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Sustainable mathematics education in a digitalized world

Proceedings of MADIF 12
The twelfth research seminar of the Swedish Society for Research in Mathematics Education
Växjö, January 14–15, 2020

Editors:
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Preface

This volume contains the proceedings of MADIF 12, the twelfth *Swedish mathematics education research seminar*, held in Växjö, January 14–15, 2020. The theme for this seminar was *Sustainable mathematics education in a digitalized world*. The MADIF seminars are organised by the *Swedish society for research in mathematics education* (SMDF). MADIF aims to enhance the opportunities for discussion of research and exchange of perspectives, amongst junior researchers and between junior and senior researchers in the field. The first seminar took place in January 1999 at Lärarhögskolan in Stockholm and included the constitution of the SMDF. The list shows all MADIF seminars.

MADIF 1, 1999, Stockholm
MADIF 2, 2000, Gothenburg
MADIF 3, 2002, Norrköping
MADIF 4, 2004, Malmö
MADIF 5, 2006, Malmö
MADIF 6, 2008, Stockholm
MADIF 7, 2010, Stockholm
MADIF 8, 2012, Umeå
MADIF 9, 2014, Umeå
MADIF 10, 2016, Karlstad
MADIF 11, 2018, Karlstad
MADIF 12, 2020, Växjö

Printed proceedings of the seminars are available for all but the very first meeting. This volume and the proceedings from MADIF 9, 10 and 11 are also available digitally.

The members of the MADIF 12 programme committee were Yvonne Liljekvist (Karlstad University), Johan Häggström (University of Gothenburg), Hanna Palmér (Linnaeus University), Linda Mattsson (Blekinge Institute of Technology), Lisa Björklund Boistrup (Malmö University), and Oduor Olande (Linnaeus University). The local organisers were Hanna Palmér and Miguel Perez (Linnaeus University). The programme committee was, during the autumn, extended with three extra editors from the SMDF board: Anette Bagger (Örebro University), Cecilia Kilhamn (University of Gothenburg), and Maria Johansson (Luleå University of Technology), to handle the large amount of contributions to the seminar.

The programme of MADIF 12 included two plenary lectures by invited speakers: Professor Dame Celia Hoyles, University College in London, held a talk titled *Programming and mathematics: insights from research in England*, and Professor Paul Drijvers, Freudenthal Institute, Utrecht University and HU University of Applied Sciences in Utrecht, held a talk titled *Computational thinking in the mathematics classroom*. As before, MADIF works with a format of full ten page papers and with short presentations. This year the number of full papers was 27 which is nine more than in MADIF 11, the number of short presentations were 23, which is twice as much as in MADIF 11. The seminar also had two symposia, where three papers on a common theme were presented and discussed. As the research seminars intend to offer formats for presentation that enhance feedback and exchange, the paper presentations are organised as discussion sessions based on points raised by an invited reactor. The organising committee would like to thank the following colleagues for their commitment to the task of being reactors and moderators: Andreas Borg, Anna Teledahl, Anna Ida Säfström, Anna Wernberg, Anneli Dyrvold, Ann-Marie Pendrill, Camilla Björklund, Eva Norén, Ewa Bergqvist, Helena Grundén, Helén Sterner, Iben Christiansen, Johanna Pejlare, Jonas Bergman Ärlebäck, Jorryt van Bommel, Jöran Petersson, Laura Fainsilber, Linda Marie Ahl, Lisa Östling, Maria Fahlgren, Maria Johansson, Ola Helenius, Olov Viirman, Per Nilsson and Robert Gunnarsson.

This volume comprises 24 research reports (papers) and one symposium report. Furthermore, the volume also contains abstracts that present three research reports, one symposium and 23 short reports. In a rigorous two-step review process for presentation and publication, all papers were peer-reviewed by three researchers. Short presentation contributions were reviewed by members of the programme committee. Since 2010, the MADIF Proceedings have been designated scientific level 1 in the Norwegian list of authorised publication channels available at http://dbh.nsd.uib.no/kanaler/

The editors would like to express their gratitude to the following colleagues for reviewing submitted reports: Alexandra Hjelte, Allen Leung, Andreas Borg, Andreas Eckert, Anna Ida Säfström, Anna Lind Panzare, Anna Teledahl, Anneli Dyrvold, Anette Hessen Bjerke, Antti Viholainen, Arne Engström, Camilla Björklund, Carina Zindel, Cecilie Carlsen Bach, Charlotta Andersson, Christina Svensson, Daniel Clarke Orey, Ester Levenston, Eva Jablonka, Eva Norén, Ewa Bergqvist, Frithiof Theens, Frode Rønning, Hanna Fredriksdotter, Hanan Innabi, Hanna Viitala, Helena Johansson, Helena Roos, Iben Christiansen, Ida Bergvall, Ingela Bjursjö, Ingi Heinesen Højsted, Jan Olsson, Jane Tuominen, Jannika Lindvall, Johan Lithner, Johanna Pejlare, Jonas Bergman Ärlebäck, Jonas Dahl, Jonas Emanuelsson, Judy Sayers, Jöran Petersson, Karen Givvin, Kajsa Bråting, Kirsti Hemmi, Kristina Juter, Laura Caligari, Linda Marie Ahl, Lennart Rolandsson, Lui Albaek Thomsen, Magnus Österholm, Malin Albinsson, Malin Gardesten, Maria Fahlgren, Maria Alessandra Mariotti, Martin Nyman, Mats Brunström, Miguel Perez, Mirela Vinerean Bernhoff, Morten Elkjaer, Morten Misfeldt, Natalia Karlsson, Nils Buchholtz, Ola Helenius, Ove Gunnar Drageset, Per Nilsson, Peter Frejd, Peter Markkanen, Peter Nyström, Petra Svensson Kjällberg, Ray Pörn, Reidar Mosvold, Rickard Wester, Robert Gunnarsson, Sara Engvall, Sikunder Ali, Suela Kacerja, Tamsin Meaney, Timo Tossavainen, Thomas Hillman, Tomas Bergqvist, Troels Lange, Uffe Thomas Jankvist, Ulla Runesson Kempe and Ulrika Ryan.

The organising committee and the editors would like to express their gratitude to the organisers of *Matematikbiennalen 2020* for financially supporting the seminar. Finally, we would like to thank all participants of MADIF 12 for their engagement in an intense scholarly activity during the seminar with its tight timetable, and for contributing to an open, positive and friendly atmosphere.

The Program comittee

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Designing a tool for exploring toddlers' numerical competencies in preschool

CAMILLA BJÖRKLUND AND HANNA PALMÉR

There is a growing consensus in research that children's numerical competencies start to develop at a very early age. However, less is known how toddlers (1–3-yearolds) learn about numbers and few tools are developed to make their progress in learning visible and researchable. In this methodological paper, we present the process of designing such a tool to be used in a combined research-development project. The focus of the paper is on the process of designing the tool that is based on theoretical principles, founded in the preschool traditions and attract young children's attention.

In this paper, we present the process of designing a tool for exploring numerical competencies among young children. The expedience of the tool is important to discuss, since its' purpose is to reveal how children develop their ways of experiencing numbers. Research has given us a quite good idea of what numerical and arithmetical skills to expect from preschool children. Far less is however known about *how* toddlers (1–3-yearolds) learn the complex meaning of numbers and *how* they learn to use numbers in basic arithmetical problem solving. To address this lack in knowledge we conduct a combined research-development project DUTTA (Educational studies of toddlers' number sense and emerging arithmetic skills, funded by the Swedish Institute for Educational Research, grant no. 2018-00014), in which we explore the numerical development and emerging arithmetic skills among toddlers at three preschools over a period of three semesters. The project is conducted in close collaboration between researchers and three preschool teachers in Sweden, aiming to empirically investigate what constitutes toddlers' learning of numbers and emergent arithmetic skills and to elaborate on how preschool education can facilitate this development.

Contemporary pedagogies emphasize the importance of taking the perspective of the learners when attempting to facilitate learning. Further, education should commence from where the learners stand in their knowledge development emphasizing the learners as autonomous and intellectual individuals (van Oers, 2018). This requires ways to establish the learner's perspective and competence. In Swedish preschools, this establishment is made both spontaneously

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in everyday activities and more systematically based on local templates. However, to establish internal and external validity in our research-development project we needed to develop a joint tool to be used for exploring participating toddlers' numerical competencies at the involved preschools. In this methodological paper we present and discuss the process of designing this tool aimed at making numerical competencies and development among toddlers both visible and researchable. The specific methodological question is whether the tool makes possible to explore *how* aspects of numbers and arithmetics are discerned by the toddlers at different times of the project.

First in this paper we make a short review of the theoretical foundations of the tool. After that, we report on the development of the tool and how we have strived for reliability as well as internal, ecological and external validity of the included tasks. This process is illustrated by analyzing one of the tasks and how children may respond to the task. In the paper, preliminary insights that will serve as a basis for further development and fine-tuning of the instrument are presented.

Necessary aspects for number knowledge

As for now, there is in research support for four fundamental aspects of numbers that are necessary for children to learn about in order to develop their numerical understanding and arithmetic skills: representations, cardinality, ordinality and part-whole relations (Baroody & Purpura, 2017; Carpenter et al., 1982). These aspects of number have in a large body of research proven to be essential for successful use of numbers in arithmetic problem solving (Baroody & Purpura, 2017; Carpenter et al., 1982; Fuson, 1992). Further, interventions directed particularly at these four aspects of numbers have been found to be successful with older (4–5-yearolds) preschool children (Björklund et al., 2021; Kullberg et al., 2020). Thus, it seems reasonable to focus on these four aspects as a basis for numerical development among young children:

Representations (including number words and finger patterns): As numbers are abstract to their nature they have to be represented in some way (Goldin & Shteingold, 2001). Children are often seen using fingers to illustrate a number, which in turn can be either iconical (seeing the full hand as an image that is called ”five”) or symbolical (the whole hand has the meaning of five, e.g. as an answer to the question ”how many”).

Ordinality (the relation between objects in a sequence): Ordinality is directed towards single entities within a set, which means that every item, situation or number word has its exclusive position in a sequence and is thus related to the other ones in the same sequence (Fuson, 1988). Most preschool children learn to recite the counting rhyme up to 20 or higher and thus experience numbers as an ordered sequence of words, but relating ordered entities to each other is however a more advanced knowledge of ordinality.

Cardinality (numbers seen as one set of units): Cardinality is generally defined as numbers referring to comprehensive sets of units. ”The cardinal word principle” means that the last uttered number word in a counting act includes all the counted items (Gelman & Gallistel, 1978). This relies on counting on the sequence to determine the cardinality of a set, while studies on subitizing (recognizing small sets without counting; see Wynn, 1998), indicate that cardinality must not rely on counting, but on a perception of numbers as a composite set.

Part-whole relations (for example, 3 and 5 are parts of 8; 8 objects can be rearranged as 4 + 4 or 6 + 2): Learning to understand arithmetical principles is based on the child's ability to handle the part-whole relations of numbers (Piaget, 1952; Starkey & Gelman, 1982). Children's understanding of such structural relationships allows them to develop ideas about addition and subtraction, not as strategies but as units to operate on. Such an awareness of basic and general relationships among numbers will eventually allow the child to recognize and make use of more general mathematical relationships, such as decomposition, commutativity or the compensation principle (Venkat et al., 2019).

In previous studies, these four aspects have mainly been studied as separate skills or knowledge domains. In our project, the intention is instead to integrate all four aspects as one construct. Further, there is a broad spectrum of research on cognitive abilities and arithmetic strategies conducted during the last 40 years (see Carpenter et al., 1982; Baroody & Purpura, 2017 for overviews). However, our focus is not on children's cognitive abilities or their use of strategies per se, but instead on *how* the *mathematical content* is experienced and understood by the toddlers in terms of which aspects of numbers they experience in a given situation and how these experiences constitute their numerical competencies.

Exploring early mathematical knowledge and development

There seems to be a consensus in research on early mathematical development that number knowledge is a complex construct. However, how to investigate young children's learning of this complex phenomenon and skill varies. Most tools developed are directed towards children older than three years, and studies who include toddlers are either very limited or broad in their scope. This as there are certain challenges in exploring young children's numerical competencies as verbal utterances cannot be taken as the primary source for understanding. Instead, observations of children's acts have been used in some tools directed at the youngest preschoolers (e.g. MIO, see Davidsen et al., 2008). However, some of these tools suffer from ceiling effects (Reikerås & Salomonsen, 2019) or provide mixed results due to differences in methods for conducting the observations (Tudge et al., 2008).

In experimental studies children's competencies are studied in isolation and in controlled forms (e.g., Sarnecka et al. 2017). Such studies as well as observations of children's spontaneous numerical interactions (Davidsen et al., 2008) have indeed given valuable knowledge about numerical skills among toddlers. However, *how* toddlers' learn the complex construct of numbers cannot be studied comprehensively within experimental designs (Donaldson, 1983) but asks for a more complex task situation that reminds of naturalistic settings, yet still used within a systematic design with theoretical underpinnings to ensure validity and consistency in the findings. Another critique addressed by Donaldson (1983) is that children seem to show different skills depending on the context. Thus, results from experimental studies may have low external and ecological validity, not necessarily transferable to educational settings. Further, many studies on early mathematical competencies and development focus on one specific ability, most often a cognitive ability, such as comparing set cardinality (Sarnecka & Carey, 2008), arithmetic expectations (Wynn, 1998) or children's dispositions towards certain features, such as SFON tendencies (Spontaneous Focusing on Numerosity, see Hannula, 2005), with or without making connections to later mathematical achievements.

A consequence of the few and often limited studies (in range, reliability and validity) on toddlers is that the current knowledge of their numerical competencies is fragmented and seldom include longitudinal dimensions and/or attention to individual differences. Individual differences is however of importance since the variation in mathematical experiences within the age span 1–3 years is very broad (Doverborg & Pramling Samuelsson, 2009). At the same time as research has provided us knowledge of children's competencies and general learning trajectories (Sarama & Clements, 2009), the methodologies for how to explore toddlers' numerical competencies and development are still in need of advancement. We know, from the earlier studies and discussion above that when developing a tool aimed at establishing the perspective and competencies of children, the contexts ought to be familiar to the children at the same time as reducing interference of irrelevant elements.

Also, in the DUTTA project we need a tool making it possible to explore how children develop one way of understanding numbers to more advanced ways of understanding and using numbers. Developing such a tool is a theoretical as well as a methodological question.

Theoretical framework

Our theoretical framework when developing the tool is Variation theory of learning (VT) which implies an interest in children's different ways of experiencing numbers (Marton, 2015). According to VT, children need to "see" or expe-rience several necessary aspects of a phenomenon simultaneously to be able to understand and handle the phenomenon in a prosperous way. In our study this means that if the child experiences the above mentioned four aspects of numbers simultaneously, it is possible to learn to use arithmetic strategies successfully and with flexibility (Björklund et al., 2021). For example, a child needs to ”see” the meaning of numbers as cardinal values as well as their ordinality in order to add units in a counting task, thus coordinating the cardinality and ordinality properties of numbers. Discerning more aspects of numbers, liberates the child to see numerical tasks in new ways that allows more advanced strategies to be used. In our study, how a child handles and responds to a situation involving numbers is interpreted as expressions of certain aspects being discerned and some perhaps are not yet discerned. When considering number knowledge as a complex construct of several aspects, VT provides a framework to describe this construct as constituting necessary aspects and particularly the relation between these aspects as fundamental for number learning.

The tool developed within the DUTTA-project

In the development of the tool we directed specific attention to enable children to express different *ways of experiencing numbers*. The tool consists of seven tasks. In this context, the notion of task does not imply a written assignment to be solved by the children, but are instead play and games that the children are invited to participate in. Based on previous research and the theoretical foundation (VT) the tasks were designed in accordance with the following three principles: 1) children of a very young age ought to be able to relate to and reason about the content based on their previous experiences, 2) necessary aspects for developing numerical competencies and basic arithmetic skills are covered, and 3) children can express different ways of understanding, allowing both a variety of experiences between children and within the same child over the prolonged period of time to be studied. Thus, each task has five levels of difficulty to avoid ceiling effects. Further, even though one task has a specific aspect of numbers foregrounded, they are not mutually exclusive to a certain aspect. When designing the tasks the intention was also to provide the best possible conditions for the children to express their numerical competencies, but without losing internal and external validity. When exploring the toddlers’ numerical competencies and development we do not expect them to act and reason in one specific way but instead a variation of actions and reasoning is expected.

Trying out the tool

The project, including development of tools and methods has been approved by the *Swedish ethical review authority* (Dnr: 2019-01037). Each task has been tested in authentic situations with 13 children between 18 and 41 months. The pilot-testing was conducted by the children’s preschool teachers who before were trained in the art of conducting task-based interviews with very young children. The interviews are designed as a situation where the preschool teacher invites the child to take part in play and games framing the seven tasks. The interviews were all video-recorded with the children's legal representatives' written consent. The expedience of the tool was evaluated based on the video-recordings, according to the three principles described above. Specific consideration was given to the materials and manipulatives, in that earlier research has shown critical differences in what the children perceive in a task, depending on the materials at hand (Björklund, 2014).

Below we illustrate the two basic levels of the seventh task in the tool and thereafter discuss how the responses given by children comply with the three principles for designing the tool (bold text is what the teacher says, in brackets how the teacher is supposed to act).

1. **The kitties (3) are playing hide-and-seek. Can you count while they are hiding?** (three identical toy cats are first shown to the child and then put in a box where the child cannot see them)
   Here is a kitty (the teacher takes one cat out of the box, showing it to the child)
   Are there any left in the box? How many are left? (the teacher takes another cat out of the box)
   Are there any left? How many are left in the box? (the teacher takes the last cat out of the box)
   Are there any left? How many are left? (the teacher and child check to see that the box is empty)

Figure 1. *The toy cats*

2. (check that the box is empty). **The kitties (3) are hiding once more in the box.** (the teacher hides the toy cats in the box)
   Here is a kitty (the teacher takes one cat out of the box)
   Are there any left? How many are left? Another kitty comes out. (the teacher now shows two cats)
   Are there any left? How many are left? One kitty is hiding again (the teacher puts one back into the box)
   How many are in the box now? Here comes a kitty out again (the teacher takes one cat out again)
   Are there any left in the box? How many? Another kitty comes out (the teacher takes out the third cat from the box)
   Are there any left in the box? How many? (the teacher and child check to see that the box is empty)

The *first principle* stated that children of a very young age ought to be able to relate to the content of the task and to reason based on their previous experiences. This task was framed as "hide-and-seek" that many children have experiences of. Also children without such experience often find "hiding" to be thrilling. Kitty the cat is part of all tasks, only differing in sizes. In the hide-and-seek play the three smaller cats are used as players. The choice to use identical toys is based on the theoretical conjecture (VT) that what varies against an invariant surrounding is what can be discerned (Marton, 2015). That is, in order to afford the children to discern the numerical relations and how they alter throughout the play, the toys are kept invariant to reduce attention to individual features. In an early version of the task, different toy animals were used. This however attracted the children's attention to the kind of animals, asking for instance for a preferred animal ”the bear” rather than attending to number. Thus, by using identical props we were able to reduce attention to some of the non-numerical aspects of the play.

According to the *second principle* the tasks were to cover the four aspects of representations, cardinality, ordinality and part-whole relations. The aim with the hide-and-seek activity was to provide the child with the opportunity to express his/her understanding of cardinality in general and numbers' part-whole relations in particular. Also, in the beginning of the play the child is encouraged to count on the number sequence while the cats were hiding, which is a basic feature of numbers' ordinality. The activity thereby has multiple levels for analysis; the question ”are there any left” may reveal the child's understanding of the cardinality of three and the follow-up question ”how many” opens up for numerical representations of the experienced number. Number relations are changing within the whole of three objects, making the relations between one, two and three explicit (part-whole relation). In the piloting of the task we observed opportunities to explore children's awareness of cardinality of numbers, for example in the following excerpts with Jamil (2 years 10 months)

Jamil: Can I have Kitty? (Jamil gets the toy)

Teacher: Are there any left in the box?

Jamil: Maja, can I have Maja? (Jamil gets the second toy) Now I have two! (Jamil hugs the toys)

This child expresses him possibly discerning the aspect of cardinality as he concludes having ”two” when given the second toy. Another example with David (3 years 5 months).

David: Where are all my buddies? (Holding the larger Kitty the cat, talking with a play voice)

Teacher: Look, here is one (Puts one toy on top of the box) Are there any kitties left in the box?

David: Two (Simultaneously showing index and middle finger)

This excerpt is a short but illustrative example of a child discerning the numerical relations in the given numbers. He answers with both number words and finger representation how many toys are *hidden* when one appeared on top of the box. To respond in this way, the child is assumed to have discerned the numerical relation between the visible toy, the two hidden toys and that they form a whole of three together. These two excerpts are examples of how we find that childrens' understanding of numbers is possible to explore in this task. The changes between hidden and visible toys provide a variation of numerical relations. This variation opens for the child to express him/her discerning the part-whole relation of 1-2-3, which is also an indication of discerning cardinality.

The *third principle* was to allow children to express different ways of understanding. Such expressions are considered important as units of analysis, allowing both the variety of experiences between children and within the same child over a prolonged period of time to be studied. In other words, we aimed to develop tasks that would allow the children to express themselves in ways that could be analysed in depth rather than merely ”can-cannot” categories. The following two ways of responding to the beginning of the ”hide-and-seek” activity where the child is asked to count on the counting sequence while the toy cats are hiding illustrate one such variation we seek to explore.

Ines (1 years 11 months): One. Three. Six.

The child saying random numbers can be interpreted as knowing that number words are words connected to ”counting”, as a string of words. In addition, she says three words, which corresponds with the number of items to be counted. She may thereby experience the one-to-one correspondence between the sets of kitties and number words. The random order indicates that numbers' ordinality are not yet discerned by the child, but experienced as ”words” of a certain kind. A different way of experiencing numbers is expressed in the following excerpt with Olivia (3 years 4 months).

Oliva: One, two, three, four, five. (The three toys are visible on top of the box. Olivia points irregularly at the toys from one side to the other. She stops counting when the toy furthest to the right is pointed at)

The act of pointing and stop counting when the items run out could be interpreted as the child experiencing that counting has to do with determining the number of a set. Saying the correct sequence of number words is also a sign of discerned features of ordinality. However, it could also be a procedural act as in ”something you do when asked 'how many' ”. Since the child is not pointing in one-to-one correspondence between object and number word said it is unlikely that the child experiences numbers as means to determine the quantity of a set (not coordinating the ordinal and cardinal meaning of numbers). In this particular case the counting is however meant to be a measurement of time as part of the play. Thus, numbers can be experienced in different ways according to their meaning and purpose in different situations, which is critical to make use of numbers in proficient ways in everyday interaction and play, as well as arithmetic problem solving. As the children were enabled to express different ways of understanding, the third principle for our design of the tool seems to be fulfilled.

Conclusion

In this methodological paper, we have presented the process of designing a tool to be used in a combined research-development project exploring the numerical knowledge and development among toddlers. Our main intention was to find out whether the tool could reveal aspects discerned by the toddlers that are central for developing numerical competencies (e.g. Baroody & Purpura, 2017; Fuson, 1988, 1992). The design process included trying out tasks and props with children in various ages from different preschools. The evaluation of reliability, internal, external and ecological validity showed that the assessment tool can make visible different ways of understanding numbers, which presumably will allow both the variety of experiences between children and within the same child to be observed and analyzed. Ways of experiencing numbers are assumed to constitute those aspects that the child is able to discern and differentiate at a certain moment and situation, which gives us a framework for interpreting learning in terms of which aspects and how the child expresses him/herself discerning at different occasions. Gaining such data for qualitative analyses is critical for forthcoming development of preschool education in our project. By designing the tool where several aspects of numbers are discernable, we open up for an analysis of the complex construct of numbers. In developing the tool we became particularly aware of the importance of reducing irrelevant features (see also Björklund, 2014), but still trying to frame the tasks as meaningful from the child's perspective (van Oers, 2018). The pilot study indicates that the tool is sufficient enough to be used in the primary study. Further studies using this tool with a larger sample will be conducted and evaluated to confirm whether the tool is robust enough for exploring toddlers' learning of number and emerging arithmetic skills also in a longitudinal perspective.

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Connected classroom technology to monitor, select and sequence student responses

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This paper reports a study of teachers' use of *Connected classroom technology* to prepare for whole-class discussions building on students' computer-based work in mathematics. The study investigates four upper secondary school teachers' management of time and progression during the phase of the lesson where students are working in pairs. The findings highlight various didactical choices made by the teachers. These choices and some related challenges are discussed.

In concluding a survey on technology use in upper secondary mathematics education, Hegedus et al. (2017) raise the question: "How can the teacher make best use of student created contributions?" (p. 32). A typical response has been that, supported by technology, teachers can develop more formative practices in which instruction is shaped by analysis and assessment of these contributions (e.g. Cusi et al., 2017). However, according to Drijvers (2011), it is more challenging for teachers to survey students' work with a computer than with conventional textbooks using paper and pencil. Moreover, we have found that even if students produce paper-and-pencil responses (to computer-based activities) that reveal their understanding (including basic mathematical misunderstandings), these are most often not registered by teachers during the lesson (e.g. Brunström & Fahlgren, 2015). This highlights the questions of whether and how technology can be used to give teachers more insight into students' mathematical thinking, in real time, to inform subsequent teaching activities so as to create a formative teaching approach.

Support for formative practices

In the field of technology and mathematics education, there is growing interest in how technology can support formative practices in mathematics. When referring to formative assessment practices, we use the definition by Black and Wiliam (2009, p. 9).

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> Practice in a classroom is formative to the extent that evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited.

One type of technology that is used to support teachers to achieve this type of classroom practice is often referred to as *Connected classroom technology* (CCT). CCT is defined as ”... a networked system of personal computers or handheld devices specifically designed to be used in a classroom for interactive teaching and learning.” (Irving, 2006, p. 16).

Earlier studies in this field have reported on the use of systems that connect students' handheld graphical calculators with the teacher's computer, e.g. *TI-Nspire navigator* (Irving, 2006). For example, Clark-Wilson (2010) reported on a project investigating secondary school teachers' practices using this system. There was one feature of the system, the Screen capture, that the teachers found particularly useful. Through this feature, the teachers can view all students' hand-held screens on their own computer. Clark-Wilson found several ways of using this feature, e.g. ”... monitoring students' activity during the lesson; supporting teachers to know when to intervene; promoting and initiating whole-class discourse ...” (p. 752). Another popular feature was the Live Presenter, through which the teacher could share interesting student screens with the whole class. This provided a ”shared learning space” where students' own suggestions were discussed with the teacher and with their peers (Clark-Wilson, 2010).

In recent years, CCT appropriate for one-to-one settings where students are equipped with a personal computer has been developed. Cusi et al. (2017) report on a study that used a set of digital worksheets embedded in a specific CCT, *IDM-TClass*, through which students' computers are connected to the teacher's computer. They found how various types of digital worksheet enhanced formative assessment strategies in whole-class activities. In the type called ”problem worksheet”, students worked in pairs or small groups on open-ended tasks and they were prompted to submit written responses as they progressed. This allowed the teacher to survey their answers (in real time) and to select answers to use as a basis for a whole-class discussion. In contrast to the use of Screen Capture, described above, where the teacher could survey students' ongoing work on their calculator, the CCT in this case only displays submitted answers.

However, there is a challenge for teachers to survey multiple student answers (in real time) to use as a basis for subsequent instruction (Olsher et al., 2016). One example of an ongoing project that addresses this issue, is the development of the online assessment platform, STEP (Seeing The Entire Picture). The aim of this project is to support teachers by automatically categorizing student submissions. This CCT goes beyond just categorizing the responses as being right or wrong, providing the teacher with information about students' mathematical understanding at a group level (Olsher et al., 2016).

Another challenge for teachers is to plan whole-class discussions based on students' computer-based work (Cusi et al., 2017). Cusi et al. found it helpful to use the five practices proposed by Stein et al. (2008, p. 321).

1 Anticipating likely student responses to cognitively demanding mathematical tasks,

2 monitoring students' responses to the tasks during the explore phase,

3 selecting particular students to present their mathematical responses during the discuss-and-summarize phase,

4 purposefully sequencing the student responses that will be displayed, and

5 helping the class make mathematical connections between different students' responses and between students' responses and the key ideas.

In the study to be described, the focus is on teachers' use of CCT to monitor, select, and sequence student responses in preparation for a whole-class discussion. However, implementing this kind of technology-supported practice is a complex undertaking, and there are several didactical choices to consider among which many relate to the issue of time management and lesson flow.

Time management and lesson flow

It is well established in the literature that time plays a critical role in reform-oriented teaching, e.g. integration of technology (Assude, 2005; Leong & Chick, 2011). Ruthven (2009) includes "time economy" as one of "five key structuring features of classroom practice" in relation to teachers' use of computers in school mathematics lessons. Assude (2005) investigated teachers' time management strategies when integrating dynamic geometry in the primary school. She observed how the teachers in her study used some time saving strategies that might be useful for others to consider. One strategy is to avoid unnecessary disruptions during the activity, another strategy is to make sure that the students are already familiar with the mathematical objects needed (Assude, 2005).

Investigating Japanese mathematics teachers' conception of high-quality teaching practice, Corey et al. (2010) reported that they gave a great deal of attention to The Flow Principle. Of particular interest, for this paper, is the aspect of flow that "... deals with time allotment to different segments of the lesson and transitions between these sections." (p. 454).

So far, however, there seem to be few empirical investigations of how teachers manage their time when using CCT in their orchestration of mathematics lessons. This paper reports the findings from a study looking at mathematics teachers' implementation of a designed computer-based lesson, consisting of three stages: *introduction, pair work,* and *whole-class discussion*. In particular, this paper aims to investigate time management and progression during the phase of the lesson where students are working in pairs on activities developed for a dynamic mathematics software (DMS) environment. The research questions are: While using CCT during students' pair work on computer-based activities, how do teachers manage, in general, (a) the lesson flow, and, more specifically, (b) their time to monitor students work and to select and sequence student responses in preparation for a whole-class discussion?

Method

The present paper reports on a study of four upper secondary school teachers' performance of a lesson using a specific CCT, *Desmos classroom activities*. Since this is a case study, the intention is not to provide generalizable results, but to identify some didactical choices appearing when teachers utilize this type of technology. Although the participating teachers were all familiar with the use of DMS, the use of CCT was new for them.

As a basis for planning the study, we used data, in terms of student responses to an explanation task, from a study with 229 students (Fahlgren & Brunström, 2018). Our mathematical-conceptual analysis of these student responses (which space does not permit us to report here) provided key formative information about what kind of response categories to expect during this particular activity, i.e. the first stage in the Stein et al. model (2008). Guided by the Stein et al. model, we developed step-by-step guidance for a lesson consisting of three stages: *introduction, pair work,* and *whole-class discussion*. For a detailed description of the theoretical framing behind the design, see Fahlgren and Brunström (2019). The guidance included a suggestion of response categories (to the explanation task) to search for among the student responses. Moreover, it provided a recommendation on sequencing consideration of these responses during the whole-class discussion as well as suggesting some questions to pose.

This paper focuses on the pair-work stage, and specifically how the teachers utilized the CCT for monitoring, selecting and sequencing student responses to the specific explanation task (denoted ”1c”). Particular attention was paid to teachers' utilization of the following types of CCT view:

Summary. This view provides the teacher with an overview of all students' progression, i.e. how many items they have done (see figure 1).

Specific item. It is possible to survey all student responses to a single item at the same time and to select specific responses by using ”snapshots”.

Presentation preparation. All snapshots taken are automatically placed in this view. The teacher can sequence the selected student responses by dragging them to different presentation views for display (in whole class). The ordering of the presentation views could easily be changed and it is possible to show several student responses on the same presentation.

During the pair work, the students used two computers; one displaying an e-worksheet (Desmos classroom activities) and one displaying the DMS environment, in this case GeoGebra. The students were prompted to submit responses, in terms of descriptions and explanations, to each item as they proceeded. It was these responses that the teachers had access to (and not the GeoGebra displays).

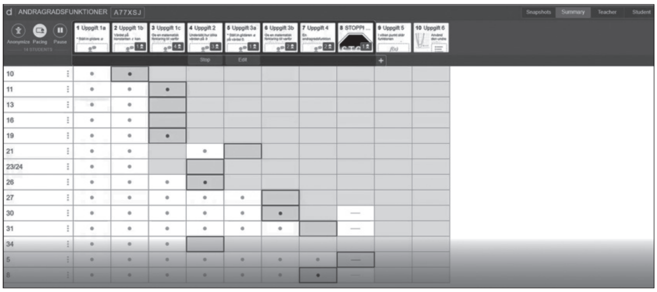

Figure 1. *A screen capture of the Summary view from one of the teachers' screen*

Data collection and analysis

The main data consists of screen recordings of the teacher's computer providing information about the teacher's options and choices. In addition, each lesson was audio recorded, and field notes were made through classroom observation by two researchers focusing on which students the teacher interacted with. Finally, a joint meeting with the teachers afterwards, where some observations by the researchers were presented and discussed, was audio recorded.

The data from the screen recordings (during the pair-work stage) were time coded as follows. First, each time that the teacher shifted the type of CCT view was indicated which resulted in several ”time spans”. Next, each time span was analysed to indicate instances where the teacher actively used the CCT for monitoring, selecting or sequencing. Data from both screen recordings and classroom observations were used in this phase. Finally, the teachers' uses of the CCT during the pair-work stage were compared and contrasted. This resulted in the identification of ”didactical variables” (Ruthven et al., 2009) and possible values of such variables. Put simply, a didactical variable (DV) is any aspect of the task, the task environment, and the teachers' management of them which may influence the unfolding of the expected trajectory of learning. In this paper we characterise a DV in terms of the way in which a teacher might ask about that variable.

The joint meeting with the teachers afterwards provided useful information about affordances and constraints experienced, didactical intentions behind various choices as well as suggestions for improvement.

Unfortunately, the screen recording of one of the teachers (Teacher A) was interrupted after 14 minutes. Accordingly, for this teacher, we only have data from the classroom observation (including audio recording) and the joint meeting.

Results

Table 1 provides an overview of the time devoted to each stage of the whole lesson by the four teachers. In this section we focus on two main types of result relating to the time management and progression of the pair-work stages: Managing lesson flow and Monitoring, selecting and sequencing.

Table 1. *The duration of each stage of the lesson in the classes*

	Teacher A	Teacher B	Teacher C	Teacher D
Introduction	12:45	10:43	6:45	16:53
Pair work	19:15	31:38	31:08	33:09
Whole class	12:17	10:35	20:33	18:02
Total time	44:17	52:56	58:26	68:04

Managing lesson flow

Within the pair-work stage, we noted that the time from the first pair completing the computer-based activity to the beginning of the whole-class discussion was 26 min. (Teacher B), 24 min. (Teacher C), and 24 min. (Teacher D) respectively. In the meeting with the teachers afterwards, this issue was discussed. The idea was that the students, when they had finished the activity, should continue with their work in the textbook, while waiting for the rest of the class to complete the activity. However, the teachers found that several students did other things, i.e. they felt that they had finished as the activity had been performed. Thus, the lesson flow was disrupted for these students. One alternative discussed is to design activities that include an initial ”core” which all students would complete prior to the class discussion, plus some ”extension” to be tackled by those students that finish the ”core” early.

One reason that the pair-work activity took so long for some students was that notions that were new for them appeared in the activity (as observed in two classes). Another reason was that some students got stuck on the explanation task (1c), probably because this was an unfamiliar type of task for them. In this way, the lesson flow was disrupted for these students as well.

Since it is important to minimize the ”waiting time” for students that finish the activity early, and at the same time provide all students sufficient time to adequately engage with the task to be discussed, we suggest the following DV (and possible values of it): *When should the whole-class discussion start?* (DV1): (i) When all pairs have finished the whole activity, (ii) When all pairs have finished the task to be discussed, or (iii) When all the expected answer categories have been generated by at least one student pair. Three of the teachers chose (i) while one of the teachers chose (ii). They all used the Summary view to decide when to start the whole-class discussion. Indeed, it may be that the availability of the Summary view via the CCT encouraged the teachers to wait for all students to complete the task or activity. If a teacher would prefer (iii), the Specific item view would be useful.

Monitoring, selecting and sequencing

This section provides results about the teachers' use of the various types of CCT view (introduced above). One of the teachers whose screen was recorded did not utilize the Specific item view. Instead, to survey the students' responses to a specific item, the teacher looked at different groups' responses to the item one at a time (by clicking on the corresponding square in the Summary view). As a consequence, the teacher looked at the same responses several (up to six) times while there were responses that s/he did not view at all. This result, we argue, highlights the usefulness of the Specific item view. This (unexpected) behavior of one of the teachers limited the data available concerning the use of the Specific item view. Equally, it affected that teacher's opportunity to exploit the full potential of the CCT. Consequently, the detailed analysis of teachers' utilization of all three CCT views comprises only two teachers.

The time diagrams in figure 2 show the various CCT views displayed on these two teachers' screens during the pair-work stage: Summary (S), Specific item – explanation task (1c), Specific item – other tasks (O), and Presentation preparation (P). Moreover, the diagrams show the (relative) length of each time span as well as whether the teacher actively used the CCT (grey) or not (black). The numbers in the diagrams indicate the numbers of student responses selected (1c) or dragged for presentation (P) during the particular time span.

When analysing and comparing the time diagrams in figure 2 several didactical variables were identified in relation to the teaching practices of monitoring, selecting, and sequencing (Stein et al., 2008).

One form of monitoring is to use the Summary view to examine all students' progression. One teacher (Teacher D) only used this form once, while the other teacher (Teacher C) used it frequently (eight times, for short time spans). Together with the field notes taken during classroom observations, the time

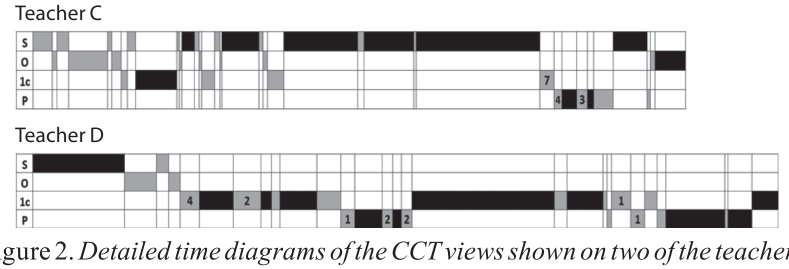

Figure 2. *Detailed time diagrams of the CCT views shown on two of the teachers' screens during the pair-work stage*

diagrams made it obvious how Teacher C often utilized the Summary view to detect students who needed help, i.e. students who were stuck or had skipped an item. Classroom observations and the joint meeting provided evidence that Teacher A and Teacher B also used this view in the same way. In the discussion during the meeting afterwards, all teachers agreed that this CCT view was useful for this purpose.

Another way of monitoring is to use the Specific item view to monitor all students' responses to a particular item. When comparing the time diagrams in figure 2, it seems that Teacher D used this feature more than Teacher C. The reason for this might be that (as the further evidence below indicates) Teacher D wanted to start the selecting process quite early.

In relation to the selecting and sequencing process, the time diagrams in figure 2 show that the two teachers used quite different strategies. Teacher D started selection after seven and a half minutes of the pair work and took all but one snapshots within four minutes. Three and a half minutes later, the teacher started sequencing. After another ten minutes, the teacher took a further snapshot (the last one) and completed the sequencing by adding this snapshot to the first presentation view. Teacher C, on the other hand, started selection after twenty seven and a half minutes of the pair work, and used just over a minute to take all (seven) snapshots. Then s/he immediately started sequencing, which was finished within less than one and a half minutes.

The screen recording shows that Teacher B, like Teacher D, started selection quite early (after four and a half minutes of the pair work), and took the last snapshot almost twenty eight minutes later. Sequencing started after seventeen minutes of the pair work, and was finished immediately after the last snapshot was taken. Data from classroom observations revealed that Teacher A, like Teacher C, started selection at the end of the pair work and took all snap shots and prepared all presentation views within a few minutes.

Some other issues were also raised in the reflection meeting. Three teachers pointed out the challenge of helping students when needed, and at the same time preparing for the whole-class discussion. Further, they found it challenging to identify the student answers in terms of the response categories.

To summarise, while two of the teachers started the selecting and sequencing processes quite early and had several periods of interaction with students before they completed the presentations, the other two teachers conducted selection and sequencing in a focused manner at the end of the pair-work stage. The screen recording from Teacher D revealed that a consequence of starting selection early might be that some students revise their responses after the teacher has taken the snapshot. This resulted in the following didactical variables being identified, and possible values of these variables. *When should the selecting process start?* (DV2): (i) As soon as some relevant answer has been produced, (ii) When all students have finished a specific task, or (iii) When all the expected answer categories have been produced. And *When should the sequencing process start?* (DV3): (i) As soon as a student response has been selected or (ii) When the selecting process has been finished.

Discussion

This study set out to investigate possible didactical choices related to the management of time and lesson flow that teachers have to consider while using CCT to support formative practices during students' computer-based work. Didactical variables provided a useful tool to identify situations where the participating teachers made various choices. Although the study is a case study, the findings can provide some guidance for future practice and research on the use of CCT to prepare for a whole-class discussion based on students' computer-based work.

Although the teachers did indeed find the CCT features supportive, it was challenging for them to orchestrate the pair-work stage, i.e. both to provide help to students and to prepare for whole-class discussion. One way of reducing the workload for teachers while students are working on their computers is to avoid new mathematical notions in the tasks or to introduce such ideas to the class in advance. This might also reduce the length of the pair-work stage, and hence the waiting time for students that finish the activity early. This aligns with one of the time saving strategies found in the study by Assude (2005).

These findings also raise questions about whether technology can support teachers further in their work of monitoring, selecting and sequencing student responses. This issue is addressed by the ongoing work with the STEP platform in which student responses are automatically categorized to off-load from teachers this time-consuming task (Olsher et al., 2016). However, it is a challenge to design tasks that can be automatically assessed and categorized. Thus, we suggest task design as a fruitful area for further work in relation to automatic categorization of (digitized) student responses.

When to start the whole-class discussion (DV1) is a crucial question influencing the lesson flow (Corey et al., 2010). Three of the teachers started when all students had finished the whole activity, which resulted in waiting time for several students. In this way, the lesson flow was disrupted for these students. On the other hand, starting too early with the whole-class discussion might disrupt the lesson flow for those students that are still working on the activity.

When reflecting on the didactical variables identified, they all relate to the optimal timing of key steps in preparation for (DV2, DV3), and initiation of (DV1), the whole-class discussion. A natural progression of the work reported in this paper is to investigate the pros and cons of choosing particular values of the identified didactical variables.

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A pre-study of grade 6 students' orientation to social and sociomathematical norms during mathematical problem solving in groups

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This paper presents an investigation of mathematical problem solving in two small groups of students in grade 6, as well as the groups' orientation to social and sociomathematical norms. The interaction within the groups was analysed with proof schemes as analytical tool, in combination with principles and procedures of *Ethnomethodological conversation analysis* (EMCA). The analysis showed that one group oriented to social and sociomathematical norms that gave rise to a potentially positive learning opportunity, whereas the other group primarily oriented to a social norm of equality that overshadowed the mathematical discussion. This study serves as a pre-study for the analysis of larger material, where EMCA appears as a promising methodological contribution.

Within the field of mathematics education, there is a consensus that collaboration is beneficial for students' mathematical development (e.g. Wood & Kalinec, 2012). However, letting students take part in group discussions and collaborative tasks does not automatically lead to productive mathematical work; sometimes participation as such is favoured, prior to discussions regarding the mathematical content of the activity (e.g. Kilhamn et al., 2019).

This paper presents an investigation of grade 6 students' mathematical problem solving in small group interaction. The study is based on the *ethnomethodological approach*, which is infrequently applied to previous research within the field of mathematics education (Ingram, 2018). According to ethnomethodology, social interaction is a process where participants orient to shared norms of conduct, and where actions are organised in recognisable patterns (Heritage, 1984). By organising actions in patterns, the participants contribute to the establishment of norms (Ingram, 2018), which in the mathematics classroom consist of both general *social norms* and *sociomathematical norms* that are specific to mathematical activities (Yackel & Cobb, 1996). According to Yackel and Cobb (ibid.), the development of sociomathematical norms creates a "taken-as-shared" sense of when and how to contribute to mathematical

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discussions, as the norms concern both the content and the process of participating in a mathematical activity. For the same reason, McClain and Cobb (2001) argue that the development of sociomathematical norms is closely related to the students' mathematical development. Kazemi and Stipek (2008), who compare learning opportunities in varying classroom practices, confirm McClain and Cobb's statement. However, further analysis of the role of social interaction in the mathematics classroom is valuable, particularly regarding how teachers' and students' turns build upon each other, and contribute to patterns that organise mathematical classroom communication (Drageset, 2015). Wood and Kalinec (2012) also call for the analysis of the relation between academic and social aspects of students' collaboration in the mathematics classroom. The aim of this study is thus to analyse both mathematical and social aspects, and their relationship, of grade 6 students' mathematical problem solving in small group interaction. Our research questions are as follows.

1. What characterizes explanations and solutions that students consider mathematically acceptable?

2. What social and sociomathematical norms do students orient to, when engaging in mathematical problem solving?

As only two groups were observed, this study serves as a pre-study for the analysis of larger material.

Theoretical framework

This study is based on a combination of *ethnomethodology* and the *emergent perspective*. The aim of ethnomethodology is to analyse what people, who participate in various kinds of everyday activities, do in order to make those activities meaningful (Heritage, 1984). The emergent perspective shares this aim, with a specific focus on mathematical activities (Cobb & Yackel, 1998).

According to the emergent perspective, the relation between the individual student and the social context of the mathematics classroom is central. As the constitution of norms is an important factor of the classroom culture, Yackel and Cobb (1996) developed the concept of *sociomathematical norms*, in connection to the formulation of the emergent perspective. Sociomathematical norms are normative aspects that specifically concern mathematical activities, whereas general social norms apply to relations between participants. One example of a social norm that is applicable in all classrooms, regardless of subject matter, is that students are expected to explain and account for their solutions to a problem. The corresponding sociomathematical norm is to expect that an explanation of a solution to a *mathematical* problem is *mathematically acceptable*. Similarly, in any discussion, a new suggestion of a solution to a problem should be different from what has already been suggested, but in a discussion of a *mathematical* problem, the suggestions have to be *mathematically different*. Students should also be able to recognize and appreciate the variety in *sophistication* of different mathematical solutions (McClain & Cobb, 2001).

What counts as mathematically acceptable, different and sophisticated solutions varies between classrooms, as sociomathematical norms emerge through interactive processes between the teacher and the students (Yackel & Cobb, 1996). For instance, the teacher contributes to the development of sociomathematical norms by explicitly asking her students to present different solutions to a given task, as shown by McClain and Cobb's (2001) analysis of discussions in primary school classrooms. The teacher's way of asking questions, or giving attention to certain explanations, also implicitly contributes to the development of normative patterns in the social interaction; this is revealed by Partanen and Kaasila (2015), in their investigation of the development of norms during upper secondary school students' collaboration in small groups.

McClain and Cobb (2001), as well as Kazemi and Stipek (2008), investigated norms of entire classrooms, whereas we, like Partanen and Kaasila (2015), focus on small group work. However, we do not analyse the actual establishment of norms over time; instead, we focus on how participants orient to norms during the course of group interaction. Sociomathematical norms were part of the theoretical framework in Levenson, Tirosh and Tsamir's (2009) investigation of the discrepancy between the norms teachers endorse and students perceive, as well as in Wester's (2015) analysis of the tension between the teacher's intentions and students' perception of norms. The difference between these two studies, and the study reported in this paper, is that our aim is to analyse students' interaction, rather than the interaction between students and teachers. Students' interaction was also the focus in Tatsis and Koleza's (2008) identification of norms in students' problem solving in pairs. However, Tatsis and Koleza performed their study as an experiment, whereas ethnomethodological investigations concern naturally occurring activities (Heritage, 1984).

Empirical material and method for analysis

The empirical material of this study consists of video recordings of two heterogeneous groups of students in grade 6, solving a mathematical problem. The observed group work represents a natural and authentic classroom situation, as the teacher of the class often let her students collaborate in small groups. The two groups (group A and group B) were observed during one lesson where they solved a combinatorial problem that was formulated as follows.

> There is a line-up at the bus stop. In how many different ways can:
> a) 2 persons stand in line?
> b) 3 persons stand in line?
> c) 4 persons stand in line?
> d) Try to find a rule for calculating the number of line-ups.

At the beginning of the lesson, the teacher instructed the students to start out by working individually with the problem. After a few minutes, the teacher asked the students to turn to the classmate sitting next to them, and agree on a ”pair-solution”. Thereafter, the dyads formed groups of four, and were told to agree on a ”group-solution”.

In order to characterize the explanations and solutions that the students considered mathematically acceptable (RQ1) we used the concept of *proof schemes* as analytical tool. According to Sowder and Harel (1998), proof schemes serve as a classification of what makes people convinced that an assertion is true. In mathematics, the three main classes of proof schemes are *analytic, empirical* and *externally based*. Students who are able to reason in a logical and general manner demonstrate *analytic proof schemes*, whereas *empirical proof schemes* consist of explanations that rely on the perception of examples or concrete objects. The *externally based proof schemes* describe situations where the convincing factors are located outside of the student, such as what an authority (for instance the teacher) has stated. The initial stage of the analysis was thus to code the students' utterances with regard to demonstrated proof schemes.

To further analyse the social interaction in the groups, we used principles and procedures of *Ethnomethodological conversation analysis* (EMCA). One principle of EMCA is to analyse naturally occurring data. Another principle is to treat talk-in-interaction as contextually embedded, in the sense that participants' utterances and actions only can be understood in relation to what other participants say and do (Heritage, 1984). One major finding in EMCA research is that *repair practices* (that is, participants' handling of various kind of trouble during talk, such as problems of understanding) is interactionally organized in at least three parts: *trouble source, initiation of repair* and *repair* (Sidnell, 2010). Each spoken turn in the analysed sequences was therefore related to the preceding and the following turns, and their contributions were interpreted with regard to how the participants displayed an understanding of the turn (or not). As norms can be identified as patterns in social interaction (Ingram, 2018; Yackel & Cobb, 1996), we also analysed sequences of turns that demonstrated the groups' orientation to social and sociomathematical norms (RQ2).

Below, in the analysis of the interaction in the two groups, dialogue excerpts are organised in turns at talk, including relevant embodied actions and handling of artefacts. Descriptions by the transcriber are marked with ((double parenthesis)) and speaker's emphasis with <u>un</u>derlining. The students have been given fictitious names: Alice, Anna, Alan and Andy in group A, and Bea, Bibi, Benny and Billy in group B.

Analysis of the interaction in group A

During individual work, Andy wrote a table of all possible combinations of line-ups for three persons, where the digits 1, 2 and 3 represented persons standing in line. Thereafter he wrote two sets of six combinations, all beginning with number 1 and 2 respectively, representing four persons in line, as shown in figure 1.

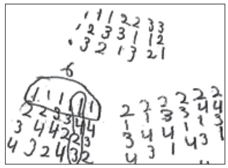

Figure 1. *Excerpt from Andy's notes*

Using the two tables with six combinations, Andy was able to calculate the total number of possible line-ups for four persons. He explained his calculation to his ”pair-partner” Alan in the following way:

> if the same person is in the front then there are six different ways for the ones behind to stand and then there are four persons who can be in the front so I just figured six times four is twenty four.

Andy reduced the problem from four to three persons, by fixating one person in the first place, and was then able to calculate the number of line-ups: ”six times four”. As Andy formulated a solution based on logical reasoning about a general character of the problem, he demonstrated an *analytic proof scheme*.

At first, the other group members could not quite understand Andy's explanation. In the following sequence, Andy initially repeats his analytic explanation, which Anna and Alice respond to.

41 Andy:	if the 1 is in the first place all the time then there are six different ways if there are three persons behind
42 Anna:	yeah but what if everyone else is in the first place
43 Andy:	((writes combinations with number 4 in the first position)) as there's a 4 you have to take 4123 4132 42 well you get it
44 Anna:	yes but all the others can be in the front
45 Andy:	but check this out if there's only the 4 in the front
46 Alice:	yeah
47 Anna:	yeah
48 Andy:	with these three persons behind they can move about so there are six different ways
49 Alice:	and then we can do the same thing with the others
50 Andy:	yes and then y'know there are four persons
	six different ways six times four is twenty four
51 Alice:	mm ((nods)) good let's do this as our group's

Anna's "yeah" (42) indicated that she accepted Andy's solution (41), but her following question also revealed that she did not understand his explanation. Andy treated Anna's question (42) as a *trouble source* when he switched to an *empirical proof scheme* and provided an example (43), as an attempt to *repair* her difficulties to understand. However, Anna showed that she still did not understand Andy's general explanation, and extended the repair sequence, when once again stating that everybody can be in the first position (44). Andy then continued to explain, by stressing the importance of keeping the same number in the first place: "only the 4" (45). Anna and Alice showed that they listened, by saying "yeah" (46, 47). Alice also showed that she now could follow Andy's reasoning (49), which Andy confirmed by repeating his initial way of explaining his solution (50). This also closed the repair sequence. Thereafter, Alice made it public that the group had agreed on Andy's suggestion as the group's joint solution (51). Andy's group mates did not simply accept his solution, but strived to understand his general explanation, as shown by Anna, raising the same objection twice (42, 44). Together with Andy's thorough explanation of his solution (43, 45, 48), the repair sequence displayed an orientation to a social norm to *strive for joint understanding*.

Soon after the sequence presented above, Alan managed to intuitively formulate the factorial function when telling his group how the number of line-ups for three and four persons can be calculated.

> one times two is two times three is six
>
> one times two is two times three is six six times four is twenty four

The group assessed Alan's formulation as "cool", "smart" and "magic", and told the teacher that their solution was "great" and "awesome". Together with the effort to understand Andy's logical reasoning, the group's response to Alan's formulation of the solution displayed an orientation to a norm that *solutions to mathematical problems should be formulated on a general level*. This corresponds to the primal sociomathematical norm of recognizing the sophistication of a specific solution.

Analysis of the interaction in group B

In group B, Bibi initially wrote all possible combinations of three persons in line. Thereafter, she let the digits 1, 2, 3 and 4 represent four persons in line, and presented her solution as one list of six combinations, all beginning with 1, as shown in figure 2.

Just like Andy, Bibi formulated a solution based on logical and general reasoning when explaining her calculation to her "pair-partner" Bea:

> here are all the ways with a 1 in the front and all the ways with a 2 in the front that should get exactly the same number as there are just as many figures [so] with four figures I just did six times four and that's twenty four .

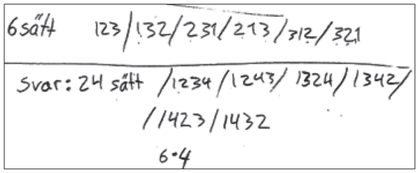

Figure 2. *Excerpt from Bibi's notes*

Bibi demonstrated an *analytic proof scheme* in formulating a general character in her solution: "with a 2 in the front, that should get exactly the same number". Benny and Billy, on the other hand, presented their solution as tables of letters and dots, thus demonstrating an *empirical proof scheme*, as shown by figure 3.

Figure 3. *Excerpt from Billy's notes*

Benny and Billy assumed that everybody had to change places in order for the new combination to be "different", and when the dyads came together as a group, they instantly realised that they had interpreted the problem in completely different ways. However, the students did not discuss the mathematical content of their solutions. Instead, Billy asked the teacher to join them in order to assess Bibi's solution of the number of line-ups of three persons.

43 Billy: ((points at Bibi's solution)) can you can you do it this way what's it now eh 123 132 231 213

44 Teach.: ((interrupts Billy)) you have kind of drawn how these different people stand

45–46 ((omitted talk about who wrote the solution))

47 Teach.: yes you could do it that way

48 Bea: yes

49 Benny: okay

Billy's request for the teacher's assessment (43) demonstrated an *externally based proof scheme*, which also Bea and Benny accepted in their responses to the teacher's positive answer (48, 49). The possible problem of having two solutions was, however, not resolved. Shortly after the sequence presented above, Benny turned to the teacher, again demonstrating an externally based proof scheme, in asking which one of the dyads' solutions the group should choose.

71 Benny: which one should we have as the group's eh joint

72 Teach.: ((to Benny and Billy)) well since you thought about it in a different way ((omitted talk about different parts of the task)) maybe you could present a

	and b in your two different ways of thinking ’cause you just thought about it in a different way but you haven’t like done anything wrong
73–76	((omitted talk about which worksheet to write on))
77 Benny:	we write both of them
78 Bibi:	we like write both of the ways
79 Teach.:	you could do that
80 Bea:	mm
81 Teach.:	it can be quite interesting there could be others in the class who thought about it in this way

The teacher’s responses (72, 81) resemble the sociomathematical norm that suggested solutions to a mathematical problem have to be mathematically different (cf Yackel & Cobb, 1996). However, the teacher also oriented to a social norm of equality, in telling Benny and Billy that they ”haven’t, like, done anything wrong” (72). Together with the statement that they ”just thought about it in a different way” (72) the teacher’s stance towards the dyads’ differing solutions became guiding for the group’s subsequent assessment.

98	Bibi:	but your idea was also very well thought out
99	Benny:	right
100	Bibi:	because it is it depends both of them can be the correct answer y’know
101	Billy:	well both none of them is actually the correct one
102	Bibi:	no exactly
103	Benny:	it all just depends on how you think about it
104	Bibi:	exactly
105	Billy:	both are just as correct
106	Benny:	both were just as good

Initially, Bibi assessed Benny and Billy’s solution positively (98), which Benny agreed with (99). The group also agreed that both solutions were equally correct (100, 101, 105, 106) and Benny repeated that the only difference between the two solutions was ”how you think about it” (103). Although the students talked about their mathematical solutions, this sequence demonstrates an orientation to a social norm that *both parties are equal*.

Conclusions and discussion

Many studies regarding classroom norms (e.g. Kazemi & Stipek, 2008; Levenson et al., 2009; McClain & Cobb, 2001; Yackel & Cobb, 1996; Wester, 2015) analyse whole class interaction, and interaction between teachers and students. The study reported in this paper adds to previous research in that we focus mainly on social interaction within small groups of students, engaging in mathematical problem solving.

The students in the observed groups had belonged to the same class, with the same teacher, for two years. It is therefore reasonable to assume that they had taken part in the same mutual processes of developing classroom norms. Nevertheless, our results show that there were significant differences regarding which social and sociomathematical norms the groups oriented to. As the development of sociomathematical norms ”gives rise to learning opportunities” (Yackel & Cobb, 1996, p. 466), our results imply that students' in the very same classroom create and experience a variety of learning opportunities, within different groups.

The solution that group A considered to be mathematically acceptable was characterized by an analytic proof scheme, as Andy's explanation was based on logical reasoning about general features of the problem. By engaging in a collaborative repair sequence, the students oriented to a social norm to strive for joint understanding. In positively assessing Alan's formulation of how to perform calculations, the students also displayed an orientation to a sociomathematical norm that solutions should be formulated on a general level. The group therefore (intuitively) created a potentially positive learning opportunity, characterized by what Kazemi and Stipek (2008) denote *press for conceptual learning*.

