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Problem solving as a learning activity – an initial theoretical model

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Problem solving has been considered the gold standard of mathematical activity. It is a goal of mathematics education that students become problem solvers, and it is suggested that problem solving is a superior method for learning mathematics. However, the arguments supporting the claim that problem solving leads to better learning are often vague. In specific studies, problem solving often constitutes mere one part of a compound design, making it difficult to determine the specific contribution of problem solving. The aim of this paper is to develop an initial theoretical model for problem solving as a learning activity, based on existing frameworks and previous research. Suggestions for how this model could be empirically tested are also discussed.

Introduction

Problem solving has a special status in mathematics education. Problems are termed "the heart of mathematics" (Halmos, 1980, cited in Schoenfeld, 1992, p. 339) and it is stated that "We do mathematics only when we are dealing with problems" (Brousseau, 1997, p. 22). Developing problem solving competence is seen as a key goal for learners (NCTM, n.d.; OECD, 2019; Schoenfeld, 1992; Skolverket, 2019) and problem solving is seen as, and shown to be, essential for developing mathematical knowledge and thinking (Brousseau, 1997; Cai, 2003; Downton & Sullivan, 2017; Ridlon, 2009; Schoenfeld, 1985; Jonsson et al., 2014).

However, explanations for *how* problem solving leads to better learning are often vague or lacking, which is evident in the studies in Sidenvall's (2019) review of research on teaching designs based on learning by problem solving. Some studies lean on a general and implicit argument that you learn what you do (Abdu et al., 2015; Csíkos et al., 2012; White et al., 2012). In some designs, additional characteristics of tasks are emphasised, such as real-world basis (Bonotto, 2005; Schukajlow & Krug, 2014), contrasting examples (Coles & Brown, 2016) or explanatory prompts (Swan, 2007). In some studies, problem solving is mere one design element, which is justified as an enabler of other elements, such as discussion of different solutions (Coles & Brown, 2016; Kotsopoulos & Lee, 2012;

Pang, 2016; White et al., 2012). While good teaching naturally involves multiple elements that interplay, this constitutes a scientific difficulty, as it obscures the contribution and value of different elements. This in turn creates obstacles for practice. If we do not know how design elements affect learning, the adaptations that are necessary in implementation risk altering integral elements while preserving extraneous features.

One important clue in understanding how problem solving works as a learning activity is that it promotes *creative mathematically founded reasoning* (CMR). CMR is reasoning that is constructed by the learner (novelty) and supported by arguments (plausibility) that are founded in intrinsic mathematical properties of the components of the problem (anchoring) (Lithner, 2008; 2017). However, as with problem solving, the more specific mechanisms of CMR that enhance learning are yet unknown (Lithner, 2017).

In sum, there is a need for clarification of how characteristics of problems and problem solving contribute to the development of specific learning goals. The first step in that direction is to formulate an initial theoretical model for how problem solving works as a learning activity, which is explicit and structured enough to be empirically tested. The aim of this paper is to develop such an initial model by linking the CMR framework to previous research and other frameworks regarding problem solving.

Characteristics of problems and problem solving

Problems are usually defined as a subset of mathematical tasks carrying specific characteristics. These characteristics are supposed to lead to a specific type of mathematical work on tasks, called problem solving. It is then argued that it is the characteristics of this kind of mathematical work that facilitates the better type of learning associated with problem solving. Since the definition of problem varies, so does the foregrounded characteristics of problem solving and the arguments for how problems lead to the activities with those characteristics.

In this paper, we focus on frameworks characterizing problems as involving something unknown. Schoenfeld (1985) defines problems as tasks where the problem solver does not have easy access to a procedure giving a solution. Beghetto (2017) states that problems entail uncertainty regarding how to think and act. In stronger formulations, the unknown is specified as knowledge that the problem solver does not yet have (Brousseau, 1997; Hiebert & Grouws, 2007), but is needed in order to solve the problem. Sometimes, this unknown knowledge is restricted to a method (NCTM, 2000; Skolverket, 2019). It follows that whether a task is a problem depends on the person attempting to solve it, and the context in which it is presented. For example, in everyday life a task may evoke other resources and strategies than if it is presented in a classroom, rendering it routine in one context, but not in another. However, the relations between task and context are complex, and a thorough elaboration on these relations beyond the scope of this paper.

