

Using a semiotic chain, a sequence of abstractions is created while preserving the important relationships from the everyday practices of the students. This chain has at its final link some mathematical concept that is desirable for the students to learn. Using this process, a teacher can use the chain as an instructional model that develops a mathematical concept starting with an everyday situation and linking the two. The example in figure 1 is a chain that was developed and used with three practicing fourth grade teachers (Hall, 2000), for the purpose of exposing them to this notion. In three phases, Hall gave the teachers increasing autonomy to construct their own chains based on the experiential realities of their students, and to use these chains in their classroom practice. The everyday practice in the figure 1 was chewing gum and the mathematical concept being developed was base five addition.

This chaining process involves metonymy - as indeed all signifiers are in a sense metonymic in a semiotic model (Presmeg, 1998a), since they are “put for” something else - and also reification, since each signified in turn is constructed as a new object (Sfard, 1991) that is symbolized by the new signifier. Chaining thus casts light on both processes as they are implicated in the construction of mathematical objects. Changes in discourse in moving through the chain also exemplify the negotiation of mathematical meaning through social interaction that Sfard (2000) and Dörfler (2000) both regard as central in “symbolizing mathematical reality into being” (Sfard’s title).

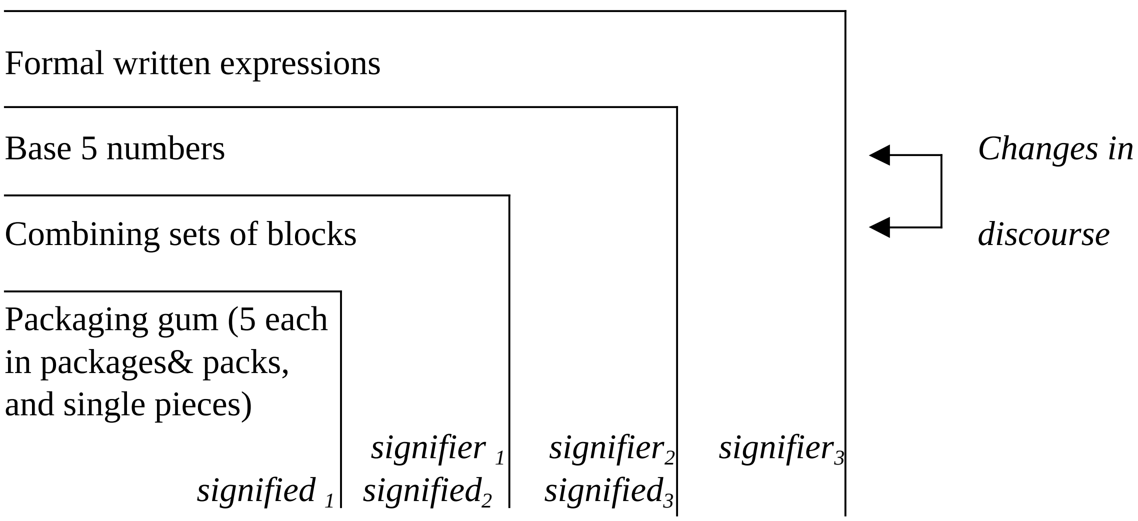

Figure 1. An example of a semiotic chain used by Hall (2000).

Chaining using this model proved to be a useful tool in enabling these elementary school teachers to link activities from the lives of their students, in a series of steps, with the mathematics of the classroom (Hall, 2000). Hall investigated the process of constructing and using such chains in two modes. Firstly, one might start with an everyday practice that is meaningful to the participants, and then see what mathematical notions grow out of the chaining as it is developed. Secondly, one might focus on a mathematical concept that is to be taught, and then search for a starting point in the everyday practices of students that can lead to this concept in several links of the chaining process. Not surprisingly, Hall's teachers found the second mode to be more suited to their classroom practice, giving them more control over the syllabus. They were able to use the chaining process successfully in their own classroom mathematical practices. Chains constructed by the teachers were designated as either *intercultural* - bridging two or more cultures, or *intracultural* – having a chain that remained within a single culture. Examples of the first type involved number of children in students' families, pizzas, coins, measurement of students' hands, linking in a series of steps with classroom mathematical concepts. These are intercultural because the cultures and discourse of students' homes or activities are linked with the different discourse and culture of classroom mathematics, for instance, the making of bar graphs. Manipulatives were frequently used as intermediate links in these chains, as in the following general model (Hall, 2000, p. 174).

Mathematical concept to represent the manipulative
Manipulative to represent the specific activity
Specific activity within the everyday activity
Everyday activity

Figure 2. A general model of an intercultural semiotic chain.

The second intracultural type, involving chains that were developed within the culture of a single activity, was evidenced in a chain involving baseball team statistics. The movement along the chain could be summarized as follows:

Baseball game → Hits vs. At Bats → Success Fraction → Batting Average

It was not the activity that was preserved throughout the chain, but merely the culture of baseball within which the chain was developed. Preliminary results from a study by Cheryl Hunt Adayemi (in progress) suggest that the model is also viable for use with pre-service elementary teachers, facilitating their use of students' own activities through chains that link with the school mathematics curriculum.

However, as in my previous research with graduate students in my *Ethnomathematics* class (Presmeg, 1998b), this conceptual model was not completely adequate as an explanatory lens. Hall's dissertation research grew from his participation in this *Ethnomathematics* course and attempts in that course to use this model of semiotic chaining to link cultural practices in a series of steps to formal, abstract mathematics (the intercultural model described in the foregoing). But when Hall (2000) in his dissertation research taught the elementary school teachers to construct and use their own semiotic chains in their mathematics classrooms, starting from activities in which their students were engaged, it became apparent in his analysis of the data that a more complex model would have provided a better lens for understanding the processes involved. In one instance there was the phenomenon of a sign within another sign (two related and nested signifiers for the same signified), which did not fit the pattern of the dyadic chain. The lesson involved the children counting the number of boys and girls in the class and constructing pictographs, using stick figures to stand for two children each. Both the stick people and the pictograph could be thought of as signifiers for the same signified, the students in the class. Hall (2000, p. 181) represented the situation as follows.

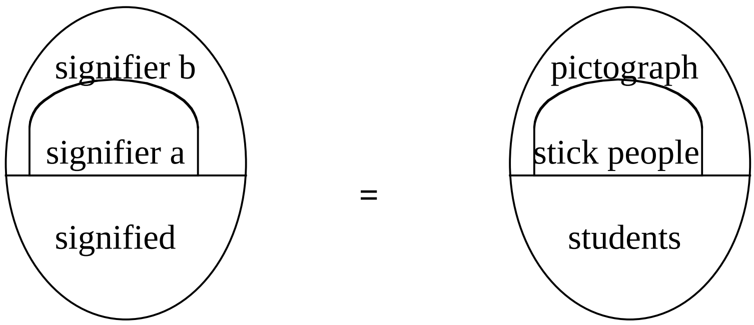

Figure 3. A signifier within another signifier.

The evident need to take not only this kind of nestedness but also the construction of meaning into account, resulted in further development of the theory, as described in the next section.

A Peircean nested model of signs

Each of the rectangles in figure 4 represents a sign consisting of the triad of object, representamen, and interpretant, corresponding roughly to signified, signifier, and a third interpreted component, respectively. This interpretant involves meaning-making: it is the result of trying to make sense of the relationship of the other two components, the object and the representamen. Note that the entire first sign with its three components constitutes the second object, and the entire second sign constitutes the third object, which thus includes both the first and the second signs. Like Russian nested dolls, sign 1 becomes an object in its own right (O_2) and resides within the second sign; similarly sign 2 becomes an object in its own right (O_3) residing within the third sign. Each object may thus be thought of as the reification of the processes in the previous sign. Once this reification occurs, this new object may be represented and interpreted to inform the creation of yet another object.

Key: O = Object (signified)
R = Representamen (signifier)
I = Interpretant

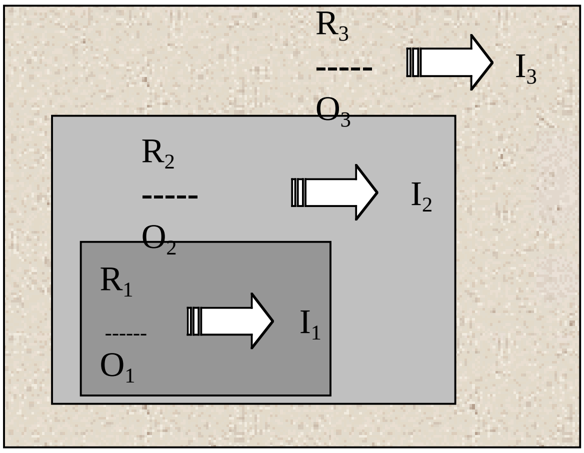

Figure 4. A Peircean representation of a nested chaining of three signifiers

In his writings, and as I have chosen to do in this model, Peirce sometimes referred to *sign* as the whole triad of object, representamen, and interpretant. But more often in his published work, *sign* is the word he used when referring to the representamen, the signifier. The meaning can usually be inferred from the context, avoiding ambiguity. However, it is necessary to be aware of this double usage of the term. In Peirce's "trichotomy of signs," as he stated, there are three kinds of signs (referring to the representamen):

> Firstly, there are *likenesses*, or icons, which serve to convey ideas of the things they represent simply by imitating them. Secondly, there are *indications*, or indices, which show something about things, on account of their being physically connected with them. Such is a guidepost, which points down the road to be taken, or a relative pronoun, which is placed just after the name of the thing intended to be denoted, … Thirdly, there are *symbols,* or general signs, which have become associated with their meanings by usage. Such are most words, and phrases, and speeches, and books, and libraries (Peirce 1998, p. 5; his emphasis).

It is the likenesses, or *icons*, which are metaphors. A metaphor implicitly compares two domains of experience, giving meaning to elements of one of these domains by reference to structural similarities in the other. Peirce's *indices* are metonymies rather than metaphors. In a metonymy (e.g., "Washington is talking with Moscow") a representamen stands for an object in such a way that the context is needed for its interpretation (e.g., not the cities, but the governments centered in those cities, are communicating). All metaphors and metonymies may also partake, to a greater or a lesser extent, in being *symbols* in Peirce's sense, depending on the degree to which they depend on general usage or convention. (For a fuller treatment of metaphors and metonymies, including examples, see Presmeg, 1992 &1998a)

Patterns of interaction in mathematics classrooms

Åse Streitlien, Høgskolen i Telemark

I will concentrate my contribution on the social norms and patterns of interaction in mathematics classrooms. Different ways the teacher asks and challenges her students open up for different mathematical contributions from the students. The students need to recognise both the mathematical demands as well as the social demands in the discourse. Some of the rules and routines for participation are openly expressed; others are more hidden or ambiguous and difficult for young students to understand. A question I would like to raise is how social rules affect what counts as knowledge in mathematics and what occurs as learning

The background for my reflection on this issue is my study of communication in mathematics classrooms in primary school. Attending two classrooms as a participant-observer over a year provided the opportunity to view the meanings teachers and students constructed through formal and informal processes. Extensive field notes and audiotapes of classroom talk captured the dialogue between the teacher and her students and the activities going on. When I listened to my tapes and read my transcripts, the social norms and rules of the classroom discourse became a dimension of vital interest. I became increasingly aware of the children's difficulties in interpreting the rules for participating in whole class teaching.

Classroom discourse is a form of institutional talk, and as such has certain characteristics. Firstly, it is oriented to pedagogical goals, and the participants are interacting for the specific purpose of learning. Secondly, the participants take the roles of "instructor" and "instructed", and therefore have unequal rights of participation. Finally, there is a certain amount of centrally focused attention with basic rules of participation; either one person speaks at a time, or multiple speakers say more or less the same thing. As a result of these characteristics it is generally the teacher who initiates the interaction, introduces the topic and decides who can talk and when. Instruction is usually characterised by a three-part-dialogue– often described as the IRF-structure: The teacher initiates (I), a student responds (R), whereby the teacher gives feedback, e.g. by evaluating the answer, by “uptaking” the pupil’s answer in the instruction, or by inviting the students to explain their solution. The IRF-structure was first identified by the sociolinguists Sinclair and Coulthard (1975). Later on Mehan followed up with a study in 1979. Mehan found that the students have the rights to take initiatives on their own, but they have to recognise when such initiatives are welcome. For many young students these unwritten rules of interaction are complicated to understand.

Another study I would like to mention is Nystrand’s “Opening dialogue” (1997). He and his co-researchers meant to identify two main types of instruction described as monologically and dialogically organised instruction. Nystrand describes the differences between these two types of instruction when it comes to epistemology, communication model, source of valued knowledge and texture of the lesson. In the case of *monologically* organised instruction knowledge is given, and its source is the teacher or the textbook, not the students. The teacher prescribes and monitors the answers. The teacher initiates all topics of discussion and determines what is worth knowing. Knowledge is treated as fixed, objective and autonomous. Recitation involves interaction, but this is superficial and procedural prescribed by the teacher. In the case of *dialogically* organised instruction students’ interpretation and personal experience are included. Knowledge is something generated and constructed. Both teacher and students participate in this construction. Students figure out, not just remembering. Consequently, these two types of instruction open up for quite different students’ positions in the learning process.

One of the rules that run through most of the lessons is the rule of “raising hands”. Raising hands in the classroom is a way of displaying to the teacher and to other students that you know the answer. Mostly, the teacher can assume that when a child is showing a hand, he also knows the answer. Thereby, there is less risk for asking someone who does not know the answer. From the teacher’s perspective this is a way of distributing the talk in a fair way.

I will now give you a short cut of a transcribed dialogue from the mathematics classroom in grade two. The class is working with numbers between ten and twenty. The teacher has drawn two columns on the blackboard where the tens and units should be marked. These young children are extremely eager to participate. Each of them wishes so strongly that the teacher will choose her/him for answering. Before the following episode takes place, the teacher has reminded the students over and over again of raising hands and not saying anything before they are invited.

Let us have a look at the excerpt[1].

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[1] Pause < 2 secs. = /
Pause > 2 secs = //

01. Teacher: How many squares do we need to colour on the other column in order to get fifteen together?
02. A pupil: Hh-mm!
03. T: How many squares do we have to colour here (POINTS AT COLUMN NUMBER 2) / Per?
04. Per: Eh //
05. T: How many squares do we have to colour on the other column in order to / it is fifteen we should get
06. Per: Is it / I think it is / five / five or something?
07. Tor: I know!
08. T: There is ten / here
09. Tor: I know!
10. T: Yes / that's good / I can see that you have raised your hand // there are many more besides // how many squares do I have to colour on the other column in order to get fifteen? // Then we should listen to someone else / Kari?
11. Kari: Ten and five
12. T: Ten and five / do you agree with Kari?
13. Students: Yes / yes / yes!

The teacher repeats the same question four times (01, 03, 05, 10) before she gets the answer she wants. The question is linked to what the students are supposed to learn in mathematics. However, in utterance number 10 this aim is mixed with the expectation of social behaviour. On one hand, the students have to find the right answer very quickly in competition with fellow students. They have, on the other hand, to control their behaviour, raising hands and keeping quiet about what they know.

Later on one of the students, Eva, is fetching a deep sigh, and the teacher asks her:

14. T: Yes / was this a little bit difficult?
15. Students: No / no!
16. Eva: Teacher / I did not get my turn in spite of putting up my hand constantly!

This episode shows us that what seems obvious for adults is not always comprehensible for children. There is a complex relation between form and function in the teacher's use of language. Tor does not understand why his knowledge is not valued. He is confused by the hidden rules. Why should not the teacher give turn to him since Per seems so doubtful about his answer? In other situations Tor's initiative would be valued as a helping hand, but not so in classroom discourse. When he says he knows, the teacher answers that many others also know. Tor and Eva have raised their hands, but the underlying meaning is that raising hand is no guarantee for being the chosen one. Eva has done exactly what the teacher had asked her to do – but in spite of that, she never got the opportunity to answer.

In the episode the social rules and routines seemed to disturb the mathematical content in the discourse. So my last question is: How concerned should we be with discourse patterns and structures in mathematics classrooms? The children I have referred to in the example are young children, 6 and 7 years old. Their school career is, so far, a short one. Nevertheless, their meeting with the discourse of mathematics in early school years will affect their attitudes and beliefs of mathematics for years to come. If students have to struggle in concentrating on the rules of participation, less attention will be given to the mathematical content. If a student like Tor will gain several experiences of being rejected by the teacher, it is reasonable to assume that he will stop taking initiatives.

The general emphases on the importance of good lesson planning and teacher control tends to put pressure on follow lessons plan as closely as possible, and to avoid any unplanned learning activities. Consequently, the discourse structure becomes less flexible, and the teacher and the textbook are resources of knowledge. The aim is to find the "right" answer. In this context there will be less time for discussion and negotiation of meaning. This is in conflict with the view that student initiative, participation and involvement in instruction represent an important aspect of learning processes. The young students need space and time for expressing what they know and manage on their own. Greater variability in the patterns of communication should create more opportunities for student participation in the learning process. One way to do this is to use learning opportunities created by the students themselves, picking up topics introduced by the students, let them reflect and figure out or allowing them to decide how to develop a particular activity. As Nystrand (1997) says:

> When teachers ask questions about what students are thinking, and when they ask questions about students previous answers, they promote fundamental expectations for learning by seriously treating students as thinkers, that is, by indicating that what students think is important and worth examining (s. 28).

References

Cobb, P., Gravemeijer, K., Yackel, E., McClain, K., & Whitenack, J. (1997). Mathematizing and symbolizing: The emergence of chains of signification in one first grade classroom. In. D. Kirshner & J.A. Whitson (Eds.), *Situated cognition: Social semiotic, and psychological perspectives* (pp. 151-233). Hillsdale, NJ: Lawrence Erlbaum Associates.

Cobb, P., Yackel, E., & McClain, K. (Eds.) (2000). *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design.* Mahwah, New Jersey: Lawrence Erlbaum Associates.

Dörfler, W. (2000). Means for meaning. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design* (pp. 99-131). Mahwah, New Jersey: Lawrence Erlbaum.

Hall, M. (2000). *Bridging the gap between everyday and classroom mathematics: An investigation of two teachers' intentional use of semiotic chains.* Unpublished Ph.D. dissertation, The Florida State University.

Mehan, H. (1979). *Learning lessons: Social organization in the classroom.* Cambridge: Harvard University Press.

Nystrand, D. M. (1997). *Opening dialogue. Understanding the dynamics of language and learning in the English classroom.* New York: Teachers College Press.

Peirce, C. S. (1992). *The essential Peirce. Volume 1*, edited by N. Houser and C. Kloesel. Bloomington: Indiana University Press.

Peirce, C. S. (1998). *The essential Peirce. Volume 2*, edited by the Peirce Edition Project. Bloomington: Indiana University Press.

Presmeg, N.C. (1992). Prototypes, metaphors, metonymies and imaginative rationality in high school mathematics. *Educational Studies in Mathematics*, *23*, 595-610.

Presmeg, N. C. (1998a). Metaphoric and metonymic signification in mathematics. *Journal of Mathematical Behavior, 17*(1), 25-32.

Presmeg, N.C. (1998b). Ethnomathematics in teacher education. *Journal of Mathematics Teacher Education*, *1*(3), 317-339.

Presmeg, N. C. (2002). *Semiotic perspectives in mathematics education research*. Presentation in the research presession of the 80th Annual Meeting of the National Council of Teachers of Mathematics, Las Vegas, April 20, 2002.

Saussure, F. de (1959). *Course in general linguistics*. New York: McGraw-Hill.

Sfard, A. (1991). On the dual nature of mathematical conceptions. *Educational Studies in Mathematics, 22*(1), 1-36.

Sfard, A. (2000). Symbolizing mathematical reality into being – or how mathematical discourse and mathematical objects create each other. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design* (pp. 37-98). Mahwah, New Jersey: Lawrence Erlbaum.

Sinclair, J. & Coulthard, R. (1975). *Towards an analysis of discourse. The English used by teachers and pupils.* London: Oxford University Press.

Whitson, J. A. (1994). Elements of a semiotic framework for understanding situated and conceptual learning. In D. Kirshner (Ed.) *Proceedings of the 16th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* Baton Rouge, Nov. 5-8, 1994, Vol. 1, pp. 35-50.

Whitson, J. A. (1997). Cognition as a semiosic process: From situated mediation to critical reflective transcendance. In D. Kirshner & J. A. Whitson (Eds), *Situated cognition: Social, semiotic, and psychological perspectives* (pp. 97-149). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Reflekterande samtal för pedagogisk utveckling
Lärare och specialpedagog i samverkan om lärande i matematik

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Studiens övergripande syfte var att studera hur specialpedagogen och lärare arbetar tillsammans för att utveckla undervisningen i matematik. Forskningsintresset var riktat mot att studera handledningssamtalen mellan lärare och specialpedagog med syftet att granska om samtalen skapar en gemensam referensram som ger lärarna verktyg att förstå, förklara och utveckla den egna undervisningspraktiken. Vidare var syftet att beskriva eventuella förändringar i lärarnas förståelse och förhållningssätt till den egna undervisningen och till elever som har eller kan tänkas få behov av särskilt stöd i matematik. Undersökningen har en explorativ karaktär där olika datainsamlingsmetoder kommit till användning. Det empiriska materialet består av handledningssamtal, intervjuer, informella samtal, deltagande observationer, lärares och elevers dokumentation samt officiella skoldokument. Resultaten visar att samtalen bidrar till att starta processer, vilka hjälper lärarna att synliggöra sina ställningstaganden och bli medvetna om sina värderingar. Samtalen underlättar för lärarna att utveckla ett reflekterande förhållningssätt som ger dem handlingsberedskap och kunskaper att bättre förstå och förändra den egna praktiken.

Bakgrund och forskningsanknytning

Studien handlar om hur två lärare tillsammans med en specialpedagog arbetar för att utforma undervisningen i klassrummet för att främja alla elevers matematiska lärande[1]. En central fråga i detta sammanhang är vad syftet med undervisningen är och vilka kunskaper eleverna skall utveckla. Enligt kursplanen i matematik (Skolverket, 2000) skall undervisningen karakteriseras av ett lärande som innebär upptäckter och utveckling som att se mönster, relationer och samband. Eleverna skall resonera och kommunicera, värdera och göra bedömningar samt upptäcka och förhålla sig till matematiken i vardagslivet. Matematiken i skolan har emellertid inte alltid denna inriktning. I undervisningen handlar det ofta om att eleverna skall ge rätt svar, prestera bra på prov och arbeta i läroboken. De

[1] Studien är genomförd inom projektet ”Matematik i en skola för alla” med forskningsanslag från Skolverket. Tidigare dokumentationer i projektet är ”På spaning efter en skola för alla” (Ahlberg, 1999) och ”Lärande och delaktighet” (Ahlberg, 2001). Den aktuella studien är dokumenterad i rapporten ”Reflekterande samtal för pedagogisk utveckling” (Ahlberg, Klasson & Nordevall, 2002).

elever som inte ”svarar rätt” får oftast träna ”mer av samma sort”. För att det skall vara möjligt att uppfylla läroplanens mål borde perspektivet vidgas och matematikens språkliga och sociala karaktär lyftas fram. Detta gäller också elever i behov av särskilt stöd. Även dessa elever borde få tillfälle att möta matematiken på olika sätt, få tillfälle att ”tala matematik” och upptäcka mönster, relationer och samband (Ahlberg, 1999).

En stor del av den forskning som bidragit till kunskapsutvecklingen om lärande och undervisning i matematik är grundad i konstruktivismen med rötter i Piaget (1969). Det som utmärker en konstruktivistisk syn på kunskap och lärande är att kunskap konstrueras aktivt av den lärande människan (Cobb och Bauersfeld, 1995; Engström, 1997; von Glasersfeld, 1993; Jaworski, 1994). Lärande i matematik tilldrar sig stort intresse även inom det sociokulturella perspektivet (Lave, 1988; Wyndham 1993). Enligt Säljö (2000) är skolans verksamhet inte primärt inriktad mot att utveckla ett matematiskt kunnande som gör det möjligt för eleverna att klara av vardagens problem. Istället står abstrakt reflektion och hantering av symboler i fokus. Skolan som institution utvecklar egna perspektiv och normer för prioritering och bedömning. De förhållningssätt som framträder i skolan skiljer sig därför från dem som utvecklas i vardagliga situationer där ett matematiskt kunnande ingår på ett naturligt sätt.

Forskning om lärande och undervisning i matematik har även genomförts inom den fenomenografiska forskningsinriktningens ram (Ekeblad, 1996; Neuman 1987; Runesson, 1999). Som en plattform för den aktuella studien tjänar några undersökningar av Ahlberg (1992, 1999, 2001). I dessa betonas vikten av att variera undervisningskontexten. I en studie i skolår tre i grundskolan fick eleverna möta olika typer av matematiska problem som anknöt till deras föreställningsvärld. Det matematiska innehållet presenterades på skilda sätt och eleverna fick även tillfälle att variera sitt perspektiv på problemen genom att använda skilda uttrycksmedel som att rita, tala, skriva och räkna. Problemen kunde lösas på olika sätt och det fanns inte bara ett enda rätt svar. Eleverna diskuterade med varandra hur de gått tillväga och fick på så sätt ta del av olika sätt att tänka kring problemens lösningar. Resultaten av studien visar att innehållet och utformningen av undervisningen befrämjar elevernas matematiska lärande. Det framkom att elever utvecklade sin matematiska förståelse när läraren målmedvetet lyfte fram matematikens sociala och språkliga karaktär och skapade inlärningssituationer där olika aspekter av matematik synliggjordes. Samtliga elever drog fördelar av arbetssättet, men framför allt utvecklade de lågpresterande flickorna sin matematiska förståelse (Ahlberg, 1992).

Reflektion och specialpedagogik

För att utveckla skolan och lärares professionalitet är det betydelsefullt att skapa arenor för lärares reflektion (Alexandersson, 1998). Specialpedagogens handledning med sina kollegor kan vara en sådan arena som skulle kunna bidra till att utveckla lärarnas matematikdidaktiska kompetens. Ett viktigt inslag i denna kompetens är att läraren är medveten om de val hon gör och att det finns alternativa vägar. Detta är särskilt betydelsefullt då det gäller undervisningen med elever i behov av särskilt stöd. Persson (2001) visar att skolans sätt att bemöta dessa elever ofta följer ett välkänt och inarbetat mönster som domineras av organisatorisk differentiering eller stöd av en speciallärare/specialpedagog.

Det betonas mycket starkt i propositionen om den förnyade lärarutbildningen (Prop 1999/2000:135) att samtliga blivande lärare skall tillägna sig special-pedagogiska kunskaper för att få beredskap att möta alla barns och elevers behov. Ahlberg (2001) beskriver en rad aspekter som är betydelsefulla för elevernas lärande och delaktighet. Det handlar om samhälls- och organisationsaspekter, demokrati- och likvärdighetsaspekter, sociokulturella aspekter, kommunikativa och språkliga aspekter, socio-emotionella aspekter, kognitiva och perceptuella aspekter, fysiska aspekter samt didaktiska aspekter. Ett komplext samspel mellan dessa aspekter påverkar elevers lärande och det är därför inte möjligt att med en enkelriktad orsaksmodell förklara varför en elev lyckas eller misslyckas med matematiken.

Ett kommunikativt relationsinriktat perspektiv

Den genomförda studien är grundad i ett kommunikativt relationsinriktat perspektiv där begrepp som delaktighet, kommunikation och lärande är centrala. Perspektivet har starka influenser från vissa riktningar inom det sociokulturella perspektivet och den fenomenografiska forskningsinriktningen. Liksom i det sociokulturella perspektivet ses de som bundna till kultur, kontext och situation. Människors kompetens och kunnande är beroende av den historiska, sociokulturella miljö som de är en del av. Trots gemensam kultur och sammanhang förekommer dock individuella variationer i människors sätt att erfara världen. Varje individ har sin särprägel. På grund av skiftande erfarenheter, olika perspektiv och sammanhang, erfar, uppfattar och förstår människor saker och ting på olika sätt, även om de ingår i en social praktik. Kunskapsbildning och meningsskapande förläggs därför inte till människan *eller* till den praktik som människan deltar i. Istället betonas relationen mellan människan och de sammanhang i vilka hon ingår. Det handlar om att samtidigt se till individen, den sociala praktiken och till strukturella aspekter som formar den enskildes lärandemiljö (Ahlberg, 2001).

Studiens syfte

Studiens övergripande syfte är att studera hur specialpedagogen och lärare kan arbeta tillsammans för att utveckla undervisningen i matematik. Forskningsintresset är riktat mot att studera handledningssamtalen mellan lärare och specialpedagog för att granska om samtalen skapar en gemensam referensram som ger lärarna verktyg att förstå, förklara och utveckla den egna verksamheten.

Vidare är syftet att beskriva eventuella förändringar i lärarnas förståelse och förhållningssätt till den egna undervisningen och till elever som har eller kan tänkas få behov av särskilt stöd i matematik.

Metod och genomförande

Aktionsforskning har influerat studiens uppläggning. En huvudtanke i denna är att kunskapen inom verksamheten ska genereras inifrån – från lärarna själva. Detta utesluter inte extern hjälp av forskare och andra, men den placerar de verksamma i centrum (Elliot, 1991; Mc Nieff, 1988).

I undersökningen etablerade två lärare, en specialpedagog och två forskare ett samarbete för att stödja lärarnas reflektion och försök att utveckla den egna verksamheten. Lärarnas och specialpedagogens arbete följdes under ett läsår. De båda lärarna undervisade i var sin åldersblandad grupp med elever i skolår fyra och fem. Sara, som är mellanstadielärare hade varit på skolan sedan den startade för drygt tio år sedan. Emma är grundskollärare med inriktning mot skolår 1-7 inom svenska och samhällsorienterande ämnen. Hon hade varit anställd vid skolan ett år när studien genomfördes. Karin hade arbetat ett år som specialpedagog vid skolan och är ursprungligen mellanstadielärare.

Det empiriska materialet

En mängd olika datainsamlingsmetoder utnyttjades för att dokumentera studien. *Handledningssamtal*en med specialpedagogen och lärarna är centrala i datamaterialet. Sammanlagt genomfördes 13 samtal. De spelades in på band och transkriberades till skriven text. Samtalen varade i genomsnitt en timma. Vid dessa samtal styrde lärarnas och specialpedagogens upplevelser och tankar samtalens innehåll. Utgångspunkten vid samtalen var ofta olika dilemman som lärarna ställdes inför i det dagliga arbetet. Under det läsår studien pågick genomfördes 22 *klassrumsobservationer* i varje klass. Dessa hade till stor del karaktären av deltagande observationer. Forskarens intresse riktades såväl mot lärarnas som mot elevernas arbete med ett särskilt fokus på de elever som var i behov av särskilt stöd. Även *intervjuer* och *informella samtal* med lärare, specialpedagog, rektor och elever utgör ett väsentligt bidrag till det empiriska materialet.

Analys och tolkning

Analysen och tolkningen har syftat till att försöka förstå och förklara lärarnas och specialpedagogens ord och handlingar för att ge en sammanhängande bild av handledningssamtalen och den dagliga praktiken. Intresset har varit inriktat mot att studera hur lärarna och specialpedagogen reflekterar över sin undervisning och om samtalen på något sätt visar spår i verksamheten. Vid analyserna av handledningssamtalens innehåll och struktur används en modell av det reflekterande samtalet som utvecklats i en tidigare studie (se Ahlberg 1999).

Resultat

Resultatredovisningen inleds med en tematisk beskrivning av samtalens innehåll. Beskrivningen av lärarnas ord och handlingar har delats in i en inledande fas, en fortlöpande fas och en avslutande fas. Avsikten är att rikta ljuset mot såväl lärarna som specialpedagogen och belysa hur de erfar det vardagliga arbetet med matematiken i skolan. Därefter ges en jämförande beskrivning av lärarnas erfarande av praktiken i den inledande och den avslutande fasen.

Perspektiv och lärande i den inledande fasen

Lärarna menar att på grund av att skolans profil är riktad mot samhällsorienterande ämnen och svenska har utvecklingen av matematikundervisningen vid skolan fått stå tillbaka. Det gemensamma arbetet i den inledande fasen innebär att Karin i egenskap av specialpedagog ansvarar för ”sina” barn när hon är i klassrummet. Både Sara och Emma följer i sin undervisning läroboken, som de själva benämner som hastighetsindividualiserad. Karin talar också om läroboken som hastighetsindividualiserad och tror att det är ett genomgående drag för hela skolans matematikundervisning. ”Undervisningen är lättare när barnen arbetar på sin nivå.” Det uppfattas således som mer arbetsamt för lärarna att försöka variera och planera för olika innehåll vid lektionerna. Det kräver tid och lärarna anser sig inte ha den tiden.

Karin har dock noterat att i synnerhet Sara ibland använder sig av egenhändigt producerade matematikuppgifter. Hon har låtit eleverna arbeta med matematik kring en klassfest, det politiska valet och matematikintegrering i temat ”katter”. Båda klasslärarna påtalar de nationella provens betydelse. Emma känner sig stressad och orolig för en del elever och inför allt som skall läras till dess. Både lärarna och specialpedagogen belyser tiden som en betydelsefull faktor för vad man gör i klassrummet. Utifrån observationerna kan man ana en viss stress hos Emma när hon väljer att låta eleverna arbeta med egenhändigt producerade matematikuppgifter. Dels kan detta vara en följd av att hon har som mål att alla elevers uppgifter skall presenteras och bearbetas gemensamt, dels som en följd av att eleverna är ovana att arbeta med denna typ av matematikuppgifter och en stund in på lektionen börjar visa oro. Även Emmas egen bristande rutin vid denna typ av arbetssätt kan till viss del påverka skeendet.

Perspektiv och lärande i den fortlöpande fasen

Under den fortlöpande fasen påtalar alla tre lärarna att det är viktigt att vara insatt i organisationsfrågor, såsom skolans resurser och dess fördelning. Behovet blir mycket tydligt i och med ökade krav på måluppfyllelse, samtidigt som antalet barn i behov av särskilt stöd ökar. Emma är bekymrad över de minskade resurserna i skolan och frågar sig hur man skall lyckas med att fler elever når målen. Hon ger som exempel att hon fått en ny elev i sin klass. Denna elev är i stort behov av stöd, men det finns inte tillgång till något extra stöd. Hon berättar också om Nina som hellre vill rita och läsa än att arbeta med matematik. Nina tycker att allting är svårt. När hon ska lösa ett problem använder hon genast några tal i problemet för att göra en uträkning, utan minsta tanke på rimlighet. Vid eget arbete väljer Nina aldrig matematik.

> Om man inte är hos Nina direkt, då åker matteboken iväg eller ner i bänken och hon tjurar. Innan hon har öppnat matteboken vet hon eller tar för givet, att hon inte klarar den sidan. Hon är så bestämd. Det är jättesvårt. Men om man är i närheten och ser att hon spänner sig i ryggen då kan man lirka. Men det är inte möjligt att vara hos henne så ofta som man skulle behöva. (Emma)

I det enskilda samtalet kring matematik nämner Nina att matematik är ”tråkigt och jobbigt”. Emma har haft samtal med föräldrarna. Nina skall få gå till Pia som undervisar i svenska som andraspråk. Hon har för tillfället få elever och kan därför ta emot henne. Denna lösning visar sig fungera bra, då det är lugnare där och Nina har lättare att koncentrera sig i den lilla gruppen.

Emma och Karin har även svårt att hinna med Saida och funderar över hur man kan förändra detta. Saida har svårigheter med mönster och begrepp, bl. a. hälften och dubbelt. Vid det enskilda samtalet med Saida framkommer att hon uteslutande ser matematik som proceduriell och kopplad till de fyra räknesätten. Hon anser själv att hon är ”bra på plus och minus”. Hon behöver öva mer på att ”ställa upp – över tusen med plus och minus”.

Under läsårets gång har samarbetet mellan Emma och Karin kommit igång. De konstaterar att det gemensamma reflekterandet kring elever och innehåll ger en tydligare struktur, överblick och utveckling av klassrumsarbetet. Samtal kring elever i svårigheter bidrar till att utveckla och förändra deras förhållningssätt. Ett gemensamt ansvarstagande utvecklas, vilket blir särskilt tydligt hos Emma och Karin i deras arbete kring eleven Saida. Emma menar att det är viktigt att Saida får lära sig ty sig till andra än Karin och känna ”att ej alltid vara sist på skalan”. Hon har axlat ett ansvar för en kamrat, som har det svårt med språket. Vid något tillfälle hade hon yttrat till Sara, som har gruppen i engelska; ”Du förstår, hon (Fatima) förstår inte när du pratar engelska hela tiden”. Enligt Emma hade Saida vid ett annat tillfälle i ett samtal mellan Emma och Fatima, med en självklar auktoritet tolkat till elevernas gemensamma hemspråk och sedan tillbaka till svenska.

> Det är ingen katastrof om hon kör fast i någon uppgift och får försöka flera gånger. Hon fixar det ändå. Då tar hon det lugnt eller tar hjälp av kompisar. (Emma)

Sara menar att det underlättar om Karin och hon kan utbyta tankar och idéer. Hon nämner att hon kan se att en förändring har skett inom henne själv. Det har blivit enklare att omsätta sina matematiska idéer till praktiska handlingar i klassrummet.

> Jag har ofta vetat att så och så skall jag lägga upp mattearbetet. Det har dock tagit alltför lång förberedelsetid. Nu har det fantastiska inträffat att det har blivit lättare att snabbt passa varje tillfälle till matteprat, t ex när jag läste i högläsningsboken och upptäckte matteproblem… diskuterade och räknade detta istället för att diskutera händelseinnehållet. Det har blivit lättare och naturligare att hitta matematiken i allt vi gör. (Sara)

Lärarna diskuterar även grupparbetets betydelse och menar att det har betydelse för lärandet men att det också har en social dimension. De belyser vikten av utbyte av matematiska strategier för att undvika att bokens lösningsmodell blir den enda och riktiga. Samtidigt påtalar de att både elever och föräldrar oftast fokuserar kvantitet och produkt. ”Var är de i matteboken?” Lärarna är överens om att de vill ha mer utbyte av idéer mellan sig, avseende litteratur, material och undervisning. De föreslår vidare att deras samtalsmodell borde ges utrymme inom den egna organisationen för fler lärare, eftersom de själva ser vad den ger.

Perspektiv och lärande i den avslutande fasen

Under den avslutande fasen har lärarnas uppmärksamhet i större utsträckning riktats mot att variera matematikundervisningens innehåll och utformning. De konstaterar att matematikämnet även inrymmer en social dimension, i vilken eleverna kan mötas för ett utbyte av strategier och lärande av varandra. En bidragande orsak till detta är enligt Karin att de i ökad utsträckning diskuterar undervisningens innehåll i förhållande till elevernas behov. Därtill har Emmas och Karins samarbete utvecklats till ett gemensamt ansvar kring elever i behov av särskilt stöd. Karin har fungerat som direkt stöd i klassrummet och som stöd i gemensamma pedagogiska diskussioner. Emma har förändrat sitt tänkande kring detta. Från att i början ha varit tveksam till om det gemensamma reflekterandet kunde leda till utveckling i klassrummet påtalar hon nu fördelarna med att få arbeta i nära samarbete med specialpedagogen. I sitt arbete med elever i behov av särskilt stöd ser Emma vilken påverkan hennes och Karins arbete haft för en av eleverna. Flickan har ”blommat upp” socialt i klassen och visar på en egen initiativförmåga i sitt kunskapande. Emma ser också en förändring hos sig själv i förhållande till matematiken. Hon ”tänker oftare matematik” och hon drar paralleller till språkutveckling och betonar betydelsen av elevers olikheter och dess konsekvenser för undervisningen.

Emma ger uttryck för en viss besvikelse över att inte samarbetet med klasslärarkollegan kommit igång i den utsträckning som hon förväntat sig. Något som bekräftas av Sara, som menar att hennes sjukdom hade lagt hinder i vägen. Hon poängterar dock att det är viktigt att stötta och hjälpa varandra. Det är något som hon vill utveckla, ”annars orkar man inte”.

Forskarnas närvaro och de pedagogiska diskussionerna under läsåret har bidragit till att lärarna fokuserar matematikämnet i högre grad i olika undervisningssammanhang. Sara menar att det har hjälpt henne att naturligt integrera matematiken i hennes totala undervisning. Karin menar att hon ser en förändring i sitt samarbete med Sara och Emma. De samtalar och diskuterar i en helt annan utsträckning än tidigare. Alla tre önskar att det reflekterande pedagogiska samtalet ryms inom den ordinarie organisationen. Emma menar att det egna arbetet underlättas om flera vuxna samtalar och upplever ”igenkännandets gemenskap”.

Reflekterande samtal för pedagogisk utveckling

En slutsats som kan dras av studien är att samtalen kännetecknas av en riktning mot problematisering och perspektivseende. Samtalen är inte riktade mot att direkt finna ett svar på det diskuterade dilemmat. Istället är intentionen att försöka förstå olika skeenden samt söka och pröva olika vägar för att komma tillrätta med olika problematiska situationer. Handledningssamtalen har därmed karaktären av reflekterande samtal.

Då lärare samtalar om sitt dagliga arbete refererar de enligt Ahlberg (1999) på en övergripande nivå till *lärande, skolan som social praktik* samt *skolans mål och värdegrund.* Lärarna talar även om elevens förutsättningar, kunskaper och behov, undervisningens innehåll och organisering samt skolans organisation och kultur. Dessa referenser tas som utgångspunkt för en jämförande analys mellan de två polerna ”den inledande fasen” och ”den avslutande fasen” i det följande. En översikt ges i tabell 1 och 2. I varje modul beskrivs kärnan i det sätt på vilket lärarna erfar sin praktik.

Referensområdena lärande, skolan som social praktik samt skolans mål och värdegrund har berörts. I de samtal då något referensområde inte har behandlats har forskarna ofta fört in detta och därmed vidgat samtalets rörelse. I första hand gäller det skolans mål och värdegrund vilket forskarna, men även specialpedagogen uppmärksammar i större utsträckning än lärarna. Samtalen har därmed vidgats till att se lärande, skolans sociala praktik samt mål och värdegrund genom flera nivåer inom skolans verksamhet.

Tabell 1. Lärarnas erfarande av praktiken i den inledande fasen

Referenser	*Elevens förutsättningar, kunskaper och behov*	*Undervisningens innehåll och organisering*	*Skolans organisation och kultur*[2]
Lärande	Läroboken är utgångspunkten i undervisningen Samtliga elever arbetar enskilt i samma lärobok. De arbetar i sin egen takt på sin egen nivå	Undervisningen huvudsakligen inriktad mot kvantitet Specialpedagogen förser klassläraren med uppgifter för elever i behov av särskilt stöd	Skolans verksamhet och intresse huvudsakligen riktat mot svenska och samhällsorienterade ämnen
Skolan som social praktik	Specialpedagogen arbetar individuellt med elever. I klass-rummet eller i enskildhet	Bristande tid för samplanering och samverkan Varierande uttrycks-former och samverkan mellan elever i begränsad omfattning	Accepterande av elevers olikheter Bristande tid för pedagogiska samtal och reflektion
Mål och värdegrund	Alla elever ska känna samhörighet och delaktighet i skolan Specialpedagogen stöder elever i svårig-heter genom att arbeta individuellt med dem	De nationella proven och uppnåendemålen styr undervisningen	Strävan mot att möta alla elevers olikheter Specialpedagogen har det huvudsakliga ansvaret för elever i behov av särskilt stöd

Även i den avslutande fasen berörs samtliga referensområden. En beskrivning av samtalens innehåll i den avslutande fasen ges i tabell 2.

[2] Skolans organisation och kultur bedöms i denna undersökning med utgångspunkt från vad som framkommer vid samtalen med klasslärare och specialpedagog samt vid intervjun med rektorn.

Tabell 2. Lärarnas erfarande av praktiken i den avslutande fasen

Referenser	*Elevens förutsättningar, kunskaper och behov*	*Undervisningens innehåll och organisering*	*Skolans organisation och kultur*
Lärande	Läroboken är en av många redskap för eleverna att lära Elevernas olika sätt att tänka och resonera uppmärksammas och tas som utgångspunkt i undervisningen Balans mellan krav och förmåga	Undervisningen i ökad utsträckning inriktad mot förståelse Specialpedagogen diskuterar med klassläraren omkring uppgifter för elever i behov av särskilt stöd Matematikens kommunikativa och sociala aspekter fokuseras	Verksamhetens intresse riktad mot svenska och samhällsorienterade ämnen och matematik Kopplingen mellan skolämnena uppmärk-sammas
Skolan som social praktik	Organisering av lärandemiljöer som befrämjar den enskilde elevens lärande	Varierande uttrycksformer och samverkan mellan elever i ökad utsträckning Gruppen ses som en tillgång i lärandet Specialpedagogen deltar i planering och genomförande av undervisningen	Uppmärksamheten riktad mot elevernas olikheter Inriktning mot samtal och samarbete
Mål och värdegrund	Elever i svårigheter deltar i den sammanhållna gruppen	Elevers olikheter ses som en möjlighet Elevers erfarenheter tas som utgångspunkt för undervisningen och ses som möjligheter	Medvetet arbete mot att möta alla elevers olikheter Specialpedagog och klasslärare tar gemen-samt ansvar för elever i behov av särskilt stöd

Vid en jämförelse mellan faserna framkommer att de samtal som förts om lärande, social praktik och mål och värdegrund har lett till en förändring som är kopplad till elevens förutsättningar, kunskaper och behov, undervisningens innehåll och organisering samt skolans organisation och kultur. Det handlar oftast om en gradskillnad där vissa referensområden under studiens fortlöpande får en ökad fokusering och uppmärksamhet

Förändringar i verksamheten sker genom att lärarna reflekterar och införlivar idéer och tankar i den egna erfarenhetsvärlden. Samtalen ger konsekvenser för lärarnas tänkande och handlande och förändrar till viss del lärarnas syn och förhållningssätt till ämnet matematik och matematikundervisningen. Studiens resultat visar att handledningssamtal kan bidra till lärarens möjligheter att distansera sig och utveckla ett kritiskt förhållningssätt till den egna undervisningen. Samtalen verkar stödjande vid dilemman som uppkommer i det dagliga arbetet, och bidrar dessutom till att skapa ett gemensamt språk och gemensamma referensramar hos de deltagande lärarna. Lärarens förhållningssätt och förmåga till flexibilitet i tänkande och inlevelse i elevens situation har avgörande betydelse för hur en elevs skolsituation kommer att gestalta sig. Resultaten visar att handledningen i vissa avseenden leder till en utveckling i den enskilde lärarens tänkande som också lämnar avtryck i det dagliga arbetet.

Referenser

Ahlberg, A. (1992). *Att möta matematiska problem. En belysning av barns lärande.* Göteborg: Acta Universitatis Gothoburgensis.

Ahlberg, A. (1999). På spaning efter en skola för alla. *Rapport 1999:08*. Institutionen för pedagogik och didaktik, Göteborgs universitet.

Ahlberg, A. (2001). *Lärande och delaktighet*. Lund: Studentlitteratur.

Ahlberg, A., Klasson, J-Å. & Nordevall. E. (2002). Reflekterande samtal för pedagogisk utveckling. Lärare och specialpedagog i samverkan om lärande i matematik. *Insikt 2002.2*. Högskolan för lärande och kommunikation. Högskolan i Jönköping.

Alexandersson, M. (1998). Styrning på villovägar. Perspektiv på skolans utveckling. *Rapport nr 14*. Institutionen för metodik i lärarutbildningen, Göteborgs universitet.

Cobb, P. & Bauersfeld, H. (Eds.) (1995). *The emergence of mathematical meaning: Interaction in classroom cultures*. Hillsdale, NJ: Erlbaum.

Ekeblad, E. (1996*). Children learning numbers*. Göteborg: Acta Universitatis Gothoburgensis.

Elliot, J. (1991). *Action research for educational change*. Milton Keynes: Open University Press.

Engström, A. (1997). *Reflektivt tänkande i matematik. Om elevers tankekonstruktioner.* Stockholm: Almqvist &Wiksell International.

von Glasersfeld, E. (1993). Reflections on Number and Counting. In S. T. Boysen and E. J. Capaldi (Eds.), *The Development of Numerical Competence. Animal and Human Models.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Jaworski, B. (1994). *Investigating mathematics teaching. A constructivist enquiry*. London: The Falmer Press.

Lave, J. (1988). *Cognition in practice*. Cambridge: Cambridge University Press.

Mc Nieff, J. (1988). *Action research principles and practice*. London: Routledge.

Neuman, D. (1987). *The origin of arithemetic skills. A phenomenographic approach*. Acta Universitatis Gothoburgensis.

Persson, B. (2001). *Elevers olikheter och specialpedagogisk kunskap*. Stockholm: Liber.

Piaget, J. (1969). *The child's conception of number*. London: Routledge & Kegan Paul.

Prop.1999/2000:135. *En förnyad lärarutbildning*. Stockholm: Utbildningsdepartementet.

Runesson, U. (1999). *Variationens pedagogik. Skilda sätt att behandla ett matematiskt innehåll*. Göteborg: Acta Universitatis Gothoburgensis.

Skolverket. (2000). *Grundskolan Kursplaner och betygskriterier*. Stockholm: Fritzes offentliga publikationer.

Säljö, R. (2000). *Lärande i praktiken*. Stockholm: Prisma.

Wyndham,J. (1993). *Problem-solving revisited. On school mathematics as a situated practice*. Linköping: Linköping Studies in Arts and Science, 98.

On Reasoning Characteristics in Upper Secondary School Students’ Task Solving

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Introduction

The background to this report is the concern that the Swedish school system cannot help sufficiently many students to reach a desired level of mathematical competence. One of the main purposes for mathematics at upper secondary school in Sweden is students’ ability to analyse and solve problems. We decided to focus on problem solving and students’ reasoning, a central component in mathematics.

Our starting-point was the earlier research by Lithner (2000) that indicates that undergraduate students in problematic situations tend to rely on their, often mathematically superficial, experiences. In very few cases their strategies are grounded in relevant mathematical concepts. In these cases, the reasoning is still dominated by the individuals’ memory images and familiar routines. The purpose here was to make a study similar to Lithner (2000), focusing on upper secondary school students. We suspected that this kind of behaviour is also common at that level.

We claimed that students in upper secondary school in Sweden spend a large part of their time in class solving different types of mathematical tasks, mainly textbook exercises (Bergqvist et al, 2003). In this activity, they often meet problematic situations where it is not obvious how to proceed. When a student is facing a problematic situation he or she must make a strategy choice concerning what to do in order to solve the problem, and then implement this strategy. In our study, we were interested in on what bases these choices and implementations are made.

The general question was *What makes students succeed or fail in a problematic situation?* We investigated this when students worked with mathematical task in a test like situation: alone and without other aid than a calculator. When students have other aids, for example peers or textbooks, the reasoning may be radically different (Lithner, 2003).

Theoretical framework

In this study we used to a large extent the framework presented in Lithner (2003). The first part of the framework is to structure the data by describing problematic situations, using a four step reasoning structure:

1. A *problematic situation* is met where it is not obvious how to proceed.
2. *Strategy choice:* Try to choose (in a wide sense) a strategy that can solve the problematic situation.
3. *Strategy implementation.*
4. *Conclusion:* A result is obtained.

In step two, a predictive argumentation can support the strategy choice and in step three, a verifying argument can support the implementation of the strategy. The argumentation is the most important part, since it makes it possible to discuss why the student acts in a certain way.

The second part of the framework is to classify the argumentations concerning the strategy choice and the conclusion. In the framework four different reasoning types were presented. These types are results from analysis of empirical data (Lithner, 2003). The types were:

- *Plausible Reasoning* (PR). Reasoning mainly based on intrinsic mathematical properties without having to be complete or fully correct.
- *Reasoning based on Established Experiences* (EE). Reasoning based on previous experiences from the learning environment.
- *Algorithmic Reasoning* (AR). Recalling a certain algorithm that will probably solve the problematic situation.
- *Piloted Reasoning* (PdR). Someone else controls all the strategy choices that could have been problematic to the solver.