In group B, the dyads presented two solutions, characterized by analytic and empirical proof schemes respectively, but instead of discussing each other's suggestions, the students invited the teacher's authority in choosing which one to present as the group's joint solution. The teacher then conveyed that it is interesting to present different solutions, which is in line with the sociomathematical norm that suggestions to a mathematical solution have to be mathematically different (Yackel & Cobb, 1996). However, the fact that the dyads' suggestions actually were solutions of two different problems, due to differing interpretations of the task, was never discussed. Instead, both the teacher and the students primarily oriented to a social norm of equality, which overshadowed the mathematical discussion. This is problematic, as discussions that lack assessments of the content of the solutions do not contribute to students' mathematical development (cf. McClain & Cobb, 2001).

Wood and Kalinec (2012) suggest that researchers should focus on both the mathematical activities and the social talk, in order to better understand how teachers' arrangements of groups, and the design of tasks, might support students' group work. Kilhamn et al.'s (2019) finding that the mathematical concepts and ideas of collaborative tasks do not always get adequate attention underlines the importance of investigating academic as well as social aspects of students' collaboration. Kilhamn et al. also encourage continued investigation of how sociomathematical norms can be made explicit to students, to support their mathematical development. Our analysis on a turn-by-turn basis (as called for by Drageset, 2015) of students' demonstrations of proof schemes, and orientation to social and sociomathematical norms, showed how differing conversational patterns may shape mathematical content as well as social practices in students' group work. EMCA therefore appears as a promising methodological contribution to the analysis of collaborative problem solving.

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Five roles of the designer

Miguel Perez

This discussion paper raises some of the ethical issues related to different ways of interacting with teachers in educational design research. A categorisation of five possible roles that the designer may assume is proposed. These different roles will be framed from a perspective that may be referred to a systems approach to design. Focus is therefore put on methodological issues with the purpose to stimulate reflection on matters of design ethics that go beyond anonymity and informed consent. Finally, the different roles are related to the current tradition in educational design research.

Educational design research may be described as ”a family of methodological approaches in which instructional design and research are interdependent” (Cobb & Gravemeijer, 2008, p. 68). This family of approaches explicitly shares a twofold goal. The goal is about addressing real-life problems in classrooms and in teachers' everyday practices, as well as about contributing to theory and our understanding of the processes involved (Barab, 2014; Cobb et al., 2003; Collins et al., 2004; Lesh & Sriraman, 2005; McKenney & Reeves, 2012). Although the field has provided valuable insight into the complexity of education, we are all aware of the continuous difficulties concerning the dissemination of research results. For this reason, researchers within mathematics education have expressed the need for more encompassing design approaches in which the roles of teachers and other actors are more clearly considered (Cobb et al., 2017; Hoyles & Noss, 2015). As Van den Akker and Nieveen (2017, p. 76) explain:

> A crucial challenge for more successful innovation in education is to build bridges and more interaction between many levels, factors and actors. One of the most promising strategies is to strive after more frequent and direct interaction between teachers, developers and teachers in educational design, development and research activities.

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Although the idea of increased collaboration with teachers seems good, there are also difficulties associated with connecting two different communities – research and practice. But as noticed by Lester & Wiliam (2002), researchers in their reports rarely discuss the difficulties and the ethical issues involved when interacting with teachers and other stakeholders. The aim of this discussion paper is therefore to raise some aspects of design methodology – referred to as design ethics (Devon & van de Poel, 2004) – that are seldom discussed within the field of educational design research in mathematics education and that go beyond issues of anonymity and informed consent. Design ethics relates to the way the decision-making process is organised and how different tasks are divided between different actors. A second aspect concerns the role of stakeholders in the design process and the way in which they are included or even excluded (ibid.). In particular, I will suggest a categorisation of five possible roles that the designer may assume. The roles are called *designer as artist, designer as expert, designer as facilitator, designer as provider,* and finally *designer in service*. These different roles will be framed from a perspective that may be referred to a systems approach to design.

A systems perspective on design

This section provides the philosophical foundation for the elaboration of the five roles of the designer. But before we continue, the meaning of the word design needs to be clarified. In a systems approach, to design means to be involved in goal-seeking, or teleological, behaviour that aims at creating change, such as improving something or making something more usable or more sustainable, but without the idea of the existence of a final end (Bereiter, 2002; Churchman, 1971; Simon, 1996). Creating change is a process that assumes that something exists first as a given even if the situation or the task may be unclear from the beginning. Although others may be involved, the designer is often the one who initiates and brings change into a situation by introducing artefacts, such as teaching artefacts or design principles, developed to attain goals.

The researcher as designer

In design research the researcher and the designer are often the very same person (or a group of persons). Still, it may sometimes be convenient to speak of them as though they are two separate persons in order to recognise the kind of considerations that need to be made as many of the design aspects that are involved in design research may relate in different ways to the researcher and to the designer. For example, while the researcher is expected to meet scientific demands from research communities, the designer is expected to meet pragmatic demands in practice. The researcher is expected to be rigorous and the designer is expected to be creative. In particular, the designer is involved in creative processes that strongly depend on subjective judgements. These judgements are established in proven knowledge about the domain of investigation, in this case, mathematics education. In other words, while the designer needs to be immersed in the social context, the researcher has a different role. He or she is expected to provide a rational explanation for the judgements made. Thus, in order to put things into perspective, the researcher may need to momentarily detach himself or herself from the very source of creativity that the designer draws from.

Another dilemma relates to the nature of the designed solutions. While the researcher may require the solution to be innovative with a high degree of complexity, practitioners may prefer a simple solution that works and is easy to implement. However, such conflicts may in fact be productive: ”Creative design arises when there is a conflict to be solved between the designer's high-level problem, and the client's standards for an acceptable or useful solution" (Cross, 2006, p.72). Nevertheless, not knowing or having a good idea of how to balance multiple demands may result in neither being met.

Didactic systems

A system can be seen as a set of related elements. Elements can be concepts, objects, subjects or a combination of these. Language is an example of a conceptual system. A falling apple under the influence of gravity is an example of a physical system where concepts and objects are connected through the laws of Newtonian mechanics. Systems are also made up of other systems which are called subsystems. An important decision is therefore how large or small set of components that needs to be considered and how to conceptualise the relation between them.

Some systems can be called teleological systems, meaning systems that pursue goals, like a soccer team that trains for an upcoming game or like a teacher who wants a group of students to learn something. Didactic systems are systems that involve persons with a didactic intent. These systems consist of subjects that pursue goals, which in this case, is defined by the learning objectives. Furthermore, didactic systems are not restricted to formal learning spaces such as classrooms. However, the didactic systems that are mainly considered here are those that exist in formal settings, for example, in institutions such as schools.

The client, the decision-maker and the designer

To create change through design, there must be a purposive individual who can produce alternatives that can potentially lead to his or her objectives (Churchman, 1971; Nelson & Stolterman, 2012). This must be someone with the ”ability to imagine that-which-does-not-yet-exist to make it appear in concrete form as a new, purposeful addition to the real world" (Nelson & Stolterman, 2012, p. 12).

To set the stage, we can imagine three such characters: the client, the decision-maker and the designer. In this casting, the client is the one whose desires should be served by the system and who can be described in terms of his or her objectives or goals. The client's interest in these goals can be described by a "trade-off" principle that tells us about the priority of different possible futures. The designer's responsibility is to imagine the client's goals, but as the designer's resources are limited, the client cannot expect to have it all. Instead, all the designer can do is to provide close approximations to these idealistic desires (Churchman, 1971).

While the designer and the client need to share the same value structure, the client and the decision-maker do not necessarily do so. The decision-maker character has a different role because he controls the resources within the systems environment. By this, he is part of creating the real design. The relationship between these three characters becomes even more intricate when realising that, in real life, "both client and decision maker are highly complex entities, made up of interacting forces" (Churchman, 1971, p. 48). In addition, it is possible that all three characters reside in one person.

The designer engages in design efforts for the purpose of bringing change to an existing didactic system. The didactic system is also a teleological system concerned with learning. It includes teachers and students, and although learning in the didactic system is not restricted to students, it is the students that are the main targets of the didactic intent. In this sense, the didactic system serves the students, meaning that the students can be conceived of as the clients of the didactic system. The educational designer may share the didactic intent towards the students, but her efforts may not necessarily focus on the students. The designer may choose to address any actor of the didactic system (e.g. teachers). By including the didactic system as a teleological component, the students may remain as clients also within the teleological system of the designer. However,

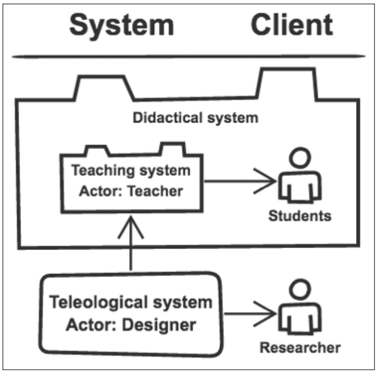

Figure 1. *A design system and its actors*

the teleological system must also be designed so it can serve the researcher as a client, as in, the researcher herself and her ambition to contribute to scientific knowledge. In this sense, the designer may interpret the situation not only as if there were two connected systems but also two clients, in particular, the students and the researcher (figure 1). Although teachers may not be explicitly considered as clients, they nevertheless have important roles in the teleological system, for example, as actors in the didactic system – in particular, the teaching system.

In this conceptualisation, the didactic system is considered as part of the teleological system in such way that the didactic system preserves its natural function even when embedded within the larger system of the designer. An alternative way is to conceptualise the didactic system as part of the environment of the teleological system. In this latter case, students are no longer clients unless the designer and the teleological system take over the didactic intent. However, this means shortcutting the teachers' natural role within the teaching system.

Five roles of the designer

We can imagine that the designer's interaction with the client and the decision-maker can take different forms. Inspired by Nelson and Stolterman (2012), we consider five roles of the designer: the designer as artist, as expert, as facilitator, as provider, and as designer in service. The first four roles have individual merits and shortcomings that the fifth role, designer in service, attempts to exploit by assuming different roles during the design process. We proceed by presenting the four basic roles. Later in this section, we elaborate on the fifth role, designer in service.

Designer as artist

In this role, the designer acts as the sole owner of the design process. The client has little or no influence other than providing the designer with a relevant context. Furthermore, the designer as artist may not even be very interested in the desires or needs of the client. In this situation, the designer acts in a fashion more or less sufficient unto himself. The design solution is based on the designer's own judgements as an instance of artistic expression.

This means that in the role of designer as artist, the designer controls both the goals and the design process. Even if the designer and potential clients may share similar goals, we can assume that they are viewed from different perspectives, at least if we assume that the designer and the client are not the same person. There is no active match-making between the designer and the client. Any potential relevance or utility for a client would be due to the experience and skills of the designer to produce such results. In the worst case, relevance for practice would be more or less accidental or only due to chance.

Designer as expert

In this role, the client is not given the possibility to contribute to the design process. Although the client's goals may be considered, the designer is the one with predetermined insights and design solutions. No customised interactions with the client are needed, as the client merely acts on behalf of the initiatives presented to him/her.

In this case, the designer helps the client in the way that the designer believes is the most effective or worthwhile. The designer finds the mandate to do so from the effort that the designer has put into analysing the context and the design problem as well as the scientific knowledge that the designer brings into the situation. However, this is, by the very nature of helping, a one-sided relationship (Nelson & Stolterman, 2012). Furthermore, if the problem is correctly analysed according the designer's methods, the conclusions and solutions cannot be regarded as incorrect. If the designer should fail to produce any useful and relevant outcomes, it could be blamed on interfering variables or on shortcomings in the client's implementation. If it is the latter, then the immediate solution could be to educate the client so he or she will become more proficient regarding the operationalisation of the designer's proposal.

Designer as facilitator

When the designer acts as facilitator, the client is expected to decide what goals should be pursued and what should be done. The designer acts merely to organise and to support the design process and does not contribute with new perspectives or new ideas.

In the role of designer as facilitator, the designer considers the perspective of the client. The designer allows the client to make use of his or her wide experience and ”real” knowledge in the design process. The client decides what goals to pursue for the purpose of achieving results with relevance for local practices. In this case, solutions reside within the current environment represented by the client. The designer's primary role is to facilitate any effort that the client suggests. The designer addresses issues of relevance and utility by giving the client authority to control goals and processes. In other words, the client is made responsible for strategic decision-making.

Designer as provider

In this role, the designer refrains from participating in the design process. As a provider, the designer acts only as an instrument by answering questions from an intentional client. In this case, the designer does not contribute intentionally to any part of the design process.

In this case, the client assumes total control over goals and processes. The designer provides support only when asked and in well-defined and limited issues (i.e. technical and scientific support). The designer relies on the client to know how to use the knowledge and tools that the designer has provided. Also, in this case, relevance seems to be secured, as the client is made responsible for deciding on both strategies and tactics.

Ethical considerations

Both the designer as artist and the designer as provider require only minimal interaction between the designer and the client. This minimal interaction may cause difficulties for the designer to develop sufficient understanding for making informed ethical design judgements, as the designer works in isolation with limited interaction with the client and other actors of the system.

Furthermore, on one hand, the designer in the role of the designer as artist does not make the assurance that the knowledge produced can be understood or be used by a specific client for his or her purposes. On the other hand, in the role of the designer as provider, the designer does not have insight in the client's design process and cannot be expected to judge if the solutions are generalisable beyond the immediate environment. In both cases, the knowledge produced is either the property of the designer or of the client. Each role focuses on esoteric knowledge rather than exoteric knowledge. In other words, the roles of the designer as artist and the designer as provider are self-serving rather than other-serving (Nelson & Stolterman, 2012). Thus, if dissemination is regarded as an important value of design, it may be lost with little chance of recovery.

In comparison, the role of designer as an expert or a facilitator may appear more appealing, as the control of goals and processes are shared between the designer and the client. However, the expert and the facilitator face other difficulties. The designer in the role of the designer as expert merely "accesses the voice" of the client and does not pay sufficient attention to the client's needs. And the designer as expert could be accused of acting in a superior way that could cause a conflict between the values of the expert and the values of the stakeholders involved. If the designer as expert fails to produce results that the client understands, he or she may be criticised for using an insensitive top-down approach that does not account for the specifics of the situation.

The designer as facilitator also faces other problems. Unlike the designer as expert, the facilitator does not contribute with new perspectives because he relies on the judgement of the client to know what to do. The facilitator's may suggest design strategies based on the client's request. The client is responsible for risk-taking regarding design tactics and design solutions. Nevertheless, along with all the other characters, the designer as facilitator is responsible for how design activities may affect others; however, the facilitator does not engage in redirecting the client's actions. The designer as facilitator believes that no matter what happens, he or she should not interfere. Therefore, if something goes wrong, the facilitator could be accused of being, although presumably scientific objective, socially irresponsible. Furthermore, as no new inputs are introduced, the facilitator accepts the client's formulation of the situation. But the client's intuition may be misleading: The solution that the client implements may come at the expense of other goals that the client is not aware of and also not prepared to sacrifice. The passiveness of the designer as facilitator may worsen the situation for the client. The facilitator chooses to accept the client's perspective and does not engage in analysing the consequences of the client's design solution. This approach may risk contributing to the establishment of a recurring problem instead of engaging in resolving the problem and improving the design solution. The client is fully responsible for evaluating the design solution and its consequences.

The designer in service

In summary, none of these power relations between the designer and client are fully satisfactory, but nevertheless, they should not be discarded so easily. Thus far, we have examined four out of five possible roles of the designer. The last role is the designer in service. This role represents an intricate relationship where both the designer and the client are engaged dynamically in the design process. This relationship involves switching between the four previous roles. In this sense, it is a balanced relationship between the designer and the client but still with the tensions of the other roles. The role of designer in service does not mean unconditionally accepting either proposed problem formulations or any initial ideas for solutions, as presented by the client or by other "experts" of the environment (Nelson & Stolterman, 2012). The designer in service switches carefully between the four roles, purposively and intentionally rather than by decree. For some specific purposes or in some phases of the design process, the designer may momentarily assume the role of the artist, the expert, the facilitator or the provider to better understand and deal with a situation – to rock the boat, so to speak, but not in a harmful way. Rather, it is done carefully with the intention of negotiating and developing a mutual understanding of the design objectives and emerging issues in the design process. The tactic of merely asking what the client wants may not be sufficient, as it cannot always be expected that the client will know what he or she exactly wants or is capable of expressing it explicitly. Furthermore, "[...] the statement of needs and wants is often confused and frequently wrong, simply because statements of wants and needs serve so many different purposes for the individual" (Churchman, 1968, p. 181). Instead, the designer may act like the expert or the artist and introduce artefacts or other arrangements and, by getting his client to react to them, form a preliminary understanding of how well the design proposal fits in the environment. Which roles are relevant to assume in order to satisfy a design goal is the responsibility of the designer to decide in collaboration with the other actors of the teleological system. The designer and the other actors of the system may bring their own perspectives into their partnerships. In this sense they are equal partners, but as their access to social and or scientific resources may be different, they cannot always be expected to contribute in the same way.

The current educational design research tradition

In educational design research within mathematics education, the standard procedure is that "a research team assumes responsibility for a group of students' learning" (Cobb & Gravemeijer, 2008, p. 68). A team design and implement teaching activities based on a selection of principles. Empirical outcomes are then analysed by the researchers by utilising predefined theoretical frameworks, with focus on describing and evaluating learning effects of the implemented activity (e.g. Cobb & Gravemeijer, 2008; McKenney & Reeves, 2012). Although teachers may sometimes be part of the research team, this standard procedure suggests that the dominant role assumed in this tradition is designer as expert. This role is used throughout the design process, including for the dissemination of research results. However, this strategy has not been effective for improving relevance for practice. As Dewey (1929, p. 19) asserted:

> No conclusion of scientific research can be converted into an immediate rule of educational art. For there is no educational practice whatever which is not highly complex; that is to say, which does not contain many other conditions and factors than are included in the scientific finding.

Thus, in order to improve the current situation, perhaps other roles should also be considered. As expressed by Cross (2006): "Design knowledge resides firstly in *people*: in designers especially [...] secondly in *processes*: in the tactics and strategies of designing [...] Thirdly [...] in *products* themselves: in the form and materials and finishes which embody design attributes" (p. 100–101).

In the role of designer in service the participating teachers are more clearly positioned as designers and agents of change. On one hand, the designers' subjective judgement is invited as an essential creative feature in the research process. On the other hand, inviting flexibility and subjectivity can obscure the rationale for the research process thus making it harder to understand and follow. For this reason, the role of designer in service may be methodologically challenging, as the research process may not longer follow a predetermined trajectory to the same extent as in the role of designer as expert. Nevertheless, by giving the teachers extensive responsibilities in the design process, they acquire design knowledge and control of the process. This may increase the possibility that people other than the researcher will also consider the design a good design.

Note

This paper is an adapted excerpt from the thesis by the author. See Perez (2018) for original publication.

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Components of knowledge in solving linear equations

Jöran Petersson

This article identifies knowledge components needed for successfully solving linear equations. Data for this purpose is 359 Swedish year 9 students' written responses to the test task "solve the equation $2x + 3 = 11$". The following set of knowledge components were identified; arithmetic knowledge, parsing knowledge, balancing equations, giving a value to the unknown, not omitting parts and the habit of verifying the solution. This paper discusses for which of these knowledge components, students could discover and correct their own errors if they would both solve an equation and verify its solution.

In mathematics textbooks, a standard method for teaching how to solve linear equations is the *canonical method* (Buchbinder et al., 2015). This method includes the steps of first simplifying on each side of the equals sign arriving at the form $ax+b=cx+d$. After this follows inverse operations by making appropriate addition and/or subtraction operations arriving at the form $ex=f$ thus isolating the unknown on one side and finally multiplying and/or dividing in order to identify the value of 1 (one) unit of the unknown. Moreover, it seems that many students use the canonical method as a mechanical procedure (Huntley et al., 2007).

Though solving linear equations has been taught and learnt since Babylonian time (Friberg, 2005), and there now is a large body of research on teachers teaching and students learning how to solve them, Otten et al. (2019) yet made a call for further research in this area. The reason for their call is that they found the often-taught balance model (Andrews & Sayers, 2012; Marschall & Andrews, 2015), be it physical, digital or drawn, to be complex as a didactical tool in the presence of negative numbers (Vlassis, 2002). Nevertheless, even when only positive numbers are present, students make errors that seem difficult for the student to identify as errors if simply applying the canonical method as a procedure (Petersson, 2018b). The aim of the present study is to explore the role of verifying a solution in relation to students' responses to the task of solving a linear equation.

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Components of prerequisites for solving linear equations

As stated above, the standard tool for solving linear equations is the canonical method (Buchbinder et al., 2015) and that students seem to too often use this as a mechanical procedure (Huntley et al., 2007). Now, one component in the canonical method is that it rests heavily on that the students view the equals sign as relational (in contrast to operational), which Knuth et al. (2006) found to be crucial for success in linear equation solving tasks. Another way to say this is that the equality must be kept balanced. A second component in the canonical method is arithmetic knowledge. For example Hall (2002) and Petersson (2018a) found that in subtractions, students may sometimes use counting down strategies where they include or exclude both the starting and ending number thus getting, for example, the difference $11-3$ one unit too small or too large. A third component in the canonical method is parsing algebraic expressions correctly, for example $2x$, as a multiplication and not as an addition (Humberstone & Reeve, 2008; MacGregor & Stacey, 1997; Petersson, 2018b). A fourth component in the canonical method is the concept of the unknown. Asquith et al. (2007) described perceptions of the unknown as a hierarchy of seeing unknowns as a multiple number, a specific number and an unknown digit. This includes giving a value to one unit of the unknown; that is setting ”$x=\ldots$”.

Research question

A lot of research on solving linear equations explores some single component, such as those mentioned above. Less research seems to have explored several components simultaneously. Moreover, less research seems to have explored verifying a solution. Hence, the research question in this study is to explore how often students use verifying and what role explicitly verifying a solution hypothetically could play in helping students identifying and correcting their own errors made in some component of the canonical method.

Methods

To answer the research statement, the author collected 359 Swedish year 9 students' written responses to one task on linear equations given in a mathematics test. The students' responses were analysed with respect to the knowledge components described above.

The test task

Vlassis (2002) separated between what he called arithmetic linear equations, having the unknown on only one side, and non-arithmetic equations, having the unknown on both sides. The explored test task was ”Solve the equation $2x+3=11$”. There are two reasons for choosing this arithmetic equation as test task. The first reason is that this task is simple enough to get many responses, since a difficult equation might result in many blank responses not contributing to the result. Still, it is possible to, in this task, make errors corresponding to all the components mentioned earlier. The second reason for choosing this task is that it is similar to linear equation tasks in released Swedish year 9 national tests in that the unknown occurs on only one place (see table 1). More-over, table 1 shows that in all years except 2006 and 2012; two arithmetic operations with different priority are present, the unknown has a positive integer solution and the coefficients are integers in the sense that it is natural to view for example $x/2$ as dividing by an integer rather than multiplying by the decimal number 0.5. The exceptions are 2006 and 2012, where the equation task contained only addition. On the other hand, in 2006 the solution was a negative integer and 2012 the solution was a decimal number.

Table 1. *Linear equation tasks on released national tests (Prim-gruppen, 2019)*

Year 9 national test task	Task formulation
2014 part B task 7	Solve $25-5x=10$
2013 part B task 9	Solve $x/2+1=5$
2012 part B1 task 9	Solve $2{,}35=0{,}5+x$
2010 part B1 task 10	Solve $13-3x=7$
2009 part B1 task 7	Solve $17=3x+5$
2008 part B1 task 8	Solve $x/3+\ 2=5$
2006 part B1 task 6	Solve $x+6=\text{-}2$

Analysing the students' responses

Each student's response to the task was categorised with respect to verifying or not verifying the solution explicitly in the written response. In addition, each student's solution to the equation was categorised as correct or incorrect. Now, a response with an incorrect solution could contain several simultaneous errors. For example, a single response could contain both arithmetic errors such as setting $11-3=9$ and parsing errors such as interpreting $2x$ as $2+x$. Thus, each incorrect response was analysed with respect to each of the components of balancing, arithmetic, parsing, explicitly giving the unknown a value and a fifth category that was found while examining the responses, namely omitting parts of the equation.

The students

359 Swedish year 9 students agreed to participate in this study and the linear equation task was given in a teacher administrated classroom test in mathematics. To check the generalisability of the sample, the students in this study were

compared with a sample from whole of Sweden with respect to the achievements on the national test part B, which is the part in which linear equations of the studied type occur, as seen in table 1. The author received achievement data from the whole Sweden sample from the National Test Team. In their sample, percentage of correct responses were 46 % for the second language students and 60 % for the first language students. The author got national test achievement data from five of the six schools that participated in the present study and these second language students ($n = 146$) achieved on identical level as the national sample while the first language students ($n = 113$) achieved 56 %, which a little lower than the national sample. This latter difference is likely due to residential segregation effect (Hansson, 2010, 2012) since the students sampled were from schools with a high proportion of second language students. The similarity between the sample in this study and the national random sample with respect to achievements on the national test, suggests that the knowledge in mathematics of these two samples are similar, which in turn indicates high reliability of the student sample used in this study. It should also be noted that the small difference between the two samples is not crucial for the validity of the present study not comparing achievements quantitatively but instead qualitatively identifying knowledge components that the students need for mastering solving linear equations. In fact, a lower achievement might mean a larger proportion of students not giving an answer to the task but it likely also implies a larger proportion and thus richer mix of incorrect responses, which should increase the saturation (the actual occurrence) of the different knowledge components of the canonical method found in the literature. This should contribute to a higher validity of the results.

Results

Balancing as a component in solving linear equations

Figures 1a and 1b exemplify unbalancing the equation. In figure 1a the student tried to isolate $2x$ on the left side by subtracting the number ”3” on one side and adding the same number on the other side. Since the student in the next two lines did identical operations on both sides, though interpreting the square root sign as halving, it seems as if the student has confused ”doing the same operation on both sides” with ”change sign when moving a term to the other side”. In that sense, this error is neither an arithmetic error nor a parsing error but an erroneous balancing of the equation. The same holds in figure 1b where a student swapped $2x$ and 11 with each other in order to isolate $2x$ on one side. Moreover, the student in figure 1b used the equals sign to mark the transformation of an equation into another equation, which clearly indicates an operational use of the equals sign as ”becomes” instead of ”equals”.

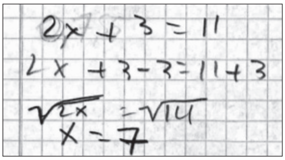

Figure 1a. *Opposite operations*

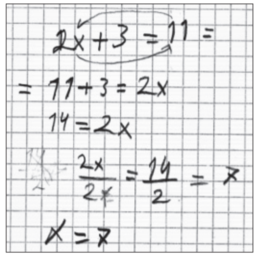

Figure 1b. *Terms moved around*

Omitting parts or terms when solving linear equations

In figure 2a, the student in the response explicitly verified that $8+3=11$ but concluded that $x=8$ instead of $2x=8$ thus seeming to ignore the factor 2. In figure 2b, the student subtracted 3 on only one side but apart from that did arithmetically correct operations (though in an odd way by dividing by 11 instead of by 2), parsed the symbols correctly and correctly balanced the equation through the rest of the solving process. Together this indicates that the student may simply have forgotten to subtract 3 from the right hand side.

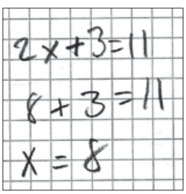

Figure 2a. *Omitted dividing by 2*

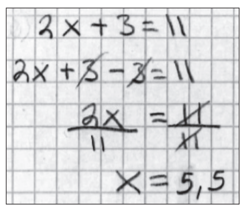

Figure 2b. *Omitted subtracting 3*

The arithmetic component in solving linear equations

In the responses in figures 3a and 3b, it seems as if the students viewed the equals sign as relational since they consequently did the same operations on both sides. Despite the correct balancing of the equations, the two responses contain arithmetic errors. Indirectly we can also assume that their calculations

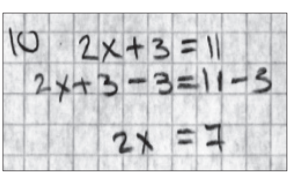

Figure 3a. *Calculates 11 – 3 to 7*

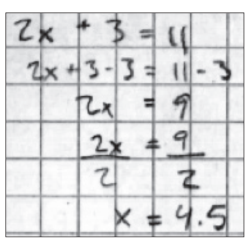

Figure 3b. *Calculates 11 – 3 to 9*

of $11-3$ leading to wrong differences 7 and 9 instead of 8 indicate that they did not use number facts when calculating the difference. Instead, they likely used counting down strategies where the student in figure 3a excluded the starting and ending number when counting down from 11 to 7 while the student in figure 3b instead included the starting and ending number when counting down from 11 to 9.

The component of giving a value to the unknown

In figure 3a, the student responded with ”$2x=7$” but did not proceed to determine a value of one single x. In this category of responses there were also a few cases of responses ”$2x=8$” and those that stated ”$8+3=11$” without explicitly giving a value to the unknown.

The parsing convention component in solving linear equations

The only error in the calculations in figures 4a and 4b are that these students parsed the original equation $2x+3=11$ in ways that differ from what is endorsed in mathematics, namely seeing $2x$ not as a multiplication, but as an addition in figure 4a and as a power in figure 4b. Else, the arithmetic calculations in both figures 4a and 4b are correct with respect to the parsing error that each student made. When it comes to the students' view on balancing the equation, the calculations in figure 4a shows that this student consequently did the same operation on both sides until getting some solution. From this, we conclude that this student views the equals sign as relational and knows balancing as a way to solve equations. From the arithmetic statement in figure 4b we can see that the left hand side evaluates to the right hand side of the equality thus indicating at least an operational view of the equals sign while figure 4b does not give any information about if the student mastered the equals sign as relational though an operational use is evident. Finally, these two students treated the unknown as a variable, whose value should be determined, which they did explicitly in figure 4a and implicitly in figure 4b.

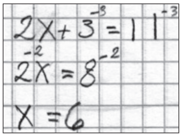

Figure 4a. *2x parsed as* $2+x$

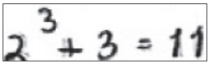

Figure 4b. *2x parsed as* 2^x

The component of verifying a solution

In figures 4b and 5, the students explicitly verified that a specified value of the unknown satisfies the equation.

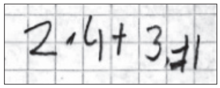

Figure 5. *Verify a solution*

Frequency of various solution components

There were 255 (71 %) correct responses. Of the correct responses there were 121 responses using the canonical method, 83 just responded $x=4$ while 50 wrote the explicitly verified solution as $2 \cdot 4+3=11$ and only 1 (one) both gave a solution via the canonical method *and* verified the solution. A note is that we do not know if these 83 students mentally did or did not verify their solution. We only know that they did not verify their response on paper. There were 56 students, who did not answer the task. Among the 48 (13 %) erroneous responses, there were three non-classified responses. These were the responses ”8*x*”, ”18” and ”24”. Of the remaining 45 responses given in table 2, 30 made one single type of error. There were 10 responses coded in two error categories and among these, it was most common to combine unbalancing the equation with some other error. In 5 cases, there were responses coded in three error categories. A clarifying note is that counting the number of errors in each category instead of the number of responses explains why the sum 65 of the content in table 2 exceeds the 45 responses categorised.

Table 2. *Frequency of components*

Component	Frequency (relative frequency)
Arithmetic error	6 (2 %)
Parsing error	19 (5 %)
Balancing error	12 (3 %)
Omits part	13 (4 %)
Unknown gets no value	15 (4 %)

Discussion

One focus in this study is students’ habit of explicitly verifying solutions when working with equation tasks. One striking observation in the data was that only one single student out of 359 responded with both solving the equation and explicitly verifying the value of the unknown by inserting it into the original equation. Else, it was common to *either* use the canonical method for solving *or* simply insert the solution into the original equation. For the students who came up with an incorrect solution to the equation, a combination of the canonical method and explicitly verifying the solution should help at least some students to discover their own errors and give them a chance to self-correct their responses.

Another focus in this study is errors in components of the canonical method. Students that respond as in figures 3a, 3b and 4a demonstrate knowledge about the canonical method (Buchbinder et al., 2015) and thus uses the equals sign as relational (Knuth et al., 2006). However, the two students responding as in figures 3a and 3b, made the same kind of arithmetic errors as described in Hall (2002) and Petersson (2018a). Their responses show that also arithmetic knowledge is an indispensable component for correctly solving linear equations. Moreover, the arithmetic component in responses 4a and 4b and the balancing component in response 4a are correct with respect to the erroneous parsing described in research literature (Humberstone & Reeve, 2008; MacGregor & Stacey, 1997; Petersson, 2018b). This means that also correct parsing is an indispensable component when solving equations.

Of the balancing errors in table 2, more than half of them were similar to those in figures 1a and 1b and could be related to a non-relational view of the equals sign as described in Asquith et al. (2007) or due to a diffuse conception about the canonical method. One assumption is that students might discover errors due to both arithmetic mistakes and incorrect balancing if a task on equations asks for *both* a solution procedure *and* an explicit verification of that solution as in figure 5. This might also hold for students that omits parts of the equation during the solving procedure as in figures 2a and 2b. We can only speculate why some students omitted terms. A guess is that it might relate to the working memory of the individual student and thus not the cause of being careless.

However, students making paring errors as in figures 4a and 4b will not discover their errors by verifying their solution. For example, the response in figure 4b is an arithmetically correct and explicit verification. Instead, they need instruction on the sanctioned parsing rules. Neither might asking for verification help students that do not give a value to the unknown as in figure 3a. On the other hand, a hypothesis is that learning the habit of explicitly verifying solutions should raise the awareness of actually assigning a value to a single unit of the unknown, which might make them elaborate their incomplete solutions into complete solutions.

A third focus in this study is how explicitly verifying solutions hypothetically could help students identifying and correcting their own errors when solving equations. Since only 1 (one) student of 359 both explicitly solved and explicitly verified the solution while several students hypothetically should have discovered their own errors if they had verified their solution, one conclusion for the teaching and testing of solving equations is to encourage and remind students to also verify their solutions. This would help, in particular, students that else make frequent errors when solving equations. Moreover, it is likely that these students often are low-achievers and thus would benefit from this. One way to promote students' habit of verifying is to, in tests, explicitly ask for both solving and verifying, since it is commonly known that what is examined to a higher extent also is learnt.

A suggestion for further research is to, through an intervention study, explore if teaching and examining both solving and explicitly verifying would help students discovering their own errors themselves. Furthermore, would this at the same time would support them in building a relational view of the equals sign and learning correct balancing of equations? Such a study should, of course, include non-arithmetic equations, and if applicable also quadratic equations and systems of linear equations.

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A qualitative analysis of students' use of base-ten material

Robert Gunnarsson and Malin Albinsson

Previous research indicates that manipulatives, like base-ten blocks, not necessarily strengthen students' understanding of numerical place-value and the decimal numeral system. This study takes its starting point in the hypothesis that to create functional teaching situations with base-ten blocks, it is necessary to first know students' prior understanding of such manipulatives. Therefore, here we present an analysis of students' understanding when using such manipulative material to visualise multidigit numbers. The data was collected from individual interviews with 58 students in grade 1 (6–7 years old). Using methods borrowed from phenomenography, we identify six qualitatively different categories of students' understanding, and, based on these, suggest implications for the design of teaching situations.

For many young students, the structure of multidigit numbers is, in itself, difficult to learn (see e.g. Fuson, 1990). Adding a manipulative material is not necessarily enough to support students' understanding of how numbers are structured. This has long been recognized as a problem. Ball (1992, p. 46) asks for: ”a lot more opportunity to discuss and develop ways to guide students' use of concrete materials in helping students learn mathematics”, and continues that we ”need to listen more to what our students say and watch what they do”. Hence, one way to read this is: to make a concrete material really supporting the learning process, one should start from the students' understanding. In this paper, we take exactly this position; to design a functional teaching-intervention, we first need to analyse students' understanding of the manipulative at hand.

The manipulative material we are studying is the base-ten material. Base-ten material has been used for many decades in mathematics teaching and learning (Kim & Albert, 2014). It is a type of manipulative that consists of blocks of different shapes, see figure 1, where each block type represents ones, tens, hundreds, or thousands, respectively. The intention is that students, by using these blocks, should learn the structure of the decimal numeral system and the place-value of digits.

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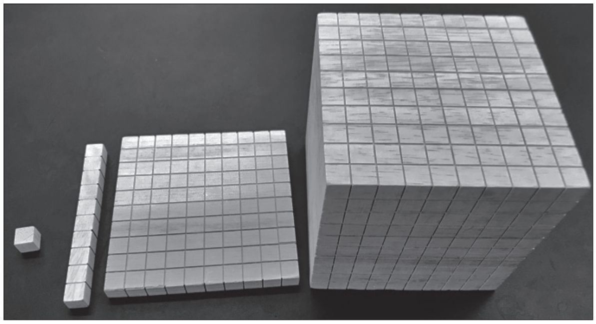

Figure 1. *The base-ten material*
Note. The material consists of (from the left) small cubes (representing ”ones”), rods (representing ”tens”), squares (representing ”hundreds”) and large cubes (representing ”thousands”)

However, there are indications that base-ten-manipulatives accompanied with instruction does not help students to a more profound understanding of place-value (Osana et al., 2017; Puchner et al., 2008). Unfortunately, it seems as if the negative result is not limited to this specific manipulative material. A meta-analysis of the efficacy of different concrete materials shows that students who have used manipulatives when working on their number sense in some studies did perform better, and in some cases equally good, but in many studies the students performed worse on a retention-test, compared to students who had trained with a textbook (Carbonneau et al., 2013).

Going beyond the issue whether base-ten material facilitates students' understanding or not, the question remains *how* students understand the material. There are a few studies that have been devoted to this more specific issue, and to illustrate the development in this area, we select two studies in particular. Early on, students' understanding of base-ten blocks was studied within a Piagetan framework by e.g. Labinowicz (1985). Such studies sort students' understanding into Piaget's stages of intellectual development. More recent studies analyse the understanding based on other frameworks, e.g. Nurnberger-Haag (2018). These discuss how students operate with the material, and study students' understanding of base-ten manipulatives from the perspective of embodied cognition. However, there are, to our knowledge, very few studies using a phenomenographic approach on students' understanding of base-ten materials. As phenomenography can be used to identify what students need to discern to build a solid understanding of a phenomenon (Pang, 2003), it can also give information when designing functional teaching interventions.

Aim of this study

The aim of this study is to use a qualitative approach with phenomenographic features to describe the characteristics of how students in primary grades understand a supporting material – in this case the base-ten material – used to illustrate multidigit numbers.

Theoretical framework

For this study, where we analyse students' understanding of a manipulative material, and how it relates to the properties of numbers, we chose to employ features from the framework of phenomenography. As described by Marton (1981), phenomenography is a research orientation to describe how concepts are understood by, for instance, a learner. Phenomenographic studies have been conducted in many different areas, particularly in mathematics. One example is Neuman's (1997) study on numeracy and number sense. Studies based on the phenomenographic approach typically result in categories of description of the conception (Marton, 1981). According to Pang (2003), what separates phenomenographic categories can often be interpreted as critical aspects that students need to discern in order to advance the understanding of a phenomenon. We too are interested in what qualitatively different categories of understanding students exhibit. Hence, a qualitative approach where we borrow the analysis method from phenomenography can be a suitable step in making informed choices when planning teaching situations.

Methods

For this study, 58 Swedish students in their fall semester of grade 1 (age 6–7 years) were individually interviewed. The interviews were part of a larger intervention study (a learning study project) to be presented elsewhere. In the interviews the students were asked to tell the value of different Pokémon-cards and represent that value with base-ten blocks. The students were chosen from a screening of all preschool students (with the material ”Blå lådan”) of age 6 in a municipality in Sweden. Based on the test results, three schools with particularly low scores were chosen. In a learning study processes with young students, interviews are often used to collect pretest and posttest data. However, in this study the interviews constitute the data.

The interviews comprised mainly two main tasks: (1) To tell the value of given Pokémon-cards (with the numbers ”13”, ”42”, ”117” and ”258” written on them, respectively), and show that number using base-ten blocks. (2) To suggest a value of a fifth Pokémon-card, write the value with numbers, and show it using the base-ten blocks. During the entire interview the student had access to a large set of base-ten blocks. The interviews were documented by capturing video data. This procedure copies, in many ways, the process of extracting data for phenomenographic studies in other areas (see e.g. Han & Ellis, 2019). However, as the resolution of the video is too poor to make high-resolution images, the figures below are reconstructed as close as possible from what students showed during the interviews.

The data from the interviews were verbatim transcribed and the transcripts coded in such a way that the parts where the students could have revealed their understanding of the number system could be extracted. These extracts were then subject to analysis where qualitative similarities and differences were identified, which, in turn, lead to qualitatively different categories of students understanding and use of the material. Hence the categories emerged from the data. The data were jointly analysed by the authors.

Results

The analysis resulted in six qualitatively distinct categories of students' understanding of how the base-ten material can be used. In defining the categories, we have made a selection; students' answers that were not indicating an understanding of the base-ten material as representing a quantity, for instance by using the blocks as mere building blocks, have been omitted from the result. The final categories are presented below.

Category 1. Blocks can represent numbers

Students' responses that are indicating some form of recognition of the blocks as representing different numbers have been assigned to this category. One example is a student taking the thousand-cube and says: ”this is two-hundred fifty-eight”. This is, of course, not correct but answers that fall into this category are typically guesses or estimations. The essential is that the material can represent numbers. No counting seems to be involved for answers within this category.

Category 2. One block is equal to ”one” [counting one-by-one]

To this category we refer students' answers where counting one-by-one is used in spite of the blocks' size and intended value. The counting does not necessarily need to be correct. As when one student counts like: ”thirty-five, thirty-two, thirty-three, ... no, thirty-six ...” and picks another block for each number the student says. In addition, there are students counting faster or slower than the pace with which (s)he picks the blocks. But there are also students who do count correctly. As in this case where 107 small single cubes are counted one-by-one:

Teacher: What is the value of this Pokémon-card? [shows a card with the number 117]

Student: Hundred ... eh ... one hundred seven.

Teacher: One hundred seven. Can you show how much that is?

Student: Oh, it will be a lot.

Teacher: Yes, it will be a lot.

Student: one, two, ... [counts loud one-by-one up to 100] ... one hundred. [looks at the teacher]

Teacher: One hundred.

Student: [counts on one-by-one up to 107. Again looks at the teacher]

Teacher: One hundred seven.

Student: Yes.

This conversation was an example of a student answer where each block is a ”one”. The counting one-by-one resulted in a pile of small cubes, see figure 2a.

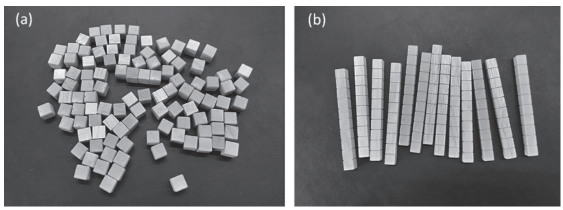

Figure 2. (a) *A student has represented a number (107) with the blocks in the base-ten material by counting them one-by-one* (b) *Another student counted one-by-one up to 13, but with rods*

Typically, the smaller unit-cubes were considered within this category. But students can also pick any other block to count as one. As shown in figure 2b, a response we could assign to this category is a student showing the number thirteen using ten-rods. In addition, we also assign student answers that represent, for instance, the number 258 with three piles of only small unit-cubes (one pile with two, one with five and one with eight cubes). This is of course correct in some sense, although not fully in line with the intention of the base-ten material. Anyhow, these students seem to have perceived the cardinality principle, or at least the one-to-one principle. Hence, in this category, the blocks seem to be regarded as numbers, but only one by one.

Category 3. Compound blocks represent the combined value

Answers that indicate an understanding of the compound-ness of the blocks, for instance, that a ten-rod has the same value as ten single unit-cubes, is assigned to this category. An example of understanding can be seen in this excerpt.

Teacher: You can decide the value [of the card] all by yourself.

Student: Two thousand nine hundred

Teacher: Can you show me with the blocks how much that is?

Student: How much it is worth? [picks a hundred-square and counts silently, but points at one row at a time.] Ten. One hundred. [picks more hundred-squares] Two hundred, three hundred. ... Nine hundred. [picks a tenth hundred-square] This is one thousand. [pushes the hundred-squares aside and takes a thousand-cube] Two thousand.

Teacher: How much did you say he should be worth?

Student: Two thousand

Category 4. Blocks can be used both as value and as number

Some student answers indicate an understanding of the blocks as if they can represent either number or value. In some cases, the same type of block is representing *both* number and value in the same example, as, for instance, shown in this excerpt.

Student: Two hundred fifty-eight.

Teacher: Yes, good. Can you show it also with the blocks?

Student: Two [takes two ten-rods] ... hundred [takes a hundred-square]. Fifty [takes five rods]. Eight [takes eight small cubes]. There.

This student's understanding resulted in the blocks shown in figure 3a. Here, the student seems to use ten-rods to represent five tens (that is, one ten-rod represents the value ten, or possibly the number one) and, at the same time, two rods times one hundred (hence, two ten-rods represent the number two). In any case, the hundred-square seems to represent a value.

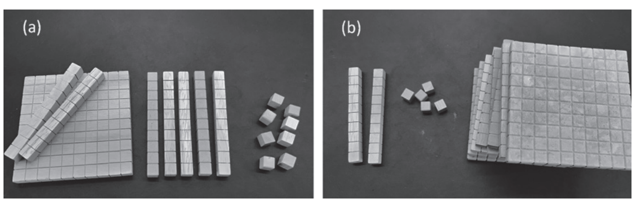

Figure 3. (a) *One students' representation of the number 258* (b) *Another students' representation of the number 852 (the student read the numbers backwards). In both cases, the blocks are used as both values and numbers*

Another example within this understanding of the blocks is shown in figure 3b. Here, the student's reply indicates that the digits in each position is right but where it seems the student did not see the value of each block. Another example from this category is when a student continued counting to the aimed value but with different blocks, as illustrated in figure 4 and in the following excerpt.

Student: Eh, thousand. [points at the one-thousand cube]

Teacher: And ... all this one thousand, or? [move a hand over the blocks]

Student: Thousand, eh ... [selects hundred-squares and counts on] Two thousand and three thousand.

Teacher: So, your Pokémon-card has the value of three thousand?

Student: [nodding]

In the example in this excerpt and in figure 4, the student indicates an understanding that one value can be represented by different blocks. Here, both thousand-cube and hundred-squares are representing one thousand, each.

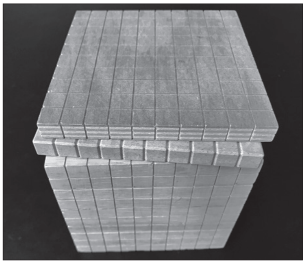

Figure 4. *A student's representation of the number 3000 – first a thousand block, and then two hundred blocks on top*

Category 5. The blocks in each position is of the same type

The main thing in this category is that the blocks are picked and counted from larger to smaller value. Student answers assigned to this category is based on piling blocks with higher value, and blocks with lower value, separately. However, the number of blocks is not necessarily limited within each decade (i.e., there could be more than nine blocks in each pile). The answers can give an incorrect illustration of the number, but nevertheless, they still indicate an understanding of the manipulative material that seems rooted in the place-value principle that the larger number are counted first and then the smaller numbers.

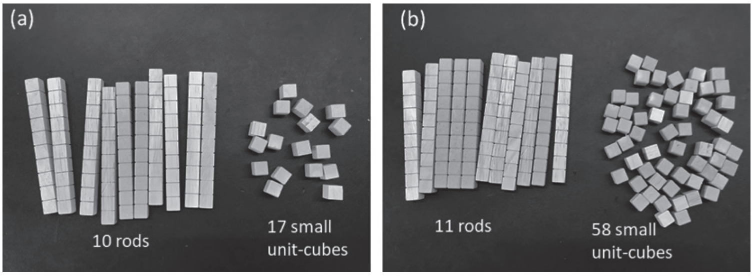

Figure 5. (a) *A student's representation of the number 117 by 10 ten-rods and 17 unit-cubes* (b) *Another student's illustration of 258 using 11 ten-rods and 58 unit-cubes*

In figure 5 we see two examples of students' answers that are using some kind of place-value-principle. Here the students are considering the blocks as representing one-hundred and seventeen (100 + 17), and two-hundred and fifty-eight (200 + 58), respectively. The blocks are positioned in such a way that the hierarchical structure of the place-value system is evident. This can be compared to figure 3b, where different blocks in different positions are used. Hence, the answer in figure 3b indicate that the blocks are used to mark numbers regardless of the size of the blocks. In this category (category 5) of students understanding of the blocks, we instead place answers that are indicating an understanding that the shape of the blocks symbolize some kind of value.

Category 6. In each position there is a maximum number of blocks

In this category we place student answers that indicate an understanding of place-value with a limit to the number of numbers in each position (maximum nine numbers). One example that beautifully illustrates an understanding of the maximum number of blocks is this excerpt.

Teacher: You can make up a number [of the card].

Student: Then it should be one-thousand nine-hundred and ninety-nine. [makes the configuration shown in figure 6]

Teacher: Ok, and if you now should write this number with digits, could you do that on this paper?

Student: Yes, easy. [writes 1999]

Teacher: Good. What would happen to your number if you just added one more unit?

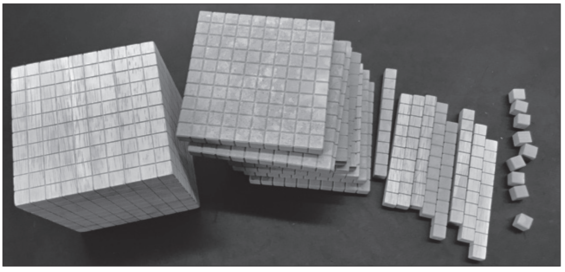

Figure 6. *A student's representation of the number 1999. It could be the maximum number (s)he could make in this system (although there were more blocks)*

Student: Then it would ... one hundred ... one thousand ... Oh, it will be two thousand.

The student in this excerpt appears to have an understanding of a maximum, both as s(he) seems to look for the maximum number overall (and hence chooses the number 1999), and seems to realize that when adding one, there must be a shift to higher values in the adjacent higher position.

In addition to the categories described above, we also identified a group of answers where the blocks were used merely as a set of building material. Although they do show some kind of conception of the base-ten material in itself, these answers do not involve the value or cardinality of numbers and are therefore not considered as a category of its own.

Discussion and educational implications

Based on the claim of Pang (2003), we can use the categorization to give input for the design of teaching sequences based on the base-ten material. These implications are then in terms of *hypothetical critical aspects*, deduced from the qualitative differences between phenomenographical categories.

From our data, we can speculate about hypothetical critical aspects. One such is that the number of blocks should mirror the number to be represented (the cardinality principle of the material). As an example, to represent the value of 200, one should represent this with an equivalent number of small cubes (compound or not). We note that understanding of base-ten blocks as units that, independent of size and shape, could represent "one" was noted already by Labinowicz (1985). However, we see a difference between counting blocks one-by-one independent of size (category 2) and perceiving blocks as representing value as well as number (category 4).

We also deduce the hypothetical critical aspect that each block (except the unit-cubes) represents a value, not a number, and that blocks of the same shape and size represent the same value. In relation to this, we identify a hypothetical critical aspect to be that the different size and shapes of the blocks can be used to represent the different digits in a number and that the hierarchy from smaller to larger blocks mirror the digits from right to left in a multi-digit number. The category which suggests that students use the blocks both as value and as number (category 4), indicates that it is important to make the students aware of the additive property of the material. Naturally, there is both a multiplicative and an additive component in the place-value issue, for instance as in $258 = 2 \times 100 + 5 \times 10 + 8 \times 1$. But traditionally, the intention of the material is that the student should be aware of the *additive* component between the different orders of magnitude. All-in-all we conclude that the qualitative categories can help make informed choices/decisions when designing teaching sequences, also regarding base-ten materials.

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Practices in multilingual mathematics classrooms: word problems

Eva Norén and Laura Caligari

This is a study in multilingual mathematics classrooms where Swedish is the language of instruction. Our aim is to explore what troubles do appear when students work with mathematical word problems and how teachers provide scaffolding for students' learning. Classroom observations and student interviews were conducted. The lessons followed the structure of introducing, modelling, jointly practicing and individually performing. Students' understanding of how to go about in the mathematical word problem genre advanced when they became familiar with the context and worked together by explaining, communicating in pairs and constructing individual word problems.

In this paper, we present a study conducted in multilingual mathematics classrooms where Swedish is the language of instruction. Languages have been seen as resources in multilingual mathematics classrooms since the 1990s (Barwell, 2009; Moschkovich, 2007; Prediger & Schueler-Meyer, 2017). However, few studies in classrooms with multiple languages represented have been pursued. Moreover, multilingual students in Swedish mathematics classrooms still underachieve (Skolverket, 2019). Thus, there is a need of more research in multilingual mathematics classrooms where students' and teachers' linguistic resources are drawn on. Another rational for such research is that multilingual classrooms have become more common in Sweden. Today, numerous mathematics classrooms consist of many mother tongues spoken by students and teachers. These students are all second language learners in mathematics. Research reveal that it can be difficult for students to communicate mathematical ideas and concepts when instruction is in their second language (Barwell, 2009). Furthermore, research) show that word problems are particularly difficult for second language learners (Clarkson, 2007).

Sweden of today is a multilingual country. A little more than 20 % of the students in compulsory school use other first languages than Swedish, Arabic being the most common (SvD, 2018). Mathematics teachers are supposed to support second language learners' acquisition in mathematics through content and language integrated teaching (van Eerde & Hajer, 2009; Hajer & Norén,

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2017), meaning that teachers' instruction have to focus on specific language requirements of mathematics. The overall aim for this study is to explore how teachers provide scaffolding when second language learners encounter difficulties when solving mathematical word problems. In this paper we more specifically ask:

1 What difficulties can be discerned when second language learners read and solve mathematical word problems?

2 What strategies do teachers use to scaffold second language learners' proficiencies in solving word problems?

This study, reports from a project that explored and tried out content and language integrated teaching of mathematics in school year 4 and further in year 5, from October 2017 until December 2018. We present students' classroom work with mathematical word problems and concepts, as well as interviews with students.

Word problems and multilingual classrooms

As noted above, studies in multilingual mathematics classrooms have shown that proficiency in the language of instruction relates to attainment in mathematics, and proficiency in two or more languages makes a difference for students' attainment. Nevertheless, strong proficiency in a language that is not the language of instruction has also been shown to have an impact for students' mathematical attainment (Clarkson, 2007).

Gerofsky (1996) defined word problems as a certain genre of mathematical literacy, comparing it with other spoken and literary genres. Word problems are often associated to as real-world problems that students can relate to, though they have been criticized to be too artificial (De Corte et al., 2000) even saying that "word problem solving" is disconnected from the real world. For example, in Greer (1993) 13–14-year-old students in Northern Ireland ignored aspects of the real world when answering word problems. One explanation is that students relate to contexts familiar to them. Barwell's focus on word problems and its genre in some of his research in multilingual mathematics classrooms (2009), showed that multilingual students relate to their own cultural experiences and home culture when solving and constructing word problems of their own. Another clarification is that second language learners often draw informally on their mother tongue when solving word problems, in order to increase their learning (Clarkson, 2009; Planas & Civil, 2013).

Theoretical considerations

We acknowledge the sociocultural nature of the resources second language learners and teachers bring to the mathematics classroom (Moschkovich, 2007).

Furthermore, also in line with the sociocultural tradition deriving from Vygotsky, language and content-based integrated teaching of mathematics and language supportive theories are adopted (Gibbons, 2002; Smit, 2013; van Eerde & Hajer, 2009). This tradition is related to scaffolding processes through challenges and support to promote students' autonomy. Smit (2013) writes that scaffolding is "a certain kind of support provided by teachers to help students move forward" (p. 14). Scaffolding is temporary and will gradually be removed as the structure being fostered becomes more solid and more reliable. Scaffolding in multilingual mathematics classrooms means:

– making the mathematical content understandable by putting it in contexts that the students can relate to,
– promoting students' active language use both orally and in writing in the mathematics classroom,
– offering varied and long-term linguistic support.

Drawing on sociocultural theories, the practices in content and language integrated classrooms often adapts scaffolding (Vygotksij, 1999) in a teaching and learning cycle model presented by Gibbons (2002). The cycle was further developed for mathematics teaching and learning by Smit (2013) in a project on second language learners' reasoning about line graphs. The teaching and learning cycle involves a series of four phases in which a specific genre of text required in a school context is introduced, modelled, together practiced and individually performed by the students. According to Gibbons (2002), the idea is that second language learners have to progressively acquire skills in the language of instruction along a continuum from every-day language to more academic language, from spoken to written language and connected by literate spoken language. However, second language learners don't acquire academic language skills through classroom discourse, like first language students often do. Second language learners need scaffolding in relation to the academic language of each school subject, in this paper mathematical language, says Gibbons (2002).

Methodology

The empirical data is ethnographic (Hammersley & Atkinson, 2007) and derives from classroom observations, fieldnotes, audio recordings of classroom interaction, collections of students' materials, and student interviews. In two of the classrooms (Red School) in this study, besides Swedish, there were nine first languages spoken by the students[1]. In the third classroom (Blue School) there were 13 first languages spoken besides Swedish[2]. Students' parents in both schools were informed about the ethical issues and signed consent forms. All names of schools and students are pseudonyms.

The study draws on 14 participant observations in the Red school and the two classes respectively, in which 40 students are learning mathematics. To verify findings about how students perceived mathematical word problems in the Red school we interviewed students in another school, the Blue School, where students solved some of the mathematical word problems that the students in the Red School had solved. Interviews with 8 students were recorded individually. The students solved word problems, answered questions and talked about their experiences of solving word problems. They took on the tasks by reading the word problems and solving them while ”thinking” aloud.

The methodology is interpretative and relays to knowledge building and cultivates research capability through collaborative analysis and critical reflection of students learning and classroom practices (Calder & Murphy, 2018). Regarding difficulties discerned when students solve mathematical word problems, the analysis of students' interaction with teachers and classmates identified themes (Braun & Clark, 2006). The themes were thoroughly linked to the practice in the classrooms. The analysis also showed that teachers' scaffolding strategies, depended on the difficulties experienced by their students.

Participant observations

Participant observations were conducted in the Red school where two teachers started to change their mathematics teaching towards using more content and language integrated methods. So far, their mathematics teaching hadn't helped their students achieve as expected. Word problems in mathematics were the most challenging area for the students to work with, as well as for the teachers to teach. For example, when the students got the assignment to work with a thematic chapter "The Kolmården Zoo" with a lot of word problems from their textbook, the classroom became "chaotic" (teachers' expression). Besides the two teachers there were mother tongue supervising personnel. Thus, students had opportunities to use their first languages alongside Swedish in the mathematics classroom.

Interviews

Students in the Blue school expressed that the word problems they worked with in the classroom were the ones from their mathematics textbook and that they worked individually with them. The student also explained that it should be "quiet so you don't disturb each other" (student quote). They also said that there were some occasions when they worked in small groups, for example, if the word problem was difficult, then they read the text with the teacher and focused difficult words.

During the interviews in the Blue School, the students said that word problems were different from mathematical assignments without words in the mathematics textbook. The word problems required an understanding of their own reading and they had to decide on what calculation methods to use.

