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Polysemy and the role of representations for progress in concept knowledge

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Mathematics is a polysemic enterprise, where the same name is given to different things. Polysemy is present whenever mathematical patterns, identified in different circumstances, share the same structure. This structure will be subsumed under the same mathematical symbolism. Therefore it is of interest to theorize upon progress in concept knowledge, with respect to the polysemic nature. We argue that concepts can be brought together through an epistemological shift that occurs when the meaning introduced by situations and iconic representations is replaced by meaning residing in non-iconic representations that exists independent of the situations and iconic representations. Our theorization can be used in teaching design and by curriculum developers.

Introduction

Every mathematical concept has a mathematical definition. It is only through carefully formulated formal expositions that we can ensure that the mathematics we construct is logically coherent and free of contradictions. Structuralistic expositions of mathematics came in vogue through the formation of the group of mathematicians writing under the pseudonym Nicolas Bourbaki. The Bourbaki group started their quest in first half of the 20th century, interestingly in part as a reaction to the work and style of the French mathematician and polymath Henri Poincaré, who tended to tone down rigor in his presentations to instead stress intuitive connections (Senechal, 1998). The tension between formalism and intuition is not only a matter of style, but is built into the mathematics itself. Poincaré allegedly formulated this as: *Mathematics is the art of giving the same name to different things*.

Poincarés dictum concerns polysemy: that the same lexical item stands for a family of related senses. In mathematics not only lexical items, but also mathematical symbol systems and the concepts themselves regularly have several but related senses. The symbol 1/3 can mean both one divided by three and the rational number one third. In this strong sense, mathematical concepts are polysemic.

Polysemy is not synonymous with ambiguity. In mathematics education research literature, however, polysemy has often been treated as something creating ambiguity and hence difficulties for students (Janvier, Girardon &

Morand, 1993; Zazkis, 1998). As we see it, polysemy is instead an essential feature of mathematics. It is polysemy that allows mathematics to be extremely compactly formulated and still widely applicable. Whenever mathematical patterns identified in different circumstances share the same structure, this structure will be subsumed under the same mathematical symbolism, creating polysemy.

Polysemy is very widespread in mathematics. Even the basic mathematical concept of whole numbers is conceptually polysemic. This has been thoroughly investigated by Lakoff and Núñez (2000), using the terminology of conceptual metaphors from cognitive linguistics. They describe how the concept of whole numbers and basic arithmetic is structured by means of four distinct experiential realms: object collection, object construction, using a measuring stick, and moving along a path. These are initially psychologically distinct, but by means of conceptual blending, they together form one single concept of whole numbers. An example from higher mathematics is given by Thurston (1994) in a famous article on the nature of mathematical work. Thurston refers to a list with at least 37 different ways of conceiving the derivative of a function and even claims "The list continues; there is no reason for it ever to stop" (Thurston, 1994, p. 164).

Our aim is to theorize polysemy and discuss implications for mathematics learning. In line with Vergnaud (1998) we see mathematical concepts as psychological constructs, but analysis of conceptualization "must be made in mathematical terms, since there is no way to reduce mathematical knowledge to any other conceptual framework" (p. 167). We will reflect on elementary mathematical concepts, and will argue that they are typically born out of classes of situations or from pictorial representations, that we will treat by the term iconic representations. Such elementary concepts are later often subsumed under the same concept by means of being labeled by the same words and by being handled by the same symbol system, creating polysemy. We will argue that it is not until we move to symbol system representations, that certain problems generated by polysemy can be resolved.

The paper is hence prescriptive, normative and theoretical. We declare certain aspects of mathematics as essential. Then, we draw conclusions about what this will mean for conceptualization and instruction. We complement this theoretical endeavor with examples from an empirical analysis of Swedish textbooks as a way to see to what extent the vision we present is realized in a present practice.

