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Students' use of written and illustrative information in mathematical problem solving

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This study investigates how elementary students use written or illustrative information in their mathematical problem solving. A previous study indicated that students who focus on illustrative information in task solving are more successful than those who focus on the written. Our study expands this idea, suggesting that there are different ways of attending to illustrative and written data. Students can treat the two sources of information as isolated or trying to connect and combine them in order to verify or test solution ideas but also to generate new ideas. This may have implications for teachers seeking to support students in their problem solving. Encouraging students to make productive use of written and illustrative information may assist them in overcoming obstacles.

Introduction

Learning mathematics through problem solving has long been considered a productive way for students to learn mathematics. Teachers and researchers have tried out and investigated various approaches to instruction that promotes problem solving in mathematics. One of the dilemmas with problem solving to learn mathematics is that it is difficult for students to solve mathematical problems. This is somewhat of a paradox because if a task fails to be challenging to the student it also loses some of its potential as a tool for learning. Earlier research has shown that students who solve mathematical problems by creatively constructing new solutions are more likely to solve similar problems at a later stage than students who are given instructions on how to solve the problem (Lithner, 2008). This points to a crucial junction in mathematics teaching, students need to meet challenging problems, but it is to be expected that many of them will need help in getting past some of the challenges. This help however should not remove the challenges by introducing a method with which the problem can be solved but rather provide clues on how to overcome obstacles without a complete description of a solution method. Providing feedback that helps the student to proceed with her problem solving without giving her too much information is a demanding task for teachers. It is unlikely that there will ever be a best practice in the form of a fixed set of strategies for teachers who engage in interactions with students. There are however, several general ideas on what teachers should to consider in their interactions with students who are stuck in problem solving, examples include asking students to explain their reasoning, to encourage them to develop and justify their reasoning and to test their conclusions (Olsson & Teledahl, 2018, 2019).

Beyond such general approaches, research on the ways in which students approach certain problems may provide further clues on how teachers can assist students in overcoming stages that are problematic, in their problem solving. A recent study (Norqvist, Jonsson, Lithner, Qwillbard, & Holm, 2019) investigated what items of information that students focused on, while solving a non-routine task. The study used eye-tracking techniques and found that students who focused their attention on pictures that illustrated the mathematical problem were more successful than their peers in a post test. The present study aims to investigate this idea by examining students' reasoning in problem-solving situations that contain both written and illustrative data. An investigation of the ways in which students consider different data in a mathematical task may provide valuable information on how teachers can assist students in proceeding with mathematical problem solving in situations where they are stuck. The research question that guides the investigation is: In what ways are students using written and illustrative data in their problem solving?

Background

In school mathematics, teachers are often providing the students with procedures, which, if performed correctly, will solve tasks. When solving non-routine tasks this may foster strategies of recalling memorized procedures possible to use when constructing the solution. Lithner (2008) defines this approach to reasoning as algorithmic reasoning (AR). Why teaching mathematics this way, by providing algorithms, is a prevalent practice may be explained by the fact that it is relatively easy for the teacher to prepare, and the students are often successful in solving tasks (Blomhøj, 2016). However, a wide range of research has stated that teaching in which the teacher provides instructions on how to solve tasks, is not an efficient way to teach mathematics (Hiebert, 2003). Students will engage in rote learning, which is focused on executing steps in a procedure, without understanding the intrinsic mathematics. This behaviour excludes, what many studies suggest as important for learning; engaging in constructing and justifying solutions (Brousseau, 1997; Lithner, 2008). Brousseau (1997) claims that to learn mathematics one needs to construct solutions using mathematics, something which Lithner concretizes further with the definition of creative mathematical reasoning (CMR). That is, when solving non-routine tasks, for which students do not know a solution method in advance, they engage in constructing solutions and formulating arguments (Lithner, 2008). While they construct the solution method themselves, they must assess whether the method will solve the task or not. In this process, the mathematics will gain meaning for the student and she will learn. Such an approach to mathematics teaching requires a different teacher role. Instead of explaining how to solve tasks the teacher should prepare suitable tasks, encourage the students to use their mathematics resources and ask them to justify their solutions (Brousseau, 1997).