Findings

In the thematic analysis of fieldnotes and audio recordings (from classrooms and interviews), three main themes were discerned regarding what seemed to create difficulties for students when solving word problems: ”difficult” words, ”difficult” contexts and ”conceptual understanding”. In various ways, the themes relate to each other. The analysis also displays strategies for how teachers scaffold their second language learners. The strategies were promoting students to: use their first language, ask questions on the word problems presented, mark ”key words” in the word problems and to actively use Swedish orally and in writing. Teachers' scaffolding was sometimes planned ahead, building on earlier lessons, results on diagnostic tests or text material that was going to be used. At other occasions the scaffolding unfolded while teaching, building on interaction in the classroom, for example students' questions and students answers to teachers' questions. The scaffolding was often relating to students understanding of mathematical concepts.

”Difficult” words

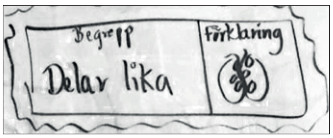
A mathematical word can be difficult, but the reason may be that the concept the word represents may not be understood. It is sometimes hard to define if students have troubles with the mathematical words, their second language or if a student has missed the understanding of a concept. There are always linguistic challenges in learning mathematics in a second language (Schleppegrell, 2007).

The students in the Red School had regularly failed when solving word problems in the classroom. Therefore, the teachers started systematically to diagnose the students on mathematical word problems One diagnostic test was taken from McIntosh (2008). Firstly, students solved the diagnostic test in Swedish, thereafter, they were offered to solve the same test in their mother tongue. A word problem that was difficult for the students to solve in both languages was:

> Bo cut his apple in half, and then cut one of the halves in half again [*Bo delar sitt äpple i halvor. Sedan delar han ena halvan mitt itu*].
>
> a) How many pieces of apple does he now have?
>
> b) What fraction of the whole apple is one of the small pieces?

The task was formulated in Swedish, though here from the original source in English. The most difficult part of the wording was neither the first part of the sentence, nor the next. However, there was a wording in Swedish that seemed to be difficult for most students, *mitt itu* [3] (it is not a straightforward translation of ”one of the halves again”). The students had no difficulties when cutting the apple in half, but had to stop reading at *mitt itu*, get an explanation, and then go on cutting one of the halves in half again. Regarding the questions it was obviously a complex word problem to solve. Only four students out of 39 could solve the (b) question in Swedish. One teacher strategy for scaffolding students was that mother tongue speaking teachers explained the words and the mathematical phenomena to the students on their mother tongue. This was promoted by the Swedish teachers.

When solving the word problem in their mother tongue some students found it was no difference or it was easier, but some students felt more comfortable using their mother tongue. Another scaffolding strategy was to draw and write on the white board and grouping students in smaller groups, giving them follow up tasks to work actively with words earlier articulated in the diagnostic test. One example is shown in figure 1.



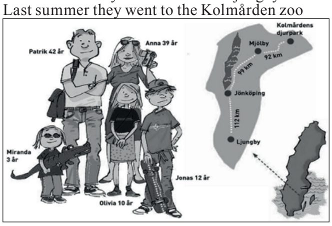
Concept	Explanation
Sharing equally	Drawing of halt an apple

Figure 1. *Photo from the white board*

It was chosen by one teacher because she wanted her students to elaborate more on the Swedish wording *mitt itu*[4] relating it to ”share equally in two parts”.

When interviewing Blue Schools students, some words in the word problems seemed to be difficult to read and understand. Students got stuck on words and it showed as they slowed down their reading, sounded out the words aloud or reread words. The most difficult words were not always the mathematical words, but names of cities and people. Mathematics words like *sträcka* [distance] on a map caused trouble, because it has the same pronunciation and almost the same spelling as *streck* [line]. Additionally, lines were drawn on the map to show distances (see figure 2). Another example, “there are different words in the same assignment” (student quote) in one of the word problems and the picture that followed the assignment.

This is the family Svensson from Ljungby in Småland.
Last summer they went to the Kolmården zoo



164. How much older is a) Daddy than mummy? b) Jonas than Miranda?
165. Look at the map. The journey started in Ljungby. The first day they went to Jönköping and then to Mjölby. How long was it?

164. Hur mycket äldre är a) pappa än mamma? b) Jonas än Miranda?
165.Titta på kartan. Resan började I Ljungby. Första dagen åkte de till Jönköping och sedan till Mjölby. Hur lång *sträcka* blev det?

Figure 2. *Kolmården zoo tasks (Undvall et al., 2011, p. 43)*[5]

In the example in figure 3, two students reacted to the words *serietidningar* [comic books] and *tidningar* [magazines] in the word problem but in the illustration, it said *serier* [series] “is it the same thing? It can’t be series you watch on TV?”, a student in the Blue school reasoned out loud. The three words all referred to the same comic books, thus, confusing the students[6].

I ett stånd såldes gamla serietidningar. Peter köpte 16 tidningar. Hur mycket fick han betala?

In one stand, old comic books were sold. Peter bought 16 magazines. How much did he have to pay?

Figure 3. *The Market theme, the word* serier *up to the left (Undvall, et al, 2011)* [5]

”Difficult” contexts

In the Red school, the teachers returned to theme about a trip to Kolmården in the textbook. When it was first introduced a month before, students skipped the tasks. Kolmården is a big park with wild animals from different parts of the world. The tasks to solve included a family going there, word problems on the family members’ ages and their trip by car to the zoo.

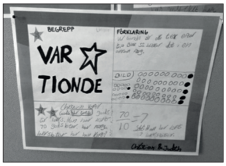
None of the students were familiar with the Kolmården context, thus, a lot of explanations were needed. The scaffolding strategy followed a structure in which one of the teachers started with bringing the first two pictures up on the smart board. The teacher inventoried what the students knew by letting them ask questions and collaboratively communicate about the pictures and it’s mathematical content. Teachers wrote on the smart board. Both text and pictures were carrying mathematical meaning. Finally, students solved the word problems and later, in pairs, construed their own word problems for other students to solve. When working in the textbook with other themes, like the Market (figure 3), the context was familiar to most students, which helped them solve the word problems.

Conceptual understanding

When students worked by themselves in the textbook, in the beginning of school year 4 in the Red School, many of them tried to skip the word problems. Students were saying “the word problems are too difficult” and “I don’t know what to do”. When elaborating with the Kolmården theme the analysis showed it was obviously difficult for the students to understand the context and the wording. One example: “Look at the map”. The picture of the map (figure 2) carries meaning to the word problem, thus, not understanding the map is a drawback. The wording ”mitt itu” when cutting an apple, and other examples from word problems like ”every tenth”, serier [comic books] and ”addition”, also created troubles relating to concepts.

Before starting to solve word problems and in line with planned scaffolding, the teachers taught strategies like, “look for difficult words and words you don’t understand, mark the numbers, mark the words, underline the question/s, what information can you find”? Teachers and students together defined which words were mathematical concepts and which were some kind of key words or every day words used for constructing word problem. Words to define were picked from the word problems. Students categorized the words and teachers wrote them in lists on the white board, one for concepts, and one for key words. The teachers often led collaborative classroom talk with students' questions as starting points.

Successively, students were assigned concepts from the list. In pairs they



Concept EVERY TENTH	Explanation Every 10th, for example, after ten flashes, it turns into a different color.
Task Christian buys candy. Every tenth candy is a piece of liquorish. He bought 70 pieces of candy. How many pieces of liquorish did he buy?	Solution $\frac{70}{10} = 7$ Answer: He bought 7 pieces of liquorish

Figure 4. *Student material*

elaborated on the concepts by writing explanations, so that others would understand the concept, while also constructing word problems for others to solve. Figure 4 is an illustration of that certain activities, such as collaborative work, encouraged students to talk to each other and to actively take on various assignments. The regular work in pairs made students exchange mathematical knowing. In activities like the exploration and construction in figure 4 of the concept ”every tenth”, students got used to examine concepts, explain to each other and communicate. This scaffolding strategy helped students solve and construe word problems of their own. Thus, it became obvious that the scaffolding strategies used, motivated students to solve word problems at the end of the school year 4.

Discussion

This study shows the importance of examining, on the one hand, what difficulties second language students encounter when working with word problems and, on the other hand, how teachers can scaffold and support second language learners' mathematical word problem solving skill, by reducing difficulties observed.

In the Red School classrooms, second language learners worked more engaged when familiar with the context and when they had been given time to work systematically with word problems. For example, the word problems relating to Kolmården, compared to the word problems relating to the Market, created challenges for the second language learning students. One reason could be that students related the Market to their experiences and thus their home culture and mother tongue. The Kolmården theme was the opposite, students couldn't relate to their experiences (Clarkson, 2009; Planas & Civil, 2013). Students in the Blue school also got stuck on certain words when reading word problems. Those words, like *sträcka* and *streck* [distance and line], are usually learned in relation to everyday language, hence not familiar to all second language learners. In the Red school, students' active use of language was promoted both orally and in writing. For example, when elaborating on mathematical concepts or when construing word problems for their peers. Students were offered linguistic support, mostly in Swedish but also in their mother tongue, scaffolding them to negotiate meaning (Gibbons, 2002). In our analysis we noted that even though the teachers in the Red school were not deliberately adapting the teaching and learning cycle (Gibbons, 2002; Smit, 2013) their lessons followed the structure of introducing, modelling, jointly practicing and individually performing. In other words, our study indicates that second language learners' understanding of how to go about in the mathematical word problem genre advance when they become familiar with the context, get access to their mother tongue and are promoted to work together by explaining, communicating in pairs and construing individual word problems.

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Notes

1 Arabic, Dutch, French, Persian, Polish, Russian, Spanish, Tigrinya and Turkish.

2 Albanian, Arabic, Bengali, Chaldean, English, Greek, Igbo, Kurdish, Moroccan Arabic, Romani, Somali, Turkish, and Persian.

3 The wording *mitt itu* would probably not have caused the same troubles for a native Swedish speaking student because the wording is part of the everyday language, and not the academic language.

4 ”Mitt itu” can in Swedish also be to ”dela lika” or ”dela i två lika delar” *share equally* in two parts.

5 Illustration by Johan Unenge, who has given his permission to use it in this paper.

6 It is complicated to explain this in English, it makes sense in Swedish but it doesn't in English. In English comic books and magazines doesn't mean the same kind of publication, but in Swedish *serietidningar, tidningar* and *serier* can refer to same publication, here, the ones soled at the market.

Children’s awareness of numbers’ part-whole relations when bridging through 10

Camilla Björklund

This paper reports findings from an assessment of fifty-one 6–7-yearold children’s ways of solving arithmetic tasks. In particular, ”double counting” strategies were found to be enacted when bridging through 10 even though the children evidently had learnt conceptually more powerful ways of encountering arithmetic tasks in the number range 1–10. In this paper the reasons for this outcome are analysed and discussed in terms of children’s ways of experiencing numbers’ part-whole relations.

That young children change strategies between arithmetic tasks is not unknown to the field of mathematics education, but the explanations why children abandon successful and conceptually logic strategies for more primitive ones differ. The aim of this paper is to contribute one way of explaining such strategy changes in terms of children’s ways of experiencing the part-whole relations of numbers. In order to make this contribution, a specific research question was raised: What awareness of numbers’ part-whole relations is reflected in the children’s ways of handling arithmetic tasks? This question is answered through a qualitative analysis of 51 children’s strategies in solving addition and subtraction tasks within the number range 1–10 and bridging through 10.

Background

This study is part of the research project FASETT, which aimed to deepen the knowledge about young children’s learning of elementary arithmetic. The project took its’ departure point from earlier research in which children’s arithmetic skills have been studied in terms of their ways of experiencing numbers (Neuman, 1987). An intervention program was conducted in preschool where children were afforded to experience numbers in such ways that in particular numbers’ part-whole relations were coming through (Björklund et al., 2020). This, based on the assumption that children who learn to discern numbers’ structural relationships (see Venkat et al., 2019) would be better prepared to handle elementary arithmetic tasks and also apply this knowledge to a larger number range. A follow-up assessment one year after the intervention was

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conducted to evaluate any long-term effects of the intervention. Results from this follow-up assessment is the object of inquiry in this paper.

Research on early arithmetic skills

There is an abundance of studies describing children's strategies in solving arithmetic tasks (e.g. Baroody & Purpura, 2017; Fuson, 1988). Much is learnt from these studies about more or less prosperous strategies and what cognitive abilities that influence the child in making use of certain strategies. A cognitive perspective in research induces that early arithmetic understanding is local to particular principles, proceeding through more and more complex levels (Starkey & Gelman, 1982). There are claims that children are "forced" to develop more advanced strategies when the number range exceeds 10 and children can no longer rely on their fingers to represent units (Carpenter & Moser, 1982). This leads for example to a Counting-on strategy rather than the less advanced Counting-all strategy when exceeding 10. Counting strategies are thus ways to keep track of counted units often referred to a number line (mental or physical) and are commonly observed among young children (Laski et al., 2014).

Furthermore, the strategy "double counting" is by Fuson (1988) defined as a way of keeping track where the sequence of words are entities to be counted. When solving e.g. $15-7=\_$ the child has to keep track of how many units are taken away and how many are left, at the same time, usually by raising one finger for each counted counting word and then counting the raised fingers, or by indexing with number words: "15, 14 (1 taken away), 13 (2), ... 8 (7)", thus the numbers are double counted. This is considered a normal step in the development of arithmetic skills, by Steffe (2004), as an extension of counting and thus a higher level of functioning. Neuman (1987) also observed this strategy in children's arithmetic problem solving, but concluded it being a strategy invented by the children, when they experience arithmetic tasks as operating with single units and not as a flexible part-whole relation. Double counting is according to Neuman in this sense not a powerful strategy since it is cognitively demanding and put a heavy load on the working memory.

A contrast to the dominance on seeing counting strategies as the outset for learning arithmetic is given by researchers who advocate that a structural approach, attending to numbers as part-whole relations, should be emphasized already in the early years (Brownell, 1935; Davydov, 1982; Neuman, 1987; Schmittau, 2003). This based on the fact that counting strategies may help children solve simple arithmetic tasks but do not support children in recognizing the numerical relations between and within numbers that more advanced tasks presupposes. An emergent awareness of structural relationships appears among young children as they analyse and discern local relationships of numbers in arithmetic tasks. This will eventually allow them to identify more general mathematical relationships and properties that enable more advanced arithmetic strategies such as decomposition and commutativity to be used (Venkat et al., 2019).

Gray and Tall (1994) argue that individual differences in arithmetic skills development and enactment are related to preferences of counting procedures, or ability to derive from what is already known. Most children have both alternatives available and can choose the most convenient way to handle a task. Nevertheless, Gray and Tall show that children who primarily rely on procedural strategies, such as counting single units, do not relate a task to earlier solved tasks as known facts, which in turn is not provoking a need to remember facts, since the counting procedure provides a security. Methods based on deriving from what is already known, on the other hand, enhance the ability to remember facts and support children in using those facts; ”I know 4 plus 4 is 8, then 4 plus 3 must be one less, 7”.

In a long-term perspective the strategy preference in solving arithmetic tasks becomes critical when the number range increases and more advanced arithmetic is introduced. According to Ostad (1998), a strategy learnt in isolation from its conceptual foundation induces more errors (see also Geary et al. 2004) and is hard to transfer to novel problems. Furthermore, if a single unit counting strategy (such as double counting) has been established as the preferred one, it is not easily abandoned (Cheng, 2012). Thus, it seems that children's use of strategies is more complex than learnt ways to solve problems. Consequently, for children to advance their arithmetic skills they need to develop a conceptual understanding of numbers, or in other words, to see numbers' part-whole relations in ways that allow them to act in accordance with a structural relationship identified in an arithmetic task.

Methods

A structural approach in teaching elementary arithmetic to preschool children was implemented in the FASETT intervention program with the intention to afford a conceptually solid basis for arithmetic skills development. The structural approach meant directing attention to numbers' part-whole relations as an outset, rather than counting single units. This was enacted in designed activities conducted by preschool teachers in five preschools with children attending their last preschool year (as 5-yearolds). To evaluate the outcomes of the intervention we investigated children's ways of solving arithmetic tasks and interpreted their actions and solutions as expressions of ways of experiencing numbers and number relations, in line with the theoretical framework Variation theory of learning (Marton, 2015). Assessments were done before, right after and delayed one year after the intervention. All assessments were video-recorded, with the children's legal representatives' informed consent. The third (delayed) assessment is the object of inquiry in this paper, Video-documentation was a criterion for inclusion in the analysis, since children's ways of using their fingers were considered important data, which could not be collected in sufficient details by any other method. 51 children were included. The assessments were done individually by members of the research team in the children's own school settings.

Items in the assessment were designed to evoke the possibility to enact different strategies both in the number range 1–10 and bridging through 10. The tasks were given as oral word problems and the children were asked to explain verbally or in other ways show how they had arrived at their answer. No manipulatives were available but the children were encouraged to use their fingers if they thought it would be helpful. The target tasks for this particular study are:

A Your friend has 2 shells and you have 5. How many do you have together?

B On Saturday you get 10 candies. You eat 6 of them. How many are left?

C Today you are going to set the table. There are 3 glasses already on the table but there will be 8 persons eating. How many more glasses do you need to get?

D On your birthday, you are blowing up balloons. After the party, 3 are broken and 6 are whole. How many balloons did you blow up from the beginning?

E You have 8 marbles and your friend 5. How many do you have together?

F If you have 15 stickers and give 7 to your friend. How many are left?

To answer the research question *how is the awareness of numbers' part-whole relations reflected in the children's ways of handling arithmetic tasks* a total of 306 observations across the tasks A–F (51 children x 6 tasks), were analysed. Each answer was coded as correct/incorrect and according to the children's different ways of handling the tasks, first considering the strategies observed among all children and then regarding the relation to tasks below or bridging through 10.

The analysis of children's expressed awareness of part-whole relations was done using Variation theory as theoretical framework. According to Variation theory (Marton, 2015), children can only act in accordance with their way of experiencing a phenomenon, that is, in this study the children's actions and explanations are interpreted as expressions of their way of experiencing the task and the numbers given in the task. Some aspects (e.g. ordinality of numbers) are prominent when enacting counting single unit strategies and others are indispensable to discern in order to be able to re-group or decompose number sets (e.g. numbers' part-whole-relations). Aspects that are discerned thus determine different ways of experiencing (and thus acting on) the same task. If the child is only able to see some, but not other necessary aspects, it is assumed to limit what strategies the child is able to enact. Based on these theoretical principles, an analysis of the strategies enacted by the children can reveal which aspects they have discerned and which are yet undiscerned, in particular when some tasks are provoking difficulties and the observations show children encountering such tasks in different ways. This theoretically driven approach allows us to describe what aspects in particular the children need to ”see” in order to make use of powerful strategies. The following excerpts are answers given to the task C (3 + _ = 8) and will illustrate the difference between how a child who is interpreted as experiencing numbers as part-whole relation acts and thus enacts a structuring strategy, and a child who sees numbers as single units, thus enacting a counting strategy, acts:

> *Counting*: The child starts with three fingers unfolded on the right hand (thumb, index and long finger), unfolds the two other fingers and two more in consecutive order on the left hand, moving the lips silently. Then moving each finger from the right thumb in the same order and unfolds the long finger on the left hand as well. Starts moving each finger on the right hand again, then says ”I think I need eight, no”, counts by pointing at each finger on the right hand, saying ”I think I need six more”.

> *Structuring*: The child unfolds three fingers on the left hand and three more on the right hand, saying ”five, there were three glasses and it should be eight, then I have to get five more to make eight” raises the two folded fingers on the right hand showing eight unfolded fingers.

The different ways of handling the same task reveal differences in the children’s ways of experiencing numbers: the first child creates sets by adding ones but fail to coordinate different sets since the counted units are not seen as separated from the whole. This makes it difficult for the child to find out what constitutes the missing part. The second example shows on the other hand a child handling the part and the whole simultaneously, he sees ”the five in the eight” and his actions reflects an awareness of the part-whole relations of the numbers in the task. In the results section below, these differences in awareness of numbers are discussed in terms of discerned aspects of numbers.

Results

The observations of the strategies enacted by the children reveal a variation of ways to solve the tasks. The strategies found in the tasks below 10 are:

Known facts. The child knows the answer, retrieved from memory or as a result from mental arithmetic.

Structuring. Transforming a problem into a simpler one, attending to number relations, by decomposing numbers or structuring the task with finger patterns.

Counting. Using a strategy that includes counting single units, such as Counting-all, Counting-on or Counting-down.

Guessing. Answering without apparent counting or reasonable explanations.

In the tasks bridging through 10 the same strategies were observed, with an addition of:

Structuring towards 10. Transforming a problem as above but in addition using base-10 properties of the number system.

Double-counting. The number of units taken away or added are kept track on by (mostly) fingers raised for each counting word and then identifying the counted units (fingers) to get the answer.

Among the children in the study many know number facts in the number range 1–10 and are able to retrieve from facts to find an answer. These children are considered having discerned the number relations and can attend to numbers' part-whole relations in solving the tasks. Number facts are seldom observed when bridging through 10, which means the children have to enact some structuring or counting strategy to find an answer. The dominating strategy when bridging through 10 is counting (64 of 102 observations = 63 %). However, the tasks are difficult to handle by double-counting (41 of 64 = 64 % correct answers) while the structuring strategy in the tasks bridging through 10 mostly ends up with the correct answer (17 of 19 = 89 % correct).

When further analysing the children's ways of solving arithmetic tasks an inconsistency in strategy use was discovered. Some children who were found to successfully make use of a structuring approach in the lower number range failed to use this approach when bridging through 10 and were then observed to enact a double-counting strategy in the higher number range. In the following sections a pattern of enacted strategies are presented and discussed in terms of children's awareness of the part-whole relations, based on individual children's enactment in tasks C, D, E and F (the first two tasks, $2+5$ and $10-6$ are excluded since answers were most often given as known facts). Three children were not categorized since they were only giving random answers without explanations to the tasks.

Counting as primary strategy. Fourteen children were using counting and double counting as the *only* strategy for solving the arithmetic tasks. One illustration of this strategy is the following answer to task E ($8+5$).

Child: I have 8. And then it's 5 more. Then I count 9, 10, 11, 12, 13.

Interv.: How do you know when to stop counting?

Child: I count at the same time as I count. Like this, 8 and 1 [pointing at the table when holding up his index finger], 9 is 1, 10 is 2 [holding up two fingers], 11 is 3 [holding up three fingers], 12 is 4 [holding up four fingers] and 13 is 5 [holding up five finger].

This child keeps track of the counted number words both verbally ”9 is 1, 10 is 2 ...” and by raising one finger for each added unit and succeeds in solving the task. The strategy is time-consuming and the child has to keep simultaneous attention to two parallel sequences. Enacting these kind of strategies may be due to the children experiencing numbers as single units to be added to the larger set but do not identify the relation between the given parts and the whole. The cumbersome double counting strategy becomes their only option (other than guessing) because the children do not experience the tasks constituting a structured part-whole relation. Consequently, these children may not be able to use any decomposing strategies or retrieving facts from memory – it does not make sense to them. Nevertheless, in straight forward additions like task E they quite often succeed in finding out the answer. However, in the task F (15 – 7), this strategy becomes an obstacle when having to keep track of units on the number sequence *backwards*.

Child: Fifteen, then I count down. Fifteen, fourteen [moving the thumb and index finger]. Fifteen, fourteen. Thirteen [holding the ring finger]. I don't know what comes next [counts silently]. Twelve. Is it seven then?

Experiencing numbers as single units induces these counting strategies that may solve simple arithmetic task, but when the parts exceed the number of object that the child is able to perceive or when the fingers cannot represent each unit necessary for operating on the task, the absence of structure based on number relations becomes critical.

Changing strategies when bridging through 10. Twenty-five children were found to use mixed strategies. Nine of them, who were observed to enact double-counting when bridging through 10 did seem to experience numbers' part-whole relations within the number range 1–10, as they enacted structuring strategies or known facts when solving the tasks below 10. In other words, they *did* identify number relations below 10 but could not generalize this idea to a larger number range. This brings forth an important insight: In tasks where children need to bridge through 10, quite many still seem not to experience some of the necessary aspects of the part-whole relations of numbers to enact a strategy based on structuring the part-whole relation of numbers, since they change a highly successful strategy (structuring or deriving from known facts) to a more cumbersome, error-prone and time-consuming strategy (e.g. double-counting).

Experiencing 10 as a benchmark. The analysis showed that nine children were structuring throughout all of the tasks. Based on their verbal reasoning when solving the tasks, they were experiencing 10 as a benchmark and directed their efforts to solve the task by structuring on their fingers and by decomposing numbers, as illustrated in the following excerpt from task E (8 + 5 = _).

Child: Thirteen. If you take 8, and 2 from the 5 and adding those marbles, it makes 10 here, and 3 left. If you put them together, it makes 13.

In other words, the child is able to enact the associative law of addition and seems to experience the parts in the arithmetic task in two ways, simultaneously: 8 + 5 is seen as 8 + 2 + 3 where 2 is part of a new part 10 but also a part of the decomposed 5. A similar way of reasoning about how to solve the subtraction task F (15 – 7 = _), bridging through 10 is shown in the following:

Child: Eight. First I thought like take away 7 and then I started counting 7. And 8, 9, 10, then it was 3 more and then I had 5, and 5 plus 3 is 8.

This child considers the inverse relation between addition and subtraction and adds units to 7. However, there seems to be an important benchmark when reaching 10 as the child then experiences the difference between 10 and 15 to be added to the 3. The task 15 – 7 = _ is thus experienced as a composition of 7 + (3 + 5) = 15. A closer look at these children's ways of handling the tasks reveals that they have a very clear idea of the number relations in the task, seeing them as composite sets relating to each other in a part-whole structure. What stands out is that they create units of 5 or 10 and decompose given numbers to create new units to "fill up" 5 or 10. In other terms, they experience numbers in a quite different way than children who use counting as their only strategy and also compared to the children who change strategy when bridging through 10 – they experience numbers' part-whole relations simultaneously and take hold on ten as a benchmark for their structuring and furthermore see the task as to be mathematical rather than empirical.

Conclusion

Young children's struggle with bridging through 10 is not unknown to either the research community or teachers. Laski et al. (2014) for instance showed that children's use of base-10 decomposition in arithmetic tasks was related to their knowledge of number structure and children are able to use multiple strategies depending on the task. The intervention program in FASETT emphasised numerical relations and most children learned and enacted structural strategies when solving tasks below 10. However, of interest in this particular paper is *why* children who have learned to structure and use the part-whole relations of numbers below 10 change to strategies that do not attend to the part-whole relations when tasks are bridging through 10. The results from the analysis presented here might shed light on what constitute knowledge of number structure in a general sense and consequently what teaching elementary arithmetic should consider.

Counting seems to be a safe way to find a solution among our participants and many use fingers for keeping track of counted units. Some children use (double)counting as their *only* strategy, and do in fact solve many of the tasks. However, the field of research converge on the view that structuring strategies and using retrieved facts is preferred in a long-term perspective (Baroody &

Purpura, 2017; Gray & Tall, 1994). Nevertheless, based on this investigation it can be suggested that this cannot be taught as a strategy alone, it is rather a question of which aspects of numbers that the child experiences in a task. When comparing the mixed strategy users with those who only use structuring strategies, one aspect emerges as to be critical to experience: structuring *towards* 10. Children are observed to attend to the part-whole relations of numbers and enact structuring strategies, but only those who also experience 10 as a benchmark seem to be able to generalize their seeing part-whole relations to numbers above 10 and decomposing parts to fit a 10-structure.

Fuson (1988) concluded, supported by Steffe (2004), that counting-the-count methods were advanced. Based on the results presented above, double-counting is however not an advanced strategy, even though it is cognitively demanding to keep track of parallel number lines. It rather becomes a necessary way of handling arithmetic tasks if the child does not see the task as composite sets with 10 being a critical benchmark. Strategies such as counting single units and double counting, do not support the child in experiencing the relation between and within numbers, or the base 10-structure to simplify an arithmetic task. If counting single units is the only strategy that the child is able to enact, more advanced arithmetic tasks in the number range above twenty and using multiplication will probably not be possible either. This is in line with Gray and Tall's (1994) conclusion that more able children appear to do a qualitatively different sort of mathematics than the less able, since they have alternatives and can choose the most convenient way to solve a problem. The study presented in this paper contributes that in this particular arithmetic context (below and bridging through 10) it is necessary that the child experiences number relations but also experiences 10 as a benchmark, in order to access the more powerful structuring strategies in the larger number range. In forthcoming interventions it would thus be important to direct attention to this aspect as to be critical for learning arithmetic skills, in addition to facilitating the discernment of part-whole relations of numbers as an outset.

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Towards a theoretical understanding of learning with self-explanation prompts

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Oral or written requests to students to self-explain important aspects in a task at hand (e.g. self-explanation prompts) has proven to increase learning. Research about such prompts has mainly been implemented with cognitive perspectives focused on the individual. In this paper, we suggest an alternative analytical framework grounded in a sociocultural theory. This framework is valuable because it adapts to the individual learning process as well as to the learning process that takes place in group work. In addition, this framework contributes valuable guidance to the teacher and to authors of teaching materials as well as to researchers in mathematics education. The analytical framework is explained in relation to an example task. An excerpt from student group work is also discussed.

Self-explanation prompts (SEPs), have previously been used and described as tools for teaching. SEPs are questions or elicitations that serve to induce meaningful explanations for oneself to make sense of new information (e.g. Rittle-Johnson et al., 2017). SEPs are most often used in textbooks, either as parts of the introduction when a new concept is introduced, or as a step in a step-by-step task. The SEP can for example request the reader to explain *why* something is true, *what* some aspect in a diagram means, or *how* a solution method works. SEPs can also be used orally for example as an element of a lecture. The theoretical foundation for learning with SEPs has mainly been described within the frame of cognitive theories (e.g. Nokes et al., 2011; Rau et al., 2017) or without any explicit theoretical frame (e.g. Corradi et al., 2012; Eysink & de Jong, 2011). The aim of this paper is to propose a theoretical framing of SEPs and how they can enhance learning, based on the idea of scaffolding grounded in a sociocultural tradition and Vygotskij's zone of proximal development. The theoretical explanation suggested here, contribute a solid theoretical understanding of how learning occurs for individuals as well as in group work. A second aim is to create an analytical framework to lay the foundation for well-informed teaching strategies, task design and analysis of students' learning in research on SEPs. If there is a lack of a thorough theoretical understanding of the processes involved

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when working with a teaching strategy, it is easily happened that central prerequisites for the intended learning to occur are lost.

Self-explanation prompts

In mathematics, the prompted self-explanations can for example be about the meaning of a mathematical concept or about a solution method. The purpose of the explanation is to clarify certain crucial aspects of the phenomenon of interest, connections between different parts, or links to previous knowledge, and thus strengthen one's own understanding in the learning situation (Berthold et al., 2009; McEldoon et al., 2013). Self-explanations have proven effective to enhance learning (e.g. Rittle-Johnson et al., 2017). However, self-explanations do most often not happen spontaneously (Schworm & Renkl, 2007) and therefore prompts to self-explain can be used. Despite the focus on self, SEPs can be used in group work because one person's self-explanation can constitute a piece towards the group's mutual inferences. By engaging in the explanation, central aspects are made clear to the individual, but are also made apparent to the whole group.

SEPs can be of various kinds and can be used for different purposes (see e.g. Dyrvold & Bergvall, 2019). One frequent purpose of SEPs described in previous research is as a means to foster conceptual understanding. By formulating SEPs as questions typically including a why-question or a prompt to discuss, students are encouraged to actively make inferences and construct arguments and thus strengthen their conceptual understanding. Another purpose is to support reading, mainly of multimodal texts. In this case, prompts could be designed as gap-filling tasks or questions prompting students to make inferences about the text and its content. Prompts aiming to support multimodal reading often support students in how to relate different parts of the text, such as relating quantities to bars in a diagram (Dyrvold & Bergvall, 2019). An example of a SEP aiming to foster conceptual understanding is given in the task ”The Sunflower” (figure 1).

The SEP is expressed by the sentence ”First, discuss what it means for something to grow at the same rate every week”. The SEP is supposed to support the students' understanding of ”at the same rate”, a crucial aspect of the concept of proportion. The intention with this SEP is to promote students' discussion of this central aspect and thus to support the students' development of conceptual understanding.

Theoretical perspectives

This section starts with an overview of theoretical perspectives in previous research on SEPs. Thereafter we present a substantially different theoretical

Kim grows sunflowers during the summer holidays. The summer vacation is 7 weeks long. The sunflowers break through the soil just as the summer holidays begin and then grow at the same rate every week.

First, discuss what it means for something to grow at the same rate every week.

Task: One of Kim's sunflowers is 42 centimetres after the summer holidays. How high was the sunflower two weeks after school closure if it grew at the same rate every week?

Figure 1. *The Sunflower*

perspective which highlights how SEPs function as scaffolding and nurture fruitful interrelations between thought and language, individually and also in interaction. The alternative explanation of learning with SEPs is presented in two sections. First, the fundamentals of scaffolding theory are explained within this context and second, the sociocultural tradition with emphasis on thought and language is presented. Finally, this perspective is elaborated in relation to an example of a SEP which illustrate the contribution of this theoretical view.

Theoretical framing in previous studies about SEP

In our reading of studies about SEPs, we have examined the theoretical framing in 42 recent studies focusing on SEPs, both within mathematics education and in other subjects. In these studies, a theoretical framing of SEPs is not always given or is only briefly elaborated. In studies without a pronounced theoretical argument for the SEPs, the SEPs may play a minor role in the investigation (e.g. Schalk et al., 2018) or the theoretical emphasis is laid elsewhere in the study, such as on learning with multiple representations (Corradi et al., 2012). There are also studies in which it is left to the reader to get a grip on the theoretical foundation that may be implicitly communicated through arguments for the use of SEPs such as to foster connections (Roelle & Berthold, 2013) or to support active learning and meaningful understanding (Neubrand & Harms, 2017).

In studies where the use of SEPs is based on explicit theoretical arguments, a common denominator is a cognitive perspective, a perspective that more or less turns the attention to the individual, not to the group and without any explicit explanation of the meaning of learning. The emphasis on cognitive perspectives in previous research can be traced back to the first studies within the area (e.g. Chi et al., 1989) and to several often quoted studies about SEPs, that largely have influenced the research field (Renkl, 1999). In particular, arguments for SEPs are often based on cognitive load theory (CLT) (e.g. Mwangi & Sweller, 1998; Renkl & Atkinson, 2003). In short, CLT separates between different types of cognitive processes that may impose three main types of mental effort on students' working memory when they work with some learning material: intrinsic, extraneous and germane cognitive load. Intrinsic cognitive load stems from the inherent nature of the task at hand, a type of load that SEPs do not intend to alter. The focus in studies about SEPs are rather laid on extraneous (unwanted) cognitive load and germane cognitive load (that contributes to learning). SEPs can be used to reduce extraneous load (Sithole et al., 2017) or to induce germane load (Berthold et al., 2011) or both (Kern & Crippen, 2017).

Besides the emphasis on CLT in studies on SEP, a constructivist theoretical base does also occur in several studies. For example, in a study by Roelle et al. (2015) it is emphasized that the self-explanation activities are constructive since the learners must generate knowledge that goes beyond the provided information. Part of the goal with SEPs is also often that the students shall make inferences and revise existing knowledge, sometimes explicitly referred to as revising cognitive schemas, both in studies who have CLT as a theoretical frame and not. Cognitive schemas are also essential in Sweller's (1994) description of learning mechanisms within the CLT and accordingly the focus on schema acquisition and revision is yet another sign of the common cognitive ground within the current corpus of studies on SEPs.

In the current paper however, we suggest an alternative analytical framework based on a sociocultural theory, which is useful as it provides tools for an in-depth understanding of how learning occurs during students' individual or collaborative work with SEPs. This framework has a twofold potential to explain how learning with SEPs occurs, first as scaffolding aiming to strengthen the thought by verbalizing the understanding of the content, and second by scaffolding and directing the students' attention to crucial aspects for example of a concept. These two sides of the theoretical explanation will be further developed below.

Theory on self-explanation prompts and scaffolding

According to sociocultural theory, development takes place through collaboration and imitation of how others solve advanced tasks. If you get help and guidance through collaboration, you can soon perform the tasks that you previously did not master. This difference between content that are familiar to a student and new content can be regarded as the zone of proximal development (ZPD) for the student (Vygotskij, 1978). In short, the ZPD has been described as the space that exists between a person's achieved level of knowledge where he or she can independently solve problems, and the possible development of knowledge that can occur in interaction and with support, for example from a teacher (Bakker et al., 2015). Such support of learning in the ZPD has been denoted as scaffolding. Scaffolding are often used as a metaphor describing support given by the teacher, but can also refer to support in the form of peer learning or by artefacts. Three characteristic features central to scaffolding have been described by van de Pol et al. (2010): 1. *Contingency* – the scaffolding must be adapted to the student and his or her knowledge level. 2. *Fading* – the support provided by the scaffolding should be removed or faded as the student has attained the desired knowledge. 3. *Transfer of responsibility* – gradually, as the scaffolding is removed, the responsibility for the work is also transferred from the teacher to the student.

When taking a perspective on learning as scaffolded by SEPs, the SEPs are perceived as the means that contribute to raise understanding to a higher level. The prompt can provide students with support in identifying and directing focus to crucial aspects. In this way SEPs act as part of the teacher's, or the more knowledgeable others, support in the ZPD. When the student formulates and puts his or her explanation into words, this explanation works as a scaffolding in the ZPD (Vygotskij, 1978), for the individual, as well as for peers when thoughts are made audible. The student's response to a SEP will also constitute scaffolding for other students in group work. In combination with preceding comments from other students, these responses do together create a conversation with the potential to scaffold learning within the ZDP. The connection between language and thinking as one of the crucial aspects of learning, supported by SEPs is elaborated in more detail below.

Theory on thought and language

Vygotsky (1986) discusses language and its role for thinking and learning and emphasizes the inner language and its significance for thinking. For young children, thinking develops by speaking loudly to themselves. This phenomenon has been referred to as an egocentric language. The egocentric language eventually develops into an inner silent language that, like the egocentric language, supports the thought. This development can be compared to a student who works with a SEP, and explains crucial aspects of the concept for himself or audibly to a peer student. Gradually, thinking evolves so that the verbal explanation becomes superfluous. Then the inner silent thought suffices as support. This transformation from the verbal explanation to the inner thought is essential in relation to how learning occurs when working with SEPs.

When it comes to scientific concepts, students' conceptual understanding is most often not fully developed when a concept is introduced during a lesson or in a textbook. The student first learns to recognize a particular word and thereafter an understanding of the meaning represented by the word is developed. Thereby the understanding of a scientific concept is developed from the general by making links to the concrete and well-known (Vygotsky, 1986). By using language individually and in collaboration with others, the student can create such links between the concrete understanding of a concept and the general scientific expression. In this perspective, this link is the basis for learning of scientific concepts (Vygotsky, 1986). The function of SEPs in relation to the development of conceptual understanding is to prompt the student to use the language to explain the concept, and thus making links to the well-known.

A learning situation when working with SEPs

In this section, we illuminate our perspective by an authentic example from three grade four students' collaborative work with a SEP. The example is derived from a larger project aiming at investigating students' learning during collaborative work with SEPs. The excerpt below shows the discussion during one group of students' joint work with the mathematical task ”The Sunflower”, described in figure 1. The task contains the SEP ”First, discuss what it means for something to grow at the same rate every week”. The aim of this SEP is to support the students' understanding of the concept of proportion. The concept is new to these students since the intention is to enable students to develop knowledge in their ZPD. In the design of a task or a learning situation, it is important that the students' understanding and previous knowledge are thoroughly taken into account (van de Pol et. al., 2010). If the match between the student's level of knowledge and the requirements in the task fails, the SEP will not work sufficiently.

The SEP encourages students to explain a crucial aspect of the concept proportionality. According to the analytical framework described in this paper, the SEP aims to support the students' learning in two ways. First, the SEP fills the function of fostering a verbal discussion and explanation of the targeted aspect. This verbalizing process creates a foundation for the students' learning through the connection between thought and language. Second, the SEP supports learning by directing focus to the formulation *the same rate*, which is a crucial aspect of the concept. The following excerpt is an example of students' collaborative work with a SEP. The analysis of the excerpt, guided by the proposed sociocultural based framework, elucidates the analytical potential of this framework. Our theoretical interpretation of the students' learning is explained in relation to the analysis below the excerpt.

1	Ally	What does it mean for something to grow at the same rate every week?
2	Ben	So all ... all ... maybe not all of them get the same length every time.
3	Ally	No.
4	Ben	But they increase the same. So that each plant, like if it was 10 centimetres, then it would be 20 more, so all of them would, would be, would be, 20 10 centimetres more even though they are not quite equally long.
5	Chris	Difficult to say ... you could say that it is like a staircase [shows steps with a gesture]. So you ...
6	Ben	And each is 10 centimetres.
7	Chris	So it's like a staircase. It increases each time one step of 10.
8	Ben	Exactly! Something like that.

In line 2, Ben's utterance ”maybe not all of them get the same length every time”, reveal that the task is not too easy, and thus it is reasonable to assume that there is a potential for learning within the ZPD. If Ben was working alone, the task may have been too hard to solve, but in group work all students' verbalised thoughts became part of the scaffolding and the verbalised thoughts do therefore constitute a kind of buffer that adjusts the difficulty to the students. This can for example be seen in line 4–6 where Ben and Chris further verbalises their developing thoughts about proportional growth. Ben tries out thoughts in line 2, thoughts that develops to a preliminary definition, ”increase the same”, in line 4. In line 5 Chris uncertainty reveals that the activities are within the ZPD even for him. He is unsure and uses a metaphor to describe his thoughts. This metaphor, ”a staircase”, do thereafter constitute a part of the scaffolding that supports Ben who further clarifies ”each is 10 centimetres”. In line 7–8 Chris and Ben agrees on a summary, which is interpreted as an expression of a new level of understanding since the utterances are more developed compared to the initial statements. The students build on each other's contributions and strive against a shared under-standing. The transcript does not reveal to which extent Ally is part of the discussion. It may be that the adaption between the SEP and Ally's level of understanding is poor (ibid.).

In summary, the SEP encourages the students to express their developing thoughts, thoughts that progresses from fragments to an appropriate description of proportionality in this particular case. So, the prompt function as scaffolding as well as the developing utterances and gestures do.

Implications for mathematics education

In this paper the idea of scaffolding grounded in a sociocultural tradition is suggested as a theoretical base for the analytical framework since it captures dimensions other than those typically claimed to be explained by cognitive theories. As illustrated in table 1 the suggested framework puts emphasis on two different aspects of learning that is nurtured by the SEPs, namely to scaffold both *structure* and the relation *thought – language*. The figure also illustrates how the framework is applicable for both individual work and group work.

Table 1. *”What” and ”how” self-explanation prompts can be scaffolding*

	What	
How	structure	thought – language
individually	(1)	(2)
group work	(3)	(4)

Cell 1 and 2 capture aspects related to the support the individual receives from working with SEPs. Cell 1 concerns the mathematical structure and cell 2 is about how thinking is supported by the use of language. Cell 3 and 4 show how both the mathematical structure and the relationship between language and thought can be supported at group level.

The previous example about proportionality exemplifies an analysis of the scaffolding function of verbalised thoughts. However, a SEP does with necessity provide structure, by directing the attention to the aspect the students are supposed to develop their understanding about. According to scaffolding theory this structure is supposed to be adapted to the learner's level of understanding, to be faded out and to be used in a deliberate way by the students. A match between the support given by the SEP and the students are briefly touched upon in relation to the example, in the interpretation of whether the students develop within their ZPD.

The suggested analytical framework opens up possibilities for understanding and analysing learning in research on SEPs, and provides guidelines in the design of tasks and learning situations. For example, SEPs can be designed to support the development of multimodal reading competence where they provide structure by focusing attention to particular features of the text. In relation to table 1 it is important to note that a learning situation can comprise several of the cells, such as group work scaffolding both on structure and thought – language. The three distinguished characteristics of scaffolding: to be contingent, to be faded out, and to be used in a manner that transfers the responsibility to the learner (van de Pol et al., 2010), can be taken into account in task design, for example by successive prompts. The students then have the opportunity to take responsibility and choose the prompts that suits their level of understanding and the scaffolding function can reach its fullest potential. A perfect match between a SEP and a student group is not easily achieved, and dynamic scaffolding given by a teacher in relation to the expressed understanding can therefore be a useful calibrator during problem solving. With a good theoretical understanding, the teacher can act flexibly and adapt a SEP to the students' needs in the current situation.

Discussion

In this paper, we have described a sociocultural based analytical framework for understanding and investigating SEPs and how they can enhance learning. We thereby highlight opportunities to recognize and analyse potentials of SEPs that might otherwise remain invisible. The framework we suggest contributes to a broaden understanding of SEPs compared to studies framed by cognitive load theory (CLT), where emphasis is put on how to provide structure to the individual, which corresponds only to the upper left cell (1) in table 1. In this way, the explanatory power of CLT is limited to an individualised learning process, and rather on how to guide the students towards what is to be interpreted (removing load), than to support learning.

We argue that with the suggested sociocultural based analytical framework, more dimensions of the potential of SEPs are highlighted, which opens up new opportunities to understand, use, and investigate this teaching tool. By using this framework, it is possible to illuminate how students can learn either individually or in groups. Learning is supported by SEPs pointing out key aspects, as well as by SEPs encouraging students to verbalise their thoughts. When students work individually with a SEP, they have to write down their answer, or answer orally for themselves instead of using language in interaction with their peer students. This is in line with the inner silent language, described by Vygotskij (1978). The analysis of the example also shows how this theory can provide a basis for understanding students' learning while working with SEPs. In summary, the described framework provides improved possibilities for teaching, learning, and research on SEPs as described by the four cells in the model in table 1.

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”Programming is a new way of thinking” – teacher views on programming as a part of the new mathematics curriculum in Finland

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Programming has recently been included as a part of the mathematics curriculum throughout the grades in several countries. This is also the case in Finland. In this explorative study, we focus on Finnish Swedish primary school teachers' views of programming in school mathematics and on the connection that they spontaneously draw between mathematics and programming. Most teachers connect programming in primary school to the explicit activity of writing, giving or following instructions and to different aspects related to logical thinking. In addition, some teachers consider programming as an important problem-solving tool and still some of them mainly as an activity in mathematics. Only a few teachers connect programming to central computer science concepts as algorithms and abstractions and to specific mathematical areas.

In line with several other countries, Finland has recently included programming as a part of mathematics curriculum in primary school (Duncan & Bell, 2015; Hemmi et al., 2017). This implies that primary school teachers (grades 1–6) have to integrate programming in their mathematics lessons. Primary school teachers as generalists need widespread professional development concerning technical skills and understanding of suitable pedagogies to successfully implement new curriculum ideas (Benton et al., 2017). Moreover, it is not quite clear what exactly is to be focused on at different grades as the Finnish national core curriculum is written in a very general way. The path from inclusion of programming in the mathematics curriculum to enacting lessons targeting it in a relevant manner is complex (Mannila et al., 2014). The authors point out there are several issues to be discussed and defined to succeed in the implementation process of programming in the primary school classroom.

The present paper contributes by reporting the findings of an explorative study among primary school teachers the first year after the introduction of the new national core curriculum in Finland. We focus on what primary school teachers spontaneously ascribe to programming, and investigate particularly features connected to different aspects of computational thinking and the

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connections they make to topics typically included in mathematics learning. The research question is the following: How do primary school teachers view programming as a part of the new mathematics curriculum?

With teachers' views, we refer to ways in which teachers describe programming and the teaching of programming.

The Finnish context

The general task of mathematics education as stated in the national core curriculum (applied from 2016) is to develop students' logical, accurate and creative thinking (FNBE, 2014). Programming is included in the content of Mathematical thinking skills and applies to all students from first grade up to the end of grade nine (see Hemmi et al., 2017). Learning programming in mathematics starts in grades 1–2 with constructing simple algorithmic instructions by using symbols in written or oral form and testing them. During grades 3–6, the emphasis is on formulating instructions in a graphical programming environment.

There have not been any national efforts to systematically offer all teachers in-service education in programming, but different agents, such as regional authorities, universities, the National Board of Education and private companies, have frequently organized courses for teaching programming. The courses offered for class teachers have focused for example on visual programming with block-oriented tools like Scratch, code.org and various applets, educational robotics, algorithmic thinking and elements of programmable electronics and making. Teaching programming has often been technology-driven and enthusiastic teachers and other actors have considered what they can do with a particular tool. Therefore, there might be a danger that a holistic picture of the learning path of children is not so clear for primary school teachers (cf. Hemmi et al., 2017).

Relevant literature

Computational thinking

The origins of computational thinking in early mathematics education can be traced back, more than thirty years, to the work of Papert who developed computer software to facilitate children to engage and explore computer programming as a natural problem-solving tool in their mathematics studies (Papert, 1980, 1996). Later, Wing (2006) defined computational thinking as "representing a universally applicable attitude and skill set involving solving problems, designing systems and understanding human behavior, by drawing on the concepts fundamental to computer science" (Wing, 2006, p. 33). After that, several organizations and authors have presented different definitions of computational thinking (Grover & Pea, 2013). Many of these definitions are quite general and may indeed involve activities not necessarily directly connected to programming and coding.

Brennan and Resnick (2012) introduced a framework with three dimensions of computational thinking: computational concepts, computational practices and computational perspectives. The first dimension includes common concepts that programmers use as they develop programs, such as sequence, iteration and function. Computational practices reflect different activities and problem solving practices that occur in the programming process, such as planning, testing, debugging, reusing and remixing. The third dimension, perspectives, involves the programmer's connection and relationship to other members of the programming community and to the surrounding technological world. These dimensions are appropriate for understanding how K–12 students approach and connect to programming with Scratch and they are well in line with the programming content and ambition of the newly launched curriculum in Finland. In our study, we used Brennan and Resnick's (2012) dimensions as starting point for the data analysis, but broadened it taking an open iterative approach as our study is of explorative character.

While various studies about the relation between programming and mathematics in school curriculum have been conducted, the possible effects of programming on the learning of mathematics have not been clearly stated (Benton et al., 2017). However, many researchers highlight the critical role of the teachers in making explicit and systematic links between programming and students' existing and developing mathematical knowledge (Benton et al., 2017).

Teachers' views on programming

There is little knowledge in the field of mathematics education about teachers' views of programming and teaching of programming as a part of mathematics curriculum and even less on how primary school teachers cope with the recent reforms in different countries. The paper by Mannila et al. (2014) surveyed teachers' experiences about and perceptions of computational thinking in five European countries. Hijón-Neira et al. (2017) investigated primary school teachers' views on programming in schools in one region in Spain through a questionnaire and they analyzed the responses of 46 teachers. The teachers agreed on the benefits that programming provides in several areas, for example the development of thinking skills, the organization of ideas, the ability of abstraction and problem solving, motivational aspects, and the opportunities offered by teaching through games. The respondents remarked the importance of having properly trained teachers to teach this subject. Funke et al. (2016) interviewed six primary school teachers about their opinions on computer science and the findings pointed out that the teachers had no clear image on what computer science in school is, but they highlighted the importance of implementing computer science at an early educational stage. Recently Nouri et al. (2019) investigated which skills 19 teachers interested in programming themselves aimed to develop among pupils. Apart from Brennan and Resnick's (2012) dimensions, they found some general skills related to digital competency and 21st century skills.

Pointing out the earlier concerns, for example about children bypassing mathematical ideas within less structured learning activities and without teacher guidance, Benton et al. (2017), examine the relationship between learning to program and learning to express mathematical ideas through programming with Scratch on primary level mathematics (age 9–11). The teachers in the study expressed that they needed the powerful ideas of mathematics curriculum to be clearly connected to the programming aspects of the computing curriculum. The study shows that it is possible to connect programming (with Scratch) to mathematical learning among students of different abilities and it had a positive effect on students' motivation. Yet, the study raised a number of concerns with respect to teachers' confidence and subsequent use of the technology within their teaching to support the learning of both computational and mathematical concepts.

Data collection and analysis

The target group of this study is primary school teachers that are working in schools where the instructional language is Swedish.[1] The empirical data for this study was obtained using a web-based survey that was distributed to teachers through Swedish Finnish primary school principals (190) during the spring term 2017. The survey contained 34, mostly multiple-choice questions and took about 30 minutes to complete. The final group of respondents consisted of 91 teachers, 70 female and 21 men. The age and regional distribution of the respondents were satisfactory. Of the 91 participants, 71 had participated in at least one in-service training course in programming. In this paper, we solely explore and analyze teachers' answers to one open question in the questionnaire: "What is programming? You should focus on programming in primary school but you can also relate to programming in general." From the context of the questionnaire, it is clear that this question is directly related to how the current change of the national curriculum (inclusion of programming) affected the mathematics content. The teachers were asked to focus on programming in school and to reflect on what they found important. The focus is on the teachers' spontaneous reflections about programming. We did not want to probe them in anyway and the aim was to investigate if they spontaneously mentioned connections between programming and mathematics as it is a part of mathematics in the Finnish curriculum.

Possibly, due to the current nature of the topic, many respondents gave relatively rich answers. The number of words in the different answers varied from one word to 108 words and the mean number of words in the answers were 25. The teachers' responses (in Swedish) were first read and interpreted separately by two first authors of this paper. We initially started the analysis with the categories of Brennan and Resnick (2012) in order to identify what kind of computational thinking teachers' utterances were possibly expressing. Yet, these categories were not helpful in identifying other important views, such as curriculum issues and views on connections between programming and mathematics. Therefore, we conducted a data-driven iterative analysis (e.g. Bryman, 2001) starting by identifying certain similarities and generalities among the answers. We found six categories suitable for the final analysis (table 1). Due to the openness of the question, one answer could be assigned to several categories. In the Results section below, we describe the categories in more detail and exemplify them with teachers' expressions translated to English in order to make the analysis transparent.

Results

The distribution of the teachers' views on programming in relation to the six analytical categories can be seen in table 1.

Table 1. Distribution of teachers' views on programming

Category	*n*
1. Writing, giving and following instructions	59 (65 %)
2. Logical thinking and identifying patterns	29 (32 %)
3. Algorithms, abstractions, modularization and testing	10 (11 %)
4. Problem solving	17 (19 %)
5. Use of modern technology and digitalization	9 (10 %)
6. Curricula, progression and future aspects	15 (16 %)
Total	139

The number of answers assigned to different number of categories are 0 (6), 1 (48), 2 (24), 3 (10), 4 (2), 5 (1). That is, six answers were uncategorized and 24 answers belonged to two different categories. No answer was assigned to all six categories. Below we describe and exemplify the categories identified for teachers' spontaneous views on programming in school.

Writing, giving and following instructions

This category is the most common among the answers as 65 % of the teachers connected programming to the explicit action of writing, giving or following instructions. The next extract shows a typical teacher answer in this category.

> In primary school education, it is important to let students test to program a computer, give instructions to another person or to a robot and try to make it complete the desired task. (Teacher 10)

As shown below, several teachers connect these kind of actions to activities associated to spatial thinking.

> [...] A simple way is to say; Go two steps to the right, one backwards and then five steps forward. Then you have come to the finish. (Teacher 33)
>
> For example, to be able to get a Beebot to go from one place to another by programming it. (Teacher 78)

A step-by-step procedure is sometimes connected to teachers' interpretations of instructions.

> Programming is to give detailed step-by-step instructions that do not offer space for misinterpretations or ambiguity. (Teacher 72)

Several teachers pointed out that the instructions need not to be given to a computer or robot, but equally well to a fellow student.

Logical thinking and identifying patterns

This category was the second most common as 32 % of the teachers point out that programming is connected to logical thinking and/or the identification of patterns. Most responses in this category state that programming promotes learning of logical thinking as shown in the extract below.

> Programming is, for example, to split a problem in to smaller parts, to see relations, to learn to think logically, to create something new." (Teacher 64)

Others express that programming is very similar to logical thinking.

> I think programming is very much about logical thinking and recognizing patterns. (Teacher 45)

Most of the teachers connect programming to a combination of handling instructions and applying logical thinking

Algorithms, abstractions, modularization and testing

These programming terms are common concepts in computer science and 11 % of the teachers claim that a central aspect of programming is the explicit constructing of algorithms, abstractions and the modularization or testing of a program.

> It is about coding, solving complex problems by splitting them into smaller pieces, identifying patterns, creating abstractions and writing algorithms. (Teacher 16).

Teachers that connect to these concepts are likely to have more in depth knowledge of the process of applying programming to solve problems.

Problem solving

The important problem solving aspect of programming is highlighted in 19 % of the answers.

> Programming is a really good activity that trains the ability to solve problems. (Teacher 43).

Another teacher who connects programming to ”logical thinking, ability to solve problems, systematics and creativity” concludes with ”Programming is mathematics.” (Teacher 72). Despite the close and important connection between mathematical problem solving and programming, no teacher answer is giving any explicit example of such a problem solving activity.

Use of modern technology and digitalization

A few teacher descriptions (10 %) consider programming in school from a more general perspective. Some responses address the importance of understanding the relation between human and machines.

> […] to realize that everything a machine can do is due to a human that has programmed it. (Teacher 10)

Others stress that programming is a part of modern technology.

> Several things in our close environment work with help of programming, e.g. machines, computer games and telephones. Industry uses robots that have been programmed. (Teacher 75)

This category captures more general aspects of programming not directly related to a school context.

Curricula, progression and future aspects

The aspects of curriculum and progression concerning programming in primary school are the focus in 16 % of the answers. Several teachers saw programming as a positive element in mathematics lessons and important for all students to learn, for example to prepare for future work life.

> We have to prepare them for the working life after school when they must be prepared to think creatively. (Teacher 58)

On the other hand, there were teachers who were not convinced about the importance of learning programming for all students.

> I think programming is fun, but I do not see it as a useful ”subject”. That type of thinking can be acquired in many other ways. (Teacher 22)

Some of the teachers express a concern about the lack of information about the progress throughout the grades 1–6 concerning programming.

> Interesting, but I would like to have a clearer plan about what to do each school year. (Teacher 69)

Several teachers also stated that they lacked relevant curriculum materials for teaching programming.

Variation in teachers' descriptions

The range and the qualities in teachers' responses varied a lot. Some of the teachers touched several categories while others only responded with short sentences categorized into one category. The following extract is an example of the former and was coded into categories 1, 2, 3 and 6.

> Programming is a working process were you construct an algorithm, a hypothesis or a plan of how something should be executed or work. This plan is then tested and updated in order to work correctly. On a basic level, it can be as easy as working with numbered instructions. For older students it proceeds to the creation of block-based events using apps and computer programs and then finally in the highest grades by coding using a text-based language. (Teacher 62)

This teacher captures several important concepts and practices in computational thinking, such as instructions, events, algorithm, planning and testing as well as the progression of the topic. The second example is coded into categories 2, 3 and 4.

> Programming is all about logical thinking and problem solving. It is about coding, solving complex problems by splitting them into smaller pieces, identifying patterns, creating abstractions and writing algorithms. You can practice programming using different programs, games and languages. Programming is a new way of thinking. (Teacher 16)

The focus in this answer is on problem solving, the thinking aspect and the creation of algorithms and abstractions. The teacher also claims that programming really adds a new color to the classroom activity palette. The last example is coded into categories 2, 4 and 5.

> Programming is a way to teach students logical thinking, understanding of relations and problem solving. They develop both cognitively and linguistically. In time, they will understand that all new technology they use is based on programming. (Teacher 40)

This teacher specifically lifts logical thinking and problem solving as important learning goals and the technology connection is mentioned. The answer also highlights the communicative aspect of programming as being important.

Discussion and conclusion

The aim of this study was to explore and describe the way Finnish Swedish-speaking primary school teachers' view programming as a part of the new mathematics curriculum. Several teachers saw programming as a positive element in mathematics lessons and important for all students to learn. Programming was primarily connected to the explicit writing, giving or following of instructions. It was also considered to contribute to the development of logical thinking, serve as a valuable tool in problem solving and be a useful skill in future work life.