Previous research on theories for conceptualization

Conceptualization in mathematics has often been studied though examining a shift in development where the conceptual entity is first an action or process, and later becomes a mathematical object in an individual's mind. The cognitive process of forming an object, a static conceptual unit, from a dynamic process is referred to as reification (Sfard, 1991). The well-established APOS theory addresses reification by describing the process of conceptual understanding as evolving from actions to schemas (Asiala et.al, 1996). In APOS, a shift in conceptual understanding occurs through a cognitive organization of actions; processes and objects in schemas that form the framework for understanding concepts in new related problem situations. Sfard (1991) describes the development of mathematical concepts from process to object as a hierarchy in three stages: interiorization; condensation and reification. Sfard's claim that mathematical concepts start their lives as processes and her theory assumes that an ontological shift is needed for the concepts to become objects. “Only when a person becomes capable of conceiving the notion as a fully-fledged object, we shall say that the concept has been reified.” (Sfard, 1991, p. 19).

Skemp (1976), Tall and Vinner (1981), Hiebert and Lefevre (1986), and also Vygotsky (1962) introduced theoretical notions that are not based on hierarchy, but rather provide labels and classifications for knowledge. Development of knowledge is framed within dichotomies. Concept development is synonymous with closing the gap between divergent or flawed notions of mathematical concepts. A prominent example, the dichotomy between procedural and conceptual knowledge, has since its introduction by Hiebert and Lefevre (1986) been recontextualized several times. Today the labeling have moved from a clear cut between procedural and conceptual knowledge to a more nuanced view, where the two knowledge types cannot be seen in isolation from the other. How this knowledge shall be framed is still under discussion. “Although there is broad consensus that conceptual knowledge supports procedural knowledge, there is controversy over whether procedural knowledge supports conceptual knowledge and how instruction on the two types of knowledge should be sequenced” (Rittle-Johnson, Schneider, & Star, 2015, p. 587). Skemp’s (1976) notions *instrumental* and *relational* understanding represent a labeling of how students understand concepts as instrumental understanding (knowing what to do) and relational understanding (knowing what to do and why). Tall and Vinner (1981) have a similar approach; namely, the difference between the formal definition of a concept versus all the images of a concept a individual may have, that can be successful in some situations and limiting in others. Also Vygotskian theorization includes a labeled dichotomy between spontaneous concepts and scientific concepts that require teaching to be understood. Vygotsky emphasizes, “the very notion of scientific concept implies a certain position in relations to other concepts, i.e., a place within a system of concepts” (Vygotsky, 1962, p. 93).

As observed by Vygotsky (1962) all higher thinking is mediated by systems of signs. Duval (2006) characterizes semiotic system by referring to multi-functional systems, like spoken or written natural language, and mono-functional systems, like most specific mathematical symbol systems. We will be concerned with mathematical symbol systems. Nunes (1999) calls such representations compressed, indicating that they contain hidden culturally coded information not visually distinguishable in the symbols themselves. She contrasts this with extended representations, where the form of the representations itself contains the necessary information to decode the mathematical information at hand. We will use the term iconic (a term we will define more carefully later) for such representations and refer to other representation as non-iconic.

From the theories above we learn that conceptual progress is complicated psychological processes. However, to examine polysemy and analyze the role played by iconic and non-iconic representations we need to consider connection between concepts in different situations. This is why we pay special attention to work by Vergnaud.

Concepts cannot exist in isolation from other concepts. For that reason the French psychologist Gérard Vergnaud developed the *Theory of conceptual fields* (2009) to provide an explanation for that mathematical conceptualization psychologically is very complex, despite that formal mathematical expositions normally are clear and hierarchical. A conceptual field consists of a set of concepts tied together and a set of situations where the concepts apply. According to Vergnaud (c.f. 1998, 1999, 2009) the meaning of a concept comes from a variety of situations and, reciprocally, a situation cannot be analyzed with one concept alone, but only with several concepts, forming systems. Conceptual fields consist of such clusters of situations and concepts.

