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Conceptual knowledge in mathematics – engaging in the game of giving and asking for reasons

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In this study I re-analyse a transcript from Kazemi and Stipek (2001) in order to show how constructs of the semantic theory of inferentialism can be used to give account of conceptual knowledge in mathematics. In mathematics education research connections are crucial to conceptual knowledge. Inferentialism provides a theoretical conceptualization of connections, in terms of inferential relationships and moves in the language practice of giving and asking for reasons. Based on this conceptualization, the present study shows how constructs of inferentialism can facilitate a fine-grained analysis of conceptual knowledge in mathematics and provide insight on teachers' actions in pressing for conceptual knowledge in teacher-student interaction.

Introduction

There are several frameworks for describing knowledge in mathematics. Across different frameworks, research seems to agree on *conceptual knowledge* as one core component of mathematical knowledge (e.g., Kilpatrick, Swafford, & Findell, 2001; Niss, 2003). Conceptual knowledge corresponds to relational knowledge (Skemp, 1976) and is thought of as "a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (Hiebert & Lefevre, 1986, pp. 3-4).

Since Hiebert and Lefevre's (1986) seminal work, conceptual knowledge has come to have a prominent position in defining and characterizing mathematical knowledge (Star, 2005) and in capturing the learning of significant concepts within different mathematical domains (Baroody, Feil, & Johnson, 2007). However, it has been observed an ambiguity in how conceptual knowledge is understood and used (Baroody et al., 2007; Star, 2005). In line with Hiebert and Lefevre's (1986) definition, research often emphasizes connections as crucial to conceptual understanding but, what Crooks and Alibali (2014) observe is that the meaning and structure of connections are often described in vague or general terms, with limited theoretical grounding.

The aim of the present study is to show how constructs of inferentialism (Brandom, 1994, 2000) provide opportunities for fine-grained analyses of conceptual knowledge and of teachers' actions in pressing for conceptual knowledge in teacher-student interaction.

To make my analytical point, I re-analyse a transcript from Kazemi and Stipek (2001) where one teacher is trying to press two students to extend their reasoning on addition of fractions. Kazemi's and Stipek's study is representative for many studies on conceptual knowledge in mathematics, where conceptual knowledge is described in general terms, with no explicit theoretical grounding (Crooks & Alibali, 2014).

Conceptual knowledge – type and/or quality of mathematical knowledge?

Despite a clear movement in both research and educational practice toward emphasizing conceptual knowledge it does not appear to be a clear consensus in the literature as to what exactly conceptual knowledge is and how best to measure it (Crooks & Alibali, 2014). The term "conceptual knowledge" has come to denote a wide array of constructs. Hiebert and Lefevre (1986) define *conceptual knowledge* as:

> …knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information (pp. 3-4).

Conceptual knowledge is keyed on connections, but the definition says nothing about whether the connections relate to mathematical procedures or concepts. Star (2005) argues for the need to treat type and quality as two independent dimensions in the conceptualization of conceptual (and procedural) knowledge in mathematics. Star proposes that conceptual knowledge is the knowledge type of concepts or principles and describes the quality of conceptual knowledge in terms of connections. Commenting on Star's proposal, Baroody et al. (2007) suggest an alternative in which they describe quality as a matter of connections along the Likert scale of *not*, *sparsely, somewhat, well* and *richly connected.* However, what is meant by, e.g., sparsely connected, or how this differs from somewhat connected, is not theoretically underpinned or described in any detail (Nilsson, in press).

The suggestion of the present study is to redefine conceptual knowledge as exclusively a knowledge type and to use the theory of inferentialism to extend our understanding of the role and meaning of connections as a means for giving account of qualities of conceptual knowledge. To this end I follow Anderson et al. (2001) and define conceptual knowledge as the knowledge of *classifications, structures* and *principles*.

An inferentialist account of conceptual understanding

Inferentialism constitutes a semantic theory (Brandom, 1994) in which inferential relationships (connections) in concept use are considered a necessary and inseparable part of knowledge and meaning-making (Bakker & Derry, 2011). Following Sellars (Sellars, Rorty, & Brandom, 1997), Brandom (1994) argues that we, as human beings, possess the ability to make judgments and form reasons for how and why things happen as they do (Bakker & Derry, 2011; Brandom, 2000). A belief or an action of a person has a position in the space of reasons as the person is aware that their belief or action could be different and can reflect whether their belief or action should be as it is (Mackrell & Pratt, 2017).