As a result of extensive empirical work, Schoenfeld (1985) presents a framework for the knowledge and behaviours required for problem solving. The framework comprises four categories: resources, heuristics, control and belief systems. In order to reach a solution, the problem solver needs to evoke mathematical *resources*, such as facts, algorithms, standard procedures, understandings and intuitions. She also needs *heuristics*, i.e. strategies for making progress on unfamiliar tasks, such as drawing figures and investigating examples. In order to plan, monitor and evaluate her process, she needs to use *control*, which can be seen as a subset of what other researchers have called metacogntive or self-regulative skills (Schoenfeld, 1992). Finally, she needs a belief system allowing her to think that her mathematical knowledge is useful and that she can make progress if she tries.

Beghetto (2017) argues that problem solving is characterised by creative thought and action, while Hiebert and Grouws (2007) stress that problem solving involve effortful struggle, as the problem solver "grapples with key mathematical ideas that are comprehendible but not yet well formed" (p. 387). Brousseau (1997) states that problem solving entails overcoming an obstacle by constructing a specific piece of knowledge.

An initial theoretical model for problem solving as a learning activity

The model incorporates the idea that the knowledge and behaviours coming into play when solving problems can be captured in Schoenfeld's (1985) four categories: resources, heuristics, control and belief systems. While Schoenfeld (1985) views this set as prerequisites for problem solving, we view this set as problem solving competences that are not only applied and used during problem solving, but also developed and improved. This two-fold function is conveyed in the term *exercising* (Säfström, 2013). Therefore, this set is also an adequate categorisation of plausible learning goals of problem solving.

For each competence, we will describe how problem solving is hypothesised to work as a learning activity developing this competence in relation to characteristics of problems, connecting existing frameworks and current research. For each competence we will also consider the meaning of two important conditions for learning: time and success. Regarding *time* we will elaborate on whether and how problem solving is likely to enhance learning in the moment and by sustained activity over longer periods of time. Regarding *success* we will describe whether and how learning is likely to be affected by whether the student successfully solves the problem.

Developing resources by problem solving

In problem solving, resources are exercised in a different way than when working on routine tasks. The unknown of the problem requires novelty (Lithner, 2008). It is not possible to find the solution by mere guessing (although guessing may be a constructive part of the solution), therefore the solver needs some kind of argumentation supporting the construction of the solution. This argumentation needs to be mathematically plausible and anchored in intrinsic mathematical properties of relevant components of the task (Lithner, 2008). It is this particular consideration of both known and new mathematical properties of the particular resources in question, not required and usually absent in routine task solving (Norqvist, 2017; Norqvist et al., 2019), that reinforces, expands and connects the individual's resources. In other words, problem solving requires resources to be organised in a way that enables the student to identify intrinsic mathematical properties of components and construct a line of arguments following, or possibly creating, a path of connections. Therefore, mathematical knowledge cannot be used purely atomistically, probing memory of facts and algorithms. Instead, problem solving leads to learning mathematics with understanding, i.e. "making connections, or establishing relationships, either between knowledge already internally represented or between existing networks and new information" (Hiebert & Carpenter, 1992, p. 80). The idea that problem solving results in a more connected and efficient organisation of resources is supported by Karlsson Wirebring et al. (2015) who showed that students who learned by problem solving used less brain activity on post-test while performing better, compared to students who learned by routine tasks.

The activation of resources is immediate, and new connections can be made within the solution process of a single task. Therefore, learning with respect to this competence can take place within a single learning session, as previously shown (Jonsson et al., 2014). Over time, exercising of connected resources may both densify and strengthen connections. On the one hand, consideration of intrinsic mathematical properties and attempts at constructing arguments exercise resources even if problem solving fails. On the other hand, the number of completed tasks is not correlated to increased learning *per se* (Jonsson et al., 2014). Therefore, unsuccessful problem solving may still be a better learning activity for resources than successful work on routine tasks.