Theory

Our theorization will be formulated in the language of conceptual fields. Vergnaud (1999) defines a concept C = C (S, I, R) as a triplet of three related sets. S: the set of situations where the concept is relevant. I: the set of operational invariants that can be used by an individual to deal with these situations, and R: the set of representations, symbolic, verbal, graphical, gestural etc. that can be used to represent invariants, situations and procedures (for more information see Vergnaud, 1999). Note that in this definition, S and I are psychological categories, that is, mental constructions, while R can be both mental and physical/external. Since we are not here interested in explaining the thinking of particular individuals but to analyze mathematical conceptualization in general, we will deal with S and I from the point of an observer (Maturana, 1988). This means that we will assume that in educational and mathematical settings, enough individuals will form S and I similar enough so that makes sense to talk about them as phenomena in themselves, independent of a particular individual holding them.

We will extend and specify Vergnaud's definition. To avoid a lot of technical detail, we will abuse the notation of representation and call an image (◐), symbol (1/2), word (one half) or other combination of signifiers a representation, without specifying what it represents and for whom. Any of the three symbols above can in some certain circumstance represent any of the others, as well as some underlying abstract idea or invariant. We need to distinguish two particular types of representations, iconic and non-iconic. We call a representation iconic when some observable patterns in the representation correspond to some structure in the represented idea or invariant. The images ◐ and ◕ are iconic representations of multiplicative part-whole relationships. In non-iconic representations, the denotation instead builds on convention, where the typical examples are spoken language or letters being combined into words in written language. We will be interested in mathematical symbol systems and our main example is a/b. The non-iconic symbols 1/2 and 3/4 are not just composite symbols that can represent the same part-whole relationships as the partly colored circles above. They belong to a system with a set of transformation rules, governing how changes to a symbol relate to changes in the represented invariant.

Our theorization stipulates three things:

1. The origin of concepts. Vergnaud's view on concepts means that situations, invariants and representations are conceptually intertwined. But for concepts that are introduced in schooling, the initial invariant, from which the concept is bootstrapped, must come from either a situation or a representation (see footnote 2 in Duval, 2006, for an elaboration). Combining this with our characterization of two types of representations creates three essential ways of generating concepts. First, concepts can be connected to the invariants in a class of situations, like when meaning of division is given through describing numbers of things to be divided in a number of bags. Second, concepts can be connected to iconic representations, like when fractions are given meaning by an image of a partly colored circle. Third, concepts can be connected to mathematical relations formulated in non-iconic symbol systems, like when division is described by saying that a/b is a number c such that $a=b\cdot c$.

2. The umbrella effect. Like we exemplified with whole numbers, mathematical concepts are regularly subsumed into more general concepts. When concepts generated from situations or iconic representations are subsumed under more general concepts there will be invariants from the original concept that will cease to be invariant under the new umbrella. In the equal sharing situation, used to explain division above, division of a by b results in a number c that is smaller or equal to a, but this does not hold for division in general. Likewise, when part-whole relationships are iconically represented by circle sectors, no fraction can be bigger than the whole circle. But, in general a/b can have any size. Note that the three examples in the previous paragraph are all denoted by the same symbol system, a/b. Even though we in this case continue to talk about division and fractions as different things, we can from a mathematical point of view subsume both concepts under the umbrella of quotient constructions.

3. Contradiction of invariants. One iconic representation that generates fractions is part whole relationships of whole numbers. For $a \leq b$, a/b is associated to a colored beads out of b total beads, like 2/3 is a symbolic representations of the image ●●○. But another iconic representation that also generates fractions is when a certain part of one circle is taken to represents fraction, like when ◐ represents 1/2. Both these representations individually form straightforward one-to-one correspondences with the symbol system. But when the part-whole representation is extended to also denote numbers bigger than one, this model is mixed with the previous model. The representation is normally extended so that ●◐ is taken to mean one whole and one half that is 3/2. But when the ideas from the two representations are mixed, then ●◐ can just as well mean 3/4. The one-to-one relationship is hence broken. We claim that this case is typical for when different iconic representations or situations are used to generate the same concept.