The teacher's answers reveal that programming is interpreted in different ways. Most teachers view programming as an activity whereas others interpret programming from a subject point of view. The activity view is typically more related to categories 1–4 and the subject view is more visible in categories 5 and 6.

We next briefly comment on the relation of the answers to the framework by Brennan and Resnick. Many Finnish primary school teachers use Scratch as a programming tool and many of them have attended in-service training courses addressing Scratch. When they are to describe what they consider as programming, it might be that they view programming mainly through the lens of Scratch. Some answers that belong to category 1 and a few answers connected to category 2 and 3 are closely related to Brennan and Resnick's concept dimension. Common words in the teachers' answers are instruction, command, sequence, variable, conditional and loops. However, most answers relate to the practices dimension. The teachers typically describe how a certain programming activity is conducted and how it connects to some other learning aspect such as logical thinking and problem solving. They focus on how the students are working and learning. As one answer reveals, ”Programming is, for example, to split a problem in to smaller parts, to see relations, to learn to think logically, to create something new”. The perspectives dimension was visible in a few answers in category 5, with focus on modern technology and social aspects.

The explicit connections to specific mathematical content were scarce. For example, no teacher answer relates to the application of programming to arithmetic, algebraic expressions nor probability. The examples given can only be connected to spatial thinking and geometrical shapes. Along the lines with the concerns mentioned by Benton et al. (2017), it might be that primary school teachers do not fully apprehend the interplay between mathematical and computational content and learning. This also relates to some teachers' concerns about lacking of information and relevant materials to be able to concretize the general goals of the national core curriculum. One suggestion for future action could be to organize educational courses for teachers that focuses on explicit connections between programming and mathematical content and learning.

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Note

1 Finland has two official languages and approximately 5 % of students in compulsory education attend a school where Swedish is the language of instruction.

Integrating programming in Swedish school mathematics: description of a research project

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This paper describes a new research project investigating the implementation of programming in Swedish school mathematics, specifically in relation to algebra learning. Based on Chevallard's framework of transposition of knowledge, the project investigates what types of activities and systems of representations are introduced and argued for as programming makes its way into mathematics teaching, and what these may entail. Tentative results reveal syntactic and semantic differences between programming and algebra that may cause problems for students. Interviews with teachers show that they seek inspiration and activities from social media and internet rather than textbooks and other published teaching material.

During the past five years programming and computational thinking have emerged as new skills in several countries' national school curricula. Although computational thinking was introduced already in the 1980s (Papert, 1980) it did not become widely adopted, possibly because digital technology did not have the impact it has today through the internet and digital devices (Kotsopoulos et al., 2017). However, about thirty years later, Wing returned to the term computational thinking arguing that it should be taught in schools alongside reading, writing and arithmetic (Wing, 2006).

The integration of programming and computational thinking in school curricula has been done in various ways (Mannila et al., 2014). For instance, in England, programming was made part of a whole new subject, ”Computing” (Berry, 2013), while Finland and Sweden adopted a blend of cross-curriculum and single subject integration with the strongest link to mathematics (Bocconi et al., 2018). Unlike other countries, Sweden included programming in the mathematics curriculum in close connection to algebra through all grade levels, which makes the Swedish case unique in an international perspective. Until now, research on computational thinking and algebraic thinking has run on separate tracks, but the Swedish case offers a great opportunity to investigate the intersection of these two research domains.

The overall aim of the project described in this paper is to contribute to the international research field concerning the complex issue of implementing

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programming and computational thinking in school mathematics (grades 1–9), specifically in relation to the learning of algebra. Based on two interrelated studies, the project attempts to explore how computational thinking is connected to algebraic thinking and, ultimately, how this connection may create opportunities, challenges or pitfalls for student learning of algebra. The aim of this paper is to describe the project as well as report some tentative results concerning what aspects of programming and computational thinking that have made their way into school practice. However, we commence with a brief survey over the two research fields of computational thinking and algebraic thinking. Both are fairly new as research fields and therefore attempts at defining and conceptualizing what is meant by these types of thinking is still an ongoing process, which we have described in more detail elsewhere (Kilhamn & Bråting, 2019).

Computational thinking

Over the past decade, computational thinking (CT) has been paid increased attention in education at all levels. The significance of CT can be explained by the fact that it supports cognitive development and creative problem solving, as well as by the growing interest in artificial intelligence (Nouri et al., 2020). In the Nordic countries, CT and programming are included in an evolving definition of digital competence (Bocconi et al., 2018). In Sweden, programming was introduced as a new content in the national syllabus for mathematics in 2017, a revision which was expected to be fully implemented in August 2018 (Swedish National Agency of Education, 2018), giving schools a short time frame for teacher training and preparation.

Often, the concept represents a product-oriented perspective where various tools are applied in order to form CT skills (e.g. Grover & Pea, 2013). Aho (2012) defines CT as ”the thought processes involved in formulating problems so their solutions can be represented as computational steps and algorithms” (p. 832). Problem solving and algorithmic thinking are thus considered fundamental aspects of CT (Futschek, 2006). Developing CT also requires students to deal with the sometimes counterintuitive notations and conventions in the syntax of different programming languages.

Generally, CT is considered a more extensive concept than programming, although teaching and learning programming requires the use of CT (Hickmott et al., 2018). Furthermore, Brennan and Resnick (2012) highlight the appropriate role of programming as a means to develop CT, identifying three dimensions of CT: i) *concepts* such as sequences, loops and data; ii) *thinking practices* such as debugging, remixing and abstracting; and iii) the *perspectives* expressing, connecting and questioning. These dimensions come to the fore in programming activities at a school level, and form a useful framework for both teaching and assessing CT. An early initiative to use this framework has been taken by Nouri et al. (2020) in a thematic analysis of teacher interviews.

In a literature review of studies on CT in mathematics classrooms, Hickmott et al. (2018) found few studies that explicitly linked the learning of mathematics concepts to CT. Even when concepts involving numbers, operations or algebra were present, the primary intention was always the introduction of programming concepts.

Algebraic thinking and algorithms

Researchers have suggested several frameworks for conceptualizing algebraic thinking in elementary grades. Many of these draw on Kaput's (2008) description of early algebra in terms of three content strands; the study of structures and relations; the study of functions; and the application of a cluster of modelling languages. Although a conclusive definition of algebraic thinking (AT) does not [yet] exist, most definitions include the two core aspects of expressing generalizations and symbolizing in formal or informal systems of representation (for several examples see Kieran, 2018).

The introduction of algebra in school mathematics was traditionally assumed to build on a thorough knowledge of arithmetic and was therefore not introduced until secondary school. For some decades, however, this division between arithmetic and algebra has been rejected, and it is now generally accepted that supporting algebraic thinking and the use of algebraic tools already in the early grades is beneficial to the learning of both arithmetic and algebra (Kieran, 2018).

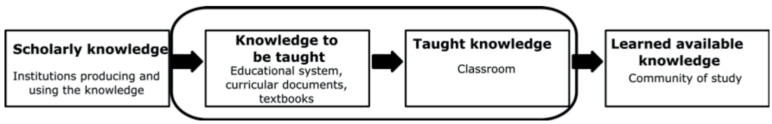
Before the 1980s, traditional algorithms were seen as a cornerstone of arithmetic, but following the invention of pocket calculators a debate flourished on the necessity of these algorithms (Kamii & Dominick, 1997). Traditional algorithms were replaced by an increased emphasis on number sense and conceptual understanding. In Sweden, the term algorithm was removed from the description of arithmetic in the national curriculum in 1994, but re-inserted in 2018 as a core concept in algebra in connection to programming, implying a shift of emphasis from a procedural use of algorithms to a conceptual understanding of algorithms. In mathematics education, an algorithm is defined as a finite sequence of executable instructions which allows one to find a definite result for a given class of problems (Brousseau, 1997). The general structure of an algorithm connects algorithmic thinking to AT, so potentially, algorithms and algorithmic thinking may lie in the intersection of AT and CT.

Theoretical frames of the project

Within the research project we use the theory of transposition of knowledge (Chevallard, 2006) in order to study the implementation of programming in school mathematics, see figure 1. While the relevance of *Scholarly knowledge* is what is achieved by professional programmers, *Knowledge to be taught* is made legitimate by different actors making decisions about what, when, and why to teach, and is thus broken into smaller parts (Kang & Kilpatrick, 1992). When computer programming is transposed to become a body of teachable knowledge in school, decisions are made for example concerning where to place the topic in relation to other topics, what kinds of activities and symbolic representations to use, as well as how and in what order different aspects of programming and CT are to be taught at specific age levels.

The process of transposition of computer programming has already started in Sweden through the choices made in the revision of the national curriculum (Swedish National Agency of Education, 2018). The next step of didactic transposition will happen as the national curriculum is interpreted and operationalized in textbooks and teaching materials, and further by teachers, to become *knowledge taught* in the classroom. The project described in this paper offers a unique opportunity to study the transposition of knowledge related to aspects of computer programming, computational thinking and algebra as it occurs in the implementation of the revised national curriculum in Sweden.

Figure 1. *The four phases in the didactic transposition process*



Note. The encircled processes appear in focus of the project described in this paper

To deepen our analysis, we identify and describe the systems of representations that appear in different semiotic representations (Duval, 2006), how and why these are chosen, discarded or taken for granted in each phase of the didactic transposition. Especially, we investigate how computer-related representations interact with already present algebraic systems of representation.

Method

Within the project, we conduct two studies that both separately and related to each other help us to discern important aspects of the issue of integrating programming in school mathematics. Here we briefly describe the studies and exemplify what has been done so far.

Study 1. Teaching materials and textbooks

This study focuses on the transposition from *Scholarly knowledge* to *Knowledge to be taught* in the didactic transposition process. We investigate the current steering documents in mathematics education, government produced teaching materials, and commercially produced mathematics textbooks including teacher guides. The selection of textbooks is made according to their popularity and diversity due to earlier studies showing that there are substantial differences between textbook series (Bråting et al., 2019). At present, four textbook series from three different publishers have been chosen for analysis. We are currently conducting a qualitative content analysis of the textbooks' tasks with respect to the programming content. The first two of Brennan and Resnick's (2012) dimensions have been used as a base for an analytical tool.

In this paper, we will report some results from our initial analysis of textbook series for grades 1–6 as well as our investigation (Kilhamn & Bråting, 2019) of programming activities suggested in a government-provided teaching material available online [1] based on the 2018 revised curriculum in Sweden. In the latter, we have utilized Duval's (2006) framework to highlight syntactic and semiotic aspects of algebraic concepts that appear in both algebra and programming, such as the equal sign, variable, algorithm and function.

Study 2. Teachers' voices

The second study focuses on the transposition from *Knowledge to be taught* into *Taught knowledge* (figure 1). Data is gathered from teachers who are in the process of implementing programming in their mathematics classrooms. In the analysis we focus on what didactical choices teachers make and why, as well as what opportunities, challenges and pitfalls they identify, in particular in relation to different semiotic registers. At present, we have data from two sources; a) teachers' written documentations of lesson studies (c.f. Fernandez & Yoshida, 2012) and b) individual teacher interviews with early adopters.

Lesson study plans and *teachers' written reports* of enacted lesson studies were collected from 24 groups of teachers, attending an in-service development programme; in total approximately 135 primary and secondary school teachers. The teachers involved in the programme came with mixed ability and motivation towards teaching programming as part of the mathematics curriculum. Each lesson study consisted of two or three cycles of co-planning, enacting and revising a lesson about programming in mathematics. In some of the lesson studies the lesson was taught by the same teacher in each cycle, in others by different teachers. The written documentations have been scrutinized to identify types of activities and programming environments used, as well as challenges and questions raised by the teachers. These results will be instrumental in helping us identify themes for focus group interviews further on.

Individual semi-structured interviews have been made with "early adopters", i.e. teachers who actively teach programming at an early stage of the implementation and who identify themselves as enthusiastic about bringing programming into mathematics lessons. The early adopters were recruited through our teacher education networks. At present 20 interviews of around 30 minutes each, covering teachers from grades one to nine, have been conducted, audio-recorded and transcribed. Some of the teachers will later participate in a second round of interviews. Using NVivo software, a content analysis is being conducted, relating interview data to theoretical definitions of AT and CT. What mathematics and what types of activities teachers choose to present in their classrooms as well as how they justify their choices will be analysed as a means of understanding the transposition of knowledge (Chevallard, 2006). We will describe some tentative results that indicate issues that we need to analyse in a more systematic way.

Tentative results of study 1

The tentative results of Study 1 show that there are differences between programming and algebra, regarding both syntax and semantics, that may cause problems for students' algebra learning. In our initial analysis of government-provided teaching material (Kilhamn & Bråting, 2020), the different meanings of the equal sign and the concepts variable and algorithm have been discussed in terms of Duval's (2006) framework of different systems of representations. For instance, in algebra the equal sign is used as a *relational* operator. Therefore, it would be meaningless to write $a=a+1$ since it is not true for any value of a. Meanwhile, in programming the same expression is understood as the *assignment* "add 1 to the value a" which is often used when a program needs to loop through a range of consecutive integers. Instead, the double equal sign (==) holds a relational meaning in programming. These kinds of differences can afford the development of algebraic thinking through contrasting examples and awareness of accuracy, or constrain it if the teacher is unaware of the different experiences students have. In addition, we need to take into account that the equal sign already causes problems in school mathematics since students tend to interpret it as an operator symbol (4 + 3 *make* 7) rather than a relation (Kieran, 1981). In Duval's (2006) terminology, one may argue that there are differences between the two systems of representations as well as within the same system of representation.

The initial analysis of textbooks in mathematics for grades 1–6 reveals that the implemented content is similar in the different textbook series, although it has been included in different ways. While some textbook publishers have revised all textbooks in mathematics in order to include programming and digital tools, others have offered most of the new content online as supplementary material and in teacher guides. Regarding the programming content, our initial results show that the most common concepts included in the textbooks' tasks are stepwise instructions, algorithms, iterations and repeated patterns. For instance, more than half of the tasks in textbooks for grades 1–3 focus on stepwise instructions and about a third on iterations. There is a high correspondence between the textbooks' content and the prescribed content in the revised 2018 curriculum document. However, the connection between algebra and programming in the textbooks is vague. Programming content is either added as separate chapters, or integrated in already existing chapters of arithmetic, statistics, geometry or problem solving. We find this result interesting, given that a major part of the programming content in the 2018 curriculum document is included within the core content of algebra.

Tentative results of study 2

The lesson studies show, as expected, that activities vary according to grade in the school system, but also seem to vary depending on the programming environment and language that teachers have chosen to use. Teachers express two issues of concern when teaching with programming environments, and some uncertainty about what to teach and why.

Teachers in the lesson studies express how block programming environments offer a variety of features that could become distractive for some students, with an abundance of side effects and aesthetics to control. In contrast, they highlight syntax issues as challenging in environments for textual programming. For example, a conditional statement in programming can be written in many ways enhancing students' creativity and motivation, while in textual settings students have to follow the rules of the syntax and pay attention to many different signs. The two types of environments seem to offer quite different representations of concepts that teachers have to take into consideration when teaching.

Another finding relates to semantic differences in different programming environments. For example, in unplugged activities arrows pointing up, down, left and right signify points on the compass while on a robot they signify forward, backward, and which way to turn.

The types of activities and choices of programming environments described in our interviews with early adopters mirror those found in the lesson studies. Scratch, Code.org and Python are most frequently used. However, they have quite different ideas about what to teach and why. They express uncertainty about what types of activities and environments to include in the teaching of programming, for example, if activities with spreadsheets or interactive geometry programs such as GeoGebra should count as programming or not. They all agree that programming increases motivation in mathematics, which will hopefully affect students' learning of mathematics. The activities they describe include a limited variety of mathematical content, the most common being movement or lines and figures in a coordinate system, calculations, probability or statistics. Working with patterns is quite common, but mostly those activities are limited to finding a repeated pattern that could be coded as a loop. Connections to algebra are scarce in the activities described, and not particularly highlighted in the interviews. When specifically asked, variables are brought up by some as the link to algebra.

The preferred source of inspiration and ideas for most of the early adopters is social media, where they participate in special groups for mathematics teachers. Rather than using textbooks or publisher-produced digital teaching aids, they choose internet-based environments that are free of charge and easily accessible. Although most of the early adopters had taken a basic course of 7.5 ECTS credits in programming for teachers, only a few of them had a profound skill in programming themselves, enabling them to create their own programming activities.

Discussion

In conclusion, it could be said that teachers are in a challenging situation. As students move through the grades different programming environments will be used, causing them to learn and relearn the semantics and syntax of symbols. In addition, we identify challenges connected to aspects of learning symbols and concepts that appear in both programming and mathematics with slightly different meanings and syntax. We therefore consider teachers' knowledge about these matters tremendously important, and we intend to make use of Duval's (2006) framework to further investigate and better describe such features. Our preliminary analyses of teaching materials, textbooks, lesson study documentations and interviews indicate that the didactic transposition (Chevallard, 2006) of knowledge concerning aspects of programming and computational thinking is shaky and diversified. Teachers take an active part in it through social media, but at the same time expose themselves to influences they may not be able to view critically since the difference between commercial marketing and peer support is not always clear.

The emphasis on social media as a source for ideas and inspiration about how to fulfil the new curricular demands implies two things. One is that there is obviously a very active community of teachers who interact and help each other. The other is that there is a lack of authority in decisions about what to teach and why to teach it.

The teachers who now teach programming in schools are educated mathematics teachers with no, or very limited, programming skills. From the interviews with early adopters it is clear that such skills are necessary in order to see potentials and pitfalls well enough to create activities suitable for specific learning goals. In particular, moving from a procedural use of algorithms to a conceptual focus on the structure of algorithms, i.e. connecting CT with AT, is not possible with limited knowledge of programming. Furthermore, the difference between ”using digital tools” and ”teaching programming” is not always clear. For some teachers, programming essentially means coding, focusing the first two dimensions in Brennan and Resnick's (2012) framework; computational concepts and thinking practices. For others, it seems to be an overarching concept incorporating the third dimension, that is the computational perspectives expressing, connecting and questioning, along with skills in handling digital tools.

We also note that the integration of programming into the core content of algebra in the Swedish curriculum is not mirrored in the transposition process from knowledge to be taught to taught knowledge (figure 1). Neither textbook authors nor teachers seem to see the connection clearly, and programming activities in mathematics more often deal with other mathematical content. In addition, the focus on programming per se seems stronger than the focus on mathematical concepts in our study, in a similar way as described by Hickmott et al. (2018).

According to Nouri et al. (2020) and the teachers they interviewed, the so called 21st century skills could be taught with block programming, i.e. Scratch, but according to the teachers in our study, it is not that simple, as block programming also brings a tension between students' self-expression and teachers' management. We agree with Nouri et al. (2020) that teachers need increased knowledge about different programming concepts, but would also like to add the need of a thorough understanding of how different mathematical concepts, e.g., variables can be used and denoted differently in different programming environments. Therefore, in the next stage of the project, we will scrutinize to what extent teachers are aware of the affordances of mathematical symbols and concepts in relation to Brennan and Resnick's (2012) framework.

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Notes

1 https://larportalen.skolverket.se/#/moduler/1-matematik/alla/alla

Signs emerging from students' work on a designed dependency task in dynamic geometry

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This paper reports on the design and implementation of a didactic sequence in the frame of a design-based research study. We elaborate on the task design principles and present the analysis of two Danish grade 8 students (age 13–14) working on the very first task of the sequence. The Theory of semiotic mediation frames the analysis of data, which was collected in the form of screencast, video and written products. The results indicate that students expect dependencies to be non-hierarchical in DGE; that specific prompts may be needed to shift students' attention to specific elements of constructions; and that asking the students to explain unexpected observations seems to be necessary for active reflection.

In the vast research literature on dynamic geometry environments (DGE hereafter), several studies deal with the relation between DGE affordances and students' mathematical reasoning, conjecturing and proof (e.g. Sinclair & Robutti, 2013). A seminal affordance of DGE, is that dynamic geometrical figures may be constructed, which may be manipulated by *dragging*, while certain properties remain invariant. The relationship between the elements of the figure is locked in a hierarchy of dependencies, determining the outcome of a dragging action (Leung et al., 2013). These dependencies are linked to the theoretical properties of the figure, which are decided by the construction method, by the theory of Euclidean geometry governing the system and by software design choices. Although the research literature on DGE affordances is comprehensive, Sinclair et al. (2016) state that task design and teacher practice remain understudied, a statement, which was reiterated by Komatsu and Jones (2018). An influential contribution in this domain is the *Theory of semiotic mediation* (Bartolini-Bussi & Mariotti, 2008), which provides a framework for describing the complex relation between tasks performed with artefacts, such as DGE, and students' development of mathematical meanings, as well as the role of the teacher in this regard.

In this paper we report on a didactic sequence, which was carried out in lower secondary school in Denmark, focusing especially on the task design related to a "dependency task" (we elaborate on this notion to in the task design section).

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In the next section, we briefly introduce the key concepts from the Theory of semiotic mediation, after which the research question of this study is formulated. Afterwards, we describe the objectives, hypotheses and choices concerning the task design of the initial tasks in the didactic sequence. Then we present data analysis from a pair of students working on the very first task in the sequence.

Theoretical approach

The *Theory of semiotic mediation* (Bartolini Bussi & Mariotti, 2008) characterizes how students form personal meanings in relation to the use of an artefact. The initial personal meanings developed by the student, may not match the mathematical meanings an expert mathematician (the teacher) would recognize. However, through the didactical intervention, the evolution into mathematical meanings may occur. Bartolini Bussi and Mariotti (2008) coined the term ”semiotic potential” of an artefact to express the duality of possible personal meanings and mathematical meanings, which may be evoked by using an artefact in the solution of a specific task. Awareness of this potential enables the teacher (or researcher) to design tasks aiming to promote certain mathematical learning. Such cognitive development is described as a process of internalization, which has ”two main aspects: it is essentially social; it is directed by semiotic processes. In fact, as a consequence of its social nature, external process has a communication dimension involving production and interpretation of signs.” (Bartolini Bussi & Mariotti, 2008, p. 750). Therefore, the analysis of the internalization process may be oriented towards the analysis of the use of signs in social activities. In other words, the evolution of student meanings may be analysed by interpreting the signs the students produce, e.g., gestures, verbal utterances, written signs or DGE actions in social activities. The analysis may unveil to what extent the didactic sequence, including the task design, foster the production of signs with underlying meanings that are in alignment with the aim of the sequence. The development of meanings can be highlighted by identifying specific semiotic chains, e.g. chains of relations of signification (Bartolini Bussi & Mariotti, 2008, p. 756)

The teacher plays an essential role in supporting the evolution of personal meanings toward mathematical meanings. However, interpreting and reacting in classroom discussions and on the spot to signs produced by the students may be challenging for the teacher. Therefore, it may prove useful to accompany the design of the tasks both with the analysis of the semiotic potential and the description of the possible ”unfolding” of such a potential. In this way, the task design may integrate types of signs, which can be expected to emerge as the students work on specific artefact tasks. In this light, the following research questions arise: As students work on a *designed dependency task*, which type of signs emerge that are related to the use of the dragging tool and can be seen as evidence of students’ awareness of the logical relationship between the geometrical properties in play? How may choices in the task design be revised in order to support the intended learning aim more effectively?

Method

The study is part of a larger project in which design-based research methodology is applied (Bakker & van Eerde, 2015). The aim of the project is to develop design principles for didactic sequences that utilize the potentials of DGE in relation to students' development of reasoning competency[1] (Højsted, 2018). Based on an initial theoretical analysis, a 15 lesson didactic design was developed and tested in three design iterations in three different Danish 8th grade classes (age 13–14). The students worked in pairs using one computer. Data was gathered in the form of screencast recordings from all groups and collection of students' written products. In addition, three groups were chosen in each class for external video recording to allow for a richer collection of emerging signs. The groups were chosen in collaboration with the teacher to comprise a range of low to high attaining groups concerning mathematics achievement. All students had previous GeoGebra experience, and knew the basics, e.g. using commands for construction.

The data is analysed by identifying the emerging signs/semiotic chains of the students, in order to make a synthesis of possible personal signs that the teacher may expect, and to review to what extent the meanings are aligned with the expected outcome of the task design. Finally, the design is evaluated in light of the analysis and some refinements of the design are proposed. Due to space limitation, we only present data from one medium-high achieving group in this paper; we do however refer to data from other groups in the analysis and concluding discussion.

Task design

The design of the initial tasks in the didactic sequence can be decomposed into three related dimensions. At the macro level, there is an *objective*, which describes the students' learning aim. Then there is a *hypothesis* about the types of tasks, which may support the students to achieve the aim. Finally, there are *choices* made in the micro level of design, such as formulations in the task and descriptions of student activity. To ensure alignment, each choice should be coherent with the hypothesis, which in turn should be coherent with the objective. This structure is homologous of that of the design-based research (Bakker & van Eerde, 2015) and consistent with the predictive and advisory nature of the research project.

The learning *objectives* are twofold: (1) that the students develop an awareness of the logical relationship between geometrical objects in GeoGebra, which are perceptually observable as invariants during dragging. That involves being able to discern free and locked objects in GeoGebra, and to be aware of the fact that it is these relations between objects, which decide the outcome of dragging. (2) That they are able to interpret the construction dependency geometrically as logical dependency. This requires geometrical attention to the theoretical relations induced in the construction procedure.

The *hypotheses* concern both these objectives and are related to the semiotic potential of a DGE with respect to the logical dependency between geometrical properties of a constructed figure (Leung et al., 2013; Mariotti, 2014). The semiotic potential already described in relation to a construction task, is now reformulated from the perspective of the task design in terms of objective, hypotheses and choices. The hypotheses are based on the previous literature concerning the semiotic potential of tools in a DGE. Hypothesis (1): Since any constructed figure behaves according to the geometrical relationships defined by its construction procedure, students acting on a figure produced by a construction command can observe the difference, and may realize the dependency induced by a command, i.e. the perceptual result of dragging may be related to the input commands. We denote this type of task, which encourages the construction of a figure and consequent guided exploration of the dependencies in the figure, a ”dependency task”.

Hypothesis (2) concerns the semiotic mediation process. The students' perception of the phenomena observable on the computer screen may be linked to a geometrical interpretation. Partly, this geometrical interpretation may occur spontaneously if the students utilize their previous geometrical knowledge, but in particular, it may evolve through social interaction, and most essentially, through the mediation of the teacher in classroom discussion. Even though hypothesis 2 is of utmost importance, we will primarily focus on hypothesis 1 in this paper.

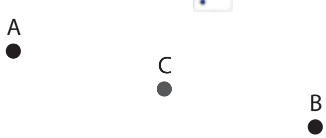
Besides the general choice of proposing the exploration of a constructed figure, four *choices* are made at the micro level of the task design, in alignment with the hypotheses. (i) *The students are encouraged to make a construction, which contains certain dependencies between objects*. The choice reflects that the goal is to foster awareness of properties in the construction. Therefore, by selecting the construction commands themselves, instead of being handed a ready-made construction, the students may reflect on how to make the construction appear in the DGE. In addition, they may interpret the behaviour of the construction during dragging, as a consequence of their construction method. However, some guidance was given in the form of accompanying pictures of commands, which may be useful to complete the construction, as well as a picture of the required construction. Choices (ii–iv) are related to what White & Gunstone (2014, p. 44–65) refer to as Prediction-Observation-Explanation: The students are required to predict the result of an event, and to justify their prediction. Afterwards, they report what they observe and resolve any differences between prediction and observation. (ii) *The students are encouraged to predict,* *before they drag objects, what will happen on the screen when they drag certain points, and to justify their prediction to the co-student they are working with*. This choice reflects the aim of directing the students' reflections onto theoretical properties of the construction. Asking the students to predict the properties of the diagram before they drag, may give rise to conflict, if what they observe does not coincide with their prediction. The conflict can provoke intellectual curiosity (Laborde, 2003). Encouraging students to justify their prediction supports them in becoming able to justify claims to others, which is a characteristic of the reasoning competency (Niss & Højgaard, 2019, p. 16). The aim is that the justifications become anchored in the theoretical properties, which they have just induced. (iii) *The students are encouraged to drag certain points and to describe what happens*. This step is added so that the students can confirm the expected outcome, or wonder why it did not go as expected and try to figure out why. Again, with the goal of students becoming aware that the theoretical properties induced in the construction are responsible for the outcome of dragging. (iv) *The students are encouraged to give an explanation concerning certain essential relations in the construction*. This step may direct the students' attention to certain essential properties of the construction, again to pursue the main goal of developing awareness of the theoretical properties of the constructions, which decide the outcome of dragging. Steps (ii–iv) are sometimes repeated for different elements of the same construction.

According to the Theory of Semiotic mediation, the request of discussing and writing that accompanies each task constitutes the semiotic component of the design related to hypothesis (2); it is expected to trigger the semiotic mediation process that is rooted in the use of the artefact and will be further developed in the collective discussion. The specific choices concerning the role of the teacher in the classroom discussion, will not be elaborated upon in this paper.

The unfolding of the semiotic potential

In this section, we analyse part of the data from two students, Sif and Ole, working on the very first task of the sequence. First, we introduce the task formulation, then we present the emerging student signs and analyse them using a semiotic perspective. The first part of task 1 is described in figure 1.

1.a. Construct two points A and B in GeoGebra and the midpoint C between them. Use the command 



1.b. What do you think happens to the other points when you drag point A? Guess first and justify your guess to your partner. Investigate afterwards, what happened?"

Figure 1. *The first part of task 1*

The teacher described Sif as a high achiever and Ole as an above average achiever in the mathematics classroom. They are about to guess to question 1.b:

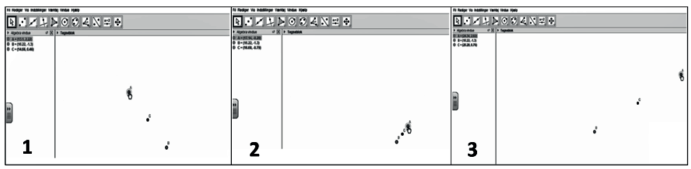
18 Ole: Ehm, I think B stays in the same place and A will be moved and C will still be in the middle. [Marking the points onscreen with his finger while explaining]

19 Sif: Eh, B stays in the same place. And then.

20 Ole: C remains in the middle.

21 Sif: [Writing the answer and talking] C will move, depending on A's ... will move to stay in the middle.

22 Sif: [Reading the text out loud] Investigate afterwards what happened. [Drags point A]



26 Sif [Writing and talking] We guessed right.

The pre investigation written answers was (the guess): ”B stays the same, C will move in order to remain in the middle” and the post investigation answer was: ”we guessed right”.

Analysing 1b

Sif and Ole describe the expected movement of each point in the construction. Their expressions indicate an awareness of the fact that the relations induced in the construction will be maintained. The description is at the local level of points of the construction: ”B stays in the same place”, ”A will be moved”, ”C will still be in the middle” (line 18). We can also notice that the utterance about point C is elaborated upon to highlight that it will not stay in the same position but rather that ”C *remains* in the middle” (line 20). Further, they indicate that C is dependent on A ”C will move, depending on A's ...” (line 21). We may interpret that the personal meanings underlying the expressed signs are coherent with the meanings that the activity aims to foster, namely that there are relations between the geometrical objects, that these relations determine the dependency between the points and such relations decide the outcome of a dragging action. The students do in fact explicitly express such a dependency as the final result of a semiotic chain that evolves in the dialog between the two students. We can see how the meaning of dependence becomes more and more explicit in the semiotic chain: ”still be”, ”remains”, ”move depending on A and move to stay in the middle”. In the last formulation – that is not reported in the written report – both the dependence relation and the specific property originating the dependence are made explicit.

Considering the aim of step (ii), the students do make a prediction based on the theoretical properties, which they induced in the construction. Sif also offers an explanation to the expected movement of point C ”C will move, depending on A's... will move to stay in the middle.”, but not for the other points. The students drag and confirm their prediction, which was the aim of step (iii).

Step (ii) and (iii) on different elements of the construction

Task 1d was as follows: ”What do you think happens with the other points when you drag point C? Guess and justify first. Investigate afterwards, what happened?” The following occurred when Sif and Ole worked on task 1d.

95 Sif: [Reading the text] What do you think happens to the other points when you drag point C? Guess and justify first.

96 Ole: It's all moving together.

97 Sif: Then everything moves because C must be in the middle. Then they will move in relation to C? [The tone indicates a question and she looks at Ole]

98 Ole: I think so.

99 Sif: Then one could imagine that it was just a line moving around.

[Sif gestures with her hand a line going through the three points. It looks like she moves her hand so that it remains parallel to the initial line]

100 Ole: Yes exactly, in parallel.

101 Sif: Okay, so we just say that everything will move relative to point C. [Writing]

103 Sif: Yes, because it must be in the middle in relation to C. [Sif tries to drag point C]

104 Ole: Oh!

105 Sif: One cannot move C. [Sif writes down]

109 Sif: Ehhh, and why can't you? ...

[The teacher (T) has stood next to them for a while, and decides to intervene]

110 T: Why can't you move C?

111 Ole: Eh, I don't know

112 T: Why can't you move the midpoint?

113 Sif: It is perhaps because it is the midpoint in relation to the other two points.

114 T: What do you think you can move if it was? [It seems he is asking them what is possible to move]

116 Sif: A and B.

117 T: Yes you can move on A and B because that was what he said [”he” means the researcher, which did an introduction to the material at the start of the lesson], you know, it’s a dynamic program. That is, C will always be the midpoint, that is, C is automatically moved if A and B are moved.

118 Ole: Yes exactly.

119 T: If you do not move A and B, then C stays the same.

120 Sif: Okay. [They move on to the next task]

The pre investigation written answers was (the guess): ”Both points will move in relation to C” and the post investigation answer was: ”You cannot drag C”.

Analyzing 1d

We may interpret from line 96 ”It’s all moving together” that Ole intuitively expects that the construction will move as a Rigid/solid structure. This immediate expectation is observed in several other groups too. Sif understands Ole’s suggestion, elaborates upon it and justifies why it may be so, based on the construction. However, she is not completely sure (line 97). Although she says that the line moves ”around”, her gesture suggests a movement remaining parallel with the initial position in an orthogonal translation (line 99), Ole notices the meaning of the gesture and makes it explicit (line 100).

The students predict what will happen and justify their predictions, based on theoretical properties, which was the aim of step (ii). The description of the expected global movement of the construction is interwoven with justifications based on local elements such as ”everything moves because C must be in the middle” (lines 97, 101, 103). The surprising result of their dragging investigation leads them to wonder why C cannot be moved. It seems plausible that they would continue to work on this question if the teacher did not intervene, hence step (iii) worked according to plan in this case. The initial teacher action is promising. First, he asks ”why can’t you move point C” followed by a reformulation into ”why can’t you move the midpoint”. This highlights the theoretical status of point C. The intervention of the teacher moves from a general to a more specific question. Such a shift leads the student to immediately grasp the suggested geometrical perspective and guess (line 113). However, the teacher then shifts the focus from non-draggable to draggable points, to which the students correctly answer A and B. He explains what they already know, that C remains the midpoint, referring to an authority argument and to the software ”that is what he said, you know, it’s at dynamic program” (line 117), and finally ”If you do not move A and B, then C stays the same” (line 119). His explanation does not make it any clearer why C cannot be moved. What is observed can be explained both by geometrical reasons and by software design reasons. In GeoGebra, it is not possible to drag locked objects, which are derived from other objects. However, other DGE[2] allow dragging the dependent points too. The teachers action seems to close the door on the students’ investigation. It can be considered a missed learning opportunity.

Concluding discussion

In our analysis of the type of signs produced by the students, we notice that several students, including Sif and Ole, expect the construction to behave as a rigid/solid structure during the dragging of derived points. This finding indicates that students do not immediately expect the relations between elements to be hierarchical. In fact, the students signs suggest that they intuitively to view the construction as a rigid/solid structure with non-hierarchical dependencies. The finding is consistent with previous results with junior high students and graduate students in mathematics education (Talmon & Yerushalmy, 2004). This knowledge may be useful to the teacher if such dependency tasks are to be introduced in the classroom, or if other tasks are introduced, in which the hierarchical nature of the environment is expected to be exploited. The fact that the teacher only refers to the geometrical property may limit the interpretation of the phenomenon to geometrical reasons. What can be observed is explained both by geometrical reasons and by ”software design” reasons. What the students are to become aware of refers precisely to both.

From the analysis of the signs emerging from other groups, we see that, in order to explain the on screen phenomena, some students refer neither to the construction process nor to geometrical properties. They instead use a global description of the construction, e.g. the signs ”they move along in parallel” or ”the sides move along like a stick”. They seem not aware of the necessary attention to relations between local elements of the construction, and that they should interpret what happens on the screen in relation to the construction process. It may be useful to revise the task formulation concerning choices (ii–iii) to ask more directly about each element in the construction pre investigation, and post investigation, in order to support a geometrical interpretation of the phenomenon that can be observed, i.e. instead of asking ”what happens when you drag point A”, the question could be more focused, aimed at directing the attention on specific elements of the figure, e.g. ”what happens *to point B* when you drag point A” etc.

In our interpretation of the students' signs produced in relation to task 1d, we found that even though the students are surprised and intellectual curiosity arises (Laborde, 2003), they may just write what happened, and quickly move on to the next task (this happened in some cases). The task may be reformulated so that, in case the construction does not behave as predicted, then the students are encouraged to explain why. This may lead us to the hypothesis that a good task choice would encourage the students observe a situation where they expect something to unfold and on the contrary, this does not happen. Afterwards they *must explain why*. The general hypothesis could be: In front of something unexpected, an explanation rises …

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Notes

1 The reasoning competency is one of eight mathematical competencies in the Danish KOM framework (Niss & Højgaard, 2019).

2 E.g. Geometer Sketchpad 3, as mentioned in Talmon & Yerushalmy (2004).

Mathematical communication competency in a setting with GeoGebra

Cecilie Carlsen Bach

The core of mathematical communication competency is the ability to interpret others and the ability to express oneself mathematically. This paper examines the activation of this competency and its interplay with digital technologies in an 8th-grade classroom (age 14–16), focusing on two students' interaction with GeoGebra. The aim is to investigate and discuss the stated interplay from two different lenses: discourse and instrumental genesis. The lenses bring different perspectives. On the one hand, the results show that using GeoGebra increases the complexity of students' mathematical communication because the students need to switch between discourses. On the other hand, to support communication competency, GeoGebra must be an instrument for the students.

In mathematics education, the focus on both digital technologies and mathematical competencies is increasing (Trouche et al., 2013; Niss & Højgaard, 2019). However, a significant problem arises when implementing these two paradigms, which seem to run separately. This paper focuses on the interplay between digital technologies and the mathematical communication competency (as described in Niss & Højgaard, 2019) in an 8th grade classroom (students aged 14–16) in Denmark.

In Denmark, it is meaningful to investigate this interplay because of the implementation of both digital technologies and mathematical competencies within the national curriculum in both primary and secondary education (UVM, 2017; 2019). However, research on the mathematical communication competency is limited, and research on its interplay with digital technologies does not exist.

This paper aims to investigate and discuss the interplay between the use of digital technologies and mathematical communication competency using two theoretical dualities: visual mediators-routines (Sfard, 2008) and artefact-instrument (Guin & Trouche, 1998). This paper asks how the two theoretical dualities of visual mediators-routines and artefact-instrument contribute

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to understanding the activation of students' communication competency in a setting in which students work with GeoGebra.

This study investigates the interplay using an illustrative case of two 8th grade students' problem solving as data. The mathematical problem, which the students worked with, was partly designed to activate communication competency and partly to the use of GeoGebra. To understand how the communication competency is activated when using of GeoGebra, the competency description is not enough. To understand the mathematical communication aspect, Sfard's (2008) perspective on discourses is utilised as an analytic tool. Also, the theoretical concept of instrumental genesis is used as an analytical tool to examine the use of GeoGebra (Guin & Trouche, 1998). Two transcripts of the two students' work are presented, followed by an analysis from a discourse perspective and then, an analysis from the perspective of instrumental genesis. The discussion combines the analyses with the competency perspective.

Mathematical communication competency

In the Danish competencies framework (KOM), general mathematical competency is defined as "someone's insightful readiness to act appropriately in response to *a specific sort* of mathematical challenge in given situations" (Niss & Højgaard, 2019, p. 14). KOM consists of eight mathematical competencies, all of which are related yet different, and one of them is mathematical communication competency. Having mathematical communication competency means having

> […] ability to engage in written, oral, visual or gestural mathematical communication, in different genres, styles, and registers, and at different levels of conceptual, theoretical and technical precision, either as an interpreter of others' communication or as an active, constructive communicator. (Niss & Højgaard, 2019, p. 17)

Frequently, mathematical communication includes one or more mathematical representations and generates the use of particular mathematical concepts, results, or theories (Niss & Højgaard, 2019).

Communication in mathematics

This section introduces theoretically to mathematical discourse, focusing on visual mediators and routines.

Sfard (2008, p. 146) defines communication in mathematics as "[...] a rule-driven activity in that discursants' actions and re-actions are from certain well-established repertoires of options and are matched with one actioner in a nonaccidental patterned way".

This definition regards communication as a patterned activity between individuals who act on others' actions based on rules. According to Sfard (2008), mathematical sentences give an impression of handling material things, but mathematical nouns replace the names of the material objects. In colloquial discourses, the objects exist independently of the discourse involved in the communication, but this does not apply in mathematical discourse. Visual mediators might represent objects in mathematics, but the actors of the mathematical discourse never gain access to the actual objects (Sfard, 2008).

Sfard (2008) includes four elements in the definition of mathematical discourse: word use, visual mediators, narratives, and routines. First, an important characteristic is the use of words that are distinct for mathematics. These words could be ”equation”, ”slope”, ”piecewise function”, and ”graph” when working with functions. Second, a visual mediator in a mathematics discourse is ”a visual realization of the object of a discourse” (Sfard, 2008, p. 302). Particularly in mathematics, this involves mathematical symbols, but a realization could also include graphs, gestures, words, and drawings. Third, the narratives are descriptions and explanations of mathematical objects and activities. Forth, routines are ”repetition-generated patterns of our actions” (Lavie et al., 2019, p. 153).

Based on Sfard (2008), having mathematical communication competency (Niss & Højgaard, 2019) means that a student must be able to engage in a mathematical communication situation based on particular rules for the activity. The student must be able to use mathematical words; use and understand different visual mediators existing in the communication situation and understanding rules of engaging in a communication situation (Sfard, 2008). In a mathematical communication situation in which students work with GeoGebra, GeoGebra becomes a mediator of discourse. The students then both act and react on each other and the in- and outcomes provided by GeoGebra (Antonini et al., 2020).

Instrumental genesis

When students use digital technology within the classroom, the software influences the mathematics content that the students are learning, because new opportunities to interact with mathematical objects emerge (Guin & Trouche, 1998). The use of GeoGebra is no exception. Distinguishing between artefact and instrument is key when using instrumental genesis to look at students' use of technology. An artefact is regarded as a material object; an instrument, on the other hand, does not exist in itself, but an artefact becomes an instrument for a person when she can use it in an activity (Verillon & Rabarbel, 1995). When an individual manipulates an artefact into an instrument, instrumental genesis happens (Guin & Trouche, 1998).

The complex process of instrumental genesis leads to the reorganization of activity – in this way, the student can manipulate the artefact, and it becomes an instrument. The student needs to acquire new knowledge to be able to make new procedures. At the same time, ”the features of instrumented activity are specified” (Guin & Trouche, 1998, p. 201) concerning both the constraints and the possibilities inherent to the artefact. The constraints and the possibilities are subsequently connected to the new procedures of the artefact. Then, the student encounters the artefact, and the student can ”identify, understand and mange in the course of this action” (Guin & Trouche, 1998, p. 201). Thereby, the instrumental genesis occurs when the student has new possibilities to use the artefact, and when a reorganization of the instrumented activity takes place. The artefact becomes an instrument for the student (Guin & Trouche, 1998). In an instrumented activity, it is essential to state the close relationship between the students' mathematical knowledge and knowledge about use of the instrumental (Lagrange, 2005)

The use of the digital tool offers different representations and words (Guin & Trouche, 1998), which is interesting when looking at mathematical communication (Niss & Højgaard, 2019).

Method

The aim of this study is reached by investigating the communication between two students that are solving a mathematical problem concerning piecewise functions, with the help of GeoGebra. The data consists of the students' speech, their actions, and the reactions from GeoGebra. Data is presented as transcripts of a discussion between the two students.

This section presents the educational context, the design of the mathematical problem, and the transcript used as data in the analyses.

Design of the mathematical problem

The students solve a task, based upon a released PISA task about newspaper sellers. Two job advertisements describing sellers' pay per week are presented from two different newspapers: Zedland Star and Zedland Daily. The Zedland Daily pays 60 Danish kroner every week and, additionally, 0.05 Danish kroner per sold newspaper. Zedland Star pays 0.2 Danish kroner per sold paper, and then 0.4 additional Danish kroner per newspaper sold after selling 240 newspapers in one week (OECD, 2012, p. 70, task no. PM994Q).

The designed lesson lasted 90 minutes. The students worked together and shared one computer to promote communication. The task offered the use of different representations, essential for both mathematical discourse (Sfard, 2008) and the process of instrumental genesis (Guin & Trouche, 1998). In the original PISA test, the task was a multiple-choice question – the students had to choose between four pictures, which all contained both a graph for Zedland Star and Zedland Daily. In the present study, the students were guided by instructions about the newspapers' pay, and they were asked to draw the graphs themselves:

> Use GeoGebra to calculate the differences between how the newspapers pay the sellers.
> – Compare the graph and solutions for the two newspapers.
> – Choose the newspaper you would prefer to work for and explain thoroughly why.
> – Prepare arguments for a discussion in plenum using your results, including different representations.

Zedland Daily is defined as $F_D(x) = 0.05x + 60$. Zedland Star is defined as $F_S(x) = 0.02x$ if $0 \leq x \leq 240$ and $F_S(x) = 0.6x - 96$ if $x \geq 240$ (OECD, 2012, p. 70, task no. PM994Q).

Educational setting

Data collection took place in an 8th grade (students aged 14–16) classroom in Denmark. The students usually worked with GeoGebra in school, but they were not used to solve tasks concerning piecewise functions in GeoGebra. Learning about piecewise linear functions is mentioned in the Danish curriculum (UVM, 2019). However, working with piecewise linear functions demands a more comprehensive understanding of functions than just working with linear function (Bayazit, 2010).

Transcripts of the students' dialogue

The data is presented as transcripts of dialogue between two students (S1 and S2). Two transcripts function as extracts and focus on the students' process of the mathematising of Zedland Star when using GeoGebra. Zedland star is the piecewise linear function.

The following is Transcript 1. S1 has, just before the transcript started, written $fx = 0{,}4x + 48$ in the algebra window in GeoGebra. The use of commas is essential here: In Denmark, the rule is to use decimal *commas* instead of decimal *points* in decimal numbers. (GG = GeoGebra).

Transcript 1. Decimal commas versus decimal points

S1 This is what we agree on, right? For every *x*, we have *0,48*? [S1 looks at S2] Then we are pressing "enter".

S2 Enter, enter, enter.

GG Please check your input.

S1 What? No! Oh, it is because. I know what I did wrong. [S1 points with the mouse on *fx*] A parenthesis is missing there. There is a parenthesis missing there! [S1 controls the computer and writes $f(x)$ instead of *fx*]

S1 Otherwise, we will change it into *y*. [S1 presses enter again]

GG Please check your input.

S1 What? [S1 changes $f(x)$ into *y*] How bad are we at this?

GG Please check your input.

S1 Oh, it is because you need to use a period instead of a comma. It still needs to be *y*. [S1 changes the comma into a decimal point]

S1 Wouldn't it [=GeoGebra] show it? [S1 right-clicks with the mouse in the algebra window; she does not understand why it cannot be shown in the graphical window]

S1 Show object. Oh, it is because it starts at *48*, we are so stupid. [S1 zooms out]

Transcript 2, below, is an extract of the dialogue when the students figure out that Zedland Star is a piecewise function.

Transcript 2. Understanding piecewise functions

S1 Ups. We did something wrong. This is not right, S2. [S1 and S2 look at the face of GeoGebra] That one is not right. It says that when you have sold zero newspapers, you have *48* DKK (=danish valuta), but that is not true? When you sold that many when you have sold *240* newspaper, you have *48* DKK – then, you are over here, but I do not know how to do that. [S1 points at (240, 48) in the coordinate system]

S2 Okay. [S2 takes on the control of the computer, deletes $y=0.4x+48$ in the algebraic window. Now it says, "$y=$"]

S1 Okay. We will try to change it again.

S2 Okay. So, if you have sold *240*, you get *48* DKK, okay?

[…]

S1 This is really how you are supposed to write it, *x* ... This is how it is until we reach *240*, then we have to ... [S1 writes "$y=0.2x$", but does not press enter]

S2 Should we try something else? [S2 adds "0.4" to the equation]

S1 Okay 0,4*x*. I think that we are missing something on that one [pointing at their present graph for the Zedland Star]. There must be a limit, a parenthesis, or something else. Is there supposed to be something else, when it reaches *140*? [meaning *240*]

S2 Yes.

Analyses of the students' communication

This section presents two analyses of the presented transcripts. First, focus is on the students' communication. Secondly, focus is on the use of GeoGebra.

Communication when using GeoGebra

In this section, transcript 1 and 2 are analysd using Sfard's (2008) concepts *visual mediators* and *routines* to understand the mathematical communication between the students. The analysis focus on identifying different uses of *visual mediators*, which is uses of various mathematical realisations of mathematical objects, in this case piecewise functions, and students' different *routines*, that is, patterns of visible actions relating to the mathematical content.

In transcript 1, the students' communication involves different visual mediators such as ”0,48”, the equation written in GeoGebra, and the graphical representation of the function. Taking decimal numbers as visual mediators, a discrepancy between the discourse between the students and the discourse between the students and GeoGebra emerge. In the discourse between the students, decimal numbers are written using decimal comma (e.g., ”0,48”). In the discourse between GeoGebra and the students, decimal numbers are written using decimal points (e.g., ”0.48”). In the discourse between the students and GeoGebra, GeoGebra becomes a mediator of the mathematical discourse between the students. The students then need to switch between discourses and the different visual mediators used in the two discourses. Looking at transcript 2, a development of the students' use of decimal points and commas appears when the students *say* ”comma”, but uses decimal points when writing in GeoGebra. The students then know how to use the visual mediators of both discourses.

The students' use of a decimal comma when writing decimal numbers can also be described as a routine in their mathematical discourse between the students (i.e., when they write decimal commas in transcript 1 and say comma in transcript 1 and 2). In the discourse mediated by the students and GeoGebra, the use of decimals points when writing decimal numbers in GeoGebra is regarded as a meta-rule mediated by GeoGebra. The use of a decimal point is a pattern, repetitive in their action when working with decimal numbers. Mainly in transcript 2, this pattern of action appears when the students repeatedly keep on the writing the decimal points due to GeoGebra's demands.

Summing up, two different mathematical discourses exist when students use GeoGebra in class analysing the visual mediators and the routines in transcript 1 and 2.

The use of GeoGebra when communicating

In this section, the students' uses of GeoGebra in transcript 1 and 2 are analysed, utilising two concepts from instrumental genesis: *artefact* and *instrument*. The concepts help to understand how GeoGebra influences the mathematical communication between the students. The analysis focuses on identifying if GeoGebra is (still) an *artefact* for the students, which is when the program is just a material thing, or if GeoGebra has become an *instrument*. GeoGebra is regarded as an instrument if the students can use GeoGebra in an activity (Verillon & Rabardel, 1995). The process of *instrumental genesis* is identified by looking at reorganisations of the students' problem solving activity (Guin & Trouche, 1998).

At the beginning of transcript 1, GeoGebra is an artefact for the students since the students' ability to use GeoGebra is limited. The students keep using a comma instead of a point, which indicates that the students have not yet acquired the knowledge needed for the use of GeoGebra involving decimal numbers. Because of the feedback provided by GeoGebra, the students reorganize their activity to solve the task. Thereby, the feedback increases knowledge acquisition and the students slowly begin to understand the constraints of GeoGebra.

At the end of transcript 1, the students have learned how to use; that is, the need to use decimal points instead of decimal commas. The process of instrumental genesis has begun: at the very end of transcript 1, further possibilities for the use of GeoGebra arise, which is the zooming feature. This development indicates that the students' acquisition of knowledge is an ongoing process throughout the activity.

In the middle of transcript 2 show the reorganization of activity when writing decimal numbers within GeoGebra. GeoGebra has become an instrument for the students. Although the students have instrumented GeoGebra working with decimal numbers, the students experience new constraint when aiming at making a piecewise function at the end of transcript 1. Elements of GeoGebra remains artefacts.

The students' understanding of functions to include piecewise functions are lacking, which could be a reason why the students try to construct Zedland Star as one function for the whole interval. The mathematical knowledge concerning the object related to the artefact is essential for instrumental genesis. However, the students' mathematical understanding increases – or, their knowledge about representing piecewise functions in GeoGebra improves – as the students work in the teaching session.

Discussion

In this section, the activation of the mathematical communication competency from the two dualities, visual mediators-routines and artefact-instrument, is discussed.

As stated earlier, the communication competency consists of both the ability to express oneself and the ability to interpret others' communication (Niss & Højgaard, 2019). From a mathematical discourse perspective, the students need to master the different discourses that they participate in (Sfard, 2008). As seen in transcript 1 and 2, the students need to be able to switch between the two discourses. In a situation, in which the students activate their communication competency, they must understand how the routines and visual mediators of the particular discourses they participate in. The situation depends on who is engaged in the communication situation – a student taking the person that he communicates with into account master the communication competency better

(Niss & Højgaard, 2019). The students must be able to switch between different discourses communities. For instance, the discourse changes occur between peers, in classroom discussions, and when working with software (Sfard, 2008; Antonini et al., 2020). Using the former analysis combined with the aspects of the communication competency, the students not only need to express themselves to other *people* without the use of software, but also to *software*, such as GeoGebra, and they need to express themselves through *software* to others (Niss & Højgaard, 2019). When students communicate to *software*, such as GeoGebra, they communicate based on GeoGebra's rules of discourse (transcript 1). The students express themselves through *software* when they use the results from the software in communication with others, depending on the current discourse (transcript 2). In the activation of the communication competency, the students express themselves to others based on rules of discourse outside the software, but the students must understand how to switch between discourses (Sfard, 2008; Antonini et al., 2019; Niss & Højgaard, 2019). If the students do not understand the different visual mediators and routines for each discourse, the communication is not as effective as it could be, and their communication competency would seem less developed (Niss & Højgaard, 2019).

Using instrumental genesis to analyse the students' activation of their communication competency, the students need to have GeoGebra as an instrument (Guin & Trouche, 1998). When GeoGebra functions as an instrument, the students understand and control the communication and the rules for the use of decimal points versus decimal commas. If students' attention is on how to write decimal numbers, their focus moves from the mathematical aspects of the activity (i.e. piecewise functions) to the features of GeoGebra, because GeoGebra then is yet an artefact (Verillon & Rabardel, 1995). At the end of transcript 1 and beginning of transcript 2, the students' have been through the process of instrumental genesis, which means that the students' problem solving activity has been reorganised (Guin & Trouche, 1998). After the reorganisation, the students can focus on representing Zedland Star using both equations and graphs. Knowledge of various representations makes the students more competent in mathematical communication when they express their mathematical ideas and solutions to the tasks (Niss & Højgaard, 2019).

The students' knowledge about the mathematical content (i.e. functions), seems to be very relevant not only for the use of GeoGebra but also to show mathematical communication competency. When the students do not know that Zedland Star must be formulated differently for the two intervals, the students' seems less competent when communicating because they express an incorrect understanding of piecewise functions (Niss & Højgaard, 2019; Bayazit, 2010).

Summing up, this study indicates that mathematical communication competency is more complex when students use GeoGebra to solve mathematical problems. Both perspectives indicate that the students must be familiar with the tool to support the competency. Using Sfard (2008), students must understand the different discourse they participate in and they must switch between them. In this case, GeoGebra mediates the discourse – students' then must acts based on GeoGebra rules. Also, instrumental genesis must happen, involving a reorganisation of activities for the students (Guin & Trouche, 1998), when the students understand the constraint and possibilities.

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Problem solving as a learning activity – an initial theoretical model

Anna Ida Säfström and Johan Lithner

Problem solving has been considered the gold standard of mathematical activity. It is a goal of mathematics education that students become problem solvers, and it is suggested that problem solving is a superior method for learning mathematics. However, the arguments supporting the claim that problem solving leads to better learning are often vague. In specific studies, problem solving often constitutes mere one part of a compound design, making it difficult to determine the specific contribution of problem solving. The aim of this paper is to develop an initial theoretical model for problem solving as a learning activity, based on existing frameworks and previous research. Suggestions for how this model could be empirically tested are also discussed.

Problem solving has a special status in mathematics education. Problems are termed ”the heart of mathematics” (Halmos, 1980, cited in Schoenfeld, 1992, p. 339) and it is stated that ”We do mathematics only when we are dealing with problems” (Brousseau, 1997, p. 22). Developing problem solving competence is seen as a key goal for learners (NCTM, n.d.; OECD, 2019; Schoenfeld, 1992; Skolverket, 2019) and problem solving is seen as, and shown to be, essential for developing mathematical knowledge and thinking (Brousseau, 1997; Cai, 2003; Downton & Sullivan, 2017; Ridlon, 2009; Schoenfeld, 1985; Jonsson et al., 2014).

However, explanations for *how* problem solving leads to better learning are often vague or lacking, which is evident in the studies in Sidenvall's (2019) review of research on teaching designs based on learning by problem solving. Some studies lean on a general and implicit argument that you learn what you do (Abdu et al., 2015; Csíkos et al., 2012; White et al., 2012). In some designs, additional characteristics of tasks are emphasised, such as real-world basis (Bonotto, 2005; Schukajlow & Krug, 2014), contrasting examples (Coles & Brown, 2016) or explanatory prompts (Swan, 2007). In some studies, problem solving is mere one design element, which is justified as an enabler of other elements, such as discussion of different solutions (Coles & Brown, 2016; Kotsopoulos & Lee, 2012; Pang, 2016; White et al., 2012). While good teaching

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naturally involves multiple elements that interplay, this constitutes a scientific difficulty, as it obscures the contribution and value of different elements. This in turn creates obstacles for practice. If we do not know how design elements affect learning, the adaptations that are necessary in implementation risk altering integral elements while preserving extraneous features.

One important clue in understanding how problem solving works as a learning activity is that it promotes *creative mathematically founded reasoning* (CMR). CMR is reasoning that is constructed by the learner (novelty) and supported by arguments (plausibility) that are founded in intrinsic mathematical properties of the components of the problem (anchoring) (Lithner, 2008; 2017). However, as with problem solving, the more specific mechanisms of CMR that enhance learning are yet unknown (Lithner, 2017).

In sum, there is a need for clarification of how characteristics of problems and problem solving contribute to the development of specific learning goals. The first step in that direction is to formulate an initial theoretical model for how problem solving works as a learning activity, which is explicit and structured enough to be empirically tested. The aim of this paper is to develop such an initial model by linking the CMR framework to previous research and other frameworks regarding problem solving.