Developing heuristics by problem solving

Non-problems can be solved by standard procedures, and if a procedure is known it suffices to apply it or, at most, determine which procedure is suitable from a delimited list. If possible, many students choose simple and routine methods (Downton & Sullivan, 2017; Norqvist et al., 2019). However, the unknown of problems makes resources insufficient for attaining a solution. Therefore, problem solving entails exercising heuristics, i.e. strategies for constructing one's own solution. Single problems are not enough to develop heuristics. A method becomes a heuristic only when it is found successful on a set of similar but unfamiliar problems (Schoenfeld, 1985). This implies that learning of single heuristics requires both extended time and success. In order to learn a set of heuristics and when they are suitable, one needs to encounter different types of problems, requiring additional time and failure as well as success.

Developing control by problem solving

Control is exercised when the task requires strategic decisions and evaluation of advances and setbacks. Therefore, the faculty of problem solving as a learning activity for control depends on the level of uncertainty and complexity. Uncertainty requires evaluation: if the interpretation of the problem is dubious, if it is unclear what to do or whether what was done is correct, one needs CMR to construct arguments for strategy choices and conclusions (Lithner, 2008). Complexity requires strategic decisions: if the problem requires multiple steps and consideration of numerous details, one needs to monitor the process (Schoenfeld, 1985).

If consistently prompted for, control can be enhanced over the matter of months (Schoenfeld, 1992; Shilo & Kramarski, 2019). It is likely to develop more slowly by problem solving alone. This combination of requirements—sufficiently difficult problems and prolonged commitment—may prove insurmountable for many learners. While occasional failure is likely to stimulate exercising of control, repeated failure may not, implying that the development of control requires a subtle balance between success and failure. Indeed, previous research show that higher demand of CMR may result in students resorting to imitative reasoning (Boesen et al., 2014; Sidenvall et al., 2015). Therefore, it is of outmost importance for each individual student to encounter problems that are challenging enough to activate control and at the same time reasonable to solve.

Developing belief systems by problem solving

If given problems on an appropriate level, learners can take responsibility for the process and construct their own solutions for problems of increasing difficulty (Brousseau, 1997; Lithner, 2008). This is hypothesised to establish the belief that one's own arguments anchored in intrinsic mathematical properties can solve problems, promoting the learner's own mathematical authority. Problem solving is connected to different norms than imitation of procedures, and there is a reflexivity between socio-mathematical norms and beliefs (Cobb & Yackel, 1996). Frequent problem solving may therefore affect the learner's beliefs regarding whether solving problems is viable and what level of effort is expected when dealing with mathematical tasks.

It is well-known from both research and practice that beliefs can be difficult to change and take time to develop (Hannula, 2006). We also propose that the development of mathematical authority is success sensitive. At least in relation to some beliefs, the choice of task may be crucial. If the student holds the belief that she cannot solve mathematical problems, the most important design decision may be to choose a problem that the student will in fact solve.

Discussion

In this paper we have presented an initial theoretical model for how problem solving works as a learning activity. The main rationale for this model is that it can serve as a starting point and guidance for further empirical research on mathematics teaching and learning. The model provides direction for potentially fruitful experiments, regarding duration of interventions and how problems should be designed. However, we want to stress that the aim is not to determine necessary and sufficient causes for learning. While such causes would no doubt be valuable if discovered, we acknowledge that "causes" are often more accurately described as "conditions" (Shadish et al., 2002). For example, resources could be learnt with understanding in other situations than problem solving, but tasks will not develop resources unless the task involves something unknown. However, problems will neither develop resources unless additional conditions are fulfilled, e.g. that the problem provides an opening for the learner to start working on the problem.