Here, reasoning had the same meaning as in Lithner (2003): "the line of thought, the way of thinking, adopted to produce assertions and reach conclusions". In this definition, reasoning doesn't necessarily have to be based on formal deductive logic, and it may even be incorrect. Mathematical reasoning is any type of reasoning in mathematical task solving.

Due to the limited space we will not present the definitions of all concepts involved, and we will not present all the details of the full framework. The definitions can be found in Lithner (2003).

Research questions

This study was based on the following research questions:

Q1: In what ways do students manage or fail to engage in PR as a means of making progress in solving tasks?

Q2: What are the roles of EE, AR, PdR or other types of reasoning in these situations?

Method

Three different upper secondary school study programmes were represented in the study: the natural science programme (NV, the most mathematically intense program), the social science programme (SP, also a programme meant to prepare the students for higher studies just as the natural science programme, but not focusing on mathematics) and the hotel, restaurant and catering programme (HR), the latter a vocational programme and one of the least mathematically intense. Five students from each programme participated, and the students were in their eleventh school year (age 17 - 18 years). The sessions were video taped, using a camera placed directly above a sheet of paper and each session was limited to 40 minutes. The recordings showed the students' written work and use of a calculator. The students were asked to 'think aloud' and solve the task in a 'test-like' situation. To each programme three to six mathematical tasks were selected (Bergqvist et al, 2003). The tasks dealt with mathematics recently covered by the teachers in each class and similar tasks could in most cases be found in the students' textbooks.

After each session we tried to interpret the students' work. The goal was to make a first description of what was taking place, and also to speculate why this was happening and how the students were thinking. A second meeting with the students took place three or four days after the first session. The aim was to increase the reliability of the interpretations. The students were invited to comment on their own performance and we also suggested possible reasons why they were acting in a certain way. The impression we got was that the students commented our suggestions in a clear and honest way. Seven out of approximately fifty task solving attempts were chosen for a more detailed analysis. The choice was made to be a representative selection of all problematic situations.

A quick description of the students that participated in each programme (Bergqvist et al, 2003):

- Natural science programme (NV): the students that participated tried to a large extent to figure out or remember the appropriate method. They were rather good in the carrying out of the attempted procedures, but they had problems in reviewing their work.
- Social science programme (SP): the students from this programme were very much focused on finding correct algorithms, but they all had severe difficulties in analysing the mathematics involved in the algorithms.
- Hotel, restaurant and catering programme (HP): The students we studied from this programme had a tendency to guess what algorithm should be used, but they seemed not to remember the algorithms very well.

Results and discussion

In this study we analysed seven situations, but many other situations were also considered. In many situations the students simply solved the task by applying a correct algorithm. We picked problematic situations where it was not clear to the students how to proceed. In the data almost all students failed or had severe difficulties, but many of the chosen students solved other tasks correctly. We rejected tasks where the students didn't meet any problematic situations after choosing an algorithm (Bergqvist et al, 2003).

In the analysis we found that six out of the seven students chose their strategies on only or mainly surface property considerations, and they focused on using more or less well-mastered algorithms (Bergqvist et al, 2003). It was a high dependency on finding relevant complete algorithms, or at least algorithms possible to use. In most situations, the choice of method was trying to remember something related to the situation at hand. Sometimes, this may be a reasonable strategy, but it is often insufficient when meeting different kinds of problematic situations (Bergqvist et al, 2003).

When the students for some reason failed to carry out the chosen algorithm, two main different approaches, often combined with questions or comments to the interviewer, were found (Bergqvist et al, 2003):

- To quickly change to another algorithm chosen from a 'toolbox' of possible alternatives, and the decision whether an algorithm was appropriate or not was based on surface considerations.
- To simply stop working.

The questions or comments that the student proposed to the interviewer were made in order to get some kind of hint or guidance about what to do next. There were hardly any situations where the students made an attempt to do some kind of evaluation of the chosen algorithm, or to reconstruct or modify the algorithm to the situation at hand.

One important result from this study was the identification of a description of a new reasoning type, a repeated AR that will be called Repeated Algorithmic Reasoning (RAR), with the following definition:

Repeated Algorithmic Reasoning (RAR)

The reasoning in a task solution attempt will be called *repeated algorithmic reasoning* (RAR) if the reasoning fulfils both of the following two conditions:

1. The general strategy choice is to repeatedly apply algorithms, where each local strategy choice is founded on recalling that a certain algorithm will (probably) solve a certain task type. The algorithms are chosen from a set of (to the reasoner) available algorithms that are (to the reasoner) related to the task type by surface properties only.

2. The strategy implementation is carried through by following the algorithms. No verifying argumentation is required. If an implemented algorithm is stalled or does not lead to a (to the reasoner) reasonable conclusion, then the implementation is not evaluated but simply terminated and a new algorithm is chosen.

RAR does not differ from AR only in the sense that it is repeated. The main difference between these two is that to be classified as RAR, the chosen algorithms must be chosen on surface property considerations only. It must be stressed that, especially if the set of algorithms to choose from is small and each one is relatively simple, RAR often works well.

Only one clear example of PR was found among the chosen situations (Bergqvist et al, 2003). In several situations, a possible (and elementary relative to the course taken) PR seemed close, for instance asking one self about the appearance of a graph or the meaning of an equality sign. It was striking that even a limited amount of reflection or afterthought could have led the students to far more positive results (Bergqvist et al, 2003).

One purpose in our work was to make a study similar to Lithner (2000). We made the following comparisons (Bergqvist et al, 2003):

- The same lack of PR was found, but when in Lithner's study it was replaced by EE (reasoning based on established experiences), it was here replaced by AR (algorithmic reasoning) and RAR (repeated algorithmic reasoning). One reason behind this difference might be that AR and RAR are not effective choices in undergraduate mathematics. Due to the much larger range of available algorithms and procedures, some other means to guide the superficial strategies must be found.
- In both studies there was a dominance of superficial reasoning strategies. This was in both studies found to be one of the main reasons behind the students' difficulties.

The research questions that served as a basis for the analysis were:

Q1: In what ways do students manage or fail to engage in PR as a means of making progress in solving tasks?

Q2: What are the roles of EE, AR, PdR or other types of reasoning in these situations?

To answer the questions we summarised and discussed the types of reasoning that were found.

The students in the Social Science programme (SP) and the Hotel, Restaurant and Catering programme (HR) to a large extent used algorithmic reasoning (AR) in their work. They tried to find a suitable algorithm, often by trying to remember or on other superficial grounds. The algorithm was chosen because it had something to do with the situation at hand, and (occasionally) that it seemed possible that it would solve the task. The algorithm was then carried out, step by step, mostly without attempts to verify or evaluate the algorithm or the result. It is possible that AR is an effective method in school mathematics, at least on lower levels (year 6 - 10), where the number of possible algorithms in each area is very small. If there are two alternatives, the chance of picking the right one is rather good, especially if you take surface `keyword-like' characteristics into consideration.

Repeated algorithmic reasoning (RAR), where new algorithms were chosen on more or less surface considerations, was tested when no progress is experienced. That type of reasoning was found in the work of a few students in the Natural Science programme. They changed quickly between algorithms, and appeared rather skilful in their work. The problem was that they rarely had any intrinsic property considerations as a ground for their choices of algorithms, and therefore the possible success depends on two things: if the chosen algorithm will solve the task, and if the algorithm is mastered good enough by the student. In the most clear example of RAR, one student tries four different algorithms in only a few minutes. The first choice would have solved the task if she had mastered the whole algorithm and not only the first part.

RAR might need a slightly higher level of understanding compared to AR, since it requires some kind of evaluation of the algorithm at hand, even if this evaluation can be made on very superficial grounds. If so, RAR would have had a better chance of success than AR, since almost all available algorithms could be tested. RAR should also be more valuable at upper secondary school (year 10 - 12) where each mathematical area contains several algorithms, and not only two or three as in earlier years. The definition of RAR is a result of this study.

Among the situations where the students used algorithms that failed, the most notable part was the almost total lack of attempts to understand why the algorithm failed, or if it could be modified to the situation at hand. This lack also seemed to be one major reason behind their difficulties.

There are also several situations, mainly in the Social Science programme and in the Natural Science programme, where students rely heavily on their interaction with the interviewer. This is classified as piloted reasoning (PdR), since all important strategy choices are either made by the interviewer, or a result of a question or a comment from the interviewer. Piloted reasoning is in line with the concept of the didactical contract (Brousseau, 1997). The students listen carefully to what the teacher says, and acts according to the conversation. For the students, this is a way to get correct answers in a very large part of the tasks. To the teacher, it means a quick and manageable way to guide almost a whole class through the textbook. Brousseau (1997) claims that as long as both teacher and student follow the didactical contract, no learning occurs. However, piloted reasoning can also be something positive, a way to help a student to reach understanding in an area. If the student is uncertain at a specific level, it is possible for the teacher, by piloted work at a higher level, to help the student to strengthen his or her understanding on the lower level. In that case, it is crucial that the guidance is not too extensive so that the guider does not resolve all problematic situations for the student.

One student in the Natural Science programme used PR in his work. He analysed the shape of a graph and could from that decide where to find the smallest value. Traces of PR was also found in a few situations with other students. In several situations plausible reasoning seemed near at hand to the students, but for various reasons, the students refrained from using it.

It appeared that sometimes students did not attempt to use PR, and sometimes their conceptual understanding was not sufficient for PR. The two competences conceptual understanding and PR ability are probably connected, since PR requires basic conceptual understanding, and the latter may not be possible to develop by only solving routine, non-PR, exercises where the main goal is to practice algorithms (Lithner, 2003). It should be stressed that the conceptual understanding that is indicated in the analyses as missing are quite elementary relative to the courses the students have taken. Students who are encouraged and able to use plausible reasoning in their mathematics learning, will stand a better chance to avoid what Ross (1998) calls "a matter of following a set of procedures and mimicking examples without thought as to why they make sense".

References

Bergqvist, T., Lithner, J. & Sumpter, L. (2003). Reasoning characteristics in upper secondary school students' task solving. *Research reports in Mathematics Education, 1,* Department of Mathematics, Umeå University.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer Academic Publishers.

Lithner, J. (2000). Mathematical reasoning in task solving. *Educational Studies in Mathematics*, *42*, 165-190.

Lithner, J. (2003). Students' mathematical reasoning in university textbook exercises. *Educational Studies in Mathematics*, *52*, 29-55.

Ross, K. A. (1998). Doing and proving: the plave of algorithms and proof in school mathematics. *American Mathematical Monthly*, 252-255.

Teachers' Preparedness for 'Modern Mathematics' in Iceland

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Introduction

This study is concerned with how the Icelandic school system and, in particular, teacher education, was prepared for the radical alteration of 'modern' mathematics and its language in the 1960s and onwards. Four measures, taken or not taken by the authorities, hindered the development of mathematics teaching in Iceland:

- Mathematics was not among the main subjects in the single education institution in Iceland, the Reykjavík Learned School/Gymnasium, for four decades in 1877–1919, when there was no mathematics department in the school.
- Admission to the Reykjavík Gymnasium was restricted for two decades, 1928–1946, shortly after the establishment of its mathematics department.
- Teaching of mathematics at the Teacher Education College, established in 1908, decreased during four and a half decades in 1922–1967.
- The part of the education bill in 1946 about further education of teachers was not adopted.

The study considers whether the compound effect of these four events influenced the capacity of the Icelandic school system to cope with the modern mathematics reform movement and its special emphasis on formal symbolic mathematical language. Or did other circumstances, such as the Icelandic inheritance of home- and self-education mend the situation?

The Learned School

The roots of the conditions of mathematics education for teachers in Iceland can be traced back to the 19th century. At the beginning of the 19th century, Icelandic students alone, graduating from the sole learned school in Iceland, were exempt from requirements in mathematics at the University in Copenhagen, Iceland being a colony of Denmark and Copenhagen thus its capital. However, during 1822–1862 the Learned School, first situated in Bessasta_ir and later in Reykjavík, enjoyed excellent teaching from Björn Gunnlaugsson, a mathematician who had earned two gold medals for mathematics at the University in Copenhagen. The students studied arithmetic, algebra, geometry, stereometry and trigonometry. (Skólask_rsla, 1847–1862).

Discussions about teaching modern languages in learned schools were intense in Denmark and other European countries during the decade after Björn Gunnlaugsson's retirement. In 1871 the Danish parliament passed legislation on division of Danish learned schools into two departments, a language and history department and a mathematics and science department. In 1875, following Iceland's own constitution in 1874, a committee was appointed in Iceland to prepare a regulation for the Icelandic school. In 1876 it presented a proposal where three modern languages were introduced: French and English as compulsory and German optional. Hebrew was eliminated, Greek and a couple of other subjects were to be reduced while Latin and mathematics would keep their previous status. (Álitsskjal, 1877). Letters from the governor of Iceland in Reykjavík to the minister of Iceland in Copenhagen and from the minister to the king of Denmark, found in the National Archives of Iceland, reveal interesting intrigues resulting in the sacrifice of mathematics education in the last two school-years for increased Danish education. (The National Archives of Iceland. Íslenska stjórnardeildin. S. VI, 5. Skólamál. Isl. Journal 15, nr. 680. Skjalasafn landshöf_ingja LhJ 1877, N nr. 621). The Reykjavík Learned School became a language department school. Stereometry and trigonometry were dropped and mathematics lost its place as a prestigious subject in the Icelandic school system for a period of 40 years.

The problems dealt with during that time were possibly reflected in the following final examination item from 1914 (Skólask_rsla, 1914):

$$\left(\frac{1\frac{92}{689}\cdot 4\frac{15}{142}}{\frac{32}{39}\cdot 1\frac{1}{14}} - \frac{\frac{707}{871}\cdot\frac{91}{101}}{\frac{21}{134}} \right) \cdot \frac{8190\cdot 0{,}05^2+5{,}04}{11{,}34:\frac{3}{4}}\cdot\frac{29\cdot 0{,}12\cdot 4{,}6+58\cdot 7{,}164}{10{,}788}$$

The Danish school system was reformed in 1903, resulting in greater coherence. In 1904 a new regulation about the Icelandic school was adopted, where it kept its learned school's characteristics to a larger extent than the Danish schools. It remained as a 6-years language department school with Latin as the main subject as before, and it did not have any direct connections to other schools (Torleifsson, 1975, p. 70–72).

The primary teacher education

Following Iceland's home rule in 1904, the first legislation on a public school system was adopted in 1907. Children of ages 10–13 years were to be educated in schools and 7–9 years old children should be educated at home upon the responsibility of the homes under the supervision of communal authorities.

From 1746 regulations had prescribed that the homes were responsible, under the supervision of the parish priests, for children acquiring knowledge in reading and Christendom. Only in 1880 was a similar regulation implemented for writing and arithmetic. The regulations were effective concerning reading and even writing while arithmetic education was beyond the capacity of many homes.

For this reason many of the first students attending the Teacher Education College had not been to school before and had to be taught basic arithmetic skills. The college was fortunate to have a doctorate mathematician, Ólafur Dan Daníelsson, for its first decade (1908–1919) and mathematics had its due place as a main subject. For most of the following 40 years, until the 1960s, the college did not have a full-time tenured mathematics teacher. The mathematics' share of teaching hours decreased from 8% of the total teaching hours to 5%, while the school was lengthened from three years to four. The syllabus, up to 1946, did not reach introduction to algebra. Thus the graduated teachers were, by and large, not familiar with the special symbolic language of advanced mathematics. (Sk_rsla um Kennaraskólann 1908–1962).

As was the case in the single learned school/gymnasium in the country in 1805–1919 there was only one mathematics teacher in the Teacher Education College in 1908–1967. The responsibility and the respect of the subject greatly depended on whether there was a respected scholar at work, such as the gold medalists Björn Gunnlaugsson and Ólafur Daníelsson, or persons deeply engaged in other obligations or even having personal problems.

In 1946, following Iceland's independence in 1944, new education legislation was adopted, establishing a coherent school system from the age of 7 up to university. Education from 7–15 years was now compulsory and a national examination at the age of 16, held in many schools, opened the door to a four years' gymnasium.

A part of the 1946 legislation bill was aimed at ensuring further education of teachers in the University of Iceland, established in 1911, i.e. to educate teachers for the lower secondary level. However the University rejected the idea of accepting students from the Teacher Education College and the bill was never presented for final discussion in the parliament. The bill only allotted for one new professor, in pedagogy and didactics, and the University seems to have been expected to take over further education of teachers in Icelandic and mathematics at no extra cost, neither in housing nor teaching. The University referred to 'another kind of preparation'. (Al_ingi. Dagbók 45–46. 615). The Teacher Education College graduates would not have fitted into the mathematics teaching for the engineer students with the preparation they had. The Teacher Education College was therefore a dead end to the primary teacher education until 1963. Teachers had to seek further education abroad, in teacher university colleges in the Nordic countries, England and the United States, which only a small fraction did.

Before 1946 teacher graduates only had had arithmetic but after 1946 they were provided with first courses in algebra and classical geometry. Graduation from the college only provided rights to teach in the primary schools, including the first year of the lower secondary school up to the age of 14 and possibly the second year. However in the time of a great shortage of lower secondary school teachers the entire period 1946–1977 until the basic school was established, primary teachers were often appointed to teach in the lower secondary level, though preferably not for the national examination. In 1957–1958 there were 760 tenured primary teachers in the country, whereof 604 had teacher education, or 79 %. In the same year, tenured teachers at the lower secondary schools were 275, whereof about 70 had primary teacher education, or 25 % (Gunnarsson, 1958, p. 141).

The lower secondary teacher education

In 1919 a mathematics department was established at the gymnasium in Reykjavík and the mathematician Ólafur Dan Daníelsson was hired to teach there. Shortly thereafter, in 1928, the minister of education concluded that the increasing number of applicants to the Reykjavík gymnasium, (_orleifsson, 1975, p. 74) which still was the only upper secondary school, would result in a too numerous class of professional men and officials. The minister initiated the establishment of a number of district schools in the following decade to educate young people to become good farmers and housewives, which indeed increased greatly the general education in the country, while, at the same time, he decided to restrict the admission to the gymnasium. Even if Akureyri gymnasium was established in the more sparsely populated Northern Iceland, the restriction resulted in that only approximately 370 students graduated from the mathematics department in Reykjavík Gymnasium (Skólask_rsla, 1919–1946), and a little over 100 in Akureyri Gymnasium in the period 1919–1946 (Jónsson, 1981, p. 27–34).

According to the 1946 legislation the requirements for teachers at the lower secondary level were 1–2 years of study in the respective subject and the equivalent of a one year course in general pedagogy and didactics. However there was no institute in the country to train mathematics teachers and the Second World War had cut Iceland off from the university in Denmark.

As the Teacher Education College had had no algebra and mathematics was first taught in the University of Iceland from 1941 to students in the newly established engineering department, the only competent candidates for mathematics teaching for the national examination were among the less than 500 students from the gymnasium in 1919–1945, most of whom had become professional men and officials.

A short survey of five teachers, born in the 1920s, renowned for being ‘good’ mathematics teachers, shows that all of them had graduated from one of the gymnasiums’ mathematics department. Thereafter their background was studies of engineering, medicine, law or economy for 1–3 years or else one year in the Teacher Education College (Kennaratal).

The University of Iceland established a BA-education programme for mathematics teachers for the lower secondary level in 1952, mainly as a part of the engineer training. Approximately 30 students graduated from the programme in the period 1952–1972 (Árbók Háskóla Íslands, 1952–1972). Only 15 of these ever taught at the lower secondary level (Kennaratal). As we shall see they proved good servants to the education system. They were trained by dedicated teachers, such as Gu_mundur Arnlaugsson, who shared with them their mathematical skills. Most of the students themselves had begun teaching and therefore received the instruction with that in mind.

The 'modern mathematics' reform

Iceland was a founding member of OEEC, later OECD. However, probably for the reasons of small population and relative isolation at that time, Iceland did not participate in 1959 in the important OEEC seminar on 'modern mathematics' teaching in Royaumont, France, nor did it participate in the work of the NKMM, the Nordic committee for modernizing mathematics teaching. During the next few years, though, Icelandic educators participated in several follow-up meetings about modern mathematics teaching, arranged by OECD, and in 1967, the modernizing of mathematics teaching was in action at all school levels in Iceland, mainly by the initiative of the university and gymnasium teacher Gudmundur Arnlaugsson.

The situation in teacher education up to the mid 1960s when the 'modern mathematics' reform a kind of swept over Iceland at all school levels has been described. In addition to the lack of teachers with adequate education there was lack of textbooks. Textbooks for the lower secondary level in arithmetic and algebra were either written by Ólafur Daníelsson in the 1920s or based on his books. Textbooks for the primary school were also written in the 1920s by one of Ólafur Daníelsson's student teachers. Impatience for innovation was in the air.

Reform in the Lower Secondary Schools

In the lower secondary school the reform began with a textbook written by Gu_mundur Arnlaugsson for the selective group heading for national entrance examination into the gymnasium (Arnlaugsson, 1966). In 1968 all these pupils were studying modern mathematics (Ministry of Education, 1968). In the early 1970s modern mathematics was introduced to all levels of the lower secondary school.

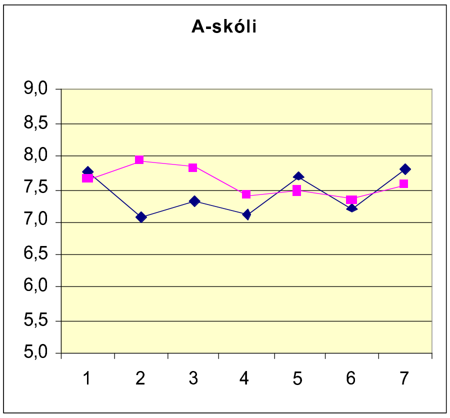
Headmasters of the lower secondary schools, running the national examination all around the country, put their pride into finding 'good persons' from the small group of eligible people, to take care of the mathematics. A small survey shows how the national examination results in mathematics were dependent on stability in the teacher force and the education of the teachers.

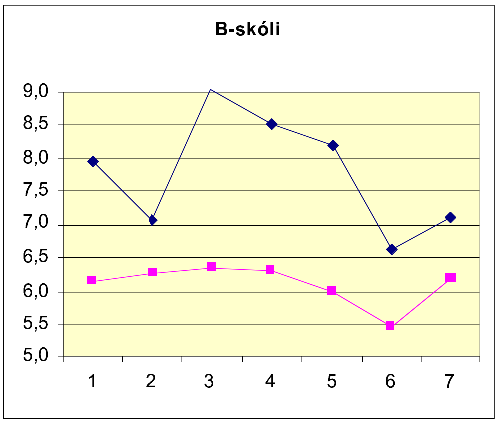
The results in the national examination in 6 schools in the period 1967 to 1973 are shown in the graphs below. The red curve shows the general average of grades in 9 subjects: Icelandic, Danish, English, history, geography, natural sciences, physics and mathematics, while the blue curve shows the average for mathematics (Ministry of Education, National Examination Board 1967–1973).

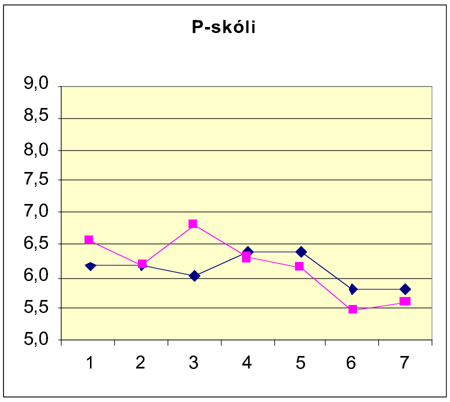
Schools A and P were situated in the capital area, School A was a selective school, attended by especially able pupils.

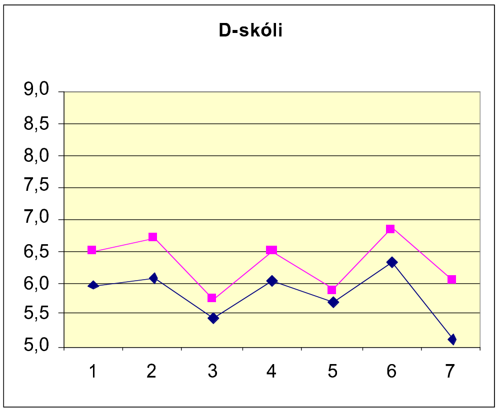
Schools B and R were typical fishing town schools.

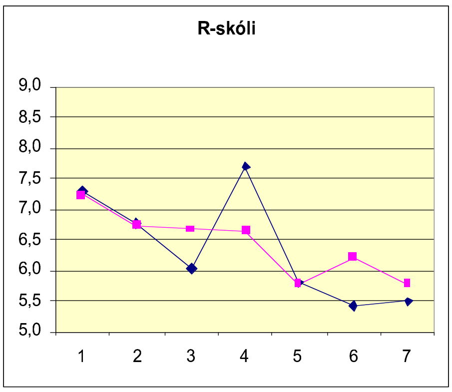
Schools D and S were boarding schools in rural areas.

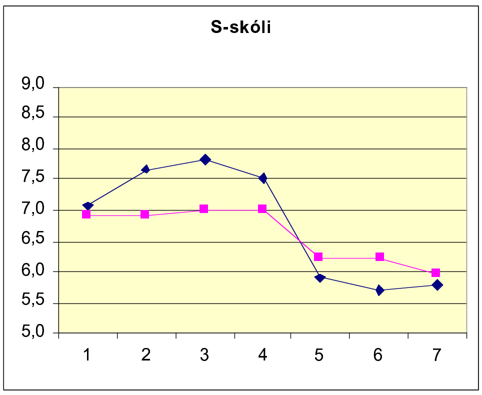












In 1967 several schools began teaching modern mathematics as a part of the syllabus and in 1969 all schools had modern mathematics as a part of their syllabus. In most years mathematics was 0,2 below the general average on a scale 0–10, while in the years 1969 (year 3), possibly as all schools were now obliged to take up the modern syllabus, and in 1971 (year 5) the average for mathematics was considerably lower (0,6 and 0,8) than the general average. The graphs have been corrected for this by adding the difference to the mathematics grades.

In school A the mathematics teacher, a lawyer, was on leave in 1967 to 1969 while non-mathematics university students had taught mathematics. After the lawyer came back and shared the mathematics teaching with a BA-educated teacher the results became on equal terms with other subjects.

In school P a BA-educated mathematics teacher was on sickness leave in 1967 and 1969. Otherwise the results were above average. The headmaster said (in 2002) that the results had been better earlier when the entrance to the national examination had been restricted. The results with the present mathematics teacher had been affected by his illness. He had not realized that they were above the average.

In school B a BA-educated teacher taught the pupils for all three years of lower secondary education and the results were by far the best in the country. In 1969, when the mathematics average in the country was 0,6 below the general average, it did not affect his pupils. The headmaster said (in December 2003) that he knew that the results in mathematics were better than in the neighbourhood schools. However he believed that this had been the case in most subjects in his school, as he had had excellent staff. The mathematics teacher though had managed to motivate his pupils to work hard. The headmaster had not connected the results with the mathematics teacher education.

School R suffered from frequent changes of teachers. In 1968–1970 a BA-educated teacher taught mathematics. The first year the results dropped while in the second year the results were exceptionally good.

School D was a typical rural school with some changes of mathematics teachers with no special training, even if they avidly attended in-service courses in ‘modern mathematics’. The results were generally about 0,4–0,6 below average.

School S was also a rural school where a BA-educated teacher died suddenly in 1970. After that university students took over. The headmaster said in a conversation (in 2002) that mathematics had been taken good care of in his school and it always had had extra hours compared to the standard number recommended by the ministry of education. The teacher had been an excellent person while the headmaster did not connect the results in mathematics with his special education in mathematics.

Before the survey only the exceptional results in school B were known. The good results in school S were a surprise as was the one year in school R. They show that a teacher with good education and teaching skills, given time, can obtain good results. The sample of teachers is not typical for schools at that time and by coincidence more schools with BA-educated teachers were chosen than the average. The Reykjavík schools, with the majority of pupils taking the national examination, were not included in the survey as the examination moved between institutions within the national examination period 1946 – 1976.

The survey shows that in this period of ‘modern mathematics reform’ with new and unfamiliar language the few BA-educated teachers seem to have had an advantage of easier adaptation to new syllabus.

Reform in primary schools

In 1966 a set of textbooks by the Danish teacher Agnethe Bundgaard in Frederiksberg, Copenhagen, (Bundgaard et al, 1966–1968) was chosen by the initiative of Gu_mundur Arnlaugsson (Gíslason, 1978), and translated and tested in six groups in the first grade of primary school. The following year, schools and teachers could apply to join the test groups. Over 80 teachers applied and began to teach the new material. In 1969 approximately 50% of the age cohort of seven years old pupils in Iceland had the new material (Kristjánsdóttir, 1977). A great number of in-service courses in mathematics, lasting a couple of weeks, were held, a novelty for most teachers.

Soon after 1970 problems arose. From grade 3 the mathematical language became increasingly formal. When new teachers, who were not familiar with the extremely formal mathematics language in this set of textbooks, took over the classes 4 to 6, problems arose. It became clear to parents and the public that school mathematics teaching was changing radically. This aroused discussions and reactions which are difficult to trace as very little is found in newspapers, journals or letters to the ministry. One teacher dared to express his view in the teacher journal that in no other country would it have been allowed to pour such a new syllabus over the whole population (Sveinsson, 1972). The problems, however, were mainly discussed within the teacher rooms in schools and certainly the teacher consultants must have heard more than a rumour.

Teachers, interviewed now 30 years later, remember this time vividly, all of them for hard work, some of them as an exciting and rewarding time while others thought the syllabus too difficult, both for the teachers and the pupils. All agreed that it was rather aimed at the more able pupils.

The ministry of education hurried to have a new set of textbooks written, a kind of synthesis, with less emphasis on set theoretical language and notation, yet a more varied coverage than the traditional material. This changed the lower mathematics syllabus in Icelandic schools permanently from simple arithmetic to a varied material with geometry, introductory statistics and number theory in addition to arithmetic. The arithmetic skills may never have reached the same level as before the reform, but then one should keep in mind that a new era of hand calculators was on its way.

Conclusions

'Modern' mathematics was a great challenge for both primary and secondary school teachers. The tradition of mathematics education in Iceland was weak for the four reasons cited in the introduction. They certainly influenced the capacity of Icelandic teachers to cope with the 'modern mathematics' reform movement, as only few teachers in compulsory schools could acquire adequate training in mathematics in contemporary sense. Yet the reform opened up a whole new area of previously unfamiliar mathematics for them. Icelandic education tradition was self- and home-education, a tradition which may have helped many teachers over the most difficult obstacles.

However, it was shown that it was easier for well educated lower secondary teachers to guide their pupils to the core of the reform, deeper understanding, while others may have looked at the new syllabus as just a set of a new type of problems. Those primary teachers, who were inclined for mathematics, enjoyed this opportunity for further education through this work, while others were overwhelmed and gratefully accepted the synthesis syllabus, something they might not have done without the previous experience.

Lastly it is remarkable to notice how the history of mathematics education in Iceland depended on single individuals; first Björn Gunnlaugsson in the middle of the 19th century, then Ólafur Dan Daníelsson in the first half of the 20th century and thirdly Gu_mundur Arnlaugsson in the second half. They are examples of how, in a small community as Iceland is, an individual can have a strong influence on the whole community.

References

Al_ingi. Dagbók 45 – 46, nr. 615.

Arnlaugsson, G. (1966). *Tölur og mengi*. Reykjavík.

Álitsskjal nefndarinnar í skólamálinu (1877). Reykjavík.

Árbók Háskóla Íslands (1953–1973). Reykjavík.

Bundgaard, A. & Kyttä, E. (1966–1968). *Stær_fræ_i. Reikningur*. 1a–2b. Reykjavík.

Bundgaard, A. (1968–1972). *Stær_fræ_i. Reikningur*. 3a–6. Reykjavík.

Gíslason, K. (1978). "*N_ja stær_fræ_in (The New Mathematics)*". An unpublished manuscript. Reykjavík.

Gunnarsson, F. (1958). *Kennaraskóli Íslands 1908–1958*. Reykjavík.

Jónsson, G. (Ed.) (1981). *Saga Menntaskólans á Akureyri 1880–1980, 2*.

Kennaratal, I–V. (1958–1965, 1985–1988). (A biographical dictionary of teachers). Reykjavík.

Kristjánsdóttir, A. (1977). *Um stær_fræ_inám í grunnskóla skólaári_ 1977–1978*. (A letter to mathematics teachers). Ministry of Education. School Research Department. Reykjavík

Ministry of Education, National Examination Board (Landsprófsnefnd) (1968). *Drög a_ námskrá í landsprófsdeildum mi_skóla*. Reykjavík .

Ministry of Education. National Examination of Middle School. *Examination Books 1967–1973*. Reykjavík.

Private conversations with headmasters in schools B, P, R and S and teachers in schools A and D in 2001–2003.

Sk_rsla um Kennaraskólann í Reykjavík 1908–1962.

Skólask_rsla fyrir Reykjavíkur lær_a skóla (1847–). (A school report for the Reykjavík School which from 1946 was called Hinn lær_i skóli í Reykjavík (Lær_i skólinn), from 1904 Hinn almenni menntaskóli í Reykjavík and from 1936 Menntaskólinn í Reykjavík).

Sveinsson, M. (1972). Huglei_ingar um stær_fræ_inám fyrr og sí_ar. *Menntamál, Vol. 45* (2). Reykjavík.

The National Archives of Iceland. Íslenska stjórnardeildin. S. VI, 5. Skólamál. *Isl. Journal 15*, nr. 680.

The National Archives of Iceland. Skjalasafn landshöf_ingja LhJ 1877, N nr. 621. Tillögur rá_gjafans um reglugjör_ fyrir hinn lær_a skóla.

Torleifsson, H. (1975). *Saga Reykjavíkurskóla, I*. Reykjavík.

Teachers and Assessment –
A Description and Interpretation of Teachers' Actions Connected with the Mathematics National Test for School Year 5

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Lärarhögskolan i Stockholm

In Sweden there is no external examination of pupils' knowledge - the class teacher does all the assessment. To assist in this assessment there are National Tests and diagnostic materials. In this paper I focus on the ways in which teachers assess and describe pupils' performances in mathematics. In our work in PRIM-gruppen[1] with the National Test for school year 5, we continuously collect data from pupils and teachers from all of Sweden and this data is used for the analyses in this paper. These inquiries then form a picture of teachers and their assessment.

Theoretical perspective, purpose and questions

This paper is mainly based on the theoretical perspective entailed in pedagogical assessment (Gipps, 1994). The purpose is to deepen our knowledge about how teachers in school year 5 assess pupils' knowledge achievement in mathematics. The following questions are being discussed:

- Do teachers verbalise their observations and assessments of pupils' knowledge achievement in the competency profile enclosed in the National Test for year 5? What comments do they write about the pupils' competencies?
- How do teachers assess pupils' work in one part of the test?
- What attitudes do teachers have towards the National Test in mathematics and towards pedagogical assessment?

Background

Theories regarding assessment of knowledge

Pedagogical assessment in the 2000s must be based on contemporary knowledge about how learning takes place. We find ourselves now in a cultural transition period, from a testing culture to an assessment culture (Gipps, 1994). Shepard (2000) proposes a similar viewpoint and says that today we are largely centred on constructivism and socio-cultural theory when dealing with theories

[1] See www.lhs.se/prim

of learning. If we intend that all students should be able to investigate, analyse, reason and interpret, then our assessment must reflect this (Gipps, 1994). Also Leder (1992) emphasises the relationship between teaching and assessment and she states that assessment in mathematics does not always consist of the same type of questions and tasks that the teaching of mathematics is expected to have, that is, assessment will include tasks that give the pupil opportunities to use general solutions and to use more unconventional thinking.

Leder also says that it is precisely this joint development of teaching in collaboration with the development of assessment that will influence pupils' learning of mathematics in a positive direction.

De Lange (1992) describes the following principles for the construction of test items:

1. Assessment should be an integrated part of the learning process, i.e. the assessment should support pupils' learning.
2. Assessment should give the pupils the opportunity to demonstrate what they can do rather than what they cannot do.
3. Assessment should reflect and concretise all the course goals.
4. The quality of an assessment does not primarily depend on how objectively pupils' results can be assigned numerical scores.
5. Assessment should fit well into the school environment and programme.

Pettersson (2001) emphasises that assessment is characterized by considerable flexibility. The pupils should be put in different situations so that they can demonstrate their competence in various ways.

Gipps (1994) describes in summary a definition of pedagogical assessment that includes the following points:

- Pedagogical assessment is based on the view that assessing performances is not an exact science; and that the relationship between student, task and context is complex.
- In pedagogical assessment clear goals/criteria are set up for the pupils' performances.
- Pedagogical assessment encourages pupils to think rather than listing or "regurgitating" facts.
- Assessment that elicits the best possible performance of the pupil requires tasks that are concrete and are experienced as being a relevant part of the pupils' world. The tasks are presented clearly and the assessment situation must not be felt as threatening to the pupils.
- With regard to grading and/or scoring the assessor/teacher needs guidance to ensure good comparability with other assessors. This can be achieved by providing examples of assessments/evaluation.

- In pedagogical assessment we should move away from evaluations in the form of one single number/letter. Instead we ought to strive towards other ways of describing the achievement of pupils that include more comprehensive descriptions of their knowledge and profiles of various types.

The National Test in Mathematics for school year 5

In Sweden the teacher has the exclusive responsibility of assessing the pupils' knowledge throughout all school years, that is, there is no external examination. To help the teachers in their assessment the National Agency of Education provides national tests and diagnostic materials.

Every year, subject tests in English, Mathematics and Swedish are given in school year 5. The national tests in mathematics are constructed by PRIM-gruppen at The Stockholm Institute of Education. The test for school year 5 is a service offered to the schools, but most municipalities have decreed that all schools must participate in these assessments. The main purpose of the subject test is to help the teachers to assess the development of knowledge in relation to the goals stipulated for attainment by school year 5. Another aim is to enable the identification of both the strong and weak points of the pupils. The test is carried out during the major part of the spring term each year and is one component in the entire assessment of the student's knowledge made by the teacher. There are no fixed test dates, rather the teacher/school plans and administers the test when it best suits the local conditions. The tests are intended to be integrated into the ordinary teaching and each pupil is given sufficient time to work on the various tasks in the test. Teachers are encouraged to respond to a questionnaire and also to submit student solutions for students born on a certain birth-date. The submitted material can be used for studies of various kinds (Skolverket, 2000).

Not all the goals to be attained by school year 5 are tested by one single test. Instead some specific areas are focused upon in one given year. These mathematical areas are tested by means of tasks of various kinds, from short individual questions to more comprehensive group tasks.

Besides test questions, the testing material also includes information for the teachers to assist them in their assessment. An example of such information is the provision of guiding norms for each question with examples of acceptable answers and descriptions of misconceptions that students might show. The teacher is expected to collate his/her assessment of the knowledge of a pupil in various areas using a competency profile (see appendix). As a guide for this, each part of the test contains a description of what knowledge the pupil ought to demonstrate for each part. As a guideline for the assessment, various acceptable answers that the pupil should have on each part are presented. When the teacher fills in the competency profile table, consideration should also be given to the pupils' performance on other occasions besides the test situation itself.

The subject test for school year 5 conforms well to the majority of the criteria that characterize pedagogical assessment. The pupil has opportunities to demonstrate knowledge in various situations, e.g. short answer questions, longer questions and larger group tasks. In the assessment guidelines there is a clear connection with the stipulated goals to be attained and goals to strive towards and in the competency profile table the goals are clearly presented for the pupil. There are also two different self-assessment parts in which the pupil is given the opportunity to reflect upon his/her own ability (Björklund, 2004).

Method and implementation

The empirical work reported here includes qualitative and quantitative analysis of about 200 competency profiles from the National Test in Mathematics for year 2000, qualitative and quantitative analysis of 200 student solutions from the National Test in Mathematics for year 2000, as well as an analysis of the teachers' answers to the teacher questionnaires. The data concerning the competency profiles and student solutions consists of materials which include both student solutions and competency profile. The test for school year 5 is not compulsory, and it is not possible to draw any general conclusions from these data. I still think that it is possible to get a hint about the general situation, as the test is carried out in more than 90 % of the municipalities.

Results

How teachers fill in the competency profiles

It is not common for the teachers in the survey to make use of the opportunity to comment on/describe verbally their assessments in the pupils' competency profile (see appendix).

In 66% of these profiles such comments are made in two or less of the spaces provided. The percentage of the profiles in which teachers made no comments is 45%. To a much greater extent the teachers make use of the option of indicating using an "X" on the existing arrows. On 88% of the profiles 6, 7, or 8 such marks were indicated. It is clear that the teachers prefer indicating their evaluation using an X rather than describing it in words, when both methods can be used. On the competency profiles where the teacher has filled in one or more spaces, almost one third (31%) have only written a negative comment, e.g. what the pupil is unable to do, general negative comments etc. Seventeen percent describe only strong points. There is a difference between the goals to be attained that are tested in the test for year 2000 and the other goals, in regard to the number of filled-in spaces. Teachers verbalise their assessments for the goals tested by the test to a much greater extent than for the other goals. The same is true for the indication "X" at the arrows provided, but not quite as significant as in the former case.

How teachers assess each task

The test includes tasks of various kinds with respect to assessment. In this paper I have analysed one section of the test. In one of the tasks, pupils are to choose the missing puzzle piece. In this case there is only one correct answer. For several questions there are two or more correct answers. The assessment guidelines give examples of different possible answers to these questions. For some tasks there is a risk that the teacher makes a negative evaluation because of a follow-up error. A follow-up error can arise when a pupil uses the result of a previous question to solve a later one. If the answer to the earlier question is wrong, but the solution of the later one is correct, the answer should be considered correct. For such questions there is a description in the teacher's guide to prevent the student suffering a deduction for such a follow-up error. In some tasks, accuracy is an important aspect of the assessment. In one case the teacher must use a ruler to check an answer. Teachers have followed the assessment guidelines on all questions in 67 % of the student solutions. The types of mistakes that teachers make in assessment agree in part with the character of the tasks described earlier:

- Follow-up errors. The teacher looks blindly at the “correct” answer given in the guidelines, not noticing that the pupil’s apparent error depends on an error in an earlier question.
- Shortage of accuracy. In some cases a student’s solution is assessed as correct though it was inadequate, and vice versa.
- Other errors. Sometimes it is difficult to understand why a teacher makes the assessment he/she does. One such example is when the pupils’ answer is judged as incorrect because the answer is not located at the right place.

Teachers attitudes to assessment

Teachers are generally pleased with the structure and content of the test. They think that the coverage on the national test is acceptable. Most of the comments on the various parts of the test are positive. Teachers also appreciate the competency profiles. The teachers feel that the test gives them support in assessing the performances of the pupils and that the assessment guidelines provide sufficient help. They also feel that the recommendations regarding what pupils should be able to achieve are reasonable in relation to the goals to be attained, as expressed in the study curriculum (Alm & Björklund, 2000).

The fact that the teachers are positively inclined towards the test in general and that they are specifically positive towards the assessment guidelines, makes it evident that they are also positive towards pedagogical assessment – at least in connection with the National Test for School year 5.

Conclusions

In summary, based on previously described analyses, I come to the following conclusions:

- Most teachers in the survey do not verbalise their assessments of pupils' knowledge in the various areas of the subject. Many teachers that in fact do make verbal comments on their assessment reports tend to focus on weaknesses of the pupils.
- Most teachers in the survey, with the help of a subject test in school year 5, have sufficient competence to assess pupils' performance on individual tasks in one section of the test. Some teachers make mistakes in their assessment. This can be of consequence for some groups of pupils.
- Most teachers are by and large positive towards assessment using the test material. Thus I conclude that most teachers are positive towards pedagogical assessment in connection with the National Test for school year 5.

Discussion

Looking back at the findings presented in this paper, my dominating feeling when it comes to the teachers is that they really want to do a good job in assessing their pupils' performances. They also want to document their assessment in a professional way. Thus the first conclusion presented above (saying that most teachers do not verbalise their assessment and that when they do, they tend to focus on weaknesses) don't come from the teachers being lazy or not willing to be supportive to the pupils. When teachers do not verbalise their assessment they probably find the marking with X on the arrows sufficient. When they verbalise the weaknesses to a higher extent than the strengths they probably want to focus on what the pupil has to learn. All this is from the teachers' point of view. From the pupils' point of view it is better when the teachers do verbalise their assessment and when they describe the pupils' strengths. Black & Wiliam (2001) put it this way:

> Feedback to any pupil should be about the particular qualities of his or her work, with advice on what he or she can do to improve, and should avoid comparisons with other pupils.

The Swedish model – how can it be improved?

In my opinion the Swedish model is mainly a good one and should be retained. With the teacher as assessor the pupil has many opportunities both in the testing situation and in teaching situations to demonstrate his/her abilities. But I also think that teachers must get help to improve their competency in assessing. One way towards this end is to provide more time for teachers to do assessment and above all, time for discussions about assessment. It has been shown that the quality of assessment is significantly enhanced if teachers discuss together assessment in concrete terms (Gipps, 1994). Black & Wiliam (2001) also emphasise the importance of the competence of teachers in formative assessment. They state that this in addition leads to better achievement levels for the pupils. They further state that they see no other measure that is so certain to increase the level of performance of the pupils. In view of the findings of this paper, professional development of teachers in the area of assessment would be a desirable measure.

References

Alm, L. & Björklund, L. (2000). Matematik. In *Ämnesprov i svenska, engelska och matematik för skolår 5 vårterminen 2000*. (Subject tests in Swedish, English and mathematics for school year 5, spring term 2000). Stockholm: Skolverket.

Björklund, L. (2004). *Lärare och bedömning*. (Teachers and assessment). Stockholm: Lärarhögskolan i Stockholm.

Black, P. & Wiliam, D. (2001). *Inside the black box. Raising standards through classroom assessment.* London: Kings College.

Gipps, C. V. (1994). *Beyond testing. Towards a theory of educational assessment.* London: The Falmer Press.

de Lange, J. (1992). Critical factors for real changes in mathematical learning. In G. C. Leder (ed), *Assessment and learning of mathematics*. Australia: Acer.

Leder G (1992). Curriculum planning + assessment = learning? In G. C. Leder (ed). *Assessment and learning of mathematics*. Australia: Acer.

Pettersson, A. (2001*). Bedömning av 15-åringars kunskaper i matematik*. (Assessment of achievement in knowledge of mathematics for 15 year olds). Stockholm: Lärarhögskolan i Stockholm.

Shepard, L. (2000). The Role of assessment in a learning culture. *Educational Researcher, Vol. 29*, No. 7, 1-14. http://www.aera.net/pubs/er/arts/29-07/shep01.htm

Skolverket (2000). *Ämnesprov i engelska, matematik, svenska, skolår 5*. (Subject tests in Swedish, English and mathematics for school year 5, spring term 2000). Stockholm: Skolverket.

Competency profile in mathematics year 5 for ____________________

The purpose of the competency profile is to show your *overall assessment* (assessment of Subject Test 5 + other work) of the pupil's merits and weaknesses in mathematics.

Competency areas	Teacher's comments	Attainment of objectives for year 5
Concrete problems in the immediate environment		Satisfactory
Arithmetic basic number perception and number concepts for natural numbers calculating in natural numbers: mental calculation, written calculating methods		
understanding and being able to use calculating methods calculating with a calculator		
numerical patterns, unknown numbers number perception – simple numbers in the form of fractions and decimals		
Geometry comparing, estimating, measuring: length, areas, volumes, angles, mass		
basic spatial perceptions geometrical figures and patterns scale		
time and time differences		
Statistics diagrams, tables, measures of central tendency		

On the Role of Problem Solving and Assessment in Swedish Upper Secondary School Mathematics in Finland

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In this article I will focus on teachers' use of problem solving and of different kinds of assessment in Swedish upper secondary school mathematics in Finland. Information about the teachers' use of problem solving and assessment methods as well as possible signs of their willingness to change the instructional practice in these respects are gathered as a part of a survey, directed to all the mathematics teachers in Swedish upper secondary schools in Finland.

Problem solving as a part of instruction

My starting point for a conceptual understanding of problem solving is found in Mason and Davis (1991), according to whom the Problem Solving Theme Group report from the Fifth International Conference on Mathematics Education (Burkhardt et al., 1988) "took as the salient characteristic of a problem that the problem solvers face an unfamiliar task and they do not know an immediate path to a solution". They continue with pointing out important aspects of mathematical thinking like specialising, generalising, conjecturing and convincing. Furthermore, they find a supportive atmosphere in the classroom and challenging questions important for problem solving instruction and state that "an atmosphere of constant questioning and conjecturing is more likely to be effective than the occasional burst of problem solving within an otherwise formal framework".

To meet new challenges in work, school, and life, students will have to adapt and extend whatever mathematics they know. Doing so effectively lies at the heart of problem solving. A problem solving disposition includes the confidence and willingness to take on new and difficult tasks. By learning problem solving in Mathematics, students should acquire ways of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations that will serve them well outside the mathematics classroom. (NCTM, 2000)

In Principles and Standards there is generally a strong emphasis on the processes in mathematics instruction and these words tell us about the direction of many American efforts in the area of problem solving in the last decades, also stated by others, for instance Lambdin (1999). Earlier Schoenfeld (1985), having performed a study of "heuristic training", made the conclusion that problem solving practice is not enough but explicit training is required, and that much work has to be done in order to find out an appropriate classroom instruction.

Two years earlier Frank Lester (1983) had summarised four basic principles about problem solving and asserted them to have emerged from research literature over about fifteen years:

- pupils must solve many problems in order to improve their problem-solving ability
- problem solving ability develops slowly over a prolonged period of time
- students must believe that their teacher thinks problem solving is important in order for them to benefit from instruction
- most pupils benefit greatly from systematically planned problem solving instruction

Lester's conclusions will serve as my starting point as concerns problem solving in this article. It should be said that although my text so far could have been based on work from other parts of the world as well, the progress in the area of problem solving and instruction in USA is quite outstanding, all the way from Polya's (1957) fundamental work (Black & Atkin, 1996).

My perspective on problem solving is to great extent that of the teacher, which makes it quite natural to pose questions to teachers in a survey. The role of problem solving in mathematics instruction was discussed by Ernest (1998), who made a classification in four perspectives. His third perspective is the problem solving perspective, which has its origins in the work of Polya (1957). In accordance with Ernest, Björkqvist (2001) has formulated that the perspective is the problem-solving perspective when the most important to learn is considered to be the acquisition of problem solving strategies and heuristics.

With at least partly a problem solving perspective, it is desirable to use the concept problem solving in a wide sense. It could mean, for instance, an ability to handle questions such as “Can you find a way to handle this task?” when posing an almost but not quite familiar task to the pupils. It could also mean that the pupils are expected to find a solution to a problem that is in a way similar to what they have seen but requires another strategy, or that is not so similar at all but can be solved by a known strategy. The tasks could also take more or less time to solve, from some minutes to several hours, when talking about investigations or modelling projects. Modelling projects can also be related to Ernest's first perspective, which is the applications perspective. Björkqvist (2001) finds Ernest's first perspective to be near the origin of what is often called realistic mathematics instruction, when sub-processes in the transfer between reality and the abstract world of mathematics are formalized and a mathematical modelling of phenomena in the real world is accomplished. Björkqvist also considers it reasonable to regard mathematical modelling as the most profound kind of problem solving in mathematics.

Assessment as a part of instruction

Charles et al. (1987) state that problem solving has to be a regular and frequent part of the instructional program if the students shall consider problem solving to be important. They also think that assigning grades to progress in problem solving makes problem solving more important for the students and they give guidelines like "Advice students in advance that their work will be graded" and "Use a grading system that considers the process used to solve problems, not just the answer".

In the Principles and Standards (NCTM, 2000), under the headline "The Assessment Principle", the authors make the following statement as concerns the important purposes of assessment: "Assessment should be more than merely a test at the end of instruction to see how students perform under special conditions; rather, it should be an integral part of instruction that informs and guides teachers as they make instructional decisions. Assessment should not merely be done *to* the students; rather, it should also be done *for* the students, to guide and enhance their learning." They continue with several considerable research-based statements about assessment enhancing the learning:

- research indicates that making assessment an integral part of classroom practice is associated with improved student learning
- the tasks used in an assessment can convey a message to students about what kinds of mathematical knowledge and performance are valued
- feedback from assessment tasks can also help students in setting goals, assuming responsibility for their own learning, and becoming more independent learners
- a focus on self-assessment and peer-assessment has been found to have a positive impact on students' learning.