The character of mathematics we have described and the theoretical conclusion we draw has important consequences for mathematical conceptualization. It is simply inevitable that when mathematics is grounded in iconic representations and situations, there will be cognitive conflicts whenever several subconcepts are gathered under the same concept, something we have described as an essential aspect of mathematics. These cognitive conflicts will be unresolvable as long as the meaning of mathematical concepts continues to be grounded in the iconic representations and situations. This is because there are invariants inherent in iconic representations (or situations) that are coherent within such a representation and in relation to the mathematical concept, but that are not coherent between iconic representations. When corresponding mathematical relations are represented and dealt with by a mathematical symbol system, the above-described incoherence are no longer present. Therefore, a necessary consequence of these theoretical observations is that it is important that teaching is designed to overcome the incoherence created by iconic representations. The meaning of concepts and the relations they entail are, wisely, initially drawn from situations and iconic representations, but must at some point be placed in the mathematical relations and the symbol systems. The icons and situations, that before generated the meaning needs to get a new role as just being examples, or concrete realizations of the mathematical.

An empirical example from Swedish textbooks

To what extent do the types of representations we have dealt with occur in practical mathematics teaching? As a proxy for how concepts might be taught, we examined two common Swedish textbook series, covering grades 1 to 6 and grades 7 to 9[1].

[1] The series are Favorit Matematik from Studentlitteratur, second edition for grades 1-6 by Karppinen, Kiviluoma, & Urpiola and Matte direkt from Sanoma utbildning, grades 7 to 9 by Carlsson, Hake, & Öberg.

Due to space limitations we will not dwell on our method and results in this theoretical paper. In brief, we selected all introductions of concepts, notations and procedures concerning multiplicative relations expressed in fraction notation, i.e., division, ratios and rational numbers. In total there were 56 such instances that were categorized according to a two dimensional typology concerning the form and function of arguments. The form was categorized according to if arguments referred to invariants in situations, in iconic representations or in a symbol system. The function was categorized according to if the argument was explanations of concepts, of connections, or of procedures or a description of notations.

The gist of the material is that while reasoning in terms of symbol systems at times are used to explain procedures, the typical case is that reasoning in terms of symbol systems is not used to explain connections or motivate concepts until grade 8. Concepts are instead explained by situations and iconic representations. Invariants in the situations and iconic representations are then labeled by the symbol system and it may be explained what aspect of the symbol that relates to what aspect of some invariant. The first real exception comes in grade 8, where reasoning in terms of relations in the symbol system is used to introduce the inverse. We discuss a representative case below.

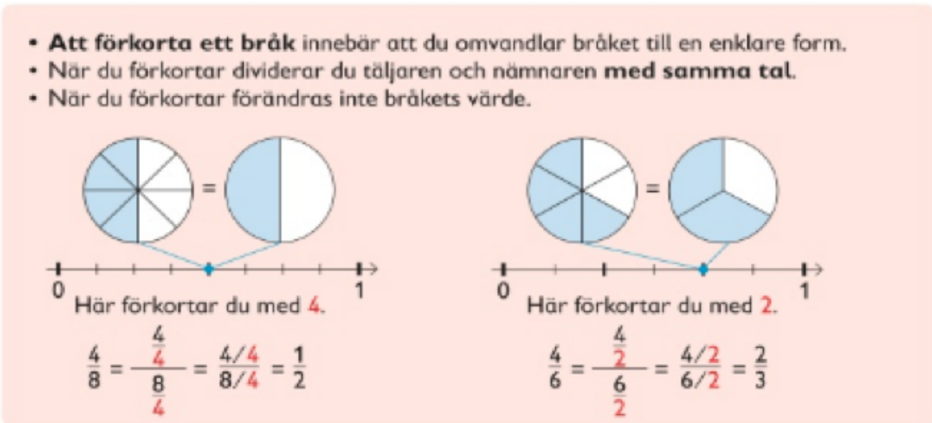

Figure 1: Introduction to reducing fractions in grade 5. The text marked with bullets in the top of the figure says: ***To reduce a fraction*** *means that you transform the fraction to a simpler form. When you reduce you divide the numerator and the denominator* ***with the same number****. When you reduce, the value of the fraction does not change.*