Characteristics of problems and problem solving

Problems are usually defined as a subset of mathematical tasks carrying specific characteristics. These characteristics are supposed to lead to a specific type of mathematical work on tasks, called problem solving. It is then argued that it is the characteristics of this kind of mathematical work that facilitate the better type of learning associated with problem solving. Since the definition of problem varies, so do the foregrounded characteristics of problem solving and the arguments for how problems lead to the activities with those characteristics.

In this paper, we focus on frameworks characterizing problems as involving something unknown. Schoenfeld (1985) defines problems as tasks where the problem solver does not have easy access to a procedure giving a solution. Beghetto (2017) states that problems entail uncertainty regarding how to think and act. In stronger formulations, the unknown is specified as knowledge that the problem solver does not yet have (Brousseau, 1997; Hiebert & Grouws, 2007), but is needed in order to solve the problem. Sometimes, this unknown knowledge is restricted to a method (NCTM, 2000; Skolverket, 2019). It follows that whether a task is a problem depends on the person attempting to solve it, and the context in which it is presented. For example, in everyday life a task may evoke other resources and strategies than if it is presented in a classroom, rendering it routine in one context, but not in another. However, the relations between task and context are complex, and a thorough elaboration on these relations beyond the scope of this paper.

As a result of extensive empirical work, Schoenfeld (1985) presents a framework for the knowledge and behaviours required for problem solving. The framework comprises four categories: resources, heuristics, control and belief systems. In order to reach a solution, the problem solver needs to evoke mathematical *resources*, such as facts, algorithms, standard procedures, understandings and intuitions. She also needs *heuristics*, i.e. strategies for making progress on unfamiliar tasks, such as drawing figures and investigating examples. In order to plan, monitor and evaluate her process, she needs to use *control*, which can be seen as a subset of what other researchers have called metacognitive or self-regulative skills (Schoenfeld, 1992). Finally, she needs a *belief system* allowing her to think that her mathematical knowledge is useful and that she can make progress if she tries.

Beghetto (2017) argues that problem solving is characterised by creative thought and action, while Hiebert and Grouws (2007) stress that problem solving involve effortful struggle, as the problem solver ”grapples with key mathematical ideas that are comprehendible but not yet well formed” (p. 387). Brousseau (1997) states that problem solving entails overcoming an obstacle by constructing a specific piece of knowledge.

An initial model for problem solving as a learning activity

The model incorporates the idea that the knowledge and behaviours coming into play when solving problems can be captured in Schoenfeld’s (1985) four categories: resources, heuristics, control and belief systems. While Schoenfeld (1985) views this set as prerequisites for problem solving, we view this set as problem solving competences that are not only applied and used during problem solving, but also developed and improved. This two-fold function is conveyed in the term *exercising* (Säfström, 2013). Therefore, this set is also an adequate categorisation of plausible learning goals of problem solving.

For each competence, we will describe how problem solving is hypothesised to work as a learning activity developing this competence in relation to characteristics of problems, connecting existing frameworks and current research. For each competence we will also consider the meaning of two important conditions for learning: time and success. Regarding *time* we will elaborate on whether and how problem solving is likely to enhance learning in the moment and by sustained activity over longer periods of time. Regarding *success* we will describe whether and how learning is likely to be affected by whether the student successfully solves the problem.

Developing resources by problem solving

In problem solving, resources are exercised in a different way than when working on routine tasks. The unknown of the problem requires novelty (Lithner, 2008). It is not possible to find the solution by mere guessing (although guessing may be a constructive part of the solution), therefore the solver needs some kind of argumentation supporting the construction of the solution. This argumentation needs to be mathematically plausible and anchored in intrinsic mathematical properties of relevant components of the task (Lithner, 2008). It is this particular consideration of both known and new mathematical properties of the particular resources in question, not required and usually absent in routine task solving (Norqvist, 2017; Norqvist et al., 2019), that reinforces, expands and connects the individual's resources. In other words, problem solving requires resources to be organised in a way that enables the student to identify intrinsic mathematical properties of components and construct a line of arguments following, or possibly creating, a path of connections. Therefore, mathematical knowledge cannot be used purely atomistically, probing memory of facts and algorithms. Instead, problem solving leads to learning mathematics with understanding, i.e. "making connections, or establishing relationships, either between knowledge already internally represented or between existing networks and new information" (Hiebert & Carpenter, 1992, p. 80). The idea that problem solving results in a more connected and efficient organisation of resources is supported by Karlsson Wirebring et al. (2015) who showed that students who learned by problem solving used less brain activity on post-test while performing better, compared to students who learned by routine tasks.

Learning with respect to this competence can take place within a single learning session, as previously shown (Jonsson et al., 2014). This indicates that the activation of resources can be immediate, and new connections made within the solution process of a single task. Over time, exercising of connected resources may both densify and strengthen connections. On the one hand, consideration of intrinsic mathematical properties and attempts at constructing arguments exercise resources even if problem solving fails. On the other hand, the number of completed tasks is not correlated to increased learning *per se* (Jonsson et al., 2014). Therefore, unsuccessful problem solving may still be a better learning activity for resources than successful work on routine tasks.

Developing heuristics by problem solving

Non-problems can be solved by standard procedures, and if a procedure is known it suffices to apply it or, at most, determine which procedure is suitable from a delimited list. If possible, many students choose simple and routine methods (Downton & Sullivan, 2017; Norqvist et al., 2019). However, the unknown of problems makes resources insufficient for attaining a solution. Therefore, problem solving entails exercising heuristics, i.e. strategies for constructing one's own solution. Single problems are not enough to develop heuristics. A method becomes a heuristic only when it is found successful on a set of similar but unfamiliar problems (Schoenfeld, 1985). This implies that learning of single heuristics requires both extended time and success. In order to learn a set of heuristics and when they are suitable, one needs to encounter different types of problems, requiring additional time and failure as well as success.

Developing control by problem solving

Control is exercised when the task requires strategic decisions and evaluation of advances and setbacks. Therefore, the faculty of problem solving as a learning activity for control depends on the level of uncertainty and complexity. Uncertainty requires evaluation: if the interpretation of the problem is dubious, if it is unclear what to do or whether what was done is correct, one needs CMR to construct arguments for strategy choices and conclusions (Lithner, 2008). Complexity requires strategic decisions: if the problem requires multiple steps and consideration of numerous details, one needs to monitor the process (Schoenfeld, 1985).

If consistently prompted for, control can be enhanced over the matter of months (Schoenfeld, 1992; Shilo & Kramarski, 2019). It is likely to develop more slowly by problem solving alone. This combination of requirements – sufficiently difficult problems and prolonged commitment – may prove insurmountable for many learners. While occasional failure is likely to stimulate exercising of control, repeated failure may not, implying that the development of control requires a subtle balance between success and failure. Indeed, previous research show that higher demand of CMR may result in students resorting to imitative reasoning (Boesen et al., 2014; Sidenvall et al., 2015). Therefore, it is of outmost importance for each individual student to encounter problems that are challenging enough to activate control and at the same time reasonable to solve.

Developing belief systems by problem solving

If given problems on an appropriate level, learners can take responsibility for the process and construct their own solutions for problems of increasing difficulty (Brousseau, 1997; Lithner, 2008). This is hypothesised to establish the belief that one's own arguments anchored in intrinsic mathematical properties can solve problems, promoting the learner's own mathematical authority. Problem solving is connected to different norms than imitation of procedures, and there is a reflexivity between socio-mathematical norms and beliefs (Cobb & Yackel, 1996). Frequent problem solving may therefore affect the learner's beliefs regarding whether solving problems is viable and what level of effort is expected when dealing with mathematical tasks.

It is well-known from both research and practice that beliefs can be difficult to change and take time to develop (Hannula, 2006). We also propose that the development of mathematical authority is success sensitive. At least in relation to some beliefs, the choice of task may be crucial. If the student holds the belief that she cannot solve mathematical problems, the most important design decision may be to choose a problem that the student will in fact solve.

Discussion

In this paper we have presented an initial theoretical model for how problem solving works as a learning activity. The main rationale for this model is that it can serve as a starting point and guidance for further empirical research on mathematics teaching and learning. The model provides direction for potentially fruitful experiments, regarding duration of interventions and how problems should be designed. However, we want to stress that the aim is not to determine necessary and sufficient causes for learning. While such causes would no doubt be valuable if discovered, we acknowledge that ”causes” are often more accurately described as ”conditions” (Shadish et al., 2002). For example, resources could be learnt with understanding in other situations than problem solving, but tasks will not develop resources unless they involve something unknown. However, problems will neither develop resources unless additional conditions are fulfilled, e.g. that the problem provides an opening for the learner to start working on the problem.

Some researchers assert that mathematics can only be learnt by problem solving (e.g. Brousseau, 1997). At the same time, mathematical knowledge and skills are sometimes described as prerequisites for problem solving (Bergqvist et al, 2010; Schoenfeld, 1985). We argue that the notion of CMR (Lithner, 2008) provides a means for unifying these views: problem solving gives opportunities to exercise resources in different ways than routine tasks, since the absence of a predetermined method demands constructing arguments anchored in intrinsic mathematical properties. [1] Therefore, resources are structured and connected in problem solving, and this structure provides a means for further reasoning and problem solving.

The model presented here considers problem solving as a learning activity in itself. This focus also reveals potential difficulties and shortfalls in problem solving, e.g. that developing belief systems may require success, while developing heuristics and control may require failure. Such insights can guide the combination of problem solving with other design elements. As previous research shows, problem solving is better realised if combined with other activities. Brousseau (1997) emphasises the function of institutionalisation in valuing and reformulating students' own constructions of resources and heuristics in culturally accepted terms. This can be achieved in whole class discussions, as suggested by others (Kotsopoulos & Lee, 2012; Pang, 2016; Stein et al., 2008). Acquiring and using a heuristic vocabulary during problem solving has been shown to further development of heuristics (Koichu et al., 2007). With respect to control, Schoenfeld (1985) supports the Vygotskyean hypothesis that individual reflection is preceded by social reflection, and has shown empirically that questions prompting for reflection develop students' metacognitive behaviour.

It is probable that the teacher-student interaction during problem solving influences norms and practices in other parts of the classroom context, and that the norms and practices in the classroom context at large influence problem solving (Cobb & Yackel, 1996). Attending to certain aspects of the students' work and thinking lades these aspects with value, affecting beliefs. For example, asking what the problem is about signals that this is something important to consider. Asking about the students' thinking and work signals that this is of value, besides the result and final answer. It is therefore a reasonable hypothesis that interaction concerning heuristics, control and resources also influences belief systems, but possibly over longer periods of time. This method may be more effective for changing belief systems than trying to affect them directly.

To be useful for practice, further elaboration on how other activities support or hinder learning during problem solving is needed, and indeed providing theoretical and empirical bases for such elaborations is a key concern for our continued work. It is, however, beyond the scope of this paper and our current understanding of the complex phenomena involved.

Suggestions for empirical studies

We know from Schoenfeld's (1985) work that the four categories of knowledge and behaviours are detectable when observing students' problem solving processes. We also know that mathematical reasoning can be studied by means of observation and interviews (Lithner, 2008). While these methods provide a foundation for how learning and development of problem solving competence can be studied, they are not sufficient in themselves. Problem solving as a phenomenon gives rise to specific challenges when it comes to studying development over time. By definition, a task cannot be used to test the problem solving competence of a student twice. If different tasks or activities are used at different occasions, performance can be profoundly affected by the characteristics of the tasks. Hence, validity for problem solving measures is a delicate issue.

In addition, some aspects of problem solving may be specifically difficult to study. To a large extent, the processes involved in problem solving are taking place in the mind, hidden from observation. While the observer can access some of those processes by asking questions, such methods are always interventional. This is especially true for control, as it is epistemologically debatable how accounts given when requested relate to peoples' actual rationale (Edwards, 1997). This issue is lessened, but not removed, by studying the exercising of control communicated by group members in group work.

As most existing methods for studying problem solving involve time consuming methods such as observation or interviews, it may be difficult to conduct studies at scale. While surveys and questionnaires, e.g. for self-reported beliefs and metacognitive skills, are available, such methods can suffer from poor validity (Veenman & van Cleef, 2019).

It is clear that additional work is needed in order to empirically test our initial theoretical model and further understanding of how problem solving functions as a learning activity. Undoubtedly, we will have reason to reconsider and revise our model as our work proceeds. Nonetheless, we believe that the formulation of initial models serves an important purpose in guiding and evaluating empirical studies, and for interpretation of both failure and success.

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Note

1 In some cases connections and arguments can obstruct learning (Phenix & Campbell, 2004), but we believe those cases to be few and rather peripheral to a general theory of mathematics learning.

Conceptual knowledge in mathematics – engaging in the game of giving and asking for reasons

Per Nilsson

In this study I re-analyse a transcript from Kazemi and Stipek (2001) in order to show how constructs of the semantic theory of inferentialism can be used to give account of conceptual knowledge in mathematics. In mathematics education research connections are crucial to conceptual knowledge. Inferentialism provides a theoretical conceptualization of connections, in terms of inferential relationships and moves in the language practice of giving and asking for reasons. Based on this conceptualization, the present study shows how constructs of inferentialism can facilitate a fine-grained analysis of conceptual knowledge in mathematics and provide insight on teachers' actions in pressing for conceptual knowledge in teacher-student interaction.

There are several frameworks for describing knowledge in mathematics. Across different frameworks, research seems to agree on conceptual knowledge as one core component of mathematical knowledge (e.g. Kilpatrick et al., 2001; Niss, 2003). Conceptual knowledge corresponds to relational knowledge (Skemp, 1976) and is thought of as "a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (Hiebert & Lefevre, 1986, pp. 3–4).

Since Hiebert's and Lefevre's (1986) seminal work, conceptual knowledge has come to have a prominent position in defining and characterizing mathematical knowledge (Star, 2005) and in capturing the learning of significant concepts within different mathematical domains (Baroody et al., 2007). However, it has been observed an ambiguity in how conceptual knowledge is understood and used (Baroody et al., 2007; Star, 2005). In line with Hiebert's and Lefevre's (1986) definition, research often emphasizes connections as crucial to conceptual understanding but, what Crooks and Alibali (2014) observe is that the meaning and structure of connections are often described in vague or general terms, with limited theoretical grounding.

The aim of the present study is to show how constructs of inferentialism (Brandom, 1994, 2000) provide opportunities for fine-grained analyses of

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conceptual knowledge and of teachers' actions in pressing for conceptual knowledge in teacher-student interaction.

To make my analytical point, I re-analyse a transcript from Kazemi and Stipek (2001) where one teacher is trying to press two students to extend their reasoning on addition of fractions. Kazemi's and Stipek's study is representative for many studies on conceptual knowledge in mathematics, where conceptual knowledge is described in general terms, with no explicit theoretical grounding (Crooks & Alibali, 2014).

Conceptual knowledge

Despite a clear movement in both research and educational practice toward emphasizing conceptual knowledge it does not appear to be a clear consensus in the literature as to what exactly conceptual knowledge is and how best to measure it (Crooks & Alibali, 2014). The term ”conceptual knowledge” has come to denote a wide array of constructs. Hiebert and Lefevre (1986, pp. 3–4) define *conceptual knowledge* as ”[…] knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information”.

Conceptual knowledge is keyed on connections, but the definition says nothing about whether the connections relate to mathematical procedures or concepts. Star (2005) argues for the need to treat type and quality as two independent dimensions in the conceptualization of conceptual (and procedural) knowledge in mathematics. Star proposes that conceptual knowledge is the knowledge type of concepts or principles and describes the quality of conceptual knowledge in terms of connections. Commenting on Star's proposal, Baroody et al. (2007) suggest an alternative in which they describe quality as a matter of connections along the Likert scale of *not, sparsely, somewhat, well* and *richly connected*. However, what is meant by, e.g. sparsely connected, or how this differs from somewhat connected, is not theoretically underpinned or described in any detail (Nilsson, 2020).

The suggestion of the present study is to redefine conceptual knowledge as exclusively a knowledge type and to use the theory of inferentialism to extend our understanding of the role and meaning of connections as a means for giving account of qualities of conceptual knowledge. To this end I follow Anderson et al. (2001) and define conceptual knowledge as the knowledge of *classifications, structures* and *principles*.

An inferentialist account of conceptual understanding

Inferentialism constitutes a semantic theory (Brandom, 1994) in which inferential relationships (connections) in concept use are considered a necessary and inseparable part of knowledge and meaning-making (Bakker & Derry, 2011).

Following Sellars (Sellars et al., 1997), Brandom (1994) argues that we, as human beings, possess the ability to make judgments and form reasons for how and why things happen as they do (Bakker & Derry, 2011; Brandom, 2000). A belief or an action of a person has a position in the space of reasons as the person is aware that their belief or action could be different and can reflect whether their belief or action should be as it is (Mackrell & Pratt, 2017).

Inferentialism takes a social and pragmatic stance on meaning-making and understanding. Rather than considering the space in which thought moves, Brandom suggests looking at inferences from the perspective of playing a language game (Wittgenstein, 1968). Brandom (1994) introduces the *Game of giving and asking for reasons* (GoGAR) as a metaphor to describe how knowledge and meaning-making emerge inferentially within a social and pragmatic practice of reasoning. The principle idea is that we, as human beings, negotiate meanings in the way we use concept [1] (Seidouvy et al., 2019). In the words of Brandom (2002); concepts become what they are according to how they are used, in ”being a move in the ’game of giving and asking for reasons’ ” (p. 528).

Inferentialism is resolutely holistic (Bakker & Derry, 2011). One needs many concepts in order to have any, since the content of each concept is constituted by its inferential relationship to other concepts. Think of the situation of having practical mastery of the concept of ”probability,” coming into articulation in the claim, ”the probability of six is 1/6 when rolling a die.” Claiming this implies, among many things, to know that the claim is based on a perfectly symmetric die, that the relative frequencies of sixes stabilize around 1/6 as we increase the number of rolls and that the probability of not having a six is 5/6. This example involves a GoGAR of many reasons related to the concept of probability, of which only a few been made explicit here. The main point, however, is that these reasons are relevant and become contentful due to their inferential connections. At least four types of inferences can be delineated (Brandom, 2000; Nilsson, 2020).

Identity. Identity inferences are probably the most common inference in language games. It speaks to our ability to make classifications. For instance, pointing to the picture of a quarter-shaded circle one could write ”This is 1/4” and someone else could say ”This is 25 percent”.

Negations. Negation inferences speak to comparison and contrasting. We understand what something is when we are able to infer what it is not (cf. Marton et al, 2004).

Conditionals. Conditionals take the form of ”if-then” clauses and are probably the prime construct of inference. With conditionals, we focus on the circumstances needed for something to happen or to be. The inference, ”*If* there are 50 black marbles in a bag with a total of 100 marbles *then*, it is a fifty percent chance to pull a black marble from the bag” exemplifies a conditional.

Counterfactuals (Brigandt, 2010). Counterfactuals relate to conditionals, as they deal with circumstances in an explicit way. Significative of counterfactuals is clauses involving the expression of ”had been”. Say that you encounter a parallelogram (P) for the first time. Expressing, ”if all four angels of P would have been 90 degrees, then P *would have been* a rectangle” exemplifies a counterfactual, which adds to your understanding of when to use the concept of rectangle and the concept of parallelogram.

I re-analyse a transcript from Kazemi and Stipek (2001) to show how an inferentialist account of conceptual connections can facilitate a fine-grained analysis of conceptual knowledge in classroom talks in mathematics.

An inferentialist re-analysis of conceptual knowledge

Kazemi's and Stipek's study is cited 480 times (Google Scholar 25 September, 2019) and is representative for many studies on conceptual knowledge in mathematics, where conceptual knowledge is described in general terms, with no explicit theoretical grounding (Crooks & Alibali, 2014).

Kazemi's and Stipek's (2001) study involved four teachers in grades 4 and 5, all teaching the same lesson on the addition of fractions. The aim of the study was to ”analyse and provide vivid images of classroom practices that create a press for conceptual learning” (Kazemi & Stipek, 2001, p. 78).

Kazemi and Stipek (2001) analysed and compared episodes from two lessons of high press and two lessons of low press. In the present study I will focus on one transcript on high press interaction (Ms. Carter), since it provides most vivid images of classroom talks in mathematics for the support of conceptual learning. The episode is from a fifth-grade classroom.

It is important to understand that the purpose is not to question the accuracy of Kazemi's and Stipek's analysis. According to Kazemi and Stipek, their analysis is more extensive and detailed than what is common in many studies looking at conceptual knowledge in mathematics. So, in some sense their study is used as a critical case (Flyvbjerg, 2006). In other words, if I manage to show that inferentialism can provide support to elaborate further on Kazemi's and Stipek's analysis on conceptual knowledge, there is reason to believe that this would be possible in many other cases.

I begin the analysis below by briefly presenting the conclusions made by Kazemi and Stipek. I then turn to the inferentialist analysis, which took place in the following steps. Firstly, in order to account for conceptual learning, I searched for inferential patterns enacted, according to identity inferences, negation inferences, conditionals and counterfactuals. Secondly, I looked at the teacher's role. Kazemi and Stipek focused on how the teachers pressed for conceptual learning. Looking at the episode from an inferential lens, in the second step I was searching for instances where the teacher missed opportunities to press students' reasoning further, making the GoGAR more explicit. Thirdly, I compared the outcomes of Kazemi's and Stipek's analysis with the inferentialist analysis.

Results

Episode 1 in Kazemi and Stipek (2001)

Ms. Carter asked Sarah and Jasmine to explain how they divided nine brownies equally among eight people and why they chose particular partitioning strategies.

Sarah: The first four, we cut them in half. [Jasmine divides squares in half on an overhead transparency]

Ms. C.: Now as you explain, could you explain why you did it in half?

Sarah: Because when you put it in half, it becomes ... eight halves.

Ms. C.: Eight halves. What does that mean if there are eight halves?

Sarah: Then each person gets a half.

Ms. C.: Okay, that each person gets a half. [Jasmine labels halves 1–8 for each of the eight people.]

Sarah: Then there were five boxes [brownies] left. We put them in eighths.

Ms. C.: Okay, so they divided them into eighths. Could you tell us why you chose eighths?

Sarah: It's easiest. Because then everyone will get ... each person will get a half and [whispers to Jasmine] How many eighths?

Jasmine: [Quietly to Sarah] 1/8.

Ms. C.: I didn't know why you did it in eighths. That's the reason. I just wanted to know why you chose eighths.

Jasmine: We did eighths because then if we did eighths, each person would get each eighth, I mean 1/8 out of each brownie.

Ms. C.: Okay, 1/8 out of each brownie. Can you just, you don't have to number, but just show us what you mean by that? I heard the words, but ... [Jasmine shades in 1/8 of each of the five brownies not divided in half]

Jasmine: Person one would get this ... [Points to one eighth]

Ms. C.: Oh, out of each brownie.

Sarah: Out of each brownie, one person will get 1/8.

Ms. C.: 1/8. Okay. So how much then did they get if they got their fair share?

Jas./Sar.: They got a 1/2 and 5/8.

Ms. C.: Do you want to write that down at the top, so I can see what you did? [Jasmine writes 1/2 + 1/8 + 1/8 + 1/8 + 1/8 + 1/8 at the top of the overhead projector]

Kazemi's and Stipek's presented analysis

In this situation of high press exchanges, Kazemi and Stipek claim that students went beyond descriptions or summaries of steps to solve a problem. Instead, the students linked their problem-solving strategies to mathematical reasons and, in that sense, Kazemi's and Stipek's approach parallels the inferentialist idea that conceptual content is constituted in the game of giving and asking for reasons. Kazemi and Stipek then infer that, the exchange among Sarah, Jasmine and Ms. Carter highlights a conceptual focus.

> Ms. Carter asked Sarah to explain the importance of having eight halves and the reason why the partitioning strategy using eighths made sense. After Jasmine gave a verbal justification, Ms. Carter continued to press her thinking by asking her to link her verbal response to the appropriate pictorial representation by shading the pieces, and to the symbolic representation by writing the sum of the fractions.(Kazemi & Stipek, 2001, p. 65)

Some reflections on Kazemi's and Stipek's analysis/conclusions. In the first line we see that ”reasons” are part of their analysis. However, what they ascribe as a reason in the exchange is not clear and not is the meaning of reasons, in terms of what follows from it and what it follows from. Next, we note that ”link” is central to their analysis. It is claimed that links are made between verbal responses and pictorial representations and symbolic representations. But, the nature of the links (connections) are not made explicit. Are the links just acts of classification, according to identity inferences, there is a low degree of inferential reasoning involved and so of conceptual knowledge.

Inferentialist analysis

Kazemi and Stipek (2001) used the episode of Sarah, Jasmine and Ms. Carter as an example of high-press interaction that moves the talk beyond descriptions or summaries of steps to solve a problem. However, scrutinizing the interaction by an inferentialist lens, we are provided more detailed information of the content of the talk and of missed opportunities for the teacher to press the students' conceptual understanding on fractions further.

Figure 1 present a summary of the inferentialist analysis of the episode with Sarah, Jasmine and Ms. Carter. From a GoGAR-perspective we can say that the episode centers around a ”four-brownie task” and a ”five-brownie task”. The four-brownie task is about explaining why each person gets a 1/2 of a brownie from four brownies and the five-brownie task is about explaining why each person gets 5/8 of a brownie from 5 brownies. The GoGAR is then about making sense of these two tasks and the solution of them.

The solution to the four-brownie task is structured in two conditionals expressing *quotitive division*, whereas the solution to the five-brownie task is structured in two conditionals expressing *partitive division* (figure 1). Making

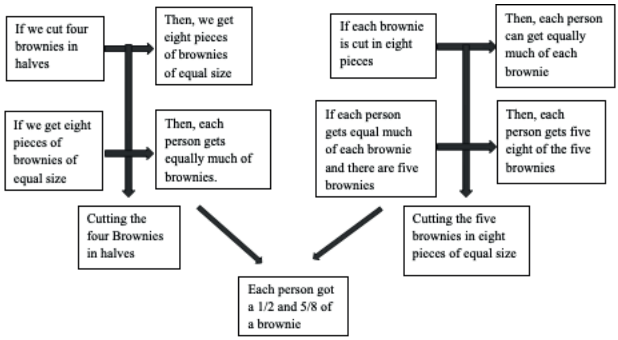

Figure 1. *A summary of the inferentialist analysis of the solutions to the four-brownie task (to the left) and to the five-brownie task (to the right)*

explicit their solution to the four-brownie task Sarah claims, ”The first four, we cut them in half”. If this claim comes alone, one could infer that the answer applies to partitive division, resulting in ”two”: half of four is two. However, the phrase involves ”cut” and is accompanied with Jasmine dividing each of the four squares in half on an overhead transparency. So, the inferential meaning of half is not constituted from how half takes a position in ”half *of a set* (partitive division)” but, from how it takes a position in ”how many half brownies *are contained in* four (whole) brownies (quotitive division)”. Further, to make sense of Sarah's and Jasmine's reasoning, one may infer that, what they present is not the actual solution but a reconstruction of a solution. In other words, Sarah and Jasmine knows that each person will receive half of a brownie out of the four brownies and, what they do is to show that four brownies contains, or can be split into, eight halves. An alternative interpretation could be that the students infer from the implicit equation $4 \cdot x = 8$ that each person will have half of a brownie. So, to understand more about the students' reasoning to make the content more explicit, and so, accessible to the rest of the class, there were reasons to press Sarah and Jasmine further on the four-brownie task.

The solution of the five-brownie task follows a two-step structure reflecting partitive division. Sarah claims, ”We put them in eighths”. Ms Carter then adds to the GoGAR, ”Okay, so they divided them into eighths”. In partitive division you want to find out the size of each part if you have a whole that is to be divided into a given number of parts. The brownies are to be distributed over eight persons so, the brownies should be divided into eight parts of equal size. However, now the total amount of brownies was not cut directedly in 5/8. In a two step-procedure, each brownie was first cut in eights. Each of the five brownies then contributed with 1/8 of a brownie so every person received 5/8 of a brownie.

I agree with Kazemi and Stipek that Ms. Carter presses the talk beyond descriptions or summaries of steps to solve a problem. However, from the inferentialist analysis, we can see several situations where the teacher is not taking advantage of opportunities to develop the GoGAR on fractions further. For instance, Ms. Carter leaves the solution on the form of two separate fractions 1/2 and 5/8. The GoGAR could have been extended by pressing the students to make explicit the *identity* inferences from ”1/2” to ”4/8” and then, consequently, from ”1/2 + 5/8” to ”4/8 + 5/8= 9/8”. The GoGAR could then have been further extended by making explicit the identity inference from 9/8 to 1 1⁄8.

Ms. Carter could also have developed the GoGAR by means of negotiation inferences. In other words, she could have pressed the students to elaborate on differences between the conditional patterns in the solution of the four-brownie task (quotitive division) and the conditional patterns in the solution of the five-brownie task (partitive division). The fundamental negation is that, the principle by which Sarah and Jasmine distribute the first four brownies is not the same as the principle they use to distribute the last five brownies.

Concluding discussion

In this study I have re-analysed a transcript from Kazemi and Stipek (2001) in order to show how constructs of the semantic theory of inferentialism (Brandom, 1994, 2000) can be used to give account of conceptual knowledge in mathematics. Research emphasizes connections as crucial to conceptual knowledge (Hiebert & Lefevre, 1986). However, Crooks and Alibali (2014) observe that the meaning and structure of connections are often described in vague or general terms, with limited theoretical grounding. On a theoretical level then, the significance of the present study should be seen according to how it provides a theoretical conceptualization of connections, in terms of inferential relationships and moves in the language practice of the game of giving and asking for reasons. On an empirical level, the significance of the study should be seen according to how this inferential conceptualization of connections facilitates a fine-grained analysis of conceptual knowledge in mathematics and provide insight on teachers' actions in pressing for conceptual understanding in mathematics in teacher-student interaction.

Kazemi and Stipek (2001) used the episode presented above as an example of a teacher creating high-press interaction. However, looking at the episode by inferentialist means, we came to see that many inferential relationships were left implicit in the interaction. There were several opportunities to push the students further, making the content even more explicit and accessible to the class. But, of course, inferentialism does not do all of the job itself. Being able to perform the above inferentialist analysis requires that the analyst is knowledgeable in mathematics. So, on a theoretical level, the inferentialist perspective suggests the need for further research on investigating and/or trying to developing teachers' inferentially structured knowledge in mathematics.

There was no counterfactual in play in the episode above. Nor, was it, in the same way as was the case with a negation inference, easy to see how a counterfactual could be brought to the fore as an explicit topic of a GoGAR. It was actually difficult to give account of any counterfactual at all, when looking at the entire empirical material presented in Kazemi and Stipek (2001). This is probably representative for many classrooms. However, if conceptual knowledge is about making semantic connections, there is reason to engage students in a variety of contexts in which such connections can be exercised. On this account, the present study suggests research to explore the design of tasks and activities that challenge students and teachers to engage in GoGARs that allows for counterfactuals.

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Note

1 Brandom makes no significant distinction between terms, words and concepts so in the rest of the article I will use them interchangeable.

Primary students' expressed emotions towards mathematics education

Martin Nyman

A body of research highlights factors relating to students' emotions – towards themselves, the social environment, learning, and the subject itself – as being of pivotal importance for learning. This paper reports on a study where students in grades two and five were interviewed about their experiences with mathematics, especially focusing on expressed emotions. Using a combination of deductive frameworks and an inductive search, nuances in students' expressed emotions were revealed, with tentative results indicating that issues of control are significantly important and that boredom conceals emotionally important complexities.

Previous research has indicated that emotion is an intrinsic part of every learning situation, including mathematics learning (e.g. Hannula, 2006; Radford, 2018; Ryan & Deci, 2000; Schukajlow et al., 2017). Emotions ”simultaneously emerge from, and shape experience" (Liljedahl, 2014, p. 27) and thus, play a part in the individual's structuring of future action through the interrelationship between emotion, motive and action (Leont'ev, 2009).

It appears that student interest in, motivation for and engagement with mathematics is inversely proportional to years of schooling (e.g. Blomqvist et al., 2012; Hannula, 2006). However, available studies on affective factors like emotion primarily cover teenagers or adults (Dowker et al., 2019) and focus on mathematics anxiety in relation to solving tasks without explaining how these phenomena develop (Batchelor et al., 2019). Thus, there is reason to focus more on capturing the nuances within emotions, as well as the supposed link and interplay between different affective factors, to better understand the mechanisms behind student action. Therefore, the aim of this study is to explore student emotions in relation to mathematics and in particular nuances in expressed emotions by addressing the following research questions:

1 How do students express emotions in relation to mathematics and what are the characteristics within these expressed emotions?

2 How and by what mechanisms are students' emotions linked to expressed motives?

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Background

The neurological way of describing emotions is to view them as bioregulatory reactions made up of chemical and neural responses that the brain produces, and that this production is performed automatically and in steps (Damasio, 2004). The initiation of the emotion is followed by biochemical changes in the body and brain and finally the emotion is made conscious, resulting in the person ”feeling the emotion”. This mechanism places emotion in the middle of the physiology-psychology divide as the mechanism that connects the two (Damasio, 2004). Understanding this connection is important for education. Since the aim of the present study is to explore nuances in students’ expressed emotions, Schirmer’s (2015) operationalisation of the concept is used. She defines emotion as ”conscious or unconscious mental states elicited by events that we appraise as relevant for our needs and that motivate behaviours to fulfil these needs” (Schirmer, 2015, p. 26).

The division between emotion and feeling has been explored and discussed for mathematics education purposes by other researchers (e.g. Sumpter, 2020). One conclusion being that emotions include both bodily experiences and, sometimes but not always, a cognitive interpretation and/or expression of these experiences. Another conclusion is that the analysis of emotional nuance must expand beyond a positive-negative dichotomy.

Further, a body of research in the field of psychology has tried to establish whether the number of different emotional sensations are infinite or limited to a finite set of basic emotions (Shirmer, 2015). The theoretical starting point for this study comes from Löwheim (2011). He advocates a finite set of eight emotions, making the neurologically grounded argument that this figure represents the maximum number of configurations one can derive from the three synaptic amino acids involved in the process of producing sensations in the brain. These eight emotions are: excitement, joy, surprise, distress, fear, shame, disgust and anger. Though the number is fixed, the individual experience of emotional sensations may be more varied, since sensations occur to various extents and can be more or less intense. Contrasting this fixed number, Lewis (2013) presents a different – and larger – set of emotions at play in educational situations. He also discusses an interrelation between the concepts emotion and motivation. Theories for describing this interrelation are important for understanding the mechanisms for student action.

To further understand the role of emotion in student action in relation to teaching, it is important to acknowledge the different positions of the aquisitionist and participationist paradigms respectively (Liljedahl, 2014). The acquisitionist paradigm implicitly treats emotion as a psychological phenomenon that is a reaction to (interpretation of) the individual’s experience and which regulates their future actions. In the participationist paradigm, the role of emotion is intertwined with the individual’s actions through their motives which in turn are created by their needs. Since motives, hence also needs, can be unknown to the individual, the emotions work as regulators of actions by feeding back the fulfilment of the individual's needs. As a consequence of the causal chain – needs-motives-action – the importance of motivation is significantly reduced in the participationist paradigm (e.g. Arievitch, 2017). It is understood that the individual is not required to have motivation (i.e., pleasure, salary or grades) in order to do something, but is required to have a need (personal, social or other), which creates a motive for action The individual is understood not required to have motivation (i.e., pleasure, salary or grades) in order to do something, instead they are considered to have a need (personal, social or other), which creates a motive for action (Leont'ev, 2009). Independent of the paradigm there is still a need for expanding the multitude of affectively relevant dimensions even further. One attempt is made by Hannula (2006, 2012), who combines eight dimensions, grouped together in three dimensions on the sides of a cube – emotion, cognition and motivation make up the first side of the cube; psychological, physiological and social the second; and the two temporal dimensions state and trait the third side.

Looking at empirical studies conducted within the Swedish context, one indication reported by Blomqvist et al. (2012) is that students' emotional dispositions turn from positive to negative around the age of nine; but, due to methodological constraints, the study neither discusses nuances within the group positive versus negative, nor potential causes for these emotions. Another indication is the connection between emotion and motivation (Nyman & Sumpter, 2019) and between emotion and achievement (Palmér & van Bommel, 2018). These studies, conducted with youngsters ranging from six to 18, imply an interconnection between emotion and other affective and cognitive factors that is stable and established early. Karlsson (2019) discusses the issue of anxiety towards mathematics expressed by poorly achieving students and points to a strong social link to negative emotions, but also suggests that emotional sensations can be either the cause or the effect of a situation. These examples depict emotion as a concept both interconnected with other psychological and physiological constructs as well having both an inhibitory and a promotive function (see also, e.g. Hannula, 2015; Dowker et al., 2019).

Methods

Since the aim of this study involves capturing nuances in student expressions, semi-structured interviews were chosen as the method for data collection. This method combines providing the respondents freedom to elaborate on their thoughts with the structured format of a questionnaire. It also allows the interviewer to pose follow-up and clarifying questions and for the respondent to do the same. The interview guide was based on a seven-item questionnaire instrument developed by Dahlgren Johansson et al. (2010) and later also used in a study by Blomqvist et al. (2012). The questionnaire combined closed items using a four-step-based likert scale in the form of happy and sad faces, with free-text items and a drawing task at the end. In the present paper, focusing on emotion, questions 1, 2 and 5: ”How do you like math?”, ”How do you feel before a math lesson?” and ”How do you feel when you do math?”, respectively, will be discussed. All 19 interviews were conducted by the author with the interviewees face-to-face, one-on-one, in a room near the classroom during lesson time. Each interview was audio-recorded and lasted between 20 and 30 minutes.

Participants

Data were collected at three schools in an urban area, for convenience and to remain within the same municipality. One inner city and two suburban schools, located in opposite areas in the the municipality were chosen. Students between the ages of eight and 11 in grades two and five, respectively, were chosen to enable comparisons with the previous studies (Dahlgren Johansson et al. (2010) and Blomqvist et al. (2012)). Due to the respondents relative youth, it was necessary to spend some time with each class prior to data collection, and thus, for practical reasons, the number of participating schools had to be limited to three. Ethics considerations stipulated by the Swedish Research Council through Codex (Vetenskapsrådet, 2017) were followed, so every participant had written parental consent and were informed that participation was voluntary and that they could stop the interview without reason at any time. The aim of this study was to capture general nuances in students' emotions towards mathematics rather than emotions related to extremely high or low levels of achievement, while the number of interviews had to be limited. Therefore, on the day of the interviews, the teachers were asked to pick out, from among the volunteering pupils, individuals that they considered to be neither extremely proficient in nor having serious difficulties with mathematics. This was done hoping to reach representatives from the presumably large and often self-sufficient group of students that ” just go about their business” during mathematics lessons. The teachers were also asked to consider pupils who would manage the interview situation comfortably without feeling stressed or uneasy. In total, 19 pupils were interviewed – 10 in grade two and nine in grade five.

Analysis

Prior to analysis, the interviews were transcribed by the author. The data were transcribed verbatim, including non-verbal communication like exclamations and extended pauses. Questions number 1 (”How do you like maths?”), 2 (”How do you feel before a maths lesson?”) and 5 (”How do you feel when you do maths?”) were those that explicitly framed emotions or feelings. Therefore, as a first analytical step, the transcripts were marked where the responses to the selected questions appeared. In addition, a second reading of the transcripts was made, looking for instances where respondents had made additional references to emotions or feelings in the exchanges that were the result of follow-up on questions other than the selected three. The first step in the subsequent analysis was then carried out and the instances were coded Positive or Negative with utterances like ”I like math when …” and ”It makes me stressed when …”, respectively. An instance was coded Neutral when the respondent referred to factors that were emotionally neutral, such as hunger. The second step used Löwheim’s (2011) theory of basic emotions to divide the three initial categories into sub-categories. Finding these categories involved paying close attention to the words or expressions used by the respondents. For example, ”I feel relaxed” instances were coded under the subtheme Relief, while ”fun”, ”like”, ”happy” were coded under the subtheme Joy. This example also illustrates a decisive difference in emotional strength or intensity where ”happy” is considered to be stronger than ”like”.

In order to expand the analysis beyond the descriptive, a framework for analysing data by the types of motives or justifications in which respondents framed their responses was adopted as a third step. This framework was developed by Hannula (2012) and describes eight themes, four of which (cognitive, motivation, social and physiological) could be found in the data, while one is the construct which is the focus of this article – emotion. Here the coding focused on the specific ways the respondents expressed their experiences. For example, ”I’m challenged by it” connects to the theme Motivation, whereas ”I don’t want my friends to laugh” connects to the theme Social. This example also highlights that each theme ranges over positive as well as negative emotions. The mapping of categories over Hannula’s framework resulted in a number of responses, for example, the category Content carried properties that were inconsistent with any of Hannula’s themes. This called for a fourth analytic step where the remaining responses were weighed against each other and the rest of the responses. This aimed to discern additional themes that captured the same level of explanatory dimension that the rest of the themes did. This step was an inductive search for similarities, inspired by the approach of thematic analysis (Braun and Clarke, 2006).

Results

The results are summarised in two tables – table 1, followed by the analysis focusing on the first research question, and table 2, building on this and focusing on dimensions of the second research question.

Table 1 shows the different emotional themes developed from data and an example for each theme. The right hand column shows the eight basic emotions listed in the theory section.

As table 1 shows, there are discrepancies between the themes derived from the expressions in the data and the theoretical constructs. The most salient of these discrepancies is the theme Relief. Students clearly expressed having feelings of relief in relation to managing an activity or coping with their tasks

Table 1. *Excerpts and themes mapped against constructs from theory of Basic emotions*

Excerpt	Expressed emotions	Basic emotions
It's fun and exciting	Joy	Excitement
It feels like you're on top of things	Content	Joy
If feels safe to have a kind teacher that helps you, if you need	Relief	–
–	–	Surprise
Sometimes it's a bit tiresome	Discontent	Distress
You get stressed because if it's correct or not	Stress	Fear
I'm ashamed to ask	Shame	Shame
–	–	Disgust
	–	Anger

Table 2. *Student's expressed justifications for emotions experienced*

	Cognitive	Motivational	Technical	Personal (relates to self)	Social
Positive (91)					
Joy (34)	Process (7)	Challenge (19)			
	Position (8)				
Content (15)			Situation (14)	Autonomy (1)	
Relief (20)			Control (20)		
Negative (64)					
Discontent (1)		Boredom (1)			
Stress (52)			LC Temporal (7)	LC Personal (45)	
Shame (11)				Personality (8)	Social (3)

Notes. Number of instances within brackets. LC = Lack of Control.

on time. This approaches being something of a double negative – the responses are positively loaded but the argument is based on the absence of something negative. The emotion Distress manifests primarily as discontent; however, the boundary with stress is not crystal clear; in fact, the emotion Fear manifests as something closer to stress. It seems that fear of not being able to solve problems, answering questions from the teacher, etc, causes stress. Looking at the rest of the themes, we see that full strength of the emotions are generally not expressed both, either on the positive or the negative side. And, the strongest emotions, especially the negative, are not present in the data.

Further, table 2 summarises how expressed motives are distributed over the expressed emotions listed in table 1. In the left column the expressed emotions are listed and the table shows the different ways these emotions are justified by the respondents, structured under five themes. The results are discussed with a focus on qualitative differences between categories and themes even though number of instances is presented (in brackets) to provide an overview of the relative frequencies between different themes.

Among the Positive emotions, the first subtheme is Joy where Process and Position are closely related and both link to Cognitive dimensions but differ in relation to outcome, whereas Challenge links to Motivational dimensions. In all three categories, respondents express enjoyment or happiness originating in the cognitive development they feel in relation to mathematics. When an utterance expresses this sense of development and cognitive expansion as being sufficient in themselves it is coded Process. When this shifts more towards the result of the work, which shows a linkage to the person appreciating their moving position on a ”mathematical ability scale” this is coded Position.

Content is emotionally weaker than joy – positive, but with less intense sensations. The two concepts in this category both frame the working situation but for different reasons.

Tania: Solving the maths tasks, and talking to your friend, sometimes we can […] talk a bit about maths and I think that's fun.

Matteus: You can count in your own way. No one says you have to write this way or that.

The Situation theme, illustrated by Tania, is linked to technical issues around the working situation such as where to sit, being allowed to listen to music, and similar expressions of mathematics being a safe and comfortable activity. The Autonomy theme is also linked to the working situation, however, here the positive feelings are associated with the possibility of doing it ”in your own way” as Matteus puts it. The Control theme also links to the working situation, but the feeling is expressed as the relief of being able, knowing how to do the work. Matteus again.

Matteus: I feel calm, I can work at my own pace, not anybody else's pace. And it's not a contest about being first and so on.

Comparing the positive subthemes with each other indicates two main sources of emotion – one related to inherent mathematical properties and another to factors outside the mathematical content. And even though the emotional subcategories can be placed along an intensity continuum the different motive themes do not appear to form any similar simple pattern.

Before turning to the negative emotions, we see that the total number of positive responses is considerably larger than the sum of the different categorised positive responses (91 versus 69), and that negative responses have no such ”overflow”. Thus, there is a difference between how positive and negative emotions are expressed. The 22 positive, non-categorised responses contain those saying ”Good”, without any further comments. This is not the case among the negative responses, which are always motivated. Among the negative responses, we see that the Boredom theme only has one instance, which calls for caution in drawing conclusions – Chris (Y2) takes an unusually long time to answer the question, ”How do you feel before a maths lesson?”, but eventually chooses the happy-face card, however, with the following remark.

Chris: Because it's a bit, like, I get tired from math but when I know we are going to do fun pages or something that perhaps is fun then it's number one [the happiest-face card].

Chris goes on to explain that he likes ”when it gets difficult also” because ”it ain't fun when you have really easy stuff, like two plus two and stuff”. After this, the interviewer asks Chris what it is that makes him tired:

Chris: It's when you must do super many pages, […] all the way to, like, page 80 when you're on page 65, then my arm gets really tired.

Chris describes mathematics as being fun; however, for it to be fun, some level of challenge (i.e. working on non-routine tasks) is necessary. In the light of this, Chris' tiredness is interpreted as negative and categorised as Discontent. Chris never explicitly says maths is boring or uses explicitly negative words to describe it. However, adding other instances where students generally describe ”zoning out”, the negative valence as well as its strength is clear, for example.

Marcus: Because when I start working in the maths book it gets a little messy […] so I can't concentrate because I just look at other stuff”.

Among the other negative subthemes, Stress is the most commonly expressed emotion and its motive is related to lack of control, divided between a Temporal and a Personal subcategory (LC_T and LC_P, respectively). Not having enough time to finish tasks during an exam or keeping up with the pace of the class connects the emotion to temporal factors. The superficially similar category of personal lack of control (LC_P), is probably a more disadvantageous emotion because it links to issues of self-efficacy, self-control and potentially also self-worth. Chris' introspection is a typical LC_P-coded response.

Chris: Like, ”Oh no, I don't know how to do it!” and I bet it'll take a long time to learn.

Even though this kind of data are too small to draw quantitative conclusions from it, it is worth noting that this group is the largest among the negative responses, in fact the largest altogether.

Looking further at Shame, the results signal that the social dimension can come from feeling ashamed in relation to others, or from being outside a group. The category splits into two.

Vera: It feels really difficult because my friend sitting next to me is quite fast and so then I feel a bit stressed – that I'm quite far behind then.

Vera describes a shameful feeling, however, not connected to herself or her abilities, but her concern is that she will be left behind, that she will literally be on her own. The decision is therefore to code this type of instance as Sociality. Now compare this to what Jane (Y5) is saying:

Jane: Sometimes, some say, like, ”that one is really easy” – just blurts it out. Then I feel ashamed for asking. Because it feels like they might say that I'm bad at maths.

Jane's comment has the same social dimension, but this feeling originates from beliefs of self-worth. Therefore, this different type of shame-connected motive is coded Personality. Thus, a hierarchical structure seems to exist between the concepts: Personality – Sociality – LC_P – LC_T, in descending order of emotional strength and separated in levels of interaction.

Discussion

Given the important role emotion plays in learning situations (e.g. Hannula, 2012; Radford, 2018) and that emotion reciprocally both shapes and is shaped by a person's experience of education (Liljedahl, 2014), this paper contributes to describing and analysing these connections and to revealing some of the nuances within emotion and other affective constructs. This is relevant since emotion is a major ingredient in the individual's motives for engaging in education, and is therefore linked to achievement (Leont'ev, 2009; Schukajlow et al., 2017). When discussing the results, it is important to bear in mind that since this study is a small-scale interview study its primary contribution lies in offering avenues for continued research . However, from a methodological perspective, it is interesting to note that eliciting responses using likert-scale ”faces” turned out to be more fruitful than anticipated when it prompted respondents to reflect on situations they associated with happy and sad faces, respectively. This resulted in every respondent contributing very rich data. Thus, the findings of this study can be sen as a qualitative continuation of the more quantitative studies by Blomqvist et al. (2012) and Dahlgren Johansson et al. (2010).

In relation to RQ 1 the results indicate that students are indeed emotional about mathematics, and table 1 contains a summary of the ways these emotions are expressed. Looking at the characteristics and comparing the data with Löwheim's (2011) basic emotions we see that both positively and negatively loaded emotions are represented: joy, excitement, shame and distress. Factors of fear and distress manifest themselves more like stress and discontent, respectively, and occur both in test/exam situations and during ordinary lessons. These factors are linked to personal or social dimensions in line with what others have reported (e.g. Karlsson, 2019; Samuelsson, 2011). Even though the strongest negative emotions of anger and disgust are not explicitly present in the data, it is a strong possibility that these emotions, together with fear, are all expressed as stress. This suspicion implies the need for further research. However, instances of ”weaker” responses that are clearly emotional in nature are also present, like this feeling of relief expressed by Matteus: ”I feel calm, can work at my own pace”. In conclusion, even though considering feelings as conscious and emotion as unconscious (cf. Damasio, 2004; Schirmer, 2015) implies that respondents express thoughts about feelings rather than emotions, as educators we need to take these emotion-related experiences (both strong and weak) into account.

In relation to RQ 2, the ways students justify emotions, shown in table 2, can be understood as a starting point for understanding the mechanisms connecting emotion to other affective concepts. For example, it seems that issues of motivation emerge when students are asked to describe emotion, and vice versa. This is parallel to the interplay between motivation and emotion reported by Nyman and Sumpter (2020), and Hannula (2012), where Motivation and Cognition, as well as social dimensions, are part of his model. Further, the Personal dimension appears to be very similar to Samuelsson's ”self-concept” (2011). However, the Technical factors, contributing to negative as well as positive emotions, do not seem to be part of any previous theory. Considering these points and the limited size of this study, further research in this field is required – preferably using a participatory methodology – to capture the fleeting emotion-connected dimensions occurring momentarily during lessons. When looking at implications for teaching, the participationist perspective's connection (e.g. Arievitch, 2017) to action, through student emotions and motives, places the power to influence students more in the hands of the teacher – through well planned teaching – than a more psychological and individualistic view on emotion and affect does. The design of this type of teaching should also be the subject of further research.

One additional, and peculiar detail from this study is the absence of instances of the emotion Surprise. Shirmer's (2015) definition of emotion rests on the concepts of appraisal and need implicitly connects emotion to motive (e.g. Leont'ev, 2009). Inspired by Liljedahl's research on the ”AHA!-experience” (2005), and since surprise can be connected to a motive for understanding (the reason for one's surprise), implies that surprise is a dimension of learning situations that could be made fruitful in order to enhance student learning. Thus, potentially a starting point for further research.

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On the notion of ”background and foreground” in networking of theories

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In this paper, we report on a finding in an ongoing literature review on *Networking of theories*. As theories are the focus of networking practices, discussion of what is meant by theory is an ongoing debate. In our reading of these discussions, we experience a discrepancy in the use of the notion of background theories and foreground theories, which can be related to an absolute or a relative understanding of these notions. We account for this discrepancy and discuss potential consequences of each perspective to argue that a new notion ”framing theories” or a distinction between ”background theory inside mathematics education” and ”background theory outside mathematics education” may accommodate these consequences.

The term ”networking of theories” stems from the thematic working group (TWG) on theoretical perspectives and approaches in mathematics education research (MER) at the *Congress of European Research in Mathematics Education* (Kidron et al., 2018). The group confronts the issue of the diversity of theories in mathematics education, and claims that ”theoretical approaches can *only* become fruitful *if* connections between them are actively established” (Bikner-Ahsbahs et al., 2014, p. 8). Taking this stance, the group has embarked on the challenge of how to establish connections between theories by developing ”networking of theories” as a research practice. Several important questions and issues have been discussed over the years. Kidron and colleagues (2018) state the following examples: ”What are the aims of connecting theories? […] To what extent does the networking depend on the theories that are considered?” (p. 257); ”To what extent do we share the same notion of theory? (p. 257); ”What are the different aims of networking?” (p. 258); ”What do researchers do when they use more than one theory? Do the different approaches use the same words with different meanings?” (p. 258). Such questions have been addressed in the

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literature on networking of theories, e.g. Bikner-Ahsbahs and Prediger (2006) the ZDM article ”Diversity of theories in mathematics education – How can we deal with it?”, the ZDM issue ”Comparing, combining, coordinating – networking strategies for connecting theoretical approaches” edited by Prediger et al. (2008), and not least in the recent book ”Networking of theories as a research practice in mathematics education” edited by Bikner-Ahsbahs and Prediger (2014). Surely, the potential answers must to some extent draw on a common notion of ”what theory is” – we return to this below. For now, we draw the attention to the observation that in the available literature on networking of theories, there are often references to the notion of background theories and foreground theories (to be explained in more depth below) – this often occurs with specific reference to Mason and Waywood (1996), who initially introduced the terms into MER. Our ongoing review, which so far encompasses 96 publications on networking of theories, reveals the observation that the use of these two terms in more recent literature do not necessarily align with the original description by Mason and Waywood. More precisely, although some theoretical perspectives are attributed the role of background theories; these are not necessarily used in the sense of Mason and Waywood. Hence, there is a discrepancy between the descriptions and the actual use. In this paper, we ask the question: *How are the notions background theories and foreground theories used in the literature on networking of theories?*

We do not provide a full account of the 96 publications due to the space limitations of this paper. Instead, we present and discuss our finding through two carefully selected illustrative cases, showing the discrepancy in the use of background theory. Before we get to these cases, we briefly discuss the notion of theory itself and explicate the original notion of background and foreground theories as defined by Mason and Waywood (1996).

What is ”theory” in mathematics education research?

In networking of theories, a minimum requirement must be that we can agree on what is and what is not *a theory*. The literature – not only in mathematics education – is rich on various attempts of coining what theory is. For the reader who is unfamiliar with this discussion, we provide a brief account in this section. The reason we do this is not to apply this in our further analyses, but rather as a general comment to the ongoing discussion on what a theory actually is, and not least what a theory must be described by in order to be networked with other theories. We shall consider *a theory* from the perspective of networking theories, not least, with reference to what has taken place in this literature.

Kidron et al. (2018) state that the questions of what a theory is and how theoretical frameworks shape MER ”came into play when comparing or just talking about theories is the heterogeneity of what is considered as a theoretical framework in MER and the consequent possible incommensurability of the investigations that are carried out in different theories" (p. 261). Radford (2008, p. 320) suggested that a theory is a way of producing understanding and ways of action based on a triplet PMQ:

> A system, P, of basic principles, which includes implicit views and explicit statements that delineate the frontier of what will be the universe of discourse and the adopted research perspective.
>
> A methodology, M, which includes techniques of data collection and data-interpretation as supported by P.
>
> A set, Q, of paradigmatic research questions (templates or schemas that generate specific questions as new interpretations arise or as the principles are deepened, expanded or modified).

Around the same time, Prediger et al. (2008) surveyed different notions of theory found in the literature. This led them to distinguish between static and dynamic notions of theory, eventually pleading for a dynamic understanding: "theories or theoretical approaches are constructions in the state of flux" and they "consist of a core, of empirical components, and its application area. The core includes basic foundations, assumptions and norms, which are taken for granted" (p. 169). Niss (2019), however, notes: "The fact that theories or theoretical approaches are in a state flux doesn't mean that the definitions of the concepts are as well". We agree with Niss (2019) that: "Anything called a theory (or theoretical framework, construct etc.) is a *theory of something*! I.e. it deals with certain sorts of *objects* and *phenomena*, as well as *terms* for these". Mason and Waywood (1996) define such *objects* as the "sorts of things that are studied, even if they are not perceived as 'things' in any material way" (p. 1058). From Radford's (2008) account, it is unclear where these objects reside, although several researchers in networking of theories seem to consider them as part of the principles (P).

Foreground and background theories

As mentioned in the introduction, Mason and Waywood's (1996) distinction between foreground and background theories is often referred to in the discussion of the concept of theory. In this section, we outline our interpretation of the distinction as a basis for further discussion. Mason and Waywood (1996) present theory as a "hypothesis, or possibility such as a concept that is not yet verified but that if true would explain certain facts or phenomena" (p. 1055). They define foreground theory as *explicit* hypothesising based on the process of asking and answering questions within mathematics education, because ”[…] the foreground aim of most mathematics education research is to locate, precise and refine theories *in* mathematics education about what does and can happen within and without educational institutions” (Mason & Waywood, 1996, p. 1056).

Thus, from the process of questioning ”things” within a local or specific area of mathematics education research gives rise to new theories in forms of explicit hypotheses about what is happening, or what can happen under certain circumstances. The foreground theories are generated *within* mathematics education and can have one or more of four different functions: descriptive; explanatory; predictive and informing practice. Conversely to foreground theory, Mason and Waywood define background theory as *implicit* hypothesising or as a belief that guides behaviour. They consider that ”every act of teaching and of research can be seen as based on a theory *of* or *about* mathematics education” with reference to Thom (1976), who puts it as ”all mathematical pedagogy, even if scarcely coherent, rest on a philosophy of mathematics” (quoted in Mason & Waywood, 1996, p. 1056). In this sense, the theory remains in the background and implies an implicit way of action or behaviour of the teacher or researcher, but is not used with an explicit aim. It is important to notice that a background theory does not become a foreground theory, just because the hypothesis becomes explicit. Mason and Waywood (1996) emphasise that as a researcher, it is important to be aware and explicit about one’s own background theories and their implicit assumptions and hypotheses. They explain:

> Background theories encompass an object (aims and goals of the research, including what constitutes a researchable question [...]), objects (what sorts of things are studied, [...]), methods (how research is carried out, validated and applied), and situation (as perceived by the researcher), and provide a language for discussing these. The situation necessarily assumes, manifests, encompasses, and is constituted through a philosophic stance manifested in the discourse and in other practices. (p. 1058)

This implies that the activities of research, such as framing researchable questions, using an appropriate method, collecting data, using analytical tools and looking at results as well as the validation hereof, are all determined and constructed by the background theory. This is elaborated with examples of how theoretical positions such as post-modernism, phenomenology and different directions within constructivism stress different ways and methods to investigate sociological and psychological dimensions and phenomena in educational research. Hence, we understand background theory as the theory that affords the conditions for the structure of the research, but it is *not* a theory generated within mathematics education research (MER). In addition, MER draws on theories from domains such as psychology and sociology, and their philosophical positions as well as their methods (Mason & Waywood, 1996). Accordingly, we understand Mason and Waywood's (1996) explanation of background theories as theories establishing the view by which we look at mathematics education, for example critical theory, constructivism, social-constructivism, phenomenology or ethnology. It also follows that we understand their term of foreground theory as the theoretical constructs generated and developed by research in mathematics education that have explicit aims in forms of describing, explaining, predicting and/or informing specific situations, concepts and practices happening or possible to happen in the teaching and learning of mathematics.