Assessment is also a valuable tool for making instructional decisions, and a wide variety of assessment methods as well as purposes for assessment are indicated:

- assessment and instruction must be integrated so that assessment becomes a routine part of the ongoing classroom activity rather than an interruption
- in addition to formal assessments, such as tests and quizzes, teachers should be continually gathering information about their students' progress through informal means, such as asking questions during the course of a lesson ...
- assessment … should focus on students' understanding as well as their procedural skills
- to make effective decisions, teachers should look for convergence of evidence from different sources
- over-reliance on formal assessments may give an incomplete and perhaps distorted picture of students' performance

- assessments should allow for multiple approaches
- although less straightforward than averaging scores on quizzes, assembling evidence from a variety of sources is more likely to yield an accurate picture of what each student knows and is able to do (NCTM, 2000).

Problem solving and assessment in upper-secondary schools

Although Finland, in comparison to most other countries, gives less time to Mathematics in the upper secondary school, the Finnish results in Third International Mathematics and Science Study (TIMSS) have been widely noticed. However, Näätänen (2001) points out that the results indicate that it has not been paid enough attention to the fundaments of mathematics in Finland. Especially the way of practicing problem solving is not sufficient to build a systematic knowledge in mathematics and to develop abstract thinking. In the bases for upper secondary school curricula (1994), however, it is in the general part stressed that the pupils should learn to examine all information critically and to apply their knowledge to different problems and practical situations. Furthermore, in the part for mathematics it is strongly emphasized that instruction should give the pupils abilities to make applications in problem solving and also training to find solutions to mathematical problems. According to Näätänen (2001), these statements may not necessarily give the right inspiration to a sufficient instruction in problem solving.

As concerns assessment, students in upper secondary school in Finland can get two marks in Mathematics, one given by the teachers in the school and an optional one based on the result in the national matriculation examination (ME) after completed upper secondary school. In the national test in mathematics there are two tests, one based on the 10 compulsory courses in “long course” and one based on the 6 compulsory courses in “short course”. In both tests there are 10 tasks (chosen out of 15 tasks) that ought to be done in six hours. The tasks in such an exam have a very strong impact on the teaching in the upper secondary school (Burman, 2000). There are reasons to believe that the teachers think they have no time to focus on anything but the important content that is supposed to be assessed, in order to prepare their pupils for the exam in the best possible way.

In addition, the whole system makes it hard to teach problem solving in another way than with the kind of tasks that can be found in the final national assessment. Moreover, when teachers feel that there is a considerable time pressure, they stick to “good old methods” rather than to new and untried ones. Of course, there is a considerable amount of applied tasks in the final exam, but many of them are pseudo-realistic and have no close similarities with task situations in life beyond school (Palm & Burman, 2002). When a low proportion of the tasks include a relevant real-life question, students may get the feeling that solving applied tasks is a game, with rules not necessarily consistent to the rules of real-life problem solving (Palm & Burman, 2002). In this respect an alignment between curriculum, instruction and high-stakes assessment does not occur. The consequence is a conflict and we are far from assessing the specific skills of a mathematical modelling process.

Is it possible to find a way out of this conflict, at least as concerns the students' marks from the school? One possibility is to include tasks and even projects that in a better way correspond to e.g. problem-solving abilities. Another possibility is to use tests during the courses, which enables the inclusion of more varied tasks. Of course, there is not much time for assessment in mathematics in Finnish upper secondary schools, because there is not much time for instruction. Consequently, there is a need for a kind of assessment that is effective and takes as little time as possible. Therefore, one of the most interesting aims with the survey is to get some answer to the question how the teachers solve their almost insolvable time-equation under the pressure from the ME.

The research questions

On the theoretical base above and with respect to the Finnish situation, I have formulated the object of my interest in four research questions, A–D. Each question has given rise to statements and questions in the survey. The teachers are supposed to give marks to the statements on a Likert scale, from 1 = "do not agree at all" up to 5 = "totally agree". In the questions, they are supposed to give percentages between 0 and 100 to some alternatives. The four research questions are:

A. To what extent do the teachers include problem solving in their instruction?

B. Which methods do the teachers use when they assess their pupils?

C. To what extent do the teachers include problem solving in their assessment?

D. Are the teachers satisfied with their present methods in problem solving and assessment or are they willing to change their instruction in these respects?

About the sample

The participants in the survey were, in November 2002, registered as members of the Swedish Teachers' Union in Finland (Finlands Svenska Lärarförbund, FSL) and at that time also registered as mathematics teachers at the lower or upper secondary school level. Finding the addresses through the FSL was considered to be fast and reliable enough, as teachers belong to the teachers' union with a percentage close to 100%. The target group consisted of 223 teachers when mathematics teachers at the teacher training school in Vasa as a pilot group for the survey were excluded.

In Januari 2003, questionnaires were sent out to 68 teachers in upper secondary schools and 155 teachers in lower secondary schools. In the stipulated time 82 of them answered. After a follow-up letter and a reminder in the membership paper of FSL, the answers reached the very satisfying number 132 (59%). Seven questionnaires were empty, mostly because the respondent had recently left teaching because of retirement or other duties. Of the remaining 125 respondents I have picked out those 54, who had answered the questions for teachers in upper secondary school. They were 44 teachers in upper-secondary school and 10 other teachers, who had chosen to answer the questions of both levels, although they might have had their main teaching in lower secondary school.

Results from the survey

Question A. In the survey 89% of the responding teachers totally or partly agreed that there are some important strategies in problem solving to focus on in their instruction. When asked about the relation between problem solving and assessment 92% totally or partly agreed that problem-solving ability should have impact on the marks in mathematics, 98% totally or partly agreed that the students should be good problem solvers in order to get a high mark and 83% totally or partly agreed that students should be able to describe methods of solving tasks for high marks. About project work in mathematics 45% of the teachers totally or partly agreed to give projects an influence on the marks in mathematics but only 17% totally and 13% partly agreed with the issue of having at least one project in mathematics during the students' time at upper secondary school.

Question B. About assessment methods the teachers agreed totally or partly as follows:

- 55% to use tests during the courses as a part of their instruction
- 66% to usually construct at least one new task for a test in mathematics
- 77% to prefer tests with some kind of possibility to choose among the given tasks
- 100% to want the students to take responsibility for their own learning
- 91% to try to make the student aware of their knowledge
- 51% to use tests during the courses as a complement to the course test.

The teachers estimated that they in the short course in mathematics chose 28% of the tasks from previous tests, 32% from books or test banks, 16% from previous tests in the Finnish matriculation examination and that 23% of the tasks were home-made for the test. Corresponding percentages for long course in mathematics were 24%, 40%, 15% and 16%.

Question C. Most of the results as concerns including problem solving in the teachers' assessment have already been presented in A. In addition, less than 10% of the teachers said that they include tasks with new elements in their tests and the teachers estimated that 4% of the final marks in a course are based on problem solving and 2% are based on projects in mathematics.

Question D. About the teachers' willingness to make changes in their instruction, there was a difference between answers for short course and answers for long course in mathematics. As concerns short course, 65% of the teachers were in favour of decreasing the weight of the course test for the marks and 48% of increasing the weight on tests during the course, 42% of increasing the weight on problem solving and 32% of increasing the weight on projects in mathematics. The corresponding percentages for long course were 53%, 32%, 26% and 29%. On the whole, when willingness to make changes was indicated, it was often with small steps. Half of the teachers that suggested changes had at most a 10% change in the weights in assessment situations. Teachers, who were over 50 years old, or 30 years old or younger, showed less interest in making changes than the others. The percentages were, respectively, 33%, 44% and 80%. Finally, male teachers showed more interest for changes than female, as with 32 male and 22 female teachers the result was 69% to 50%.

Concluding remarks

At first it has to be said that some teachers in the actual group of respondents may know the author and the one who asks the questions as teacher educator and perhaps try to give expected answers. Moreover, it is also possible that the respondents have a different comprehension of concepts such as "test during the courses", "problem solving" and "projects in mathematics". These facts are reasons for having asked indirect questions in the survey and for making conclusions with a certain margin for security, but they are not considered strong enough to prevent from making conclusions. For instance, young teachers did not use more problem solving or projects or show more interest for making changes in their instruction than older teachers, as could have been a consequence if there were many "expected answers".

As concerns the results about problem solving, more than half of the teachers seem to think that their present instruction is enough to prepare their students in this respect without including any "explicit training", as they agree with considering problem solving important (several percentages about 90% in A) but do not intend to change their instruction in this area (42% and 26% in D). As only 30% of the teachers want to include more projects in mathematics, several teachers may have their picture of projects from other subjects and consider them to take much time and to give very little mathematical benefit. The impact of matriculation examination was confirmed although not extremely strong, as 51% of the teachers totally or partly agreed that national tests are important for their instruction and 26% totally or partly disagreed.

References

Björkqvist, O. (2001). Matematisk problemlösning. In B. Grevholm (Ed.), *Matematikdidaktik – ett nordiskt perspektiv* (pp. 115-132). Lund: Studentlitteratur.

Black, P., & Atkin, J. M. (Ed.) (1996). *Changing the subject – Innovations in science, mathematics and technology education*. London and New York in association with OECD: Routledge.

Burkhardt, H., Groves, S., Schoenfeld A., & Stacey, K. (Eds.) (1988). *Problem solving – A world view. Proceedings of the Problem solving theme group, ICME 5, Adelaide 1984*. Nottingham: Shell Centre.

Burman, L. (2000). Förutsättningarna för matematikundervisningen i gymnasiet i Finland. In J. Sjöberg & S-E. Hansén (Eds.), *Kasvatus tulevaisuuteen, Pedagogik för framtiden*, (pp. 193-203). Rapporter från Pedagogiska fakulteten vid Åbo Akademi, Vasa, Nr 22 2000.

Charles, R., Lester, F. & O'Daffer, P. (1987). *How to evaluate progress in problem solving*. Reston, VA: National Council of Teachers of Mathematics.

Ernest, P. (1998). Mathematical knowledge and context. In A. Watson (Ed.), *Situated cognition and the learning of mathematics* (pp. 13-31). Oxford: Centre for Mathematics Education Research.

Grunderna för gymnasiets läroplan (1994), Utbildningsstyrelsen.

Lambdin, D. (1999). How can research inform the preparation of standards for teaching and learning school mathematics? In Ole Björkqvist (Ed.), Quality aspects of mathematics and science education, *Reports from the Faculty of Education*, Åbo Akademi University, No 5 1999, 45-67.

Lester, F. (1983). Trends and issues in mathematical problem solving research. In R. Lesh (Ed.), *Acquisition of mathematics concepts and processes* (pp. 229-261). New York: Academic Press.

Mason, J. & Davis, J. (1991). *Fostering and sustaining mathematics thinking through problem solving*. Melbourne: Deakin University.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Näätänen, M. (2001). *Mitä TIMSS-tutkimus kertookaan suomalaisten koululaisten matematiikan taidoista ja matematiikan opetuksesta*? Netpaper No. 1 2001. [On line]. Available: http://solmu.math.helsinki.fi/2001/1/timss/ [May 12, 2003]

Palm, T., Burman, L. (2002). Reality in mathematics assessment: An analysis of task-reality concordance in Finnish and Swedish national assessments. In T. Palm, *The realism of mathematical school tasks – Features and consequences*. Doctoral Thesis No 24, 2002, Department of Mathematics, Umeå University.

Polya, G. (1957). *How to solve it*. Second Edition. Princeton: Princeton University Press.

Schoenfeld, A. (1985). *Mathematical problem solving*. New York: Academic Press.

Hamper or Helper: The Role of Language in Learning Mathematics

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Introduction

This paper reports a part of a study of how ten successful high school pupils from Denmark and England explain that they learn a mathematical concept that is new to them. It focuses on what the pupils tell about the role of language. The analysis uses various theories of learning to get a greater understanding of the pupils' explanations.[2]

Methods

Metacognition can be understood both as knowledge about and regulation of cognition. This knowledge develops with age and there is a positive correlation between performance on many tasks and the degree of metacognition. In relation to regulation of cognition, this is related to the planning of activities prior to problem solving, the monitoring as one goes along, and the checking of the outcome. The "presence of such behavior has a positive impact on intellectual performance ... its absence can have a strong negative effect" (Schoenfeld, 1985, p. 138). One might therefore assume that successful pupils know how they learn mathematics. I therefore asked five teachers to pick some of their best pupils in the classes that studied mathematics at the highest high school level. In Denmark it is *Niveau A* at the *Gymnasium* and in England it is the *AS (Advanced Subsidiary) Level Mathematics*. I named the Danish pupils: Z, Æ, Ø, Å and the English pupils: A, B, C, D, E, F. The Danish pupils were interviewed as one group and the English pupils were interviewed in pairs. Pupils D, Z, Æ, Ø, Å were girls; A, B, C, E, F were boys. The English pupils were given a sheet with some knot theory to initiate a discussion, but otherwise all the interviews were explorative.

Theories of learning that focus on language

The opinions of the role of language are divided. To some, language is a necessary thinking-tool; to others it obstructs thinking.

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[2] The study reports parts of a Ph.D. study (Dahl, 2002). Other findings are reported in Dahl (2004).

Language as helper

According to Hadamard, Polya said that "the decisive idea which brings the solution of a problem is rather often connected with a well-turned word or sentence. The word or the sentence enlightens the situation, gives things, as you say, a physiognomy" (Hadamard, 1945, p. 84). Also Russell talked in a positive way about language: "Language serves not only to express thoughts, but to make possible thoughts which could not exist without it. … I hold that there can be thought, and even true and false belief, without language. But however that may be, it cannot be denied that all fairly elaborate thoughts require words" (Russell, 1948, p. 74). I will call this a *moderate positive* view since it holds that words not only express or "mirror" thoughts but also more actively helps thinking. However, not all thoughts require words, only higher-order thinking.

Another positive view is seen in Muller who stated that "no thought is possible without words" (Hadamard, 1945, p. 66). This view is connected with Vygotsky's description of language as *the* logical and analytical thinking-tool (Vygotsky, 1962, p. viii) and that thoughts are not just expressed in words but come into existence through the words (*ibid*., p. 125). Vygotsky also said that "Language does not of necessity depend on sound" (*ibid.*, p. 38), and that: "Thought development is determined by language, i.e., by the linguistic tools of thought and by the socio-cultural experience of the child" (*ibid.*, p. 51). Thus, to Vygotsky, thoughts develop from social interaction and what we learn is inherently a product of human communication, and it would not exist for us if we were not part of the human community. These *strong positive* arguments are basically that no thought or learning can take place without the use of language and that language is an indispensable thinking-tool that makes thoughts come into existence.

Language as hamper

Berkeley argued that "words are the great impediment to thought" (Hadamard, 1945, p. 68). Galton explained that results can be perfectly clear to himself but "when I try to express them in language I feel that I must begin by putting myself upon quite another intellectual plane. I have to translate my thoughts into a language that does not run very evenly with them" (Hadamard, 1945, p. 69). Hadamard stated that "thoughts die the moment they are embodied by words" (Hadamard, 1945, p. 75), and that a thought "can be accompanied by concrete representations other than words. Aristotle admitted that we cannot think without images" (Hadamard, 1945, p. 71). Furthermore, "the more complicated and difficult a question is, the more we distrust words, the more we feel we must control that dangerous ally and its sometimes treacherous precision" (Hadamard, 1945, p. 96). But Hadamard acknowledged that "signs are necessary support of thought" (Hadamard, 1945, p. 96). Wittgenstein argued that a main source in our lack of understanding is that we do not have an overview of the use of our words, the grammar is confusing. Philosophy is a battle against the bewitchment of our mind by means of the language (Wittgenstein, 1983, §109 & §122-126). The views quoted here all centres on language as either obstructing or confusing thinking, that words are not always necessary for thoughts, or that language is not always able to express thoughts. Piaget (1970, pp. 18-19) stated that "This, in fact, is our hypothesis: that the roots of logical thought are not to be found in language alone, even though language coordinations are important, but are to be found more generally in the coordination of actions, which are the basis of reflective abstraction". Here Piaget seemed to disagree with Vygotsky who stated that thoughts develop from social interaction where the language is the thinking-tool. To Piaget, knowing an object does not mean to copy it, but to act upon it: "Knowing reality means constructing systems of transformations that correspond, more or less adequately, to reality" (*ibid.*, p. 15). An abstraction is "drawn not from the object that is acted upon, but from the action itself. It seems to me that this is the basis of logical and mathematical abstraction" (*ibid.*, p. 16). Hence, actions perform the basis of mathematical thinking, not language or interaction.

A dual nature of language

Vygotsky criticised Piaget around the concept of egocentrism[3] and egocentric speech.[4] Vygotsky stated that egocentric speech is not just accompanying the child's activity but the child uses it as a means of expression and to release tension, but it soon becomes an instrument of thought that the child uses to, for instance, plan the problem-solving (Vygotsky, 1962, p. 16). Thus, to Vygotsky egocentric speech, besides its communicative role, is an important thinking-tool and as a tool to solve problems. "Egocentric speech emerges when the child transfers social, collaborative forms of behavior to the sphere of inner-personal psychic functions" (*ibid.*, p. 19). Vygotsky then continued and stated that: "Thus our schema of development - first social, then egocentric, then inner speech[5] - contrast both with the traditional behaviorist schema - vocal speech, whisper, inner speech - and with Piaget's sequence - from nonverbal autistic

[3] The notion of egocentrism in Piaget's work is "quite unrelated to the common meaning of the term, hypertrophy of the consciousness of self. Cognitive egocentrism, as I have tried to make clear, stems from a lack of differentiation between one's own point of view and the other possible ones, and not at all from an individualism that precedes relations with others" (Piaget, 1962, p. 4).

[4] Piaget stated that "I have never spoken of speech 'not meant for others'; this would have been misleading, for I have always recognized that the child thinks he is talking to others and is making himself understood. My view is simply that in egocentric speech the child talks for himself" (Piaget, 1962, pp. 7-8). This was Piaget's response when Vygotsky wrote that Piaget's view was that "In egocentric speech, the child talks only about himself, takes no interest in his interlocutor, does not try to communicate" (Vygotsky, 1962, pp. 14-15).

[5] To Vygotsky, inner speech "is not the interior aspect of external speech - it is a function in itself. It still remains speech, i.e., thought connected with words. But while in external speech thought is embodied in words, in inner speech words die as they bring forth thought. Inner speech is to a large extent thinking in pure meanings" (Vygotsky, 1962, p. 149).

thought through egocentric thought and speech to socialized speech and logical thinking" (*ibid.*, p. 19-20). To Vygotsky, the order of the development of thinking is from the social to the individual. This internalisation process has two levels, the social and the individual: "first between people (interpsychological), and then inside the child (intrapsychological)" (Vygotsky, 1978, pp. 56-57). As a response to Vygotsky's views and critique, Piaget referred to Vygotsky's propose that:

> egocentric speech is the point of departure for the development of inner speech, which is found at a later stage of development, and that this interiorised language can serve both autistic ends and logical thinking. I find myself in complete agreement with these hypotheses. On the other hand, what I think Vygotsky still failed to appreciate fully is egocentrism itself as the main obstacle to the co-ordination of viewpoints and to co-operation. ... In brief, when Vygotsky concludes that the early function of language must be that of global communication and that later speech becomes differentiated into egocentric and communicative proper, I believe I agree with him. But when he maintains that these two linguistic forms are equally socialized and differ only in function, I cannot go along with him because the word socialization becomes ambiguous in this context: if an individual A mistakenly believes that an individual B thinks the way A does, and if he does not manage to understand the difference between the two points of view, this is, to be sure, social behavior in the sense that there is contact between the two, but I call such behavior unadapted from the point of view of intellectual co-operation. (Piaget, 1962, pp. 7-8)

Vygotsky emphasises that language is not just a means of expression; it is also an instrument of thought. Piaget agrees, but found that Vygotsky failed to understand that egocentrism itself could be a main obstacle for reaching understanding through the use of language.

Analysis

Below I will discuss three examples from the interviews.

Hamper AND helper

This example is from an interview with two English pupils, D and E. 'They' refers to the authors of the book about knot theory.

E	Don't use such big words, they are aiming to people who don't understand it and use basic. It's the way they approach it, the language, it's just too, people would struggling with the language when they are suppose to be learning the maths.
I	So is there a diff, I mean, er, so maths has nothing to do with the language? Or, can you learn maths without language.
D	Yea.
E	No, but you can use different language, simple language to convey a point.
D	Cause the maths in it is quite easy, I think, well, it's not. It is nothing really difficult what it is saying is this is what a knot is, this is (E: Yea) what a link is, and, OK, that really really simplistic. It took me a long time to work out what they were trying (E: Yea what they were explaining) whereas the fact as soon as I, kind of translated it, I thought oh well, that's what a knot is, find that's easy.
I	What did you translated it.
D	Into simple language [laughs] er, it er.
I	You translate it before you understand it, er, so if you have understand, then, you don't need to translate it.
E	I think it here would be easier if the author translated (D: Yea [laughs]) rather than er leaving the reader to do it, I mean.
D	You have to do the two together, you have to translate while you're trying to understand.

Pupil D and E seem to agree that language can be a hindrance to learning but at the same time at least Pupil E argues that one cannot learn mathematics without using language, but he finds that one can use different language, for instance simple language. Pupil D seems to argue that one indeed can learn mathematics without the use of language. Elsewhere in the interview she argues that one can learn mathematics without language, at least when the mathematics is easy. She explains that it took her a long time to understand what the authors wrote but it was easy as soon as she translated it. It can therefore be interpreted that to Pupil D, language can obstruct thinking, but it is also a necessary translation-tool, and that simple language, after being translated from a difficult language, seems to be a thinking-tool. Pupil E also expresses that the appropriate (for instance simple) language is a thinking-tool. This might seem odd as they also state that language/notation confuses the meaning, which could be an example of the double nature of language that Piaget argued for. Pupil E does not himself directly argue for a dual nature of language whereas Pupil D directly states that one has to "do the two together, you have to translate while you're trying to understand".

Written language

This example is from an interview with two English boys, Pupil A and C.

A	If I am revising from my notes, I find that helps if I actually write the notes out again, just copying them, out from you know [C coughs] the previously. Because, if you just sort sit down to revise, you read through, even if you read it aloud, you pick up some things, remember some things. But if you write it out, you sort of read again and then write again (I: mmm) and it sort of reinforces it. And I definitely found er, if it is something where I have to memorize [1 sec silence], you know sort of examples, sort of methods, equations, how things work, I definitely find it easier if I write, er, as an aid to memory [2 sec silence] (I: mmm) er, [1-2 sec silence] Yea.
I	Do you know other tricks or do you know other people have other tricks you know about?
	[2-3 second where C and A talk at the same time].
C	Remembering or learning?
I	Er. Why is? Is there a difference between remembering and learning?
A	Yea, well [interrupted].
C	Memorizing something, then you need to know the [1 sec silence] set number of points (I: mmm) so you need to write them out, and, you know, find some sort of sequence, in to remember it. But if you are just learning, then it's about understanding, you don't need to remember (A: Yea) the detail, just (I: mmm) need to know the overall, you know, concept (I: mmm) [A tries to break in] you know, principals.
A	Yea, it's important to understand how things (C: Before) how you GET the answer (C: mmm). Because [2 sec silence] YEA, it is in math, it's going back to the same thing. It's fairly easy to learn a general formula for loads of different things like trigonometry and things like that, you can just LEARN the general formula (I: mmm), and every time you get a question, you can just sit and use it and get the answer out. But then if you come up against a problem which is SLIGHTLY different to the general [inaudible] you've learnt, if you just memorized it and you don't understand how it's got there, so you've got no change to work back, and work it out for yourself (I: mmm) whereas if you understand it [1 sec silence], you can see where it's comes from and see where you need to change it to fit what you've got. (I: mmm) Got to try and do.

Pupil A argues that written language is an aid to memory, which does not mean that he sees written language as the thinking-tool. Vygotsky's view was that thoughts come into existence through the words after the learner has participated in a social interaction where he has internalised aspects of the knowledge. To aid the memory is according to Pupil A not real learning. Real learning is to know the overall concepts and principles. Furthermore, elsewhere in the interviews there is a discussion about whether words come first or not, and here Pupil A says that he is not sure but it probably depends on the problem. If it is a visual problem where one has to think it through in 3D, he finds that it is probably better to first have the picture or a graph, but with a linear algebra problem, it might be better to have the words first and then the pictures to help one understand, because he finds that it is the words one is trying to understand. This means that although words and language have a great value they are by no means *the* way to get to understand all branches of mathematics.

Ping-pong between language and images

Below are pieces from the interview with four Danish girls, here mainly emphasising on Pupil Z's explanations. The original Danish text can be found in Dahl (2002, pp. 356, 359-361).

Z	If you for instance sit down and then say, all right, what is a triangle, and then line up what kind of concepts we are working with, it is so very important to have the concepts lined up. What is the problem really about, and what kind of area are we in, and then you can begin to work with it. And I think that it is very important to have some terms and have them worked through.
...	...
Z	I am a pupil-teacher in mathematics and I have been sitting with a pupil who get so caught in what is *f*, (Someone else: mmm) and it is idiotically to sit and wonder about this, it is not what it is all about. I want to get to the point. But this is what is relevant when one does not understand what it is that is really relevant (Someone else: mmm). And where one is going.
I	Yes OK, this sounded like you talk about a kind of bricks built on top of each other (someone else says yes yes). [Interrupted by Z]
Z	It's like the teacher speaks a different language than us (Someone else: mmm). It's a little like if you learn Italian through Danish then you would also sit and get stuck in what the teacher said instead of the meaning of it, right (I: mmm). The teacher comes with this big mathematics language and then we become so stuck with what the things are called instead of what it means.
I	Can one find the meaning outside of language? [A few seconds of silence].
Z	This is perhaps a bit philosophical?
Z	I think it is difficult. I think that it is very difficult.
Ø	I also think so. Then you would need a kind of flair for it, or a kind of feeling of in what direction it goes. Because it is almost hopeless if you do not have the ehh basics and then to understand what it is that you are suppose to reach (Someone else: mmm).

...	...
Z	It is also important, this motivation, which I also thinks lies in the confusing, there is a kind of motivation there, right. One really WANTS to learn it. (I: mmm) And this you need to have, because it does not help just sitting and learning this mathematics LANGUAGE first does it. I also think that we [inaudible] have to stick to what you [Ø] said, that perhaps a person has a kind of flair for it and is able to see this thing in another way. And I also think that this is where the distinction is between starting to see the thing FOR YOURSELF and creatively, in some kind of way, because if one really has this flair [inaudible] one can just see this spatial geometry, one can see it inside one's head the moment one is told about it. And then one does not need all these concepts, if one can see it for oneself. (Someone else: mmm), and then I think that one begins to work in quite a different way with it (Someone else: yes) [a few seconds silence]. One can also first [inaudible] concepts and examples.
I	Do you think perhaps that there is a ping-pong between that one in a way can see it and then the language?
Z	Yes.

Pupil Z explains that it is important to have some terms and have them worked through. For instance in geometry it is very important to line up the concepts one works with to be clear about what the problem is about. Pupil Z also expresses that it is difficult (but she did not say impossible) to learn without the language and as an example she mentions spatial geometry. She explains this further by saying that if one knows the notation, it is simply logical. According to her, one can perhaps learn mathematics even if one has not understood the notation. The notation is what seems relevant when one does not understand what it is that is really relevant as one becomes stocked with what the things are called, the mathematics language, instead of what it means. Learning is ping-pong between images and the language. She seems to argue that language/ notations are necessary tools in learning, but that they in themselves are not mathematics and that an unsuitable use of language can block learning. Notions can hamper learning if one does not know them, especially if one does not know them in the beginning of the learning process. It may be possible to learn mathematics even if one has not understood the notation. This seems to some extent to contradict what Pupil Z says when she states that one cannot find meaning without language, but it could also reflect that, to her, the (right) language is *necessary* but not *sufficient*.

Discussion of typologies

The ten pupils fall into various groups. Pupils C, Æ, and Å do not seem to express much here, but otherwise I will also include information about pupils not quoted above. See Dahl (2002) for full transcriptions of all the interviews:

I: Difficult to learn outside language {A, Z, Ø}.

II: Language is *the* thinking-tool {B, D, E, F}.

III: Language can hamper learning but language can also help learning {B, D, Z}.

IV: Language can hamper learning {E, F, Ø}.

The main difference between Group I and II is whether learning is only “difficult” outside language or if it is “impossible” without language. In Group I, knowing the notation is essential to be able to learn the mathematics. Language can function as a thinking-tool and it is difficult to find the meaning outside the language. They do not, however, say that it is impossible to learn outside of language. This seems similar to the moderate positive view of language. Group II has a strong positive view of the nature of language. Language is not just a means of translation, but language is a necessary thinking-tool for learning. Different languages facilitate learning. Some pupils argue that simple language is the best. In Group III the pupils describe in their own words a dual nature of language in learning. This could be interpreted to be in line with Piaget’s description of that language, due to egocentrism, can both hamper and help learning. Members of Group IV have either a more “negative” view of the role of language for learning or they elsewhere describe the dual nature of language, but without themselves reflecting directly on this. They merely at one place in the interview say one thing, at another place they say something else.

Conclusions

There seem to be various views of language in relation to how the pupils explain that they learn a mathematical concept that is new to them. Some pupils say that language is the main thinking-tool, others that it hampers thinking, and again others think that language has a dual nature as it both facilitates learning and hampers learning. Some directly indicate this through expressions such as “you have to do the two together” or agrees that there is a “ping-pong” between language and images. This also depends on the kind of language, for instance simple language or written language. The members of these groups are criss-crossing nations and gender. Referring to Piaget’s critique of Vygotsky, Piaget declared himself to be very much in line with Vygotsky about the positive role of language, but Piaget also argued that Vygotsky failed to acknowledge the obstacles language can give rise to. Thus, based on what the pupils here have explained, Piaget seemed to be right in his critique. But the quotes shown here do not reveal whether thoughts develop from social interaction and internalisation or from personal activities and construction. However, in Dahl (2004) it is discussed that the pupils are divided on this issue. The pupils do, however, in the quotes above seem to argue that thoughts (in line with Hadamard) can be accompanied by other representations than words, for instance images or, as Pupil Z puts it, by having a "flair" for it. The latter might be connected with the notion of inner speech which is thinking in pure meaning. They do not seem to argue that language is not always able to express thoughts. Hence, the pupils investigated here seem in various degrees to argue for a dual role of language, namely as language both being a hamper *and* a helper.

References

Dahl, B. (2002). A focus group study of Danish and English high-achieving high school pupils of mathematics: What can we learn from their verbalised explanations of how they learn mathematics? Ph.D. thesis. Roskilde University, IMFUFA, Denmark. (Author name: Bettina Dahl Søndergaard)

Dahl, B. (2004). Analysing cognitive learning processes through group interviews of successful high school pupils: Development and use of a model. *Educational Studies in Mathematics*. Forthcoming.

Hadamard, J. (1945). *An essay on the psychology of invention in the mathematical field*. New York: Dover.

Piaget, J. (1962). *Comments on Vygotsky's critical remarks concerning 'The language and thought of the child' and 'Judgment and reasoning in the child'*. Cambridge, MA: The M.I.T.

Piaget, J. (1970). *Genetic epistemology*. New York: Columbia University.

Russell, B. (1948). *Human knowledge - Its scope and limits*. London: George Allen & Unwin.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. London: Academic.

Vygotsky, L. S. (1962). *Thought and language*. Cambridge Massachusetts The M.I.T..

Vygotsky, L. S. (1978). *Mind in society - The development of higher psychological processes*. Cambridge, MA: Harvard University.

Wittgenstein, L. (1983). *Philosophical investigations*. Oxford: Basil Blackwell.

Deaf Children's Concept Formation in Mathematics

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The aim of the study is to illuminate deaf children's concept formation in mathematics by describing how some deaf children express themselves and act on their way towards understanding two basic concepts, the concept of multiplication with whole numbers and the concept of length.

Theories developed by Feuerstein (Feuerstein et al., 1979; 1988; 1991) are used in order to describe how deaf children develop concepts, and to investigate possibilities to help deaf children develop their cognitive potential in a more effective and adequate way. Concept maps illustrate steps and pathways taken by the pupils. The importance of language in concept formation, with a focus on sign language is illuminated.

The children in this study were pupils in a School for the Deaf, a bilingual school with the languages Swedish Sign Language and Swedish. Seven 11-year-old pupils, all the pupils in one group in grade 4, were studied. Video recordings were made of pupil-teacher interactions in problem solving situations in sign language only, with paper and pencil, with learning materials and with real things.

A large variation in the pupils' ability to solve the problems was found depending on different factors identified by Feuerstein, e.g. self-confidence, looking for meaning, search of challenge, intention to finish the work and use of known facts. No difference was found concerning the steps towards comprehension of the concepts for the deaf pupils in the study compared to those of hearing pupils. In accordance with earlier studies it was found that the deaf pupils needed more time to learn mathematics than hearing pupils normally do. As a consequence, they may learn certain concepts at a later age and the pathways towards comprehension may vary compared to those of hearing pupils. The structure of sign language and the lack of an established terminology in mathematics are also of importance.

The bilingual situation for deaf pupils is a reason for developing methods of teaching mathematics to deaf pupils, alternative to methods used today.

Introduction

This study (Foisack, 2003) is based on the question why deaf children have difficulties in learning mathematics. International research (Frostad, 1996; Magne, 1991; Moores, 2000) shows that deaf pupils achieve much lower results on tests in mathematics than hearing pupils do. On the other hand there is no research available today showing that the cognitive potential of deaf pupils differs from that of hearing pupils (Martin, 1991). If it is a fact that deaf pupils do not use their cognitive potential to a full extent, it is of great importance to investigate why this is so, and to find possible ways of improvement. Other ways of assessing than by ordinary tests might show other results than those referred to.

The children in this study were pupils in a Special School for the Deaf, a bilingual school with the languages Swedish Sign Language and Swedish. Going to school in a Special School for the Deaf is in some ways different from going to a regular school. The sign language environment is crucial as well as the bilingual approach with the two languages Swedish Sign Language and Swedish in its written form as two separate subjects (Skolverket, 2000/2002). The basic objectives are the same as in all schools in Sweden. In mathematics the deaf pupils are taught on the basis of the same curriculum as all other pupils in Sweden (Lpo94, 1994). Syllabi and grading criteria (Skolverket, 2000) are the same. In the curriculum for compulsory schools in Sweden the aim of learning mathematics is “to master basic mathematical thinking and be able to use it in everyday life”.

As to communication, and particularly language, as an important tool for developing thinking the importance of sign language for sign-language users in learning mathematics is emphasized in this study. Deaf children learn sign language in a natural way if they are exposed to it in a sign language environment. That is the case for most deaf children in Sweden today. Swedish is regarded as a second language for deaf people. Most deaf children learn Swedish at the same time as they learn to read and many of them not until they start school. Thus it is self-evident that mathematics is taught in sign language from the beginning and that sign language is the language in which they develop basic concepts. By the time the ability to read and write is developed, the pupils can find new information on their own and express themselves in written language, but there is still a need for direct communication in sign language. In the syllabus for the schools for the deaf (Skolverket, 2000/2002) there is a general text on bilingualism from which the following quotation is taken:

> Learning occurs through both languages. Bilingualism is therefore important in all school subjects, not only in the two subjects Swedish and Swedish Sign Language. Every subject has its own subject-specific concepts and a terminology that the pupils do not meet in other subjects. As a consequence it is every teacher’s obligation to make sure that the pupil masters these concepts in both languages.

Concerning mathematics there is a great amount of well defined subject-specific concepts and related terminology in spoken/written languages that pupils in school need to master. The quotation above shows that there is a need for an established terminology in Swedish Sign Language for mathematics. It does not seem fair that tests used in national assessments should be available in written Swedish only, the second language of deaf pupils. Most learning material is also in written Swedish.

Aim

The aim of the study is to illuminate deaf children's concept formation in mathematics by describing how some deaf children express themselves and act on their way towards understanding two basic concepts, the concept of multiplication with whole numbers and the concept of length.

How deaf children achieve knowledge of basic mathematical concepts is studied in order to find ways to meet the needs of deaf children learning mathematics and to promote more effective and adequate teaching in general. Questions of significance to the study are:

- How do deaf pupils, those who have already understood as well as those who are on their way to understanding the concept in question, express themselves and act when confronted with a mathematical problem of this nature?
- Does the way deaf pupils express themselves in sign language influence their concept formation?

From a mathematics education perspective the following questions are then raised:

- What steps are needed for understanding the concept?
- Are the steps the same for deaf pupils the same as for hearing pupils?

In order to follow the development of the pupils on their way towards understanding the concept, learning situations are arranged on the basis of theories developed by Feuerstein concerning mediated learning.

Theoretical framework

Two perspectives were used in the study, cognitive education and mathematics education. The cognitive education perspective is based on theories developed by Feuerstein (Feuerstein et al., 1991), Structural Cognitive Modifiability (SCM) and Mediated Learning Experience (MLE). Feuerstein has a constructivist view of learning in contrast to a behaviourist view. SCM is closely related to theories by Piaget, but the uniqueness in Feuerstein's theory is the connection with the theory called Mediated Learning Experience (MLE). Feuerstein (Feuerstein et al., 1988) has been influenced by Vygotsky (1978) as to how learning is developed in a social context. The most important characteristics of MLE are mediation of intention and reciprocity, mediation of transcendency and mediation of meaning. MLE is used in the method for assessing children's cognitive potential developed by Feuerstein, called Dynamic Assessment (Feuerstein et al., 1979) to reveal the learning potential of an individual. In dynamic assessment interaction is crucial. How much mediation is used and of what kind is registered for the purpose of the pupils to understand and solve problems on their own.

Mediation and dynamic assessment are used in this study to reveal the process of learning how to form concepts in mathematics. The theories developed by Feuerstein are used in order to describe how deaf children learn, and to investigate if it is possible to help deaf children to develop their cognitive potential in a more effective and adequate way than is usually done.

From the mathematics education perspective insightful learning, problem solving and communication are considered to be crucial in developing mathematical knowledge (Verschaffel & De Corte, 1996). Conceptual knowledge in interaction with procedural knowledge is of importance (Hiebert & Lefevre, 1986). Insight and understanding of how to solve a task must generally precede the training of the calculating skill. According to Ahlberg (1992), children need to learn to count and to solve problems. Automatised calculations are time-saving, but they do not develop the ability to formulate, understand and solve problems.

Steps children in general take on their pathways towards comprehension of the concepts were searched for and are described in concept maps (Novak, 1998). In the concept map the concepts are hierarchically ordered and are connected to each other in a network by linking words to build statements. Concept maps were used in this study to give an overarching description of mathematical concepts and their connections included in this study. Concept maps illustrate steps and pathways taken by some of the pupils in the study in forming concepts.

The importance of language in concept formation, with focus on sign language, is illuminated in the study. Since the study is theoretically based on a constructivistic view with concepts being developed in a sociocultural context, a language for direct communication is crucial. Swedish Sign Language (Bergman, 1979) is the language developed by deaf people in Sweden and is used among people who do not hear.

The empirical study

The study is focusing on the process of learning basic mathematical concepts. By choosing to concentrate on the concept multiplication with whole numbers and the concept of length in two separate parts of the study, it was possible to look into both arithmetic and geometry, since they represent different aspects of mathematical knowledge and since both number and space are mentioned as basic concepts in the current syllabus (Skolverket, 2000).

The children in the study were pupils in grade 4 in a Special School for the Deaf. There were seven 11-year-old pupils in the study. They were all the pupils in one of two parallel groups in the fourth grade. All the seven pupils had been taught in the same school since the first grade. The level of knowledge was not considered when organizing the groups. None of the pupils in the study had deaf parents. As many as half the number of the parents had another language than Swedish as their first language.

Video recordings were made of pupil-teacher interactions in problem solving situations. Communication was brought about in sign-language. The pupils met the teacher one by one several times.

Problems to be solved were generated out of a given situation. The solution of a problem presented was discussed in four different ways: in sign language with no material available, with paper and pencil, with learning materials i.e. Centimo and Cuisenaire-rods, and with real objects. In the assessment lesson it was important to start in the most abstract way and let the most concrete way be the last. To go from telling to acting was needed for the assessment of the pupil’s level of abstraction. Later on the pupils were free to chose ways to solve or to explain the problem.

In the multiplication study a problem was chosen and the difficulty of the problem was increased by changing one of two factors. The problem was to find out how many apples are needed if three/four children are to have three apples each, if seven children are to have three apples each or if one-hundred-and-three children are to have three apples each. The reason why I chose the numbers of children to be 3 or 4, 7 and 103, was that they were numbers with a certain meaning to the pupils. They were the number of children in groups they belonged to themselves. There were 7 pupils in their class, there were 3 boys and 4 girls in the class and there were 103 pupils in the school altogether at the time, a matter that had recently been focused on in a speech by the principal of the school.

In the length study the teacher and each of the pupils discussed the lengths of shelves to be put on the wall in different places. It was a problem they were familiar with from the subject of woodwork. Another reason was that we could put real shelves in front of us on the table. We could talk about lengths in a horizontal direction and the lengths could be kept shorter than one meter to focus on working with measurements in centimetres only. The pupils were challenged to estimate, measure and compare the lengths of different shelves. Four different shelves were used.

The first lesson with each pupil and in each of the two parts of the study was regarded as an assessment lesson. After the lesson a brief analysis was made from the video recordings and an assessment was made of the pupil’s ability to solve the problem presented and the level of understanding the concept in question. Assessment was also made of what else the pupil might need to know about how to learn and how to develop understanding of the concept. Dynamic assessment was used by mediation if needed for the pupil to solve the problem and to develop better understanding of the concept. How much mediation was needed and of what kind was registered.

With pupils who solved the problem on their own, attention was concentrated on how the pupil described his or her way of thinking in order to find new ways of helping other pupils to solve the problem. With pupils who needed help to solve the problem, analyses were made to find what was needed for the pupil to solve the problem and to develop better understanding of the concept. One or two lessons were consciously planned out of the needs of the pupil. The goal for the teaching was for each pupil to understand the concept to the extent it was presented in the study. The teacher interacted with the pupil using mediation if needed for the pupil to solve the problem and to develop understanding of the concept.

A test was then given to the pupils in the study containing the same kind of problems as in the lessons. The tests were presented in written Swedish and were given to the pupils individually. After the pupil stopped working on the test, the teacher initiated communication if needed for the pupil to understand the solution of the problem. Video recordings were made to be used in the analysis.

For each pupil an assessment was made of what mathematical steps he or she mastered during the first lesson and in the test situation. The results from the two occasions were compared.

Conclusion

When analysing the data from the cognitive education perspective I found a large variety in the ability of the pupils to solve the problems depending on factors defined by Feuerstein (Feuerstein et al., 1988) e.g. self-confidence, looking for meaning and search of challenge, intention to finish the work and use of known facts. They are all factors of importance to communicative competence and to problem solving.

When analysing the data from the mathematics education perspective I found no difference in general concerning steps towards comprehension for the pupils in the study compared to those of hearing pupils, as far as comparing spoken Swedish for hearing pupils and signed Swedish for deaf pupils. In the area of number sense, several pupils in the study did not master three-digit numbers, an ability usually automatised at an earlier age by hearing pupils. In accordance with earlier studies deaf pupils need more time to learn mathematics than hearing pupils. As a consequence they may learn certain concepts at a later age and the pathways towards comprehension may vary compared to those of hearing pupils.

How deaf pupils' way of expressing themselves in sign language influenced their concept formation was studied. It was analysed out of the four characteristics of sign language iconicity, simultaneity, movement and spatiality. It was found that the structure of sign language could be of help but it could also be an obstacle in mathematics. It was helpful when showing what something looked like iconically and what happened. On the contrary arguments have been raised whether the visual aspects of sign language may hamper concept formation in mathematics. In this study the possibilities of using the pupils' expressions in sign language to reveal their level of knowledge and to promote their concept development have been focused on. In constructing one's own knowledge it is essential that all possibilities are taken advantage of.

The importance of teaching mathematics by problem solving and by communication to deaf pupils as well as to hearing pupils has been emphasized in this study. For deaf pupils, a more developed terminology in sign language would make learning subject-specific concepts of mathematics less dependent on competence in spoken/written language. The pupils could then be offered better conditions to reach a more abstract level at an earlier age. The bilingual situation of deaf pupils is a reason for developing methods for teaching mathematics to deaf pupils, approaches differing from or supplementing methods used today.

References

Ahlberg, A. (1992). *Att möta matematiska problem - en belysning av barns lärande.* (The meeting with mathematical problems: an illumination of children's learning.) Göteborg: Acta Universitatis Gothoburgensis.

Bergman, B. (1979). Dövas teckenspråk – en inledning. (Sign language of the deaf – an introduction.) *Forskning om teckenspråk III*. Stockholms Universitet: Institutionen för lingvistik.

Feuerstein, R., Rand, Y. & Hoffman, M.B. (1979). *The dynamic assessment of retarded performers: The learning potential assessment device*. Baltimore: University Park Press.

Feuerstein, R., Rand, Y. & Rynders, J.E. (1988). *Don't accept me as I am.* New York: Plenum.

Feuerstein, R., Klein, P.S. & Tannenbaum, A. (Eds.). (1991). *Mediated learning experience.* London: Freund.

Foisack, E. (2003). *Döva barns begreppsbildning i matematik.* (Deaf children's concept formation in mathematics.) Malmö: Lärarutbildningen.

Frostad, P. (1996). Mathematical achievement of hearing impaired students in Norway. *European Journal of Special Needs Education, 11*(1), 66-80.

Hiebert, J. & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics.* Hillsdale, NJ: Lawrence Erlbaum.

Läroplan för det obligatoriska skolväsendet och de frivilliga skolformerna. Lpo94. (1994). (National curriculum for compulsory education and for the non-compulsory school system.) Stockholm: Utbildningsdepartementet.

Magne, O. (1991). *Dysmathematics. Facts and theories concerning mathematics learning for handicapped pupils.* (Educational and Psychological Interactions, 106). Malmö: School of Education.

Martin, D.S. (Ed.). (1991). *Advances in cognition, education and deafness*. Washington D.C.: Gallaudet University Press.

Moores, D.F. (2000). *Educating the deaf. Psychology, principles and practices*. Boston: Houghton Mifflin.

Novak, J.D. (1998). *Learning, creating and using knowledge. Concept maps as facilitative tools in schools and corporations.* Mahwah, NJ: Lawrence Erlbaum.

Skolverket. (2000). *Grundskolans kursplaner och betygskriterier*. (Syllabi and grading criteria for the compulsory school.) Stockholm: Fritzes.

Skolverket. (2000/2002). *Specialskolan. Kursplaner och betygskriterier.* (The special school. Syllabuses and criteria for grades.) Stockholm: Fritzes.

Verschaffel, L. & De Corte, E. (1996). Number and arithmetic. In A.J. Bishop et al. (Eds.), *International handbook of mathematics education.* Dordrecht: Kluwer Academic Publisher.

Vygotsky, L.S. (1978). *Mind in society.* Cambridge, MA: Harvard University Press.

Prospective Mathematics Teachers' Learning in Geometry

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Introduction

When discussing teacher qualifications in mathematics and education of future teachers in mathematics, the branch of geometry is of course only a small part of the science mathematics. Nevertheless, there exist reasons and arguments for why we should try to observe and educate competencies for teachers in mathematics with geometry in focus (Niss, 1998). This will include teachers' views and knowledge of geometry as a mathematical topic as well as the kinds of knowledge and views that they should have of the processes of learning geometry. In Sweden, as well as in other countries, geometry was an essential part of school mathematics during the period from the middle of the 19th century to the end of the 1950s. Later on, there have been discussions (Kapadia, 1985) whether the loss of geometry in the school mathematics may have caused a drawback regarding student's understanding of general concepts or not. As a result, geometry and measuring has a more prominent position in mathematics syllabi for the Swedish school today. The programmes for teacher education seem to have followed the same patterns, only with a slight displacement in time. Despite this, teachers of today are expected to fulfill the national curriculum and ensure that their students reach proficiency in these areas.

Background

In recent years, there have been results reported based on research aimed at prospective mathematics teachers' formation of concepts in more well defined fields of mathematics. The research results point out the necessity of the teacher's readiness to reconsider her or his own understanding.

> Investigations show that there are different ways for teachers to express their own appreciation of a concept. Explicit expressions based on reasons seem to be most effective. But future teachers must also learn to accept students' evaluations as expressions of their personality when they differ from their own appreciation of a concept (Vollrath, 1994, p. 64).

In a situation where the teacher's and the student's appreciation of a mathematical concept meet, demands will be raised for the teacher's ability to support the students' conceptual development. The teacher has to have both sound knowledge and a broad view with respect to the concept in point. Aspects of geometrical concepts as conceived by prospective mathematics teachers have been studied in several research studies during the last two decades. Setting out from a hierarchical classification of such concepts, Lichevsky, Vinner and Karsenty (1992) have reported their results. Hierarchical classification is a fundamental thought process in Euclidian geometry and is required in many domains, not only in mathematics. Their results show that the students' conception of the geometrical figures is global. The students relate only to the visual aspects of the figures and not to their properties, the students' conception lacks analytical elements. The researchers consider this to be connected to the students' ability to make relations to other important aspects of geometry, such as definition and proof.

Gutiérrez and Jaime (1999) presented the results of a test designed to analyse pre-service primary teachers' understanding of the concept of altitude of a triangle. Their conclusions refer to the identification of the pre-service teachers' concept images and the way they make use of their concept images and the mathematical definitions of altitude of a triangle in the resolution of specific tasks, tasks similar to those they will find in the textbooks when they begin their professional life. One of their conclusions is that prospective teachers should be given the opportunity to present, explain, and defend their particular conceptions of basic geometrical concepts. Any cognitive conflicts that arise from different concept images could be considered and resolved in the light of the formal definition.

The purpose of this study (Holmquist, 2003) is to describe prospective teachers' understanding of geometrical concepts, as it will be manifested by their explanations, recorded in words and drawings. What are the prospective mathematics teachers' answers when they are asked to give their written explanation to some basic geometrical concepts? Which are the most frequent answers and what are the characteristics of a certain answer?

Theoretical framework

Research concerning our understanding of words, objects and concepts is of great interest, not only in geometry or mathematics, and extends over a lot of different areas. Commonly used are theories in psychology or theories in linguistics and semantic structures. Theories from psychology and linguistics are used in research dealing with our understanding of mathematical words and concepts.

Our interest is in the frame of the epistemological interpretation of mathematical knowledge and we ask ourselves the question; what is a concept? According to an encyclopedia (Nationalencyklopedin, NE, 1998) a concept is

> the abstract content in a linguistic term unlike the term itself, as well as (concrete or abstract) those objects represented by the term or those the term is applied to.

This description of a concept is based on what is called the Ogden´s and Richards´ semiotic triangle or the model of signs (Ogden & Richards, 1989/ 1923). Steinbring (1994) uses the characterization of "meaning" as the "triad of thoughts, words and things" as a base for what he calls the epistemological triangle of mathematical knowledge, a model to be considered as non-static. This more dynamical way to look upon the model will make the interpretation that a concept could be understood in relation to other concepts possible.

> With regard to this epistemological triangle of "object," "sign," and "concept," it is not assumed that the relations between the "corners" of the triangle are fixed a priori, but that they must continuously be developed, installed, and eventually modified according to new prerequisites (Steinbring, 1994, p. 96).

The use of this model raises the question about demands for mathematical concepts to be unambiguous and non-contradictory. This appears distinctly in definitions of concepts made in mathematical theories and the character of mathematical concepts therefor often differs from those used in everyday communication. The role of the mathematical term in the epistemological interpretation of mathematical knowledge is in the limelight for our interest.