Several quantitative studies have confirmed that students who practice on tasks demanding CMR a week later score higher on post-tests compared to students practicing on tasks using AR (Jonsson, Norqvist, Liljekvist, & Lithner, 2014; Norqvist, 2018; Olsson & Granberg, 2019). These studies however do not go into explaining the mechanisms behind the differences. Furthermore, the findings in the studies indicate that there are many students who do not manage to solve CMR-tasks in practice. A step towards explaining some of the differences between successful and non-successful CMR with students, is discussed in a recent study which used eye-tracking techniques to explore what different data, presented in the tasks, students focused on while solving tasks (Norqvist et al., 2019). The study argues that students, when solving the tasks, extract different types of data (illustration, description, formula, example and question) necessary to solve the problem. The authors suggest that some students base their solutions on isolated examples of data from either text or illustrations, not using opportunities to combine text and illustration to verify their answers.

Visualisation in mathematics has long been acknowledged as important for students learning (Arcavi, 2003) but studies point in different directions. Some studies suggest that the combination of written and illustrative information in mathematical tasks can increase students' cognitive load, thus making it more difficult for them to solve problems (Berends & van Lieshout, 2009; van Lieshout & Xenidou-Dervou, 2018). Other studies, that have investigated students' use of carefully prepared illustrative information, have showed that this can be beneficial to students' problem solving and that productive use of visual imagery is common among expert mathematicians (Scheiter, Gerjets, & Schuh, 2010; Stylianou & Silver, 2004; Van Garderen & Montague, 2003). However, there is a need for further investigations to explain differences in success and learning, addressing students' reasoning when engaging in non-routine tasks that offer information in writing as well as through images.

In our ongoing project, we investigate teacher-student interactions aiming to support students' CMR. The approach is to iteratively establish principles for teacher action in these interactions, design mathematics activities based on the principles, and analyse the conducted activities with the purpose to develop the principles and make them useful to teachers (Olsson & Teledahl, 2018, 2019). Tasks that are used in the mathematics activities often combine written and illustrative data to instruct students. With inspiration from the study by Norqvist et al. (2019) on students use of illustrative information we revisited some of our data. A preliminary analysis indicates a pattern that appears to be common, students usually start their problem-solving process by trying to understand the written information, and then they turn to and try to understand the illustrative information. Our study is focused on the reasoning that follows this initial pattern of interpreting the problem.

Theory

Lithner´s framework for imitative and creative reasoning (2008) proposes, based on empirical studies, that a key-factor for successful learning when solving mathematical tasks is whether students engage in algorithmic or creative reasoning. Here, reasoning is defined as the line of thought adopted to produce assertions and reach conclusions in task solving (Lithner, 2008, p. 257). Algorithmic reasoning (AR) is characterized by attempts to recall a procedure that is supposed to solve the task. This includes memorized procedures from solving similar tasks and imitating teacher instructions. Creative mathematical reasoning (CMR) is characterized by the creation of a new reasoning sequence (or re-construction of a forgotten one) supported by arguments anchored in mathematics.

In our ongoing project, we have developed principles for teacher-student interactions in teaching aimed at students learning mathematics through CMR. In mathematics, teaching aiming for CMR students must have possibilities to (a) express independent reasoning, (b) develop and (c) justify their own reasoning and to (d) test their results. These principles can be used both for planning and implementing teaching, addressing both design of tasks and preparing teacher-feedback interactions.

The tasks used in this study were designed in line with Lithner´s (2017) principles: (1) creative challenge, no solution methods are available from the start and it must be reasonable for the students to construct the solution, (2) fair conceptual challenge to understand mathematical properties (e.g., representations and connections) and (3) justification challenge, is it reasonable for the particular student to justify the construction and implementation of a solution.

Method

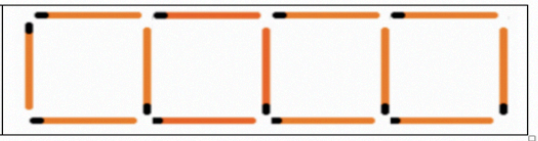
The aim of this study was to investigate students' reasoning when using textual and illustrative data in mathematical problem solving. Our study uses and re-examines research data collected continuously in an ongoing project aimed at investigating ways in which teachers can assist students in overcoming various obstacles in problem solving situations. The students and their mathematics teacher were part of the project for three years starting when the students enrolled in fourth grade and ending when they finished sixth grade. During this time, the students were regularly engaged in problem solving activities in which they worked in pairs. Problem-solving sessions were audio-recorded through a portable device placed on the students' desk. For each pair of students in this study the recordings are complemented with notes on students' body language. For every session the teacher was wearing a microphone and her interactions with every student group was also recorded. This study uses six recordings from two problem solving sessions where the mathematical task that students were working on was presented in a way that combined written and illustrative data (see fig 1 & 2).