As an example of the differences between foreground and background theories, we use Vergnaud's (2009) *Theory of conceptual fields* (TCF). As TCF is a theory developed in MER, specifically concerned with mathematical learning, it is a foreground theory. To consider the background theories of TCF, we must understand what theories precede TCF. As Vergnaud (2009) argues for his perception of schemes, he draws on Vygotsky's (1962) as well as Piaget's (1977) understanding of schemes. These two constructivist perspectives both have a broader scope on learning as they are developed outside of MER. Hence, we position them as the background theories of TCF.

A hermeneutic literature review

The following is a brief overview of our initial literature review on networking of theories. This review was conducted as a hermeneutic literature review. Due to very limited results in databases, a systematic literature review was not possible to conduct (Boell & Cecez-Kecmanovic, 2014). As a part of a hermeneutic process, the understanding of the literature is never final; a constant re-interpretation is taking place. We began by scanning CERME proceedings, relevant ZDM issues and books and reference lists for the relevant literature to expand our literature base. Furthermore, we did literature searches in MathEduc and ERIC, although this did not reveal many relevant sources. Only literature describing the practice of networking of theories in mathematics education were included in the final cohort. We described each relevant piece of literature in the following categories made our findings about background theories more explicit: 1) actual results; 2) how is networking of theories used and discussed; 3) what theories are being networked; 4) what strategies and methods are applied; and 5) perspectives with particular relevance to our overall project.

In our efforts to grasp the discussions of category 2, we compared the use of the notion of foreground and background theories in the literature on networking of theories to the original reference by Mason and Waywood (1996). Our two cases are carefully chosen to illustrate the result of this comparison: Each case utilises background theory explicitly, yet differently. But first, a further elaboration on the different uses of background theory in networking of theories.

Foreground and background theories in networking of theories

In relevant literature, the use of Mason and Waywood (1996) is widespread, both in paragraphs concerning theory and in discussions thereof. At CERME5, a communication problem within the field of MER was noticed: "Researchers from different theoretical frameworks sometimes have difficulties to understand each other in depth because of their different backgrounds, languages and implicit assumptions" (Arzarello et al., 2007, p. 1618).

This quotation emphasises the need to understand the origin and background of theories as well as their implicit assumptions and hypotheses. According to Bikner-Ahsbahs and Prediger (2006), the distinction between background and foreground theories seems applicable when analysing theories and their functions in different phases of research. This could be the characterisation of foreground theories and their respective background theories. An example is: "The theory of interest-dense situations is a foreground theory with a middle range scope (Mason and Waywood, 1996), situated in the background theoretical framework of interpretative research on teaching and learning" (Bikner-Ahsbahs & Halverscheid, 2014, p. 99).

According to Bikner-Ahsbahs and colleges (2014), the underlying theoretical assumptions must be explicit when networking theories. Bikner-Ahsbahs and Prediger (2006) point out that "the background theory and its philosophical base are deeply interwoven" (p. 53). For instance, when taking a constructivist perspective, mathematics has a philosophical view on the construction of knowledge. Nevertheless, the use of foreground and background theories is regarded neither as a definite definition of theories, nor as an absolute categorisation of theories. This leads to a more relative use of background and foreground theories, than originally intended by Mason and Waywood (1996), e.g.: "In contrast [to the absolute definition], the status of some parts of the theory can change from foreground to background theory or vice versa within the research process" (Bikner-Ahsbahs & Prediger, 2006, p. 54). We interpret this statement to mean that a theory is not only *of/about* MER or only *in* MER, but that a theory can act as either, depending on the situation. Bikner-Ahsbahs and colleges (2014) contribute to this meaning by referring to foreground and background theories as *relative distinctions*. Still, and despite the discussions of making background theories explicit, authors reporting on networking processes and results seldom explicate the distinction. Hence, the way these terms are used within research practices are less apparent that one might initially anticipate.

Examples on the different use of background theory

Our first case is an example of the relativism of the notions as presented in Bikner-Ahsbahs and Prediger (2006). Koichu (2013) describes the work of a colleague in which a selected framework is contrasted with another. The insights obtained in the contrasting process are used in a following process of unpacking a selected construct in the selected framework:

> To this end and consistently with the Bikner-Ahsbahs and Prediger's (2006) terminology, the former theory can be seen as a foreground one, and the latter – as a background one. On the other hand, they use the Hershkowitz et al.'s (2001) work as a background theory or as an overarching framework, in which their own foreground theory is embedded.
>
> (Koichu, 2013, p. 2841)

The relativism of the status of a given framework thus becomes apparent as something that emerges in particular situations in research activities expressing the relation between frameworks in use.

Our second case is an example of another use of the notion of background theory. First, Fetzer (2013) addresses a specific perspective, namely Latour's *Actor network theory* (ANT) as a background theory to understand objects in mathematics education: "*Latour's approach is fascinating and irritating and provokes the research question, if and respectively how actor network theory can be a fruitful background theory to get a better understanding about the role objects play in mathematical learning processes*" (Fetzer, 2013, p. 2800, italics in original).

Using Latour's ANT, Fetzer (2013) presents an example in line with Mason and Waywood's (1996) distinction between foreground and background theories. Latour's ANT, as a theory outside of mathematics education research, is used as a background theory determining the researchers' definition of an object, the researchable objects, methods and situations. Similar utilisations are found in Bikner-Ahsbahs and Prediger (2014) and Bikner-Ahsbahs and Halverscheid (2014). This way of using the notion of background theory implies that it is a perspective outside of MER, which allows the researcher to understand mathematics education through a particular philosophical or epistemological stance.

To sum up, our literature review on networking of theories indicates that the original terms, as defined by Mason and Waywood (1996), have undergone further development. The use of the notion of background and foreground theories in the networking of theories literature now also encompasses a more relative definition of background theory, i.e. one focusing on the relations of theories within MER.

Coexistence of two notions of background theory

In the discussion of theories related to networking of theories, Bikner-Ahsbahs and Prediger (2014) suggest to take "the notions of foreground and background theory as offering relative distinctions rather than an absolute classification, they can help to distinguish different views on theories (p. 6). This quotation clearly describes the development of the definitions of foreground and background theories. Hence, in line with the findings of our literature review, and as showcased by the two illustrative cases above (Fetzer, 2013, and Koichu, 2013), the relative and the absolute distinction of the foreground and background theories coexist in literature on networking of theories. Schoenfeld (2007) emphasizes a need for specificity of concepts in research, as loosely defined terms can produce variation in results. Looking at the absolute distinction of background theory, this satisfies Schoenfeld's criteria for specificity. However, what are the potential consequences of an absolute distinction of background and foreground theory? One consequence is that it causes a large number of foreground theories, because all theoretical developments and contributions generated inside MER are considered as such. Another consequence of the absolute distinction is an untended need for a notion that denotes the experienced distinctions between theories *inside* MER. Using Koichu (2013) as an example of Bikner-Ahsbah and colleges' (2014) relative use of the notions, theory *in* mathematics education has a similar role as a background theory. Hence, the use of foreground theory as a background theory seems to confuse the use of background theory, since *background theories inside mathematics education* and *background theories outside mathematics education* then coexist.

Moving to the relative distinction of background and foreground theories, also this might not withstand Schoenfeld's (2007) criteria for specificity. A first consequence of a relative distinction is a less clear definition of foreground and background theories. A second consequence is the existence of different utilizations of the notion of background theory. When different utilizations of background theory exist, a third consequence occurs: The importance of the background theories *outside* MER and its philosophical base might be indistinct. If researchers do not take their background theories *outside* MER into account, the implicit assumptions and hypotheses continue to be tacit.

Conclusion

Our study shows that both a relative and an absolute distinct of foreground and background theories exist in the literature of networking of theories. Koichu's (2013) uses the relative distinction when denoting the relation between theories or frameworks in use. Fetzer (2013) uses the absolute distinction when she considers the underlying beliefs or epistemological position that determines the researches' goals, aim, questions and objects. Considering both the absolute and the relative distinctions, the following consequences appear:

- Adhere to the *absolute distinction*: a need for a new notion distinguishing background theories emerges when networking.
- Adhere to the *relative distinction*: different utilizations of background theories appear.

The consequences of both reveal the need for distinguishing between foreground theories, background theories *inside* MER and background theories *outside* MER. In networking of theories, the relative distinction also builds on the changing relationship between the theories used in a research practice (Bikner-Ahsbahs & Prediger, 2006). This means that one theory may act as both foreground and background *inside* MER.

Looking at the consequences of an absolute and a relative distinction between foreground and background theories, these indicate the need for a new distinction/notion. We suggest that the *background theories inside mathematics education research* are referred to as *framing theories*. Looking at Koichu (2013), the new distinction informs and describes the different roles of foreground and background theories in networking. If the notion *framing theories* is applied, the importance of background theories *outside* MER arises and the implicit assumptions and hypotheses in background theories *outside* MER thus becomes clearer. The new notion is not needed to characterise Fetzer's (2013) networking practice and the distinction between foreground and background. However, given the use of background theory outside MER and foreground theory inside MER, the theories involved do not change between the two types in a networking practice. This means that the dynamic relationship between theories only exist between *framing theories* and foreground theories *inside* MER.

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Students' use of written and illustrative information in mathematical problem solving

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This study investigates how elementary students use written or illustrative information in their mathematical problem solving. A previous study indicated that students who focus on illustrative information in task solving are more successful than those who focus on the written. Our study expands this idea, suggesting that there are different ways of attending to illustrative and written data. Students can treat the two sources of information as isolated or trying to connect and combine them in order to verify or test solution ideas but also to generate new ideas. This may have implications for teachers seeking to support students in their problem solving. Encouraging students to make productive use of written and illustrative information may assist them in overcoming obstacles.

Solving mathematical problems has long been considered a productive way for students to learn mathematics. Teachers and researchers have tried out and investigated various approaches to instruction that promotes problem solving in mathematics (Brousseau, 1997; Cai, 2003; Hiebert, 2003). One of the dilemmas with problem solving to learn mathematics is that it is difficult for students to solve mathematical problems. This is somewhat of a paradox because tasks that fail to be challenging also lose some of their potential as tools for learning. Earlier research has shown that students who solve mathematical problems by creatively constructing new solutions are more likely to solve similar problems at a later stage than students who are given instructions on how to solve the problem (Jonsson et al., 2014; Lithner, 2008; Olsson & Granberg, 2019). This points to a crucial junction in mathematics teaching, students need to meet challenging problems, but it is to be expected that many of them will need help in getting past some of the challenges. This help however should not remove the challenges by introducing a method with which the problem can be solved but rather provide clues on how to overcome obstac-les without a complete description of a solution method. Providing feedback that helps the student to proceed with her problem solving without giving her too much information is a demanding task for teachers. It is unlikely that there will ever be a best practice

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in the form of a fixed set of strategies for teachers who engage in interactions with students. There are, however, several general ideas on what teachers should consider in their interactions with students who are stuck in their problem solving, examples include asking students to explain their reasoning, to encourage them to develop and justify their reasoning and to test their conclusions (Olsson & Teledahl, 2018, 2019).

Beyond such general approaches, research on the ways in which students approach certain problems may provide further clues on how teachers can assist students in overcoming stages that are problematic, in their problem solving. A recent study (Norqvist et al., 2019) investigated what items of information that students focused on, while solving a non-routine task. The study used eye-tracking techniques and found that students who focused their attention on pictures that illustrated the mathematical problem were more successful than their peers in a post test. The present study aims to investigate this idea by examining students' reasoning in problem-solving situations that contain both written and illustrative data. An investigation of the ways in which students consider different data in a mathematical task may provide valuable information on how teachers can assist students in proceeding with mathematical problem solving in situations where they are stuck. The research question is: In what ways are students using written and illustrative data in their problem solving?

Background

In school mathematics, teachers are often providing students with procedures, which, if performed correctly, will solve tasks. When solving non-routine tasks this may foster strategies of recalling memorized procedures possible to use when constructing the solution. Lithner (2008) defines this approach to reasoning as algorithmic reasoning (AR). Why teaching mathematics this way, by providing algorithms, is a prevalent practice may be explained by the fact that it is relatively easy for the teacher to prepare, and the students are often successful in solving tasks (Blomhøj, 2016). However, a wide range of research has stated that teaching in which the teacher provides instructions on how to solve tasks, is not an efficient way to teach mathematics (Hiebert, 2003). Students will engage in rote learning, which is focused on executing steps in a procedure, without understanding the intrinsic mathematics. This behaviour excludes students' engagement in constructing and justifying solutions, something that many studies suggest as important for learning (Brousseau, 1997; Lithner, 2008). Brousseau (1997) claims that to learn mathematics one needs to construct solutions using mathematics, something which Lithner concretizes further with the definition of creative mathematical reasoning (CMR). That is, when solving non-routine tasks, for which students do not know a solution method in advance, they engage in constructing solutions and formulating arguments (Lithner, 2008). While they construct the solution method themselves, they must assess whether the method will solve the task or not. In this process, the mathematics will gain meaning for the student and she will learn. Such an approach to mathematics teaching requires a different teacher role. Instead of explaining how to solve tasks the teacher should prepare suitable tasks, encourage the students to use their mathematics resources and ask them to justify their solutions (Brousseau, 1997).

Several quantitative studies have confirmed that students who practice on tasks demanding CMR score higher on post-tests compared to students practicing on tasks using AR (Jonsson et al., 2014; Norqvist, 2018; Olsson & Granberg, 2019). These studies indicate however that many students also fail to solve CMR-tasks in practice, but the studies do not explain the mechanisms behind these failures. Norquist et al. (2019) take a step towards explaining some of the differences between successful and non-successful CMR-students. The study argues that students, when solving the tasks, extract different types of data (illustration, description, formula, example and question) necessary to solve the problem. The authors suggest that some students base their solutions on isolated examples of data from either text or illustrations, not using opportunities to combine text and illustration to verify their answers.

Visualisation in mathematics has long been acknowledged as important for students learning (Arcavi, 2003) but studies point in different directions. Some studies suggest that the combination of written and illustrative information in mathematical tasks can increase students' cognitive load, thus making it more difficult for them to solve problems (Berends & van Lieshout, 2009; van Lieshout & Xenidou-Dervou, 2018). Other studies, that have investigated students' use of carefully prepared illustrative information, have showed that this can be beneficial to students' problem solving and that productive use of visual imagery is common among expert mathematicians (Scheiter et al., 2010; Stylianou & Silver, 2004; Van Garderen & Montague, 2003). Further investigations are needed to explain differences in success and learning, addressing students' reasoning in non-routine tasks that offer information in writing as well as through images.

In our ongoing project, we investigate teacher-student interactions aiming to support students' CMR. The approach is to iteratively establish principles for teacher action in these interactions, design mathematics activities based on the principles, and analyse the activities with the purpose to develop the principles and make them useful to teachers (Olsson & Teledahl, 2018, 2019). Tasks that are used in the mathematics activities often combine written and illustrative data to instruct students. With inspiration from the study by Norqvist et al. (2019) on students use of illustrative information we revisited some of our data. A preliminary analysis indicates a pattern that appears to be common, students usually start their problem-solving process by trying to understand the written information, and then they turn to and try to understand the illustrative information. Our study is focused on the reasoning that follows this initial pattern of interpreting the problem.

Theory

Lithner's framework for imitative and creative reasoning (2008) proposes that a key-factor for successful learning when solving mathematical tasks is whether students engage in algorithmic or creative reasoning. Here, reasoning is defined as the line of thought adopted to produce assertions and reach conclusions in task solving (Lithner, 2008, p. 257). Algorithmic reasoning (AR) is characterized by attempts to recall a procedure that is supposed to solve the task. This includes memorized procedures from solving similar tasks and imitating teacher instructions. Creative mathematical reasoning (CMR) is characterized by the creation of a new reasoning sequence (or re-construction of a forgotten one) supported by arguments anchored in mathematics.

In our ongoing project, we have developed principles for teacher-student interactions in teaching aimed at students learning mathematics through CMR. In mathematics teaching aiming for CMR students must have possibilities to (a) express independent reasoning, (b) develop and (c) justify their own reasoning and to (d) test their results. These principles can be used both for planning and implementing teaching, addressing both design of tasks and preparing teacher-feedback interactions.

The tasks used in this study were designed in line with Lithner's (2017) principles: (1) creative challenge, no solution methods are available from the start and it must be reasonable for the students to construct the solution, (2) fair conceptual challenge to understand mathematical properties (e.g., representations and connections) and (3) justification challenge, is it reasonable for the particular student to justify the construction and implementation of a solution.

Method

The aim of this study was to investigate students' reasoning when using textual and illustrative data in mathematical problem solving. Our study uses and re-examines research data collected continuously in an ongoing project aimed at investigating ways in which teachers can assist students in overcoming various obstacles in problem solving situations. The students and their mathematics teacher were part of the project for three years starting when the students enrolled in fourth grade and ending when they finished sixth grade. During this time, the students were regularly engaged in problem solving activities in which they worked in pairs. Problem-solving sessions were audio-recorded through a portable device placed on the students' desk. For each pair of students in this study the recordings are complemented with notes on students' body language. For every session the teacher was wearing a microphone and her interactions with every student group was also recorded. This study uses six recordings from two problem solving sessions where the mathematical task that students were working on was presented in a way that combined written and illustrative data (see figure 1 and 2).

The analysis of the recordings focuses on sequences where students are constructing a solution to the problem based on written and/or illustrative data. What is of interest is the way students use the data to create a reasonable solution, for example by extracting elements possible to calculate. The first part of the analysis was focused on identifying instances, in which students during their problem solving explored both written and illustrative data. In a second step, students' apparent use of the two sources of information was analysed to identify which data was used at various stages of the process and in what way.

In listening to, and reading transcripts of, students' reasoning it is sometimes difficult to distinguish the written data from the illustrative. We have relied on notes on students' use of body language and on explicit clues in their reasoning, such as "look" or "… here there are …" but we have also tried to identify clues to their focus in what is not mentioned, such as sequences in which there is no mention of any information that can be thought of as deriving from the illustration (an example of this can be found in the excerpt of the transcript, page 6, lines 4–5).

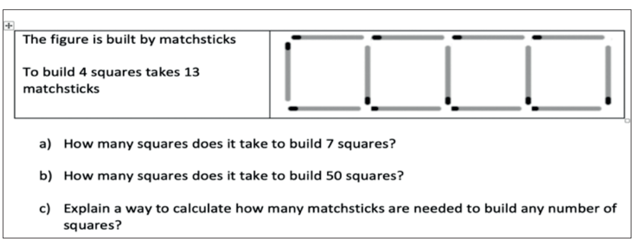

Figure 1. *The matchstick task*

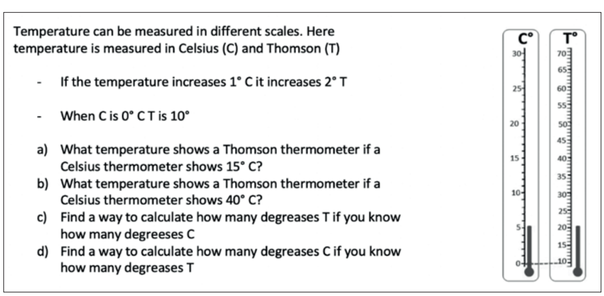

Figure 2. *The thermometer task*

Findings

When exploring students' full solution sequences of two tasks including both written and illustrative data, we observed that most students started with trying to construct a solution based on the written data. When this was not enough, they turned their attention to the picture. Then, two groups were formed, students who made connections between written and illustrative information and students who based their solutions on isolated examples, either from written or illustrative data. Students A and B's approach to the Matchstick task is an example of an attempt to construct the solution on isolated examples, first trying to construct the solution based on written data, and then by using the picture.

1 A If 4 squares are 13 sticks, we can calculate how many sticks are needed for one square … 13/4 … but 13 is not in the multiplication table of 4 …

2 B But you need one less on this side (points at the last stick in the square most far to the right)

3 A Instead we can draw or build seven squares and count the sticks

The example shows students who try to construct a solution based on the explaining text. The strategy is to calculate backwards to find out how many sticks are needed for one square. Student A realizes 13/4 will not result in a whole number. Student B observes that for the last square one less stick is needed. The students abandon the first strategy and find another one, draw and count, based on the figure. The observation on line 2 could have been used to explain why the counting backwards strategy did not work and connect the written and illustrative data for the task. Instead, the students seem to be satisfied with finding an alternative accessible strategy. This has consequences for the continuing attempts to solve the task.

4 B Okay, 22 sticks for 7 squares … how many are needed for 50 squares?

5 A uhm … we can check the calculation table for 7

6 B Yes … look … there were 22 for 7 [squares]

7 A Yes … but the multiplication table for 7 only includes 49 … we can calculate 22 x 7

8 B That is 154

9 A Yes … and then we need another square

10 B And in that one we only need 3 sticks

11 A Then it is 157

The students return to the use of a calculating strategy, even though they previously observed the problem that one square has a different number of sticks, possibly because they realize it is not possible (or at least a lot of work) to draw and count as they did in the first example. What is interesting is that they make use of their insight that the last square only needs three sticks when adding the last square (line 9–10). In addition, in this part of the solution, they use information both from text and from figure, but they do not connect them to each other. This can also be interpreted as a sign of their satisfaction with finding a strategy that seems to work.

These examples indicate that it is difficult for these students to combine information from textual and illustrative resources. Students can use information from both, but they do not connect them and draw conclusions important for the solution.

When students A and B believe they have solved the task the teacher (T) asks them to explain.

12 T Can you explain the way you were thinking when you solved the task? Can you find a way to check your answer?

13 A We were thinking that 7 squares are 22 sticks and the multiplication table for 7 goes to 49

14 A So 22 x 7 is 154 and then we needed one more to have 50

15 B And then we needed 3 more sticks.

16 A Yes, because you only need 3 sticks to build another square … but wait … we have calculated too many … we have used too many four-sticks-squares …

When students explain how they solved the task they realize that they have too many sticks because every new square only needs 3 sticks. Now they make the connections between their numerical approach and the insight that every new square adds 3 sticks. It is possible that if the teacher had not asked the students to explain they would have been satisfied with their incorrect solution.

Students C and D approached the Thermometer task (figure 2) by reading the written data. They came up with the solution to subtask *a* that 15° C equals 25° T. They are unable to figure out how to solve subtask *b* by using only information from the text, so they turn their attention to the picture and observe that for different temperatures there are different differences between T and C.

17 C 5° C is 20° T

18 D 10° C is … what are 10 ° C?

19 C It is 30° T … and 15° C is 40°

20 D But wait … we answered 25 [subtask *a*] ... 5 steps in C are 10 steps in T

After correcting subtask *a* they continued:

21 C This must be correct … look … if C increases by 1° T increases by 2° … and here there are 5 steps for 10 … T increases twice [compared to C]

22 D And while T starts at 10° … C is 0° when T is 10° and you add twice as many C to 10 [to calculate T]

Students C and D's solution of subtask *a* seems to be based on the single example that 0° C equals 10° T. When exploring the picture, they realise they are wrong (line 20). The observation that 5 steps in C equals 10 steps in T is then combined with the written information (line 21) and the solution that T always increases twice as much as C is drawn. On line 22 the conclusion on line 21 is combined with the information that when C is 0° T is 10° and a general solution to how to calculate T out of C is presented. In comparison to students A and B students C and D combines written and illustrative data, in an earlier stage of the solution.

Discussion

This study is inspired by Norqvist et al. (2019) in which eye tracking techniques were used to investigate what students focus on when they solve mathematical problems. The authors suggested that students, who focused on illustrative data in a task, when solving a problem, were more likely to solve similar tasks in a post-test. In our study, we have tried to identify not only what data the students use but also in what way it is used. Our results indicate different ways to attend to illustrative data. Students can start their construction of a solution by using only data, which is provided in writing, and then turn to the illustrative when they are unable to construct a viable solution method. This is illustrated by our first example in which students A and B search the illustrative data to find an explanation to why their proposed solution method of dividing the number of matchsticks with the number of squares, does not suffice to solve the problem. These students however turn back to their original idea, which is now modified, and abandon the picture as a source of information. Students C and D on the other hand turn back and forth between the written and illustrative data using both to verify their ideas, but also to assist them in forming new ideas. By combining the two sources of information they move away from the idea of using an isolated example to inform their reasoning. Their proposed solution method is checked repeatedly against the written data *and* the information that is derived from the image. In this way, they create arguments for their solution method that take several of the conditions of the task and the subtasks, into consideration.

It is risky to generalise based on a few examples, but it is not unreasonable to assume that the way students attend to and use illustrative and written data in mathematical problem solving may enhance their possibilities to independently construct their own solutions. Regardless of whether the solutions are correct or not, the teacher can challenge the students use (or non-use) of illustrative and written data, the idea being that increased or varied input is potentially beneficial to students learning (Brousseau, 1997; Lithner, 2008). This suggests that teachers should consider how they can design tasks and challenge students to use two (or more) data sources in productive ways. As has been shown in previous studies however (Berends & van Lieshout, 2009; van Lieshout &

Xenidou-Dervou, 2018), students' access to illustrative data can increase their cognitive load making problem solving more difficult. This suggests that teachers may play an important role in supporting students' use of information from more than one source, encouraging them to move back and forth between the two sources and to, when appropriate, use either source to verify their solution. In the case of students A and B it seems like the teacher's question ”Can you explain the way you were thinking when you solved the task? Can you find a way to check your answer?” encouraged them to combine what they had retrieved from illustrative and written data. For the students C and D, the image of the thermometers is combined with the written data, and it seems as if it is the combination that assists them in their reasoning. In this example, it is also obvious that the interaction between the two students benefits from the two sources of information as they move from discussing claims to checking them against the image as well as the written conditions. Previous studies have also suggested that using visualisations is an important part of problem-solving skills (Scheiter et al., 2010; Stylianou & Silver, 2004). Encouraging students to make productive use of different data is thus a potential new principle in our on-going project.

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Polysemy and the role of representations for progress in concept knowledge

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Mathematics is a polysemic enterprise, where the same name is given to different things. Polysemy is present whenever mathematical patterns identified in different circumstances share the same structure. This structure will be subsumed under the same mathematical symbolism. Therefore, it is of interest to theorize upon progress in concept knowledge, with respect to the polysemic nature. We argue that progress in concept knowledge involves an epistemological shift that occurs when the meaning introduced by situations and iconic representations is replaced by meaning residing in non-iconic representations that exists independent of the situations and iconic representations. Our theorization can be used in teaching design and by curriculum developers.

Every mathematical concept has a mathematical definition. It is only through carefully formulated formal expositions that we can ensure that the mathematics we construct is logically coherent and free of contradictions. Structuralistic expositions of mathematics came in vogue through the formation of the group of mathematicians writing under the pseudonym Nicolas Bourbaki. The Bourbaki group started their quest in the first half of the 20th century, interestingly in part as a reaction to the work and style of the French mathematician and polymath Henri Poincaré, who tended to tone down rigor to instead stress intuitive connections (Senechal, 1998). The tension between formalism and intuition is not only a matter of style, but is built into the mathematics itself. Poincaré allegedly formulated this as: *Mathematics is the art of giving the same name to different things.*

Poincarés dictum concerns polysemy: that the same lexical item stands for a family of related senses. Not only lexical items but also mathematical symbol systems and the concepts themselves regularly have several but related senses. The symbol 1/3 can mean both one divided by three and the rational number one third. In this strong sense, mathematical concepts are polysemic.

Polysemy is not synonymous with ambiguity. In mathematics education research literature however, polysemy has often been treated as something creating ambiguity and hence difficulties for students (Janvier et al., 1993). As

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we see it, polysemy is instead an essential feature of mathematics. It is polysemy that allows mathematics to be extremely compactly formulated and still widely applicable. Whenever mathematical patterns identified in different circumstances share the same structure, this structure will be subsumed under the same mathematical symbolism, creating polysemy.

Polysemy is very widespread in mathematics. Even the basic mathematical concept of whole numbers is conceptually polysemic. This has been thoroughly investigated by Lakoff and Núñez (2000), using the terminology of conceptual metaphors from cognitive linguistics. They describe how the concept of whole numbers and basic arithmetic is structured by means of four distinct experiential realms: object collection, object construction, using a measuring stick, and moving along a path. These are initially psychologically distinct. But by means of conceptual blending they together form one single concept of whole numbers. Another example is given by Thurston (1994) in a seminal article on the nature of mathematical work. Thurston refers to a list with at least 37 different ways of conceiving the derivative of a function and even claims: ”The list continues; there is no reason for it ever to stop” (Thurston, 1994, p. 164).

Our aim is to theorize polysemy and discuss implications for progress in concept knowledge. In line with Vergnaud (1998) we see mathematical concepts as psychological constructs, but analysis of progress in concept knowledge ”must be made in mathematical terms, since there is no way to reduce mathematical knowledge to any other conceptual framework” (p. 167). We will reflect on elementary mathematical concepts, and we will argue that they are typically born out of classes of situations or from pictorial representations, that we will denote by the term iconic representations. Such elementary concepts are later often subsumed under the same concept by means of being labeled by the same words and by being handled by the same symbol system, creating polysemy. We will argue that it is not until we move to symbol system representations that certain problems generated by polysemy can be resolved.

The paper is hence prescriptive, normative and theoretical. We declare certain aspects of mathematics as essential. Then we draw conclusions about what this will mean for progress in concept knowledge and instruction. We complement this theoretical endeavor with examples from an empirical analysis of Swedish textbooks, to see to what extent our theorization is realized in a present practice.

Progress in concept knowledge in some theories

Progress in concept knowledge in mathematics has often been studied though examining a shift in development where the conceptual entity is first an action or a process, and later becomes a mathematical object in an individual’s mind. The cognitive process of forming an object, a static conceptual unit, from a dynamic process is denoted by the term reification (Sfard, 1991). The well-established APOS theory addresses reification by describing the process of conceptual understanding as evolving from actions to schemas (Asiala et.al, 1996). In APOS, a shift in conceptual understanding occurs through a cognitive organization of actions, processes and objects in schemas that form the framework for understanding concepts in new related problem situations. Sfard (1991) describes the development of mathematical concepts from process to object as a hierarchy in three stages: interiorization; condensation and reification. Sfard claims that mathematical concepts start their lives as processes and her theory assumes that an ontological shift is needed for the concepts to become objects. ”Only when a person becomes capable of conceiving the notion as a fully-fledged object, we shall say that the concept has been reified.” (Sfard, 1991, p. 19). These theories point out the important transformation from action or process to a mental object and also that an object that has been reified (in Sfard’s terms) can later be involved in a process which in turn may be reified, creating a hierarchy of objects that all have their roots in some particular action or process. However, neither Sfard nor APOS explicitly deals with the situation when one mathematical object has its roots in several different and partly unrelated processes. That is, they do not account for the polysemic aspect of mathematical objects that we are interested in.

Another classical theory for progress in concept knowledge builds on the dichotomous classification between concept image and concept definition (Tall & Vinner, 1981). The concept image is here the formal definition of a concept, while the concept image is the total cognitive structure associated to that concept, which may include both properties, symbolic expressions, mental images and processes. The concept image may also contain what Tall and Vinner calls the concept definition image, which is a mental counterpart of a concept definition. Essentially, progress in concept knowledge within this frame of reference would mean having a concept image that, when evoked, allows the individual to deal with the concept in a way that becomes ever more in line with the concept definition. But Tall and Vinner also discussed progress in terms of the concept definition image becoming stronger, which is a metaphor for that students can rely more on formalism and be less dependent on mental pictures.

In relation to our theorization, an important aspect of a concept image associated to a concept is that it may contain conflicting components. This means that in principle, the theory of Tall and Vinner, could accommodate polysemy, where different, and sometimes possibly conflicting meanings are associated to the same concept. But the conflicts are never analyzed as an effect of polysemy, but rather as an effect on an inadequate concept image.

To examine concept knowledge for geometrical objects, Fischbein (1993) developed the theory of figural concepts. Fischbein observes that elementary geometrical objects are a symbiosis of a figural character and the formal descriptions of the object. He claims that intuition of geometrical objects comes from the figural aspect while the rigor for dealing with such objects must come from the formal conceptual form. In principle, also Fishbein's theory could accommodate polysemy, but it is not explicitly dealt with in Fischbein's original work or in papers that reference it.

In line with Tall and Vinner, progress for Fischbein would incorporate becoming more flexible in when to invoke figural thinking and formal thinking. This is reminiscent of how Vygotsky sees the transition from thinking in terms of spontaneous concepts versus thinking in terms of scientific concepts that require teaching to be understood. Vygotsky emphasizes, ”the very notion of scientific concept implies a certain position in relations to other concepts, i.e., a place within a system of concepts” (Vygotsky, 1962, p. 93). Vygotsky observes that all higher thinking is mediated by systems of signs – semiotic systems. Duval (2006) characterizes semiotic system by referring to multi-functional systems, like spoken or written natural language, and mono-functional systems, like most specific mathematical symbol systems. Nunes (1999) calls such representations compressed, indicating that they contain hidden culturally coded information, not visually distinguishable in the symbols themselves. She contrasts this with extended representations, where the form of the representations itself contains the necessary information to decode the mathematical information at hand. We will use the term iconic for such representations and refer to other as non-iconic.

From the theories above we learn that conceptual progress is complicated psychological processes. However, to examine polysemy and analyze the role played by iconic and non-iconic representations, we need to consider connections between concepts in different situations, since concepts cannot exist in isolation from other concepts. For that reason, we will use the *Theory of conceptual fields*, developed by the French psychologist Gérard Vergnaud to provide an explanation of why progress in mathematical concept knowledge psychologically is very complex despite that formal mathematical expositions normally are clear and hierarchical. A conceptual field consists of a set of concepts tied together with a set of situations where the concepts apply (Vergnaud, 2009). According to Vergnaud the meaning of a concept comes from a variety of situations. Reciprocally, a situation cannot be analyzed with one concept alone, but only with several concepts, forming systems. Conceptual fields consist of such clusters of situations and concepts and hence forms a good basis for analyzing polysemy.

Theory

Our theorization will be formulated in the language of conceptual fields. Vergnaud (2009) defines a concept as a triplet of three related sets. The set of situations where the concept is relevant; the set of operational invariants that can be used by an individual to deal with these situations; and the set of representations, symbolic, verbal, graphical, gestural etc. that can be used to represent invariants, situations and procedures. Note that in this definition, situations and invariants are psychological categories, that is, mental constructions, while representations can be both mental and physical/external. Since we are not here interested in explaining the thinking of particular individuals but to analyze progress in mathematical concept knowledge in general, we will deal with situations and invariants from the point of an observer (Maturana, 1988). This means that we will assume that in educational and mathematical settings, enough individuals will form situations and invariants similar enough so that it makes sense to talk about them as phenomena in themselves.

We will extend and specify Vergnaud's definition. To avoid a lot of technical detail, we will abuse the notation of representation and call an image (◐), symbol (1/2), word (one half) or other combinations of signifiers a representation, without specifying what it represents and for whom. Any of the three symbols above can in some certain circumstance represent any of the others, as well as some underlying abstract idea or invariant. We distinguish between two particular types of representations, iconic and non-iconic. We call a representation iconic when some observable patterns in the representation correspond to some structure in the represented idea or invariant. The images ◐ and ◕ are iconic representations of multiplicative part-whole relationships. In non-iconic representations, the denotation instead builds on convention, where the typical examples are spoken language or letters being combined into words in written language. We will be interested in mathematical symbol systems and our illustrative example is a/b. The non-iconic symbols 1/2 and 3/4 are not just composite symbols that can represent the same part-whole relationships as the partly colored circles above. They belong to a system with a set of transformation rules, governing how changes to a symbol relate to changes in the represented invariant.

Our theorization stipulates three things:

1. *The origin of concepts*. Vergnaud's view on concepts means that situations, invariants and representations are conceptually intertwined. But for concepts that are introduced in schooling, the initial invariant from which the concept is bootstrapped must come from either a situation or a representation (see footnote 2 in Duval, 2006, for an elaboration). Combining this with our characterization of two types of representations creates three essential ways of generating concepts. First, concepts can be connected to the invariants in a class of situations, like when a meaning of division is given through describing a number of things to be divided in a number of bags. Second, concepts can be connected to iconic representations, like when fractions are given meaning by an image of a partly colored circle. Third, concepts can be connected to mathematical relations formulated in non-iconic symbol systems, like when division is described by saying that a/b is a number c such that $a = b \cdot c$.

2. *The umbrella effect*. Like we exemplified with whole numbers, mathematical concepts are regularly subsumed into more general concepts. When concepts generated from situations or iconic representations are subsumed under more general concepts there will be invariants from the original concept that will cease to be invariant under the new umbrella. In the equal sharing situation, used to explain division above, division of a by b results in a number c that is smaller or equal to a, but this does not hold for division in general. Likewise, when part-whole relationships are iconically represented by circle sectors, no fraction can be bigger than the whole circle. But, in general a/b can have any size. Note that the three examples in the previous paragraph are all denoted by the same symbol system, a/b. Even though we in this case continue to talk about division and fractions as different things, from a mathematical point of view we can subsume both concepts under the umbrella of quotient constructions.

3. *Contradiction of invariants*. One iconic representation that generates fractions is part whole relationships of whole numbers. For $a \leq b$, a/b is associated to a colored bead out of b total beads, like 2/3 is a symbolic representation of the image ●●○. But another iconic representation that also generates fractions is when a certain part of one circle is taken to represents fraction, like when ◐ represents 1/2. Both these representations individually form straightforward one-to-one correspondences with the symbol system. However, when the part-whole representation is extended to also denote numbers bigger than one, this model is mixed with the previous model. The representation is normally extended so that ●◐ is taken to mean one whole and one half, that is 3/2. But when the ideas from the two representations are mixed, then ●◐ can just as well mean 3/4. The one-to-one relationship is hence broken. We claim that this case is typical for what happens when different iconic representations or situations are used to generate the same concept. We claim that the case of quotient constructions is an illustrative example of how polysemy is introduced when mathematical concepts are derived from situations or iconic representations.

The character of mathematics we have described and the theoretical conclusion we draw has important consequences for progress in mathematical concept knowledge. It is simply inevitable that when mathematics is grounded in iconic representations and situations, there will be cognitive conflicts whenever several sub-concepts are gathered under the same concept; something we have described as an essential aspect of mathematics. These cognitive conflicts will be unresolvable as long as the meaning of mathematical concepts continues to be grounded in the iconic representations and situations. This is because there are invariants inherent in iconic representations (or situations) that are coherent within such a representation, and in relation to the mathematical concept, but that are not coherent between iconic representations. When corresponding mathematical relations are represented and dealt with by a mathematical symbol system, the above-described incoherences are resolved. Therefore, a necessary consequence of these theoretical observations is that it is important that teaching is designed to overcome the incoherence created by iconic representations. The meaning of concepts and the relations they entail are, wisely, initially drawn from situations and iconic representations. But at some point, the meaning must be placed in the mathematical relations and the symbol systems. The icons and situations, that initially generated the conceptual meaning need to get a new role as just being examples, or concrete realizations of the mathematical meaning.

An empirical example from Swedish textbooks

To what extent do the types of representations we have dealt with occur in practical mathematics teaching? As a proxy for how concepts might be taught, we have examined two common Swedish textbook series, covering grades 1 to 6 and grades 7 to 9[1](Ahl & Helenius, in press). The gist of the material was that while reasoning in terms of symbol systems at times are used to explain procedures, symbol systems was not used to explain connections or motivate concepts until in grade 8 and even then, only very sparsely. Concepts were instead explained by situations and iconic representations. Invariants in the situations and iconic representations were then labeled by the symbol system whereby it may be explained what aspect of the symbol that relates to what aspect of some invariant. We discuss a representative case below.

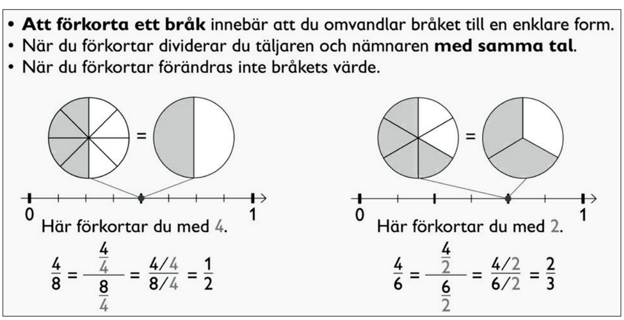

Figure 1. *Introduction to reducing fractions in grade 5*

Note. The text marked with bullets in the top of the figure says: ***To reduce a fraction** means that you transform the fraction to a simpler form. When you reduce you divide the numerator and the denominator **with the same number**. When you reduce, the value of the fraction does not change.*

That the same fraction can be represented in different ways within a symbol system captures that a fraction is a multiplicative relationship. Equality of fractions was introduced in grade 4 (Ahl & Helenius, in press). It was repeated in grade 5, together with the procedure of how to reduce fractions (figure 1). It was shown in terms of the standard symbol system for fractions how reducing a fraction can be done, by means of dividing the numerator and denominator by 2 (in the right-hand example). This is an example of polysemy, since within one calculation, the same symbolic expression is used both for representing rational numbers and division. The explanation, however, is not given in terms of mathematical relations in the symbol system. Instead, it is the geometric invariant in the iconic representation that supplies the explanation, by showing that coloring 4 out of 6 equal segments signifies the same part of the of circle as coloring 2 of 3 parts. The number line, in figure 1, iconically supplies a similar argument. Neither the circle representation nor the number line argument involves division. In the symbol system representation, it can be considered known that $4/2=2$ and $6/2=3$ but no explanation to why it is allowed to divide both the numerator and the denominator by 2 is given within the symbol system. The same argument is repeated in grades 6, 7 and 8 with similar iconic representation as above.

An argument entailing reasoning with relations in the symbol system could for example build on factorization, observing that $4=2\cdot 2$ and $6=2\cdot 3$ and that $4/6=(2\cdot 2)/(2\cdot 3)=2/2\cdot 2/3=1\cdot 2/3=2/3$. Approaching reduction of fractions from this point of view would set the stage for answering two important questions that in the illustrated explanation remain unanswered, and that in fact cannot be answered without factorization: why divide with 2? and, how do you know that you reached the simplest form? The answer to both questions builds on the concept of common divisors. In this line of reasoning it does not matter if a symbol a/b is thought of as a fraction or as a division. The symbol system approach would require that multiplication of fractions is introduced; something that is not done until grade 8 in the analyzed book series. While introducing multiplication of fractions earlier would be something we endorse, even without such a change an approach involving factorization would still be able to give suggestions on how to rearrange iconic representations when introducing how to reduce a fraction.

Implications for mathematics education

We have argued that progression in concept knowledge requires a deliberate movement from reasoning in terms of iconic representations and situations to reasoning within non-iconic symbol systems. This argument is not unique, as it could represent the classic saying: *going from the concrete to the abstract*. However, our contribution is that we build our argument on the observation that mathematics is polysemic and that the practice of generating concepts by means of situations and iconic representations inevitably generate some contradictions between different situations, requiring different interpretations, represented by the same concept and symbol system. We exemplified with concepts that can be subsumed under the concept of quotient constructions, that is, anything we denote as a/b.

We emphasize that the movement towards reasoning in symbol systems is not only about becoming versed in using symbol systems. Just as important is the epistemological development of realizing that meaning can emerge directly from relations described in symbol systems. An epistemological shift occurs when situations and iconic representations, that previously generated the mathematical meaning, shift to be representations of some meaning that exists independent of the situations and iconic representations.

We acknowledge that our theory shares properties with several previous theories that deals with progress in concept knowledge. The epistemological shift we are advocating has similarities in APOS (Asiala et al., 1996) and with Sfard (1991). But while Sfard speaks of processes that are reified and Asiala et al. about processes being encapsulated as objects, in our theorization elementary mathematical concepts can be generated both from situations (which involves processes) or from iconic representations. Our focus on iconic representations, which is something inherently figural, means that our theory also shares similarities with Fischbein's (1993) theory of figural concepts. But whereas Fischbein deal with geometrical objects, we emphasize that also many non-geometrical objects are dealt with by introducing iconic representations that have geometrical properties. As we showed in our example, even relatively elementary arguments may entail different interpretations of the same symbol, that is, polysemy. But as long as the meaning making resides in iconic representations or situations it is hard to make use of the power that the ability to go between different interpretations of the same concept allows.

A key conclusion from our theorization is the need for a particular form of epistemological shift in the progress of concept knowledge. In many previous theories, this shift involves a change in the psychological nature of the object, from process to object or from concrete to abstract. In our theorization, we instead emphasize the role of representations, with a deliberate movement from meaning making through situational and iconic representations to meaning making through relations in symbol system representations.

Nunes (1999) makes a distinction between extended and compressed representations that, as we mentioned, have resemblance with iconic and symbol system-based representations. Nunes empirical analysis indicates ”that extended representations are preferred at initial points in learning” but also that ”the analysis of conceptual relations indicates the advantages in the use of compressed representations and thus the need to cope with the move from extended to compressed representations in mathematics instruction” (Nunes, 1999, p. 38). We think this is important, in particular in relation to teaching design and production of curriculum materials. More emphasis on symbol system knowledge will obviously come with its own challenges and it is an empirical question how this can be done in practice. Our message is that the focusing on reasoning in terms of relations in symbol systems with the aim of creating learning opportunities for an epistemological shift deserves more discussion. The purpose with the present paper is to contribute to such a discussion.

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Note

1 The series are *Favorit Matematik* (2018) from Studentlitteratur, second edition, for grades 1–6 by Karppinen, Kiviluoma & Urpiola and *Matte direkt* (2011) from Sanoma utbildning, grades 7 to 9 by Carlsson, Hake & Öberg.

Positioning of programming in mathematics classrooms – a literature review of evidence based didactical configurations

Andreas Eckert and Alexandra Hjelte

This literature review looks into the roles that programming and mathematics can take in relation to each other in an educational environment. This is done by retrieving papers from Web of Science and MathEduc on mathematics and programming in education, prior to university level, and analysing their tasks through a lens of Instrumental Orchestration. The four different exploitation modes identified are manipulating a physical entity, manipulating a virtual entity, creating own interactive environment and creating, testing and refining mathematical algorithms. Depending on how mathematics and programming are positioned in the four modes, the emphasis on mathematics and programming varies, resulting in different outcomes of the lessons.

Feurzeig and Papert (1969) envisioned a future where students of all ages learn key concepts and procedures of mathematics through their newly developed programming language Logo. Now that vision seems to partly be coming true as an increasing number of countries add coding as a part of their national curriculum for elementary school to improve students' digital competence (European Communities, 2007). Feurzeig and Papert (1969), amongst others, motivates this change by highlighting the skills required to master coding, and the potential of integrating learning in other subjects in the coding process. The dual purpose opens up for different ways of introducing programming in the curriculum. Sweden for example, which is the context this paper is written within, chose to include coding in existing subjects, mathematics being one of them (Skolverket, 2018).

Mathematics curricula already includes mathematical content to be learned and mathematical proficiencies to be developed, and in countries such as Sweden programming is meant to fit into this structure. Introducing programming into the teacher's practices amplifies the complexity and challenges the way he or she has taught the subject so far (Lagrange & Monaghan, 2009). Teaching mathematics becomes more complex because of the human/machine interactions as part of the learning environment, where the teacher could be seen as a conductor creating the prerequisites for an orchestra with several different instrument to

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harmonize (Trouche, 2004). To orchestrate teaching of mathematics, we need to look into the available components, or didactical configurations, as well as how to exploit such configurations (Drijvers et al., 2010). As programming becomes a part of the didactical configuration, we need to attend to how lessons and tasks can be designed to achieve different learning outcomes.

So, how are teachers and researcher to understand the role of coding in mathematics education? We aim to identify the shifting roles of mathematics and programming in educational settings that combines the two with this literature review. It is our hypothesis that studies have utilized different didactical configurations, and exploited those configurations in various ways, in their educational designs. Our research question is *how does different didactical configurations position programming in relation to mathematics in terms of means and goals for learning?* which we intend to answer by systematically searching and analysing recent research papers.

Theoretical background

Digital competency, often highlighted in mathematics education, are versions of Wing's (2006) computational thinking (Arnulfo, 2018). It is described as problem solving proficiency where the problem solver is aware of how for example programming can support successful problem solving across disciplines. Arnulfo (2018) identifies characteristics of computational thinking that are of value for mathematics learners. They are; confidence in dealing with complexity; persistence in working with difficult problems; tolerance for ambiguity; the ability to deal with open-ended problems; and the ability to communicate and work with others to achieve a common goal or solution. These characteristics are thought of as having a positive effect on mathematical learning, and working with mathematics through programming is thought of as having a positive effect on students' development of these characteristics. Thus, showing the bi-directional intention of including computational thinking in mathematics education. Kotsopoulos et al. (2017) argue that working in a mathematics classroom to develop these characteristics is best thought of in terms of four pedagogical experiences, unplugged, tinkering, making and remixing. Tinkering is to modify premade code, making is to write your own code and remixing is to reuse and combine existing pieces of code in new ways to create programs. These three practices could be performed both unplugged and in a computer environment. However, Kotsopoulos et al. (2017) differentiates between the unplugged experience, found in curricula for younger children, and working in a computer environment. They suggest that the four practices could be used in a sequential approach for novices to develop their computational thinking and coding skill.

It is a big step to go from the unplugged experience of computational thinking to working in a computer environment. To understand the complexity of teaching mathematics with human/machine interaction, Trouche (2004) introduced the idea of instrumental orchestration. The idea is that teachers create opportunities for students to develop their computational thinking and mathematical reasoning through organizing available artefacts, i.e. didactical configuration, and determining how to utilize these artefacts, i.e. exploitation mode. Artefacts can be physical objects in the classroom, such as a computer, but also a non-physical object, such as a programming environment which is the case in the present study. Drijvers et al. (2010) compares didactical configuration with how teachers configure their teaching settings, their intentions, and what artefacts that are available. It is the *what?* of instrumental orchestration. They further compare exploitation mode with choosing when and how each instrument (or artefact) should come into play when describing how teachers exploit the didactical configuration to work toward an intended goal. It is the *how?* of instrumental orchestration. In the context of programming, choices of how to introduce, sequence, scaffold and work through a task are all part of the exploitation mode. It is our interpretation that didactical configuration and exploitation mode are key aspects of task design. Tasks are thought of as the written and/or verbal task given to the student (e.g. code a dice simulator) and its required artefacts. A task design needs both the prerequisites of a programming language, and the opportunities to use it to create a working classroom task.

Method

The research question of this paper is answered by the means of a systematic literature review. This means we systematically collect relevant research papers through database (Webb of Science and MathEduc) query and analyse the them with the purpose of answering a research question (Eriksson & Barajas, 2013).

Deciding keywords and searching for papers

We used ”math*” combined with the following phrases that complement each other; education, learning, teaching, instruction*, activity, student*, pupil* and child*. To find papers that contained mathematics and programming, those phrases were then combined with ”programming” or ”programing”, together with ”computational thinking” and ”compute”.

To get an overview over the resent research, the time for article publications were set to 2007–2018. The search and retrieval of papers were performed on August 20, 2018, therefor no papers published after that date was included in the analysis. The papers were also limited to papers that were published in English. This resulted in a hit of 318 papers from the database Web of Science. Within the database MathEduc, without the possibility of limiting the search by categories or research areas, the same search resulted in 225 papers.

Reading titles, abstracts and papers

To manually single out papers, that addressed programming and mathematics in education, relevant for this study the following inclusion criteria was used:

– The article addressed computer programming

– The article addressed mathematics education

– The article focused on programming before university level

– The article was an empirical article

The titles and parts of, or full, abstracts of the extracted papers were read by one researcher. In some cases, this was not enough to decide if the article fully satisfied the inclusion criteria. In those cases, the content of the papers was scanned, looking at research questions, method for the study and sometimes the results and conclusions to determine if the papers should be included. In a few cases there were some uncertainties if an article should be included or not. In those cases, the papers were read by both authors and discussed if it should be included based on the criteria.

Several papers addressed the combination of mathematics and programming for higher education and were excluded. Some papers only addressed programming or combined programming with other subjects but had been included in the database search due to the mention of education within STEM (science, technology, engineering and mathematics). These papers were also excluded. There were also papers that focused on mathematics education, where programming was not addressed. Often, they used the framework of computational thinking in some way, without programming, or used the phrase compute unrelated to programming. Papers that were purely theoretical or presented a task without an empirical investigation within the article were also excluded. This resulted in 15 papers from Web of Science, and 7 papers from MathEduc. However, one of the papers, written by Ke (2014), was found in both databases. This resulted in a total of 21 papers. Eight of the papers were published within journals in the area of mathematics didactics, ten papers were in journals with another didactical orientation and two papers came from journals without an explicit didactical focus.

Analysing the papers

To identify patterns, similarities and differences between tasks within the different papers the analysing tool NVivo was used to structure the data. The tasks were analysed using the lens of instrumental orchestration to identify artefacts that make out the didactical configuration (e.g. aspects of the programming environment, of devices student used, of physical objects etc.) and their exploitation modes (e.g. what the students were supposed to use the artefacts for, how the teacher organized the work with the artefacts etc.). We used the descriptions of the tasks together with the studies' research questions, methods and results in our analysis. As a first rough categorization we identified the subject content that the students worked with, the programming environment they used, different physical objects used and the context of the class (e.g. mathematics class or computer class). Aspects of the artefacts making up the didactical configurations and their exploitation modes where then summarized, categorized and presented in table 1.

Three randomly selected papers were cross-analysed by both authors to compare their interpretations. The cross-analysis ensures the coherence in the interpretation and categorization of the tasks (Bryman, 2011).

Results

To understand the shifting roles of mathematics and programming in previous research the tasks were categorized into four exploitation modes, based how the didactical configuration was exploited in the lesson to stage tasks. These exploitation modes were then worked through to identify the different roles of programming and mathematics. Three aspects of the didactical configuration that had an impact on the roles of mathematics and programming were identified. The aspects were; (1) the learning goal of the task, (2) the task given to students and (3) the tools the students used. A summary of the results can be found in table 1.

Table 1. *An overview showing exploitation modes (left) and aspects of the didactical configurations (on top) elaborated on in the result section*

	Learning goal	Task given to students	Tools used
Manipulating a physical entity	Mathematical topics (primarily geometry)	”Program this physical entity”	Programming environment and (sometimes implicitly) mathematics
Manipulating a virtual entity	Mathematical topics (primarily geometry)	”Program this virtual entity”	Programming environment and (sometimes implicitly) mathematics
Creating an interactive environment	Mathematics and programming, with programming in the foreground	”Program a (mathematical) game”	Programming environment and sometimes mathematics
Creating, testing and refining algorithms	Mathematics and programming, with mathematics in the foreground	”Create an algorithm to solve this (mathematical) problem”	Programming environment and mathematics

We expand on the exploitation modes found in the literature first, and then how mathematics and programming was positioned based the three aspects of the didactical configurations in each mode.

The didactical configurations presented in the papers resulted in four exploitation modes; manipulating a physical entity; manipulating a virtual entity; creating an interactive environment and; creating, testing and refining algorithms.

Five tasks were categorized as manipulating a physical entity, contained tasks in which the students were to program a robot to do a specific thing. The artefacts used in these types of tasks where; the mathematical topic of geometry and a robot that was used to visualize the mathematical content. The programming environment differed between programming directly on the robot (e.g. Bartolini Bussi & Baccaglini-Frank, 2015) and block programming in a computer environment (e.g. Taylor, 2018). The exploitation mode consisted of using the environment to fit the task of the moving robot and formulating tasks to make the robots do different things. The artefacts where then exploited in different ways to help the students develop their spatial understanding. For example, by letting the students discuss movements of the robots.

Our second category contain tasks coded as students working with manipulating a virtual entity. The virtual entity could for example be a programmable cat or a car on the screen that executed predetermined actions without further interaction of the user. 11 tasks were about manipulating a virtual entity. Within these 11 tasks there was an object, in a virtual setting, that the students worked with and manipulated in some way. The artefacts used in these types of tasks where the mathematical topic of geometry and functions and using a digital block programming environment. They exploited visual outputs of programming environment in tasks that consisted of either creating geometrical objects (e.g King 2015) or to understand the changing movement, speed and acceleration of objects in some way (e.g. diSessa, 2018). It was done by letting the students work in small groups, pairs or alone. Within several of these studies, mathematics is the learning objective of the task. However, they require the students to understand the structure of programming. Using the task therefore sometimes led to a situation where the tasks' mathematical learning goal was not apparent to the students. One example of this is described by Alfieri et al. (2015) where the students were supposed to move a robot through a virtual game and needed to calculate wheel rotations to make sure that they end up in the right place. The task is primarily a programming task (program the robot in the right way) in which mathematics was implicit. Mathematical knowledge was needed to solve the programming problem, which sometimes resulted in the development of students' mathematical understanding, or appreciation for the necessity of mathematics.

The third category contains tasks in which the students create their own interactive environment by programming a game, three tasks had this as a main objective. The games could contain one or more virtual entities; however, a user could interact with the game without doing changes to the code. In the description of the task the mathematical topic is not decided by the teacher. It is decided by the students in the development of their game. Because on the open character of the programming task, there was no mathematical area designated to the students. The artefact in these tasks were the block programming environment, in a computer class setting (Ke, 2014) or mathematics class setting (e.g. King, 2015). The task was to create a game for a sibling or peer. The students worked in small groups or pairs with the programming languages Scratch or TouchDevelop. It often resulted in a situation in which the students worked a lot with solving different programming problems, and developing their understanding for programming. However, Ke (2014) reported that two of the groups in their study created a game without integrating any mathematics. In the cases where students had opportunities to work with, and develop an understanding for, mathematical concepts as well as programming, students often focused on the programming.

The final category focus programming and mathematical tasks where students work with the assistance of a computer to make mathematical calculations or models. In these tasks the graphics played a minor (or no) role compared to tasks in prior categories. Six papers included students working with algorithms for calculations or creating models. Students were creating, testing, refining and interpreting different algorithms, with different purposes. The artefacts in this category were the mathematical range over different areas focusing on understanding algorithms and the programming environment. The programming environment used also differed between different text-based programming languages and block programming. The context was set in a mathematics class, informatics class or technology class. The learning goals within these types of tasks focus on understanding algorithms (Grover et al., 2015), creating algorithms to solve a mathematical problem (Psycharis & Kallia, 2017) or gaining a deeper mathematical understanding of concepts by creating mathematical models with algorithms (Kahn et al., 2011). The students worked alone or in pairs. The tasks encouraged students to exploit programming to solve mathematical problems, and as a help to visualize (Taub et al., 2015) or to generate calculations (Psycharis & Kallia, 2017) that the computer executes faster than the students.

Understanding the shifting roles of mathematics and programming

This literature review has presented different types of artefacts used in educational settings to work with mathematics and programming. We also described how these artefacts are put to work in respect to the learning goal, how they are exploited, and how this positions mathematics and programming within these contexts. Since the purpose of this study was to investigate the shifting roles of mathematics and programming the following three aspects of the didactical configurations where used to understand the role and positions of mathematics and programming in different tasks: (1) learning goal of the task, what the teacher intended the students to learn when working with the task, (2) assignment presented to the student, either a mathematical assignment or a programming assignment and (3) tools the student use to solve the assignment.

Papers with tasks of manipulating a physical or virtual entity have one thing in common, the learning goals revolve around students developing their geometric understanding. Hence mathematics has the role of the learning goal (King, 2015). The learning goal is less specific when tasked to create an interactive environment. The students are to learn mathematics when working with programming, so programming is positioned in the foreground in the activity. The learning goals also include that students should learn programming and/or develop an interest for it (Ke, 2014). Within this category, our conclusion is that both mathematics and programming are positioned as a learning goal. The tasks in the fourth task-category, like the tasks in the first two, has more explicit mathematical goals where the students are expected to work with and understand the mathematical algorithms. However, programming is also emphasized as an expected learning outcome in the tasks utilizing these exploitation modes.