Some researchers assert that mathematics can only be learnt by problem solving (e.g. Brousseau, 1997). At the same time, mathematical knowledge and skills are sometimes described as prerequisites for problem solving (Bergqvist et al, 2010; Schoenfeld, 1985). We argue that the notion of CMR (Lithner, 2008) provides a means for unifying these views: problem solving gives opportunities to exercise resources in different ways than routine tasks, since the absence of a predetermined method demands constructing arguments anchored in intrinsic mathematical properties.[1] Therefore, resources are structured and connected in problem solving, and this structure provides a means for further reasoning and problem solving.

The model presented here considers problem solving as a learning activity in itself. This focus also reveals potential difficulties and shortfalls in problem solving, e.g. that developing belief systems may require success, while developing heuristics and control may require failure. Such insights can guide the combination of problem solving with other design elements. As previous research shows, problem solving is better realised if combined with other activities. Brousseau (1997) emphasise the function of institutionalisation in valuing and reformulating students' own constructions of resources and heuristics in culturally accepted terms. This can be achieved in whole class discussions as suggested by others (Kotsopoulos & Lee, 2012; Pang, 2016; Stein et al., 2008). Acquiring and using a heuristic vocabulary during problem solving has been shown to further development of heuristics (Koichu et al., 2007). With respect to control, Schoenfeld (1985) support the Vygotskyean hypothesis that individual reflection is preceded by social reflection, and has shown empirically that questions prompting for reflection develops students' metacognitive behaviour.

It is probable that the teacher-student interaction during problem solving influences norms and practices in other parts of the classroom context, and that the norms and practices in the classroom context at large influences problem solving (Cobb & Yackel, 1996). Attending to certain aspects of the students' work and thinking lades these aspects with value, affecting beliefs. For example, asking what the problem is about signals that this is something important to consider. Asking about the students' thinking and work signals that this is of value, besides the result and final answer. It is therefore a reasonable hypothesis that interaction concerning heuristics, control and resources also influence belief systems, but possibly over longer periods of time. This method may be more effective for changing belief systems than trying to affect them directly.

To be useful for practice, further elaboration of how other activities support or hinder learning during problem solving is needed, and indeed providing theoretical and empirical bases for such elaborations is a key concern for our continued work. It is, however, beyond the scope of this paper and our current understanding of the complex phenomena involved.

Suggestions for empirical studies

We know from Schoenfeld's (1985) work that the four categories of knowledge and behaviours are detectable when observing students' problem solving processes. We also know that mathematical reasoning can be studied by means of observation and interviews (Lithner, 2008). While these methods provide a foundation for how learning and development of problem solving competence can be studied, they are not sufficient in themselves. Problem solving as a phenomenon gives rise to specific challenges when it comes to studying development over time. By definition, a task cannot be used to test the problem solving competence of a student twice. If different tasks or activities are used at different occasions, performance can be profoundly affected by the characteristics of the tasks. Hence, validity for problem solving measures is a delicate issue.

In addition, some aspects of problem solving may be specifically difficult to study. To a large extent, the processes involved in problem solving are taking place in the mind, hidden from observation. While the observer can access some of those processes by asking questions, such methods are always interventional. This is especially true for control, as it is epistemologically debatable how accounts given when requested relate to peoples' actual rationale (Edwards, 1997). This issue is lessened, but not removed, by studying the exercising of control communicated by group members in group work.

As most existing methods for studying problem solving involve time consuming methods such as observation or interviews, it may be difficult to conduct studies at scale. While surveys and questionnaires, e.g. for self-reported beliefs and metacognitive skills, are available, such methods can suffer from poor validity (Veenman & van Cleef, 2019).

It is clear that additional work is needed in order to empirically test our initial theoretical model and further understanding of how problem solving functions as a learning activity. Undoubtedly, we will have reason to reconsider and revise our model as our work proceeds. Nonetheless, we believe that the formulation of initial models serves an important purpose in guiding and evaluating empirical studies, and for interpretation of both failure and success.

Notes

1. In some cases connections and arguments can obstruct learning (Phenix & Campbell, 2004), but we believe those cases to be few and rather peripheral to a general theory of mathematics learning.

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