This will lead us to take into account a three-dimensional model, a model were the *definition* would give the epistemological triangle an additional dimension. The epistemological tetrahedron model is used to illustrate the mutual relations between concept, referent, term, and definition (Spri, 1999). In figure 1 the model is used together with the mathematical concept *diagonal*.

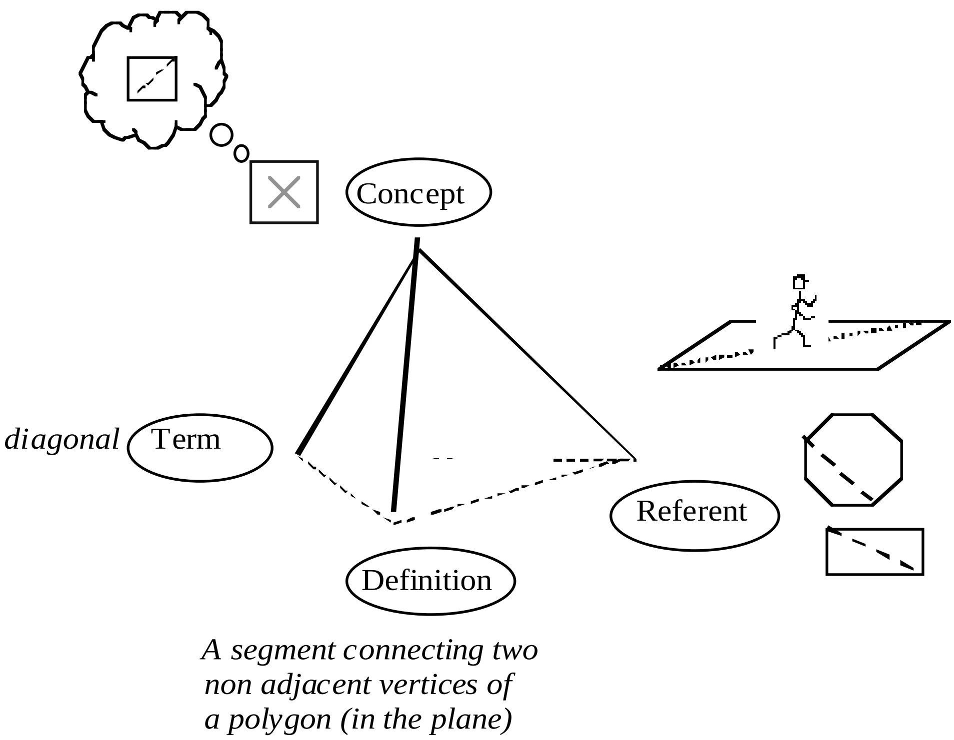


Figure 1. The epistemological tetrahedron model

The dashed and the unbroken lines in figure 1 show how every connection between terms and referents, terms and definitions, and definitions and referents take place through the concept. If there is a term, a definition or a referent, there will always be a concept. For the model, to have an epistemological function, it should be possible for the referent to be not only an object but also a relation between objects or a situation.

A theoretical model for analysis of students' answers

The well-known and discussed results in the area of conceptual understanding are about how mathematics students at different educational levels express their understanding of mathematical concepts. A theoretical structure for analysis and description of that kind of results is based on the students' *concept images*, images commonly not identical with the *concept definitions* they face in their instruction to become teachers. Vinner (1991) and Hershkowitz (1990) have reported on both results and theories in this area. Because there is no final stage of how a concept can be understood (Vollrath, 1994), there is no interest in the discussion of the students' explanations in terms of right or wrong. In relation to an established meaning of a mathematical concept, different descriptions and explanation will, of course, be found. In connection to this Vinner and Hershkowitz (1983) and Vinner (1991) give a useful theoretical structure for analysis and description of the students' explanations of the geometrical concepts included in the study. When we listen to or read the name of a more or less well known concept, or when we are solving a mathematical problem, something can be evoked in our mind. Usually, it is not the concept definition; even in the case the concept does have a definition. Instead it is something non-verbal in our mind associated with the concept name. It can be a collection of visual representations, pictures, characteristics, impressions or experiences. Such a collection of elements associated with a concept name constitutes *the concept image*.

Vinner (1991) claims that a student's experiences and those exemplifications given for a concept, either in educational situations or in other contexts, are of decisive importance for how the concept image is developed. During their time in school students usually meet just a few examples of a certain geometrical concept, examples with a common and specific visual character. Such an example then becomes the *prototypical example* and the student uses it as the frame for reference. This is called the prototype phenomenon and has been described by Vinner and Hershkowitz (1983) and Herskowitz (1990).

The study

At Göteborg University, students preparing to become teachers of mathematics and natural science for year 4 to 9 or for the gymnasium (year 10 to 12) take courses in many different branches of mathematics. One such 5-credit point (5 weeks fulltime study) course is called Geometry and Mathematical Modeling.

The students come to the course after having taken different courses in mathematics where the content varies over time. Usually the students come with courses in number theory, Euclidian geometry, linear algebra, and real analysis, which approximately correspond to 30 weeks of full time studies. Beginning in the spring of 1999, the students who came to the course were given a questionnaire, which address questions regarding how the students conceive concepts in geometry.

This study is based on written responses that have been collected from all the entrance questionnaires on geometry administered during 8 consecutive semesters, including the fall 2002 course. The students were given the questionnaire the first day of the course and there was no time limit for them working on it, although up to now no student has used more than 80 minutes. At present, 213 students have responded to the questionnaire. The issue in focus is the identification of the prospective teachers' concept images and the way they make use of their concept images and the mathematical definition of certain concepts, geometrical concepts that they will find central when they begin their professional life as mathematics teachers. The selection of questions in the questionnaire is closely related to the national curriculum for the Swedish comprehensive school and for the Swedish Gymnasium, to analysis of common textbooks that are used in Swedish schools, and to the kind of geometrical concepts that are used in national tests. The students were asked to give an explanation to the following five concepts: *diagonal*, *congruence*, *parabola*, *rhomb*, and *cycloid*:

> Explain in short terms the following geometrical concepts; please use both pictures and words. Even if your belief about the concept is vague, we appreciate if you as clear as possible will try to give an explanation.

Different responses were classified according to the critical attributes of the actual concept used by the students, but the reasons for the differences were not analyzed. With just a total of 10 % falling off it is reasonable to say that the students taking part in the study well represent the whole group of prospective mathematics teachers during the actual time period.

Results

Most of the findings correspond to these described by Hershkowitz and Vinner (Hershkowitz, 1990; Vinner, 1991). The concept image expressed by a student is often only part of or may not coincide at all with the definition of the corresponding mathematical concept. The prototype phenomenon was also observed frequently in the students' descriptions. There are reasons to separate the students´ answers in a qualitative aspect, not only in 'correct' or 'non correct'.

Diagonal

Table 1 shows the result of the analysis and categorization of the students' explanations of the concept *diagonal*.

The concept *diagonal* is a well-known concept among the students in the study. The result shows that in the main part of the explanations, the students make use of concept images in which the references are of strong visual nature.

Table 1. Distribution of student explanations of the concept *diagonal* (*N* = 213)

1 Close to a formal mathe-matical definition	2 Definition connected to a polygon	3 Definition limited to a quadri-lateral	4 Definition limited to a certain geometri-cal object (e.g. a square)	5 Divides a square or a rectangle into two equal parts	6 Just a drawing (square or rectangle)	7 …from one ver-tex to another – result of a process	8 Relates to a circle	9 No relevant answer	10 No answer
1%	2%	15%	16%	22%	17%	6%	11%	9%	1%

The overall dominating example is a *diagonal* drawn in a square or a rectangle. Those examples could be seen as prototypical examples, examples that are attained first. Among those there are notable many in the category *d*5 (*diagonal* 5) *Divides a square or a rectangle into two equal parts*.

If a description of a certain concept is bounding its meaning to a specific situation, it will constitute a concept negative example with strong irrelevant attributes.

Congruence

Table 2 shows the result of the analysis and categorization of the students' explanations of the concept *congruence*. The object for the concept of congruence is a relation between two geometrical objects and therefore this concept is different compared to the other concepts in the questionnaire.

Table 2. Distribution of student explanations of the concept *congruence* (*N* = 213)

1 Close to a formal mathe-matical definition	2 Close to a formal definitio n without certain critical attributes	3 Objects exactly covering each other (lay over)	4 Relates to characte-ristics about geometrical objects (equal angels, equal sides)	5 Simi-larity	6 It is about simila-rities between geome-trical objects	7 Relates the concept to number theory	8 Relates the concept to conver-gence	9 No rele-vant answer	10 No answer
10%	14%	3%	8%	19%	11%	1%	2%	19%	13%

For explanations in the categories *c*1 (*congruence* 1) to *c*6 it is obvious that they are based on similarities between geometrical entities. In this case the prototype example is closely related to an explanation where congruence is described as something like similarity between triangles. The reason for the students to chose an exemplification where triangles are included is most likely to find in textbooks descriptions of the concept. Among all the students there are no one using only a drawing to describe the concept congruence. On the other hand, in the categories *c*1 to *c*6 there are 30 out of 138 explanations (22 %) built solely on a linguistic statement.

Parabola

The concept *parabola* has its original definition in the geometry of conic sections. Such geometry theories are no more in the syllabus for the Swedish school system (year 1-12). Despite this, the concept parabola is used in an analytic meaning as the name for the graph of a function like $f(x) = ax^2$, and the graph is called "a graph of second degree". The more everyday meanings of the concept are related to physical phenomena like the parabolic motion of a ball projected into the air or a parabolic shaped antenna. Table 3 shows the result of the analysis and categorization of the students' explanations of the concept *parabola*.

Table 3. Distribution of student explanations of the concept *parabola* (N = 213)

1 Close to a mathe-matical definition	2 The graph of an equation of the second degree	3 Examples built on relevant attributes	4 A graph of the second degree (x^2-graph)	5 A graph	6 Just a drawing of a graph	7 Relates to a physical pheno-menon	8 No relevant answer	9 No answer
0,5%	9%	2%	3%	13,5%	8%	4%	20%	40%

With regard to the notably large amount of students' explanations in the categories *p*8 (*parabola* 8) *No relevant answer* and *p*9 *No answer,* the prototype phenomenon for the concept parabola is not so obvious. Despite this it can be of interest to look upon the prototype examples among the concept images in the remaining categories. Explanations in the categories *p*4, *p*5 and *p*6 are dominating and they are about a drawing of a graph in a coordinate system. This is also the most common description in textbooks. In the case of the concept parabola there are very few explanations based only on a linguistic approach.

Rhombus

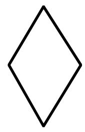
The concept *rhombus* serves as an example of how a concept can be part of an inclusive relation, the set of rhombuses is included in the set of parallelograms, which is included in the set of quadrilaterals. Table 4 shows the result of the analysis and categorization of the students' explanations of the concept *rhombus*.

Table 4. Distribution of student explanations of the concept *rhombus* (N = 213)

1 Close to a formal mathe-matical definition	2 Definition without certain critical attributes (concept negative examples)	3 Oblique square	4 Paralello-gram with angels not 90 degrees	5 Quadri-lateral (general)	6 Just a drawing	7 Polygon (more than 4 vertices)	8 A solid geome-trical object (parallel-epiped)	9 No rele-vant answer	10 No answer
23,5%	11%	21%	5%	5%	7%	2%	5%	13,5%	7%

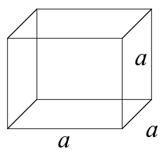
The students' explanations of the concept *rhombus* show that this is a fairly well-known concept. Besides this a relatively high amount of explanations are in category *r*1 (*rhombus* 1) *Close to a formal mathematical definition*, the highest amount in relation to other concepts in the study. The square is used as a referent in many of the students' concept images, which may connect to the fact that the rhombus is part of an inclusive relation. In this case the *prototype example* is characterized by an explanation where the square is processed to become a rhombus. They are given by the explanations in category *r*3 *Oblique Square*. The strong visual character of the square as an equilateral object, often standing on one of its sides, can lead to explanations were the relation between the rhombus and the square is somewhat contradictory.

S*r*9 *A square "standing" on one corner.*
A rhombus has not necessarily right angles.



Like in explanations of other concepts in the questionnaire, there are instances where the relation between the picture and the text is contradictory.

S*r*8 *An equilateral parallelogram*



Cycloid

The reason for the concept *cycloid* to be included in the questionnaire is quite different from what is the case for the other concepts. The concept is not dealt with in school mathematics and it is reasonable to say that it has a limited meaning in everyday language. Nevertheless, the concept is included in the terminology of mathematics and also included in the Swedish language. The concept is part of the questionnaire to make it possible to evaluate how students react on more uncommon concepts. Such an evaluation will give an indication of the students' willingness to give an explanation whether they think they have a relevant concept image or not. Table 5 shows the result of the analysis and categorization of the students' explanations of the concept *cycloid*.

Table 5. Distribution of students explanations of the concept *cycloid* (N = 213)

1 Definition related to a circle, rotating on a straight line	2 Relates to a cyclic course (iteration)	3 Relates to a circle	4 No relevant answer	5 Comment on one's own ignorance (No answer)	6 No answer
2%	4%	4%	16%	25%	49%

The students' explanations in category *c*5 (*cycloid* 5) are very interesting. They give arguments for the conclusion that the students usually indicate when they have no relevant concept image to present. In the case where the students offer an explanation they believe that they have a relevant concept image.

Discussion

The results mainly correspond with the results presented by Hershkowitz and Vinner (Hershkowitz, 1990). This is above all true for the prototype phenomenon, which frequently appeared in the students' explanations. Among the concepts used in this study, the concept of diagonal showed the strongest prototypical concept image, a diagonal traced out in a square or a rectangle. In addition the concept of diagonal generated explanations built on specific referents whose character hardly can be transferred to a more general situation. Using the vocabulary of Hershkowitz (1990), they are *concept negative examples* with strong *irrelevant attributes*.

For the students in this study the results revealed that a concept image more rarely refers to the corresponding mathematical concept definition. With the exception of some parts of the results presented for the concept rhombus, the students' concept images were mainly based on referents with weak connections to the concept definition. In other words, when asked to give a written explanation to a geometrical concept, the students in this study rarely consulted the concept definition. This can be schematically shown by a modification of the epistemological tetrahedron model (figure 2). The dashed line between a concept definition and the concept image in the student indicates the weak connection; in some cases it is questionable if the connection were at all established. It is worth noticing that the etymological aspect of the name of the concept played a subordinate role and seems to have had almost no influence on the concept images presented by the students.

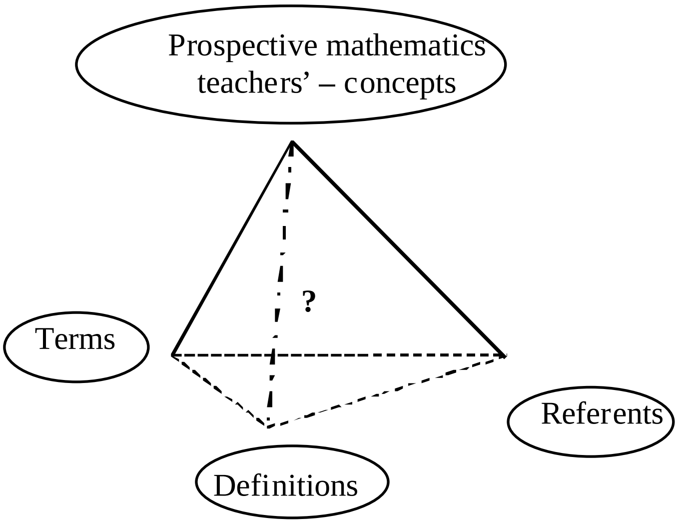


Figure 2. Modified epistemological tetrahedron model.

Usually a concept in school mathematics is synonymous with its definition. To understand a concept is then to know and be able to express the corresponding definition. This is also how the content of mathematics is presented and organized in many textbooks for school mathematics. The textbooks analyzed in this study are in general no exceptions from that.

In the light of these results, it is relevant to believe that the importance of concept images when developing understanding of a concept is underestimated in mathematics education. At the same time, as the descriptions in textbooks, education is mainly based on concept definitions which do not support wanted results. This is not, in my opinion, to say that we have to exclude the definitions. Metaphorically speaking; to reach good understanding of a concept we have to find different ways in the epistemological tetrahedron model.

In this study the visual aspect played a fundamental role in the students' way to present their explanations. Concepts whose referents have strong visual characteristic may lead to explanations were the visual objects are contradictory to the written linguistic expressions used. Geometry is with naturalness associated with drawings, which sometimes lead to negative consequences. According to Hershkowitz (1990), the ontological status of geometrical entities is a philosophical problem. The question is whether the geometrical entities are part of the physical world and if not, what are they? It is also relevant to discuss the meaning the answer of that question has for an object to be regarded as concrete or abstract.

These aspects are of particular importance when a teacher uses his or her explanations to meet and support students in their learning of geometrical concepts. It is also about the teacher's assessment and validation of students' written answers and reports, a frequent activity in school mathematics. What kind of referents and criteria are the bases for the teacher's standpoint? What kind of concept definitions and concept images are on the teacher's repertoire? The presented models may be used as a starting point for how a teacher can analyse a student's written production as an expression for understanding of mathematical content.

The results emerging from the described study will form the basis for deeper studies of what these prospective mathematics teachers learned in geometry and how their understanding of the theoretical mathematical content later on was expressed in their teacher practice.

References

Gutiérrez, A. & Jaime, A. (1999). Preservice primary teachers' understanding of the concept of altitude of a triangle. *Journal of Mathematics Teacher Education 2*, 253-275.

Hershkowitz, R. (1990). Psychological aspects of learning geometry. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and cognition* (pp. 70-95). Cambridge, UK: University Press.

Holmquist, M. (2003). Blivande matematiklärare skriver om geometriska begrepp. En studie med 213 studenter under åren 1999-2002. [Prospective mathematics teachers writing about geometrical concepts]. (Magisteruppsats). Institutionen för pedagogik och didaktik, Göteborgs universitet.

Kapadia, R. (1985). Bring back geometry. *The Mathematical Intelligencer 7*(2), 53-54, 65.

Lichevsky, L., Vinner, S., & Karsenty, R. (1992, August). To be or not to be minimal? Student teachers' views about definitions in geometry. In W. Geeslin & K. Graham (Eds.), *Proceedings of the 16th PME conference* (Vol. 2, pp. 48-55). Durham, NH: University of New Hampshire.

Nationalencyklopedin (NE) (1998). Höganäs: Bra Böcker.

Niss, M. (1998). Teacher qualifications and the education of teachers. In C. Mammana & V. Villani (Eds.), *Perspectives on the teaching of geometry for the 21st century*, (pp. 297-318). Dordrecht: Kluwer.

Ogden, C. K., & Richards, I. A. (1989). *The meaning of meaning* (8th rev ed). New York: Harvest/HBJ Book. (First published 1923).

Spri. (1999). *Metoder och principer i terminologiarbetet* [Methods and principals in the terminological work] (Spri rapport 481). Stockholm: Spri, Hälso- och sjukvårdens utvecklingsinstitut och TNC, Tekniska Nomenklaturcentralen.

Steinbring, H. (1994). Dialogue between theory and practice in mathematics education. In R. Biehler, R. W. Scholz, R. Sträßer, & B. Winkelmann (Eds.). *Didactics of mathematics as a scientific discipline* (pp. 89-102). Dordrecht: Kluwer Academic.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.). *Advanced mathematical thinking*. Dordrecht: Kluwer Academic.

Vinner, S., & Hershkowitz, R. (1983). On concept formation in geometry. *Zentralblatt für Didaktik der Mathematik 83*, 20-25.

Vollrath, H-J. (1994). Reflections on mathematical concepts as starting point for didactical thinking. In R. Biehler, R. W. Scholz, R. Sträßer & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 61-72). Dordrecht: Kluwer Academic.

KULT-projektet – Matematikundervisning i Sverige i internationell belysning

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Klassrumsstudier innebär alltid svårigheter med att tolka de skeenden som dokumenteras. Inom KULT-projektet har ett mycket omfattande datamaterial samlats in. Med utgångspunkt i en inspelad sekvens, där läraren diskuterar lösningen av en uppgift med en elev vid dennes bänk, görs ett försök att visa hur tillgången till detta datamaterial, samt möjligheter till internationella jämförelser kan bidra till en preliminär analys.

Om 'The Learner's Perspective Study'

KULT-projektet, *Svensk skolkultur – klassrumspraktik i komparativ belysning* (KULT, 2003), syftar till att beskriva svensk matematikundervisning och göra internationella jämförelser. Projektet ingår i en större internationell studie, 'The Learner's Perspective Study', (LPS study, 2003) där för närvarande 10 länder medverkar. Utifrån en gemensam design (Clark, 2000) insamlas empiri genom avancerade videoinspelningar av matematiklektioner i årskurs 8 kombinerat med uppföljande intervjuer med elever och lärare. Det som utmärker LPS studien, jämfört med andra internationella studier där videodokumentation av matematikundervisning gjorts, är att samma klass följs i minst 10 matematiklektioner efter varandra samt att tre kameror används vid inspelningarna. En rörlig kamera följer läraren som är utrustad med en trådlös mikrofon (mygga). En kamera med vidare objektiv dokumenterar hela klassrummet och en tredje kamera fokuserar en mindre grupp elever (2–4) som sitter intill varandra. På de bänkar där dessa fokuserade elever sitter finns också ett par små mikrofoner placerade så att konversationen kan fångas upp vid inspelningen. I det svenska projektet har en portabel inspelningsstudio byggts upp, vilket möjliggör digitala inspelningar direkt på hårddisk. En av de stora fördelarna vid fältarbetet har varit att elevintervjuer kunnat genomföras i direkt anslutning till de inspelade lektionerna. Vid dessa intervjuer har intervjuaren och eleven tittat på och kommenterat en film där fokuseleverna kan följas samtidigt som inspelningen av läraren syns i en mindre ruta uppe i ena hörnet. Den digitala tekniken möjliggör en mycket snabb synkronisering av de båda inspelningarna till en mixad film, vilket varit värdefullt.

I vart och ett av de medverkande länderna, som förutom Sverige är Australien, Filippinerna, Hong Kong, Israel, Japan, Kina, Sydafrika, Tyskland och USA, görs inspelningar av minst 10 lektioner i följd i tre åttondeklasser. Varje lektion följs upp med intervjuer av minst två av fokuseleverna. Dessutom intervjuas läraren några gånger under inspelningsperioden. Eleverna svarar på enkäter om matematikundervisning samt gör ett allmänt matematiktest.

Fältarbetet i Sverige är avslutat och totalt har 48 lektioner i tre klasser spelats in. Eftersom varje lektion filmats med tre kameror innebär det totalt 144 lektionsfilmer. Därutöver finns också 75 elevintervjuer och 12 lärarintervjuer, om vardera 40–80 minuter, dokumenterade på videofilm. Av de lektioner som följts kommer data från 10 sammanhängande lektioner i varje klass att organiseras enligt de krav som ställs inom det internationella projektet. Det innebär bl.a. att all konversation ska transkriberas på ett standardiserat sätt och dessutom översättas till engelska. Allt ska sedan tidskodas och infogas i ett speciellt format så att man i en löpande textremsa kan följa med i det som sägs på respektive film. På det här viset skapas ett datapaket från varje klass, med videofilmer från lektionerna och intervjuerna, alla transkriptioner, inskannat elevmaterial, svar på enkäter och test, utdrag från läromedel mm. Vid University of Melbourne har inrättats ett internationellt centrum för klassrumsstudier, ICCR - the International Centre for Classroom Research, där allt material kommer att samlas och vara tillgängligt för besökande forskare.

LPS-studien avser att studera matematikundervisning på ett mer uttömmande sätt än andra liknande internationella studier. Genom den design man valt är detta möjligt. Dokumentationen av inte bara läraren utan också hela klassrummet samt en mindre grupp elever, som efteråt ges möjlighet att kommentera vad som händer under lektionen, genererar ett mycket informationsrikt datamaterial. Det går att både följa och studera skeenden under lektionen och att beakta flera av de medverkandes tolkningar, förståelse och uppfattningar av det som sker, så som de ger uttryck för detta i de efterföljande intervjuerna.

KULT-projektet

Det svenska KULT-projektet bedrivs i samarbete mellan forskargrupper från Uppsala universitet och Göteborgs universitet. Analyserna kommer därvid i det första skedet att ske med metoder som är etablerade i och tidigare använda av respektive forskargrupp. I den ena traditionen studeras interaktionsmönster i klassrummet och i den andra hur det matematiska innehållet behandlas och förstås. En ambition inom det svenska projektet är dessutom att de båda teoretiska perspektiven ska relateras till och berika varandra.

Ett av syftena med det svenska projektet är att studera relationen mellan undervisning och lärande av matematik. Frågor som hur matematikinnehållet hanteras i undervisningen och hur eleverna förstår detta innehåll, kommer att belysas. Analyserna kommer bl.a. att göras utifrån ett variationsteoretiskt perspektiv (Marton & Booth, 1997; Runesson & Marton, 2002).

När detta skrivs har endast preliminära analyser kunnat göras. Fältarbetet har nyligen avslutats och organiseringen av det omfattande datamaterialet är mycket tidskrävande. De första internationella jämförelserna har dock påbörjats mellan en av de svenska skolorna och en skola från USA, där samma matematikinnehåll (elementära linjära funktioner) är föremål för undervisning. En poäng med dessa jämförelser är att synliggöra aspekter på undervisningen i det svenska klassrummet, som annars lätt kan tas för givna av svenska forskare.

Preliminära resultat

Den först inspelade svenska klassen, SW1, följdes under 16 konsekutiva matematiklektioner. Det matematiska innehållet som behandlades under dessa lektioner var Linjära ekvationer med en obekant, Förenkling av uttryck, Koordinatsystem och Samband. Vid lektion 11 har man kommit in i det sistnämnda avsnittet. Lektionen inleds med en lärarledd genomgång som behandlar linjära funktioner och dess grafer, samt avläsning och konstruktion av diagram som beskriver ”verkliga händelser” t.ex. kostnad–antal–diagram och väg–tid–diagram. Vid den lärarledda inledningen på lektionen diskuteras begreppet ”lutning” hos en linje utifrån en OH-bild (se appendix 2). I Emanuelsson m.fl. (2003a) presenteras preliminära resultat från de inledande analyserna av denna lärargenomgång samt av en ”bänksekvens” lite senare samma lektion, där läraren och en av eleverna diskuterar en uppgift i läroboken. De preliminära analyserna pekar på att det som behandlades under genomgången inte var tillräckligt för att ge eleven möjlighet att lösa uppgiften i läroboken.

Vid en jämförelse av lektionssekvenserna från SW1 och US2 (den andra skolan från USA) på makronivå framträder tydliga skillnader. I SW1 ges eleverna stort utrymme att interagera med läraren och delta i konstituerandet av det matematiska innehållet. Läraren i SW1 har på det viset mycket mindre ”kontroll” över innehållet än läraren i US2. I Emanuelsson m.fl. (2003b) pekar man också på skillnader i det språk som används. I SW1 används till stor del ett vardagsspråk och ”krångliga” matematiska termer undviks till skillnad från US2, där stor vikt läggs vid att korrekt matematisk terminologi används. En liknande skillnad är också tydlig beträffande de uppgifter man arbetar med. Uppgifterna i SW1 handlar alla om något som eleverna kan relatera till, t.ex. cykelturer i form av väg–tid–diagram och liknande. Uppgifterna i US2 är däremot helt och hållet ”inom-matematiska” – algebraiska uttryck på formen $y = f(x)$, värdetabeller och grafer – och gör sällan anspråk på att beskriva något verkligt förlopp eller liknande. I en första analys av bänksekvenser med elev-elev interaktion visas att detta kan få konsekvenser för hur grafer tolkas. I Emanuelsson, Sahlström & Liljestrand (2003) beskrivs hur den ”realistiska” kontexten i en uppgift, där en cykeltur beskrivs i ett diagram, kan bidra till att grafen tolkas som att ”hon cyklar upp” och ”hon cyklar ner”. I den amerikanska klassen finns exempel på hur eleverna tillsammans skapar en egen terminologi för att beskriva egenskaper hos olika kurvor.

Förutom de lärarledda delarna av lektionerna arbetar eleverna enskilt med (oftast olika) uppgifter i läroboken i SW1, medan de i US2 arbetar i grupper om ca 4 elever med en gemensam uppgift. Uppgifternas karaktär är också olika. Uppgifterna i de svenska läroböckerna är till stor del ”korta” i den meningen att de inte ska ta så lång tid och att eleverna förväntas göra ett flertal varje lektion. I US2 arbetar man med samma ”långa” uppgift under flera lektioner. Slutligen är det matematiska innehållet tydligt sekvenserat i SW1, både beträffande lärarens genomgångar och elevuppgifterna. Exempelvis behandlas först olika lutning (koefficienten k i linjens ekvation $y = kx + m$), sedan skärningen med y–axeln (koefficienten m i ekvationen). Innehållet i de amerikanska lektionerna präglas av en större samtidighet – både lutning och skärning med y–axeln hanteras vid samma tillfälle. De preliminära resultaten från denna mer övergripande analys sammanfattas kortfattat i tabell 1.

Tabell 1. Preliminär jämförelse av SW1 och US2 på makronivå.

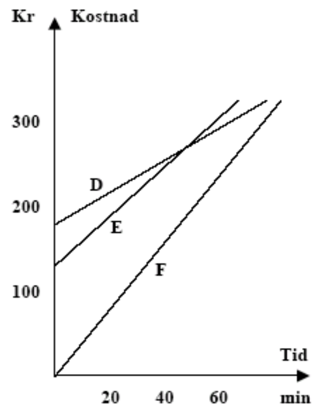
SW1	US2
Vardagligt språk	Matematisk terminologi
Realistisk kontext	Matematisk kontext
Elevmedverkan	Lärarkontroll
Enskilt arbete	Arbetet i grupp
Korta uppgifter	Långa uppgifter
Sekvensering	Samtidighet

SW1 – Lektion 11 – Uppgift 37

På matematiklektioner i Sverige används en stor del av tiden till att eleverna arbetar med uppgifter i läroboken och läraren handleder en eller ett par elever i taget. Nedan beskrivs ett sådant tillfälle och ett försök till analys. Handledningen äger rum under lektion 11 i SW1 och varar i ungefär två och en halv minut (appendix 1). Både läraren och eleven verkar ha en uppriktig vilja att komma till rätta med svårigheterna och läraren lämnar inte eleven förrän denne säger att han förstått.

Lektionen inleds med en diskussion kring ett ”disco” som eleverna arrangerat. Efter ca 7 minuter börjar en lärarledd genomgång utifrån en OH-bild (se appendix 2). Lutningen hos två linjer, samt relationen till motsvarande formel ($K = kx$) diskuteras. I intervju beskriver läraren målet som att eleverna ska kunna ”se lutningen på formeln”. Ungefär 32 minuter in på lektionen avslutas genomgången och eleverna börjar arbeta med uppgifter i läroboken. Efter ytterligare några minuter kommer läraren till eleven Faros bänk. Faro har kört fast på uppgift 37.

37. I diagrammet finns 3 olika linjer.
    Vilken av linjerna visar sambandet
    a) $K = 125 + 3x$
    b) $K = 4x$
    c) $K = 175 + 2x$



Figur 1. Uppgift 37.

Faro börjar med att referera till något läraren sagt under genomgången (samtalet finns i sin helhet i appendix 1), nämligen att man ska ”kunna se” linjens utseende på uttrycket.

1. FARO: Kolla … a. [pekar med pennan på uppg. 37a som lyder: ”a) K=125 + 3x”]
2. L: //Aaa..
3. FARO: //Vil- … vilken av linjerna visar sambandet?
4. L: Ja just //det.
5. FARO: //Du sa du sa att man kunde se lin-//kurvan av bara den här.
6. L. //Ja … a den här-
7. L: Ja just det den där lutar mest, den lutar lite mindre, å den lutar minst. [pekar på de tre funktionsuttrycken i uppgiften]
8. FARO: Mm..
9. L: Men här [L pekar eventuellt på ”175” i 37c] har dom ju slängt in en grej framför också, vad betyder det här … om X är noll … då blir det där noll va … men då börjar den på hundrasjuttifem. [visar i diagrammet]
10. FARO: Mhm..
11. L: Så det är den där [pekar i diagrammet] … C … det är den där … K fyra X den börjar från origo //och går rätt upp … den där börjar på hundratjugofem där har vi hundratjugofem … (//Och där har vi hundrasjuttifem) (…)
12. FARO: //Ja.
13. [Här reser sig läraren upp och tittar irriterat ut över klassen]
14. FARO: Är B noll här … är det den B [L tittar ned mot FARO igen] … nej B är den största eller hur?
15. L: Ja de- de- den lutar mest … titta den här lutar ju brantare än den och än den … [L sätter pennan mot ”b) K=4x” i boken] … men sen börjar dom här på lite olika … //värden. [L pekar ut kurvorna]

Läraren verka tolka Faros uttalande om att B ($K = 4x$) är den *största*, som att B svarar mot den linje som har *störst lutning*. Det är osäkert om det är vad Faro avser med att B är den största. Han verkar först låta linje F vara uttryck B (*B är noll*), för att sedan ändra sig (*nej B är den största*). En möjlig tolkning är att när Faro använder *störst* har det betydelsen *ligger överst* eller *högst upp* i diagrammet i det här fallet. Alltså att uttrycket B inte svarar mot linje F som ju ”ligger underst”.

16. FARO: Men alltså är det här fyra X. [FARO pekar på linjen ”F” i grafen]
17. L: Va vad är det du ska- den där är fyra X ja [L pekar på linjen ”F”] … aa … mm för no- om du sätter in noll där … så //blir kostnaden noll.
18. FARO: //Alltså kan jag skita i dom här som är framför?
19. L: Nej för dom gör ju att du börjar där … det har vi inte gått //igenom än.
20. FARO: //Aha do- do- dom visar var jag börjar.
21. L: Precis … och här står det egentligen noll plus.
22. FARO: Men vad //visar dom här då //förutom att de- denna visar vilken som är //störst och minst?
23. L: //Lut- … (aa) lutar [L lägger pennan längs linje i diagram] den visar hur de lutar och //hur mycket varje sak kostar.
24. FARO: //Men jag kan- jag kan inte bara se bara så här.

Det verkar som att Faro har svårt att skilja på de olika betydelser koefficienterna 125 och 3 har i ett uttryck som $K=125 + 3x$.

25. L: Joo. [till FARO] [L reser sig]
26. FARO: Jamen om inte dom här två skulle finnas så skulle ja aldrig kunna ha vetat vilken- vilken tre X är av dom här. [pekar på linjerna i diagrammet]
27. L: Nej fast du kan veta vilken hundratjugofem plus tre X är.
28. FARO: Hur //gör man det då? [till L]
29. L: //För- för att den börjar på Y-axelns på hundratjugofem.
30. FARO: Aa den börjar på hundratjugofem sen tre X, var ska jag hitta det då?
31. L: Jo och det är för varje sån steg du går där så ökar den tre … så hade du gått tio ut så kommer den å öka tretti. [pekar i diagrammet]
32. FARO: … Aa
33. L: Hängde du med?
34. FARO: Nej.
35. L: Den här [pekar på OH] tvåan sa vi d//en ökar på varje steg du går på X-axeln.
36. FARO: //Mm..
37. FARO: Mm..
38. L: Så får du Y två.
39. FARO: Aa..
40. L: Och ett till då ökar vi Y med två till.
41. FARO: Aa okej.
42. L: Den där den är ju- ökar för varje steg där … tre steg. [L ser mot FARO och illustrerar en trappa i luften]

43. FARO: … Aha … jag tror jag fattar … det är som en trappa. [FARO tecknar själv en trappa]
44. L: Ja det blir det. [L på väg bort från FAROs bänk]

Hur är det möjligt att förstå Faros svårigheter med uppgiften? Även efter att ha tittat igenom den här videosekvensen ett stort antal gånger är det inte lätt att reda ut på vilket sätt Faro uppfattar uppgiften och vad han egentligen behöver hjälp med. Ännu svårare är det naturligtvis för en lärare att snabbt (helst direkt) komma fram till hur man på bästa sätt kan hjälpa en elev i svårigheter.

I rad 19 konstaterar Läraren att de inte gått igenom uttryck av typen $y = kx + m$ ($m \neq 0$). Den lärarledda genomgången under första delen av lektionen behandlade endast uttryck av typen $y = kx$. Faro har därmed endast haft möjlighet att erfara en möjlig variation när linjer med olika lutning behandlats, men inte att linjernas skärningspunkt med y-axeln kan variera. Detta i kombination med den hastighetsindividualisering som förekommer, gör att den sekvensering som läraren kan ha planerat för innehållet – att först behandla uttryck av typen $y = kx$, där betydelsen av parameter k tas upp, följt av uttryck av typen $y = kx + m$, där m hanteras – omintetgörs då Faro ”kommit längre” i boken och möter uppgifter som han inte är förberedd på (se även Emanuelsson m.fl., 2003a). Faros uttalanden i rad 18 och 22 kan möjligen styrka tolkningen av att han inte vet hur han kan hantera uttryck med flera parametrar givna.

I den lärarledda inledningen om relationen mellan uttrycken $K = 15x$, $K = 10x$ och $K = 2x$ och respektive grafs utseende (se figur 8 i appendix 2) försöker läraren ett flertal gånger poängtera relationen mellan koefficienterna 15, 10 och 2 och *lutningen* på respektive linje. Eleverna uttrycker sig inte lika tydligt om lutningen utan använder formuleringar som ”ligger högre” och ”ligger lägre”. Vid några tillfällen under genomgången accepterar läraren detta sätt att beskriva förhållandet mellan de tre linjerna. Ett exempel på det är i slutet av genomgången då själva poängen ska sammanfattas.

T	//Va kan man alltså utläsa av den termen eller den- den siffra som står i samband med X i en sån här formel?
[…]	
T	Vi har tre stycken formler [börjar peka ut formlerna på OH-bilden]
[…]	
T	Vad kan man alltså säga om femton tio två direkt när ni får en sån där formel då kan ni säga nåt om dom här i alla fall hur dom är inbördes?
S	Eh
Martina	Vilken ordning dom är eller?
T	Ja- nej inte vilken ordning men hur dom lutar [börjar luta sin linjal snett i luften]
Johan	Hög låg eller eh … mitt emellan

T	visst … om du får tre kurvor … så här … vi säger att du får den och så har vi inte några siffror eller nåt på axlarna men vi har dom här tre- tre l- linjerna … kan ni säga vilken ekvation som hör till vilken lin//je då?
[…]	
Annette	//Den högsta är högst och den lägsta är lägst [mycket sorl]
T	Aa … ssh ssh aa
Martina	Men du ser ju det
T	Ja hur ser du de då?
Martina	Jo för de-
Annette	Den högsta är högst och den lägs//ta är lägst
Martina	//Ja precis
T	Visst de //här [pekar mot ”K=2x” på OH-bilden]… den eh het- den har ett namn den kallas lutningskoefficient
Martina	//(De-)
S	Ååh
Veronica	[Smackar irriterat] ååh [lägger huvudet på bänken]
T	Men- men de spelar ingen roll men- de- det som är viktigt är … vi vet hur mycket dom lutar [pekar på alla funktioner snabbt stegvis]

Uttalandet, ”den högsta är högst och den lägsta är lägst” accepteras till sist av läraren som en korrekt beskrivning av sambandet mellan formel och graf. En möjlig tolkning av vad eleven egentligen menar är att ”den formel som har den högsta koefficienten svarar mot den högsta linjen i koordinatsystemet”. Så länge man endast jämför proportionaliteter, som under denna lärarledda genomgång, blir skillnaden mellan ”lutar mest/är brantast” och ”är högst/ligger överst” inte märkbar då man avser samma linje med båda sätten att uttrycka sig. I Uppgift 37 blir däremot skillnaden betydlig, linjen F som ”lutar mest” är den som ”ligger underst”. Faro verkar ha svårt att hålla isär lutningen och placeringen (y-led) och det kan möjligen förklara hans osäkerhet i rad 14.

SW1 – Lektion 11 – Elevintervju

Under intervjun efter lektion 11 framkommer det att Faros förståelse av de algebraiska uttrycken är oklar. Efter att Faro tittat på sekvensen där han och läraren diskuterar uppgift 37 stoppas bandet.

Int	Du *fattar* du detta? … förklarar han *bra* här?
FARO	Nej sådär
Int	Sådär?
FARO	Ja … jag fattar fortfarande inte varför man har X och massa grejer där det är onödigt tycker jag
Int	X blir problemet
FARO	Ja

[…]

Int	Vad står det här för någonting? K är lika med 125 plus tre X vad- vad- vad- vad betyder det här? … har du någon idé om vad det betyder?
FARO	Nej
	[…]
Int	Ja … men X blir lite knepigt
FARO	Mm
Int	Vad *är* det då?
FARO	Någonting säger han. X är någonting // man vet inte vad det är
Int	// X är … någonting [skrattar]
FARO	Ja någonting säger han
Int	Någonting precis vad som helst?
FARO	Någonting
Int	Atomer eller spöken eller?
FARO	Ja
Int	Nej det kan det väl inte vara?
FARO	Nej men någonting säger han- något tal man vet inte vad det är
Int	Aha
FARO	Det sa han i alla fall på al- algebran

Innebörden i variabeln *x* är inte klar för Faro utan ställer till besvär, ”*x* är onödigt”. Faro tolkar i bästa fall *x* som ett okänt tal, vilket är den betydelse *x* haft i avsnittet om ekvationer och ekvationslösning som man arbetat med någon vecka tidigare. För att kunna förstå formler och hur de är relaterade till grafer räcker dock inte den uppfattningen till. Nu kan vi bättre förstå de svårigheter Faro har med Uppgift 37. Både vid diskussionen om uppgiften och vid lärarens genomgång krävs att *x* förstås som en variabel, en symbol för en hel uppsättning tal på samma gång. Med den begränsade uppfattning av *x* som Faro har måste en stor del av diskussionen vara utomordentligt svårbegriplig.

Vid den lärarledda genomgången växlar man dessutom perspektiv på ett annat sätt ett flertal gånger. Ibland betraktas hela uttrycket/formeln som eget objekt som är relaterat till en ”hel linje” i diagrammet, ibland betraktas uttrycket för ett *x*-värde i taget och en punkt i taget på linjen. Liknande skifte i perspektiv gör läraren obehindrat under ”bänksekvensen”. I rad 7 (se transkript) betraktas hela linjer och dess lutning, i rad 9 ett enda värde på *x*. I rad 31 börjar läraren låta variabeln *x* anta flera olika värden, ett i taget. Att Faro inte hänger med i det resonemanget förstår vi lättare efter att ha tagit del av hans uppfattning om vad bokstaven *x* kan ha för betydelse.

Sammanfattning

En design där matematikundervisning dokumenteras på ett omfattande sätt med flera kameror under en lång följd av lektioner och med videoinspelning av uppföljande intervjuer i snar anslutning till respektive lektion, där elever och lärare kommenterar lektionsförloppet, ger stora möjligheter till en djupare förståelse av den studerade undervisningen och relationen mellan undervisning och lärande. Dessutom kan internationella jämförelser bidra till att synliggöra karakteristiska drag i undervisningen på makronivå, vilket underlättar analysen av förlopp på mikronivå. De mycket preliminära tolkningar av ”bänk-sekvensen” som presenterats här hade varit svåra att göra utan den stora tillgången på data som insamlats på många olika sätt. Trots det är det svårt att dra några säkra slutsatser om något så komplext som lärande och undervisning i matematik.

Referenser

Clarke, D. (2000). *The learner's perspective study. Research design.* Melbourne: University of Melbourne.

Emanuelsson, J. m.fl. (2003a). *The way we do things in Sweden: On school cultures and maths lessons from different perspectives.* Paper presented at the NFPF meeting in Copenhagen, March 6-9 2003. [En tidigare version är tillgänglig på: www.ped.uu.se/kult, 030814]

Emanuelsson, J. m fl (2003b). *Interaction and variation of content in a Swedish and a US mathematics classroom: Talking the talk or getting it right?* Rapport presenterad vid symposiet, "Social interaction and learning in mathematics classrooms in Australia, Germany, Hong Kong, Japan, Sweden, and the United States" vid EARLI–konferensen 26-30 augusti. Italien: Padua.

Emanuelsson, J., Sahlström, F. & Liljestrand, J. (2003). *Contrastive perspectives on constructing learning – in classrooms and in research.* Rapport presenterad vid LPS Team Conference, 1-3 december, Melbourne.

KULT (2003). *Svensk skolkultur – klassrumspraktik i komparativ belysning.* [Elektronisk] Tillgänglig: <http://www.ped.uu.se/kult/default.asp>. [2003-10-29]

LPS study (2003). *The learner's perspective study.* [Elektronisk] Tillgänglig: <http://www.edfac.unimelb.edu.au/DSME/lps/>. [2003-10-29]

Marton F. & Booth S. (1997). *Learning and awareness.* Mahwah, NJ: Lawrence Erlbaum Associates.

Runesson, U. & Marton, F. (2002). The object of learning and the space of variation. In F. Marton & P. Morris (Eds.), *What matters? Discovering critical conditions of classroom learning.* Göteborg Studies in Educational Sciences 181. Göteborg: Acta universitatis Gothoburgensis.

Ett stort tack till övriga medlemmar i KULT-projektet, Josefin Häggblom, Johan Liljestrand, Sverker Lindblad, Fritjof Sahlström vid Uppsala universitet och Jonas Emanuelsson, Ference Marton, Livia Norström, Ulla Runesson vid Göteborgs universitet.

Appendix 1

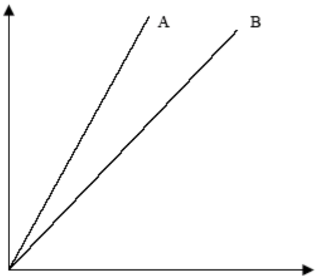
SW1-L11 [39:40 – 42:15]

FARO	Kolla … A [pekar med pennan på uppg. 37a som lyder: ”a) K=125 + 3x”]
T	//Jaa..
FARO	//Vil- … vilken av linjerna visar sambandet?
T	Ja just //det.
FARO	//Du sa du sa att man kunde se lin-//kurvan av bara den här.
T	//Ja … a den här-
T	Ja just det, den där lutar mest, den lutar lite mindre, och den lutar minst. [pekar på de tre funktionuttrycken i uppgiften]
FARO	Mm..
T	Men här [T pekar eventuellt på ”175” i 37c] har dom ju slängt in en grej framför också vad betyder det här … om X är noll … då blir det där noll va … men då börjar den på hundrasjuttifem. [visar i diagrammet]
FARO	Mhm..
T	Så de är den där [pekar i diagrammet] … C … det är den där … K fyra X den börjar från origo //och går rätt upp … den där börjar på hundratjugofem där har vi hundratjugofem … (//och där har vi hundrasjuttifem) (…) [T reser sig upp och tittar irriterat ut över klassen]
FARO	//Ja.
Jessica	//John när får vi byta platser? [till T]
FARO	Är B noll här … Är det den B [T tittar ned mot FARO igen] … när B är den största eller hur?
T	Ja de- de- den lutar mest … titta den här lutar ju brantare än den och än den … [T sätter pennan mot ”b) K=4x” i boken] … men sen börjar dom här på lite olika … //värden. [T pekar ut kurvorna] [Bänkgranne Alva vänder sig mot FARO och T]
Rodi	//John! [ropar på T]
FARO	Men alltså är det här fyra X. [FARO pekar på linjen ”F” i grafen]
T	Va vad är det du ska- den där är fyra X ja … aa … mm. [T pekar på linjen ”F”]
ALVA	John jag behöver hjälp! [till T]
T	För no- om du sätter in noll där … så //blir kostnaden noll.
FARO	//Alltså kan jag skita i dom här som är framför?
T	Nej för dom gör ju att du börjar där … de har vi inte gått //igenom än.
FARO	//Aha do- do- dom visar var jag börjar.
T	Precis … och här står det egentligen noll plus.
FARO	Men vad //visar dom här då //förutom att de- denna visar vilken som är //störst och minst?
Hannes	//Miraz … här ( ) är det femton … kilometer. [åsyftar uppg 24 sid 178, pratar till Miraz som kommer in i klassrummet]
T	//Lut- … (aa) lutar [T lägger pennan längs linje i diagram] den visar hur det lutar och //hur mycket varje sak kostar.

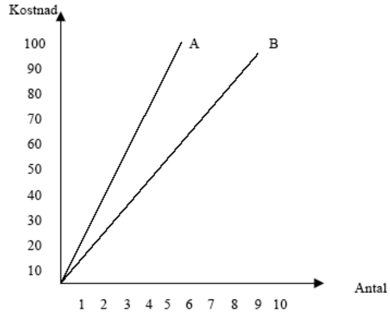
FARO	//Me jag kan- jag kan inte bara se bara så här.
T	Joo! [till FARO] [T reser sig]
FARO	Jamen om inte dom här två skulle finnas så skulle ja aldrig kunna ha veta vilken- vilken tre X är av dom här. [pekar på linjerna i diagrammet]
T	Nej, fast du kan veta vilken hundratjugofem plus tre X är. [till FARO]
FARO	Hur //gör man det då? [till T]
T	//För- för att den börjar på Y-axelns på hundratjugofem. [till FARO]
FARO	Ja den börjar på hundratjugofem sen tre X var ska jag hitta det då? [till T]
T	Jo och det är för varje sån steg du går där så ökar den tre … så hade du gått tio ut så kommer den att öka tretti. [pekar i diagrammet]
FARO	… ja..
T	Hängde du med?
FARO	Nej.
T	Den här [pekar på OH] tvåan sa vi d//en ökar på varje steg du går på X-axeln.
FARO	//Mm..
FARO	Mm..
T	Så får du Y två.
FARO	Ja.
T	Och ett till då ökar vi Y med två till.
FARO	Ja okej.
T	Den där den är ju- ökar för varje steg där … tre steg. [T ser mot FARO och illustrerar en trappa i luften]
FARO	… aha … jag tror jag fattar … det ärsom en trappa. [FARO tecknar själv en trappa]
T	Ja det blir det. [T på väg bort från FAROs bänk]

Appendix 2

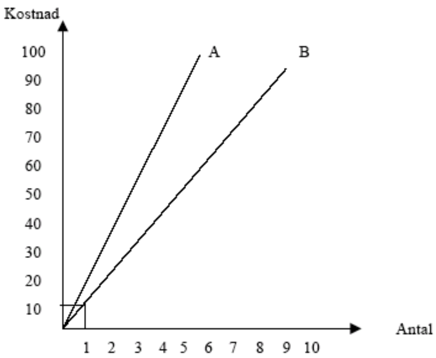
Grafer som visas med OH under den lärarledda inledningen av lektionen.
Läraren börjar med figur 1 och kompletterar den successivt till figur 8.