Inferentialism takes a social and pragmatic stance on meaning-making and understanding. Rather than considering the space in which thought moves, Brandom suggests looking at inferences from the perspective of playing a language game (Wittgenstein, 1968). Brandom (1994) introduces *the Game of Giving and Asking for Reasons* (GoGAR) as a metaphor to describe how knowledge and meaning-making emerge inferentially within a social and pragmatic practice of reasoning. The principle idea is that we, as human beings, negotiate meanings in the way we use concept[1] (Seidouvy, Helenius, & Schindler, 2019). In the words of Brandom (2002); concepts become what they are according to how they are used, in "being a move in the 'game of giving and asking for reasons'" (p. 528).

Inferentialism is resolutely holistic (Bakker & Derry, 2011). One needs many concepts in order to have any, since the content of each concept is constituted by its inferential relationship to other concepts. Think of the situation of having practical mastery of the concept of "probability," coming into articulation in the claim, "the probability of six is 1/6 when rolling a die." Claiming this implies, among many things, to know that the claim is based on a perfectly symmetric die, that the relative frequencies of sixes stabilize around 1/6 as we increase the number of rolls and that the probability of not having a six is 5/6. This example involves a GoGAR of many reasons related to the concept of probability, of which only a few been made explicit here. The main point, however, is that these reasons are relevant and become contentful due to their inferential connections.

At least four types of inferences can be delineated (Brandom, 2000; Nilsson, in press):

- *Identity*. Identity inferences are probably the most common inference in language games. It speaks to our ability to make classifications. For instance, pointing to the picture of a quarter-shaded circle one could write "This is ¼" and someone else could say "This is 25 percent".
- *Negations*. Negation inferences speak to comparison and contrasting. We understand what something is when we are able to infer what it is not (cf., Marton, Tsui, Chik, Ko, & Lo, 2004).
- *Conditionals.* Conditionals take the form of "if-then" clauses and are probably the prime construct of inference. With conditionals, we focus on the circumstances needed for something to happen or to be. The inference, "*If* there are 50 black marbles in a bag with a total of 100 marbles *then*, it is a fifty percent chance to pull a black marble from the bag" exemplifies a conditional.
- *Counterfactuals* (Brigandt, 2010). Counterfactuals relate to conditionals, as they deal with circumstances in an explicit way. Significative of counterfactuals is clauses involving the expression of "had been". Say that you encounter a parallelogram (P) for the first time. Expressing, "if all four angels of P would have been 90 degrees, then P *would have been* a rectangle" exemplifies a

[1] Brandom makes no significant distinction between terms, words and concepts so in the rest of the article I will use them interchangeable.

counterfactual, which adds to your understanding of when to use the concept of rectangle and the concept of parallelogram.

I re-analyse a transcript from Kazemi and Stipek (2001) to show how an inferentialist account of conceptual connections can facilitate a fine-grained analysis of conceptual knowledge in classroom talks in mathematics.

An inferentialist re-analysis of conceptual knowledge

Kazemi's and Stipek's study is cited 480 times (Google Scholar 25 September, 2019) and is representative for many studies on conceptual knowledge in mathematics, where conceptual knowledge is described in general terms, with no explicit theoretical grounding (Crooks & Alibali, 2014).

Kazemi and Stipek's (2001) study involved four teachers in grades 4 and 5, all teaching the same lesson on the addition of fractions. The aim of the study was to "analyse and provide vivid images of classroom practices that create a press for conceptual learning" (Kazemi & Stipek, 2001, p. 78).

Kazemi and Stipek (2001) analysed and compared episodes from two lessons of high press and two lessons of low press. In the present study I will focus on one transcript on high press interaction (Ms. Carter), since it provides most vivid images of classroom talks in mathematics for the support of conceptual learning. The episode is from a fifth-grade classroom.

It is important to understand that the purpose is not to question the accuracy of Kazemi's and Stipek's analysis. According to Kazemi and Stipek, their analysis is more extensive and detailed than what is common in many studies looking at conceptual knowledge in mathematics. So, in some sense their study is used as a critical case (Flyvbjerg, 2006). In other words, if I manage to show that inferentialism can provide support to elaborate further on Kazemis and Stipek's analysis on conceptual knowledge, there is reason to believe that this would be possible in many other cases.