That the same fraction can be represented in different ways within a symbol system captures that a fraction is a multiplicative relationship. Equality of fractions is introduced in grade 4. It is repeated in grade 5, together with the procedure of how to reduce fractions (figure 1). It is shown in terms of the standard symbol system for fractions how reducing a fraction can be done by means of dividing the numerator and denominator by 2 (in the right hand example). The explanation, however, is not given in terms of mathematical relations in the symbol system. Instead, it is the geometric invariant in the iconic representation that supplies the explanation by showing that coloring 4 out of 6 equal segments signifies the same part of the of circle as coloring 2 of 3 parts. The number line iconically supplies a similar argument. In the symbol system representation, it can be considered known that 4/2=2 and 6/2=3, but no explanation to why it is allowed to divide both the numerator and the denominator by 2 is given within the symbol system. This is why we do not interpret this as an argument explained in terms of relations within a symbol system. Instead, the symbol system transformations are denotations of the reasoning given by the iconic representations. The same argument is repeated in grades 6, 7 and 8 with similar iconic representation as above.

An argument entailing reasoning with relations in the symbol system could for example build on factorization, observing that $4=2\cdot 2$ and $6=2\cdot 3$ and that $4/6 = (2\cdot 2) / (2\cdot 3) = 2/2\cdot 2/3=1\cdot 2/3=2/3$. Approaching reduction of fractions from this point of view would set the stage for answering two important questions that in the illustrated explanation remain unanswered and that in fact cannot be answered without factorization. Why divide with 2 and how do you know that you reached the simplest form? The answer to both questions depends on common divisors. It is worth noting that the symbol system approach would require that multiplication of fractions is introduced, something that is not done until grade 8 in the analyzed book series. While introducing multiplication of fractions earlier would be something we endorse, even without such a change, an approach involving factorization would still be able to give suggestions on how to rearrange iconic representations like the circle model or number line model when looking for how to reduce a fraction.

Implications for mathematics education

We have argued that progression in concept knowledge requires a deliberate movement from reasoning in terms of iconic representations and situations to reasoning within non-iconic symbol systems. This argument is not unique, as it could represent the classic saying *going from the concrete to the abstract*. However, our contribution is that we build our argument on the observation that mathematics is polysemic and that the practice of generating concepts by means of situations and iconic representations inevitably generate some contradictions between different situations, requiring different interpretations, represented by the same concept and symbol system. We exemplified this by looking at concepts that can be subsumed under the concept of quotient constructions, that is, anything we denote as *a/b.*

We emphasize that the movement towards reasoning in symbol systems is not only about becoming versed in using symbol systems. Just as important is the epistemological development of realizing that meaning can emerge directly from relations described in symbol systems. An epistemological shift occurs when situations and iconic representations, that previously generated the mathematical meaning, shift to be representations of some meaning that exists independent of the situations and iconic representations.

While we acknowledge important properties from several previous theories on conceptualization, like metaphors (Lakoff & Núñez, 2000), reification (Asiala et al., 1996; Sfard 1991), dichotomies in images (Tall & Vinner, 1991) instrumental and relational understanding (Skemp, 1976) and semiotic registers (Duval, 2006), we also add a focus on the shift to a change in representations, rather than a change in the nature of the object. What we call iconic representations could be elements of multifunctional registers, but the problem we observe is not about seeing the right thing, rather about the complications when different subconcept are merged (c.f. Duval, 2006). Nunes (1999) makes a distinction between extended and compressed representations that, as we mentioned, have resemblance with iconic and symbol system based representations. Nunes empirical analysis indicates "that extended representations are preferred at initial points in learning" but also that "the analysis of conceptual relations indicates the advantages in the use of compressed representations and thus the need to cope with the move from extended to compressed representations in mathematics instruction" (Nunes, 1999, p 38). We think this is important, in particular in relation to teaching design and production of curriculum materials. More emphasis on symbol system knowledge will obviously come with its own challenges and it is an empirical question how this can be done in practice. Our message is that the focusing on reasoning in terms of relations in symbol systems with the aim of creating learning opportunities for an epistemological shift deserves more discussion. The purpose with the present article is to contribute to such a discussion.

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