The second aspect is the assignment of the task. Students work with tasks encouraging them to program something. It could be a robot, a figure, a game, or an algorithm. In that sense the assignments are programming assignments. However, some of the tasks in the third and fourth category explicitly said that mathematics was a part of the task; e.g. create a mathematical game (Ke, 2014) or program an algorithm to solve [this mathematical problem] (Psycharis & Kallia, 2017). That was not the case when asked to manipulate a physical or virtual entity

The final aspect consists of the tools the students need, and use to solve the given assignment. Within all the exploitation modes the students are using programming tools to solve their assignments, which is expected taking our search into account. They are also using mathematics in some way, however to what extent and in what way differs both between and within the different exploitation modes. The mathematics is needed to solve the problem *how to move the robot through an environment* (Alfieri et al., 2015) or *make the cat hit the basket case* (Sung et al., 2017). However, the mathematics may not be explicit to the students, and even sometimes lost. Within Creating an interactive environment, when the students worked with games, the mathematical content was not decided by the teacher or researcher. The (mathematical) content was set by the students, depending on how they designed their game and which mathematical problems they encountered. This sometimes meant that the students worked with mathematics and sometimes they did not, as in the case of creating an interactive environment by programming a game. Ke (2014) concludes that two of the groups created a game without having mathematics incorporated, and so the role of mathematics was lost.

Discussion and conclusions

In this paper we have identified different roles that mathematics and programming can adopt when working with the two simultaneously in an educational environment. The results indicate that the roles are not fixed but can take a variety of different positions in relation to each other. Programming can be used for working with mathematics. Mathematics can be used for working with programming. Programming and mathematics can also complement each other in solving different tasks and problems. Our results suggest that these positions are set by learning goals, formulation of the assignment, the environment in which the students can work with the problem and the tools they need to solve it. These shifting roles risk placing one in the foreground and one in the background, both in expected learning outcomes and in the actual activity that the students are working with. Without an explicit mathematical topic, aim and formulation of task, programming tends to be positioned in the foreground (Ke, 2014). But it is possible to balance both mathematics and programming so that they complement and giving each other meaning and purpose.

This paper has shown that awareness of the mathematical content that the students are expected to work with, is an important aspect to take into consideration when working with programming and mathematics. There is however a need for more research that focuses on the relationship between the two subjects. Research that can further investigate how these two subjects can be used in a way that is beneficial for both the development of mathematical understanding and knowledge of programming. The direction takes us deeper into understanding pedagogical aspects of computational thinking (Arnulfo, 2018) and how the structure of programming can be combined with mathematics. Research might bring us closer to the ideas of Feurzeig and Papert (1969), where programming can be used as a way to gain an understanding for mathematical concepts, but where mathematics also can be used to understand programming. The research project this paper is written within will use the outcomes to design research interventions on the topic, further exploring how different task designs work in a school context.

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On pre-service secondary teachers' mathematical content knowledge in statistics

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In this paper we report on the use of a pre-post-test design to study pre-service secondary mathematics teachers' mathematical content knowledge in statistics before and after their first university course in statistics. The results show that the participants were successful in the pre-test on items related to sampling, probability and the general logic of making formal statistical inferences, but struggled with items concerning distributions. Comparing the pre- and post-test reveals an increasing average of the participants' scores in most statistical areas, but that topics like informal statistical inferences and distributions remain challenging for the majority of the participants.

The technologies of today collect and make vast quantities of data easily available. However, data itself does not tell us anything, but requires being organized and looked at using models to provide information and knowledge. Now more than ever models are needed in private and professional settings to interpret and make sense of data in various forms (Manyika et al., 2011). In this context, understanding a range of statistical topics and learning statistical reasoning are invaluable tools for all students to become proficient with, in order to enable them to interpret and make sense of data (Franklin et al., 2007; OECD, 2013). In Sweden, students in grades 7–12 learn about randomness, probability, descriptive statistics, measures of spread, correlation, causality, regression, and the normal distribution (Skolverket, 2011a; 2011b). However, learning statistics has proven to be challenging. Research has shown that students, as well as teachers, often poorly understand the statistical procedures they learn, and additionally have difficulties in interpreting graphs and making inferences from data (Bakker & Derry, 2011; Batanero et al., 2011; Shaughnessy, 2007). In line with the argument by Ball at al. (2008), it is important that teachers have a solid understanding of statistical content knowledge to be able to teach the statistics syllabus in schools successfully. Although Batanero et al. (2011) highlighted the lack of adequate research related to both pre-service and practicing

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teachers' statistical content knowledge, some studies suggest that pre-service secondary mathematics teachers (P-SSMTs) struggle with learning statistics (Lovett & Lee, 2018).

In this paper we contribute to the understanding of P-SSMTs' statistical content knowledge, and investigate if there is any merit to the claim that P-SSMTs struggle with the statistics course(s) in their teacher training programmes as noted by Lovett and Lee (2018). To do this, we investigate a class of Swedish P-SSMTs' statistical content knowledge using a pre-post-test design.

Aim and research question

The aim of the study presented in this paper is twofold: (i) to provide a snapshot of P-SSMTs' statistical content knowledge when they are admitted to the teacher training programme and before having taken any university courses in statistics; and (ii) to identify how the P-SSMTs' strengths and weaknesses of their statistical content knowledge found in (i) change (if at all) after their first university course in statistics. To this end, we investigate the following two research questions: (RQ1) *What statistical content knowledge do P-SSMT display before their first university course in statistics?* and (RQ2) *How does P-SSMTs' statistical content knowledge change as the result of participating in a university course in statistics?*

Previous research

In their review of the literature on teachers' and pre-service teachers'statistical content knowledge, Lovett and Lee (2018) conclude that research to date primarily focused on (pre-service) elementary teachers' statistical content knowledge, whereas literature investigating P-SSMTs' statistical content knowledge is spares. Within the limited literature on secondary (pre-service) teachers' statistical content knowledge, Lovett and Lee (2018) identifies 3 main research areas focusing on: (1) computations, algorithms and procedures; (2) insufficient reasoning skills; and (3) obstacles around interpreting and developing graphical representations. The main results from these areas are that P-SSMT are well equipped when it comes to using procedures and algorithms for computations, such as calculating mean values. However, the repetitive focus of standard procedures in mathematics and statistics courses in the teacher training programmes tend to have a negative impact on (pre-service) teachers' statistical content knowledge in terms of their statistical reasoning skills and abilities to make interpretations of graphical representations. In particular, understanding and interpreting box plots and histograms, analyzing skewed distributions, sampling distribution, variability, confidence intervals and p-values, as well as reasoning about, and making inference between, sample- and population distributions pose difficulties (Lovett & Lee, 2018).

The general remark above regarding the spare research on teachers' and P-SSMTs' content knowledge in statistics is also valid in the Swedish context. However, Nilsson and Lindström (2013) profiled 43 (whereof 18 + 6 secondary teachers) Swedish teachers' knowledge base in probability. They found that the teachers' ”knowledge profile is more computationally oriented than conceptually oriented” (p. 61), and identified five knowledge profile patterns showing (1) a base level understanding of the classical interpretation of probability; (2) challenges concerning the structuring of compound events; (3) issues with conjunction and conditional probability; (4) having a less degree of specialized content knowledge than common content knowledge (cf. Ball et al. (2008)); and (5) problems with random variation and principles of experimental probability.

Theoretical framework

In this paper we are interested in mapping and assessing P-SSMTs' content knowledge in statistics. Hence, we generally situate our work in the research field of mathematics education as investigating the *common content knowledge* (CCK) within the framework of *mathematical knowledge for teaching* (MKT) by Ball et al. (2008), or more specifically within statistics education research as CCK as understood in the *statistical knowledge for teaching* framework (SKT) by Groth (2013). We use the statistical content in itself as the organizing framework for the analysis, and to be able to capture more nuanced aspects of P-SSMTs' CCK in statistics, we structure the content within the CCK in statistics by departing from the five ”big ideas of statistics” discussed by Pfannkuch and Ben-Zvi (2011): *data, patterns in data, variability, distributions,* and *inference*. These five content topics are not disjoint, but rather important and intertwined facets of what it means to engage in statistical inquiry. For example, and as discussed by Franklin et al. (2007), *probability* and *sampling* are important aspects permeating and connecting all these five ”big ideas”. In addition, and in light of the recent developments within the statistics education research community, we also consider it productive to divide inference to be either *formal* or *informal* (cf. Makar & Rubin, 2009). Hence, we in this paper conceptualize and structure CCK in statistics into 5 areas related to *probability, sampling, distributions, informal inference,* and *formal inference*.

Methodology, method and research setting

To answer the two research questions we draw on previous, but largely unpublished, research experiences based on a research instrument constructed from previously published research and well-documented instruments. The general idea behind this compiled instrument, which we call CIiS (Concept Inventory in Statistics), was to provide a snapshot of the test-taker's CCK in various areas and topics within statistics. Hence, we designed our study around the adaptation and use of the CIiS as a pre- and post-test given to P-SSMTs before and after their first university course in statistics. Before elaborating on the CIiS instrument and its construction further, we will first briefly describe the research setting.

The research setting

The P-SSMTs participating in the study were enrolled in a teacher education programme at a Swedish university. Before taking the pre-test, the P-SSMTs had studied one semester of mathematics covering topics such as algebra, linear algebra, calculus and mathematics education. None of these courses included any statistical content. However, all the P-SSMTs had completed a section on statistics as part of their upper secondary schooling. The upper secondary level mathematics syllabus (Skolverket, 2011b) includes topics such as: *Statistical methods for reporting observations and data from surveys, discussion of correlation and causality, methods for calculating different measures of central tendency and measures of dispersion including standard deviation,* and *properties of normally distributed material*.

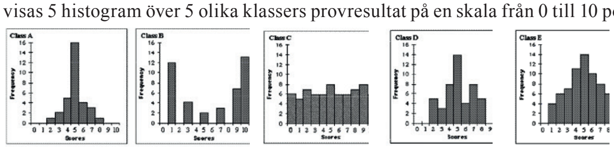
The pre- and post-test were administrated before and after the P-SSMTs took a 8-week course in statistics (among other courses), that was composed of 12 lectures, 12 lessons and 1 laboratory activity using statistical software. The course used and was structured around the textbook by Britton and Garmo (2012), and covered content such as *stochastic variables, probability distributions, expectation values, variance, covariance and correlation, normal and binomial distributions, uncertainty associated with parameter estimation* and as *confidence intervals*.

The pre-test was given to $n_{pre} = 30$ P-SSMTs the day before the first lecture in the statistics course. The time allocated for the pre-test was originally two hours, but since all P-SSMTs finished the CIiS within an hour, only one hour was allocated for the post-test. The post-test data was collected before the final written exam in an extra and voluntary review session for the final exam. Hence, participating in the post-test was not mandatory for the P-SSMTs, and resulted in $n_{post} = 17$ P-SSMTs taking the post-test.

The instrument

The CIiS instrument used for the pre- and post-test was originally compiled as a preliminary diagnostic and design tool for two in-service courses on the teaching and learning of statistics for teachers in the US at the upper secondary level with various backgrounds (Lee et al., 2013). The instrument consists of 30 multiple-choice items organized in 20 question selected from other prior validated instruments. Each item only had one correct answer, but some items have two alternatives (8 items), others three (8 items), four (12 items), or five alternatives (2 items). Of the 30 items on the CIiS, 19 come from the CAOS 4 instrument [*Comprehensive assessment outcome in statistics*] (delMas et al., 2007), eight from the instrument ARTIST [*Assessment resources tools for improving* *statistical thinking*] (http://app.gen.umn.edu/artist/), two from Zeiffler et al. (2008), and one item was added by the instructors of the US in-service teacher course as a complement to one of the CAOS 4 items. See delMas et al. (2007) for a detailed discussion how the items in CAOS 4 and ARTIST were developed, tested and validated. All the 30 items, originally written in English, were translated into Swedish by one of the authors and then proof-read and validated by the other author and an experienced department colleague. Examples of multiple-choice items from the CIiS is presented in figure 1 and 2 below.

Nedan visas 5 histogram över 5 olika klassers provresultat på en skala från 0 till 10 poäng.



Vilken av klasserna har störst standardavvikelse och varför?

Klass A, eftersom detta histogram har störst skillnad i höjd mellan staplarna.
Klass B, eftersom många av provresultaten ligger långt ifrån medelvärdet.
Klass C, eftersom detta histogram har störst antal olika resultat.
Klass D, eftersom fördelningen av resultat är spretig och ojämn
Klass E, eftersom histogrammet har stor spridning av resultat och ser normalfördelat ut.

Figure 1. *Item 5 on the CIiS pre- and post-test: reading and describing a distribution (using standard deviation)*

To investigate the P-SSMTs' CCK in statistics, seven subscales were compiled based on: (1) our conceptualization of CCK in statistics (*probability, sampling, distributions, informal inference, formal inference*); (2) the specified measured learning outcomes of the individual CAOS 4 items in delMas et al. (2007); and (3) the explicit organization and categorization of the items in the ARTIST material – see table 1 below.

Ett förtag som håller på med familjespel har tagit fram en liten plasthund som man kan kasta som en tärning. Hunden kan antingen landa med alla fyra benen mot underlaget, helt på rygg, eller på sin högra- eller sin vänstra sida. Företaget vet dock inte hur stor sannolikheterna är för de fyra olika utfallen. Vilken av nedanstående metoder är mest lämplig om företaget vill uppskatta de olika sannolikheterna för hur hunden kan landa?

A. Eftersom det finns fyra olika utfall, så kan har varje utfall tillskrivas en sannolikhet av 1/4.
B. Kasta hunden många gånger och beräkna hur många procent av gångerna som de olika utfallen inträffar.
C. Simulera situationen och ta fram data baserat på en modell som har fyra lika sannolika utfall.
D. Inget av ovanstående alternativ.

Figure 2. *Item 7 on the CIiS pre- and post-test: probability*

Result

Table 2 in the Appendix shows the P-SSMTs' result on the 30 items on the CIiS pre- and post-test. We first summarize the P-SSMTs' CCK in statistics in terms of the pre-test scores for all the $n_{pre} = 30$ P-SSMTs, and then discuss the scores of the $n_{post} = 17$ P-SSMTs completing both the pre- and post-test.

Overall scores on the CIiS pre-test

The distribution of the $n_{pre} = 30$ P-SSMTs' overall score on the CIiS pre-test is displayed in figure 3. Out of a maximum score of 30, the $n_{pre} = 30$ P-SSMTs achieved in average a score of $\bar{x} = 17.13$. The standard deviation was $\sigma = 3.46$. The 25th percentile is 14.75 and the 75th percentile is 20.00.

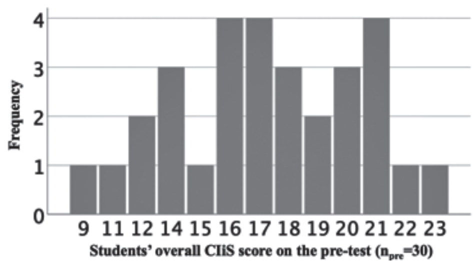


Figure 3. *P-SSMTs' overall CIiS scores on the pre-test*

The three items on the CIiS pre-test on which the P-SSMTs were most successful focused on the *probable sampling outcomes given an explicit distribution of a population* (93 %, item 12B), *interpreting a probability statement in a real world context* (90 %, item 8), and *drawing an inference based on the outcome of a described experiment* (87 %, item 9A). The four items on which the P-SSMTs scored the lowest involved *how errors due to sampling affects inferences about a population mean* (13 %, item 11), *understanding the graphical description (histogram) of a quantitative variable* (20 %, item 2), and *interpretations of confidence intervals* (27 %, both item 17B and 17D respectively).

CIiS subscales scores on the pre-test

Table 1 below displays the P-SSMTs pre-test scores on the CIiS seven subscales, and shows that the P-SSMTs were most successful in items related to *probability* (77 % success rate), *issues involving sampling* (71 % success rate) and the *general logic of making statistical inferences* (67 % success rate). The areas in which the P-SSMTs struggled the most were *confidence intervals within making statistical inferences* (41 % success rate), *informal inference* (47 % success rate) and *distributions* (49 % success rate), which also is reflected in the success rate of the individual items pointed in previous section.

A pre- and post-test comparison on the CIiS overall scores

Looking at the overall score on the pre- and the post-test for the $n = 17$ P-SSMTs who completed both tests, the P-SSMTs' average scores changed

Table 1. *P-SSMTs' results on the seven subscales on the CIiS pre- and post-test*

Subscale (max subscale score: items on test)	**Pre** (n=30) % ($\bar{x}$; σ)	**Pre** (n=17) % ($\bar{x}$; σ)	**Post** (n=17) % ($\bar{x}$; σ)
Probability (2: 7, 8)	77 % (1.53; 0.63)	82 % (1.65; 0.61)	88 % (1.71; 0.50)
Distributions (4: 1–4)	49 % (1.97; 0.93)	70 % (2.18; 1.07)	44 % (1.76 ;1.03)
Sampling (7: 5,6,12A-B,13,14A–B)	67 % (4.80; 1.40)	71 % (4.94; 1.39)	89 % (6.24; 0.90)
Informal inference (3: 10A-B,11)	47 % (1.40; 0.73)	49 % (1.47; 0.72)	39 % (1.18; 0.73)
[a]SI: general logic (3: 9A-B, 20)	71 % (2.13; 0.82)	65 % (1.94; 0.82)	80 % (2.41; 0.80)
[a]SI: confidence intervals (6: 17A-D; 18, 19)	41 % (2.47; 1.38)	51 % (3.06; 1.30)	60 % (3.59; 1.73)
[a]SI: *p*-values (5: 15, 16A-D)	57 % (2.83; 1.18)	39 % (1.94; 0.83)	48 % (2.41; 0.80)

Note. [a]SI is and abbreviation used for *Statistical Inference*

non-significantly ($t(16)=0.566, p=0.579$) from ($\bar{x}=18.12; \sigma=3.18$) to ($\bar{x}=19.65$; $\sigma=3.78$). The individual score on 18 of the items increased on the post-test, whereas 11 scores decreased and the score on one item stayed the same.

A large gain can be seen on item 6 focusing on *expected patterns in sample variability* were 53 % of the P-SSMTs who answered incorrectly on the pre-test got the answer right on the post-test (more than doubling the success rate from 36 % to 77 %). Also on item 17D (on *confidence intervals*) a large portion (44 %) of the P-SSMTs who got this wrong on the pre-test answered correctly on the post-test, resulting in an overall increase from 35 % to 70 % on the item. Large gains can also be found (see table 2) for items 17B (*confidence intervals*), 14A, 14B (on *sample size*) and 5 (*sample variability*), and it is notable that the score went up for four of the six items focusing on *confidence intervals*. The gains in item 6, 14B and 17D are all statistically significant on the $p=0.05$ level.

Among the 11 items on which the score was lower on the post-test compared to the pre-test, is item 11. Item 11 is about *informally rejecting a null-hypothesis*, and the success rate on this item went down from 24 % to 6 %. Indeed, 77 % of the P-SSMTs answered incorrectly on item 11 on both tests. In table 2 one can also see what portion of the P-SSMTs changed from correct answers on the pre-test to incorrect answers on the post-test (and in this context items 1, 4 (both *describing a distribution*) and 10A (*informal inference using boxplots*) are notable). In addition, one can note that the score for four of the six items focusing on *p*-values became lower on the post-test. The only statistically significant negative change in score on the $p=0.05$ level was found for item 1.

A pre- and post-test comparison on the CIiS subscale scores

Table 1 above shows that the P-SSMTs increased their CCK in statistics as mea-sured by the CIiS in five of the seven subscales. The largest gain ware in the subscale *Sampling* (from 71 % to 89 %) and in *Statistical inference: general logic* (from 65 % to 80 %). However, the P-SSMTs' CCK in statistical went down in the subscales *Distributions* (from 70 % to 44 %) and *Informal inference* (from 49 % to 39 %). Only the increase measured in the subscale *Sampling* was statistically significant ($t(16) = -3.096$, $p = 0.007$).

Discussion

Regarding RQ1 our result shows that the P-SSMTs participating in the study had relatively good CCK in statistics with respect to *probability, sampling* and *the general logic of making statistical inferences*, but poorly handled *confidence intervals, informal inferences* and *distributions*. The latter is in line with previous research results (Lovett & Lee, 2018) and is perhaps not surprising since neither confidence intervals, informal inferences and distributions normally are part of the P-SSMTs prior educational experiences. The P-SSMTs relative high average score on items related to probability is in line with the result of Nilsson and Lindström (2013) regarding that the participants displayed a basic understanding of the theoretical interpretations of probability. However, the CIiS subscale measuring probability is composed of only two items, and hence only provide a crude and selected snapshot of the participates CCK with respect to probability. It is interesting that *p*-values, otherwise pointed out as troublesome for P-SSMTs in the research (Lovett & Lee, 2018), not stood out as difficult in the pre-test.

After having participated in the course in statistics, the results show that with respect to RQ2, the P-SSMTs CCK in statistics increased between the pre- and post-test in all areas except those related to *distributions* and *informal inferences*. Although somewhat speculatively, given the limited amount of data, the decrease in the subscale *informal inference* taken together with the increase in the three subscales of *formal statistical inference*, indicate that the course in statistics favor the formal aspects of statistics over more informal ways of making inferences. From a general CCK perspective focusing on the formal aspects of mathematics, this makes sense. However, in light of more recent discussions within the statistics education community, informal inference is suggested to be more important and productive for the teaching and learning of statistics (i.e. an important component of so-called *specialized content knowledge* (SCK) to be develop within the MKT and SKT frameworks). The low scores on the subscale *informal inference* point to the need to provide the P-SSMTs with learning opportunities to develop their informal inference reasoning skills, either in the statistics course, or in an accompanying mathematics education course. In such a (re-)design and development project, a research-based extension of the CIiS scale on informal inference might be a useful tool, which in addition would provide interesting research opportunities. In its present form the CIiS only provide a selective snapshot of the CCK in statistics, with a small number of items in each subscale. Hence it is hard to generalize the findings in this study, and the result must rather be interpreted in relation to the particularities of the research settings at hand.

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Appendix

Table 2. *P-SSMTs' results on the pre- and post-test*

Item #	$Pre_{n=30}$	$Pre_{n=17}$	$Post_{n=17}$		Item response pattern[a]			
	% correct	% correct	% correct	*n*	Incorrect (%)	Decrease (%)	Increase (%)	Pre & Post (%)
1	56.7	76.5	*47.1*	16	18.8	31.3	0	50
2	20.0	29.4	*17.6*	16	62.5	18.8	12.5	6.3
3	46.7	47.1	**58.8**	17	23.5	17.6	29.4	29.4
4	73.3	64.7	*52.9*	16	12.5	31.3	18.8	37.5
5	56.7	64.7	**88.2**	17	5.9	0	35.3	58.8
6	40.0	35.3	**76.5**	17	11.8	11.8	52.9	23.7
7	63.3	70.6	**76.5**	17	11.8	11.8	17.7	58.8
8	90.0	94.1	94.1	17	0	5.9	5.9	88.2
9A	86.7	76.5	**94.1**	17	0	5.8	23.5	70.6
9B	66.7	58.8	**76.5**	16	6.3	12.5	31.2	50
10A	63.3	64.7	*35.3*	15	13.3	46.7	20	20
10B	63.3	58.8	**76.5**	15	6.7	6.7	26.7	60
11	13.3	23.5	*5.9*	17	76.5	17.7	0	5.9
12A	83.3	94.1	*88.2*	17	0	11.8	5.9	82.4
12B	93.3	94.1	94.1	17	0	5.9	5.9	88.2
13	83.3	82.4	**88.2**	17	0	11.8	17.7	70.6
14A	66.7	64.7	**94.1**	17	5.9	0	35.3	58.8
14B	56.7	58.8	**94.1**	17	5.9	0	35.3	58.8
15	83.3	82.4	82.4	17	0	17.7	17.7	64.7
16A	60.0	52.9	**64.7**	16	18.8	12.5	25	43.8
16B	26.7	29.4	*23.5*	16	56.3	18.8	18.8	6.3
16C	76.7	88.2	*82.4*	16	6.3	6.3	6.3	81.3
16D	36.7	35.3	*23.5*	16	56.3	18.8	12.5	12.5
17A	60.0	58.8	*47.1*	15	26.7	20	13.3	40
17B	26.7	29.4	**52.9**	15	33.3	6.7	40	20
17C	36.7	52.9	**64.7**	15	20	6.7	26.7	46.7
17D	26.7	35.3	**70.1**	16	25	0	43.8	31.3
18	66,7	82.4	*70.1*	16	0	25	18.8	56.3
19	30.0	47.1	**52.9**	16	31.3	12.3	18.8	37.5
20	60.0	58.8	**70.6**	14	0	21.4	28.6	50

Note. [a] Following DelMas et al. (2007) item response pattern reported are: Incorrect = incorrect on both pre- and post-test; Decrease = correct pre-test, incorrect post-test; Increase = incorrect pre-test, correct post-test; Pre & Post = correct on both pre- and post-test (**bold** indicate an increase; *italic* a decrease).

They saw and dared to call things mathematics: facilitators' views on an online mathematical professional development module

Troels Lange and Tamsin Meaney

Although much money is expended on developing professional development resources, little is known about how facilitators, who often mediate the materials for teachers, evaluate them and how these evaluations compare with those of the teachers. To provide input into this area, the results from a survey completed by 59 facilitators of an online mathematics module for preschool teachers are described and compared with those of preschool teachers. Although the facilitators gave similar responses about the three design elements of the module – content, tasks and relationships – they also identified areas, which were not covered by what was in the module itself.

In this paper, we analyse the results of a survey completed by 59 facilitators about their experiences with online, professional development (PD) modules for preschool teachers to gain insights into the design of these materials. We compare the facilitator results with those of teachers who had undertaken the course. Identifying differences as well as similarities in views provides insights into whether the facilitators' understandings of the materials are shared by the teachers and if the materials should be altered to increase the impact on teachers' learning.

Recently, preschool teacher PD has received attention with a special issue on this topic being published in 2017 in *Mathematics teacher education and development*. Most of these articles have focused on changes to teacher practices or knowledge. Of the few papers on facilitators of PD for preschool, the focus has been on the training of the facilitators (Hassidov & Ilany, 2014). Nevertheless, calls have been made to focus more on the important role of facilitators in mathematics PD (Lange & Meaney, 2013). This is particularly necessary when facilitators mediate online PD materials developed by others as "little is known about best practices for the design and implementation of these oTPD (online teacher professional development) models" (Dede et al., 2009, p. 9). Hill et al. (2013) called for an evaluation of PD design elements from a range of teachers and facilitators to build up a body of knowledge that moves beyond local, idiosyncratic approaches to implementation. As more online PD materials are made

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available in the future (Dede et al., 2009), there is a need to better understand from users, such as facilitators and teachers, what are the design features that effectively support PD.

Background

In 2012, Sweden initiated a large-scale PD programme for mathematics teachers. Administered by the Swedish Agency for Education (Skolverket), teachers in collegial groups read, discussed, planned and tried activities and reflected on their experiences. The input for these tasks comes from web-based PD modules, which for preschool teachers have 12 parts to be completed over 18 months. All modules have the same structure with each part comprising four sections: (A) individual studies; (B) group discussion and planning; (C) enactment/observations in own teaching situations; and (D) group discussion and follow-up.

In previous research (Helenius et al., 2015), we described a design model for PD materials based on our decision-making during the production of the first third of the PD-module materials. The design model includes 3 elements: content deemed as important for teachers to understand; tasks for the teachers to undertake; and the relationships that needed to be nurtured. We consider that the choice of content is the driver of the other two elements. For the preschool module, the content was based on Bishop's (1988) six mathematical activities, because curricula and policy documents indicated that this was the knowledge teachers needed (Skolverket, 2011; Utbildningsdepartementet, 2010).

The 12 parts of the preschool module that provide input on the content are: (1) Introduction to Bishop's (1988) six mathematical activities; (2) Playing; (3) Explaining; (4) Documenting what the child can do; (5) Introduction to Locating and Designing; (6) Locating; (7) Designing; (8) Documenting for teacher planning; (9) Introducing quantifying; (10) Measuring; (11) Counting; and (12) Documenting for supporting the work environment. Each set of four parts had an introduction, two parts related to Bishop's six mathematical activities and a summary part which discussed different aspects of documentation.

Although Skolverket financed facilitators to work with school teachers, similar funding was not made available for preschool teachers. The teachers were instead expected to organise themselves into groups, which would then work through the online materials. However, when the materials were made available, many municipalities funded facilitators to organise PD sessions for groups of preschool teachers. However, at this point in time, there was no training specifically for these facilitators.

We collected data through an online survey for teachers and one for facilitators, from March to May 2016. The surveys included similar questions, modified to suit the different roles of teachers and facilitators. Contact was made with municipalities, across Sweden, where it was known that their preschool teachers had completed at least 4 out of the 12 parts of the module (the equivalent of at least 6 months' worth of work). The municipalities then forwarded the link for the online survey to teachers and facilitators.

Both surveys had 27 questions, 21 were multiple-choice and the other six were open-ended. Both surveys asked for information about the mathematics that participants had in their teacher education and their years of experience working in preschools. The open-ended questions provided comments from the facilitators about the three elements of the design model; content, tasks and relationships.

Results and discussion

In this section, the results from the facilitators are compared with the results from 267 preschool teachers, published in Helenius et al. (2017). We do this to gain insights into whether the facilitators' understandings of the materials were the same as the teachers and if the materials could or should be altered to increase the impact on teachers' learning.

The facilitators and the teachers (Helenius et al., 2017) had similar amounts of mathematics in their teacher education, similar amount of experience of working in preschools, and had completed similar amounts of the online module. Of the 59 facilitators, 8 (14 %) had less than five years' experience working in preschools, 8 (14 %) had between five and ten years and 43 (73 %) had more than ten years. 29 (49 %) of the facilitators had no mathematics in their teacher education, 27 (46 %) had 15 ECTS and 2 (3 %) had 30 or more ECTS; one did not respond to the question. 50 (85 %) of the facilitators had completed eight or more parts of the module and the rest at least four parts. A similar proportion of teachers had finished at least four parts, but 12 % had competed all twelve parts. In the next sections, we describe the results for the three elements: content, tasks and relationships.

Content

The facilitators provided information on content by answering questions about which parts of the module they appreciated the most and the least. The questions allowed for more than one part to be nominated. As was the case for the teachers, far fewer facilitators indicated a part of the module that they appreciated the least than a part that they appreciated the most.

The parts that were nominated by more than 20 % of the facilitators as being appreciated the most, were: (2) Playing, (6) Locating, and (7) Designing, which was nominated by 30 %. These results are similar to those of the teachers (Helenius et al., 2017) and indicated that the parts appreciated the most included content specifically about Bishop's (1988) six mathematical activities.

The parts that the facilitators liked the least were (5) Introduction to Locating and Designing, (6) Locating, and (8) Documenting for teacher planning, which were nominated by 14 % of facilitators. Although these results differ from those of the teachers (Helenius et al., 2017), the numbers are too small to indicate a clear trend. However, far fewer facilitators, like the teachers, nominated a part they liked least compared to the percentages who nominated parts they liked the best.

Locating featured in both the most and least appreciated part, which was also the case for the teachers (Helenius et al., 2017). This indicates that more information is needed about why Locating produced such divided views. In the open-ended question, one facilitator (F48) explained why they liked Locating the most, ”jag uppskattade mest ’lokalisera’, det satte igång både lärarens och barnens fantasi” (I appreciated ”Locating” the most, it initiated both the teacher's and the children's imagination). However, there were no comments about why it was liked the least, indicating that further research is needed.

It could be that the facilitators needed more time to better understand the ideas to do with Locating. Like some teachers (Helenius et al., 2017), several facilitators indicated that it was only after completing several parts that they could understand how the parts were related. For example, F15 wrote ”det har varit trögt innan begreppen lagt sig och fått förståelse för arbetslaget. pga att det går fort fram blir det rörigt varje gång en ny [del] inleds”. (It has been slow before the concepts have settled and gained understanding in the work team. Because it progresses quickly, it gets messy every time a new [part] is started).

Other comments, similar to F15, suggest that the facilitators found the content to be challenging for the teachers in that it extended their combined understandings about what children could do mathematically in preschools. F37's response to the question, about which part they appreciated the most, illustrates this point:

F37: Väldigt svårt att välja en! Spontant så skulle jag säga att delen om leka och förklara är det som gett oss mest. Detta med tanke på att man ofta inte tänker de delar som matematik”

Very difficult to choose one! Spontaneously, I would say that the part about playing and explaining is what has given us the most. This is because you often do not think of these parts as mathematics.

Other facilitators talked about witnessing ”aha experiences”, suggesting that while the content may have been challenging, it provided opportunities for the teachers (and the facilitators) to gain new insights. Many also indicated that the teachers were able to see mathematical learning opportunities in their work and were more willing to discuss mathematics together, than they had been earlier. The results in table 1 showed the responses the facilitators gave to the questions about why a part was appreciated the most or the least.

More facilitators indicated that it was the tasks, including the discussions, rather than what the teachers learnt that made them appreciate a part (this is discussed in more detail in the next section). Half of the facilitators indicated that they chose a part as being least appreciated if they considered the written

Table 1. *The reasons for why a part was most or least liked by the facilitators*

	The teachers learnt the most / least from it	The written texts and films made it easier/ harder to understand the message	The activities with children clearly showed/ did not show how much mathematics they can do	The discussions with colleagues facilitated (not) understanding of the message in the part
Most (*n*=53)	15 (28 %)	12 (23 %)	34 (64 %)	28 (53 %)
Least (*n*=44)	15 (34 %)	23 (52 %)	6 (14 %)	8 (18 %)

texts and films in sections A and B to be difficult to understand. This is perhaps not surprising as the texts and films conveyed the content. In a response to the open-ended question, F30 wrote, ”Tycker att alla moduler varit bra, med en del texter med svåra ord att bena ut kanske uppskattades minst” ([I] think all parts have been good, with some texts with difficult words to work out maybe being appreciated the least).

Tasks

The module tasks were situated in the four sections, A, B, C, D, in each of the 12 parts. As designers, we considered that the tasks should be specific and connected to the context. To do this, we considered: How can the affordances of context and artefacts be utilised to support content delivery? Why would teachers want to engage in these activities? Therefore, we asked the facilitators about what they considered supported or hindered the teachers' learning in two survey questions where they could choose more than one answer. Tables 2 and 3 show their responses as well as the compatible teacher responses from Helenius et al. (2017).

Table 2. *The tasks that the facilitators considered to contribute the most to teachers' learning*

	Materials	Discussion with colleagues	Trying out activities with children	Documentation of own and children's learning
Facilitators (*n*=59)	18 (30 %)	51 (86 %)	47 (80 %)	26 (44 %)
Teachers (*n*=255)	53 (21 %)	201 (79 %)	141 (55 %)	76 (30 %)

Table 3. *The difficulties that hindered learning from the PD tasks*

	Time to do PD	Texts too hard	Film not relevant	Activities too difficult to implement	Activities not appropriate for children's group	Discussions not helping learning
Facilitators (*n*=56)	47 (84 %)	30 (54 %)	20 (36 %)	8 (14 %)	17 (30 %)	2 (4 %)
Teachers (*n*=245)	204 (83 %)	105 (43 %)	73 (30 %)	51 (21 %)	52 (21 %)	3 (1 %)

The facilitators and the teachers valued the tasks to a similar degree (see table 2). Although the facilitators appeared to be generally more positive, the differences were not statistically significant using a chi-square test. The facilitators considered that discussions with colleagues and trying out tasks with children contributed the most to the teachers' possibilities to learn.

In contrast, table 3 shows that a lack of time was considered the biggest hindrance to teachers' possibilities to learn by the facilitators and the teachers (Helenius et al., 2017). However, as F13 noted, facilitators acknowledge that a lack of time was often compounded with other issues outside the scope of the module:

> För att kunna förändra den rådande praktiken måste man förstå. För att kunna och vilja förstå behöver man vara intresserad. Fokuserar man på annat t.ex. tidsbrist stänger man in sig i ett hörn, tyvärr. Olika utbildningsnivå har påverkat väldigt mycket.
>
> In order to change the current practice, one must understand. To be able and willing to understand, you need to be interested. Focusing on other things e.g. lack of time, pushes one into a corner, unfortunately. Different levels of education have had a great impact.

Of the tasks that the teachers were expected to engage with, the facilitators, as had the teachers, highlighted the impact of texts, which were too hard to read, on teachers' possibilities for learning. Facilitators would be expected to support the teachers to understand the text, but with similar education and background to the teachers, they may have struggled with the texts in the same way. As designers of the module, it is important to consider how to provide texts that better supported teachers to engage with the content. Materials written specifically for facilitators, to help them support teachers' academic reading were made available later.

Figure 1 shows that more than 80% of facilitators considered that all the tasks contributed ”quite a lot” or ”very much” to the teachers' learning. This included reading texts and watching the videos, which teachers had also

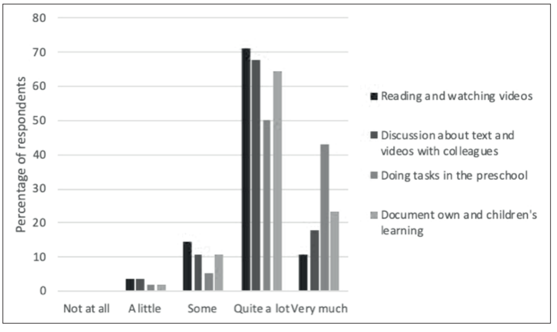

Figure 1. *Graph of contribution to learning by percentage of facilitators*

nominated to the same degree in their responses to this question (Helenius et al., 2017). Some of the facilitators suggested that the concepts in the texts and the films became understandable when they were discussed with the teachers. For example, F46 stated, ”Genom att de diskuterade texterna och filmerna med varandra och försökte hjälpas åt med vissa begrepp som kunde ibland uppfattas vara svåra att begripa” (By discussing the texts and films with each other and trying to help each other with certain concepts that could sometimes be perceived as difficult to understand). When the teachers engaged with the concepts, the facilitators noted that they learnt. F21 summarised the impact of the PD tasks on the teachers' learning by stating ”De såg och vågade att kalla saker för matematik” (They saw and dared to call things mathematics).

In some responses to the open-ended questions, documentation – when completed – was highlighted as a valuable tool for supporting conversations. For example, F35 stated, ”Dokumentationerna gör att vi kunde få syn på sådant som vi inte sett under själva aktiviteten samt kunna diskutera hur vi går vidare” (The documentation means that we could see things that we did not see during the activity itself and are able to discuss how we are going to proceed). However, some comments suggested that the documentation was not completed or only connected to being able to see mathematics everywhere (see Helenius et al., 2017). Not completing the documentation activities could affect teachers' relationships with each other, the facilitators and parents, as discussed in the next section.

Relationships

In designing the module, we recognised that there were a number of relationships that needed to be supported through the materials. One of these was the relationship between us and the teachers. However, when there were facilitators, then they and not the materials mediated the content and tasks to the teachers, with our relationship to both groups having a different role.

Table 4 outlines what the facilitators considered to be the most important part of their role (they could give more than one answer). It is unclear whether the facilitators who nominated ”ensuring the teachers had access to the materials” simply meant distributing the materials or making them understandable to the teachers. However, almost all the facilitators saw supporting discussions, presumably on the materials, as important.

In discussing their role in the open-ended question, the facilitators acknowledged that it was sometimes hard to get teachers to engage in the different

Table 4. *The tasks that the facilitators considered to be most important*

	Ensure teachers have access to the material	Support discussions in sections B and D	Support the practical tasks in section C	Evaluate the documentation
Facilitators (*n*=59)	37 (63 %)	51 (86 %)	18 (31 %)	23 (39 %)

tasks. For example, F38 wrote ”försökte men svårt då jag har kollegor som inte tycker att detta varit så roligt” (tried but hard when I have colleagues who do not think this was so fun). In this case it seemed that the online materials were not in themselves sufficient stimulus for the teachers to engage with the PD. In contrast, other facilitators mentioned that the materials seemed to exert pressure on the teachers to engage in the tasks. For example, F18 stated, ”Man ’tvingas’ in i uppgifter – bra att få lite press på sig så att det blir gjort. De har ändå sett vad mycket matematik det finns och går att få in i verksamheten” (One is ”forced” into tasks – good to have some pressure put on you so that it gets done. They have nevertheless seen how much mathematics there is and that can get into the work situation). When the teachers did engage with the materials, the facilitators considered they learnt more.

As the designers of the PD materials, we had tried to engage the teachers by ensuring that the materials made use of the teachers' previous experiences. 41 out of 56 facilitators (73 %) considered that the materials did this quite a lot or very much. However as noted earlier, the facilitators identified difficulties with comprehending the texts, which could be considered as occasions where we, as the designers, had misunderstood the teachers' competencies. To overcome these difficulties, the facilitators often mentioned the role of discussions.

F37: Att få höra vad andra fastnat för i texten, både det som är lätt och svårt har gett en ökad förståelse för textens innehåll. Att få förklara det man kan för någon annan ger en större förståelse. Kollegialt lärande är toppen, tillsammans är vi starka!

Being able to hear what others got stuck with in the text, both what is easy and difficult, has given an increased understanding of the content of the text. Explaining what you can [i.e. understand] to someone else gives you a bigger understanding. Peer learning is the top, together we are strong!

As the designers of the module, we had wanted the materials and tasks to support the teachers to develop their relationships with children, colleagues and parents. Figure 2 suggests that the facilitators indicated that this was the case, although the impact was the most with children and the least with parents. Similar results arose from the teacher survey (Helenius et al., 2017).

Many of the facilitators commented about how teachers' relationships with children could be extended by challenging the children mathematically, ”Att lärarna tar tillfällen i akt att lyfta barnens matematiska kunnande, mycket mer spontant än tidigare” (The teachers take the opportunity to raise the children's mathematical knowledge much more spontaneously than before) (F56). For this facilitator, the materials provided support for teachers to identify mathematical learning opportunities for the children, which may have gone unnoticed earlier.

It may be that teachers saw increasing their relationships with children as their main role, but had difficulty considering how mathematics education could increase their relationship with parents. Therefore, there was a possibility for

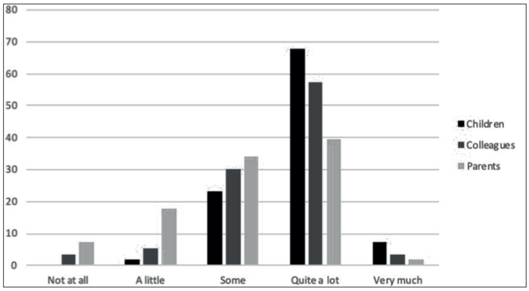

Figure 2. *Facilitator views on how the module affected teachers' relationships*

facilitators to emphasise this through mediating the materials, which might have been missed by teachers, focused on what they considered to be their main roles. Nevertheless, some facilitators noticed that working with the module did provide teachers with new ways to engage parents in discussions about their children's mathematics. For example, F20 stated, "Vi har blivit bättre på att tala om för föräldrarna hur och varför vi arbetat med matematiken i vår vardag" (We have become better at telling parents how and why we worked with the mathematics in our everyday). Some facilitators noted that documentation tasks in the module supported teachers to have materials that facilitated conversations with parents. However, other facilitators noted that it was not always easy to talk about the children's mathematics with their parents, especially when they had another mother tongue, "de har i princip annat hemspråk allihopa och har svårt att förstå" (They basically all have different home languages and have difficulties understanding) (F19). The materials did not provide suggestions for overcoming this problem.

Conclusion

Hill et al. (2013) challenged the PD community to gather evidence from a range of different contexts to consider how design elements operate in different local situations. The Swedish PD programme for preschool teachers provides one such context. Comparing the results from the facilitators with those of the teachers (Helenius et al., 2017) indicates that there were many similarities in the responses. In regard to the content, the facilitators like the teachers considered that the content gave the teachers new insights into mathematics education in preschool. However, sometimes time was needed for the teachers to understand how to connect the new ideas to what their experiences in preschools were and it was the facilitators who had to keep the teachers motivated, while they adapted their thinking.

In regard to the tasks, the facilitators noted that discussions with colleagues were very useful in supporting the teachers to make sense of their reading. Doing things with children also resulted in the teachers experiencing

”aha-moments”. Although the reading of the texts was seen as an important part for gaining these moments, there is some research to be done on how to support the facilitators to do this. In particular, more needs to be known about how facilitators mediate the materials as well as how we as the designers can rewrite some of the texts to make them more easily understood, but ensure they introduce the teachers to new ideas. This could ease the work of the facilitator in motivating the teachers to persevere in reading and making sense of the texts. According to the facilitators, doing activities with the children seemed to support teachers to elaborate on their relationships with children. To a lesser extent, the facilitators considered that the materials supported the teachers to engage with parents about the mathematics education the children were doing. Again, research is needed to investigate how to improve the possibilities in the materials to better support relationships with parents but also to provide support to facilitators so they can better support what opportunities that were already there.

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Critical aspects of equations when explored as a part-whole structure

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The aim of this paper is to present critical aspects that were identified when students explored equations as a part-whole structure with negative numbers included. Students in grades 3, 8 and 9 participated in a ”theoretical work”. Learning study was used as a research approach and learning activity theory constituted a guiding principle when designing research lessons. According to the analysis, five critical aspects were identified, regardless of grade. The critical aspects are: there is a relationship between all the numbers in an equation; two parts together equals a whole with the same *value*; what constitutes the parts and the whole, respectively; the same relationship can be formulated in four different ways; the whole can assume a lower value than the parts.

The aim of this paper is to present critical aspects that were identified when students explored relationships, as a part-whole structure, between numbers in equations (e.g. Schmittau, 2005). Accordingly, the critical aspects concern what students need to discern in order to learn how the numbers in an equation relate to each other (cf. Davydov, 2008). The equations consisted of additive structures (Vergnaud, 1982) and included negative numbers (integers). One challenge concerning teaching negative numbers may be that it is not straightforward to explore them empirically by quantities (Schubring, 2005). A reason for extending the numbers to negative numbers in this study was to challenge an assumption that subtraction tasks always lead to the difference consisting of a lower value than the minuend, and that addition tasks lead to the sum consisting of a higher value than the addends (Bishop et al., 2014). Another reason for extending the numbers was to challenge students not to just ”know” or ”see” the ”answers”. Challenging these assumptions was taken as a way to afford a strong focus on the relationship between the numbers in an equation rather than a focus on calculations.

One theme discussed in the literature is that when the focus is only on calculations based on rules and procedures, this may lead to students not being given an opportunity to reflect on mathematical structures beyond the rules

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and procedures at hand (Kilpatrick et al., 2001; Mason et al., 2012). Although students can solve routine tasks, based on rules and procedures on some occasions, challenges may occur when handling similar tasks in new situations when underlying structures are not discerned (Brown et al., 1988). Knowing about the inverses of addition and subtraction may be of importance, since subtraction is frequently declared in previous research as more difficult arithmetic than addition (e.g. Baroody, 1984; Brissiaud & Sander, 2010). Previous research has focused on addition and subtraction tasks, based on general structures as a part-whole structure (e.g. Carpenter et al., 1981). A part-whole structure can be depicted as the whole is built up by parts (Carpenter & Moser, 1982; Schmittau, 2005). Attributing a part-whole structure as a relationship between numbers by general symbols and not by specific values, means there is nothing to calculate, which may support students to focus on the general structures (Davydov, 2008). Focusing on general structures requires first and foremost to notice the structure and to analyse relationships between quantities and between numbers (Cai & Knuth, 2011; Kieran, 2018).

In this paper, we will answer the research question *What do students need to discern in order to master equations based on relationships between the numbers?* The findings discussed in the paper are in conjunction with two other articles dealing with contents closely related to each other (Andersson & Tuominen, in progress; Tuominen et al., 2018).

Methodology

The study was conducted with learning study as research approach, since it offers a basis for interventions building on systematical and iterative processes (Marton, 2015). This was of importance in order to identify what students need to discern in order to master equations based on relationships between the numbers. In order to address the research question, the notion of critical aspects was adopted (cf. Marton, 2015; Marton & Booth, 1997). *Critical aspects* is a core concept deriving from variation theory, a theory of learning. In order to learn the intended, in this case, relationships between numbers in equations (the object of learning), there are necessary aspects – critical aspects – to be discerned. Critical aspects are relational, which means there is an interconnection between the students, the object of learning, and in which ways the students experience the object of learning (Pang & Ki, 2016). In this study, we coordinated variation theory (mainly critical aspects) with learning activity theory (Davydov, 2008). Eriksson (2017) claims that the two theories are possible to combine in relation to their focus on what students need to learn and how they manage the content. Learning activity theory was used as a theoretical guiding principle when designing the research lessons, and it also provided a lens with regards to the focus when identifying the critical aspects (see Analysis section).

Learning activity theory suited our research interest since it addresses the development of students' consciousness and thinking regarding the theoretical knowledge accomplished through theoretical work. This kind of work is characterized by what Davydov (2008, p. 115) defines as ”[...] contentful abstraction and generalization and theoretical concepts, taken as a unity [...]”. Based on learning activity theory, a starting point is to introduce mathematical content based on general structures and subsequently to exemplify the content with specific numbers though still based on general structures (Davydov, 2008). In the case of our study, the theoretical work concerned relationships between numbers in equations, and, in order to capture and visualize the abstract properties, a learning model (see figure 1) was used (Davydov, 2008; Gorbov & Chudinova, 2000).

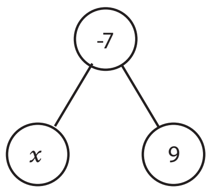

Figure 1. *The learning model used in our study, inspired by Davydov (2008)*

The intention with using the model was to enable students to identify the relationship between a whole and two parts, and how the same relationship can be formulated in four ways (cf. Davydov, 2008). This, in turn, may enable students to discern addition and subtraction as inverses (Greer, 2012).

Students, 149 in total, from grades 3, 8 and 9, attending compulsory school, participated in the study. The different grades were chosen based on the researchers' experiences as teachers. Initially there were two different projects, but as the researchers collaborated they noticed that the same tentative critical aspects were identified, regardless of grade. Due to this similarity, the researchers decided to collaborate in one research project. In total, nine video-recorded research lessons were conducted, while each student participated in one research lesson. According to the students' teachers, as well as findings from interviews with the students (see Tuominen et al., 2018), the students had no experiences of teaching based on relationships between numbers in equations, regardless of the students' different ages. None of the students in grade 3 had previous experiences of negative numbers as operands.

The data material consists of transcribed video recorded research lessons and audio recorded interviews, as well as the students' written expressions from the pre- and post-tests, and lessons. The pre- and post-tests were identical and consisted, largely of the same content, regardless of grade. Some of the students were interviewed after the pre- and post-tests. The intention was to explore how students grasped and managed the different tasks.

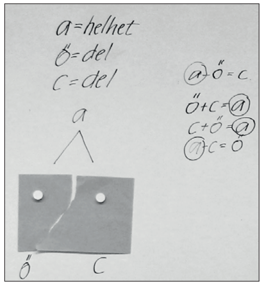

Figure 2. *A relationship between quantities, formulated by general symbols*

Initially, during the research lessons, a relationship between quantities was explored as a theoretical work. As depicted in figure 2, the whole *a* was divided into two parts by empirical material.

The relationship between the quantities was formulated with general symbols, in this case, with *a, c,* and *ö*. The intention with using general symbols initially was to enable students to focus on *the relationship*, not on something to calculate. The model in figure 2, constructed by a participating teacher and some of the participating students, was used as a learning model. Additionally, during each research lesson, the model in figure 1, constructed by Tuominen and Andersson, was used when the students explored relationships between *numbers* in equations, since it is not straightforward to explore negative numbers using *quantities*. This means that the teaching went from the general to the specific, though still as a theoretical work. For example, the students in grades 8 and 9 explored equations such as $-7 - x = 9$ and subsequently formulated the same relationship in further three ways (see figure 3). Based on a part-whole structure, ”x” and ”9” constitute the parts and ”-7” constitutes the whole; that is, ”-7” is built up by ”x” and ”9” (see the second and the third formulations in figure 3). This relationship applies regardless of the four formulations. In this relationship, the whole assumes a lower value than one of the parts, which is valid when negative numbers are included.

$$-7 - x = 9$$
$$x + 9 = -7$$
$$9 + x = -7$$
$$-7 - 9 = x$$

Figure 3. *A seemingly difficult equation, reformulated in three ways*

Discerning that one relationship can be formulated in several ways is about the need to discover that ”[...] any mathematical operation has an unambiguous structure [...]” (Davydov, 2008, p. 148). When students discern that the same relationship can be formulated in four ways, it may enable them to choose a more convenient equation to solve. This can be particularly advantageous when negative numbers are included.

Analysis

In order to identify critical aspects, an initial analytical question was formulated: *What are indications of critical aspects when exploring equations as a part-whole structure in theoretical work?* During the process of analysis, the focus was on what students expressed, orally and in writing. It showed that there was an interplay between the data and previous research concerning critical aspects (Marton, 2015; Pang & Ki, 2016), concerning relationships between numbers (e.g. Davydov, 2008; Schmittau, 2005), and concerning the theoretical work (Davydov, 2008). The analysis was conducted as follows. *First*, the video and audio recordings were listened to several times in order to distinguish expressions regarding relationships. *Second*, the recordings were transcribed verbatim. *Third*, the transcripts were read several times and students' expressions regarding equations and numbers, known or unknown, and mathematical symbols were highlighted. Also the students' written expressions from the pre- and post-tests, and lessons were analysed according to the same criteria. *Fourth*, whether, and in what way, the students expressed a relationship between the numbers in equations (cf. Davydov, 2008; Schmittau, 2005) was analysed, interpreted, and categorized. The guiding principle for this was how, in excerpts and elsewhere, there were qualitatively different ways (see Marton, 2015) of expressing a relationship. This process provided a basis for us to identify critical aspects regarding relationships. *Finally*, the analytical question *What do students need to discern in order to manage numbers in equations as relational?* was adopted. This was to support the process of an additional analysis, where excerpts from students' qualitative different expressions were compiled into five categories where each, ultimately, represented a critical aspect.

An example of how we interpreted different expressions is how excerpt 1 below was read as that Elli did not perceive that a relationship can be formulated in different ways, but rather that the two formulations were two separate equations with no connection between them. The excerpt was regarded as critical aspect number 4. Elli's expression in excerpt 2 below was placed in the critical aspect number 5, since it was interpreted that she did not discern the relationship between the numbers. Further, it was interpreted as if she supposed that equations with addition always lead to a sum with higher value than the addends. The expression in excerpt 2 differs qualitatively from the expression in excerpt 1. The five critical aspects are described in the Findings below.

Findings

The five critical aspects will be presented and exemplified below, through descriptions, excerpts, or figures. The critical aspects concern Davydov's (2008) discussion of students' consciousness and thinking regarding theoretical knowledge and work; in the case of this study, exploring equations as theoretical work. Based on the fact that critical aspects do not concern what students struggle with, but with what enables them to discern necessary aspects (Marton, 2015), there are examples in this section of when students discerned and when students did not discern necessary aspects. The order in which the critical aspects are presented does not imply the need for them to be discerned in that particular order.

1. There is a relationship between *all* the numbers in an equation

The critical aspect *there is a relationship between* all *the numbers in an equation* was manifested in different ways. An instance of when this aspect was possible to identify in the data was that several of the students, regardless of grade, did not express anything regarding the relationship between *all* the numbers. Rather, the students focused on the numbers in relation to the mathematical signs or the position of the numbers in an equation, without consideration of the other numbers included. One example of that is when students in the pre- and post-tests were supposed to formulate the equation $x-5=3$ in several ways. Sometimes, students placed the numbers as, for example, $5-3=x$, which resulted in a different relationship. In these two equations, ”*x*” consists of different values. In the analysis, it was interpreted that the students in the theoretical work were not conscious of, and did not experience, the importance of focusing on *all* the numbers, simultaneously.

2. Two parts together equals a whole with the same value

This critical aspect is based on a critical aspect identified in an analysis by Tuominen et al. (2018); *two quantities together (two parts) build up a third quantity (the whole) with the same ”value” as the two parts together.* The critical aspect identified in Tuominen et al. concerns quantities (in the form of volume). An equivalent critical aspect was also identified in this study. Because it concerns numbers in equations, instead of quantities, it is consequently formulated as *two parts together equals a whole with the same value*. In this analysis, we also formulated the same critical aspect, as *if one of the parts is taken away from the whole the other part is what remains*. Formulating the same critical aspect in different ways means that various perspectives are adopted, from parts or from a whole. An instance of when this critical aspect was possible to identify in the video data was when a teacher and students explored the relationships between quantities by using pieces of paper (figure 2) (cf. Davydov, 2008). During a research lesson in grade 3, students denominated the whole by *a* and the two parts by *c* and *ö* (the letters were chosen by the students) and one student suggested expressing the relationship as $c+c=a$. In the analysis, this was interpreted as the student not experiencing how the whole and the parts, the quantities *a*, *c* and *ö*, were related to each other. Another student in grade 3 expressed: ”if you take *ö* plus *c* it is equal to *a*” and later the same student expressed: ”*a* minus *ö* is equal *c*”. In the analysis, this was interpreted as the student discerning a relationship between the three numbers and how the same relationship can be formulated in two different ways.

3. What constitutes a whole and parts, respectively

Another critical aspect identified in the analysis, is *what constitutes a whole and parts, respectivel*y. An instance of when the critical aspect was found in the data was when students in grade 8, were encouraged in the post-test to mark ”the whole” in four different equations. Below, there is a solution from one student, who marked the ”answer” as the whole. The student's markings are depicted in bold and underlined (figure 4). Although the answer is the whole when it comes to addition, that is not the case when it comes to subtraction.

2 + 7 = **<u>9</u>**
(-7) + 2 = **<u>(-5)</u>**
2 – 7 = **<u>(-5)</u>**
(-7) – 2 = **<u>(-9)</u>**

Figure 4. *The whole experienced as the answer*
Note. The equations are reconstructed due to the poor quality of the original.

In the analysis, the markings in figure 4 indicate that the student did not experience the whole based on a part-whole structure. The student rather experienced the whole as the *answer* in the four different equations.

4. The same relationship can be formulated in four different ways

A further critical aspect is *the same relationship can be formulated in four different ways* (cf. Davydov, 2008; Schmittau, 2005). An instance of when this critical aspect was present in the data was when students in grade 3 were exploring a relationship between the numbers ”*x*”, ”2” and ”3”. On the whiteboard, the teacher wrote two equations showing the same relationship as $x+3=2$ and $2-x=3$. The students were encouraged to identify the whole and the parts, supported by the model (figure 1) and to formulate the relationship in four ways.

Excerpt 1, grade 3

Teacher:	What is the whole? [The teacher points to the whiteboard]
Elli:	Ah, wait ... Are we talking about the first [equation] ?

The communication in excerpt 1 is an instance of this critical aspect. The example demonstrates indications of a student not expressing that the two equations represent the same relationship and thus, that the whole and the parts are the same regardless of the two shown equations. What the student expressed was interpreted in the analysis as the student rather experiencing the two separate equations as two different relationships.