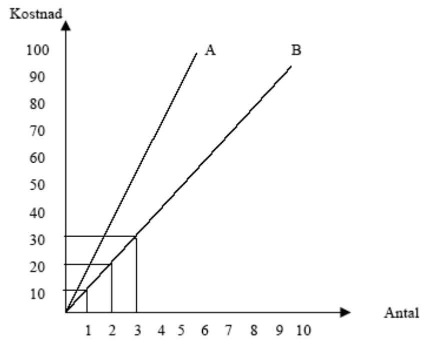
Figur 1



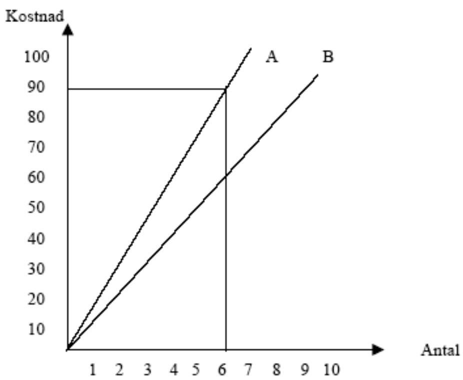
Figur 2



Figur 3

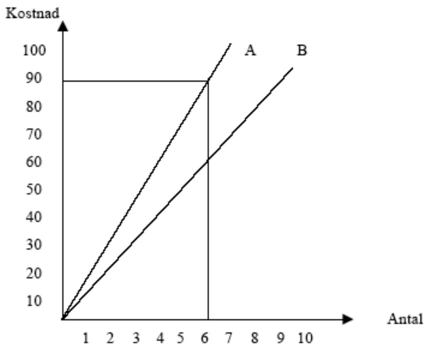


Figur 4



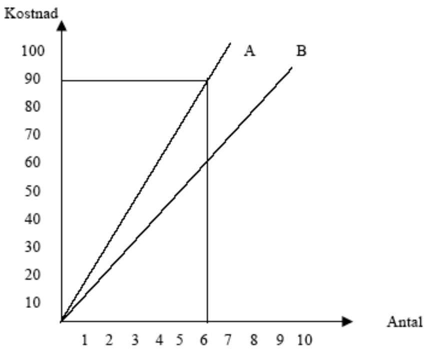
K = 15x

F



K = 15x
K = 10x

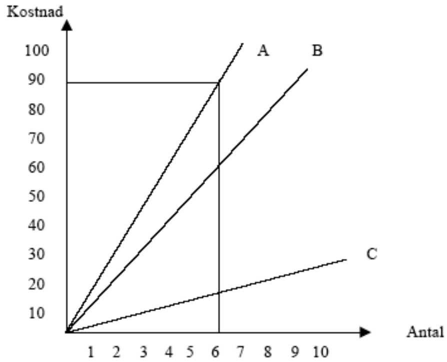
Figur 6



K = 15x
K = 10x
K = 2x

x	K	koor.
0	0	(0; 0)
1	2	(1; 2)
5	10	(5; 10)

Figur 7



K = 15x
K = 10x
K = 2x

x	K	koor.
0	0	(0; 0)
1	2	(1; 2)
5	10	(5; 10)

Figur 8

Limits of Functions – How Students Solve Tasks

Kristina Juter
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Introduction and questions

This study was conducted to reveal how students at university level justify their solutions to tasks with various degrees of difficulty. The study is part of a larger study of students' concept formation of limits of functions. The study was carried out at a Swedish university at the first level of mathematics. Two groups of students taking the same course in successive semesters have been solving tasks. Their solutions are here categorised and analysed to create a picture of how students reason about limits.

The questions are: How do students solve problems with limits? How do they explain their solutions?

Theoretical background

About problem solving in general

Mathematics is often expressed with symbols operated by certain rules. The rules have to be known to an individual engaged in mathematical activity and they can be memorised. This is not enough, though, if he or she wants to understand mathematics. Instead of only memorising formulas and procedures, the individual needs to have an exploring attitude to problem solving (Schoenfeld, 1992). Pólya (1945) describes a way to go about it in terms of decomposing and recombining. The problem is at first considered as a whole. Then details are examined to give more information for the solution process. The details are combined in different ways and this may give a new perspective to the problem as a whole.

Students learn new and improved methods for problem solving as they take courses in mathematics. This means that they eventually can have quite a few methods to choose from, both new and old ones. When an individual encounters a problem he or she might not fetch the optimum solution method from the mind (Davis & Vinner, 1986; Pólya, 1945). This is not the same as saying that the student can't solve the problem in a better way. We can't know what strategies are available in an individual's mind, but we can see the chosen method. The students' actions are shaped by their abilities (Star, 2000). However effective and numerous methods a person has in his or her mind, if they are unreachable at the time they are needed they are of no use.

A concept image is the total cognitive representation of a notion that an individual has in his or her mind (Tall & Vinner, 1981). It might be partially evoked and different parts can be active in different situations leading to possible inconsistencies. An individual's concept image might differ from the formal concept definition or the concept image in itself can be confusing or incoherent.

Lithner (2003) has studied students solving tasks at university level. His study showed that almost all the time the students spent on mathematics at home was devoted to exercises. The students' preferred way to reason was: look for surface properties in other solutions or theorems and use the same method in the new problem. The students compared the problems with solutions in the textbook to the problems they should solve and used the strategy from the textbook. One problem for the students was to identify for them useful surface properties to select the correct procedure for the solution.

About problems related to limits

Infinity is a notion that can cause trouble. It is something an individual has one or several intuitive representations of (Tall, 1980). If there are multiple different representations evoked simultaneously the result might be erroneous. When dealing with limits of functions one has no specific method or algorithm as one has, for example, for Diophantine equations. The limit process appears potentially infinite and students can get the impression that there is no end to it (Tall, 2001). It can be hard for them to work with items that are confusing in identity. Is it an object or a process?

A common error in students' concept interpretations of limits of functions is that functions do not attain their limit values (Cornu, 1991; Tall, 1993; Tall, 2001; Szydlik, 2000). There is also a possible mix-up of $f(a)$ and $\lim_{x \to a} f(x)$ (Davis & Vinner, 1986). These two flaws combined can totally block students in their struggle with tasks that could easily be solved, for instance, with an equation.

When students meet the concept of limits at universities for the first time, they have already been working with functions and their graphs. The goals in the curriculum for upper secondary school in Sweden do not mention limits explicitly, but the students are expected to learn about derivatives and integrals (Skolverket, 2003). This implies that limits of functions are discussed in some form. The students at universities therefore have an existing concept image of limits of functions that has been satisfactory in the contexts they have been in so far. Hence there is no need to learn the formal limit concept to be able to analyse functions (Williams, 1991). The students have to experience the need for further sophistication in their mathematical development to adjust their existing mental representations of limits.

The definition of the limit concept often causes difficulties for the students (Cornu, 1991; Juter, 2003; Vinner, 1991). The students' concept definitions are not always compatible with the formal concept definition and that can cause an incoherent concept image with different rules for different situations. If a problem is stated in a manner that is not specifically represented in the students' concept images, there can be more than one representation evoked in the effort to solve the problem. This can make the students confused and unable to proceed.

There are many things that can disturb the solving process. The goal of this study is to find out more about the students' solution strategies when solving problems involving limits of functions and the justifications of their choices.

The study and the students

The students in this study were enrolled in a ten weeks full time course in mathematics. They were learning analysis and algebra at basic university level.

In the spring semester of 2002 111 students solved *Task 1* to *Task 3* described below. They had treated limits of functions in the course and it was nothing new for them. 11 days later they got *Task 4* and *Task 5*. 87 students participated in that session. The last set of tasks provided solutions that could be wrong or incomplete. This was stated on the sheet with the tasks and the students were to give a complete and correct solution to each task. The solutions were given to provoke the students to either agree or find the errors instead of just leaving the tasks or giving a brief answer.

A new group of students got the same tasks the following semester (autumn of 2002). They got all the five tasks at the same time, after the notion of limits of functions had been dealt with. 78 students took part in this study. They were fewer than the previous semester and one reason is that the students who had biology or chemistry as main subject were offered another course more suitable for them and this was not an option for the students in the spring study.

In both cases the tasks were part of three questionnaires with questions about limits and attitudes towards mathematics in general. Two interviews with each of 18 of the students were conducted in the autumn study (Juter, 2003).

The first three tasks in this presentation were slightly altered in the second study since many of the students misinterpreted or did not understand what the tasks were about. The change was from "Can the function f(x) = $2x+3$ attain the limit value?" to "Can the function f(x) = $2x+3$ attain the limit value in 1a?" with respective functions in *Task 1* to *Task 3* below.

All students in both groups got the questionnaires, so no selection was done.

Method

The tasks were constructed to focus on different aspects of the limit concept. The difficulty varied in order to identify the level the students could handle. I explained what I wanted the students to do at each session they responded to the questionnaires to make sure that it was clear to them.

The collected data has been rewritten and categorised with the aid of the computer program NUD*IST (N6, 2003). The categories were decided from the raw material. I did not create them in advance other than that the right and wrong answers made different categories. There were subcategories in each of them based on the students' justifications of their responses. They were chosen from the different reasons the students used for their solutions. This process led to a number of categories. Categories with similar types of reasoning were merged together to make the presentation more accessible. Some solutions are in more than one category since some students gave more than one solution or solutions that fitted more than one category for other reasons. This way to work with the data gave different types of category systems for the different tasks.

Empirical data

Examples of typical student answers are provided in a table after each task. The tables also include the number of students from each semester in each category. The numbers within brackets are percentages of participating students in each class. (R) indicates that the answer is right and (W) denotes a wrong answer.

Task 1: a) Decide the limit: $\lim_{x \to 3}(2x+3)$.

b) Explanation.

c) Can the function f(x) = $2x+3$ attain the limit value in 1a?

d) Why?

The first two parts of the task were solved mainly in two ways. The students' either just replaced x with 3 or they solved the task by letting x tend to 3 and state that the function gets close to 9. Almost all students answered correctly. The last two parts of the task resulted in a more varied set of solutions presented in *Table 1*.

Table 1. Typical student answers in the different categories for *Task 1c – d.* Number of students (% of the students).

Category	**1c**	**1d**	**Spring 2002**	**Autumn 2002**
Theory (R)	Yes	The function is continuous at the point	22 (20)	23 (29)
Replace x by 3 (R)	Yes	$2x+3=9$ for $x=3$	22 (20)	21 (27)
No explanation (R)	Yes	-	15 (14)	8 (10)
Limits not attainable (W)	No	A function does not attain the limit value, it only comes very close, it is in the definition	9 (8.1)	10 (13)
No reason (W)	No	-	3 (2.7)	3 (3.8)
Empty or misinterpretation		The answer has no connection to the question or is missing	40 (36)	16 (21)

Task 2: a) Decide the limit: $\lim_{x\to\infty}\frac{x^3-2}{x^3+1}$.

b) Explanation.

c) Can the function f(x) = $\frac{x^3-2}{x^3+1}$ attain the limit value in 2a?

d) Why?

Most students solved this task by locating the dominant terms and either just reason about what happens to the function as x tends to infinity or go through algebraic calculations to find the limit value. About 15 percent of the students were unable to perform the algebraic calculations. The results of parts c and d are presented in *Table 2*.

Table 2. Typical student answers in the different categories for *Task 2c – d.* Number of students (% of the students).

Category	**2c**	**2d**	**Spring 2002**	**Autumn 2002**
$x^3-2\neq x^3+1$ (R)	No	x^3-2 can never be equal to x^3+1 for the same value of x	7 (6.3)	17 (22)
-2 & 1 (R)	No	terms –2 and 1 will always remain	7 (6.3)	5 (6.4)
No explanation (R)	No	-	18 (16)	8 (10)
Infinity reason (W)	No	Since x never attains the value ∞	16 (14)	21 (27)
Theory (W)	No	The function tends to the limit value, it does not attain it	7 (6.3)	6 (7.7)
No reason (W)	Yes	-	15 (14)	7 (9.0)
Empty or misinterpretation		The answer has no connection to the question or is missing	42 (38)	17 (22)

Task 3: a) Decide the limit: $\lim_{x\to\infty}\frac{x^5}{2^x}$.

b) Explanation.

c) Can the function f(x) = $\frac{x^5}{2^x}$ attain the limit value in 3a?

d) Why?

Almost all students were able to solve parts a and b of this task using standard limit values. A few students got it backwards. The following parts are described in *Table 3*.

Table 3. Typical student answers in the different categories for *Task 3c – d.* Number of students (% of the students).

Category	**3c**	**3d**	**Spring 2002**	**Autumn 2002**
x=0 (R)	Yes	For x=0 $\rightarrow$ f(0) = $\frac{0}{1}$ = 0	15 (14)	16 (21)
No explanation (R)	Yes	-	5 (5)	6 (7.7)
Does not reach limit (W)	No	We can only get infinitely close	14 (13)	16 (21)
$x^5 \neq 0$ or $\frac{0}{0}$ (W)	No	Then the numerator has to be zero and it never is	22 (20)	12 (15)
Right for wrong reason (W)	Yes	Because the denominator attains a much larger number for large *x*	5 (5)	16 (21)
Empty or misinterpretation		The answer has no connection to the question or is missing	47 (42)	12 (15)

Task 4: **Problem**: Decide the following limit value: $\lim_{x \to 1} \frac{x^2 + x}{x^2 - 1}$.

The students were given the following:

Solution: $\frac{x^2 + x}{x^2 - 1} = \frac{x(x+1)}{(x-1)(x+1)} = \frac{x}{x-1} \to \infty$ when $x \to 1$.

The task was for the students to decide the proper adjustments to make the solution correct:

Adjustments (What changes or complements are needed and why):

Table 4. Typical student answers in the different categories for *Task 4.* Number of students (% of the students).

Category	**4**	**Spring 2002**	**Autumn 2002**
Both sides (R)	$x \to 1$ from minus or plus	18 (21)	28 (36)
One side (R) Incomplete	As $x \to 1$ the denominator becomes negative	10 (11)	5 (6.4)
Dominant factor (W)	$\frac{x^2\left(1+\frac{1}{x}\right)}{x^2\left(1-\frac{1}{x^2}\right)} \to \frac{1+1}{1-1} \to \infty$	15 (17)	12 (15)
Reasoning (W)	*x* tends to 1 hence it can not be infinite	14 (16)	18 (23)
No change (W)	It is entirely correct!!	8 (9.2)	4 (5.1)
Empty or unclear	The answer is missing or does not make any sense	24 (28)	17 (22)

Task 5: **Problem**: Decide the following limit value: $\lim_{x\to\infty} x\sin\left(\frac{2}{x}\right)$.

The students were given the following:

Solution: $x\sin\left(\frac{2}{x}\right)=\frac{\sin\left(\frac{2}{x}\right)}{\frac{1}{x}}$. We know that $\frac{2}{x}\to 0$ when $x\to\infty$ and $\frac{1}{x}\to 0$ when $x\to\infty$.

The limit value $\lim_{x\to 0}\frac{\sin x}{x}=1$ implies that $\frac{\sin\left(\frac{2}{x}\right)}{\frac{1}{x}}\to 1$ when $x\to\infty$.

The task was for the students to decide the proper adjustments to make the solution correct:

Adjustments (What changes or complements are needed and why):

Table 5. Typical student answers in the different categories for *Task 5.* Number of students (% of the students).

Category	**5**	**Spring 2002**	**Autumn 2002**
All right (R)	$\frac{\sin\left(\frac{2}{x}\right)}{\frac{1}{x}}=\frac{\sin\left(\frac{2}{x}\right)}{\frac{2}{x}}\cdot 2\to 2$	10 (11)	16 (21)
Part right (R) Incomplete	$\frac{2}{x}$ is not the same thing as $\frac{1}{x}$ therefore you can not use limit values on limit values	9 (10)	11 (14)
Reasoning (W)	$\frac{1}{x}\to 0$ when $x\to\infty$ so the denominator has to go $\to 0$ and the expression $\to\infty$	31 (36)	18 (23)
No change (W)	Nothing	4 (4.6)	1 (1.3)
Empty or unclear	The answer is missing or does not make any sense	33 (38)	32 (41)

Analysis

The students' solutions to *Task 1a-b* are mainly correct. The correct solutions were divided into categories, with most responses falling into two categories. The second category explicitly suggests that the limit value is attainable and a large part of the students have chosen this way to solve the task. An even larger part of the students solves the task in words of approaching the limit value. The idea that limits are not attainable comes again in *Task 1c-d*, where students use it as an argument for the function not to attain the value 9. It is a documented fact that some students have this misinterpretation of the limit value definition (Cornu, 1991; Tall, 1991). Some students do not separate the part with the limit value from the part with the function, that is they mix up $f(a)$ and $\lim_{x \to a} f(x)$ as Davis and Vinner (1986) describe. If the students have the conviction that limits are unattainable, there might be problems analysing the function. This is something that follows through *Task 1* to *Task 3*. A large part of the students did not answer or answered *Task 1c-d* in a way that did not make any sense. Some wrote that it depends on what x tends to and this is the reason for the additional words in the formulation of the task in the autumn of 2002. The result of the change is that this category contained a smaller number of students the second semester. This applies for all the *c-d* questions.

Task 2 is harder for the students to handle and one serious problem is algebra. There are several students who think that x^3 in the numerator and the denominator are cancelling out in a way that erases the terms and leaves only the constants or that x^3 can be replaced by 1 with a similar reasoning. There are more students in the autumn study using algebra in a correct manner to solve the task than in the spring study. The majority of the students are unable to solve *Task 2c-d*. One problem is attainability as discussed above. Only some students regard this problem as an equation to solve and this is obvious in *Task 3c-d* as well. 22% of the students in the autumn study solved *Task 2c-d* with an equation in a correct way, whereas the corresponding figure in the spring study is 6.3%.

Task 3a-b is a standard limit value that is well known to the students apart from about 10% who got it backwards ("Infinity or no limit (W)"). There are fewer categories for this task since over 70% of all the students used the standard limit value reasoning. The problems come in *Task 3c-d* where the mix up of limits and functions is clear. Students claim that x is never zero since x tends to infinity, but this has nothing to do with the functions ability to attain the value 0. Algebra is a problem for some students here too.

Task 4 offers a challenge for many students. 46 of the 165 students were able to solve the task completely. One mistake many made was to use the dominant factor and divide, but that is not solving the problem with left and right limit value. This method is often used when x tends to infinity and the students seem to just go through the motions without considering the characteristics of the task they are involved in.

Task 5 is apparently the most demanding one since only 26 of all the students managed to solve it properly. The 20 students in the "Part right (R) Incomplete" category were also correct but they did not give the limit value, they only pointed at the inaccuracy, and thus they might or might not be able to carry out the calculations. The vast majority of the students either left the task unsolved or reasoned incorrectly.

Discussion

The tables show that the students' explanations of choices of solutions vary. The students' problems seem to come when the tasks are a bit different from what the students are used to. Parts *c* and *d* are not mathematically more demanding than the other parts of *Task 1* to *Task 3* but there is something that troubles the students. The outcome might have been different if the parts were not presented together. The students were in the context of limits when they were asked to examine the functions for attainability. The effect was that the functions were only considered locally in some cases, and the misconception that limits are unattainable (Cornu, 1991; Tall, 1993; Szydlik, 2000) made some students claim that the function could not attain the value even if it obviously could (*Task 1* and *Task 3*). The confusion of functions with limits of functions is a problem that indicates a lack of relations between the concepts. If the students were more confident about the properties and possibilities of the notions they would have a better chance to solve problems correctly. An insufficient mathematical base to work from can cause constraints on the individual in that he or she is not sure what operations are allowed and how to carry them out. This uncertainty can be the reason for the many empty answers.

Infinity is obviously an element that can cause confusion (Tall, 1980). All tasks revealed problems with infinity in different degrees. One thing that is connected with infinity is the notion of local limits in a wider context. *Table 5* indicates this by the categories "Reasoning (W)" and "No change (W)". Many students are reasoning about the local limits for the functions $\frac{2}{x}$ and $\frac{1}{x}$ separately or dissect the given function in other ways and locally consider limits. The students follow part of Pólya's (1945) model with decomposing and recombining, but the recombining to check at the whole again is overlooked. There appears to be a lack of parts in the mental web that represents this fraction of the concept image (Tall & Vinner, 1981), since they do not have access to the essential information about the properties of the limit process and functions.

Table 4 shows an example of students using surface properties (Lithner, 2003) as the students in the category "Dominant factor (W)" use a solving technique that is usually effective on rational functions as x tends to infinity. Here the students recognise the rational function but they do not consider what x tends to.

The choices of methods seem to be triggered by first sight resemblances in other cases too. There are comparisons with standard limit values. Sometimes the method is working, as it did for most students' solutions to *Task 3a-b*. Other times it does not work, as for some suggested solutions to *Task 5*. The students do not appear to have a global view of the important characteristics of the mathematics at hand. The effect of this can be that critical features are over-looked and the solution is beyond the possibility to reach for the students.

Table 2 and *Table 3* show examples of solutions with correct answer and wrong explanation. This is something that students must be confronted with to be able to repair. The textbook only gives the answer and not a full solution to the tasks so the first confrontation is in the worst scenario at the exam. If the students have used the wrong arguments for a long time, an adjustment can be hard to make. The students represented in *Table 2* who have answered correctly with no explanation, can also belong to the category of students with correct answer for the wrong reasons, since we do not know why they answered the way they did.

The students have to become aware of their problems before they feel a need to alter anything, and if the errors are not discovered nothing will happen.

References

Cornu, B. (1991). Limits. In D. Tall (ed), *Advanced mathematical thinking (*pp. 153-166). Dordrecht: Kluwer Academic Publishers.

Davis, R. & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior*, 5, 281-303.

Juter, K. (2003). *Learning limits of function, University students' development during a basic course in mathematics.* (Licentiate thesis) Luleå: Luleå University of Technology, Department of Mathematics.

Lithner, J. (2003). Students' mathematical reasoning in university textbook exercises. *Educational Studies in Mathematics*, 52, 29-55.

N6, (NUD*IST computer program) (2003). www.qsrinternational.com

Pólya, G. (1945). *How to solve it*. London: Penguin Books Ltd.

Schoenfeld, A. (1992). Learning to think mathematically: problem solving, metacognition and sence-making in mathematics. In D. Grouws (ed), *Handbook for research on mathematics teaching and learning* (pp. 334-370). New York: Macmillan.

Skolverket, (2003). http://www3.skolverket.se

Star, J. (2000). On the relationship between knowing and doing in procedural learning. In B. Fishman, & S. O'Connor-Divelbiss (eds.), *Fourth international conference of the learning sciences* (pp. 80-86). Mahwah, NJ: Erlbaum.

Szydlik, J. (2000). Mathematical beliefs and conceptual understanding of the limit of a function, *JRME* online, vol 31, issue 3, 258-276.

Tall, D. (1980). Mathematical intuition, with special reference to limiting processes. *Proceedings of the Fourth International Congress on Mathematical Education*, Berkeley, 170-176.

Tall, D. (1991). The psychology of advanced mathematical thinking. In D. Tall (ed), *Advanced mathematical thinking* (pp. 3-24). Dordrecht: Kluwer Academic Publishers.

Tall. D. (1993). Students' difficulties in calculus. *Proceedings of Working Group 3 on students' difficulties in calculus, ICME-7, Québec, Canada* (pp. 13-28).

Tall, D. (2001). Natural and formal infinities. *Educational Studies in Mathematics*, *48*, 199-238.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151-169.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (ed), *Advanced mathematical thinking* (pp. 65-81). Dordrecht: Kluwer Academic Publishers.

Williams, S. (1991). Models of limit held by college calculus students. *Journal for research in Mathematics Education, 22* (3), 219-223.

Learning to Communicate – Communicating to Learn in Mathematics Classrooms

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The aim of this paper is to investigate the practices of classroom learning communities whose pedagogy in the learning of mathematics draws on the socio-cultural perspective (Kaartinen, 2003). This pedagogical framework views learning as a collective process of meaning making situated in cultural contexts (Cole, 1996; Vygotsky,1962, 1978; Wertsch, 1991; Wertsch, del Rio, & Alvarez, 1995). Methodologically the paper is concerned with unravelling the dynamics of collaborative reasoning and how they give rise to the construction of diverse voices during participation in cultural activities (Kaartinen & Kumpulainen, 2001; Kumpulainen & Kaartinen, 2003). The empirical findings discussed here are derived from an in-service teacher education course.

Introduction

This study emphasizes the role of in-service teacher education for enhancing teacher participation in culturally organized activities and the development of their pedagogical thinking. The need for large-scale educational reform is currently being discussed (Kwakman, 2003; Sfard, 2001) at all levels of education. New skills addressing social and technological competencies are being embedded into curriculum in addition to traditional domain specific understanding (Kwakman, 2003), but we lack, however, research based evidence of teachers' readiness in fulfilling their new role as facilitators of students' learning processes for life-long learning. The sufficiency of traditional professional development activities such as attending courses, training, conferences and reading professional journals to refresh and update teachers' pedagogical approach, has been widely debated (Bransford, Brown, & Cocking, 1999; Darling-Hammond, 1998). Although the limitations of the traditional professional knowledge have been recognized, teachers lack the necessary support and tools to modernize their pedagogical thinking. This paper applies the concept of a "community of learners" (Rogoff, Matusov, & White, 1996) for enhancing teachers' readiness to respond to the challenges that modern society poses for mathematics education. The theoretical stance of a community of learners approach highlights transformation of participation in a collaborative endeavour (Goos, Galbright, & Renshaw, 1999). In this process, the participants build on each other's initiations and develop a joint solution for the problem. Therefore the focus of the paper is to investigate how mathematics teachers negotiate their role in joint problem solving. Specific attention will be paid to the adopted stance during problem solving and the nature of participation in the communicative activity.

Theoretical framework

In recent years cognitively-oriented approaches to education have been challenged by socio-cultural theories. The former have approached learning as an acquisition process which takes place as a result of the individual's active reconstruction of domain specific knowledge. Since the acquisition approach conceptualises knowledge as a kind of property that can be transmitted, the goal of learning is seen as the individual enrichment of domain specific concepts and procedures (cf. Sfard, 1998). The socio-cultural learning theories approach learning by examining teacher learning in its culturally situated context (Cole, 1996; Vygotsky, 1962; 1978; Wertsch, 1991; Wertsch, del Rio, & Alvarez, 1995), and hence define the learner as a cultural and historical subject embedded within, and constituted by, a network of social relationships and interactions. Learning and development, then, is explained by the changing nature of these relationships and types of participation in cultural activities (Goos, Galbraith, & Renshaw, 1999). From this perspective, teacher learning can be seen as an open-ended process with the possibility of diverse ways of acting, feeling and thinking (Renshaw & Brown, 1998). See also Kaartinen & Kumpulainen (2001; in press).

Pedagogical challenges to mathematics learning and instruction

While theoretical grounding for learning and development is being discussed (Lave & Wenger, 1991; Wertsch, 1991; Wenger, 1998), there is also a growing interest in a sociocultural approach to mathematics education (Sfard, 2002; Hoyles, 2002). The key constructs in defining the application of the socio-cultural framework to mathematics pedagogy in this paper are the communicative approach to cognition (Sfard, 2002) and the mediational role of semiotic tools (Säljö, 1995) in the collaborative meaning making of the participants of the domain in question. The theoretical constructs represented by socio-culturalists challenge the traditional views of mathematics learning and instruction where pre-organised pieces of mathematical knowledge are being transmitted to consumers. The task of instructional design in mathematics education from the socio-cultural perspective is to give to participants the possibilities to use mathematics in structuring and re-structuring their experiences in social practices where mediational tools are put to use for specific purposes (Säljö, 1995). The adopted stance in this paper holds that promoting participatory student learning in mathematics requires also the teacher to go through participatory processes in similar types of activities. The communicative approach to cognition (Sfard,

2002) stresses the role of language in the collaborative meaning making of participants. In this paper Halliday´s (1978, p.2) formulation of "language as a social semiotic" is applied to interpret language within a socio-cultural context in which the culture itself is interpreted in semiotic terms. In the analysis of this study, the language took a specific meaning in communal discourse and was further interpreted to construct cultural meanings across contexts. The role of language for collaborative inquiry is also reflected in the writing of Järvilehto (2000), who stresses the importance of the development of joint language as a tool for collaboration and the importance of the development of consciousness in the evaluation of collaborative action.

The Study

Research goals

The goal of this study was to investigate mathematical problem solving processes in a collaborative learning situation with in-service teachers. An analytic tool for highlighting the mechanisms of collaborative problem solving was applied (Kaartinen & Kumpulainen, 2001) and further developed. The specific research goals for this study are:

- To develop an appropriate analytic tool to highlight collaborative problem solving processes in the learning of mathematics.
- To investigate the role of cultural tools in the collaborative learning of mathematics teachers.
- To investigate the processes of teacher participation in the collaborative learning of mathematics pedagogy.

Participants

The data for the study was collected from two in-service teacher education courses carried out at the Department of Educational Sciences and Teacher Education, Oulu University, Finland, during the years 2000 and 2003. Altogether twenty in-service teachers, who represented either early childhood education (10) or primary education (10), participated in this study.

Description of the professional activities

The activities presented in this study are part of a teacher education course with the aim of giving the participants tools to anchor their instruction around the collaborative application of cultural tools. The cultural tools selected for the activities were geo-boards, algebra tiles and Cuisenaire rods. The selection of these tools was due to their ability to mediate the core domain of the mathematical curriculum, such as number sense, geometry and algebra. The specific activities were: The Application of geo-boards in mathematics instruction, To Model the computational algorithms of fractions with Cuisenaire rods, and To Model the algebraic expressions of polynomies with the help of algebra tiles. All the activities involved collaborative inquiry and experimentation. During the course, the in-service teachers worked in self-selected small groups. The average size of the mixed-gender groups was four to five participants. The whole group of twenty teachers worked simultaneously in the same classroom, carrying out their research designs for executing the activities. In this paper, the empirical example of the usage of algebra tiles will be discussed.

Data Collection

The primary data for the study consist of videotaped and transcribed episodes of social interaction in collaborative problem solving situations. In the investigated activity, the role of cultural tools, such as geo-boards, Cuisenaire rods and algebra tiles, is investigated in collaborative learning activities. Specific research questions posed for the study are the following: *"How are cultural tools applied for instruction building in primary mathematics?"* and *"What is the mediational role of cultural artefacts for algebraic/arithmetic computations in collaborative meaning making?"*

Data analysis

This paper applies discourse analysis (Kaartinen & Kumpulainen, 2001, 2002) in the investigation of collaborative interactions within mathematics learning situations. The discourse analysis procedure applied in the paper draws on the ethnographically grounded approach (Gee & Green, 1998). The analysis method and its specific categories were grounded in the discourse data of the study. In the analysis procedure, the collaborative interaction is approached from two dimensions, namely from the viewpoint of *discourse moves* and from the viewpoint of *cultural focus*. The analysis of discourse moves highlights the nature of conversational exchanges between the members of the learning community, and consequently sheds light onto the participatory roles of the group members in communal activity. Moreover, the analysis of discourse moves supports content analysis by highlighting thematic patterns emerging in joint problem solving. Discourse moves identified in the discourse data are *initiating, continuing, extending, organising, agreeing, evaluating, tutoring, thinking aloud and concluding.* To highlight the interplay between problem solving elements in the collaborative activity of in-service teachers, the second dimension in the analysis method investigates the cultural focus of the social interaction on a moment-by-moment basis. The cultural focus of the interaction data consisted of the procedural, identity, material and semiotic modes. Table 1 summarises the analytic frames and categories of the analysis method.

Table 1. The analytic method for analysing collaborative problem solving in mathematics

Discourse moves	Description	Example
Initiating	Begins a new thematic interaction episode	*this is x squared*
Continuing	Elaborates either own or colleagues' reasoning	*umm and these equal four x*
Extending	Bringing in new perspectives	*yes, should we draw*
Agreeing	Accepts the ideas or explanations proposed in the previous conversational turn	*okay, clear*
Evaluating	Evaluates reasoning	*so there it is*
Tutoring	Tutors the colleagues in reasoning	*may I still advise, we had this x squared four x and three as given*
Organising	Organises the working space	*should we move this, so they don't hinder us*
Thinking aloud	Makes reasoning explicit	*this was one, two, three, four x*
Concluding	Draws together explanation building processes	
Cultural focus	**Description**	**Example**
Procedural mode	Focuses on procedural elements, such as negotiating working strategies for joint investigation	*so we should organise this with the help of one tile, four rods and three ones*
Identity mode	Highlights the evaluation of prior learning experiences in the light of new experiences through the processes of reflection, dialogue and collaborative inquiry	*I am totally unfamiliar with these*
Material mode	Concentrates on physical features of the learning situation	*should we move this, so they don't hinder*
Semiotic mode	Highlights the visibility of meaning making via mediational tools	*so this colour connected with ordering these pieces carries out the meaning*

Results

The results of this study are discussed via case-based description derived from one teacher group, to highlight joint reasoning and the application of cultural tools in collaborative learning of mathematics pedagogy.

A case-based description

This case-based description highlights the collaborative processes of one teacher group when factoring polynomials with the help of algebra tiles. The extract characterises the teachers' discourse as they negotiate and apply the usage of algebra tiles in collaborative problem solving. Table 2 shows the discourse data of the teacher group. The extract consists of 24 conversational turns in total, from a 2-minute continuous working period. The data presented in Table 2 will be discussed here by firstly summarising the findings from the analyses of the teachers' social interaction within the group. Special attention will be paid to the identification of problem solving episodes in the teachers' discourse. This is followed by a micro-level investigation of three interaction episodes in the teachers' discourse. The analysis of the teachers' discourse reveals altogether three thematic episodes in the construction of an application for the usage of algebraic tiles. The themes for episodes are problem solving with the help of algebra tiles (Episode 1), clarification through mathematising (Episode 2), and clarification through hands-on activities (Episode 3). The analysis of discourse moves shows that the thematic episodes started from the initiation, questioning and tutoring moves, leading to several conversational turns which took the form of problem solving elements such as, questioning, extending, evaluating and tutoring. The analysis of the cultural focus of the teacher participation reveals the interplay of procedural, identity and semiotic modes of interaction. In the procedural mode of interaction, the mathematical activity included problem solving and problem posing, and the symbolic nature of the interaction was grounded in the pictorial and procedural application of algebraic tiles. In the identity mode of interaction, the prior learning experiences were reflected upon through the application of new cultural tools. In the semiotic mode of interaction the problem was clarified by negotiating the nature and meaning of algebra tiles and the mathematical activity was approached through mathematising the situation either verbally, symbolically or pictorially. The material mode of interaction was seldom present and it was used for organising the working space.

Table 2: Teacher participation (Task: Factor the expression $x^2 + 4x + 3$ with the help of algebra tiles)

No	Name	Transcribed discourse	Discourse moves	Cultural focus	
Episode 1: Problem solving					
1	Maritta	so we should organise this with the help of one tile, four rods and three ones	initiating	procedural	problem posing
2	Annikki	I am totally unfamiliar with these	evaluating	identity	evaluation of one's learning history
3	Maritta	should we move this, so they don't hinder us	organising	material	organising working space
4	Liisa	so this colour connected with ordering these pieces carries out the meaning	extending	semiotic	clarifies the problem
5	Maritta	yes	agreeing		
6	Karra	this	questioning	procedural	problem solving
7	Maritta	that is how I would imagine	agreeing		
8	Maritta	this is x squared	initiating	semiotic	mathematising
9	Liisa	umm and these equal four x	continuing		
10	Karra	so there it is	evaluating		
11	Maritta	so there it is	evaluating		
Episode 2: Clarification through mathematising					
12	Liisa	but also I don't understand this	questioning	identity	evaluation of one's understanding
13	Maritta	but the length of this equals x and this one and this three	tutoring	semiotic	the meaning of algebra tiles
14	Liisa	yes, should we draw	extending	procedural	symbolic pictorial
No	Name	Transcribed discourse	Discourse moves	Cultural focus	
15	Karra	x plus one	continuing	semiotic	mathematising symbolic algebraic
16	Maritta	multiplies three x plus three	continuing		
17	Annikki	I fell off the wagon	evaluating	identity	evaluation of one's understanding

No	Name	Transcribed discourse	Discourse moves	Cultural focus	
Episode 3: Clarification through hands on activities					
18	Maritta	may I still advise, we had this *x* squared four *x* and three as given	Tutoring	procedural	Starting information of the problem
19	Annikki	yes okay but this handicraft carried out the important meaning	questioning	procedural	clarifying the meaning of action
20	Maritta	The handicraft was to collect these pieces into a connected region in a way that there are no holes	Tutoring	semiotic	the meaning of the action
21	Annikki	yes okay	agreeing		
22	Annikki	this was one, two, three, four *x*	thinking aloud	semiotic	the meaning of cultural tools
23	Maritta	and now edge multiplied by edge	tutoring	semiotic	the mathematical meaning of the action
24	Annikki	okay, clear	agreeing	identity	understanding the meaning of the action

Episode 1

In Episode 1 the teacher group started the activity by posing the problem. The episode suggests that the usage of algebra tiles was new to all of the participants. Maritta was eager (6 turns of 11) in participating and tutoring the others. Annikki (turn 2) expressed here unfamiliarity with the usage of cultural tools and Maritta and Liisa made their thinking visible in their turns so that Annikki had the possibility to follow the joint problem solving. Karra was mainly silent but when participating (turns 6 and 10) he supported the group's problem solving by questioning and evaluating. In this episode the group reached the solution to the problem.

Episode 2

This Episode 2 nicely highlights how the group of teachers deepened their understanding of the situation. Liisa starts the episode by saying *"but also I don't understand this"* referring to the mathematical meaning of algebra tiles. This turn (turn 12) raises the semiotic nature of interaction. In her turn (turn 13) Maritta explains the meaning of algebra tiles by tutoring *"but the length of this equals x and this one and this three"* Liisa (turn 14) extends the joint reasoning by suggesting the modelling of the solution by drawing. This leads Karra (turn 15) to join the discourse by writing the expression *" x plus one"* and Maritta (turn 16) continues *"multiplies three x plus three"*. The mathematical modelling of the situation doesn't help Annikki who in her turn (turn 17) says *"I fell off the wagon"*.

Episode 3

This Episode 3 starts with Maritta's (turn 18) reaction to Annikki's comment by tutoring *"may I still advise, we had this x squared for x and three as given"*. In doing so Maritta refers back to the starting information of the problem situation. In her turn (turn 19) Annikki expresses that she couldn't connect the bridge between the mathematical meaning of the situation and the hands-on activities with algebra tiles. Maritta's tutoring turns (turns 20 and 23) connected with Annikki's agreeing and thinking aloud turns nicely highlights the interaction pattern where concrete activity was connected with the abstract nature of mathematical reasoning.

Discussion and Conclusions

In this study, the interplay between the two dimensions of the analysis procedure in collaborative problem solving of mathematics teachers nicely highlights the nature of communicative meaning making of mathematics teachers. The analysis of discourse moves makes visible the nature of reasoning from the viewpoint of participation in social activity. Four different participant roles emerged in the analysis of discourse moves. These roles were the tutor, clarifier, questioner and silent supporter. From a mathematical point of view, the communicative problem solving consisted of procedural, identity, material and semiotic modes of interaction. The patterns of interaction were constructed around these modes when groups of teachers negotiated the mediational meaning of cultural tools. The analysis of these patterns revealed diverse thematic episodes in collaborative meaning making, such as problem posing, problem solving, clarification through mathematising and clarification through hands-on activities. The study suggests the power of mediational tools to make visible the abstract nature of mathematical ideas behind the computational rules of algebraic procedures. Furthermore, the analysis of the data revealed that the usage of cultural tools in collaborative problem solving of mathematics teachers aided them in elaborating their conceptual understanding of mathematical ideas. On the whole, the study yields useful information about teacher learning and development from both the social and the mathematical point of view, and provides educators and researchers with tools to develop curriculum as well as instructional solutions for mathematics classrooms, both at the school and at the teacher education level.

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References

Beattie, M. (2000). Narratives of professional learning: Becoming a teacher and learning to teach. *Journal of Educational Enquiry, 1(2),*1-23.

Bransford, J.D., Brown, A.L., & Cocking, R.R. (Eds.) (1999). *How people learn: brain, mind, experience and school*. Washington: National Academy Press.

Cole, M. (1996). *Culture in mind.* Cambridge, MA: Harvard University Press.

Darling-Hammond, L. (1998). Teacher learning that supports students learning. *Educational Leadership, 55 (5),* 6-11**.**

Gee, J. P. & Green, J.L. (1998). Discourse analysis, learning, and social practice: A methodological study. *Review of Research in Education, 23,* 119–169.

Goos, M., Galbraith, P. & Renshaw, P. (1999). Establishing a community of practice in a secondary mathematics classroom. In L. Burton (Ed.), *Learning mathematics: From hierarchies to networks*. Studies in Mathematics Education Series 13. London: The Falmer Press.

Halliday, M. A. K. (1978). *Language as social semiotic: The social interpretation of language and meaning*. London: Edward Arnold.

Hoyles, C. (2001). From describing to designing mathematical activity: The next step in developing a social approach to research in mathematics education? *Educational Studies in Mathematics, 46*, 273-286.

Järvilehto, T. (2000). Theory of the organism-environment system IV. The Problem of mental activity and consciousness. *Integrative Physiological and Behavioral Science, 35,* 35-57.

Kaartinen, S. (2003). Learning to participate – Participating to learn in science and mathematics classrooms. Acta Universitatis Ouluensis. Scientiae Rerum Socialium E 64. Oulu University Press.

Kaartinen, S. & Kumpulainen, K. (2001). Negotiating meaning in science classroom communities: Cases across age levels. *Journal of Classroom Interaction,36,*4-16.

Kaartinen, S. & Kumpulainen, K. (2002). Collaborative inquiry and the construction of explanations in the learning of science. *Learning and Instruction, 12,* 189-212.

Kaartinen, S. & Kumpulainen, K. (in press). On participating in communities of practice: Cases from science classrooms. In J. van der Linden & P. Renshaw (Eds.), *Dialogic learning*. Amsterdam: Kluwer Academic Publishers.

Kumpulainen, K. & Kaartinen, S., (2003). The interpersonal dynamics of collaborative reasoning in peer interactive dyads. *Journal of Experimental Education, 71(4),* 333-371.

Kwakman, K. (2003). Factors affecting teachers' participation in professional learning activities. *Teaching and Teacher education, 19,* 149-170.

Lave, J. & Wenger, E. (1991). *Situated learning. Legitimate peripheral participation.* Cambridge: Cambridge University Press.

Renshaw, P. D. & Brown, R. A. J. (1998). Orchestrating different voices in student talk about infinity: Theoretical and empirical analyses. In C. Kanes, M. Goos, & E. Warren (Eds.), *Teaching mathematics in new times* (Vol. 2: pp. 468-475). Brisbane: Australia: Mathematics Education Research Group of Australia.

Rogoff, B., Matusov, E. & White, C., (1996). Models of teaching and learning; Participation in a community of learners. In D. Olson & N. Torrance (Eds.), *The handbook of education and human development; New models of learning, teaching and schooling.* Cambridge: Blackwell.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher, 27(2),* 4-13.

Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics, 46,* 13-57.

Säljö, R. (1995). Mental and physical artefacts in cognitive practices. In P. Reinmann & H. Spada (Eds.), *Learning in humans and machines: Towards an inter-disciplinary learning science.* Oxford: Elsevier Science.

Vygotsky, L. S. (1962). *Thought and language*. E. Hanfmann and G. Vakar (Eds.). Cambridge, MA: MIT Press.

Vygotsky, L. S. (1978). *Mind in society: The development of higher mental processes*. M. Cole, V. John-Steiner, & E. Souberman (Eds.). Cambridge, MA: Harvard University Press.

Wenger, E. (1998). *Communities of practice. Learning, meaning and identity.* Cambridge: Cambridge University Press.

Wertsch, J. (1991). *Voices of the mind: A sociocultural approach to mediated action.* Cambridge, MA: Harvard University Press.

Wertsch, J.V., del Rio, P., & Alvarez, A. (Eds.) (1995). *Sociocultural studies of mind.* Cambridge, UK: Cambridge University.

Mathematish –
a Tacit Knowledge of Mathematics

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Purpose and method

The purpose of this paper is to highlight the symbolic notations of mathematics and to present some hypotheses. We will stress the language aspect of the symbolic notation system, and therefore we call it "Mathematish". This definition is of course rather vague, but we try to specify our use of the term in Section 2.

There is already a lot of research done about mathematics representations in mathematics education, also using linguistic tools. A short overview is presented in Brown (2001). Also in Nordic mathematics education some research is done in the semiotic field (e.g. Bergsten, 1999; Engström, 2002) and (Winsløv, submitted). Despite this our hope is that our approach to consider mathematics symbolism as a fully developed *language* could create some fruitful hypotheses for this field.

Our method is a rather speculative reasoning, but with some empirical underpinnings from the history of mathematics, from our teacher experience, and from findings in mathematics education research. Some of our arguments are supported by references to the philosophical and linguistic fields of knowledge.

In Section 1 the historical evolution of Mathematish is described briefly. Section 2 is devoted to the various properties of Mathematish as a language, by reference to semiotics and linguistics and by comparisons to natural languages. Section 3 describes aspects of mathematics content and ways that content and Mathematish interact in learning situations.

Background and need for articulation

Mathematish is a very young language compared to other languages. Many researchers have contributed to Mathematish in the form of different shorthand notations replacing expressions of natural languages. Thus Mathematish initially inherited some structure from natural languages. Mathematicians and philosophers like Leibniz and Descartes have promoted the idea of a full-blown formal language, and Mathematish has henceforth grown by further linguistic innovations by researchers fuelled by its sheer efficiency. The success for Mathematish in research and technical applications is overwhelming. This has reshaped mathematics into a subject of formal calculation. Interpretations, typical of rhetoric mathematics are often omitted, which moves "content" to the background.

David Hilbert and the formalists attempted to formalize all mathematics – for them mathematics *is* formal calculation. One central ambition of the formalists was to axiomatize mathematics, i.e. to investigate the formal foundations in order to make calculations as reliable as possible (Davis & Hersh, 1980). A special feature of Mathematish is therefore that its grammar is designed to make it possible to deduce true statements without involving content during the deduction process (see Kline, 1980; Davis & Hersh, 1980).

In 1931 Kurt Gödel demonstrated a limit of axiomatization for mathematics in his incompleteness theorem. A conclusion is that relevant parts of mathematics cannot be formalized. Nevertheless, as symbolic mathematics evolved, the dominance of formalized mathematics increased. The role of relevance or meaning of mathematical concepts and goals of mathematics research in research papers decreased, as well as explicit texts about how to construct proofs in Mathematish. In textbooks proofs often were ready-made, and intuitive and strategic aspects not expressed. It may be considered as a main part of mathematical content – strategies, ideas and methods of how to use the rules. Note that such questions cannot be answered or described in Mathematish; the symbolic notations are not constructed to have themselves as references. Mathematical content of this kind occurs today mostly verbally among researchers and experienced teachers.

The success of Mathematish has inevitably reshaped school mathematics. Industry has posed a need for engineers who can read and handle the mathematical formalism, perhaps underestimating the notion that a successful use of mathematics requires also reflection about content. The needs of mathematics researchers have formed the dominant description of mathematics today.

In this paper we focus on the Mathematish issue to develop alternative descriptions. Initially we unfold the idea that Mathematish *is* a language of its own, and thereby we obtain the possibility to use analogies with natural languages and tools from semiotics and the philosophy of language.

Purposes of Mathematish articulation

By Mathematish articulation we mean descriptions where the immediate purpose is not to provide understanding of mathematical concepts, but on mathematical symbols and their use. This includes general and specific rules and habits in the grammar of Mathematish, and how ways of writing in Mathematish correspond to mathematical ideas.

Firstly, if we regard mathematics as a natural human capability which can be expressed in many individual ways, while Mathematish is the official language, it is very natural that children when starting school do not meet mathematics for the first time in their mathematics class – they merely meet the Mathematish formulation of mathematics for the first time. Furthermore, it is well known that many people in practical occupations with no higher education are able to solve mathematical problems that occur in their occupation. Perhaps these solutions may not be regarded as mathematics since Mathematish is not used (Löthman, 1992). Sometimes the incultivation in Mathematish seems to *decrease* adult students´ capabilities to solve problems (Alexandersson, 1985). Mathematish articulation may help people to recognize their mathematical capabilities. This requires mutual translations between academic mathematics culture and informal mathematical knowledge embedded in practice.

Secondly, the lack of articulation could be a reflection of Mathematish as being an unknown foreign language for many students. Of course there is a constant struggling to learn Mathematish in the mathematics classroom, but the shortcomings may depend on that the teachers themselves have not thoroughly recognized its grammar, especially not compared its grammar to grammars in natural languages. This may depend on that mathematics teachers not so often are trained in linguistic methods. Here we can also see a mother tongue teaching paradox: the more fluently the teacher speaks the language, the more invisible (and unnecessary!) the grammar structure tends to be for him.

Thirdly, in mathematics research there is an intricate interplay between Mathematish formulation and development of both proof ideas and new concepts. Mathematish articulation may clarify this interplay. The history of mathematics gives us many examples of how Mathematish can enforce an introduction of a new concept. The mathematicians will rather stay to the symbolic manipulation rules, the grammar, and gradually accept for instance negative and imaginary number than alter these rules (Kline, 1980). Of course it is sometimes also the other way around; a new concept enforces an introduction of a new symbol and how to handle it. A historical example is the introduction of symbols in differential calculus made by Newton and Leibniz. This dynamic interplay between grammar and thought gives interesting perspectives on the history of mathematics and also on learning situations.

Fourthly, articulation of Mathematish may play a role when discussing the nature of mathematics in general. This is important in its own because of the central position of mathematics in science and society. Other properties of mathematics may become visible by means of the Mathematish-content point of view. One important aspect is the interplay between language and culture, analyzed by structuralists and poststructuralists (de Saussure, 1916; Derrida, 1976).

Properties of Mathematish

Mathematical texts are bilingual

The perspective and focus of this paper is mathematics as a subject having two sides that relate in complicated ways: its general and abstract concepts, and its special symbolic language. This is reflected in the fact that mathematical texts are *bilingual*. By this we mean that some parts are written in natural language, extended with a mathematical terminology, and some parts are written in the mathematical symbolic language, typical for arithmetic, algebra, and analysis. As a simple example we choose the following text line:

A linear equation is one that can be written ax + by + c = 0.

The first part is following ordinary language grammar and the very signs used are symbols for *phonemes*, representing the spoken language. The whole structure of the signs used is therefore phoneme based. Words, concept representing clusters of signs in ordinary language, are organized along ordinary language grammar, and the word structure is following ordinary spoken language.

The second part of the line is using other signs, not representing phonemes but mathematical concepts. The “grammar”, the rules for ordering these signs, is very different from the grammar of an ordinary language. The structure of the symbols used is following rules of mathematics, a “grammar”, especially constructed for this purpose.

Knowledge of Mathematish is knowledge of its grammar: to recognize correct formulas and correct rules to change formulas. Mathematish knowledge is knowledge in “pure formalistic manipulation”. We consider neither purposes, goals nor meanings of the manipulations as parts of Mathematish knowledge – this is content.

The idea that mathematics texts are bilingual is not new, for instance is this idea very important in Wittgenstein’s philosophy of mathematics, here cited in Waismann (1979):

> ... what is caused to disappear by (a critique of foundations) are names and allusions that occur in the calculus, hence what I wish to call *prose*. It is very important to distinguish as strictly as possible between the calculus and this kind of prose. (p. 149)

With the term “calculus” Wittgenstein included both arithmetic and algebra. We will not follow Wittgenstein all the way to his rather extreme position that “calculus” (*Kalkul) is* the real mathematics, and that “prose” (*Prosa)* is merely confusing and blurring (Marion,1998), but we find his distinction fruitful. Narrative natural language (rhetoric) as occurring in a mathematics text, extended with mathematical terminology, we will call *mathematical prose*.

Of course mathematics also has other types of representations, for instance pictures, graphs and schemas, but in this paper we focus on the two languages mentioned above, and especially on Mathematish. One reason for this focus is that both *mathematical prose* and Mathematish are established vehicles crucial for problem solving and proof activities in both school mathematics and mathematics research, and both have a language character.

The main purpose and aim of this paper is to discuss the following question: Is it fruitful for mathematics education and mathematics research to study Mathematish with similar linguistic tools that are used to study natural languages; could analogies with natural languages create interesting hypotheses?

Mathematics terminology versus Mathematish

Physics, literature, mathematics - most sciences have a terminology of its own. Specialized texts in these subjects may be unreadable for laymen. The specialized terminology is an extension of the vocabulary and is used within the grammar of the natural language. Such a specialized text may be unreadable by laymen also because of unknown figures of thoughts or unknown references. However, if the grammar is different, a person needs to learn not only new words and their meanings, but also new rules of the language. There are certainly many other specialized languages than Mathematish, such as musical notation, molecular notation in chemistry and the Labanotation in dance.

There is a mathematics terminology with is not a part of Mathematish: words such as “addition”, “real number”, “continuous”, “differential equation”, etc. But texts with no formulas, i.e. with no Mathematish, are normally not considered as mathematics texts. Conversely, mathematics texts do not consist of Mathematish only, and no natural language. They are bilingual, and switches from one language to the other are frequent and often unannounced. Some statements can be made in any of the two: Mathematish or English. Some mathematics authors use this possibility to explain Mathematish. However, in a mathematics text the two languages are mostly used for different purposes. While Mathematish is used to specify and manipulate quantitative relations, English is mainly used to describe the logic in the argument, as well as purposes, connections to other results, analogies, images, examples and applications.