I begin the analysis below by briefly presenting the conclusions made by Kazemi and Stipek. I then turn to the inferentialist analysis, which took place in the following steps. Firstly, in order to account for conceptual learning, I searched for inferential patterns enacted, according to identity inferences, negation inferences, conditionals and counterfactuals. Secondly, I looked at the teacher's role. Kazemi and Stipek's focused on how the teachers pressed for conceptual learning. Looking at the episode from an inferential lens, in the second step I was searching for instances where the teacher missed opportunities to press students' reasoning further, making the GoGAR more explicit. Thirdly, I compared the outcomes of Kazemi and Stipek's analysis with the inferentialist analysis.

Results

Episode 1 in Kazemi and Stipek (2001)

Ms. Carter asked Sarah and Jasmine to explain how they divided nine brownies equally among eight people and why they chose particular partitioning strategies.

Sarah:	The first four, we cut them in half. [Jasmine divides squares in half on an overhead transparency]
Ms. Carter:	Now as you explain, could you explain why you did it in half?
Sarah:	Because when you put it in half, it becomes ... eight halves.
Ms. Carter:	Eight halves. What does that mean if there are eight halves?
Sarah:	Then each person gets a half.
Ms. Carter:	Okay, that each person gets a half. [Jasmine labels halves 1- 8 for each of the eight people.]
Sarah:	Then there were five boxes [brownies] left. We put them in eighths.
Ms. Carter:	Okay, so they divided them into eighths. Could you tell us why you chose eighths?
Sarah:	It's easiest. Because then everyone will get ... each person will get a half and [whispers to Jasmine] How many eighths?
Jasmine:	[Quietly to Sarah] 1/8.
Ms. Carter:	I didn't know why you did it in eighths. That's the reason. I just wanted to know why you chose eighths.
Jasmine:	We did eighths because then if we did eighths, each person would get each eighth, I mean 1/8 out of each brownie.
Ms. Carter:	Okay, 1/8 out of each brownie. Can you just, you don't have to number, but just show us what you mean by that? I heard the words, but ... [Jasmine shades in 1/8 of each of the five brownies not divided in half.]
Jasmine:	Person one would get this ... [Points to one eighth.]
Ms. Carter:	Oh, out of each brownie.
Sarah:	Out of each brownie, one person will get 1/8.
Ms. Carter:	1/8. Okay. So how much then did they get if they got their fair share?
Jasm./Sarah:	They got a 1/2 and 5/8.
Ms. Carter:	Do you want to write that down at the top, so I can see what you did? [Jasmine writes 1/2 + 1/8 + 1/8 + 1/8 + 1/8 + 1/8 at the top of the overhead projector.]

Kazemi's and Stipek's presented analysis

In this situation of high press exchanges, Kazemi and Stipek claim that students went beyond descriptions or summaries of steps to solve a problem. Instead, the students linked their problem-solving strategies to mathematical reasons and, in that sense, Kazemi's and Stipek's approach parallels the inferentialist idea that conceptual content is constituted in the game of giving and asking for reasons. Kazemi and Stipek then infer that, the exchange among Sarah, Jasmine, and Ms. Carter highlights a conceptual focus:

> "Ms. Carter asked Sarah to explain the importance of having eight halves and the reason why the partitioning strategy using eighths made sense. After Jasmine gave a verbal justification, Ms. Carter continued to press her thinking by asking her to link her verbal response to the appropriate pictorial representation by

shading the pieces, and to the symbolic representation by writing the sum of the fractions." (Kazemi & Stipek, 2001, p. 65)

Some reflections on Kazemi's and Stipek's analysis/conclusions. In the first line we see that 'reasons" are part of their analysis. However, what they ascribe as a reason in the exchange is not clear and not is the meaning of reasons, in terms of what follows from it and what it follows from. Next, we note that 'link' is central to their analysis. It is claimed that links are made between verbal responses and pictorial representations and symbolic representations. But, the nature of the links (connections) are not made explicit. Are the links just acts of classification, according to identity inferences, there is a low degree of inferential reasoning involved and so of conceptual knowledge.