5. The whole can assume a lower value than the parts

Finally, the critical aspect *the whole can assume a lower value than the parts* will be presented. An example of when the whole assumes a lower value than one of its parts is from the example above when students in grade 3 explored the relationship between the numbers in $x+3=2$ and $2-x=3$. The students were encouraged to identify the relationship supported by the learning model (figure 1) and initially to identify the whole. What encouraged the students to formulate the relationship in four different ways was that the students were asked to find an appropriate operation in order to find the value of the unknown number ”*x*”. This is reproduced in excerpt 2 below.

Excerpt 2, grade 3

Teacher:	What is the whole? [The teacher points at the whiteboard] [...]
Elli:	We did so that we, the whole ... we thought the whole was five, since we turned it around so it instead became two plus three equals as something. We somehow turned direction [change the value of the whole and the parts]. We turned a little.

Excerpt 2 is an instance of the critical aspect. Based on Elli’s expression, the students in the group seemed to be aware that they had altered the equation. In the analysis, this was interpreted as if the students supposed that equations with addition always lead to a sum with a higher value than the addends. Similar examples were identified with students in grade 9. Further, it was interpreted that students did not discern a relationship. In another example, Ali, a student in grade 3, expressed: ”And that [points at *x*], ’one-minus’ plus three is equal to two”. In the analysis, this was interpreted as if the student discerned that a whole can assume a lower value than the parts, i.e. that the student discerned this critical aspect.

Summary of findings and concluding discussion

In the analysis, five critical aspects were identified when students participated in teaching inspired by learning activity theory (see Davydov, 2008). Previous research regarding relationships as a part-whole structure (e.g. Davydov, 2008; Schmittau, 2005), inspired us when exploring equations in order to identify critical aspects. So far, we have not found previous research regarding a part-whole structure where the whole assumes a lower value than the parts. For that reason, the critical aspect number 5, presented above, emerged in this empirical study. According to variation theory, an assumption is that in order to distinguish ”new aspects”, the teacher needs to take the differences between respective critical aspects into account, when designing the teaching (cf. Marton, 2015). Hence, five critical aspects could be perceived as too many. However, we argue there are justifiably five critical aspects concerning relationships and equations including negative numbers. One reason for including all five critical aspects may be that none of the students had previous experiences of being taught general mathematical structures. The assumption can be supported by Pang and Ki's (2016) emphasis that there is an interconnection between the students, the object of learning, and the ways in which students experience the object of learning.

Cai and Knuth (2011) claim there is a need for analysing relationships between quantities and between numbers and noticing structures. We argue it is not enough to *notice* structures. First, there is a need to experience that there *is* a relationship between the numbers in an equation. We also claim, that it is not enough to experience a relationship between *two* of the numbers, for example, between $x-5$ in the equation $x-5=3$. The numbers are not solitaires, which can be manipulated one by one when exploring equations as a relationship. Drawing on this, we state that students benefit from discerning the relationship between *all* the numbers in an equation.

The critical aspect *two parts together equals a whole with same value as the two parts together* is based on a part-whole structure, which in turn requires that students need to discern what constitute the parts and the whole, respectively, and further, simultaneously. When students are not already familiar with teaching based on general structures and the intention is to enable students to focus on a structure, there may be a need for initially exploring equations with quantities and using general symbols (cf. Davydov, 2008). Without anything to calculate, it may enable students to identify, for example, a part-whole structure (cf. Schmittau, 2005).

When the critical aspect *the same relationship can be formulated in four different ways* was explored, the learning model (figure 1) came to play an important role for some of the students. The learning model functioned as a mediating tool and enabled students to identify and formulate all four equations, reflecting the same relationship (cf. Davydov, 2008; Gorbov & Chudinova, 2000). Nevertheless, some of the students did not experience that $x+3=2$ and $2-x=3$ concern the same relationship. Although the critical aspects are many, they are intertwined. Maybe the critical aspect *the whole can assume a lower value than the parts*, stands out from the others.

When analysing research lessons, it became clear that the choice of values in equations was important. When the numbers in the equations were too simple, the students did not focus on general structures as relationships between numbers, and thus the part-whole structure. Students rather tried to solve the equations as they were used to doing – by rules and procedures – which was not always advantageous, since some students did not remember the rules and procedures (cf. Brown et al., 1988). Further, when the values of the numbers were too simple, there was no need for a learning model, nor the four formulations. When not challenging the students by using negative numbers, the whole always assumes a higher value than one or all of the parts. This can lead to an undesirable experience (Bishop et al., 2014). For that reason, negative numbers were of importance in this study. Visualizing a part-whole structure and that one relationship can be formulated in four different ways, has been shown to be powerful in this paper, not the least when negative numbers are included in equations (figure 3). Further, when students are proficient in addition and subtraction as inverses and in additive structures, it may enable them to choose an appropriate (for them) and convenient operation when solving equations (cf. Greer, 2012; Vergnaud, 1982).

There are limitations in the study. For example, the students participated in a teaching context concerning a content (general mathematical structures), which were unfamiliar. Despite that, we argue, there are implications for teaching. It is worth changing teaching from merely focusing on calculations based on rules and procedures, into teaching based on general structures. One reason is that the older students mostly focused on calculations based on rules and procedures, even despite (see Uziel & Amit, 2019) having attended school for many years. Another reason is that several of the younger students solved equations such as $x + 3 = 2$, supported by the learning model even though they had no experience of exploring equations as a part-whole structure or experience of negative numbers as operands. Although critical aspects are relational, teachers can use the critical aspects identified in our study as a starting point when planning lessons when teaching concerns relationships between numbers in equations. In order to determine whether the critical aspects need to be discerned in a specific order, more research is required. This can be seen as another limitation in this study.

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Eliciting pre-service secondary teachers' initial ideas of sampling

Jonas Bergman Ärlebäck and Peter Frejd

This paper investigates secondary pre-service teachers' strategies of sampling. The work of eight groups of pre-service secondary teachers asked to devise and implement a sampling strategy to answer two questions about three given populations of different sizes are analyzed. The result presented highlights the models devised and used by the groups. The paper especially discusses the pre-service teachers' models of samplings in term of how the number of samples and sample size used were decided on in the model as fixed, interval-static, or dynamic.

At the heart of statistics, or statistical thinking, "is a general, fundamental, and independent mode of reasoning about data, variation, and chance" (Moore 1998, p. 1257). A key activity giving access to, and capture, different types of variability in the phenomena being studied is *sampling* (Franklin et al, 2007). Indeed, "[t]aking representative samples of data and using samples to make inferences about unknown populations are at the core of statistics" (Ben-Zvi et al., 2015, p. 292). In addition, sampling, giving rise to various types of variability, is an important aspect to consider in relation to the emerging field of Big Data (Manyika et al., 2011).

Although statistics is part of the core content of the mathematics curriculum for the Swedish secondary school, it is surprising that the concepts *sample* or *sampling* are not explicitly mentioned or listed in connection with learning goals or examination criteria. Nevertheless, in the lower secondary curriculum (Skolverket, 2011a) the students are supposed to conduct and work with their own statistical investigations, which by necessity have to include samples and sampling. In the upper secondary curriculum (Skolverket, 2011b), the students are supposed to examine "how statistical methods and results are used in society and professional life" as well as use "statistical methods for reporting observation and data from surveys, including regression analysis", which both imply understanding and taking aspects of sampling into account.

However, as pointed out by Watson and English (2016), students' difficulties related to ideas of single and repeated sampling, as well as sampling distributions, are well documented in the statistics education research. For teachers to be

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able to support their students' learning and development to overcome their difficulties, they need solid and deep understanding of the concepts at hand (Ball et al., 2008). In order to prepare our pre-service secondary teachers (P-SSTs) for this task, a natural motivating question then becomes: "What are P-SSTs' understandings of sampling?". To investigate secondary P-SSTs' strategies and ideas of sampling, we in this paper analyze the work of eight groups of P-SSTs asked to devise and implement a sampling strategy to answer two questions about three given populations of different sizes.

Aim and research question

The study presented in this paper is part of a larger project focusing on designing sequences of classroom activities facilitating the learning of statistical ideas and concepts, such as sampling. Our goal is to answer the following research question: What sampling strategies and mathematical models do P-SSTs use and develop when working on an activity of sampling?

The aim of identifying the approaches of sampling taken by the P-SSTs, is to facilitate the design of follow-up activities that potentially can support P-SSTs and students continued development of statistical ideas.

Previous research on sampling

Generally, learning statistics has proven to be a challenging endeavor for learners of all ages, and research has shown that students often learn statistical procedures that they understand poorly and they do not know when and where and why to apply them (Batanero et al., 2011; Shaughnessy, 2007).

With respect to sampling, difficulties among teachers and students to grasp the ideas of sampling and sampling distributions is well documented in the research literature (Watson & English, 2016). For example, some students struggle with describing the relation between an estimated population statistic and an increased numbers of statistic samples, whilst other students have developed ideas about the necessity of taking multiple samples in order to be able to make statistical inferences. In addition, regarding sample size, Ben-Zvi et al. (2015, p. 296) review of the literature points at "students' statistical intuitions as not always being incorrect, but they may be crude and can be developed into correct conceptions through carefully designed instructions". Students tend to answer questions about which sample size is most accurate to use in a given setting more perspicacious, than questions on selecting a particular sample to generate a value in the tail of a population distribution. Also, difficulties in differentiating between the sample distribution and the population distribution is found among both students and teachers (Doerr & Jacob, 2011).

Using surveys and interviews from 62 students in grades 3, 6 and 9, Watson and Moritz (2000) investigated students' different sophistication in their developed concept of sampling. They identified and described six dispositions taken by the students: (i) small samplers without selection; (ii) small samplers with primitive random selection; (iii) small samplers with preselection of results, (iv) large samplers with random distribution or distributed selection, (v) large samplers sensitive to bias, and (vi) equivocal samplers. Disregarding the last disposition, Watson and Moritz found a progression from (i) to (v) with increasing grade.

In summary, the findings above show that aspects related to sampling may cause students to struggle in different ways, both concerning the concept in itself and regarding what procedures to use for determining an accurate sample size. However, progression in students' sampling techniques is also found.

Some theoretical consideration on sampling

Determining a method of sampling can be challenging (Ben-Zvi et al., 2015), and in order to establish an appropriate model for sampling, factors like sample size, population size, the aim of the study being conducted, the risks of badly selected samples as well as sampling errors, need to be considered. Miaoulis and Michener (1976) describe and discuss these factors in terms of *the level of confidence* (risk level), *the level of precision* (sampling error), and *the degree of variability* of the properties being measured (the distribution of properties in the population). All these are important to consider when determining an adequate sample size, but there is always a risk that the sample selected does not scale up to represent the population. There are basically two approaches to sampling, with clear connections to classical combinatorics and probability theory: single sample or multi-sample sampling – and in the latter case with or without replacement between samples. Single sampling methods are most common in many contexts (such as election polls) and have a theoretical basis built on proportionality or a given distribution. The mathematics of multi-sample sampling methods (with or without replacement) are technically generally much more complex, but thanks to the central limit theorem among other things, provide powerful approaches.

There are different strategies for estimating sampling size, but to calculate a representative sample size for sampling (without replacement) either large and small populations, the Cochran's formulas may be used (Cochran, 1963) (table 1).

Table 1. *Cochran's formulas for large- and small populations*

Large population	Small population
$n > (\frac{z}{m})^2 p(1-p)$	$n > \frac{z^2 p(1-p)}{m^2 + \frac{z^2 p(1-p)}{N}}$

Notes. *n* = sample size, *m* = the desired level of precision, *p* = degree of variability, *N* = size of the population, *z* = the desired level of confidence, which can be found in statistical tables connected to the area under the normal distribution curve.

An illustration of the use of Cochran's formulas to determine sample size is given below in the section *The activity Polling.*

Methodology

For this study we adopt the *models and modelling perspective on teaching and learning* (Lesh & Doerr, 2003). This theoretical framework provides us with a tool for designing activities for the P-SSTs to work on, as well as a tool for analyzing the P-SSTs' developed sampling strategies and models. Below we first elaborate on this perspective below, then describe the design of the activity, and the data collection and the analysis.

The models and modeling perspective

The models and modeling perspective (Lesh & Doerr, 2003), generally defines a model as a system consisting of elements, relationships, rules and operations that can be used to make sense of, explain, predict or describe some other system. In particular, a mathematical model focuses on the structural characteristics of the system in question. From this perspective, Lesh and Harel (2003) stress that learning is developing useful and generalized models that are made up of (1) a set of concepts used to describe or explain the mathematical objects relevant to the phenomenon studied, and (2) procedures that can be used or re-used to create useful constructions, manipulations, or predictions for achieving clearly recognized goals in a range of contexts.

Model development sequences are instructional sequence of modeling activities designed to support the development of learners' use and understanding of a given model (Lesh et al., 2003), and are constituted of three types of modelling activities: model eliciting activities (MEAs); model exploration activities (MXAs); and model application activities (MAAs). The point of departure in a model development sequence is always a MEA, which is a meaningful and realistic problem situation aiming at eliciting the ideas learners already have with respect to the learning goal of the sequence. To facilitate this eliciting, Lesh et al. (2000) developed six design principles for MEAs (*model construction-; reality-; self-assessment-; model documentation-; shareability and reusability-;* and *effective prototype principle*). In model development sequences, the initial MEA is then followed by one or more MXAs and MAAs. In MXAs, the focus is on the underlying mathematical structure of the elicited model, typically by exploring various ways of constructing, illustrating, interpreting and using different representation of the model. In MAAs on the other hand, the aim is to engage learners in adapting and applying their models in new contexts. All three types of activities are designed in such a way, that learners engage in an iterative process of expressing, testing and revising their ideas, which results in the learners developing and adapting their models (Lesh & Doerr, 2003; Lesh et al., 2003).

The activity Polling

The activity the P-SSTs worked on was designed using the six design principles for MEAs to elicit P-SSTs' approaches around sampling strategies. The activity, *Polling*, was designed as a cross-curriculum- and interdisciplinary topic with the social sciences in the context of election Gallup polls (the reality principle). The activity Polling included two questions asking the P-SSTs to devise a strategy and procedure (model construction principle) for taking samples of three different sized populations (the shareability and reusability principle) in order make an inference to the larger populations, and to write this down (the model documentation principle). In accordance with the effective prototype principle, the activity was designed around hands-on material simulating the three different population to facilitate and inspire the P-SSTs' thinking about the problem situation and the task. The activity included a closed envelope with data of the actual distributions of the three populations, which the P-SSTs were going to open in the end of the activity to be used as a reference to reflect and discuss strengths and weaknesses of their strategy and models (the self-assessment principle).

In short, the instructions given to the P-SSTs were to develop, describe and investigate statistical sampling strategies aimed at describing a given attribute in given different sized populations, in terms of answering the following questions:

Q1. How many different categories of a particular attribute are represented in a given population of size *N*?

Q2. What is the distribution of a particular attribute in a given population of size *N*?

In order to answer these questions, the activity included seven jars containing a predefined number of beads of different colors (see figure 1b). Each bead represented an individual in the population, and the different colors of the beads represented the different categories of the attribute being examined.

Three of the jars contained 20 beads, two jars contained 400 beads and two jars contained 10,000 beads. All jars had different distributions (one roughly uniform and one skew for each population size), and the population size explicitly written on the jars. The P-SSTs were allowed to freely explore and

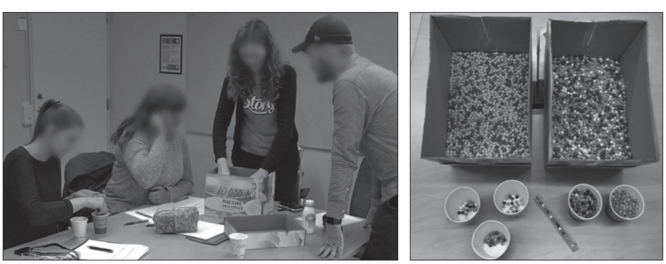

Figure 1a. *P-SSTs enacting their sampling strategy;* 1b*: The seven jars of beads and the sample taking device SaTaDev*

investigate one of the jars containing 20 beads (an ”open” jar with known distribution) before applying their developed statistical sampling strategy on the other six jars which all were ”closed jars” (with, for the P-SSTs, unknown distributions until the very end of the activity). To facilitate the actual sampling of the beads the P-SSTs could use, the sample taking device *SaTaDev*, a plastic pipe of suitable diameter sawed in half length-wised with closed ends (see figure 1b).

Returning to Cochran's formulas in table 1, we can determine the sample size for the populations of beads in the activity Polling as follows: The variability is assumed to be unknown and we adopt the maximum variability of 50 % ($p = .5$). For our purposes, we chose a 95 % confidence level ($z = 1.96$) and ± 5 % precision ($m = .05$). The populations, the number of beads in the jars, are $N = 20$, 400 and 10 000. Using the formula for *large populations* in table 1 to calculate the sample size produces $n > 385$ for all three populations. However, using the formula for *small populations* with finite population correction produces $n > 19$, 193, 370. The calculated sample sizes above show the importance of consider the finite population correction, at least in the cases where $N < 10\,000$.

The data collection and analysis

The study followed the Swedish Research Council's ethical principles with the requirements of information, approval, confidentiality and consent. Twenty-five secondary P-SSTs enrolled in 2019 in a teacher training program for secondary mathematics teachers (grades 7–9 and 10–12 respectively) participated in the study. All P-SSTs had all taken at least one semester of mathematics courses on topics such as calculus, linear algebra, statistics and mathematics education before participating in the study. As part of an ongoing course in mathematics education the P-SSTs worked on the activity Polling for 2 hours divided into eight groups of 2–4 P-SSTs in individual rooms. The P-SSTs were asked to think-out-loud what they were thinking during the activity. In addition to collecting the written work done by the groups, six of the groups were videotaped (see figure 1a) and two groups were audio recorded. In this paper we focus the analysis on the P-SSTs' written work, but consulted the video/audio when the details of the P-SSTs' writing were non-conclusive. Based on their written work, the sampling strategies and models develop by the P-SSTs were analyzed and categorized with respect to *sample procedure, sample size* and *method of inference*, as well as *if and how these varied for the different sized populations*. The emerging categories describing the variation and commonalities the P-SSTs' strategies and models, and the result of the analysis in terms of these, are presented below.

Result

The analysis of the P-SSTs' work shows that six of the eight groups used the same model to sample the populations to answer the two question Q1 and Q2 (see Q1/Q2 in table 2). Further, the fundamental strategy used by all groups to answer the questions were to make a statistical inference based on scaling up the proportions found through their sampling procedure proportionally to the whole population. Table 2 below briefly summaries the different models developed and implemented by the groups.

Table 2. *The eight models developed by the P-SSTs to answer Q1 and Q2 in the Polling! activity*

Group	Strategy/ mathematical model
1	Q1/Q2: For an *N*-sized population, take one sample of size $\sqrt{N}+5$.
2	Q1/Q2: Repeat a sample strategy of 10 beads with replacement; 4 samples for the jar with 20 beads; 8 for 400 beads and 16 for 10 000 beads.
3	Q1: One sample of maximum 200 beads. Hence: for the 20 bead jar sample all; for the 400 and 10 000 bead jars sample 200 beads (50 % or 2 %). Q2: Sampling strategy with replacement using 5 samples. The sample size for the 20, 400 and 10 000 beads jars are 5, 20 and 20 beads.
4	Q1/Q2: 10 samples with replacement of size 5.
5	Q1/Q2: Sampling with replacement. For the 20 bead jar: 4 samples á 5 beads; for the 400 and 10 000 bead jars: 3 samples using the *SaTaDev* (20–23 beads).
6	Q1/Q2: For a population ≤ 20: sample, with replacement, 10 beads (population size x 4) / 10 times. For population > 20: sample with, replacement, 20 beads (population size x 4) / 20 times.
7	Q1/Q2: If the jar contains 1–100 beads collect 100 %; 101–999 beads collect 10 %; 1000–9999 collect 5 %; and > 10 000 beads collect 1 %.
8	Q1: Five samples with replacement of size 1 % of the population (but at least 2 beads). Q2: One sample of (# colors found in Q1) x 10, but max half the population.

Table 3 below summaries the three features of the P-SSTs' models for making an inference about the given populations in terms of (1) whether or not the sampled beads were put back in the jars between samples (replacement – in the cases where this is applicable); (2) the number of samples taken; and (3) the sample size. The number of samples taken and the sample sizes in the models were both categorized as either *fixed, interval-static* or *dynamic*. In this context, *fixed* signifies that the number of samples or the sample size used in the model was fixed and independent of the population size. *Interval-static* means that pre-defined intervals for presumptive studied populations were used to determine the numbers of samples or the sample size. *Dynamic* in this context means that the number of samples or the sample size used were determined by a (non step-) function of the population size.

With respect to *the number of samples taken* (the column *# of samples* in table 3), all but three groups used a fixed number of samples in their models; two groups (2, 5) used an interval-static method, and one group (6) a dynamic method. The fixed number of samples taken varied between 1 and 10. Both the groups (3, 8) who used different models to answer Q1 and Q2 used the same

Table 3. *Identified key features of the P-SSTs' mathematical models*

Group	Same or different models			Features of model		
	Q1+Q2	Q1	Q2	Replacement	# of samples	Sample size
1	x			d.n.a.	fixed (=1)	dynamic
2	x			yes	interval-static	fixed (=10)
3	-	x		d.n.a.	fixed (=1)	interval-static
			x	yes	fixed (=5)	interval-static
4	x			yes	fixed (=10)	fixed (=5)
5	x			yes	interval-static	interval-static
6	x			yes	dynamic	interval-static
7	x			d.n.a.	fixed (=1)	dynamic
8	-	x		yes	fixed (=5)	dynamic
			x	d.n.a.	fixed (=1)	dynamic

Note. d.n.a is an abbreviation of do not apply

type of method (fixed) for determining the numbers of samples in both cases. The interval-static model of group 2 consists of a list of the number of samples for the different population sizes, whereas group 5 introduced a population size threshold dictating whether to take four or three samples. Group 6 used a dynamic model in which the number of samples taken is proportional to the population size (see table 2). All the groups taking more than one sample used replacement of previously sampled beads in their sampling models.

Turning to the *sample sizes* (the column *Sample size* in table 3) the P-SSTs used in their models, two of the groups (2, 4) used a fixed sample size regardless the size of the population, and three of the groups (3, 5, 6) used an interval-static method (including group 3 who used it twice). Three of the groups (1, 7, 8) used a dynamic model to decide what sample size to use. The interval-static model implemented by group 5 is based on the same threshold idea that the group adapted for determining the number of samples. However, in this case the group differentiated between taking samples of five beads or using the *SaTaDev*, which gave them a sample of 20 to 23 beads. The dynamic model applied by group 7 on the other hand is based on the sample size being a certain proportion of the population, but with a varying proportionality factor. In the case of group 8, their dynamic model to determine the sample size to use in answering Q2 was, so to speak, "Q1-dependent", in that their answer to Q1 was an explicit factor in their model determining the sample size used for answering Q2 (see table 2).

Looking across what approaches the groups used in their models for determining the number of samples and sample size, we found 6 different combinations. The most frequent combination of approaches taken was to combine a sampling model using a single sample and deciding the sample size using a dynamic approach (*fixed – dynamic*). The other combinations are (see table 3): *interval-static – fixed; fixed – interval-static; fixed – fixed; interval-static*

– *interval-static;* and *dynamic – interval-static*. Taken together, these models show how the P-SSTs attempted to develop a sampling method that would be sensitive to the size of the population studied. Most groups did this by adjusting *either only* the number of samples taken (groups 2, 5 and 6 did this), *or only* the sample size (groups 3, 7 and 8 did this). Group 5 developed a model that uses an *interval-static* approach to account for the population size in both the number of samples taken and the size of the samples. Only one of the groups (4) provided a model that does not at all consider the size of the studied population, but rather took 10 samples (with replacement) of sample size 5 regardless of population size.

It is interesting to note that the model for determining the one-time-use-only sample size in the sampling strategy developed by groups 1, $\sqrt{N}+5$, the correction ”+ 5” was added to their originally suggested sample size of $\sqrt{N}$, as an attempt to achieve a better prediction for small finite populations.

Conclusion and discussion

The models developed and used by the participating P-SSTs included sampling strategies based on both single and multiple samples, and considered situations both with and without replacement. The models developed to answer the two questions Q1 and Q2 were seen to be based on either *fixed, interval-static* or *dynamic approaches* to specify the number of samples to take as well as to determine the sample size. The most common sampling model the P-SSTs used ended up having a single fixed sample and the sample size was decided by using a dynamic approach (*fixed – dynamic*).

Our result shows, somewhat contrary to the claim by Watson and English (2016), that five out of the eight groups were comfortable in making an inference about the whole population based on a single sample. In addition, the models, developed by the groups using repeated sampling, did not build on ideas directly related to the central limit theorem. However, if this is due to the P-SSTs not having a solid enough understanding of the relationship between sampling distributions and population distribution (and how to exploit this) as suggested by Doerr and Jacob (2011), is an open question.

Although the activity Polling elicited a variety of models, they are all qualitatively different from more standard models used to determine sample size (such as Cochran’s (1963) formulas). Instead, the models displayed by the P-SSTs’ seem to be more ”homegrown”, but do however point to the awareness about how crucial the adaptation of the sampling procedure is for the validity of the end result. Nevertheless, given that the P-SSTs had taken a course in statistics, it is somewhat surprising that none of the important aspects according to Miaoulis and Michener (1976), and explicit in the Cochran’s (1963) formals related to sampling (i.e. *the level of confidence, the level of precision,* or *the degree of variability*), are discernable in the groups’ models. It should be noted however, that we have only analyzed the written work of the P-SSTs, and that a more detailed analysis of the groups' 2h-work and discussions most probably would provide a more nuanced picture; also with respect to what is pointed out and conspicuously absent above. This is a topic for future investigation and research.

We do not generalize our results beyond our sample, but our experience and results are in line with the argument by Ben-Zvi et al. (2015), that students in general have statistical intuitions and crude emerging ideas that can be expressed and used as a basis for designing productive learning experiences to further their understanding and abilities. Indeed, from a models and modeling perspective, the richness of ideas in the P-SSTs' developed models of sampling suggest that there are potentially many productive ways to design MXAs and MAAs to follow up the activity Polling, and to further support the development and learning of the P-SSTs. For sequential MXAs and MAAs one option could be to let the groups continue to explore their devised models by creating own simulations using programming in a suitable high-level language like Python or using a software especially developed for the teaching and learning of statistics like Fathom or ThinkerPlots.

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Full papers, abstract

Methodological reflections of repeated interviews on teaching and learning mathematics

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This paper is an exploration of what a qualitative method, interviews with students, can provide to mathematics education research and critical points important to take into consideration. In particular, if using repeated interviews with the same informants over time, what can be gained in relation to research quality? The argument made is that repeated interviews can provide with in depth knowledge and a grasp of students meaning(s). Critical points found were; person-dependency, ethical considerations, an interview re-interview effect, and a connection between interview as a method and the aim of the study. These critical points are of importance to discuss and reflect upon all through the research process. If doing so, all these critical points can be used as a quality criterion when producing in depth knowledge in qualitative research.

Sustainable assessment in mathematics: a matter of access and participation

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Success in mathematics is closely connected to participation in mandatory national test-taking. Sustainable assessment in mathematics for students with disabilities has shown to be challenging. The purpose of this article is to investigate if and how a model on participation might be useful for promoting the opportunity to display knowledge in mathematics, for all students. A conclusion drawn is that the kind of support given, the level of mathematical knowledge and participation pre-supposes each other for students with disabilities.

Unpacking “Language as resource”– the case of mathematics education in Sweden

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Lisa Björklund Boistrup

Malmö University

In this paper we unpack epistemological aspects of language and mathematics potentials embedded in the ”language as resource” discourse. We use research literature, policy, and interviews with a mathematics teacher and a multilingual student to illustrate the potentials and how they are realised in the material. We identified a ”lever potential” and ”one new whole” potential. To consider the potentials in a nuanced way, we propose an analytical model which contributes with theoretical conceptualizations that allows for grasping a relation between epistemologies of language and mathematics from the perspective of the language as resource discourse.

Evaluating numeracy apps in different cultural contexts

Aleksander Veraksa, Carme Balaguer, Silje Christiansen and Tamsin Meaney

The proliferation of mathematical apps available for young children requires early childhood teachers to be able to effectively evaluate their potential usefulness. As a result of a symposium on evaluating popular numeracy apps in three regions of the world, we describe the similarities and differences from using different evaluation tools. The results show that regardless of the evaluation tools used or the content of the local curriculum, the researchers focused on many of the same aspects, such as how the apps provided feedback to the children. Differences in the evaluations were to do with how much emphasis was placed on preparing children for school and on how mathematical understandings were represented.

In this paper, we bring together our experiences from evaluating popular numeracy apps in three different regions of the world, which we presented as a symposium at MADIF-12. Being able to evaluate apps is an important aspect of teachers' work with digital tools in early childhood institutions. Research suggests that teacher knowledge and competencies are important in determining if digital tools can "act as a tool in the learning process for the children" (Alvestad & Jernes, 2014, p. 3). If early childhood teachers lack education and experiences in using digital apps with young children, then commercial developers may gain more influence in educational practices than the curricula and policy documents intend (Alvestad & Jernes, 2014). The proliferation of digital apps, especially for young learners (Larkin, 2013), suggests that commercial developers do believe that there is a strong market for producing educational mathematical apps for young children. Consequently, understanding how to evaluate their potential usefulness is important for teachers who need to make choices in their work with children.

Currently, there are few evaluation tools available to teachers (Handal et al., 2016), particularly for early childhood settings with a focus on mathematics (Papadakis et al., 2017). Of the evaluation tools that are available, many are not specific to mathematics (Handal et al., 2016) or do not include an awareness

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of cultural differences in curricula (see for example, Papadakis et al., 2017). Thus, in this paper, researchers in three different regions of the world evaluate common numeracy apps designed for young children in those regions. Although it is difficult to compare early childhood education and care (ECEC) in different education systems (Samuelsson et al., 2018), we considered that the differences in our cultural contexts could provide valuable insights into the generic and specific aspects needed in app evaluation tools. Thus our aim for this paper, is to document the similarities and differences in our evaluations as a way of understanding which contextual factors might affect the identification of appropriate digital apps for specific situations.

Evaluating apps: issues to consider

To undertake the exploration, we focused on apps about number concepts and their relationship to regional curricula and cultural context (see figure 1) as they were considered in the evaluation tools, chosen by individual research groups. We focused on apps about number concepts, because much of the research in mathematics education for young children highlights the importance of these concepts, including in studies about digital tools (Rothschild & Williams, 2015). Although regional curricula are affected by government policy, number understandings were a focus in curricula from the three regions, Norway, Catalonia and Moscow. To allow for cultural and curricula differences, individual research groups chose an evaluation tool which they considered relevant for their contexts.

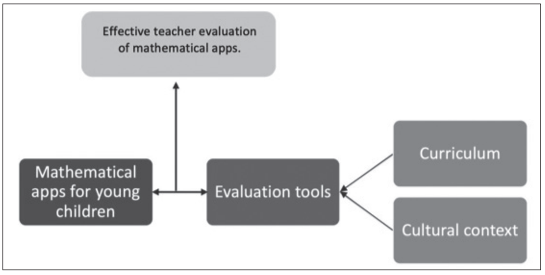

Figure 1. *analysing the alignment of early childhood curricula with digital apps and evaluation tools*

Curricula differences

The curricula used in all three regions required young children to engage with mathematical ideas. The Norwegian curriculum for early childhood, known as *The framework plan*, includes the learning area ”Quantities, shapes and spaces” (Ministry of Education, 2017), which is linked to Bishop’s (1988) fundamental mathematical activities (Reikerås, 2008). The Framework plan provides only broad guidelines for early childhood teachers, with the pedagogy for introducing ideas to children also not described in detail. However, it does state, ”Play shall be a key focus in kindergarten, and the inherent value of play shall be acknowledged” (Ministry of Education, 2017, p. 20). This suggests that formal teacher-led ”lessons” would not be appropriate.

In the Russian Federation, special emphasis is placed on the system-forming role of mathematics in education (Rasporyazhenie, 2013). Mathematical programs for preschool and primary school education, including the participation of the family, provide possibilities for mastering forms of activity, elementary mathematical notions, ideas and images, as well as the digital environment and conditions for extracurricular activities. The *Federal state educational standards for elementary education* (FSESEE) highlights the need to ensure continuity of learning experiences into the early years of school. FSESEE provides more detail that the Norwegian *Framework plan*. For example, in regard to number concepts, it states:

– To know: numbers from 0 to 9, the meaning of the signs ”+”, ”–”, ”=”, ”>”, ”<”.

– To be able: to count forward and backward up to10, to name a number within a range of 10, preceding the one named and the one following it, to indicate the quantity of objects with the help of numbers, to solve and make simple addition and subtraction tasks up to 10, to compose numbers up to 10 out of ones

A review of the main mathematical manuals used in early childhood institutions and/or those recommended by experts for this purpose in Russia (Veraksa et al., 2016; Novikova, 2018; Salmina, 1994; Fedosova, 2018) indicates that the main methods of number concept formation in preschool is:

– Visually figurative, symbolic;

– Visually active, practical;

– A sequential combination of the above methods.

Similarly, the curriculum for 3–6 years olds in Catalonia, known as *Decree 181/2008 2nd cycle (3–6 years) childhood education (structure)*, frequently refers to the importance of number knowledge as an integral part of the society. Children must recognise number (quantity and symbol) as a communicative tool for living in the world. *Decree 181/2008* provides more detail than the Norwegian Framework plan, in that it lists specific skills and objectives that children should gain in early childhood institutions. For example, one skill is, ”Think, create, elaborate explanations and get started on basic mathematical skills”. Although not stated explicitly in the curriculum, there is an expectation that children will be presented with number understandings in a specific order, similar to the situation in Russia. First, the children need to identify quantities and their ordering; secondly, to make relationships between quantities or between quantities and symbols and thirdly to undertake operations (mental changes) (Alsina Pastells, 2007). As well the sequence of activities should move from concrete (manipulation) to abstract (mental representation). There are connections to play as there is in the Norwegian Framework plan, although rather than being a source of pedagogical inspiration, the focus on its contribution to representations. For example, one objective is ”Represent and evoke aspects of the lived reality, known or imagined and express them through the symbolic possibilities offered by play and other forms of representation”.

In this brief description of the curriculum for early childhood institutions in the three regions, it is evident that historical influences (Russia), the contribution that education makes to society (Catalonia) and the importance of teachers having autonomy (Norway) are part of the cultural contexts, that early childhood teachers work in. We assume that these experiences are part of researchers' understanding of what should be highlighted in the mathematical apps.

Evaluation tools and analysis

In this section, we describe the findings from evaluating numeracy apps with the different tools. After discussing the evaluation situations separately, we then discuss the similarities and differences across the three situations.

Norway

The two common numeracy apps in Norway, which were evaluated, were DragonBox number (https://dragonbox.com/products/numbers) and Tella (http://tella123.org/#/). Both apps had been designed in Norway and were presented in Norwegian, although neither relied only on children to understand verbal language to interpret what they had to do.

The evaluation tool (available from https://dlgs.uni-potsdam.de/oer/acat-review-guide) chosen by the Norwegian group was based on *Artifact centric activity theory* (ACAT) (Ladel & Kortenkamp, 2014). This model describes five nodes (see figure 2) that could influence the learning possibilities from each app:

- a subject (usually the child/student) (S),
- an object (the mathematical content or process) (O),
- the mediating artifact (in this case an app the learner uses to familiarize themselves with the mathematical content) (A),
- the rules (how the app should behave in order so that the mathematical object can be learnt) (R), and
- the group (i.e. the total class situation in the teaching) (G).

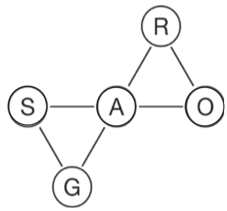

Figure 2. *ACAT model*

In the evaluation tool, each node or combinations of nodes is connected to a series of evaluative questions and a brief theoretical background is provided. The evaluation tool considers the appropriateness of the app in relationship to a specific purpose. The questions are about: the mathematical object; how children were likely to interact with the mathematical object through the app; how the app develops the interaction around the mathematical object; the suitability of the app for displaying mathematical meaning; and the use of the app in a class situation.

Table 1. *Evaluations of* Tella *and* DragonBox numbers *using ACAT*

Criteria	Tella	DragonBox numbers
Mathematical object	Quantities and understandings about number.	Number sense (part of DragonBox school).
Child(ren) interaction with mathematical object.	By dragging and tracing numerals, children complete sets of tasks (similar to activity sheet tasks)	Each number is represented by a Noom, which acts as a dynamic cuisenaire rod.
Development of the interaction	Tasks are provided in a linear order and can feel quite slow so children can become bored.	Children engage in tasks about different aspects of number sense.
Suitability of app for conveying mathematical meaning	Children are channelled into getting the correct answer so little possibility for reflection about mathematical learning.	The dynamic nature of virtual environment supports understanding the relationship between numbers
Use of app in class situation	Individual work, not tasks to develop discussion.	Individual work with children having some control.

The results from the analysis of Tella and DragonBox numbers are shown in table 1. From this evaluation, it seems that Tella is focused on number understandings but presents them in a linear fashion, providing few possibilities for play. It is difficult to determine if the children learnt something new or applied what they already knew. From watching children use the app, the ones who continued with it were the ones who already knew the answers or who were good guesses and wanted the confetti display, from getting a task correct. DragonBox numbers provided slightly more possibilities for children to play, through an activity called a sandbox where the children could explore number concepts. For example, the Noons, or animated number characters, can be sliced and squashed together showing how numbers are composed of other numbers.

From the analysis, the ACAT evaluation tool seemed to focus on the mathematical object that the children were supposed to engage with and how the app presented it. It did allow for connections to play to be highlighted, as part of identifying how the mathematical meaning was conveyed.

Russia – Moscow

Four popular apps, around number concept formation, were evaluated using four criteria: dialogue (potential for teamwork with an adult, where the app acts as a means of organizing the work); appropriateness for preschool age children; pedagogical considerations incorporated in the app; and continuity towards school education. These criteria were considered important because unlike traditional education, developmental education, based on the work of Vygotsky (1980) and his followers, occurs in the "zone of proximal development" (ZPD), that is, in the space that opens up new opportunities for learning content through interaction with an adult. Traditional education, in contrast, is considered to be based on imitation. Developmental pedagogy presupposes that number should be explored after mastering the system of relations of quantities, through measuring and counting. Each stage involves bringing the child to the ZPD connected to the

Table 2. *Evaluations of 4 apps using Vygotskian developmental ideas*

Criteria	Kids numbers and Math	Mathematics and numbers for kids Learning to count	Luntik Learning math Learning to count	Funexpected math
Having a dialogue with children.	Instructions	N/A	Expanded instructions. The dialogue between the characters in the form of a game.	Instructions
Compliance with the method of number concept formation.	Combination of a visually figurative and visually active presentation of material.	Visually figurative (symbolic) presentation	Combination of visually figurative and visually active ways of material presentation.	Visually figurative (symbolic) presentation
Methodology of concept introduction	Traditional teaching methodology	Traditional teaching methodology	Traditional teaching methodology	Traditional teaching methodology
Age appropriate	In full compliance	Does not comply	Compliance with age group	Does not comply
Continuity with school	Does not provide	Does not provide	Fully provides	Partially provides

subsequent stage and creating a need to establish a one-to-one correspondence. Number concept formation also recognises that for children of preschool age, figurative and visual thinking are the main forms of thinking, with play being the main activity for developing this. Dialogue with adults shapes the level of potential development and is, therefore, an important consideration.

Of the four apps, *Kids numbers and math* and *Luntik* were chosen by experts, whereas *Mathematics and numbers for kids* and *Funexpected math* were the most downloaded apps. In table 2, the most popular digital apps presented tasks in less appropriate ways than the apps recommended by experts. Nevertheless, the results in table 2 showed that none of the apps met the requirements for developmental pedagogy. Rather than utilising children's potential ZPD, the apps focused them on imitating actions. In addition, most of the apps did not provide sufficient continuity with school and none of them supported the stimulation of the child's dialogue with an adult.

The evaluation criteria included aspects not considered in the ACAT tool, such as continuity with school. This reflects the different histories of the evaluation tools. It also seems that the Vygotskian criteria gave more attention to pedagogical aspects, such as the need for dialogue with an adult and the presentation of the mathematical ideas to the child than the ACAT evaluation tool. However, as is the case with ACAT, the mathematical focus on the development of number understanding steered the evaluation of these pedagogical aspects.

Catalonia

The criteria to analyse commonly-used apps in Catalonia come from a digital mathematical games expert, Jean Baptiste Huynh (2015) who designed *DragonBox numbers*, which was reviewed in the Norwegian evaluation section of this article. The criteria that Huynh recommended for evaluating apps are:

1. Contexts: Are they likely to be familiar to children?
2. Digital manipulation: Are the children likely to be able to do the required actions to engage in the learning situation?
3. Educational value: Is the digital game presenting content according to the curriculum?
4. Autonomy: Is autonomy promoted? Who has the control?
5. Time on task: Are the expectations about how long the children will engage with the tasks realistic?
6. Individual learning experience: Does the digital game allow the children to learn through discovery?

Although these criteria were developed in a different context, they resonate with aspects of the Catalonian curriculum for early childhood, in that they focus on

Table 3. *Evaluations of 4 apps using Huyhn's (2015) criteria*

Criteria	Shop & math	Kids numbers & math	Fiodor	Matias wolf
Context	Not free No words About shopping.	Free with ads English Not based on a story	Medieval castle	No specific context
Digital manipulation	Easy dragging Fine motor skills needed for writing numerals	Easy manipulation, but ads need to be closed	Easy dragging but sometimes quick movements are needed	Easy tapping, dragging but writing is more difficult
Educational Content	Addition and subtraction Quantity Order Problem solving	Counting, comparing with symbols < = >), patterns, ordering, addition, subtraction	Memory with numbers, addition, ordering, quantities (0–10)	Quantity, comparing quantities and numbers, addition Three difficulty levels
Autonomy	Only correct responses accepted Some skills and knowledge required	Correct answer needed to progress Some skills and knowledge needed.	No set order, some skills and knowledge needed	Correct answer needed to progress No set order, some skills and knowledge needed
Time on task	No constraints	No constraints	No constraints	No constraints
Individual learning experience	Individuals are expected to enter all the shops and finish games No guarantee of mathematical understanding, without adult interaction	Individual No challenge.	Children can control the game but it is to be played with adults Explanations for adults about links to kindergarten curricula	Children can ”sail” through the levels easily, but some indications inform adults Some curricula connections.

the contexts in which the mathematical content are presented as well as how the children are expected to engage with the mathematical ideas.

Four apps were analysed, *Shop & math*, *Kids numbers & math learning* (not the same app as was evaluated in the previous section), *Fiodor*, and *El lobo Matias* (Matias' wolf). The first two apps were commonly downloaded apps focused on numeracy ideas while the other two apps were specifically recommended on a Catalan government website.

All the apps focused on number concepts and children progressed by getting the correct answers. The apps funnelled children towards these answers, by making no other answers acceptable or by insisting the correct answer was given. The children could play the games individually, but an adult was likely to be needed if they were to understand mathematics they engaged with. Only the app connected to shopping provide a semblance of a familiar everyday context, but as all the items cost the same, the children were likely to recognise that the app did not represent a real-life situation. None of the apps provided opportunities for learning by discovering, rather they appeared to rely on previously learnt knowledge and skills, with the app providing opportunities to practice them.

The evaluation criteria were different again, in that Huyhn's (2015) criteria considered aspects of the children's interactions with the app, such as the required digital manipulation and the time needed for the game, which were not highlighted explicitly in the criteria connected to ACAT and Vygostsky's ZPD. There was also a focus on using familiar everyday situations which did not appear in the other sets of criteria. However, these criteria can be considered as being in alignment with conveying mathematical meanings, through the interactions supported by the apps and thus connected to the pedagogy of concept introduction.

Conclusion

The results of our investigations indicate that most of the apps, either with high download figures or recommended by experts, evaluated in the three countries were unlikely to develop children's number understandings in ways that were compatible with curricula documents. Many apps expected knowledge and skills to develop in a linear manner, with few possibilities to play, something particularly important in the Norwegian curricula. Possibilities for engaging with adults, important in the Russian curriculum, were also limited. As well, the use of familiar everyday context for developing number understandings, important in Catalonia, generally lacked connection to everyday situations.

Although the evaluation tools seem to have highlighted different aspects of digital apps, they all highlighted the mathematical focus in determining the potential of the apps for supporting learning. Pedagogical considerations were also part of each of the evaluation tools, although different aspects were highlighted, such as autonomy or dialogue. However, there were differences connected to cultural contexts such as the importance of preparing children from school as well as how mathematical ideas were represented in the apps.

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Symposium, abstract

Lärarutbildningens kunskapsbas – och hur vi kanske (inte) kan känna igen den

Lisa Österling, Anna Pansell and Iben Maj Christiansen

Stockholms universitet

I Sverige förväntas lärarutbildningen ge en forskningsbaserad kunskapsbas (se till exempel, SOU 2018:19), och under VFU förväntas undervisningen motiveras utifrån teoretiska kunskaper (Christiansen et al., 2020). Samtidigt är lärare begränsade av den ”ekologi” i vilken läraren befinner sig (Pansell, 2018), där val inte alltid är explicita. Detta kan göra det svårt att känna igen i vilken utsträckning en praktik är forskningsbaserad.

Symposiets tre frågor relaterar till TRACE-projektet[1].

1 Vilken ämnesdidaktisk kunskapsbas gör vi tillgänglig för studenterna? En studie av kurslitteraturen i lärarutbildningen visar att texterna behandlar matematik i relation till undervisning och till elever men på en relativt praktisk nivå. Det är få explicit teoretiska inslag där ord som teori, definition eller analys förekommer mer sällan än ord som anknyter till undervisningsmetoder som metod, bedöma, moment.

2 I vilken utsträckning relaterar studenterna sig till denna kunskapsbas? I VFU-portföljerna väljer studenterna sällan att använda explicita teorier, och ännu mindre forskningsresultat. Ämneskunskaper verkar vara viktiga för studenternas möjligheter att resonera om undervisning, däremot verkar skolans styrdokument bli viktigare än forskningsresultat och teori.

3 Hur känner vi igen kunskapsbasen? Eftersom teorin är så osynlig finns metodologiska svårigheter. I symposiet diskuterades olika ramverk och möjligheter för att kunna göra det till exempel *kommognition* (Sfard, 2008) och *Mathematical discourses in instruction* (Adler & Ronda, 2017). Problemen med att använda intervjuer för att få syn på de nyutexaminerade lärarnas användning av teori diskuterades också.

Note

1 Ett VR-finansierat forskningsprojekt om matematiklärarutbildning. https://www.mnd.su.se/forskning/matematik%C3%A4mnets-didaktik/forskningsprojekt/trace

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Short presentations

The role of figured worlds when student teachers become teachers

Andreas Ebbelind
Linnaeus University

This short oral reports on a study concerning conflicts emerging when student teachers, Evie and Lisa, engage in a teacher education programme. The theme relates to discursive engagement, where individuals negotiate a range of other past and present social practices. In discursive engagement a text is assembled through different textual units, for example figured worlds. The aim is to illustrate the role figured worlds play in immediate social interaction. A methodological tool, with the aim at uncovering why the speaker produces a particular wording rather than any other in a specific social practice, is used. The main point is that the figured worlds about teaching and learning mathematics are critical because they discursively inform every social practice that Evie and Lisa attend.

Student teachers' content knowledge for solving elementary school fraction exercises

Anne Tossavainen
Luleå University of Technology

The aim of the on-going study is to investigate Swedish student teachers' abilities in solving elementary school fraction exercises. As a part of a questionnaire, 59 elementary school teacher programme students were asked to solve nine fraction exercises taken from national tests and support materials for mathematics teaching. The analysis of students' solutions is based on Ball et al.'s framework for mathematical knowledge for teaching. The focus is on how students' common content knowledge is and whether it corresponds the requirements of elementary mathematics teaching. The preliminary results show limited fraction content knowledge and unstable procedural abilities that do not support the deep understanding of fractions needed in teaching mathematics in a meaningful way.

Pre-service teachers' explanations of division by zero and denseness of the number line

Kristina Juter
Kristianstad University

Students' beliefs about division by zero and numbers on the number line were studied through explanations of the concepts in questionnaires and interviews during their teacher education to become primary school teachers in the years 4–6. The concepts were chosen for students' proven cognitive challenges in coping with them, with the aim to add to the existing knowledge in terms of specific and general explanation types. General and specific parts of the students' concept images were contradictory in several cases and the examples used for explaining were often based on other mathematical structures than the ones explained, e.g. 2/1 instead of 2/0 or a finite decimal recitation instead of an infinite one.

The incorporation of programming in mathematical education

Andreas Borg
Karlstad university

The presentation will describe an ongoing design research study concerning the use of programming as a mathematical tool among students in Swedish upper secondary schools. During classroom interventions, students with no prior experience of using programming in school mathematics are observed trying to solve mathematical problems with the help of programming. The Instrumental Approach is used as a conceptual framework and the concept of instrumental genesis is intended to describe the process whereby students develop (inferred) mental schemes which together with the material artefact of a programming environment may act as an instrument in order to solve mathematical problems. During the presentation, results from the first intervention will be discussed.

Algebraic thinking regarding different mathematical contents within early algebra

Helena Eriksson
Stockholm University and Dalarna University

An ongoing literature review is conducted regarding algebraic thinking and early algebra, delimitated to students younger than 12 years and to different mathematical contents. Questions asked to the review are how algebra can be used to enhance mathematical contents, and how algebraic thinking is manifested together with these young students. A tentative result shows that among 40 articles of totally 500 are presenting algebraic thinking regarding some specific mathematical content. Indications are that algebra is manifested as notations using other symbols than numbers, verbal arguments, and gestures that make these arguments more explanatory. Algebra related specifically to number sense is manifested as operations with unknown, general pattern, and variables.

Mathematical assessments for six-year-old students in Sweden and Norway

Maria Walla
Dalarna university

This paper presents a study focused on early mathematics assessments in Sweden and Norway. In many countries, including those in the Nordic region, there has been a growing trend towards measuring students' knowledge and understanding, a trend that is seen even in early education. Since 2011, a mathematics assessment tool has been available for six-year-old students in Norway. In Sweden, an assessment intended for students of the same age has become obligatory from autumn 2019. When a new assessment becomes obligatory in early mathematics, its content influences the present discourse on mathematics education. In addition, as the discourse on mathematics education changes, the content that is taught, as well as the teaching and learning of mathematics, may also change.

Kollegialt lärande kring lärsituationer för gymnasieelever med särskild begåvning

Elisabet Mellroth [1] and Andreas Bergwall [2]
[1] Karlstads universitet and [2] Örebro universitet

Design research och *Cultural-historical activity theory* kommer i denna studie användas för att studera kollegialt lärande i matematik. Studien utförs med lärare som deltar i ett skolutvecklingsprojekt med ett delsyfte att utveckla undervisningen för elever med särskild begåvning. Forskningen syftar till att bidra med kunskap om kollegialt lärande som en hållbar utvecklingsprocess i en kommun.

Number sense vocabulary: reflections from a pre-school pupil

Oduor Olande
Linnaeus university

Pupils at a very early age are exposed to vocabulary expressing mathematical concepts. It is thus imperative in a teaching and learning process to gain insight into the different ways of relating to these concepts that pupils bring with them in a formal teaching and learning situation. In the present study a case of a six-year-old's reflections on number sense is analysed with the view of discovering embedded relationships thereof. Preliminary results indicated that while the pupil's reflections are characterized by a procedural-applicational orientation, there is a provided opportunity to engage in deep aspects of number sense.

Gazing at mathematical reasoning

Mathias Norqvist
Umeå university

Eye tracking can be used to investigate how students perceive and solve mathematical tasks. This short presentation will report on two such studies where task design, mathematical reasoning and students' gaze are in the spotlight. Results show that task design has an effect on students' reasoning, as well as on how they perceive the information given in a task.

Balancing interests in a research project through internal ethical engagement

Helena Grundén and Helén Sterner
Dalarna university and Linnaeus university

In educational design research projects, there are long-term relationships between researcher and participants. Hence, in addition to external ethical engagement, researchers have to engage in internal ethical issues, which became evident when a researcher suggested mathematical content for an intervention. The suggestion was both appealing to and uncomfortable for the teachers, and this ambiguity made power relations between the researcher and the participants visible. In the moment, the researcher made decisions about the content that might not be the best. This situation made visible the importance of internal ethical engagement in advance, for example, by thinking about how we care for our participants and for what and whom we are responsible.

Sources of inequivalence in translated mathematics tasks identified with students' reflections

Frithjof Theens
Umeå university

In multilanguage assessments, the validity of the results is threatened if the different language versions are not equivalent. In this study, task-based interviews with German and Swedish students were analyzed to identify possible sources of inequivalence between the language versions of mathematics PISA tasks.

The processing of mathematical symbols in working memory

Ewa Bergqvist, Bert Jonsson and Magnus Österholm
Umeå university

This empirical study examines how different types of symbols, familiar and unfamiliar, are processed in working memory; phonologically and/or visuo-spatially.

Mathematics and physics at upper secondary school: an analysis of two lectures

Kristina Juter, Örjan Hansson and Andreas Redfors
Kristianstad university

A physics lecture and a mathematics lecture, by the same teacher and partly the same students, were studied at upper secondary school. Both lectures covered ordinary differential equations. The main aim of the present paper was to investigate the teacher's different and similar ways to handle related mathematical content in the two school subjects. The findings show a structural use of mathematics with an analytical approach in mathematics and an applied approach in relation to formulas in physics. This study is part of a larger study about mathematics in physics education funded by the Swedish research council.

Socialt risktagande vid kritiskt tänkande

Johan Prytz [1] and Helena Isleborn [2]
[1] Uppsala universitet och [2] Tiundaskolan

Vår studie handlar om hur designen av gruppuppgifter i kritiskt tänkande i matematik (åk 9) kan påverka elevernas engagemang i kritiska resonemang. Mer precist undersöks hur uppgifter kan försätta eleverna i olika affektiva situationer – socialt riskabla situationer – och hur det kan påverka elevernas resonemang och övriga beteenden. Studien baseras på videoinspelningar från två undervisningstillfällen där eleverna har arbetat med, ur affektivt perspektiv, helt olika uppgifter. Analysen tyder på att uppgifternas design och den sociala risk de medför påverkar elevernas vilja att engagera sig i kritiska resonemang.

How mathematical symbols and natural language are used in teachers' presentations

Ewa Bergqvist, Tomas Bergqvist, Ulrika Wikström Hultdin, Lotta Vingsle and Magnus Österholm
Umeå university

In this study, we examine how the use of natural language varies, considering the symbolic language in procedural and conceptual aspects of mathematics.

Relational values in inclusive mathematics classrooms – an intervention study

Malin Gardesten
Linnaeus university

The focus of this paper is on the methodological approach in a design research study. The aim of the study is to explore how primary mathematics teachers coordinate mathematical and relational proficiencies for education to make the mathematical content accessible for every student. The researcher together with the participating teachers ($n = 5$) identified the existing and desired teaching situation in two mathematical classrooms. An intervention was implemented and documented by observations, video recordings and interviews with the teachers and the students. The intervention explored the interactions between the teachers and the students, to explain possibilities of how students can be given access to the mathematical content.

Programmering för lärande i matematik – beskrivning av ett forskningsprojekt

Johanna Pejlare
Chalmers tekniska högskola och Göteborgs universitet

Här presenteras ett nyligen påbörjat forskningsprojekt med det övergripande syftet att bidra till forskningen kring den pågående implementeringen av programmering i gymnasieskolans matematik, genom att undersöka på vilka sätt programmering kan erbjuda möjligheter för lärande i matematik jämfört med en mer traditionell undervisning. Med utgångspunkt i Chevallards teori om didaktisk transposition undersöker vi dels hur verksamma lärare tillämpar programmering i matematikundervisning samt vilka möjligheter, utmaningar och svårigheter de identifierar, dels hur elevers kunskaper i matematik kan utvecklas med hjälp av programmering. Ett övergripande mål är att lägga grunden för långsiktig och hållbar samverkan mellan universitetet och skolan via lärarutbildningen.

Research on the development of junior middle school mathematics teachers' beliefs – from the perspective of history and pedagogy of mathematics

Dandan Sun
East China normal university

This research intends to explore the development of junior middle school mathematics in-service teachers' beliefs in an online programme based on the history of mathematics. More specifically, explore what is the change of the teachers' beliefs of mathematics and mathematics teaching and how this happen. Questionnaire, reflection task and interview are used to collected data. It can be seen that these in service teacher' beliefs about mathematics change, including their view on the characteristics of mathematics, the history and development of mathematics and the relevance of mathematics to society. Their beliefs about mathematics teaching change too, including their view on the goal and process of teaching, the history in teaching and so on.

Digitala verktyg som stöd för elevers sätt att uppfatta geometriska figurer

Peter Markkanen
Örebro universitet

Detta paper presenterar preliminära resultat från en större designstudie om undervisning och lärande i geometri. Designstudien fokuserar undervisning i digitala miljöer och hur den kan erbjuda elever att bygga förståelse för geometriska figurer och deras uppbyggnad för att kunna nyttja dem i problemlösande aktiviteter och i geometriska resonemang. Resultatet visar att den digitala miljön med dess dynamiska egenskaper bidrar med möjligheter till att eleverna breddar sina sätt att uppfatta figurer och utifrån figurernas uppbyggnad förmår argumentera för geometriska satser och formler.

Velocity, acceleration and the experiences of the body – derivatives and integrals in real life

Ann-Marie Pendrill
Lund university

Derivatives and integrals are often seen as abstract concepts. In this work we study how the experience of the body, combined with graphs based on smartphone data, theoretical consideration and video analysis, can support student discussions of derivatives, as well as their understanding of integrals. Examples include vertical motion during trampoline bouncing and in a small amusement ride.

How epistemological characteristics influence the design of a course in projective geometry

Olov Viirman [1] and Magnus Jacobsson [2]
[1] University of Gävle and [2] Uppsala University

In this short presentation, we report parts of an ongoing project building on the *Anthropological theory of the didactic* (ATD) to investigate how the epistemological characteristics of the topic to be taught influences the design of university mathematics courses. We focus our attention on a course in Affine and projective geometry. Analysis is ongoing, but we present some initial observations and discuss them in relation to other courses previously analysed.

Critical aspects in mathematics teacher students' writing of lesson plans

Anna Wernberg, Jonas Dahl, Cecilia Winström and Lisa Björklund Boistrup
Malmö university

This paper describes the design and preliminary findings of an ongoing study, where teacher educators tried to gain insight into their own practice, including teacher students' learning, in order for improving it. Our research interest in this short communication is to identify critical aspects in relation to Mathematical knowledge for teaching (MKT), when teacher students develop lesson plans?

How mathematical symbols and natural language are integrated in textbooks

Ewa Bergqvist, Tomas Bergqvist, Lotta Vingsle, Ulrika Wikström Hultdin and Magnus Österholm
Umeå university

In mathematical text and talk, natural language is a constant companion to mathematical symbols. The purpose of this study is to identify different types of relations between natural language and symbolic language in mathematics textbooks. Here we focus on the level of integration. We have identified examples of high integration (e.g. when symbols are part of a sentence), medium integration (e.g. when the shifts between natural and symbolic language occurs when switching to a new line), and low integration (e.g. when symbols and written words are connected by the layout).