The analysis of the recordings focuses on sequences where students are constructing a solution to the problem based on written and/or illustrative data. What is of interest is the way students use the data to create a reasonable solution, for example by extracting elements possible to calculate. The first part of the analysis was focused on identifying instances, in which students during their problem solving explored both written and illustrative data. In a second step, students' apparent use of the two sources of information was analysed in an effort to identify which data was used at various stages of the problem-solving process and in what way.

In listening to, and reading transcripts of, students' reasoning it is sometimes difficult to distinguish the written data from the illustrative. We have relied on notes on students' use of body language and on explicit clues in their reasoning, such as "look" or "…here there are…" but we have also tried to identify clues to their focus in what is not mentioned, such as sequences in which there is no mention of any information that can be thought of as deriving from the illustration (an example of this can be found in the excerpt of the transcript, page 6, lines 4-5).

The figure is built by matchsticks

To build 4 squares takes 13 matchsticks



a) How many squares does it take to build 7 squares?

b) How many squares does it take to build 50 squares?

c) Explain a way to calculate how many matchsticks are needed to build any number of squares?

Figure 1: The matchstick task

Temperature can be measured in different scales. Here temperature is measured in Celsius (C) and Thomson (T)

- If the temperature increases 1° C it increases 2° T
- When C is 0° C T is 10°

a) What temperature shows a Thomson thermometer if a Celsius thermometer shows 15° C?
b) What temperature shows a Thomson thermometer if a Celsius thermometer shows 40° C?
c) Find a way to calculate how many degreases T if you know how many degreases C
d) Find a way to calculate how many degreases C if you know how many degreases T

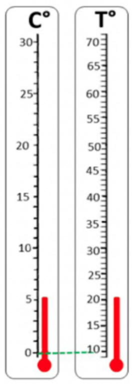


Figure 2: The thermometer task

Findings

When exploring students' full solution sequences of two tasks including both written and illustrative data, we observed that most students started with trying to construct a solution based on the written data. When this was not enough, they turned their attention to the picture. Then, two groups were formed, students who made connections between written and illustrative information and students who based their solutions on isolated examples, either from written or illustrative data. Students A and B's approach to the Matchstick task is an example of an attempt to construct the solution on isolated examples, first trying to construct the solution based on written data, and then by using the picture.

1.	A	If 4 squares are 13 sticks, we can calculate how many sticks are needed for one square… 13/4… but 13 is not in the multiplication table of 4…
2.	B	But you need one less on this side (points at the last stick in the square most far to the right)
3.	A	Instead we can draw or build seven squares and count the sticks

The example shows students who try to construct a solution based on the explaining text. The strategy is to calculate backwards to find out how many sticks are needed for one square. Student A realizes 13/4 will not result in a whole number. Student B observes that for the last square one less stick is needed. The students abandon the first strategy and find another one, draw and count, based on the figure. The observation on line 2 could have been used to explain why the counting backwards strategy did not work and connect the written and illustrative data for the task. Instead, the students seem to be satisfied with finding an alternative accessible strategy. This has consequences for the continuing attempts to solve the task.

4.	B	Okay, 22 sticks for 7 squares… how many are needed for 50 squares?
5.	A	uhm… we can check the calculation table for 7
6.	B	Yes… look… there were 22 for 7 [squares]
7.	A	Yes… but the multiplication table for 7 only includes 49… we can calculate 22 x 7
8.	B	That is 154
9.	A	Yes… and then we need another square
10.	B	And in that one we only need 3 sticks
11.	A	Then it is 157

The students return to the use of a calculating strategy, even though they previously observed the problem that one square has a different number of sticks, possibly because they realize it is not possible (or at least a lot of work) to draw and count as they did in the first example. What is interesting is that they make use of their insight that the last square only needs three sticks when adding the last square (line

9-10). In addition, in this part of the solution, they use information both from text and from figure, but they do not connect them to each other. This can also be interpreted as a sign of their satisfaction with finding a strategy that seems to work.

These examples indicate that it is difficult for students to combine information from textual and illustrative resources. Students can use information from both, but they do not connect them and draw conclusions important for the solution.