Comparisons of Mathematish* and *English

The sentence “1 + 1 = 2” is a true statement, “1 + 1 = 3” is a false statement, and “1 + 1 = +%” is no statement at all, it is meaningless. The first two follows the grammar of Mathematish, and are either true or false. The third does not follow the grammar. Then it is not a statement and can not be assigned a truth value. It is only a sequence of signs. Note that this grammar is tacit: it is not easy to say which rule is violated. A rule need to be constructed, such as: “on both sides of an equal sign there has to be symbols for numbers or variables”.

This is similar to the sentences “A frog has four legs”, “A frog has seven legs” and “A frog legs”. The first two follow the grammar of English, and we can (in principle) decide if it is true or false. The third “sentence” is meaningless.

It is important to observe that the truth of “A frog has four legs” or “A frog has seven legs” cannot be decided within the language itself. In this case one must import knowledge of biology to decide the truth/falsity. A natural language does not contain truths which they describe, with the exception of analytic truth (rhetoric logic). Note that this grammar is not tacit. “A frog legs” is no sentence since it has no verb. Someone who has English as mother tongue can probably formulate such a rule, even no explicit rule is needed to say that “A frog legs” is no sentence.

The grammar of a natural language does not follow the very structure of the empirical world, and indeed our views of that structure are changing over time. Therefore you cannot deduce new truths (except analytical) about the empirical world with natural language. Mathematish, on the other hand, has a specially constructed grammar following the structure of mathematics, which is mostly numerical and logical. If you start with true premises it is possible to deduce true mathematical sentences within Mathematish without “checking” with mathematics on an outside concept or idea level. This is for instance crucial when you are trying to prove a conjecture. In an ordinary proof, conceptions, intuition and metaphors are (afterwards) “cleaned out” and replaced by Mathematish. An important attendant question is therefore to what degree Mathematish in fact is constituting the mathematical world of concepts and theories. Could it also be fruitful to analyze the claim for using Mathematish when proving as an act of power from the established mathematical discourse, in the meaning of Foucault? See (Foucault,1961).

You could talk about “good Mathematish” in the same way as “good English”. Both good and bad Mathematish are following the grammatical rules, but good Mathematish presupposes a “cultural” knowledge and a feeling for the context. An example is that it is “better” to write $ax + by + c = 0$ instead of $xa + yb + z = 0$ for representing a line. Another example is to know that the parentheses in $f(x + h)$ and in $a(b + c)$ probably have different roles, even if you do not know the actual contexts embedding the expressions. How much do teachers take it for granted that students master not only Mathematish but also “good Mathematish” in the classroom?

You can even identify “dialects” in Mathematish; small differences in how to use symbols and following rules. This is very apparent when comparing textbooks from different countries. Is there a learning problematic with dialectal Mathematish in translated books, or for students not sharing the teachers dialect?

Mathematish – a typical language?

The theoretical background underpinning our question on the role of Mathematish in mathematics, is that the strictly regulated system of arithmetic and formula handling that has emerged in mathematics in many ways *has* the features of a language: it is using a special set of *signs*, the use of the signs is regulated by a *grammar (syntax)*, and it is possible to *produce, interpret* and *translate* propositions designed with these signs and grammar. *Mathematish* also has one of the most powerful “design features” typical for a language; the *double articulation* (or *duality of* patterning), see (Hjelmslev, 1961). This double articulation enables a semiotic code to form an infinite number of meaningful combinations using a small number of low-level units, which in themselves are meaningless. For instance is the use of x, y and z as signs for variables a mere convention started by Descartes when he chose the letters at the end of the alphabet for variable signification.

All these language features open for using methods and perspectives from well elaborated discourses in linguistics, semiotics and the philosophy of language. Our conjecture is therefore that it is fruitful to identify *Mathematish* as mentioned above, not only as representations but as a language of its own.

Tacitness of Mathematish

There is a risk that teachers, well incultivated in Mathematish, will focus merely on content presupposing that the students already master the language. As a result, the structure and the rules of Mathematish will remain largely tacit.

We use the concept "tacit" with the same meaning as in Polanyi (1967), that the knowledge is *not formulated but perhaps possible to formulate*. Some tacit knowledge is possible and also relevant to formulate by language, but other parts are better to *show in practice*. We can also imagine that there could be a kind of knowledge that is neither possible to formulate nor to show, but it is not clear if this should be called knowledge. For an elaborated analysis of tacit knowledge, see (Molander, 1996). Molander identifies a third kind of tacit knowledge, a knowledge that is suppressed to silence. A rather common experience in adults education is that adults' informal knowledge is suppressed by for instance a teacher's claim for Mathematish representation (Nunes et al, 1993).

Another relevant distinction named already in Ryle (1949) is *knowledge-how* and *knowledge-that*, both could be tacit but the latter more easy to formulate: even if you know the rules of a game (knowledge-that) it is not sure that you are an expert in playing the game (knowledge-how), and it could be hard to express this expert knowledge in words. A strong remark is made by Wittgenstein (1983), that there cannot be a rule that also includes how to use the rule.

Mathematish and mother tongues

A good knowledge of formula manipulations can be compared to knowledge of a mother tongue; it is used without any explicit translating processes. It is well known that the structure of a mother tongue is naturally tacit for the user. It is "tacit" in the meaning that it is not expressed or reflected upon, and perhaps some parts are not even expressible. If mathematics teachers use Mathematish similar to a mother tongue, they may mistakenly see the translation problem merely as a concept understanding problem.

Despite the tacitness of Mathematish, the main part of mathematics teaching is by tradition calculation with formulas. The learners are heavily confronted with a "foreign" language in the mathematics classroom. Many learners perceive mathematics as a large set of fragments with an almost non-existing larger picture about adults mathematics memories of their school time (Lindenskov, 2001). This is a natural consequence when a general description of the language Mathematish is absent. Such a general description, showing similarities between isolated calculations, constitute a grammar. Furthermore, learners often feel unfamiliar with the very symbols they use when calculating – the alphabet of Mathematish. Rather than courses in Mathematish grammar, teacher-learner dialogues could be a good tool for formulating the relevant aspects of Mathematish. In a dialogue you can "play" the language game and detect the rules in social interaction (Wittgenstein, 1967).

Learning of foreign languages

The grammars of foreign languages that are learned later in life than a mother tongue are usually not tacit. Then the learning is done with the grammar of the language, which therefore is conscious. It is known that a language learned later in life is represented differently in the human brain than a mother tongue. Furthermore, Mathematish seems to be represented in the brain differently than natural languages (Butterworth, 1999). In an example, one person, after a brain damage, could not read "54" but could read "cinque quattro", which is Italian for "five four". Sometimes it is the other way around; one patient with brain damage could not read the phoneme based words signifying a specific number, but could read (and understand) the digits signifying the same number (*ibid*.).

The development of Mathematish started to a large extent as a short hand for mathematics expressed by natural language. An example is the Italian mathematicians who started in the fifteenth century to replace standard words such as *cosa* (the unknown thing), *censo* (square), and *radice* (root) with the abbreviations *c*, *ce* and *R.* Luca Pacioli replaced *pio* (plus) and *meno* (minus) with *p* and *m* with small horizontal lines above them (Katz, 1998). Another typical example of this change is the following cite from Robert Recorde in his introduction of the equality sign (Kaplan, 1960):

> And to avoide the tediouse repetition of these woordes 'is equalle to' I will sette as I doe often in woorke use, a paire of paralleles, or Gemowe [twin] lines if one lengthe, thus = because noe .2. thynges, can be moare equalle.

Unlike natural languages, Mathematish has been *written* from the start. Being born as a shorthand for natural languages, it naturally inherits some grammatical elements from natural languages, for instance logical variables. However, due to the specific use of Mathematish, which is quantitative calculations, it has a development which differs strongly from the development of natural languages.

Mathematish is usually encountered in elementary school, however most humans develop mathematical intuition earlier in life (Clements & Sarama, 2004; Heiberg Solem & Lie Reikerås, 2004). If mathematics intuition and Mathematish connect or stay separate for students is a central question for didactics of mathematics.

It appears as if Mathematish becomes intuitive and effective as a mother tongue for a rather small minority in the population. A basic educational problem for mathematics is that mathematics teachers often come from this group, while many of the students do not. Then many learners may have problems in mathematics of a kind that represent tacit knowledge for many teachers. We regard this as a problem that must be recognized fully in the entire mathematical community. This is particularly serious in the mathematics teacher education. An example of the present weak Mathematish awareness is that there is no general agreement about a very basic language question: what is a word in Mathematish?

Mathematish and computer programming

Mathematish has symbols that are concept based, as is the case of Chinese, and not phoneme based, as in the case of English. As a result, symbols and "words" may be pronounced differently in different parts of the world, however written essentially the same way. Hence there is no need for translating the symbols, a fact that facilitates communication and mathematics development. However, a demand for translation would force clarification of the structure of Mathematish and diminish its tacitness, as has been the case for natural languages.

Computers have been constructed with mathematics and logic as its basic structure. Computer programming languages have been developed which provide alternative ways of expressing mathematical ideas, algorithms and facts. The grammar is often similar to that of Mathematish. Some computer programming languages can partially be considered as dialects of Mathematish. This allows computers to effectuate formal mathematical calculation with no regard to meaning. It appears as if mathematics content cannot be expressed by computers, in the sense that the formal calculations appear to be very inefficient once there are no clear rules for how to calculate. This can be considered as a late endorsement by computer technology of Wittgenstein's claim that there cannot be rules for using rules in the same language.

The term "vernacular" is used for a native spoken tongue as opposed to constructed or official ones. The term "mathematical vernacular" was introduced by de Bruin in 1987 in a computer science context (de Bruin, 1987). The term has been established for a formal language for writing mathematical proofs that resembles the natural language from mathematical texts. There exist several such systems today, such as Hyperproof and Mizar. These are attempts to construct new languages or representation systems for increased consistency or efficiency, while Mathematish represents the present factual use of mathematical notations and symbols.

Mathematish-content interplay in mathematics

Two kinds of mathematical knowledge

In the previous example equation, $ax + by + c = 0$, five “unknowns” (letters) are present. Most mathematics teachers probably think of x and y as parameters, and a, b and c as constants, and a mental image of a straight line given by the values of the constants a, b and c may appear. This is not at all given by the equation itself. The geometric interpretation is an example of “mathematics content”. You could also for instance interpret the equation merely as a relation between numbers. Such a concept of content is strongly culturally dependent and often personal. It is not easily formalized or defined, since it by definition is not formulas. As regards the meaning of mathematical content as knowledge that cannot be written in Mathematish, we may talk about content of two different kinds:

1. Mathematical meanings as the target of the symbols and expressions of Mathematish
2. Mathematical knowledge that cannot be expressed in Mathematish.

Examples of the first kind are applications of mathematics and geometrical figures that may be represented by formulas, where some may be rather personal. Further examples are concepts such as “oneness”, “twoness”, and so on, as properties of certain sets, represented by the symbols “1” and “2”, and so on.

Examples of the second kind are strategies for problem solving, ideas of proofs and calculations, and evaluation of models and results (Ernest, 1999).

Very often mathematical equations are starting points of mathematical thinking, and mental “anchors” for various considerations of mathematically active persons. Many of these considerations are essential for successful mathematical work, however non-formalisable and partially personal. We consider also this as part of mathematics content of the second kind.

Semiotic approaches to Mathematish

As mentioned above our perspective opens for the use of methods from discourses like linguistics and semiotics, and we will use some terms and ideas from for instance de Saussure and Peirce, and their followers often named post-structuralists and neo-pragmatists.

From de Saussure we borrow the idea that a “sign” has two parts; the *signifier* and the *signified*. Saussure himself stressed that both signifier and signified were on a *mental* level, but in accordance with many post-Saussurians we stress the signifier as a *material* entity, for instance the ink doodles constituting a text in a book. The signified, though, we claim is a *concept* or a kind of mental picture. Although the signifier is treated by its users as “standing for” the signified, Saussure emphasizes that there is no necessary, intrinsic, direct or inevitable relationship between the signifier and the signified. The link between them is quite *arbitrary:* “the signs used in writing are arbitrary, the letter t, for instance, has no connection with the sound it denotes" (Saussure 1916/1983). The links, when culturally established, become parts of a structure, and the meaning of the signs is regulated by this structure and systematic relations between the signs. No sign makes sense on its own; the meaning of "tree" is related to other signs, for instance "bush". Saussure uses an analogy with chess, noting that the value of each piece depends on its position on the chessboard. While *signification* (what is signified) clearly depends on the relationship between the two parts of the sign, the *value* of a sign is determined by the relationships between the sign and other signs within the system as a whole (Saussure, 1983). The signifiers reflect *differences* that are important for the language users; the meaning of a sign is about *what it is not*, rather then what it is.

From Peirce we use the idea that the sense-making of a sign requires an act of *interpretation* and therefore an interpreter. The interpreter produces his own "sign" of the external sign in his mind, and this sign must also be interpreted. This model is sometimes called "the semiotic triangle" with the three parts sign vehicle, sense and referent.

The process of interpretation, the *semiosis,* could be ongoing in several steps, in principle ad infinitum. A very familiar situation where the signified also could play the role of signifier is when you are using a dictionary; sometimes also some terms in the defining text must be defined. The semiosis could take a dialogic form in one person's mind or between persons. While Saussure emphasizes structure in a synchronic way, Peirce emphasizes diachronical aspects. Peirce argued that "all thinking is dialogic in form. Your self of one instant appeals to your deeper self for his assent" (Peirce 1931-58). The same idea of *dialogical understanding* is elaborated more deeply in Bakhtin (1981).

Peirce also made a typology of signifiers, depending on the grade of their arbitrariness. *Symbols* are quite conventional and have to be learned, *icons* are in some way resembling the signified, and *indices* are directly connected, like photographs, measuring instruments, and indexical words (that, this, here, there).

The Saussurian concepts stress Mathematish as a ready-made cultural phenomenon with a given structure, while the Peircian concepts stress Mathematish learning and understanding as a subjective interpreting activity, both aspects of importance for our analysis. We will also use Wittgensteins concept of *language game (Sprachspiel)* to highlight the social aspect of Mathematish, and that "understanding" is to do the right thing in this "game", see (Wittgenstein, 1967).

There is also an ontological question about Mathematish that has bearing for the philosophy of mathematics. Umberto Eco says "a symbol is a lie" (Eco, 1976), i.e. it stands for something else, but what? This is one way to answer:

> "For example...the expression $x^2 + y^2 = 1$ can be seen as mixture of numbers and letters with no particular significance, as an algebraic equation, as a representation of a circle, or *as* a circle" (Brown, 2001, p. 193).

What is Mathematish *about*? Is it about objective existing concepts now labelled, is it about constructed objects now labelled, is it the “real” mathematics, is it a template for economizing thought, or is it perhaps just a sometimes useful game?

As we have seen in the history of mathematics, and also in our teaching practice, sometimes the notation creates the concept, and sometimes the other way around. This is also a theme in mathematic education research. For instance, Sfard describes how mathematical discourse and mathematical objects are creating each other in the learning process (Sfard, 2000), and how template-driven activities create concepts.

Content - beyond the concept

There are many forms of mathematical content. The content closest to Mathematish is the set of truths, i.e. the true statements that mathematicians consider to be true in the sense of being consequences of the axioms. This kind of content can appear almost indistinguishable from its Mathematish formulation (see amalgamation below), partially since Mathematish calculation is the dominant way that is used for checking its validity. This content is defined by Mathematish calculation.

Another part of mathematical content is images and associations connected to abstract entities. It is quite possible to give a strict definition of the number 2. But the digit “2” will also have personal connotations for a student. Part of this meaning is related to experiences of this particular quantity (“2”), perhaps from a multitude of examples (two apples, two ideas, two hands,…), and from a more intrinsic mathematical direction: from knowledge of even numbers and factorization of integers. This may be parts of a content “behind” the strict concept of number 2. A mathematical sign is therefore in practice signifying not only a strict mathematical concept but also (or instead!) a big amount of personal conceptions and memories, typical for the person reading or using the sign. This is usually referred to as *concept image* (Tall & Vinner, 1981).

Concept construction

The notion of Mathematish is useful when analysing the process of students’ concept construction in cultural and cognitive aspects. As an example we chose the introduction of different kinds of numbers in school mathematics. We will analyse three aspects of this; *existence forcing, amalgamation* and *translation*. The analysis reflects the theory that a discourse constructs its objects and “reality” by introducing signifiers, and relates to Derrida (1976) and the elaborated ideas in Sfard (2000).

In everyday language a name like “table” could be introduced ostensibly (“look, this is a table”); you *point* at the signified object. Many aspects of language could be *shown* in practice and in interaction between language and action. In mathematics this is not possible, since visible objects are at most approximative examples of objects. Even in geometry the visible object is just a representation of the mathematical object: it is for instance hard to draw a line with no thickness. To discuss the nature of these objects is beyond the scope of this paper, but it is indeed an interesting ontological question. The pointing procedure must be substituted by something else. Also the *handling* of objects is invisible, and you cannot ostensibly *show* how to handle mathematical objects. You must use *material* signifiers for these purposes, for instance hands-on materials or the signifiers in Mathematish. Often operations with Mathematish are used to present and motivate new kinds of "names", for instance signifiers for numbers. In the following examples different kinds of numbers are presented by referring to Mathematish operations:

> "We have that 8 – 5 = 3 but what about 5 – 8?
>
> "We have that $\frac{15}{3} = 5$ but what about $\frac{3}{15}$?
>
> "We have that $x^2 = 4$ has the roots $x = 2$ or $x = -2$, but what about $x^2 = 3$?
>
> "We have that $x^2 - 4 = 0$ has the roots $x = 2$ or $x = -2$, but what about $x^2 + 4 = 0$?

In all these cases the operators used grammatically correct will provoke new kinds of results, and these results will in turn become signifiers for new objects. The result is transformed to a "name". Often you could still trace the operator in the signifier, for instance 3/7, a fact that is sometimes confusing for the learner: how could 3/7 and 9/21 be "the same number"? When using Mathematish grammatically correct new types of objects are *forced into existence*, often without a pre-existing learner intuition (if you are a Platonist this is of course not what is happening; instead Mathematish helps to "remember" the object). The construction of new concepts is not a process started in the learner's mind, on the contrary the language structure initiates and constructs the concepts. The concepts are not firstly existing, and than "baptized", on the contrary the names exist *before* the concepts (Wittgenstein, 1983). The same situation can be seen in the history of mathematics. For example, firstly the mathematicians constructed complex numbers by Mathematish rules, and later claimed that these should be seen as signifiers for a new kind of number (Katz, 1998).

A common tool in textbooks is to use the "number line" and put signifiers in a row along a line. Often textbooks say that "the negative numbers" are on this line, but in fact they are not. You could only see the ink doodles. Also in this case the very presentation of "names" will force objects into existence, according to Sfard. She points out that the introduction of new names and new signifiers is the beginning rather than the end of the story (Sfard, 2003). She demonstrates how the new signifiers for negative numbers appear from the very beginning: at the same time as the description of the new concepts.

A common problem in this existence forcing process is that the concepts desired never will start to exist in the learner's mind. Instead the signifier and the signified *amalgamate*; the signifier *is* the mathematical object, the object *is* the ink doodle. Mathematish may then for the learner appear as a "meaningless" procedural game with no relevance for intuitive thinking or outside school. *Translations* to mathematical prose and other types of representations are here very important to "help the object into existence". Pictures, analogies, metaphors and schemas can in a dialectical way interact with Mathematish results in order to strengthen the learner's intuition and creativity. Following Peirce, an interpretation of a signifier is an ongoing process, as a dialogue, and this dialogue is necessary also between Mathematish and mathematical prose.

A problem with many textbooks is that they are in fact encouraging amalgamation: "the *line* $y = 3x + 4$", "the *function* $y = x^2 + 3x$" and so on. These expressions in Mathematish are not presented as special representations of mathematical objects, but as the very objects themselves.

Interesting questions are for instance what the difference is between a mathematical fantasy, or "lie", and a mathematical concept forced into existence, and why students believe (or should believe) in these concepts.

Mathematics produces Mathematish rules – and vice versa

Mathematish consists of pure conventions, and of rules of calculations. Examples of pure conventions are the choice of symbols, such as "=" for equality instead of "#" or "EQ", or choices of notation such as writing a^n for n factors of *a*, and not $^n a$, $a\_^n$ or *pow*(*a*,*n*) (e.g. Bergsten, 1990; Pimm, 1987).

Logic and other truths are often formalized into rules of calculation: a *calculus*. These rules may then be used without any regard to their meaning. Examples are $x(y+z) = xy + xz$ (i.e. replacing $x(y+z)$ by $xy + xz$ is OK), $0 = 3 - 3$, $\sin(\arcsin x) = x$. Which rule is meaningful at a particular instance depends entirely on the goal and purpose of the calculations; hence on the mathematics content. Such rules of calculation take the form of grammatical rules. A counterpart in English is the statement "The horse pulls the car" that can be replaced by "The car is pulled by the horse. Hence, developments in mathematics give new Mathematish rules to use.

But sometimes it is the other way around; a calculation with mere Mathematish creates an unexpected result that afterwards has to be interpreted. This holds both for school mathematics and research.

Mathematical intuition – a human trait

A part of becoming human is learning to handle quantity, size, space and order – practical forms of mathematics that often are not formulated in Mathematish. The process of learning to walk is strongly driven by instinct, but also involves and develops the mind. Simultaneously, consciousness of the body and of three-dimensional geometry develops. One may say that every human develops mathematical intuition from the first years in life, which is formulated more or less verbally. When beginning school, this intuition meets the official language of mathematics: Mathematish.

During the first years in school, mathematics only concerns the symbols +, =, –, ·, / and the ten numerals. These symbols are certainly abstract. The abstraction lies in the generality: the same symbols are used for counting or measuring *anything*. This generality can be seen as the most basic property of the nature of mathematics: a separate formulation for calculation that is independent of application areas, and effective for all of them. It also represents the main leap of thought that challenges pupils.

Philosophy and practice

The identification of a special mathematics language may seem to be a rather philosophical endeavour, but in this paper we have tried to show that philosophy and classroom practice go hand in hand. The basis of observations about students' relations to Mathematish is our teaching practice, the teaching practice of our colleagues, and findings in mathematics education. We have described mathematics as a personal mathematics intuitive content that is both expressed and shaped by elaborated mathematics notations, called "Mathematish". We hope that our perspective, that Mathematish is a complete *language*, could create fruitful analogies with other languages.

References

Alexandersson, C. (1985). Stabilitet och förändring. En empirisk studie av förhållandet mellan skolkunskap och vardagsvetande. *Göteborg Studies in Educational Sciences, 53*. Göteborg: Acta Universitatis Gothoburgensis.

Bakhtin, M. (1981). *The dialogic imagination: Four essays* (trans. C. Emerson and M. Holmquist). Austin, TX: University of Texas Press.

Bergsten, C. (1990). Matematisk operativitet: en analys av relationen mellan form och innehåll i skolmatematik. (Mathematical operativity: An analysis of the form-content relationship in school mathematics) *Linköping Studies in Education. Dissertations No. 29*. Linköping University.

Bergsten, C. (1999). From sense to symbol sense. In I. Schwank (Ed.), *European research in mathematics education I.II* (pp. 126-137). Forschungsinstitut für Mathematikdidaktik, Osnabrück.

Brown, T. (2001). *Mathematics education and language. Interpreting hermeneutics and post-structuralism.* Revised 2nd Edition. Dordrecht: Kluwer Academic Publishers.

de Bruijn, N.G. (1987). The mathematical vernacular, a language for mathematics with typed sets. In P. Dybjer et al., (Eds.), *Proceedings of the Workshop on programming languages*. Marstrand, Sweden.

Butterworth, B. (1999). *The mathematical brain.* London: Macmillan.

Clements, D. H. & Sarama, J. (2004). *Engaging young children in mathematics.* Mawah, NJ: Lawrence Erlbaum Associates.

Davies, P.J. & Hersh, R., (1980). *The mathematical experience.* Boston: Birkhäuser.

Derrida, J. (1976) *Of grammatology* (Spivak Trans.) London: John Hopkins Press.

Eco, U. (1976). *A theory of semiotics.* Bloomington: Indiana University Press.

Engström, A. (2002). Semiotik och matematiddiaktik. En introduktion. Arbetsrapporter vid Pedagogiska institutionen, Örebro universitet.

Ernest, P. (1999). Forms of knowledge in mathematics and mathematics Education: Philosophical and rhetorical perspectives. *Educational Studies in Mathematics, 38*, 67-83.

Foucault, M. (1961). *L´histoire de la folie à l´age classique. Folie et deraison.* Paris.

Heiberg Solem, I. & Lie Reikerås, E. K. (2004). *Det matematiska barnet.* Stockholm: Natur och Kultur.

Hjelmslev, L. (1961). *Prolegomena to a theory of language.* (trans. Francis J. Whitfield) Madison: University of Wisconsin Press.

Kaplan, E., (1960). *Robert Recorde (c. 1510-1558): studies in the life and works of a Tudor scientist.* PhD thesis, New York University.

Katz, V. J. (1998). *A history of mathematics. An introduction.* Addison-Wesley.

Kline, M. (1980). *Mathematics. The loss of certainty.* New York: Oxford University Press.

Lindenskov, L. (2001). Vanskeligheder i relationen mellem matematik og virkelighed – plevt som meningslöshed. I M. Niss (ed.) *Matematikken og verden* (s. 195-217). Köpenhamn: Fremad.

Löthman, A. (1992) *Om matematikundervisning – innehåll, innebörd och tillämpning.* (Uppsala Studies in Education, 40). Uppsala: Acta Universitatis Upsaliensis.

Marion, M. (1998). *Wittgenstein, finitism, and the foundations of mathematics.* Oxford: Clarendon Press.

Molander, B. (1996). *Kunskap i handling.* Göteborg: Daidalos.

Nunes, T., Schliemann, A.D. & Carraher, D.W. (1993). *Street mathematics and school mathematics.* New York: Cambridge University Press.

Peirce, C.S. (1931-58). *Collected writings.* (8 vols.) (eds. Charles Hartshorne, Paul Weiss and Arthur W. Burks). Cambridge, MA: Harvard University Press.

Polanyi, M. (1967). *The tacit dimension*, New York: Anchor Books.

Radford, L. (2000). Signs and meanings in students´ emergent algebraic thinking: a semiotic analysis. *Educational Studies in Mathematics, 42*, 237-268.

Pimm, D. (1987). *Speaking mathematically: Communications in the mathematics classroom.* London: Routledge and Keagan Paul.

Ryle, G. (1949). *The concept of mind*. Chicago: University of Chicago Press.

Saussure, F. de (1916/1983). *Course in general linguistics* (trans. Roy Harris). London: Dockworth.

Sfard, A. (2000). Symbolizing mathematical reality into being: How mathematical discourse and mathematical objects create each other. In P. Cobb, K.E., Yackel & K. McClain (Eds.), *Symbolizing and communicating in mathematics classroom: Perspectives on mathematical discourse, tools, and instructional design.* Mahwah: Erlbaum.

Sfard, A. (2003). There is more to discourse than meets the ears: Looking at thinking as communication to learn about mathematical learning. In C. Bergsten & B. Grevholm (Eds.), *Challenges in mathematics education. Proceedings of MADIF3* (pp. 34-49). Linköping: SMDF.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics*, *12*,151-169.

Waismann, F. (1979). *Ludwig Wittgenstein and the Vienna circle.* From the notes of F.Waismann, Oxford: Blackwell.

Winsløv, C. (submitted). Semiotics as an analytic tool for the didactics of mathematics. Paper submitted to Nordic Studies in Mathematics Education.

Wittgenstein, L. (1967). *Philosophische Untersuchungen*. Frankfurt am Main: Suhrkamp Verlag.

Wittgenstein, L. (1983). *Remarks on the foundations of mathematics*. Blackwell, Oxford.

The Linguistic Side of Mathematics

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Introduction

Any discussion about different aspects of mathematics ought to start with a discussion or maybe a declaration about mathematics itself. Despite the fact that the mathematics education community often declares that mathematics is much more than just figures and calculations, most courses in teacher programmes for prospective teachers of mathematics are mainly traditional in the sense that they address calculation techniques more than other aspects of mathematics. If we on the other hand search through sources concerned with what mathematics strictly speaking is, we find very little about calculation. Courant & Robbins (1941) write in the foreword to *What is mathematics* the following:

> Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection. Its basic elements are logic and intuition, analysis and construction, generality and individuality. Though different traditions may emphasize different aspects, it is only the interplay of these antithetic forces and the struggle for their synthesis that constitute the life, usefulness, and supreme value of mathematical science.
> (Courant & Robbins, 1941, foreword)

In this statement it is difficult to see the foundation for all the computation we normally connect to the subject mathematics. Traditional philosophies, such as Platonism or intuitionism, assume that mathematics expresses eternal relationships between objects that are intuitive as well as objective. Diverging from these perspectives, philosophers, such as Lakatos (1976) and Wittgenstein (1956), considered mathematics to be also a product of social processes.

Lakatos used classroom discussions to illustrate how mathematical concepts are changed or stabilized over time through processes of agreement and refutation. From this philosophical point of view, the truth of mathematical statements is not absolute. The statements are mainly justified conjectures, which very well can fail in the future when new problems are created and new solutions are produced. Wittgenstein (1956, p. 99) described this inevitableness by his concept of a language game:

> The indelible nature of numbers and figures and the certainty of proving procedures are not expressions of the ideal existence of mathematical objects nor of the absolute validity of procedures. Because we needed such a rigid language game for various purposes, we invented it along with its grammar and its dovetailing with practice. The mathematician is an inventor, not a discoverer.

Reuben Hersh (1994) discusses mathematics as a social-cultural-historical construct, one that has been and is continuously constructed in social, cultural, and historical contexts, and consists of “truths” in those contexts. Hersh defines mathematics this way:

> The study of lawful predictable parts of the physical world has a name; the name is “physics”. The study of lawful predictable parts of the social world has a name; the name is “mathematics”. (Hersh, 1994, p. 19)

The debate is old, is mathematics invented and human-made, or is it discovered and pre-given? It seems to be both important and helpful to make distinctions between different components of mathematics. Rényi (1967) offered a distinction – the mathematician invents concepts and discovers theorems. He also compared the mathematical researcher with a seafarer who discovers unknown islands:

> If a seafarer intends to sail into a region into which nobody before has sailed he has to be an inventor. The seafarer has to construct a ship that is more storm proof than the ship of predecessors. I would like to say that the new concepts which a mathematician puts forward are like ships of a new type. These ships take the seafarer who is out for discoveries faster and faster over the stormy sea into new regions. (Rényi, 1991, p. 28)

Regardless if it is hard to find a comprehensive statement of what mathematics really is, the fact that mathematics is much more than just figures and calculations is a certainty that hopefully is well adopted by mathematicians, mathematics educators, and teachers of mathematics. Even a cursory glance at school mathematics will show that mathematics is not just about numbers. It is a system made up of letters as well as numerals, which relate to each other by operations, processes, laws and theorems. Regarded like this, mathematics is an array of symbols and laws. It is a highly symbolic art and for many students that is also where the problem of understanding mathematics begins.

A growing consensus regarding the need of a deep understanding of the language used in mathematical classrooms seems to be well founded in the mathematics education community. Many researchers have investigated the fact that students at different levels of the educational system have difficulties to understand the language of mathematics. For instance Cocking and Chipman (1988) examined the mathematical ability of language minority - particularly bilingual - students, attempting to identify linguistic and cultural variables that might explain why their mathematical ability falls increasingly behind that of students who speak English as their primary language ("majority students"). They also investigated the competencies the mathematics teachers had and stressed the importance of the educational quality.

> Math achievement is heavily dependent upon school instruction..., and it is not likely that math achievement would be related strongly to family background variables tied to socioeconomic status. Occupational expectations and information associated with socioeconomic status may affect the value assigned to mathematics study and achievement. (p. 32)

In an article in the Swedish anthology *Mathematics – a communication subject*, there is a constructed dialogue between two researchers, one mathematics educator and one language educator, where the language educator claims that our language is our *mental thumb*. In the same way as our thumb allows us to get a grip around a tool of some sort, our language can allow us to get a grip around abstractions, like mathematics. He suggests further that theoretical or abstract knowledge of any kind hardly can exist beyond or without the individual's language (Emanuelsson et al., 1996, p. 59).

As soon as we start thinking about linguistic ability, it is inevitable to touch on the notion of communication competence. The basic idea behind a language is to be able to communicate and the general notion of communication competence entails knowing how to use language to communicate in various social situations – to use language appropriate to context. Normally we judge language without meaning as useless and we automatically strive to find meaning in for instance infants' prattle.

During 2003, a debate between two Swedish mathematicians regarding the language of mathematics occurred in articles and in two public meetings. Håkan Lennerstad and Ulf Persson, both well-known mathematicians in Sweden, debated about the possibility to see a comprehensible difference between the content of mathematics and the language of mathematics. And if so, what meaning would this view have for how we should teach mathematics? The two different perspectives, on one hand the importance of letting students at different levels translate the symbolic language of mathematics to Swedish (or any other native language) and on the other hand the fact that the language of mathematics in an imprecise and vague term, came out from the oral and written debates. My interpretation of the debate is, however, that the two debaters as well as the audience considered the language of mathematics as an important concept, something proven by the theme of this conference (Lennerstad & Persson, 2003).

Teacher education in general does not explicitly stress the importance of developing a mathematical language competence as a student teacher. There are in general no courses labeled "Mathematics and language" or "The correct way to address and teach mathematical objects" or something equivalent in their teacher programs. The courses are mainly well-established courses in different branches of mathematics where the matter of mathematics as language falls far behind computational and conceptual skills.

It is a fact that students' difficulties with algebra and algebraic expressions are common, not only when students are first introduced to algebra in compulsory school, but also when they are presumed to have mastered the subject and moved on to study mathematics at the university level. All languages have grammar and meanings, syntactic and semantic components. There are mathematicians and mathematics educators who claim that many students' difficulties in algebra depend more on their struggle with the language "mathematics" than on their procedural shortcomings.

Every language has a semantic component or descriptive aspect, which is the part of the language that carries meanings – as well as a syntactic or grammatical component. In the case of the language algebra, there is an asymmetry between the semantic and syntactic component, which is well worth explicit mentioning. The language of elementary algebra, defined as being a shortened version of our natural language, borrows meaning from the natural language's semantics. The semantics of algebra is however not fixed; *d* might mean "distance" or be a parameter in a polynomial expression, or a vertex in a quadrilateral. Nevertheless, whatever special meaning we assign to *d* will be a meaning defined in our natural language.

To teach mathematics with a focus on language

In an algebra course I gave to prospective middle school teachers in mathematics at Jönköping University almost a year ago, I claimed that the algebra course, which was labeled ”Mathematics as language” instead of "Introductory algebra for Middle school teachers", indeed carried an appropriate name, since algebra can be considered a language in many ways. The students were asked to monitor and record the growing language difficulty they experienced as we proceeded towards deeper algebraic understanding, and they were also asked to do a field study where they should record and analyse the language used in classrooms they visited as part of their student teacher training.

The main question for the field study was: "How do students and teachers handle the language of mathematics and how do they change back and forth between common language and the language of mathematics."

The results from the field study were quite homogeneous, although the students visited a broad variety of schools, covering the whole range of K–12. One common finding was that the concept of "language" was not explicitly addressed or even mentioned by the teacher they followed. The students in the class were not aware of the fact that they in general used quite many words from a common mathematical language, like for instance probability, graph, percent, and median. Nevertheless, some students did find children of early age who expressed that "you have to know the mathematical language in quite many professions, and therefore you should learn it." One young girl even claimed that ”it is easier to communicate with people around me when I express myself in mathematical terms." The observed and interviewed teachers - in contrast - saw themselves mainly as teachers of mathematical methods and not as "mathematical language" teachers.

The difference between a natural, descriptive language and a normative, precise science language like mathematics is huge. Normally, we do not correct natural language the same precise way as we do with mathematical language. In fact, some of our natural language is not even considered to be correctable. Steven Pinker, a language and cognition researcher, claims that language is innate and that humans have a common "universal grammar".

> Imagine that you are watching a nature documentary. The video shows the usual gorgeous footage of animals in their natural habitats. But the voiceover reports some troubling facts. Dolphins do not execute their swimming strokes properly. White-crowned sparrows carelessly debase their calls. Chickadees' nests are incorrectly constructed, pandas hold bamboo in the wrong paw, the song of the humpback whale contains several well-known errors, and monkeys' cries have been in a state of chaos and degeneration for hundreds of years. Your reaction would probably be, What on earth could it mean for the song of the humpback whale to contain an "error"? Isn't the song of the humpback whale whatever the humpback whale decides to sing? (Pinker, 1994, p. 370)

Many students do of course not see algebra as a language, a view that is not unreasonable. Very few would call a system of symbols a language, if it had no semantic component. No real language is used only to manipulate its own words. The power of a language lies not just in the words themselves, but more in the use we can make of them when communicating with each other. The words of our language support communication since they are symbols, pointing beyond themselves to things we experience in our world. To be real, a language has to be about something.

As for the prospective teachers own learning of the language algebra, quite a few of the students in the course admitted that they experienced it easier to learn algebra when it was looked upon as both a new language and as an array of laws and symbols. Another important aspect I emphasized was the importance of different language syntax in mathematics. They were asked to look at the following different ways to present the same mathematical problem.

a) Try to find the two numbers, which add up to 78 and where the product is 1296.

b) Solve $\begin{cases} x + y = 78 \\ x \cdot y = 1296 \end{cases}$

c) Draw the graphs of $x + y = 78$ and $x \cdot y = 1296$ in the same coordinate system and determine the intersections. (adapted from Polya, 1957, p. 175)

Clearly, all the three different ways to present this same problem do actually lead us to implicit or explicit ways of moving between our natural language and the formalistic language of mathematics. If we take a), it is obvious that we will try to express the situation in formalistic language like in case b). If we are presented to case b), we are likely to try to interpret the meaning of our task in language like case a). When solving it like case c) we probably will express ourselves in both formalistic and natural language.

The view of mathematics as a language is not unproblematic and any attempt in that direction is likely to narrow the broader view of mathematics, which I advocated for in the beginning of this paper. I do not claim that mathematics should be taught as a language, more so that the language part of mathematics should be recognized and accounted for. It has frequently been pointed out that mathematics itself is a formalised language and it has been suggested that it should be taught as such. Such statements possess a degree of validity, but would appear to be somewhat dangerous and potentially confusing. Mathematics is not a language – a means of communication – but an activity and a treasure house of knowledge acquired over many centuries. (Austin & Howson, 1979, p. 176)

In order to more strongly advocate for the necessity of thinking about the linguistic aspect of the mathematics you study to become a teacher of mathematics, the following exam question was given to a group of prospective middle school teachers at the university of Gothenburg in the fall 2003.

> Describe the following statement in a ”common language” and give an example of what the statement actually says or means:

$$\forall n\left(n > 2 \Rightarrow \sum_{i=1}^{n} i^2 < n^n \right)$$

Exactly the same statement was discussed in a lecture some 4 weeks before the exam. It was given as one example of how the mathematical language becomes more and more precise and consequently at the same time becomes more difficult to interpret and hard to connect to daily language. We also discussed in what way we could define if there is anything missing in a mathematical statement as the one above, as we ordinarily can in a sentence expressed in our mother tongue. The students were in their first year of academic studies in mathematics, and had all graduated from the natural science program in the Swedish gymnasium.

It was quite a surprise to us that the vast majority of the students had major difficulties with this question in particular. Only one student out of a group of ten managed to give a full and well developed description of the statement and some examples. Many of the other students did not manage to give correct examples of for instance the summation part of the statement. Expressions like “ 1^n is always smaller than n^n and therefore $1^n < n^n$“ were common, just as mixing the meaning of *i* and *n*.

The main difficulties in translating the theorem into common language seem to be divided into the following categories:

- The order of symbols and their semantic meaning. Student's descriptions of the meaning revealed confusion about what the symbolic form says about what to do first, second, third and so on.
- What is included in the summation and what is not, i.e. should n^n also be summarized?
- The meaning of the symbol "for all *n*".

Since so many students failed to translate the symbolic statement into common language, we gave them the following statement in the re-exam:

Describe the following statement in a "common language" and give an example of what the statement actually says or means:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Even though we had had a sincere and long discussion after the first exam, the students to a large extent failed to translate this statement too. To translate from algebraic language to "common language" disclosed much more difficulties than we had anticipated. Students, who very well may master the algebraic handicraft in these two examples, demonstrated huge difficulties when trying to explain what they actually mean in a common language. These examples illuminate the difficulties we have to develop and maintain a well structured and well balanced teacher education for teachers of mathematics. As prospective teachers of mathematics, it is crucial that they are skilful in their use of the language of mathematics together with other necessary competencies. In what way should courses be changed or developed so that students, who study mathematics in order to become qualified and competent teachers of mathematics, also develop and cultivate their own mathematical language?

A couple of students claimed, that since no other courses in mathematics requires that they should be able to translate or even speak about symbolic statements, they had never thought about the vast importance of being able to speak correctly when being a mathematics teacher. Even though they complained about our way of extorting their weakness in understanding the meaning of mathematical statements, they welcomed the fact that we had set their eyes on an important competence for their future profession.

> *Student 1*: It's a true shame that we can manage to pass exams in courses about abstract algebra, calculus, discrete mathematics, and so on… and yet we are not able to explain what the symbols actually means when they are put together in a statement. It's a shame!

> *Student 2:* Why don't courses in mathematics start with for instance a list over the symbols we should use and a deep and throughout explanation and discussion about the many different meanings of the symbols? Imagine how a language teachers would start with any new language, why not do so with university teaching in mathematics (at least for prospective teachers) too?
>
> *Student 3:* In my mind, I sense that the sigma symbols is stored as a picture and I normally never attach any real meaning to it – its' just there in my memory. When there are other symbols connected to it, above, under and together with each other, it becomes just like Chinese or any other inconceivable language – a blurred scatter of images. It just doesn't make any sense to me.

Davis (1984) argued that mathematical knowledge stored in our memories is not just coded in any native language's words or sentences. He asserted the following three possible positions:

I. Knowledge is stored in the human memory in the form of ordinary native language words and statements.

II. Knowledge is stored in the human memory in the form of pictures.

III. Knowledge is stored in the human memory in a form, which is neither words and statements, nor pictures. (p. 189)

Davis acknowledged that the three positions did not need to be mutually contradictory nor mutually exclusive. It is quite possible that more than one single form of coding is used.

> We want to argue that coding mechanism II – pictures – is *not* used, although something close to it certainly is. (Indeed, many mathematics students would be better of if they could learn to make *more* use of quasi-pictorial representations.) Coding mechanism I – Words and statement – *may* be used to a modest extent (we all know some quotations, or poems or slogans form memory, *verbatim*), but does not have the dominant role that one might naïvely suppose. The main mechanism for mathematics, and probably for many other things, is coding mechanism III, which is *not* verbal, and is *not* pictorial. (p. 189)

Some of the students obviously view themselves as owners of mental objects, in some way corresponding to mathematical objects. Apparently we all have mental objects like for instance a circle that we easily can get from where it is stored in order to handle the circle is some way. The language has labelled it circle (or "cirkel" in Swedish), but what is actually stored in our mind? According to Davis it is something significantly more than a picture or image, something more like a representation of a mathematical object. Our mental circle must have the same properties and act the same as the mathematical definition of a circle. So for a sufficient understanding of mathematics the students have to construct the mental objects corresponding to the mathematical objects she or he is studying. The construction must be of such a dignity, that there is an isomorphism that holds between the two worlds in where the two different constructions exist. And the isomorphism is proven valid or unacceptable through our language.

It is important to explain that not all mathematical objects can have a true or correct mental representation; we mean an infinite line when we draw a finite line and we most likely store a finite line in our mental representation. Still we can talk about an infinite line and use our mental representation, our mental object, to support our presentation. The language consequently becomes a necessary tool for connecting our mental object with the corresponding mathematical object, when helping someone else to construct his or her mental object, or in order to help us better understand what we are observing when studying mathematics. Consequently, there are many more arguments for teaching the language of mathematics than just the fact that prospective teachers should have a language competence as teachers. All students who study mathematics should be taught the importance of the language of mathematics in order to better understand the subject.

References

Austin, J. L., & Howson, A.G. (1979). Language and mathematics education. *Educational Studies in Mathematics, 10,* 161-197.

Cocking, R. R., & Chipman, S. (1988). Conceptual issues related to mathematics achievement of language minority children. In Cocking, R. R., & Mestre, J.P. (Eds.), *Linguistic and cultural influences on learning mathematics*. Hillsdale: Erlbaum.

Courant, R. & Robbins, H. (1941). *What is mathematics?* Oxford: Oxford University Press.

Davis, R. B: (1984). *Learning mathematics. The cognitive science approach to mathematics education*. New Jersey: Ablex Publishing Corporation.

Hersh, R. (1994). Fresh breezes in the philosophy of mathematics. In P. Ernest (Ed.), *Mathematics, education and philosophy: An international perspective*. London and Washington DC: Falmer Press.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge: Cambridge University Press.

Lennerstad, H., & Persson, U. (2003). Debattforum – Matematik och språk. (Debate forum – Mathematics and language) *SMDF Medlemsblad 7,* 41 – 47.

Emanuelsson, G. et al. (red.) (1996). *Matematik - ett kommunikationsämne. Nämnaren TEMA.* Göteborg: Institutionen för ämnesdidaktik, Göteborgs universitet.

Pinker, S. (1994). *The language instinct: The new science of language and mind*. Harmondsworth: Penguin.

Polya, G. (1957). *How to solve it*. Princeton, NJ: Princeton University Press, 1957.

Rényi, A. (1991). *Dialogues about mathematics*. Basel: Birkhäuser.

Wittgenstein, L. (1956). *Remarks on the foundations of mathematics*. Oxford: Basil Blackwell.

Kategorisering av små gruppers handlingar

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Inledning

I dagens svenska gymnasieskola är det många elever som inte klarar av matematikkurserna. Enligt statistik från Skolverket var det 15 procent av eleverna på det nationella programmet som inte uppfyllde målen på kurs A i matematik på höstens kursprov 2002, medan motsvarande andel på vårens kursprov 2003 var 25 procent. När det gäller kurs B i matematik var det drygt 25 procent (Skolverket, 2004) .

Det finns många sammanhängande orsaker till att eleverna inte klarar av matematiken. En orsak kan vara hur undervisningen bedrivs och vad detta för med sig. Flera undersökningar har visat att det är svårt för skolor att hitta fungerande arbetssätt och arbetsformer som tar hänsyn till varje elevs förutsättningar och behov (Skolverket, 2003b). Det har också visat sig att matematikämnet är det ämne där undervisningen präglas av *enskilt arbete* och att läroboken är styrande (Skolverket, 2003a). Att arbeta enskilt innebär att elevens egna frågor och funderingar inte diskuteras och blir bemötta. Det innebär också att eleverna går miste om att lyssna på andra elever och kunna bemöta deras påståenden och argument.

Denna artikel presenterar en del i en studie som tar fasta på elevens möjlighet till att delta i dialoger och att använda en artefakt som stöd för lärandet. Studien som helhet har som syfte att kartlägga hur en liten grupp elever, 4 stycken, samtalar i ämnet matematik utan lärares närvaro och hur eleverna använder en artefakt, i detta fall TI InterActive[1] under samtalet.

Studien som redovisas i denna rapport har två syften:

1. Att studera hur en mindre grupp elever utan lärares närvaro samarbetar för att lösa de experiment som gruppen enats om.
2. Att studera hur gruppen utnyttjar artefakten.

Teoretisk bakgrund

Forskningen här har beröring med flera olika teoretiska och empiriska fält. Dessa är samspel i grupper, användandet av artefakter och det matematikdidaktiska fältet. Det sociala samspelet ses som den process som är bärande för elevernas utveckling och ökande förståelse för matematiken. En viktig fråga är hur en undervisning ska organiseras och genomföras för att eleverna ska utveckla en

[1] TI InterActive är en interaktivt programvara.

förmåga att kommunicera med matematikens språk och symboler. Någon entydigt formulerad teori för denna undervisning finns inte. Däremot kan man försöka hitta en balans mellan en individuell undervisning och en undervisning som bygger på samspel (Dysthe, 2003). I denna studie betraktas samspelet mellan elever och artefakten när eleverna diskuterar det matematiska begreppet räta linjen.

Den lärosyn som ligger till grund för den större studien har sina rötter i Vygotskys teorier. I denna teori ses inte bara språket och kommunikationen som ett medel för lärandet utan det är själva grundvillkoret för att ett lärande och ett tänkande skall kunna ske (Säljö, 2000).

Man kan ofta förundras över hur kort innehållet i en elevs replik är med avseende på ord och speciellt innehåll av fackuttryck. Men även om detta är fallet i många repliker uttrycks ändå en position hos den talande. En position som någon annan elev kan svara på och förhålla sig till (Bakhtin, 1997). Bakhtins tonvikt på dialogen är ett viktigt komplement (Wells, 1999) till Vygotskys ”The zone of proximal development” (Vygotsky, 1978). Det är Bakhtins tredje grunddrag för ett yttrande som är av intresse för denna studie och lärandeprocessen (Bakhtin, 1986). Detta innebär att ett yttrande inte bara påverkar mottagaren utan kommer också att påverka tillbaka till den som fällde yttrandet. När en elev uppfattar och förstår en annan elevs yttrande har denne lyssnat aktivt. Om så skett kommer eleven förr eller senare att respondera på den förra elevens yttrande. Detta innebär att elevens yttrande kommer att påverka tillbaka, vilket är en viktig ingrediens i lärandeprocessen.

Frågeställningar

I anslutning till de två syftena att studera hur en mindre grupp elever utan lärares närvaro samarbetar för att lösa de experiment som gruppen enats om och hur gruppen utnyttjar artefakten har följande frågor preciserats.

1. Hur kan man på en detaljerad nivå, kopplat till det matematiska innehåll som grupperna diskuterar, beskriva hur gruppen samarbetar för att lösa de experiment som man enats om?
2. Hur kommer artefakten in som en del i samarbetet?

Den övergripande studien

Den studie som beskrivs i den här artikeln är en del av en större studie. För den större studien har Stefan Njord följt samma elever under en längre tid (4 månader) i ämnet matematik (kurs B på gymnasiet). Den större studien bygger på klassobservationer av eleverna i helklass samt videofilmning av mindre grupper (4 elever) utan lärares närvaro. Data till denna rapport är från delar av den videofilmning som gjorts i grupp utan lärare. Innan eleverna arbetade i små grupper gjordes en enkätundersökning om elevernas attityder till skolan, matematiken, lärandet med mera. Under studien av eleverna genomfördes tre skrivningar i anslutning till deras kurs i matematik, varav en av skrivningarna var det nationella provet. Efter det att videofilmningen var slutförd gjordes en enkätundersökning om elevernas attityder kring att arbeta i grupp och hur man såg på datorns betydelse för gruppens arbete.

Studiens uppläggning

Samspelet är en viktig förutsättning enligt vår teoretiska utgångspunkt. I studien fokuseras samspelet i mindre grupper. Med mindre grupper i denna studie menas grupper om 4 elever. Grupperna studeras enskilt i ett eget grupprum utan närvaro av elevernas lärare. Varje grupp hade tillgång till en dator kopplad till en OH-kanon. OH-kanonen projicerade bildskärmen på väggen i ett format på cirka 2·1,5 meter. Detta innebar att det som gjordes via datorn blev synligt för alla i gruppen. På datorn fanns matematikprogrammet TI InterActive installerat. Gruppen videofilmades i 30 minuter. Totalt filmades 4 grupper om vardera 30 minuter.