Inferentialist analysis

Kazemi and Stipek (2001) used the episode of Sarah, Jasmine and Ms. Carter as an example of high-press interaction that moves the talk beyond descriptions or summaries of steps to solve a problem. However, scrutinizing the interaction by an inferentialist lens, we are provided more detailed information of the content of the talk and of missed opportunities for the teacher to press the students' conceptual understanding on fractions further.

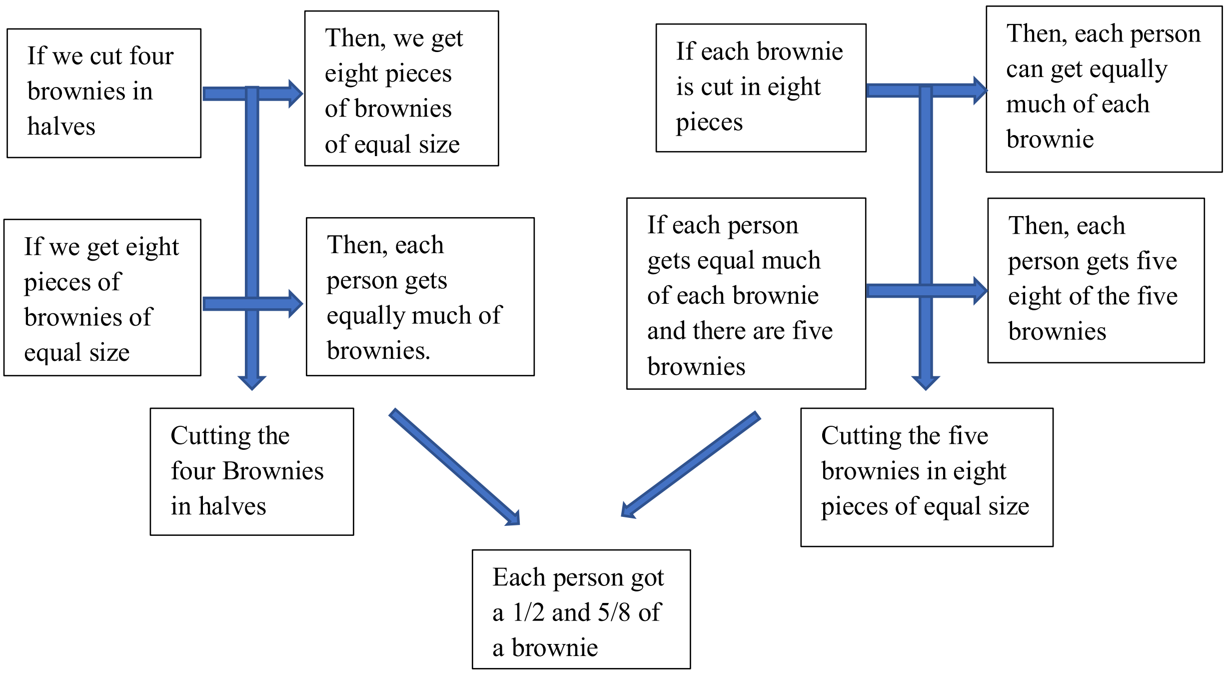

Figure 1. A summary of the inferentialist analysis of the solutions to the Four-brownie task (to the left) and to the Five-brownie task (to the right). If and Then are in italics to make explicit the conditional relationship between claims.

Figure 1 present a summary of the inferentialist analysis of the episode with Sarah, Jasmine and Ms. Carter. From a GoGAR-perspective we can say that the episode centers around a "four-brownie task" and a "five-brownie task". The four-brownie task is about explaining why each person gets a 1/2 of a brownie from four brownies and the five-brownie task is about explaining why each person gets 5/8 of a brownie from 5 brownies. The GoGAR is then about making sense of these two tasks and the solution of them.