When students A and B believe they have solved the task the teacher (T) asks them:

12.	T	Can you explain the way you were thinking when you solved the task? Can you find a way to check your answer?
13.	A	We were thinking that 7 squares are 22 sticks and the multiplication table for 7 goes to 49
14.	A	So 22x7 is 154 and then we needed one more to have 50
15.	B	And then we needed 3 more sticks.
16.	A	yes, because you only need 3 sticks to build another square… but wait… we have calculated too many… we have used too many four-sticks-squares…

When students explain how they solved the task they realize that they have too many sticks because every new square only needs 3 sticks. Now they make the connections between their numerical approach and the insight that every new square adds 3 sticks. It is possible that if the teacher had not asked the students to explain they would have been satisfied with their incorrect solution.

Students C and D approached the Thermometer task (figure 2) by reading the written data. They came up with the solution to subtask *a* that 15° C equals 25° T. They are unable to figure out how to solve subtask *b* by using only information from the text, so they turn their attention to the picture and observe that for different temperatures there are different differences between T and C:

17.	C	5° C is 20° T
18.	D	10° C is … what are 10 ° C?
19.	C	It is 30° T … and 15° C is 40°
20.	D	But wait … we answered 25 [subtask a)] ... 5 steps in C are 10 steps in T

After correcting subtask *a* they continued:

21.	C	this must be correct … look … if C increases by 1° T increases by 2° … and here there are 5 steps for 10 … T increases twice [compared to C]
22.	D	and while T starts at 10° … C is 0° when T is 10° and you add twice as many C to 10 [to calculate T]

Students C and D's solution of subtask *a* seems to be based on the single example that 0° C equals 10° T. When exploring the picture, they realise they are wrong (line 20). The observation that 5 steps in C equals 10 steps in T is then combined with the written information (line 21) and the solution that T always increases twice as much as C is drawn. On line 22 the conclusion on line 21 is combined with the information that when C is 0° T is 10° and a general solution to how to calculate T out of C is presented. In comparison to students A and B students C and D combines written and illustrative data, in an earlier stage of the solution.

Discussion

This study is inspired by (Norqvist et al., 2019) in which eye tracking techniques were used to investigate what students focus on when they solve mathematical problems. The authors suggested that students, who focused on illustrative data in a task, when solving a problem, were more likely to solve similar tasks in a post-test. In our study, we have tried to identify not only what data the students use but also in what way it is used. Our results indicate different ways to attend to illustrative data. Students can start their construction of a solution by using only data, which is provided in writing, and then turn to the illustrative when they are unable to construct a viable solution method. This is illustrated by our first example in which students A and B search the illustrative data to find an explanation to why their proposed solution method of dividing the number of matchsticks with the number of squares, does not suffice to solve the problem. These students however turn back to their original idea, which is now modified, and abandon the picture as a source of information. Students C and D on the other hand turn back and forth between the written and illustrative data using both to verify their ideas, but also to assist them in forming new ideas. By combining the two sources of information they move away from the idea of using an isolated example to inform their reasoning. Their proposed solution method is checked repeatedly against the written data and the information that is derived from the image. In this way, they create arguments for their solution method that take several of the conditions of the task and the subtasks, into consideration.

It is risky to generalise based on a few examples, but it is not unreasonable to assume that the way students attend to and use illustrative and written data in mathematical problem solving may enhance their possibilities to independently construct their own solutions. This creates situations that are beneficial to students learning (Brousseau, 1997; Lithner, 2008). If this is the case teachers should consider the various ways in which they can design tasks and assist students in using the two data sources in a productive way. As has been shown in previous studies (Berends & van Lieshout, 2009; van Lieshout & Xenidou-Dervou, 2018), students' access to illustrative data can increase their cognitive load which is something that does not necessarily support their problem solving. This suggests that teachers may play an important role in supporting students' use of information from more than one source, encouraging them to move back and forth between the two sources of information and to, when appropriate, use either information to verify their proposed solution methods. For the students C and D, the image of the thermometers is combined with the written data, and it seems as if it is the combination that assists them in their reasoning. In this example, it is also obvious that the interaction between the two students benefits from the two sources of information as they move from discussing claims to checking them against the image as well as the written conditions. Previous studies have also suggested that using visualisations is an important part of problem-solving skills (Scheiter et al., 2010; Stylianou & Silver, 2004). Encouraging students to make productive use of different data is thus a potential a new principle in our on-going project.

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