Eleverna i studien läste matematik kurs B på gymnasiet och arbetade i helklass med begreppet räta linjen. I anslutning till detta skulle grupperna diskutera något man redan gjort i helklass. Valet blev hur man med utgångspunkt från två punkter och deras koordinater kan bestämma ekvationen för den räta linje som går genom dessa punkter. Uppgiften utgår inte ifrån givna punkter utan dessa väljs av eleverna. Detta innebär att eleverna själva kan styra de olika perspektiv på den räta linjen som man vill diskutera.

Eleverna i studien hade inte före studien kommit i kontakt med pogramvaran TI InterActive. Valet av program motiveras med att 1) den har ett gränssnitt som i många avseenden påminner om elevernas miniräknare och de dataprogram som eleverna varit i kontakt med under sin skoltid och 2) att de funktioner som eleverna fick använda under studien inte gav den matematiska lösningen via en knapptryckning. Valet av dataprogrammet och möjligheten att via programmet få en koppling mellan symboler och grafiska representationer var en viktig del i upplägget. Eftersom eleverna inte använt programmet innan var det också viktigt att eleverna lätt skulle kunna lära sig programmet genom en kort demonstration och inse fördelar och nackdelar med att använda programmet.

Urval

Eleverna som medverkade i studien gick på det naturvetenskapliga programmet. Samtliga elever hade betyget godkänt från matematikkursen, kurs A. Under studiens utförande läste eleverna kursen, Matematik B. Totalt deltog 16 elever i studien, varav 4 var flickor och 12 var pojkar. Med utgångspunkt från elevernas förkunskaper definierades sedan vad som skulle diskuteras i grupperna.

Instruktion till gruppen

Gruppen informerades om att syftet med deras experiment var att gruppen skulle ha som mål att

1. alla i gruppen med utgångspunkt från två punkters koordinater skulle kunna bestämma den räta linje på formen $y = kx + m$ som går genom de två punkterna och
2. att samtliga i gruppen skulle förstå det som diskuterades.

Gruppen valde själva vilka två punkter man ville utgå ifrån. Gruppen valde också hur många gånger man behövde titta på olika punkter innan man ansåg att gruppen nått de två uppsatta målen. En förutsättning var att gruppen skulle avsluta sitt arbete efter 30 minuter.

Dataprogrammet introducerades genom att vi visade eleverna hur man via dialogrutan ”Draw line” kan ange två punkters koordinater och att programmet därefter automatiskt visar punkternas representation i ett koordinatsystem. I grafen visas också en streckad rät linje som går genom de två punkterna. Via grafen är det sedan möjligt att med hjälp av muspekaren flytta punkterna och de nya koordinaterna visas automatiskt i dialogrutan ”Draw line”. Vi visade också eleverna att man via dialogrutan ”Functions” kan ange en funktion på formen $y = kx + m$ och automatiskt få denna representerad i samma koordinatsystem som punkterna. Eleverna informerades om att via dessa två dialogrutor kan man se en koppling mellan de två punkter man väljer och den räta linje på formen $y = kx + m$ som man kommer fram till.

En förutsättning var att de punkter som valdes skulle anges i programmet, så att alla såg kopplingen mellan punkterna och den grafiska representationen av dessa. Därefter fanns det inget krav på att gruppen måste använda datorn i diskussionerna. Gruppen informerades om att någon värdering eller kommentar till vad gruppen gjort inte kommer att presenteras efteråt. Hela arbetet videofilmades och avslutades efter 30 minuter.

Analys

Analysen av studien är gjord med utgångspunkt från den videofilmning som gjordes av gruppernas arbete. Filmningen har registrerat det som eleverna gör med datorn genom att även dataskärmen filmas. Även det som eleverna skriver på sina papper har filmats. Som stöd för analysen har i vissa fall elevernas samtal transkriberats. Även utskrifter av elevernas filmade papper har gjorts. Genom att titta igenom videofilmerna ett antal gånger har analysen gjorts i tre steg:

1. Att försöka hitta beskrivande kategorier för det som sker.
2. Att försöka formulera och därmed skapa kategorier som beskriver de handlingar som respektive grupper gör (=handlingskategorier).
3. Att försöka spåra mönster för hur grupperna kombinerar de olika kategorierna.

Resultat

Analysen har resulterat i 4 handlingskategorier som mer eller mindre beskriver gruppernas process mot att lösa de föreliggande experimenten. De 4 handlingskategorierna benämns 1) *tolkning via graf*, 2) *verifikation*, 3) *trial and error* och 4) *formelräkning*. De tre första kategorierna är kopplade till användning av artefakten. Handlingskategorierna beskrivs nedan.

Tolkning via graf (T)

Via dialogrutan ”Draw line” kan gruppen ange koordinaterna för de två punkter man vill studera. När gruppen angett punkternas koordinater genererade dataprogrammet automatiskt punkterna i ett koordinatsystem. I koordinatsystemet visades inte bara punkterna automatiskt, utan även en streckad linje genom punkterna, det vill säga den räta linje vars ekvation gruppen skulle bestämma. Denna grafiska bild var hela tiden under gruppens diskussioner synlig för gruppen. Vid analysen av videomaterialet framgick att grupperna vid ett flertal tillfällen använde sig av denna grafiska representation för att tolka och resonera kring. Via analysen kan ses att grupperna använder sig av denna graf framför allt för att bestämma m-värdet ur grafen och/eller för att bestämma riktningskoefficienten (k-värdet) ur grafen. Grafen var en central utgångspunkt vid ett flertal diskussioner och uttalanden. Handlingskategorin tolkning via graf innebär att eleverna använder sig av denna graf vid diskussionen eller vid ett enskilt uttalande för att bestämma k- och/eller m-värdet.

Verifikation (V)

Som nämnts ovan skapar dataprogrammet automatiskt med utgångspunkt från att man anger två punkters koordinater i dialogruten ”Draw line” en grafisk representation av de två punkterna och den räta linje som går genom punkterna. Via dialogrutan ”Functions” är det möjligt att ange den räta linjens ekvation på formen $y = kx + m$. När man anger en rät linje i dialogrutan ”Functions” genererar dataprogrammet automatiskt dess grafiska representation i samma koordinatsystem som skapats via dialogrutan ”Draw line”. Detta innebär att gruppen får en visuell koppling mellan de två punkter man har valt att utgå ifrån och de förslag på k- och m-värden som gruppen kommer fram till. Handlingskategorin *verifikation* definieras som att gruppen har uttalat en teori/er eller hypotes/er som grund för ett angivet k- och/eller m-värde. Därefter använder sig gruppen av möjligheten att via dialogrutan ”Functions” kunna bedöma rimligheten i det angivna k- och/eller m-värdet.

Trial and error (U)

Handlingskategorin *Trial and error* beskriver samma förfarande som handlingskategorin *verifiera* med den skillnaden att gruppen *inte* uttalat någon teori/er eller egna hypoteser när man använder sig av dialogrutan ”Functions”. Handlingen tolkas mera som om att gruppen chansar på olika k- och m-värden och därefter försöker dra slutsatser om chansningen med hjälp av den graf som automatisk genereras av dataprogrammet.

Formelräkning (F)

Den fjärde handlingskategorin, *formelräkning* innebär att gruppen arbetar med papper och penna och använder sig av matematiska symboler och formler.

Hur grupperna kombinerar handlingskategorierna

Processen från det att gruppen valt två punkter tills man bestämt ekvationen för den räta linje som går genom punkterna benämnes *experiment*. I studien visade det sig att grupperna gjorde mellan 3 och 6 experiment per grupp. I tabell 1 har de 4 handlingskategorierna använts för att beskriva den eller de strategier grupperna använde sig av för att diskutera sig fram till en lösning för sitt experiment. Till höger om varje experiment anges den totaltid som experimentet tog att genomföra. Handlingarna symboliseras i tabellen av T:m som betyder ”Tolkning av m-värdet via graf”, T:k betyder ”Tolkning av k-värdet via graf”, V:m betyder ”Verifierar m-värdet”, V:k betyder ”Verifierar k-värdet”, V:k&m betyder ”Verifierar k- och m-värdet samtidigt”, U:k betyder ”Trial and error [k-värdet]”, U:m betyder ”Trial and error [m-värdet]”, U:k&m betyder ”Trial and error [k- och m-värdet samtidigt]”, F:k betyder ”Formelräkning för att beräkna k-värdet” och F:m betyder ”Formelräkning för att beräkna m-värdet”.

Tabell 1: Sammanställning av respektive grupp och respektive experiment med hjälp av de fyra handlingskategorierna.

Grupp A														Tid (min)
Experiment 1	T:m	V:m	U:k											3
Experiment 2	T:m	V:m	U:k&m	T:k	V:k	T:m	V:m	T:k	V:k	U:k	T:k			7
Experiment 3	T:k	T:m	V:k&m											2
Experiment 4	T:k	T:m	V:k&m	T:k										1
Experiment 5	F:k	F:m	V:k&m	U:m	F:k	V:k	U:k	V	F:k	F:m				15
Grupp B														Tid (min)
Experiment 1	F:k	F:m	V:k&m											3
Experiment 2	F:k	T:m	T:k	V:k&m	T:k	V:k&m	U:k	F:k	V:k&m					7
Experiment 3	F:k	F:m	V:k&m											2
Experiment 4	F:k	F:m	V:k&m											6
Experiment 5	F:k	F:m	V:k&m											2
Experiment 6	F:k	F:m	V:k&m											5
Grupp C														Tid (min)
Experiment 1	T:m	F:k	V:k&m											3
Experiment 2	F:k	F:m	T:k&m	T:m	F:k&m	V:m	F:k&m	T:k	T:m	V:k&m	U:m	F:k&m	T:k	17
Experiment 3	T:m	F:k	T:k	V:k&m	U:k&m	T:k&m	F:k&m	V:k&m	F:m					7
Experiment 4	F:k	T:m	V:k&m	F:m	V:k&m									3
Grupp D														Tid (min)
Experiment 1	F:k	F:m	V:k&m	F:m	V:m	F:m	U:k	F:k						17
Experiment 2	F:k	F:m	V:k&m											6
Experiment 3	F:k	F:m	V:k&m											5

Via tabellen kan man se att grupp A utför 5 experiment, grupp B utför 6 experiment, grupp C utför 4 experiment och grupp D utför 3 experiment. Grupp A löser sitt första experiment genom att utföra handlingarna i ordningen; en tolkning av m-värdet (T:m), en verifikation av m-värdet (V:m) och slutligen en trial and error för att hitta k-värdet (U:k).

Grupp A
Gruppen använder sig uteslutande av att diskutera fram lösningen genom att göra tolkningar via grafen (T:m eller T:k), för att därefter verifiera (V:m, V:k eller V:k&m) tolkningarna. I samtliga fyra första experimenten visade gruppen att man är säker på att avläsa m-värdet ur grafen. I två experiment, experiment 3 och 4 läste man utan problem av såväl k- och m-värdena grafiskt och verifierade dessa. Gruppen hittar här en strategi för att genomföra experimenten (T:k. T:m och V:k&m). I det sista experimentet, experiment 5 använde gruppen papper och penna för att försöka beräkna k- och m-värdet. Gruppen vet vilka formler man ska använda, men gör en felaktig beräkning som innebär att varken k- eller m-värdet blir rätt. Gruppen har inga problem med att bestämma k- och m-värdet grafiskt, däremot har man svårt att hitta det fel man gör vid beräkningarna med hjälp av papper och penna. Något stöd för att hitta de fel man gjort i beräkningen med papper och penna ger inte datorprogrammet.

Grupp B
Gruppen hittar en tydlig handlingsstruktur för att genomföra sina experiment: beräkna k-värdet med hjälp av formeln $k = \frac{y_2 - y_1}{x_2 - x_1}$ (F:k), beräkna därefter m-värdet med hjälp av formeln $y_1 = k \cdot x_1 + m$ (F:m) och avsluta med att verifiera att beräkningarna verkar överensstämma (V:k&m). Gruppen använder i stort sett beräkning via formel som verktyg för att bestämma k- respektive m-värdet. För att sedan utvärdera rimligheten i det svar man fått via beräkningarna använder man sig av datorprogrammet.

Grupp C
Gruppen visade att man visste hur man skall beräkna k-värdet (F:k) och hur man kan avläsa såväl k- som m-värde ur grafen (T:k och T:m). Däremot gjorde man felaktiga beräkningar när man använde formlerna som i experiment 2. Man insåg dock via tolkningar av grafen att man gjort fel i beräkningarna. Att hitta felet eller felen gav dock inte datorprogrammet något stöd för, precis som för grupp A. I sista experimentet beräknade man k-värdet (F:k), tolkade m-värdet (T:m) och verifierade slutligen formeln (V:k&m). Därefter beräknade man m-värdet (F:m) och avslutade med att verifiera att man hade det rätta k- och m-värdet (V:k&m).

Grupp D
I första experimentet tar det lång tid för gruppen att lösa uppgiften då man lägger ned mycket tid på att samtliga i gruppen skall förstå hur man beräknar

såväl k- som m-värdet. Metoden som gruppen använder för att beräkna uppgifterna är samma strategi som grupp B (F:k, F:m och V:k&m). Vi kan precis som för grupp B se att en grupp kan använda formelrepresentationerna av den räta linjen och därefter använda datorprogrammet för att verifiera svaren. Datorprogrammet som ett verktyg för att verifiera beräkningarna verkar vara ett viktigt moment för utvärderingen.

En sammanställning av hur många gånger respektive grupp utförde de 4 handlingarna visar på att trial and error används mycket lite i förhållande till tolkningar, verifieringar och formelräkningar.

Tabell 2: Fördelningen mellan de fyra handlingskategorierna och de fyra grupperna.

	Grupp A	Grupp B	Grupp C	Grupp D	Totalt
	Antal	Antal	Antal	Antal	Antal
Tolkningar	11	3	10	0	24
Verifieringar	10	8	7	4	29
Trial and error	5	1	2	1	9
Formelräkning	5	12	11	9	37

Sammanfattning

Resultatet visar hur man med hjälp av att skapa handlingskategorier kan på en detaljerad nivå, kopplat till det matematiska innehåll som grupperna diskuterar, beskriva hur eleverna i gruppen samarbetar för att lösa de experiment som man enats om. En av dessa handlingar (formelräkning) är kopplad till att gruppen använde sig av matematiska formler och papper och penna. De tre andra handlingarna (tolkning, verifiering och trial and error) är kopplade till att gruppen använde sig av artefakten och den inbyggda möjligheten att koppla samman matematiska begrepp med motsvarande grafiska representationer.

Totalt genomförde grupperna 18 experiment. 11 av dessa experiment avslutades genom att gruppen via artefakten *verifierade* att deras slutliga lösningsförslag överensstämde med den räta linje som gick genom de två valda punkterna. Endast 3 experiment avslutades med *beräkningar*. Inte i något av de 18 experimenten avslutade eleverna med att konstatera att man inte kunde komma fram till en gemensam lösning.

Samarbetet inom grupperna kan beskrivas med två strategier. Den första bygger på att gruppen *inte* använder papper och penna utan diskuterar sig fram till det k- och m-värde som man tror beskriver den räta linjens ekvation. Som utgångspunkt för diskussionen använder gruppen sig av den graf som genererats via artefakten. När gruppen slutligen enats om ett k- och m-värde låter man artefakten med utgångspunkt från sina k- och m-värden generera grafen av uttrycket $y = kx + m$. Genom att sedan studera grafen drar man slutsats om rimligheten i k- och m-värdet. Den andra strategin är att eleverna via formler och papper och penna bestämmer k- respektive m-värdet. Därefter använder man artefakten för att studera den grafiska representation som genereras av artefakten med utgångspunkt från formeln $y = kx + m$, och kan därmed avgöra rimligheten i beräkningarna.

Artefaktens roll i samarbetet kan beskrivas ur två perspektiv. Det första är artefaktens möjlighet att direkt när gruppen valt startpunkter ge gruppen en grafisk bild som representerar såväl punkterna som den tänkta räta linje som går genom punkterna. Denna graf får en central betydelse för två grupper (A, C) genom att många tolkningar görs av grupperna med utgångspunkt från denna graf. Det andra perspektivet är artefaktens möjlighet att via den räta linjens ekvation $y = kx + m$ automatiskt skapa dess grafiska representation. Denna möjlighet använder sig samtliga grupper av i diskussionerna.

För att få en större förståelse för den undervisningssituation som undersöks i denna studie är nästa steg att koppla handlingarna och gruppernas strategier till vad eleverna uttrycker i dialogerna.

Referenser

Bakhtin, M. (1997). Frågan om talgenrer. I Eva Haettner Aurelius & Thomas Götselius, (red.), *Genreteori*. Lund: Studentlitteratur.

Bakhtin, M. (1986). *Speech genres and other late essays*. [Trans. by Y. McGee.] Austin: University of Texas Press.

Dysthe, O. (red.) (2003). *Dialog, samspel och lärande*. Lund: Studentlitteratur.

Skolverket (2004). *Sammanfattning av resultaten från gymnasieskolans kursprov läsåret 2002/2003*. Tillgänglig: <http://www.skolverket.se /nat/resultat/kprovvt 2003.shtml>, 17 mars 2004.

Skolverket (2003a). Nationella kvalitetsgranskningar 2001–2002 Lusten att lära – med fokus på matematik. *Skolverkets rapport 221*. Stockholm: Skolverket.

Skolverket (2003b). Skolverkets lägesbedömning av barnomsorg, skola och vuxenutbildning 2003. *Skolverkets rapport 238*. Stockholm: Skolverket.

Säljö, R. (2000). *Lärande i praktiken, Ett sociokulturellt perspektiv*. Stockholm: Bokförlaget Prisma.

Vygotsky, L. S. (1978). *Mind in society, The devolopment of higher psychological processes*. Cambridge, MA: Harvard University Press.

Wells, G. (1999). *Dialogic inquiry – Toward a sociocultural practice and theory of education.* Cambridge: Cambridge University Press.

A Theoretical Framework for Analysis of Teaching – Learning Processes in Algebra

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Introduction

In this article we focus on describing a theoretical framework, which can be applied to analyse the teaching and learning processes that occur in the classrooms as seen from the student's and the teacher's point of view. This theoretical framework can be used to analyse teaching and learning processes in general, but we will apply it to teaching and learning processes in algebra. More specifically, we use it to analyse the quadratic functions and equations as an object of learning and the way in which the forming of this object of learning during classroom lessons influences the students' learning.

Historical context

The history of algebra shows that algebra in some form or other has been taught explicitly for some 1000 years, and implicitly for perhaps 3000 years. Diophantus of Alexandria wrote the first treatise on algebra in the 3rd century, and since then algebra in different forms has been developing. Eves (1990) and Burton (1991) describe three stages in the development of algebraic notation. In the first stage, rhetorical algebra, no symbols are used at all, and all equations and problems are posed and solved completely in prose form. This type of algebra was used in Europe until approximately the sixteenth century when syncopated algebra (the second stage), became standard. Diophantus is generally credited with the introduction of this algebra form that is characterized by the use of some abbreviations for the frequently recurring quantities and operations, in 240 AD. The third stage in the development of algebra is symbolic algebra, which is the algebra we use today. Symbolic algebra was developed through contributions by Francois Viète (1540-1603) and René Descartes (1596-1650), among others, and gained widespread use by the middle of the seventeenth century. With symbolic algebra, the symbols became objects of manipulation in their own right, rather than simply shorthand for describing computational procedures. The mathematical symbols were developed in parallel with the development of algebra, and in this development we can distinguish three important kinds of symbols. Symbols for numbers, quantities, variables, or objects, for example, symbols used for trigonometric functions, powers, roots and logarithms, their indices and exponents, and symbols used for variables; symbols of operation, which describe things to be performed, for example, symbols indicating addition, subtraction, multiplication, division, and their attendant grouping symbols; symbols of relation, which describe things established, for example, symbols of equality and inequality.

Sfard (1995) describes the impact of symbolic algebra as follows:

> Employing letters as givens, together with the subsequent symbolism for operations and relations, condensed and reified the whole of existing algebraic knowledge in a way that made it possible to handle it almost effortlessly, and thus to use it as a convenient basis for entirely new layers of mathematics. In algebra itself, symbolically represented equations soon turned into objects of investigation in their own right and the purely operational method of solving problems by reverse calculations was replaced by formal manipulations on prepositional formulas. (p. 24)

Important research findings

Research on teaching and especially research on mathematics teaching and learning has a long history (see, e.g., a number of “Handbooks”, which address issues of teaching or learning, such as Bishop et al., 1996; Richardsson, 2001; Wittrock, 1986). Trends today dominating the field of research on learning and teaching are constructivism, situationism and collaborative learning. More specifically, we can say that in modern education students are encouraged to construct their own knowledge, in realistic situations, together with others. Many researchers have been concerned with the move from arithmetic to algebra and in particular the cognitive gap that exists between the two (e.g. Bednarz et al., 1992). They have understandably focused on particular aspects of algebra such as the models used when solving word problems (e.g. MacGregor & Stacey, 1993, 1998), the understanding of the equality sign (e.g. Kieran, 1981), the translation from tabular form to symbolic form (e.g. Ryan & Williams, 1998), the solution of linear equations (e.g. Linchevski & Herscovics, 1994; MacGregor & Stacey, 1995), and functions and graphs (e.g. Herscovics, 1989).

Anna Sfard has outlined a theoretical framework for describing the development of understanding in algebra (Sfard, 1991, 1994, 1995; Sfard & Linchevski, 1994). This framework is based on the theory that the development of algebraic understanding in the individual student follows the same steps that can be observed in the historical development of algebra. Sfard’s theory is that the historical development of algebra from rhetorical to symbolic must be reproduced in the individual to achieve understanding of algebra. More specifically, Sfard (1991, 1995) describes three stages that characterize the development of mathematical understanding in any area of mathematics, not just algebra. In the first stage, interiorization, some process is performed on a familiar mathematical object. In rhetorical algebra, for example, numbers are effectively manipulated and those manipulations are described in prose. In the second stage, condensation, the process is refined and made more manageable, as in syncopated algebra. In the third stage, reification, a giant ontological leap is taken: "Reification is an act of turning computational operations into permanent object-like entities" (Sfard, 1994).

Researchers (e.g., Herscovics, 1989; Kieran, 1989, 1992; Sfard, 1991, 1994) describe a number of obstacles that can be connected directly to the difficulty in reification as described by Sfard. For example, children usually have difficulty in accepting an algebraic expression as an answer; they see an answer as a specific number, a numerical product of a computational operation. The equality sign is usually interpreted as requiring some action rather than signifying equivalence between two expressions. Kieran (1992) proposed that the problem with modern algebra is that we impose symbolic algebra on students without taking them through the stages of rhetorical and syncopated algebra. Thus, as many educators and students have observed, students often emerge from algebra with a feeling that they have been taught an abstract system of operations on letters and numbers with no meaning. Herscovics (1989) describes the situation by stating that the students have been taught the syntax of a language without the semantics; in other words, they know all the rules of grammar, but do not understand the meaning of the words. Sfard and Kieran would very logically argue that this situation has resulted from jumping to symbolic algebra without exploring rhetorical and syncopated algebra.

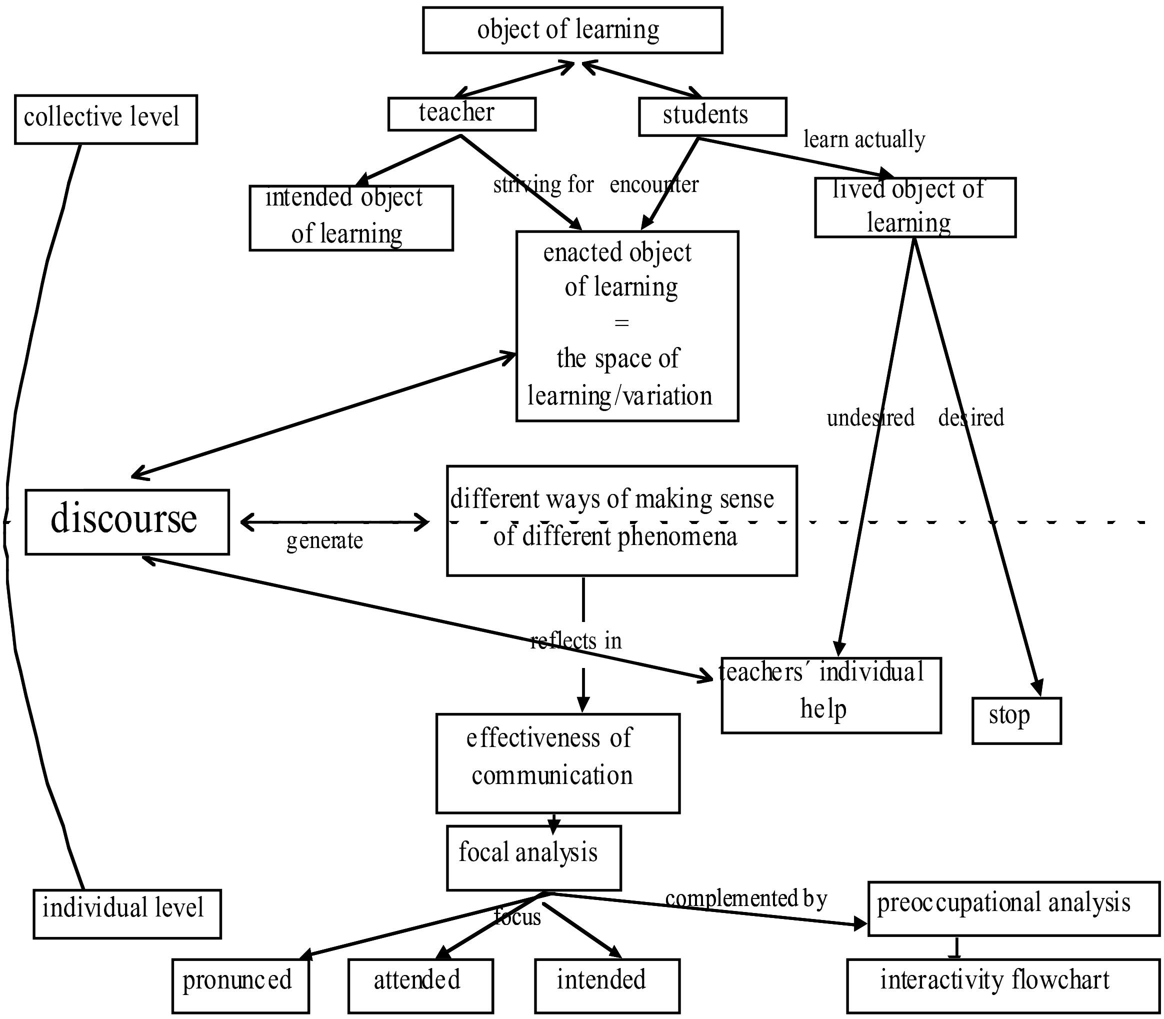
Because the students still have problems with learning algebra in school, we need to find new theoretical perspectives to understand this complicated phenomenon.

Theoretical framework

A central question around which recent research on learning and teaching revolves is how social interaction mediates the construction of knowledge in classrooms. There is research that has explored the construction of knowledge in classrooms and the ways in which meanings are socially constructed in classroom interaction (e.g. Bergqvist & Säljö, 1995; Edwards, 1993). In the light of current research it is clear that classroom interaction is seen as a valuable tool for learning, which should be studied from different perspectives in order to deepen our understanding of the practice of learning in and through social interaction.

In examining the difficulties students encounter in algebra it is necessary to study the teaching and learning that occur in the classroom. Students' opportunities to learn algebra during classroom lessons can be influenced by a variety of factors, including the knowledge and skills they already posses, as well as the activities in which they engage during the lesson. Teaching contributes to the students' learning, to the development of knowledge, but it is only one of the means by which the students develop knowledge (Marton et al., 2003).

Teaching can be analysed from many perspectives. The approach taken in this paper is to focus on features of teaching, and the way these features influence the learning opportunities for students. The way in which algebraic content is worked on during the lesson may add important information about the learning opportunities for students. In order to analyse the phenomena of the teaching and learning processes in the classroom, a tentative theoretical model was developed. We will now explain the structure of the model (see figure below) and some of its key concepts.



(Olteanu, 2004)

Observations of many different classes consistently show that teachers typically do between half and three quarters of the talking in classrooms. Talk is one of the major ways in which teachers convey information to students, and it is also one of the primary means of controlling students' behavior. Since the teachers do so much talking, it is important to analyze what they are talking about (Ottosson, 2000; Sfard, 1998; Sfard & Kieran, 2001), which means to understand the discursive construction of mathematical objects as accounting practices (Säljö, 1997). The word discourse is defined by Sfard (2002) as any specific act of communication, whether verbal or not, whether with others or with oneself, whether synchronic (like in a face-to-face conversation) or asynchronous (like reading a book). The word communication denotes an activity in which one is trying to make an interlocutor (possibly oneself) act or feel in a certain way.

The historical development of algebra shows that the development of algebraic notations and symbol use was a long and difficult process that was based on verbal descriptions that precede syncopated and symbolic notations. For this reason it is important to understand how the discourse that occurs in the classroom on both the collective and individual level, can generate different ways of making sense of different phenomena (e.g. sign system, algebra code, concepts) as a component of the object of learning (Marton et al., 2003; Ottosson, 2000). The object of learning is formed in the classroom between the teacher and the students and is something that can be identified from an observer's perspective. The term "ways of making sense" can be used for the way sense is made in a discourse as well as the way one is making sense for oneself, that is, in thinking (Ottosson, 2000). If we see thinking as communicating, the term discourse may be substituted for knowledge, and the notion of learning can be redefined to denote the activity of becoming a skilful participant of a certain specialised type of discourse (Sfard, 2002).

To investigate the variation in the different ways of making sense of the object of learning, it is necessary to study the effectiveness of verbal communication. The communication is effective if it fulfils its communicative purpose, that is, the different utterances of the interlocutors evoke responses that are in tune with the speakers' meta-discursive expectations (Sfard & Kieran, 2001). The most fundamental meta-discursive expectation of a speaker is that the conversation is coherent; that is, the respondent refers in his or her response to the same thing as the speaker has been talking about. In other words the effectiveness of verbal communication is seen as a function of the quality of its focus. Because the term focus is an interpretative concept and because it is up to an interpreter to decide what should count as a focus of a given utterance, Sfard introduced the term discursive focus with the help of three discursive components that are indispensable for effective communication: a pronounced focus, an attended focus and an intended focus (Sfard, 2002).

By understanding the effectiveness of the classroom communications, we can understand the possibilities that are given to the student so that he or she can "see algebraic expressions as objects" (Sfard, 1994). The possibilities that are offered in the classroom can be understood in terms of space of learning, and in this space we can identify the words used by an interlocutor to signal what he or she is talking about which is called the pronounced focus. The attended focus refers to what and how we are attending, looking at, listening to and so forth, when speaking. Finally the intended focus is the interlocutor's interpretation of the pronounced and attended foci and must be considered along with them. The pronounced focus marks the public (anything that is perceptually accessible), the intended focus is predominantly private (anything that is experienced), and the attended focus mediates between the two.

With the help of the three focal components, Sfard built the focal analysis, which we can use to get a detailed picture of the student's conversation (with another student or with the teacher) on the level of its immediate mathematical content. This should make it possible to understand the effectiveness of communication. The effectiveness of verbal communication is dependent upon the degree of clarity of the discursive focus. The focal analysis is complemented by preoccupational analysis, which is directed at meta-messages and examines the participant's engagement in the conversation, thus possibly highlighting at least some of the reasons for communication failures (Sfard, 2002).

In a real classroom situation the teacher observes the lived object of learning, that is, the way students see, understand, and make sense of the object of learning when the lesson ends and beyond. The students, as well as the researcher, observe the enacted object of learning, that is, what it is possible to learn in a situation from the point of view of what is meant to be learned. The enacted object of learning is the space of variation/learning constituted in the classroom. From the teacher's perspective, the object of learning is the same as the intended object of learning and it is somehow realized in the classroom in the form of a particular space of learning. This space of learning contains any number of dimensions of variation and denotes the aspects of a situation, or the phenomena embedded in that situation. From the student's perspective, the forming of the enacted object of learning should open up new ways of making sense of the presented concepts. This may be reflected in the way in which the student works with the exercises and the questions that the student asks the teacher. The notion of concept is defined by Säljö (1999, p. 81) as "repositories of human sense-making capacities and activities, they are sediments of human experiences and simultaneously tools for action."

By using the discourse defined by Sfard (2002), the teacher must, as an observer, be able to identify how the students' discourse changes. If the teacher deems that the change in the student's discourse is developing in an undesired direction, he or she must be able to present the enacted object of learning so that it can open up new possibilities to apply different ways of making sense of the discourse to the student. Varying the presentation of concepts in algebra on the basis of the three historical phases of the evolution of algebra, by changing the discourse and focusing on an effective communication, could create such new possibilities.

Concluding remarks

Even though the ontological or epistemological assumptions of the presented perspectives may differ in some ways, these perspectives can, from our point of view, be applied to understand the connection between teaching and learning algebra that occurs in the classroom. Ottosson's (2000) suggestion to change the term "ways of experiencing" to "ways of making sense" opens the possibility to analyse teaching and learning processes on both the collective and the individual level. The theoretical approach of variations may be used to empirically identify different ways in which the symbolic algebra is enacted in the classroom. In this way we can better understand the difficulties that the students encounter in the reification processes. Sfard's focal analysis and the reification theory can be applied to analyse the effectiveness of the communication between teacher and student in terms of reification process, that is, to study how the discourse focus levels the algebra from an operational to a structural level.

For teachers, to sustain a mathematical discourse of algebra in the classroom, the domain of algebra must be reflected in ways that allow students to make sense of the object of learning, see underlying patterns, and develop their own discourse.

References

Bednarz, N., Radford, L., Janvier, B. & Lepage, A. (1992). Arithmetic and algebraic thinking in problem-solving. In W. Geeslin & Graham (Eds.), *Proceedings of the 16th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 65-72). Durham, New Hampshire: Program Committee.

Bergqvist, K. & Säljö, R. (1995, August). *Social languages in classroom discourse-Identifying and appropriating knowledge.* Paper presented at the 6th European Association for Research on Learning and Instruction conference. Nijmegen, the Netherlands.

Bishop, A.J., Clements, J., Keitel, C., Kilpatrick, J., & Laborde, C. (Eds.) (1996). *International handbook of mathematics education*. Dordrecht, The Netherlands: Kluwer.

Burton, D.M. (1991). *The history of mathematics*. Boston: Allyn & Bacon.

Eves, H. (1990). *An introduction to the history of mathematics.* Philadelphia: Saunders.

Edwarts, D. (1993). Concepts, memory, and the organization of pedagogic discourse: A case study. *Educational Research*, *19*, 205-225.

Herscovics, N. (1989). Cognitive obstacles encountered in the learning of algebra. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 60-86). Reston, Virginia: National Council of Teachers of Mathematics.

Kieran, C. (1989). The early learning of algebra. A structural perspective. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 33-56). Hillsdale, NJ: Lawrence Erlbaum.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, *12*, 317-326.

Kieran, C. (1992). The learning and teaching of school algebra. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York, NY: Macmillan.

Linchevski, L. & Herscovics, N. (1994). Cognitive obstacles in pre-algebra. In J. P da Ponte & J. F. Matos (Eds.). *Proceedings of the 18th conference of the International Group for the Psychology of Mathematics Education (vol. 3, pp. 176-183).* Lisbon, Portugal: Program Committee.

MacGregor, M. & Stacey, K. (1995). Backtracking, brackets, BOMDAS and BODMAS. *Australian Mathematics Teacher, 51* (3) 28 – 31.

MacGregor, M. & Stacey, K. (1998). Cognitive models underlying algebraic and non-algebraic solutions to unequal partition problems. *Mathematics Education Research Journal*, *10* (2) 46 – 60.

Marton. F. & Tsui. A. B. M. (2003, manuscript). Classroom Discourse and the Space of Learning.

Ottosson, T. (2000, mars). *Experiences or discourse? What is studied in a phenomeno-graphic study?* Paper presenterat vid 28e NFPF-kongressen, Kristiansand, Norge.

Richardson, V. (Ed.). (2001). *Handbook of research on teaching (4th ed.).*Washington, DC: American Educational Research Association.

Ryan, J. & Williams, J. (1998). The search for pattern: Student understanding of the table of values representation for function. In C. Kanes, M. Goos, & E. Warren (Eds.), *Teaching mathematics in new times. Proceedings of the 18th annual conference of the Mathematics Education Research Group of Australasia* (vol. 2, pp. 492-499). Brisbane: Mathematics Education Research Group of Australasia.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1-36.

Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics*, *14*(1), 44-55.

Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification - the case of algebra. *Educational Studies in Mathematics*, *26*, 191-228.

Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *Journal of Mathematical Behaviour, 14,* 15-39.

Sfard, A. (2000). Steering (dis)course between metaphor and rigor: Using focal analysis to investigate the emergence of mathematical objects. *Journal for Research in Mathematics Education, 31*(3), 296–327.

Sfard, A., & Kieran, C. (2001). Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, and Activity, 8*(1), 42–76.

Sfard, A. (2002). The interplay of intimations and implementations: Generating new discourse with new symbolic tools. *The Journal of Learning Sciences, 11*(2, 3), 319-357.

Säljö, R. (1997). Talk as data and practice: A critical look at phenomenographic inquiry and the appeal to experience. *Higher Education Research and Development*, *16*, 173–190.

Säljö, R. (1999). Concepts, cognition and discourse. From mental structures to discursive tools. Paper presented at the conference on Conceptual change, Universität Jena, September 1994. In M. Carretero, W. Schnotz, & S. Vosniadou (Eds.), *New perspectives on conceptual change* (pp. 81-90). Oxford: Elsevier Science Ltd.

Wittrock, M.C. (Ed.). (1986). *Handbook of research on teaching (3rd ed.).* New York: Macmillan.

Artefacts – Instruments – Computers

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A definition of artefacts

At present, the international discussion on artefacts most often concentrates on one major type of artefact, 'artefact' is used to simply denote information and communication technology ('ICT'), sometimes it is even reduced to computers and (hopefully appropriate) software. I will try not to follow this reductionist approach - even if the discussion in (mathematics) education and didactics often takes 'artefact' in this restricted sense. In contrast to this, the paper starts from the broader definition given by Wartofsky: In the introduction to a book presenting his work, he defines 'artefact' as "…anything which human beings create by the transformation of nature and of themselves: thus also language, forms of social organisation and interaction, techniques of production, skills"; (see Wartofsky 1979, p. xiii). Taken as wide a definition as cited, artefacts related to teaching and learning mathematics can be the mathematical symbolism, ruler and compass, a probability distribution, the derivative or integral of a function, (scientific) journals, grades like "licentiate" or "phd-thesis", curricula to control the teaching of mathematics in classrooms, tests to grade students, but also institutions like Skolverket and Institutionen för matematik.

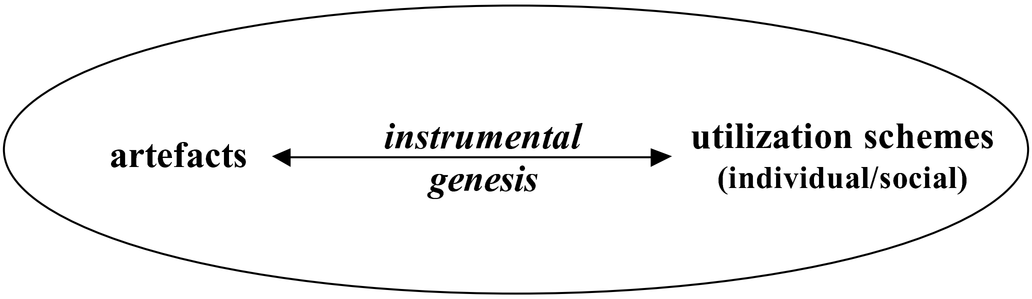
In his text, Wartofsky offers an additional classification of artefacts into three levels, which basically rests upon different uses of artefacts. He distinguishes 'primary' artefacts (those which are "directly used in the production") from 'secondary' artefacts ("used in the preservation and transmission of the acquired skills or modes of action or praxis by which this production is carried out") and 'tertiary' artefacts (those artefacts, which "constitute a domain in which there is a free construction in the imagination of rules and operations different from those adopted for ordinary 'this-wordly' praxis"; see Wartofsky, p. 202/209). Didactics (of mathematics) is mainly concerned with the analysis of 'secondary' artefacts, while mathematics (as a discipline) sometimes seems to produce 'tertiary' artefacts.

The artefact which is most used in Swedish schools is the 'secondary artefact' textbook (meant to be "… used in the preservation and transmission of the acquired skills or modes of action or praxis by which this production is carried out"). Present-day 'information and communication technology (ICT)' - like hand-held calculators, computers, software, internet - is another important example of artefacts, with the whole range of levels from primary artefacts like Computer Aided Design (CAD) to secondary artefacts like Dynamical Geometry

Software ("DGS") and tertiary artefacts like computer games. ICT is normally taken as the 'standard' example of artefacts, sometimes only ICT is discussed as 'artefacts'. To give some more recent examples of ICT-artefacts related to teaching and learning, one should mention learning in ICT-based networks, the internet and distributed learning communities.

From artefacts to instruments

The row of artefacts by Wartofsky (primary/secondary/tertiary) classifies according to different purposes of the tools. In line with this broad approach and from an ergonomic and didactical research question ("How do human beings learn to use machines?"), the French researcher Rabardel came up with an additional and helpful distinction: For an analysis of (learning with) artefacts, it is necessary to distinguish the artefact (the tool itself) from an 'instrument', " a mixed entity made up of both artefact-type components and schematic components that we call utilization schemes. This mixed entity is born of both the subject and the object" (Rabardel & Samurcay, 2001; Rabardel & Bourmand, 2003; see also graphics below).



The process, in which artefacts come to be used by human beings is called 'instrumental genesis'. It is normally starting with the development of 'utilization schemes' (ways to use the artefact) when new users begin to handle a new artefact (from the artefact to the user, this is called 'instrumentalisation' of the artefact), while 'instrumentation' is the process where persons develop new utilization schemes for given artefacts, which may not even been thought of by the creators of the artefact.

Using the mathematics textbook as an example of a widely used artefact in teaching and learning mathematics, reading a textbook problem, then solving it, then controlling the solution with the 'facit' at the end of the textbook is a good example of such an instrumentalisation, while reading a textbook problem, then deciding not to solve it, but looking up the answer in the 'facit' to impress the teacher might be an excellent example of an instrumentation of the textbook. Skolverket in its report "Lusten att lära" has offered detailed information on the normal utilisation schemes textbooks are used in Swedish classrooms (see Skolverket 2003, pp. 18f). The section on "Grundskolans senare år" gives the details: "Modellen utgörs av genomgång ibland, enskilt arbete i boken och diagnos, alternativt prov. Läraren går runt och hjälper eleverna individuellt"[1] (loc.cit., p. 20). The text not only shows the utmost importance of the textbook in Swedish classrooms, it also identifies the basic utilisation scheme of textbooks.

In line with the idea of secondary artefacts, a whole variety of instruments was and still is developed to facilitate teaching and learning mathematics. Artefacts together with utilization schemes are created to 'represent' mathematics, to study these representations (instruments) and to learn about mathematics.

Information and communication technology "ICT"

More recently, a new type of 'artefact' appeared: computers and (sometimes appropriate) software. This new type of artefact can be taken as a dynamical add-on or a real alternative to textbooks – it is the type of tool most widely discussed and researched in the didactics of mathematics. If one does not only look into the features, which are offered by the computer and the software, but also in the way, this tool is used, this entity fits into the concept of 'instrument'. To put it the other way round: For a didactical analysis, there is one general lesson to be learned immediately from our understanding of instruments: For a didactical analysis, one has to look into the software and its use in order to fully understand the instrument ICT.

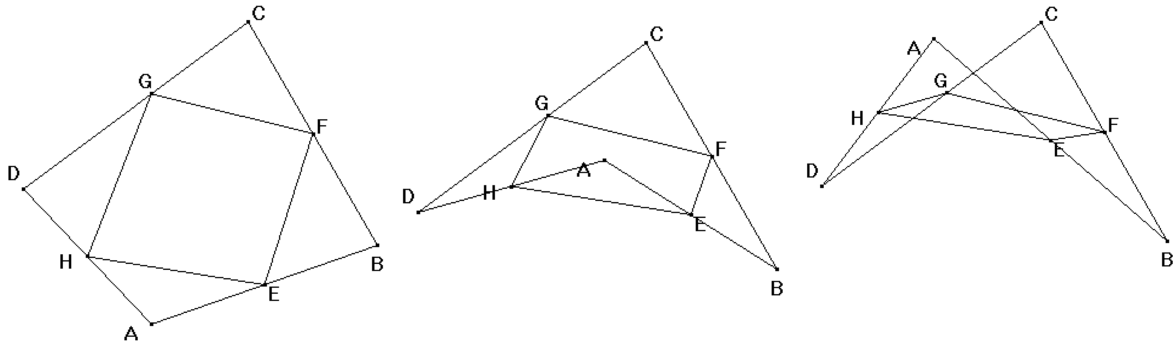
If I follow a recent phd-thesis (Samuelsson, 2003), in Sweden, 'övning'-software seems to be the predominant instrument to deliver existing knowledge, especially by using 'drill & practice' programs. Nowadays in Sweden, ICT seems not to be used innovatively. It is often simply used as a means to support an existing way of 'individualised' teaching and learning.

To illustrate my point on artefacts, I will now concentrate on Dynamical Geometry Software (often called DGS[2]). This type of software, these artefacts, can be characterised by three features, namely drag-mode, macros and locus of points. I will only look into the drag-mode (for research on macros see Kadunz, 2002; for research on locus of points see Jahn, 2002), so I first give a standard example for this DGS-feature, using a well-known geometrical statement: If you construct the midpoints E, F, G and H of the sides of a quadrilateral ABCD and join them to an "inner" quadrilateral EFGH (the 'Varignon'-quadrilateral), you will see that this inner quadrilateral is a parallelogram. Is this correct for every quadrilateral?

[1] "This teaching model consists of sometimes a presentation, then individual work with the textbook, and a diagnosis or a test. The teacher is helping the students individually." (editors' translation).

[2] For examples see 'Cabri-géomètre' at http://www.cabri.com/ or 'Geometer's Sketchpad' at http://www.keypress.com/catalog/products/software/Prod_GSP.html

Dragging the point A in a DGS, you soon get the idea, that the Varignon-quadrilateral is always a parallelogram, even if ABCD is not convex, even if two opposite sides of it intersect (see drawings further down).



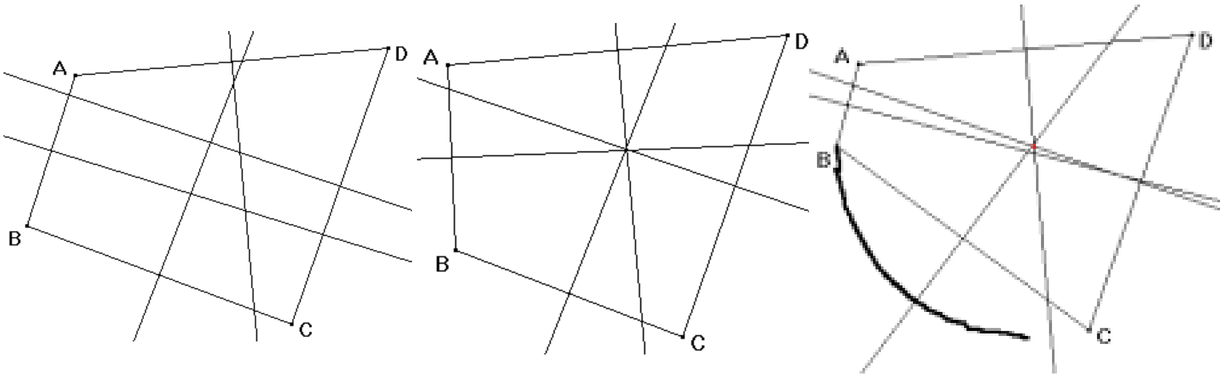
I will not add a proof of this geometrical statement, but hope to have illustrated the drag-mode as the most salient feature of DGS: the drag-mode preserves the geometrical relations used in the construction (here: E, F, G, and H being midpoints of the respective segments of quadrilateral ABCD) – even when an initial point of the construction is changed by moving it around (here: moving point A, which remains an extremity of the quadrilateral). Research on teaching and learning geometry took the ‘drag-mode’ as a most helpful learning tool in geometry. It is often even taken as the characteristic feature of Dynamical Geometry Software (DGS), as a characteristic feature of the artefact DGS.

Arzarello and his colleagues in a detailed analysis presented a whole classification of different utilization schemes of this software feature (see Arzarello, 2002, p. 67). They identify six utilisation schemes (“wandering dragging”, “guided dragging”, “dummy locus dragging”, “line dragging”, “linked dragging” and the “dragging test”), give descriptions (for examples see below) and illustrate often used ‘utilization schemes’ of DGS by a detailed analysis of tasks.

The following task and its solution was presented by the same group and analysed in a different publication (Arzarello et al., 1998).

> *Task:*
> *a) Draw a quadrilateral ABCD and the mid-perpendiculars of its sides!*
> *b) Try to find out, if it is possible to make the four mid-perpendiculars meet in one point!*
> *c) If this is possible, find out when and why!*

Task a) would be easily solved by the drawing on the left above. Dragging of point B soon shows that task b) has a positive answer – and the drawing on the right (using the ‘trace’ feature of the software) indicates, how point B has to be dragged to preserve the uniqueness of the intersections.



Arzarello and colleagues describe this prototypical solution process in more detail, showing us typical utilization schemes of DGS: At first, the students use “wandering dragging” to find out about an answer to the task b). They are “moving the basic points on the screen randomly, without a plan, in order to discover interesting configurations or regularities in the drawings” (Arzarello et al., 2002, p. 67). This is a ‘utilisation scheme’, which from the very start of DGS was mentioned by the software designers of DGS. It was the basic pedagogical legitimisation for the use of DGS in teaching and learning geometry – and it can be illustrated by the move from the first to the second drawing above.

The move from the second to the third drawing has a different quality: Here, the users develop an innovative ‘utilization scheme’, which the software designers never had planned: They are “moving a basic point so that the drawing keeps a discovered property; the point which is moved follows a path, even if the users do not realise this: the locus is not visible and does not 'speak' to the students, who do not always realise that they are dragging along a locus”, they are using ‘dummy locus dragging’ (for the citation and the name ‘dummy locus dragging’ see again Arzarello et al., 2002, p. 67).

Following Arzarello et al., the users ‘normally’ seem to add a third ‘utilization scheme’ called “dragging test” (“moving dragable … points in order to see whether the drawing keeps the initial properties. If so, then the figure passes the test; if not, the drawing was not constructed according to the geometric properties you wanted it to have”; description loc.cit.). This often is preceded by “linked dragging”, where the dragging point is linked to an object, which is thought to conserve the wanted quality (in our example: linking the point B to an appropriate circle).

In the text of Arzarello et al., the different utilisation schemes, the different drag modes seem often to be applied in a clear order: wandering dragging is followed by dummy locus dragging, which hopefully leads to a dragging test. With this standard chain of utilisation schemata, we see a more general utilization scheme for the exploration and argumentation emerge (wandering dragging – dummy locus dragging – dragging test), which can be helpful in construction and proof tasks in geometry when Dynamical Geometry Software (DGS) is available. The above-mentioned texts (Jahn 2002 and Kadunz 2002) can also be read as research on (possible) utilisation schemes of macros and locus of points when Dynamical Geometry Software is available – thus showing how helpful the idea of studying 'instruments' is. Heavily using the 'instrument' concept, Artigue (2002) describes research on Computer Algebra Systems (CAS) and gives details on utilisation schemes observed in French classrooms using CAS.

Conclusion

What has been presented can be condensed into two basic messages:

- From the concept of 'instrument', we can learn that – for research into information and communication technology – it is not enough to study the features of the tool. Only research into the ways the tool is used, a study of the 'utilization schemes' will fully inform the didactician. The cry for user studies often heard when new software for computer use is introduced somehow simpler represents the same need. To state this more general: researching instruments for teaching and learning mathematics – like information technology or textbooks – should include the identification and study of utilisation schemes to fully understand the role of the instrument under study.
- For an individual piece of software or a generic type of software, such user studies will bring to light utilisation schemes which in general have not been foreseen by the software developer. The 'instrumental genesis' will create innovative ways to use the tool (computer and software), which can be very helpful in teaching and learning mathematics. The utilisation schemes found by empirical studies can enrich the picture, which the didactician, the researcher, has of the process of teaching and learning mathematics.