The solution to the four-brownie task is structured in two conditionals expressing *quotitive division*, whereas the solution to the five-brownie task is structured in two conditionals expressing *partitive division* (Figure, 1). Making explicit their solution to the four-brownie task Sarah claims, "The first four, we cut them in half". If this claim comes alone, one could infer that the answer applies to partitive division, resulting in "two": half of four is two. However, the phrase involves "cut" and is accompanied with Jasmine dividing each of the four squares in half on an overhead transparency. So, the inferential meaning of half is not constituted from how half takes a position in "half *of a set* (partitive division)" but, from how it takes a position in "how many half brownies *is contained in* four (whole) brownies (quotitive division)". Further, to make sense of Sarah's and Jasmine's reasoning, one may infer that, what they present is not the actual solution but a reconstruction of a solution. In other words, Sarah and Jasmine knows that each person will have half of a brownie out of the four brownies and, what they do is to show that four brownies contains, or can be split into, eight halves. An alternative interpretation could be that the students infer from the implicit equation $4 \bullet x = 8$ that each person will have half of a brownie. So, to understand more about the students' reasoning to make the content more explicit, and so, accessible to the rest of the class, there were reasons to press Sarah and Jasmine further on the four-brownie task.

The solution of the five-brownie task follows a two-step structure according to partitive division. Sarah claims, "…We put them in eighths". Ms Carter then adds to the GoGAR, "Okay, so they divided them into eighths". In partitive division you want to find out the size of each part if you have a whole that is to be divided into a given number of parts. The brownies are to be distributed over eight persons so, the brownies should be divided into eight parts of equal size. However, now the total amount of brownies was not cut directedly in 5/8. In a two step-procedure, each brownie was first cut in eights. Each of the five brownies then contributed with 1/8 of a brownie so every person received 5/8 of a brownie.

I agree with Kazemi and Stipek that Ms. Carter presses the talk beyond descriptions or summaries of steps to solve a problem. However, from the inferentialist analysis, we can see several situations where the teacher is not taking advantage of opportunities to develop the GoGAR on fractions further. For instance, Ms. Carter leaves the solution on the form of two separate fractions 1/2 and 5/8. The GoGAR could have been extended by pressing the students to make explicit the *identity* inferences from "1/2" to "4/8" and then, consequently, from "1/2 + 5/8" to "4/8 + 5/8= 9/8". The GoGAR could then have been further extended by making explicit the identity inference from 9/8 to $1\frac{1}{8}$.

Ms. Carter could also have developed the GoGAR by means of negotiation inferences. In other words, she could have pressed the students to elaborate on differences between the conditional patterns in the solution of the four-brownie task (quotitive division) and the conditional patterns in the solution of the five-brownie task (partitive division). The fundamental negation is that, the principle by which Sarah and

Jasmine distribute the first four brownies is *not the same* as the principle they use to distribute the last five brownies.

Concluding Discussion

In this study I have re-analysed a transcript from Kazemi and Stipek (2001) in order to show how constructs of the semantic theory of inferentialism (Brandom, 1994, 2000) can be used to give account of conceptual knowledge in mathematics. Research emphasizes connections as crucial to conceptual knowledge (Hiebert & Lefevre, 1986). However, Crooks and Alibali (2014) observe that the meaning and structure of connections are often described in vague or general terms, with limited theoretical grounding. On a theoretical level then, the significance of the present study should be seen according to how it provides a theoretical conceptualization of connections, in terms of inferential relationships and moves in the language practice of the game of giving and asking for reasons. On an empirical level, the significance of the study should be seen according to how this inferential conceptualization of connections facilitates a fine-grained analysis of conceptual knowledge in mathematics and provide insight on teachers' actions in pressing for conceptual understanding in mathematics in teacher-student interaction.

Kazemi and Stipek (2001) used the episode presented above as an example of a teacher creating high-press interaction. However, looking at the episode by inferentialist means, we came to see that many inferential relationships were left implicit in the interaction. There were several opportunities to push the students further, making the content even more explicit and accessible to the class. But, of course, inferentialism does not do all of the job itself. Being able to perform the above inferentialist analysis requires that the analyst is knowledgeable in mathematics. So, on a theoretical level, the inferentialist perspective suggests the need for further research on investigating and/or trying to developing teachers' inferentially structured knowledge in mathematics.

There was no counterfactual in play in the episode above. Nor, was it, in the same way as was the case with a negation inference, easy to see how a counterfactual could be brought to the fore as an explicit topic of a GoGAR. It was actually difficult to give account of any counterfactual at all, when looking at the entire empirical material presented in Kazemi and Stipek (2001). This is probably representative for many classrooms. However, if conceptual knowledge is about making semantic connections, there is reason to engage students in a variety of contexts in which such connections can be exercised. On this account, the present study suggests research to explore the design of tasks and activities that challenge students and teachers to engage in GoGARs that allows for counterfactuals.

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