References

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal for Computers in Mathematical Learning (IJCML), 7*(3), 245-274.

Arzarello, F., Micheletti, C., Olivero, F., Robutti, O., & Paola, D. (1998). Dragging in Cabri and modalities of transition from conjectures to proofs in geometry. *Proceedings of the Psychology of Mathematics Education (PME-22) conference*, Stellenbosch, South Africa, vol. 2, 32-39.

Arzarello, F., Olivero, F. et al. (2002). A cognitive analysis of dragging practises in Cabri environments. *Zentralblatt für Didaktik der Mathematik, 34*(3), 66-72.

Jahn, A. P. (2002). "Locus" and "Trace" in Cabri-géomètre: relationships between geometric and functional aspects in a study of transformations. *ZDM, 34*(3), 78-84.

Kadunz, G. (2002). Macros and modules in geometry. *ZDM*, *34*(3), 73-77.

Laborde, C. (2001). Integration of technology in the design of geometry tasks with Cabri-Geometry. *IJCML, 6*(3), 283-317.

Rabardel, P. (1995). *Les hommes et la technologie. Approche cognitive des instruments contemporains*. Paris: Armand Colin.

Rabardel, P. & Samurçay, R. (2001). *From artifact to instrument-mediated learning*. New challenges to research on learning (International symposium organized by the Center for Activity Theory and Developmental Work Research, University of Helsinki – also available via internet).

Rabardel, P. & G. Bourmand (2003). From computer to instrument system: a developmental perspective. *Interaction with computers: the Interdisciplinary Journal of Human-Computer Interaction, 15*(5), 665-691.

Samuelsson, J. (2003). *Nytt, på nytt sätt? En studie över datorn som förändringsagent av matematikundervisningens villkor, metoder och resultat i skolår 7-9*. Uppsala: Uppsala University.

Skolverket (2003). *Lusten att lära - med fokus på matematik*. Nationella kvalitetsgranskningar 2001-2002. Stockholm: Skolverket.

Sträßer, R. (2001). Cabri-géomètre: Does a Dynamic Geometry Software (DGS) change geometry and its teaching and learning? *IJCML, 6*(3), 319-333.

Wartofsky, M. W. (1979). *Models. Representation and the scientific understanding*. Dordrecht: Reidel Publishing Company.

What Mathematical Ideas Do Pupils and Teachers Use when Solving a Rich Problem?

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This report is a part of a bigger study where pupils and teachers are working with rich problems. In this article, built on a social constructivist theory of learning, we will present what mathematical ideas pupils and teachers use when solving a *mathematically rich problem*. We will also show how teachers influence their pupils with their own mathematical ideas and how teachers follow up their pupils' ideas. The data consists of videotaped lessons and interviews, and of pupils' solutions. As a result from our study, we have found that

- The teachers' own mathematical ideas and solution methods direct their pupils
- The teachers sometimes have difficulties producing feedback building on their pupils' solutions.
- The teachers seldom make generalisations out of their pupils' solutions.

Background of the project

Social constructivism

The thoughts and expressed in this paper are, to a great extent, based on social constructivism, which has characterised the latest Swedish syllabi (see Hedrén, 2000). Especially, the theory that Jaworski (1994) creates by the help of grounded theory (the teaching triad) points to the teacher's three fields of responsibility in creating a mathematical classroom: the management of learning, the sensitivity to pupils and the mathematical challenge.

Problem solving

In the latest Swedish syllabus for mathematics in the compulsory school (Skolverket, 2000), it is written among other things: "To make the pupil able to practice mathematics, a balance between creative, problem solving activities and knowledge about mathematical concepts, methods, and forms of representation are called for." (Ibid. p.2, our translation.) In both the national and international research pupils' strategies in mathematical problem solving, and their ability to cope with mathematical problems, have been a rich field of research for quite a long time. In our opinion, however, the mathematical ideas that teachers and pupils are working with in problem solving activities and the occasions when the pupils seem to learn, have been much less researched.

We have used other researchers' ideas and our own experiences to define some criteria for a problem to be, in our opinion, classified as a rich problem. These criteria can be found e. g. in Taflin (2003).

Mathematical idea

The mathematical idea in a problem as an essential factor in problem solving is stressed both by Schoenfeld (1991) and by Silver & Cai (1996). Schoenfeld discusses resources as one of four important parts of knowledge for success in problem solving. As resources he assigns intuitive and informal knowledge in the actual field, facts, algorithmic procedures, routine procedures that are no algorithms, and understanding of special rules in force in the problem domain. We see these as examples of essential mathematical ideas.

Aim and questions

In this study we will try to describe some of the mathematical ideas that four teachers and their pupils are working with and giving expression to in their work with a rich problem. Our questions are:

- What mathematical ideas do teachers and pupils use when working with a mathematically rich problem?
- How do teachers' mathematical ideas influence their pupils?
- How do teachers follow up their pupils' mathematical ideas?

Method

This study is a part of a project called RIMA. We are following four classes from two schools and their four teachers during the school years 7, 8, and 9. During these three years, the pupils work with ten rich problems in all. The problems have been chosen in co-operation with the participating teachers. The teachers have been informed about the concept rich problems and are acquainted with our criteria for these problems. Researchers and teachers meet before and after each problem solving session to exchange experiences and observations made during the problem solving process and to discuss the next problem. The teachers independently design their teaching with the rich problems. On all occasions, they have chosen to use at least one lesson for the problem.

Methods to gather data

The teachers were interviewed before and after the lesson. The problem solving session was video- and audiotape recorded. The teachers carried a tape-recorder with a small microphone during the lesson. The pupils were interviewed individually in one school and in groups in the other school. These interviews were video- and audiotape recorded. The pupils' solutions were collected and examined in both schools. The problem tiles are shown below.

You lay a pattern with the help of quadratic tiles, dark and pale.
The pattern looks like this:

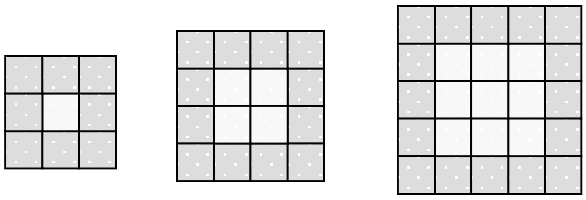


Figure 1 Figure 2 Figure 3

a) How many tiles are there in figure 5?
   How many of them are pale and how many dark?
b) How many dark and pale tiles respectively are there in figure 15?
c) How many dark and pale tiles respectively are there in figure 100?
d) How many dark and pale tiles respectively are there in figure *n*?
e) Make a similar problem. Solve it.
   (The problem has been shortened a little here.)

The problem is a pattern problem, a type of problems leading to an inductive reasoning, an attempt to generalise (often expressed as a formula where *n* is included). The solution is based on observations that should lead to the discovery of mathematical patterns.

Result with comments

In this section, we account for pupils' solutions and conversations between the pupils and their teachers during the lesson, as well as interviews after the lesson when the pupils recalled how they have been working with the problem together with their peers and their teachers.

Short account of the lessons

In the introduction of the lesson, the teachers introduced the problem to their pupils. Three of the four teachers told their pupils what mathematical ideas they should work with. They also gave clues to strategies that their pupils could use to solve the problem. The fourth teacher did not discuss what mathematics her pupils could use or learn in any way, she did not give any clues in this phase.

- Teacher 1 placed *n* and *x* on a par and gave her pupils the clue to think how they have been taught to think, when there is an *x* in their books.
  Teacher 2 told his pupils to look for patterns. He gave his pupils the clue to examine how the figures developed, figure by figure, and try to find a pattern that they could use.

- Teacher 3 connected to the problem in algebra, which his pupils had been engaged in, and mentioned that it could be useful when they solved the problem.
- Teacher 4 gave no information about the included mathematics and no clues.

During the problem solving process the pupils struggled with the problem and solved it fully or partly, and discussed it with each other in small groups or with their teacher. All teachers gave individual guidance, and when they worked with their pupils they made the suggestion to make a table. On some occasions, the teacher used the blackboard to give information to all pupils at the same time.

The lessons sometimes finished with an account of pupils' solutions at the blackboard, in some cases with the teacher's comments.

Mathematical ideas

In this study we have started from mathematical ideas as procedures and concepts in different mathematical fields and sorted the pupils' solutions with this in mind.

Count Squares

Example 1

Pupil	I cut out such squares that we counted afterwards … These are great!
Teacher	Well, you can count afterwards then. How did you do it then? … Which one are you working on?

Example 2

Pupil	We need a bigger paper if we'll be able to paint figure 100.
Teacher	Will you paint it?
Pupil	Yes, we painted figure 15, didn't we?

Comments: In example 1, the pupils have drawn the figures and cut out all squares and counted them afterwards. In example 2 they have found a method (to paint) which works and which they want to repeat. The pupils do not account for a solution to the problem.

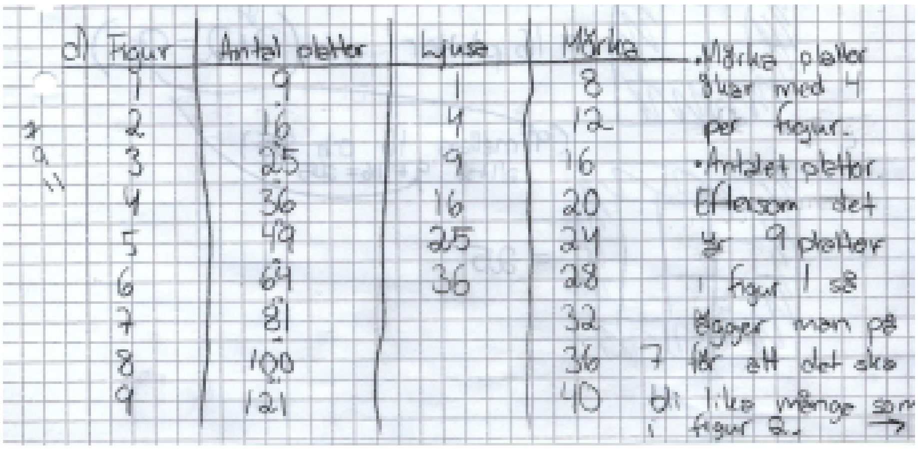
Recursion

Example 3

Pupil	Then it increases on the white ones, and then the black ones you add four each time.
Teacher	From what you had before?
Pupil	Yes.
Teacher	So to solve figure fifteen you must know the result of figure fourteen?

Pupil	I don't know, we were counting, weren't we!
Teacher	But if you add four all the time, it means to be able to calculate to fifteen you have to know what fourteen is because you have to know where you add four to, don't you? Thus to calculate figure one hundred you need to know what is the calculation to ninetynine. Then you have to calculate all of the time to ninetynine.
Pupil	That’s the only way I know.

Example 4



d) Figur	Antal plattor	Ljusa	Mörka
1	9	1	8
2	16	4	12
3	25	9	16
4	36	16	20
5	49	25	24
6	64	36	28
7	81		32
8	100		36
9	121		40

•Mörka plattor ökar med 4 per figur.
•Antalet plattor Eftersom det är 9 plattor i figur 1 så lägger man på 7 för att det ska bli lika många som i figur 2.

(Text in the figure: * Dark tiles increase with 4 per figure. * The number of tiles. Because there are 9 tiles in figure 1 you add 7 to get as many as in figure 2.)

Comments: In example 3, the pupil counted ahead figure by figure and had to know the number of tiles in the preceding figure to find out how many there are in the next one. In example 4, the pupil has made a complete table and shows the connection between the numbers.

Mathematical patterns found with the help of area

Here we found solutions building on a picture, visualisations where the pupils form associations with the concept of area. Some pupils have done this by first calculating the number of pale tiles, the inner square. They have then calculated the number of dark tiles as a frame around the inner square. In their computations, they have often used multiplication. It has been done in different ways:

a) They have calculated tile by tile without partitioning the sides.
b) They have calculated four times the side of the inner square and then added the four tiles in the corners.
c) They have calculated four times one tile more than the side of the inner square.
d) They have calculated two sides as long as the side of the inner square and two sides as two tiles longer than the side of the inner square.

e) They have calculated one side two tiles longer than the side of the inner square, two sides one tile longer than the side of the inner square and one side as long as the side of the inner square.

Example 5

Pupil	Dark ones you can take … if it is figure 100 you add 2 on the upper edge and 101 plus 101 plus 100, 102 plus 101 plus 101 plus 100.

Comments: The pupils have used two main types of area computation to find out the number of dark tiles. Either they have seen that the dark tiles make up a frame around the pale ones, or they have calculated the number of dark tiles as the difference between the total amount of tiles and the pale ones. They have used the operations addition and multiplication.

Mathematical patterns found with the help of a table

During the problem solving phase, all the teachers gave their pupils the hint to make a table.

Example 6

Pupil 1	Then she [the teacher] made up such a table, at the blackboard that we should draw, fill in and continue.
Interviewer	Mm. Did it help you in some way?
All three	Yes.
Pupil 2	It did.
Interviewer	Mm. You had not made a table yourselves?
Pupil 2	[Shakes her head]

Algebraic Expressions with *x* and *n*

What n might stand for caused many pupils a lot of trouble. Many pupils tried to find a formula for the pale tiles, $n \times n$. They could also approach a formula for the total amount of tiles, $(n + 2) \times (n + 2)$.

Example 7

Pupil	Thus, n as an equation.
Teacher	Or as an algebraic expression. That is to say that you can express it in words, too, if you ask me to take the digit of the figure, is it that one.
Pupil	Can you write the number as n then?
Teacher	That's what I mean, because it's the digit of the figure.
Pupil	Then you can write an equation like this, you know, $n - 2$.
Teacher	But it was not – *2*, was it?
Pupil	$n + 2$ …

Example 8

Teacher	No, you have a formula that you can calculate mainly what figure you want to. Then you exchange the digit for *n*. Instead of calculating the 58th figure, you calculate the nth figure.
Pupil	The *n*th?
Teacher	The *n*th figure, the *n*th figure is mathematical and it means any figure.
Pupil	Then we shall only use that one?
Teacher	No, you write *n*. Then mathematicians know that then they can use this formula for any figure they want to.
Pupil	Then you have to know a little.

Example 9

Pupil	What do you mean, *n*?
Teacher	Whichever, any figure.
Pupil	Then we use it, then we use 0!

Example 10

Pupil	Er, er, yes, but, thus, we started with to, to calculate like this, so to say, the pale ones first and call them *x*, and the dark ones were only as many as they were. /…/ The *x* stands for that the pale tiles, or, how many the pale tiles are.

Comments: In example 7, the pupil connects *n* with the concept of equation, but still he seems to think in general terms. In example 8 the teacher explains that the number is exchanged for *n* which means the *n*th figure and that *n* and formula go together. In example 9 the teacher lets the pupils choose *n* at will, and then they choose to put *n* equal to 0. In example 10, *x* is used as a mark, as a unit, $4x + 12$ means 4 pale and 12 dark tiles, which is a wrong theory, if you are searching for a general expression.

Discussion

What mathematical ideas do pupils make use of?

The mathematical concept that the pupils were mainly working with was pattern. During their search for connections between the amount of tiles and the number of the figure, the pupils made use of many different strategies.

The pupils that drew figures and cut as in examples 1 and 2, were working with a strategy that was not very successful. The pupils wanted to continue with their strategy to work out all figures. Because they had worked out 15, they would also work out 100. The strategy guided the work.

There are many different variants of how the pupils worked with recursion methods when they found out the number of tiles in all figures preceding the figure asked for. In example 3 the pupils conducted a line of argument, and in example 4 they used a table as a strategy. None of the cases accounted for leads up to a general expression. The pupil who had arranged a table found several connections, among others that the number of dark tiles increases with four for every increase in figure number.

Some pupils looked upon the connection between the total amount of tiles and the amount of pale tiles as areas and could then find the connection with the number of the figure. Other pupils saw the amount of pale tiles as area and the amount of tiles in the frame as a surrounding area. In this study, we found five different variants of the calculation of frame area. This strategy, to think geometrically to find an arithmetic connection, was a successful strategy for the pupils. Many pupils also found a general expression with this strategy.

In this study, the symbol *n* was unknown to many pupils, for instance in examples 7, 8, 9, and 10. In this case, the problem might function as an introduction to algebra and to different ways to use letters as symbols. The pupils can see how to use *n* and *x* in different ways.

In this problem, the pupils were mainly working with arithmetic, and some got on to algebraic expressions. The pupils that used the area concept to calculate the number of tiles also arrived at general expressions. Many pupils showed an instance of conceptual knowledge as well as of procedural knowledge, as Hiebert (1986) describes mathematical ideas. The pupils also showed if they could execute routine procedures and use special concepts like area, as described by Schoenfeld (1985). The problem turned out to hold a lot of different mathematical ideas that teachers and pupils could discover and work out.

What mathematical ideas do teachers use when working with a mathematically rich problem, and how do the teachers' ideas influence their pupils?

Already in the introduction of the lesson, the teachers gave clues that allowed their pupils into fixed solution strategies. It would have been more interesting to understand how their pupils would have worked if they had not been piloted. This behaviour cannot be characterised as mathematically challenging in the way Jaworski (1994) and Lester (1985) call attention to as the teacher’s task.

When the pupils were supposed to interpret and understand the problem, most of the teachers chose to give suggestions of how to solve the problem as well. The teachers' suggestions are repeated in the pupils' solutions when they arrange tables and calculate with recursion. According to Lester (1985), the teacher ought to answer their pupils' questions and only make sure that the pupils have understood what to do. We cannot judge if the pupils themselves had obtained the idea to make a table, if they had not been given that piece of advice from their teachers.

The pupils that solved the problem with the help of the area concept did not seem to have obtained the idea as a suggestion from their teachers.

All teachers suggested that their pupils should arrange a table, three of them did so in the introduction of the problem, and the fourth did so when walking around during the lesson. The pupils in example 6 told us that they obtained that idea from their teacher who drew the table on the blackboard. The pupils were helped in the right phase according to Lester's suggestion (1985). We can also hear how the pupils are constructing their own knowledge with the help of the language and their peers in accordance with the learning theories that social constructivism describes.

One teacher suggested how the pupils should think when calculating with x. For pupils working with equations, where x is a fixed number, it might cause confusion, when they see that it is about exchanging x for *different* integers, i. e. to look upon x as a variable. In their textbook, the introductory algebra with equations focuses on what the sign of equality stands for, and every x has a fixed value.

One teacher asked the pupils to think about algebra that they had just been working with. It might be a clue, but it might also mislead the pupils if the algebra tasks in the book only focus on working with letters as fixed integers.

One teacher turned his pupils on to thinking about patterns and finding connections. In our results, we found that pupils find patterns both with the help of tables and the area concept.

How do teachers follow up her/his pupils' mathematical ideas?

At the follow-up of her pupils' ideas, the teacher has a possibility to challenge her pupils mathematically as Jaworski describes it in her triad. In examples 1 and 2 the pupils show how they get stuck in slow and long-winded methods, which do not lead to general expressions. The teacher observes how her pupils are working but lets them continue.

In example 7, the teacher follows up her pupils' own thoughts, and the pupils themselves advance towards the general expressions. We conclude this dialogue as good examples of mathematical challenges, as described by Jaworski (1994).

References

Hedrén, R. (2000). *Social konstruktivism i elementär aritmetik.* Högskolan Dalarna. Kultur och lärande.

Hiebert, J. (Ed.) (1986). *Conceptual and procedural knowledge: The case of mathematics* (pp. 225-263). Hillsdale: Lawrence Erlbaum Associates.

Jaworski, B. (1994). *Investigating mathematics teaching: A constructivist enquiry*. London: The Falmer Press.

Lester, F. K. (1985) Methodological considerations in research on mathematical problem-solving instruction. In E. A. Silver, *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 41-69). Hillsdale: Lawrence Erlbaum.

Schoenfeld, A. H. (1991). What's all fuss about problem solving? *Zentralblatt für Didaktik der Mathematik, 23*(1), 4-8.

Schoenfeld, A. H. (1985). *Mathematical problem solving.* Orlando, FL: Academic Press.

Silver, E. & Cai, J. (1996). An analysis of arithmetic problem posing by middle school students. *Journal for Research in Mathematics Education, 27*(5), 521-539.

Skolverket (2000). *Grundskola. Kursplaner och betygskriterier. Matematik.* Stockholm: Skolverket. Available at http://www.skolverket.se

Taflin, E. (2003). Problemlösning och analys av rika matematiska problem. *Research reports in mathematics education, No 5*. Umeå: Umeå University.

Mathematatism and the Irrelevance of the Research Industry

A Postmodern LIB-free LAB-based Approach to Our Language of Prediction

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Mathematics education research increases together with the problems it studies. This irrelevance-paradox can be solved by using a postmodern sceptical LAB-research to weed out LIB-based mathematism coming from the library in order to reconstruct a LAB-based mathematics coming from the laboratory. Replacing indoctrination in modern set-based mathematism with education in Kronecker-Russell multiplicity-based mathematics turns out to be a genuine 'Cinderella-difference' making a difference in the classroom.

The irrelevance paradox

All over the world there seems to be a crisis in mathematics education:

> There are strong indications of increasing justification and enrolment problems concerning mathematics and physics education, as a rather international phenomenon. During recent years, reports of a significant decline in enrolments to tertiary level education involving mathematics and physics have appeared from many parts of the world, including many countries in Europe, the US, Australia, and Japan. Also at the primary and secondary school levels mathematics and physics in many countries now seem to be receiving less interest and motivation than before amongst many categories of pupils. (Jensen et al, 1998, p. 15)

In Japan Yukihiko Namikawa asks 'can college mathematics in Japan survive?'

> Actually the total education system in Japan is in crisis, and so is the case of mathematics education at universities. (..) we are facing a remarkable decline of mathematical knowledge and ability of fresh students. (..) In April 1994, we established a working group in the Mathematical Society of Japan to overcome this crisis. (..) So far we made several investigations to clarify the situations. The results were much more disastrous than imagined before start and still the problems are aggravating. (Namikawa in ICME9, 2000, p. 94)

In Denmark proposals have been made to remove pre-calculus as a compulsory subject: In their suggestions for a reform of the Danish upper secondary Preparation High School the teacher union writes that Danish must be strengthened to improve the student's ability to write and read; that English must be compulsory and so must a second foreign language; and that all students must have a basic competence in mathematics, but not all students need to take an exam in mathematics. Mogens Niss has formulated a ‘relevance paradox’

> The discrepancy between the objective social significance of mathematics and its subjective invisibility constitutes one form of what the author often calls the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics. (Niss in Biehler et al, 1994, p. 371).

The 10^{th} International Congress on Mathematical Education in 2004 shows that research in mathematics education has been going on for almost half a century. On this background I would like to supplement this ‘relevance paradox’ with an ‘irrelevance paradox’ or ‘inflation paradox’: ‘the output of mathematics education research increases together with the problems it studies - indicating that the research in mathematics education is irrelevant to mathematics education’.

A methodology:
Institutional scepticism, sceptical LIB-free LAB-research

To get an answer to the ‘irrelevance paradox’ we obviously have to use a counter-methodology. Historically research originated as bottom-up ’LAB-LIB research’ where the LIB-statements of the library are induced from and validated by reliable LAB-data from the laboratory. However the word-based ‘LIB-research’ has created a ‘LIB-LAB war’ or ‘science-war’ exemplified by ‘Sokal’s bluff’ or by the ‘number&word-paradox’: Placed between a ruler and a dictionary a thing can point to a number but not to a word, so a thing can falsify a number-statement in the laboratory but not a word-statement in the library; thus numbers are reliable LAB-data able to carry research, whereas words carry interpretations, which presented as research becomes seduction - to be lifted by the counter-seduction of sceptical LIB-free LAB-research replacing LIB-words with LAB-words being validated by being, not ‘truth’, but ‘Cinderella-differences’ making a difference. (Tarp, 2003)

The inflation in today’s LIB-research comes from library cells inhabited by persons with little or no practical classroom experience, which reminds of the production of scholastic scriptures in medieval monasteries. So a proper counter-methodology could be inspired by counter-scholasticism as e.g. the institutional scepticism of the enlightenment as it was implemented in its two democracies, the American in the form of pragmatism, symbolic interactionism and grounded theory, and the French in the form of post-structuralism and post-modernism. In America Blumer talks about practical experience, symbolic interactionism and research:

> I merely wish to reassert here that current designs of ‘proper’ research procedure do not encourage or provide for the development of firsthand acquaintance with the sphere of life under study. Moreover, the scholar who lacks that firsthand familiarity is highly unlikely to recognize that he is missing anything. Not being

> aware of the knowledge that would come from firsthand acquaintance, he does not know that he is missing that knowledge. (..) Respect the nature of the empirical world and organize a methodological stance to reflect that respect. This is what I think symbolic interactionism strives to do. (..) Sociological thought rarely recognizes or treats human societies as composed of individuals who have selves. Instead they assume human beings to be merely organisms with some kind of organization, responding to forces which play upon them. (Blumer, 1998, pp. 37-38, 60, 83)

America still has its first republic whereas France now has its fifth republic. The American settlers emigrated to avoid the feudal institutions of Europe and to install 'freedom under God'. So what Foucault calls 'pastoral power' was not present in America; but very much present both inside France and around it, and several revolutions had to be fought forcing the French republic to organise the state as a military camp where French philosophers has developed a special sensitivity towards any attempt to overthrow the democracy of 'la Republique'.

Thus the French institutional scepticism is quite different from the American by turning the question of representation upside down and focusing upon, not how outside structure installs internal concepts, but how internal concepts install outside structure; and how words can be used as counter-enlightenment to patronise and 'clientify' people by installing pastoral power.

Derrida calls the belief that words represent the world for 'logocentrism'. Lyotard defines modern as 'any science that legitimates itself with reference to a metadiscourse'; and postmodern as 'incredulity towards metanarratives' (Lyotard, 1984, pp. xxiii-xxiv). Foucault describes pastoral power:

> The modern Western state has integrated in a new political shape, an old power technique which originated in Christian institutions. We call this power technique the pastoral power. (..) It was no longer a question of leading people to their salvation in the next world, but rather ensuring it in this world. And in this context, the word salvation takes on different meanings: health, well-being (..) And this implies that power of pastoral type, which over centuries (..) had been linked to a defined religious institution, suddenly spread out into the whole social body; it found support in a multitude of institutions (..) those of the family, medicine, psychiatry, education, and employers. (Foucault in Dreyfus et al, 1982, pp. 213, 215)

In this way Foucault opens our eyes to the salvation promise of the generalised church: 'You are un-saved, un-educated, un-social, un-healthy! But do not fear, for we the saved, educated, social, healthy will cure you. All you have to do is: repent and come to our institution, i.e. the church, the school, the correction centre, the hospital, and do exactly what we tell you'.

So according to Foucault pastoral power comes from words installing an abnormality and a normalizing institution to cure this abnormality through new words installing a new abnormality etc. (Foucault 1995). Thus the pastoral word 'educate' installs the 'un-educated' to be 'cured' by the institution 'education'; failing its 'cure' it is 'cured' by the institution 'research' installing new 'scientific' words as 'competence' installing the 'in-competent' to be 'cured' by the institution 'competence development'; failing its 'cure' it is again being 'cured' by new 'research' installing new 'scientific' words etc.

Thus pastoral power is installed by a self-supporting top-down LIB-LAB-industry of research and education using self-created LAB-problems to invent new 'scientific' LIB-words that are exported to the LAB through master educated inspectors creating new problems funding new research etc.

To increase its productivity the LIB has replaced verb-based words as 'educate' with words that are not verb-based such as 'competence'. So where the 'clients' themselves knew when they were 'educating' themselves or others, they do not know when they are 'competencing' themselves or others, only the pastors know – in full accordance with the view of the inventor of pastoral power, Plato, arguing that the democracy of the sophists should be replaced by the autocracy of the 'philo-sopists' educated at Plato's academy.

By its distinction between words and numbers sceptical LIB-free LAB-research is inspired by the French postmodern scepticism by saying that 'postmodernism means institutional scepticism towards the pastoral power of words'; and by the ancient Greek sophists always distinguishing between necessity and choice, between natural and political correctness, between logos and nomos, according to the two prerequisites of democratic decisions: information and debate. Thus Plato's half brother the sophist Antifon writes:

> Correctness means not breaking any law in your own country. So the most advantageous way to be correct is to follow the correct laws in the presence of witnesses, and to follow nature's laws when alone. For the command of the law follows from arbitrariness, and the command of nature follows from necessity. The command of the law is only a decision without roots in nature, whereas the command of nature has grown from nature itself not depending on any decisions. (Antifon in Haastrup et al 1984: 82, my tranlation).

By transforming seduction back into interpretation scepticism transforms the library from a hall of fact to a hall of fiction to draw inspiration from, especially from the tales that have been validated by surviving through countless generations, the fairy tales. Hence the preferred interpretation genre in institutional scepticism is the fairytale. Grounded theory uses categorised LAB-data for axial 'fairytale-coding'. Sceptical LIB-free LAB-research looks into institutional LAB-texts to replace opponent LIB-words with proponent LAB-words found by discovering forgotten or unnoticed alternatives at different times and places inspired by the genealogy and archaeology of Foucault; and by inventing alternatives using sociological imagination inspired by Mills (1959).

The aim of sceptical LIB-free LAB-research is not to extend the existing seduction of the library, so no systematic reference to the existing 'research' literature takes place. The aim is to solve LAB-problems by searching for hidden Cinderella-differences in the LAB, i.e. by 1) identifying the pastoral LIB-word installing the problem 2) renaming the LIB-word to a LAB-word through discovery and imagination, 3) testing the LAB-word to see if it is a Cinderella-differences making a difference, and 4) publish the alternative so the problem can be decreased instead of increased.

Mathematics and mathematism

Mathematics education is an institution instituted to cure 'mathematical uneducated-ness'. Not being verb-based 'mathematics' is a LIB-word to be translated into a verb-based LAB-word by observing what goes on in the laboratory of mathematics education, the classroom. The first day of secondary school we witness a 'fraction test' as e.g.:

The teacher	*The students*
Welcome to secondary School! What is 1/2 + 2/3?	1/2 + 2/3 = (1+2)/(2+3) = 3/5
No. The correct answer is: 1/2 + 2/3 = 3/6 + 4/6 = 7/6	But 1/2 of 2 cokes + 2/3 of 3 cokes is 3/5 of 5 cokes! How can it be 7 cokes out of 6 cokes?
If you want to pass the exam then 1/2 + 2/3 = 7/6!	

Apparently we have a 'fraction-paradox' in the mathematics classroom:

Inside the classroom	20/100 = 20%	+ 10/100 = + 10%	= 30/100 = = 30%
Outside the classroom e.g. in the laboratory	20%	+ 10% or	= 32% in the case of compound interest = b% (10<b<20) in the case of the total average

20% of 300 + 10% of 300 = (20%+10%) of 300 = 30% of 300 since the common total 300 can be put outside a parenthesis. But the fraction-paradox shows that this is not always the case. So 20/100 = 20%, but no general rule says that 20%+10% = 30% or 20/100+10/100 = 30/100.

Since a part of mathematics cannot be validated outside the classroom we can distinguish between 'mathematics', which is a science that can be validated in the laboratory, and 'mathematism', which is a doctrine, an ideology, a scholasticism, that cannot be validated in the laboratory.

This gives a possible answer to the irrelevance paradox: What is disguised as ‘education in mathematics’ is really indoctrination in ‘mathematism’ teaching ‘killer-mathematics’ only existing inside classrooms, where it kills the relevance of mathematics.

As validation a killer-free LIB-free LAB-mathematics must be uncovered through a combination of concept archaeology and imagination and tested in the laboratory of learning, i.e. the classroom.

Fractions and sets - LIB-words or LAB-words?

In the laboratory we talk about ‘fractions of’ as e.g. 2/3 of 6. The textbook however talks about plain ‘fractions’ as e.g. 2/3. To see if this is a LIB-word or a LAB-word we look at its definition:

> The set Q of rational numbers is defined as a set of equivalence sets in a product set of two sets of [sets of equivalence sets in a product set of two sets of [sets of equivalence sets in a product set of two sets of [Peano-numbers]]]; such that the number (a,b) is equivalent to the number (c,d) if a*d = b*c, which makes e.g. (2,4) and (3,6) represent then same rational number. (See any textbook in modern mathematics, e.g. Griffith et al., 1970)

Since fractions are defined as examples of ‘sets’ the question is whether ‘set’ is a LIB-word or a LAB-word. To separate LIB-math from LAB-math we travel back in time in the mathematics laboratory. As to the prospects for the enlightenment eighteenth century, Morris Kline writes:

> The enormous seventeenth-century advances in algebra, analytic geometry, and the calculus; the heavy involvement of mathematics in science, which provided deep and intriguing problems; the excitement generated by Newton's astonishing successes in celestial mechanics; and the improvement in communications provided by the academies and journals all pointed to additional major developments and served to create immense exuberance about the future of mathematics. (..) The enthusiasm of the mathematicians was almost unbounded. They had glimpses of a promised land and were eager to push forward. They were, moreover, able to work in an atmosphere far more suitable for creation than at any time since 300 B.C. Classical Greek geometry had not only imposed restrictions on the domain of mathematics but had impressed a level of rigor for acceptable mathematics that hampered creativity. Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights. (Kline, 1972, pp. 398-399)

So the enormous creativity in seventeenth-century mathematics was a result of neglecting the LIB-restrictions of classical Greek geometry by practising ‘a complete disregard of logical scruples’ and instead being inspired by the laboratory’s ‘physical insight’ and ‘confidence of intuition’.

> If the seventeenth century has correctly been called the century of genius, then the eighteenth may be called the century of the ingenious. Though both centuries were prolific, the eighteenth-century men, without introducing any concept as original and as fundamental as the calculus, but by exercising virtuosity in technique, exploited and advanced the power of the calculus to produce what are now major branches (..) Far more than in any other century the mathematical work of the eighteenth was directly inspired by physical problems. In fact one could say that the goal of the work was not mathematics, but rather the solution of physical problems. (..) The physical meaning of the mathematics guided the mathematical steps and often supplied partial arguments to fill in nonmathematical steps. The reasoning was in essence no different from a proof of a theorem of geometry, wherein some facts entirely obvious in the figure are used even though no axiom or theorem supports them. Finally, the physical correctness of the conclusions gave assurance that the mathematics must be correct. (..) Lagrange wrote to d'Alembert on September 21, 1781, ‘It appears to me also that the mine [of mathematics] is already very deep and that unless one discovers new veins it will be necessary sooner or later to abandon it. Physics and chemistry now offer the most brilliant riches and easier exploitation; also our century's taste appears to be entirely in this direction and it is not impossible that the chairs of geometry in the Academy will one day become what the chairs of Arabic presently are in the universities’. (..)This fear was expressed even as early as 1754 by Diderot in Thoughts on the Interpretation of Nature: ‘ I dare say that in less than a century we shall not have three great geometers [mathematicians] left in Europe. This science will very soon come to a standstill (..) We shall not go beyond this point.’ (Ibid., pp. 614, 616, 617, 623)

The seventeenth century saw the arrival of the last form of calculations, calculus, and the eighteenth century developed the many LAB-applications of calculus within physics. Only little new mathematics was added; and around 1800 mathematicians felt that there was no more mathematics to develop as expresses by Diderot. However LIB-mathematics soon came back. In spite of the fact that calculus and its applications had been developed without it logical scruples now were reintroduced arguing that both calculus and the real numbers needed a rigorous foundation. These LIB-scruples lead to the introduction of ‘set’. So as numbers were introduced to distinguish between different degrees of multiplicity having 1 as its unit, sets were introduced to distinguish between different degrees of infinity having the natural numbers as a unit. However changing infinity from a quality to a quantity involves the question of actual and potential infinity:

> The central difficulty in the theory of sets is the very concept of an infinite set. Such sets had naturally come to the attention of mathematicians and philosophers from Greek times onward, and their very nature and seemingly contradictory

> properties had thwarted any progress in understanding them. Zeno's paradoxes are perhaps the first indication of the difficulties. Neither the infinite divisibility of the straight line nor the line as an infinite set of discrete points seemed to permit rational conclusions about motion. Aristotle considered infinite sets, such as the set of whole numbers, and denied the existence of an infinite set of objects as a fixed entity. For him, sets could be only potentially infinite. (..) Cauchy, like others before him, denied the existence of infinite sets because the fact that a part can be put into one-to-one correspondence with the whole seemed contradictory to him. The polemics on the various problems involving sets were endless (Ibid., pp. 992-993)

Kronecker objected to set theory and Russell objected to talking about sets of sets:

> A radically different approach to mathematics has been undertaken by a group of mathematicians called intuitionists. As in the case of logicism, the intuitionist philosophy was inaugurated during the late nineteenth century when the rigorization of the number system and geometry was a major activity. The discovery of the paradoxes stimulated its further development. The first intuitionist was Kronecker, who expressed his views in the 1870s and 80s. To Kronecker, Weierstrass's rigor involved unacceptable concepts, and Cantor's work on transfinite numbers and set theory was not mathematics but mysticism. Kronecker was willing to accept the whole numbers because these are clear to the intuition. These 'were the work of God.' All else was the work of man and suspect. (..) after the paradoxes were discovered, intuitionism were revived and became a widespread and serious movement. The next strong advocate was Poincaré. (..) He agreed with Russell that the source of the paradoxes was the definition of collections of sets that included the object itself. Thus the set A of all set contains A. But A cannot be defined until each member of A is defined, and if A is one member the definition is circular. (..) This idea that the whole numbers derive from the intuition of time has been maintained by Kant, William R. Hamilton in his 'algebra as a Science of Time,' and the philosopher Arthur Schopenhauer. (Ibid., pp. 1197-1200).

As to the paradoxes in set-theory even Cantor saw problems asking Dedekind in 1899 whether the set of all cardinal numbers is itself a set; because if it is, it would have a cardinal number larger than any other cardinal (1003). Another paradox is the Russell paradox showing that self-reference leads to contradiction, as in the classical liar-paradox 'this sentence is false', when talking about sets of sets as e.g. the set M of all sets that are not a member of themselves:

If $\quad M = \{A: A \notin A)\} \quad$ then $\quad M \in M \Leftrightarrow M \notin M.$

Russell solves this paradox by introducing a type-theory stating that a given type can only be a member of (i.e. described by) types from a higher level. Since fractions are defined as sets of sets of numbers they cannot be considered numbers themselves making the addition ‘2+3/4’ meaningless. Not wanting a fraction-problem modern LIB-mathematics has chosen to neglect Russell’s type-theory until computer language, needing to avoid syntax errors, has brought a renaissance to Russell’s type-theory.

To avoid the type-theory Zermelo and Fraenkel invented an axiom system making self-reference legal by not distinguishing between an element of a set and the set itself, which removes the distinction between examples and abstractions and between different abstraction levels thus hiding that mathematics historically developed through layers of abstractions; and hiding the difference between an object and its predicate or interpretation means subscribing to the logocentrism criticised by poststructuralist thinking and by the number&word-paradox.

So ‘set’ is a LIB-word derived from axioms and not abstracted from the LAB. Since the definitions of modern mathematics are based upon the concepts set, this ‘LIB-virus’ makes all definitions LIB-words different from the LAB-words of the historical LAB-definitions. Thus we can name modern LIB-based mathematics ‘meta-matics’ to distinguish it from historical LAB-based ‘mathe-matics’.

The difference between LIB-based meta-matics, LIB-MATH, and LAB-based mathe-matics, LAB-MATH, can be seen in the word ‘function’ defined by modern meta-matics as ‘an example of a set of ordered pairs where first-component identity implies second-component identity’; and defined by Euler in 1748 as a common name for calculations with a variable quantity:

> A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities. (Euler, 1988, p. 3)

Bringing LAB-based mathematics to a LIB-based academy

A LAB-based mathematics should respect two fundamental principles: A Kronecker-principle saying that only the natural numbers can be taken for granted. And a Russell-principle saying that we cannot use self-reference and talk about sets of sets. The appendix shows an example of a Kronecker-Russell mathematics based on the LAB-words ‘repetition in time’ and ‘multiplicity in space’ creating a LIB-free, set-free, fraction-free and function-free ‘Count &Add-laboratory’ where addition predicts counting-results making mathematics our language of prediction.[i]

This multiplicity-based mathematics makes a difference in the Danish pre-calculus classroom (Tarp, 2003), in teacher education in Eastern Europe (Zybartas et al., 2001) and in Africa (Tarp, 2002). Thus the irrelevance paradox can be solved if set-based mathematism is replaced by multiplicity-based mathematics. But as a pastoral power LIB-based research is interested in, not solving, but guarding the fundraising irrelevance paradox by continuing to research the indoctrination of mathematism instead of researching the education of mathematics.

To test this hypothesis I applied for a job at a LIB-based academy. The verdict of the committee[ii] shows that challenging LIB-based meta-matics with LAB-based mathematics is not considered an asset; you are only admitted to a LIB-based academy if you are loyal to its interpretation and willing and able to expand it even if it is seduction and irrelevant to the field it studies. Hence to solve the irrelevance paradox an alternative sceptical LAB-based academy has to be installed.

The MATHeCADEMY and PYRAMIDeDUCATION

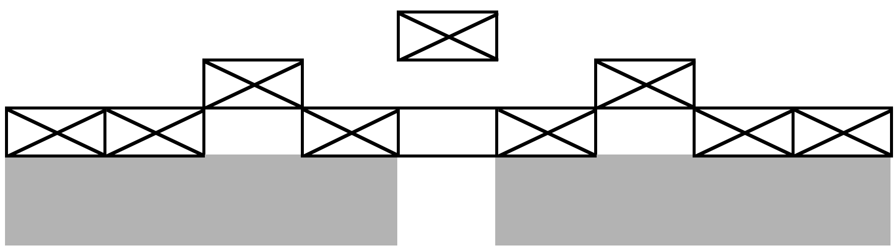
MATHeCADEMY.net is an example of an alternative sceptical LAB-based academy building on the sophist distinction between choice and necessity; and solving the irrelevance paradox by introducing a count&add laboratory posing the educational questions: 'How to count and predict multiplicity in bundles and stacks? How to unite stacks and per-numbers?'; thus respecting that 're-uniting' is the original meaning of the Arabic word 'algebra'.

At the MATHeCADEMY Primary school mathematics is learned through educational sentence-free meetings with the sentence-subject developing tacit competences and individual sentences coming from abstractions and validations in the laboratory, i.e. through automatic 'grasp-to-grasp' learning.

Secondary school mathematics is learned through educational sentence-loaded fairy tales abstracted from and validated in the laboratory, i.e. through automatic 'gossip-learning'.

In PYRAMIDeDUCATION 8 student teachers are organised in 2 teams of 4 students choosing 3 pairs and 2 instructors by turn. The coach coaches the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In each pair each student corrects the other student's routine-assignment. Each pair is the opponent on the essay of another pair. Each student pays for the education by coaching a new group of 8 students.

1 coach
2 instructors
3 pairs
8 students in
2 teams



In this way multiplicity-based mathematics will multiply as a self-reproducing virus on the Internet, until it can surface in ten years when half of the mathematics teachers have retired unable to reproduce by failing to make set-based mathematism relevant to the mathematics students.

References

Biehler, R., Scholz, R. W., Strässer, R. & Winkelmann, B. (Eds.) (1994). *Didactics of mathematics as a scientific discipline*. Dordrecht: Kluwer Academic Press.

Blumer, H. (1998). *Symbolic interactionism.* Berkely, Ca.: University of California Press.

Dreyfus, H. L. & Rabinow, P. (1982). *Michel Foucault, beyond structuralism and hermeneutics*, 2. ed. Chicago: University of Chicago Press.

Euler, L. (1988). *Introduction to analysis of the infinite*. New York: Springer Verlag.

Foucault, M. (1995). *Discipline & Punish*. New York: Vintage Books.

Griffiths, H. B. & Hilton, J. P. (1970). *A comprehensive textbook of classical mathematics.* London: Van Nostrand Reinhold Company.

Gymnasieskolernes Lærerforening (2001). *GLs hf-politik*, 2. sektion af Gymnasieskolen nr. 11.

Haastrup, G. & Simonsen, A. (1984). *Sofistikken*. København: Akademisk forlag.

ICME9 (2000) *Abstracts of plenary lectures and regular lectures*, Tokyo/Makuhari, Japan.

Jensen, J. H., Niss, M. & Wedege, T. (1998). *Justification and enrolment problems in education involving mathematics or physics*. Roskilde: Roskilde University Press.

Kline, M. (1972). *Mathematical thoughts from ancient to modern times*. New York: Oxford University Press.

Lyotard, J. (1984). *The postmodern condition: A report on knowledge.* Manchester: Manchester University Press.

Mills, C. W. (1959). *The sociological imagination,* Oxford: Oxford University Press.

Tarp, A. (2002). Killer-equations, job threats and syntax errors. In C. Bergsten, G. Dahland & B. Grevholm (Eds.), *Research and action in the mathematics classroom*. Linköping, SMDF.

Tarp, A, (2003). *Student-Mathematics versus Teacher-Metamatics*. ECER 2003, Hamburg, http://www.leeds.ac.uk/ educol/documents/00003264.htm.

Zybartas, S., & Tarp, A. (in press). *Postmodern rehumanised mathematics in teacher education, a co-operation between Lithuania and Denmark*. In Proceedings of the Norma 01 Conference, Kristianstad College, Kristianstad, Sweden.

Appendix I. A Kronecker-Russell Multiplicity-Based Mathematics

1. Repetition in time exists and can be experienced by putting a finger to the throat.
2. Repetition in time has a 1-1 correspondence with multiplicity in space (1 beat <->1 stroke).
3. Multiplicity in space can be bundled in icons with 4 stokes in the icon 4 etc.: IIII -> 4
4. Multiplicity can be counted in icons producing a stack of e.g. T = 3 4s = 3*4. The process 'from T take away 4' can by iconised as 'T-4'. The repeated process 'from T take away 4s' can by iconised as 'T/4', a 'per-number'. So the count&stack calculation T = (T/4)*4 is a prediction of the result when counting T in 4s to be tested by performing the counting and stacking.
5. A calculation T=3*4 = 12 is a prediction of the result when recounting 3 4s in tens and ones.
6. Multiplicity can be re-counted: If 2 kg = 6 litres = 100 % = 5 $ then what is 7 kg? The result can be predicted through a calculation recounting 7 in 2s:

T = 7 kg = (7/2)*2kg = (7/2)*6 litres = 21 litres	T = 7 kg = (7/2)*2kg = (7/2)*100 % = 350 %	T = 7 kg = (7/2)*2kg = (7/2)*5 $ = 17.50 $

7. A stack is divided into triangles by its diagonal. The diagonal's length is predicted by the Pythagorean theorem a^2+b^2=c^2, and its angles are predicted by re-counting the sides in diagonals: a = a/c*c = sinA*c, and b = b/c*c = cosA*c.
8. Diameters divide a circle in triangles with bases adding op to the circle circumference:
C = diameter * n * sin(180/n) -> diameter * p.
9. Stacks can be added by removing overloads:
T = 38 + 29 = 3ten 8 + 2ten 9 = 5ten 17 = 5ten 1ten 7 = (5+1)ten 7 = 6ten 7 = 67
10. Per-numbers can be added after being transformed to stacks. Thus the $/day-number 'a' is multiplied with the day-number 'b' before being added to the total $-number T: T2 = T1 + a*b.
2days @ 6$/day + 3days @ 8$/day = 5days @ (2*6+3*8)/(2+3)$/day = 5days @ 7.2$/day
1/2 of 2 cans + 2/3 of 3 cans = (1/2*2+2/3*3)/(2+3) of 5 cans = 3/5 of 5 cans

Repeated addition of per-numbers -> integration	Reversed addition of per-numbers -> differentiation
T2 = T1 + a*b T2 - T1 = + a*b $\Delta T = \sum a*b$ $\Delta T = \int a*db$	T2 = T1 + a*b (T2-T1)/b = a $\Delta T/\Delta b = a$ dT/db = a

Only in the case of adding constant per-numbers, as a constant interest of e.g. 5%, the per-numbers can be added directly by repeated multiplication of the interest multipliers: 4 years @ 5 % /year = 21.6% , since 105%*105%*105%*105% = 105%^4 = 121,6%

Conclusion. A Kronecker-Russel multiplicity-based mathematics can be summarised as a 'count&add-laboratory' adding to predict the result of counting totals and per-numbers, in accordance with the original meaning of the Arabic word 'algebra', reuniting:

	Constant	Variable
Totals m, s, kg, $	T = a*b T/b = a	T2 = T1 + a*b T2-T1 = a*b
Per-numbers m/s, $/kg, $/100$ = %	T = a^b $\sqrt[b]{T} = a$ $\log_a T = b$	T2 = T1 + $\int$a*db $dT/db = a$

The Count&Add-Laboratory

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i Through his successor-principle Peano is forcing an additive structure upon the natural numbers seducing us to believe that 2+2 = 4. However this is an example of killer-mathematics, since outside the classroom we meet many examples where 2+2 is not 4: 2*meter + 2*cm = 202*cm, 2*week + 2*day = 14*day, 2*ten + 2*one = 22*one etc.
As we can see the numbers here are per-numbers and should be added accordingly, as the integration formula 'T2 = T1 + ∫a*dx' tells us. I.e. they have to be transformed to totals first; then they can be added, but only inside a parenthesis securing that the units are the same: T= 2 3s + 4 5s= 2*3 + 4*5= 6*1 + 20*1= (6 + 20)*1= 26*1= 26/3*3= 8 2/3*3= 26/5*5= 5 1/5*5. So in this case 2+4 can give both 26, 8 2/3 and 5 1/5. Thus 2 3s + 4 5s is not 6 8s; whereas 2 3rds + 4 5ths = 6 8ths in the case of e.g. 3 and 5 bottles: 2/3*3+4/5*5 = 2+4 = 6 = 6/8*8.
Hence there is a need for a 'Peano II' giving the natural numbers a multiplicative structure so they will represent directly what they describe, i.e. stacks. And so that mathematical knowledge can grow out of the count&add-laboratory, where rules are generalised through induction and validated by counting the deduced predictions. This leads to a new kind of natural numbers, stack-numbers always having the form T = a*b = (a,b). A relation can be set up identifying stacks with identical totals by saying that the stacks (a,b) and (c,d) are identical if a*b*1 = c*d*1 as e.g. (2,6) and (3,4).
Thus a natural number becomes an equivalence class in the set of stacks where n = (a,b) if n = a*b*1 as e.g. 8 = (2,4) since 8 = 2*4*1. The natural numbers then becomes the total 'area' of a stack; identical numbers occur though a re-bundling of their stacks; and prime numbers are stacks that cannot be rebundled. This stack-representation of the natural numbers is what Kuhn calls a new paradigm. It remains to be seen if number theory will look different within this stack-paradigm, and whether special problems as Fermat's last theorem will be easier to solve within this stack-paradigm.
Reformulated as stacks the Fermat theorem a^n + b^n = c^n becomes a^n = c^n – b^n. Here a^n is an n-dimensional stack, an n-stack. And c^n – b^n is a binomial that, to become an n-stack, has to factorised as a combination of n basic binomials of the form (c-b) or (c+b). For n=2 the 2 basic polynomials can contain different signs, making it possible to reduce the product of two binomials, normally having four terms, to two terms: (c+b)*(c-b) = c^2 – b^2. But with three binomials, or more, one of the signs is repeated thus creating a trinomial, which then has to be reduced to a binomial by being multiplied with a binomial.
ii 'The applicant presents, on a normative basis referring only to sociology, an original new formulation of the specific mathematically content. However the distance is far too big to the reality and the problems that on a practical level can be connected to the teaching of mathematics. No publications show direct signs of cooperation with other research with a deviating and a more general accepted starting point, which will be a central part of the work of the applicant. On this basis the committee does not find the applicant qualified for the job'.

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