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Children's awareness of numbers' part-whole relations when bridging through 10

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This paper reports findings from an assessment of fifty-one 6–7-yearold children's ways of solving arithmetic tasks. In particular, 'double counting' strategies were found to be enacted when bridging through 10 even though the children evidently had learnt conceptually more powerful ways of encountering arithmetic tasks in the number range 1-10. In this paper the reasons for this outcome are analysed and discussed in terms of children's ways of experiencing numbers' part-whole relations.

Aim

That young children change strategies between arithmetic tasks is not unknown to the field of knowledge, but the explanations why children abandon successful and conceptually logic strategies for more primitive ones differ. The aim of this paper is to contribute one way of explaining strategy changes in terms of children's ways of experiencing the part-whole relations of numbers. In order to make this contribution, a specific research question was: What awareness of numbers' part-whole relations is reflected in the children's ways of handling arithmetic tasks? This question is answered through a qualitative analysis of 51 children's answers to addition and subtraction tasks within the number range 1-10 and bridging through 10.

Background

This study is part of the research project FASETT, which aimed to deepen the knowledge about young children's learning of elementary arithmetic. The project took its' departure point from earlier research in which children's arithmetic skills have been studied in terms of their ways of experiencing numbers (Neuman, 1987). An intervention program was conducted in preschool where children were afforded to experience numbers in such ways that in particular numbers' part-whole relations were coming through (Björklund, Ekdahl & Runesson Kempe, n.d.). This based on the assumption that children who learn to discern numbers' structural relationships (see Venkat, Askew, Mason & Watson, 2019) would be better prepared to handle elementary arithmetic tasks and also apply this knowledge to a larger number range. A follow-up assessment one year after the intervention was conducted to evaluate any long-term effects of the intervention. Results from this follow-up assessment is the object of inquiry in this paper.

Research on early arithmetic skills

There is an abundance of studies describing children's strategies in solving arithmetic tasks (e.g. Baroody & Purpura, 2017; Fuson, 1988). Much is learnt from these studies about more or less prosperous strategies and what cognitive abilities that influence the child in making use of certain strategies. A cognitive perspective in research induces that early arithmetic understanding is local to particular principles, proceeding through more and more complex levels (Starkey & Gelman, 1982). There are claims that children are 'forced' to develop more advanced strategies when the number range exceeds 10 and children can no longer rely on their fingers to represent units (Carpenter & Moser, 1982). This leads for example to a Counting-on strategy rather than the less advanced Counting-all strategy when exceeding 10. Counting strategies are thus ways to keep track of counted units often referred to a number line (mental or physical) and are commonly observed among young children (Laski, Ermakova & Vasilyeva, 2014).

Furthermore, the strategy 'double counting' is by Fuson (1988) defined as a way of keeping track where the sequence of words are entities to be counted. When solving e.g. 15–7=_ the child has to keep track of how many units are taken away and how many are left, at the same time, usually by raising one finger for each counted counting word and then counting the raised fingers, or by indexing with number words: '15, 14 (1 taken away), 13(2) … 8(7)', thus the numbers are double counted. This is considered a normal step in the development of arithmetic skills, by Steffe (2004), as an extension of counting and thus a higher level of functioning. Neuman (1987) also observed this strategy in children's arithmetic problem solving, but concluded it being a strategy invented by the children, when they experience arithmetic tasks as operating with single units and not as a flexible part-whole relation. Double counting is according to Neuman in this sense not a powerful strategy since it is cognitively demanding and put a heavy load on the working memory.

A contrast to the dominance on seeing counting strategies as the outset for learning arithmetic is given by researchers who advocate that a structural approach, attending to numbers as part-whole relations, should be emphasized already in the early years (Brownell, 1935; Davydov, 1982; Neuman, 1987; Schmittau, 2003). This based on the fact that counting strategies may help children solve simple arithmetic tasks but do not support children in recognizing the numerical relations between and within numbers that more advanced tasks presupposes. An emergent awareness of structural relationships appears among young children as they analyse and discern local relationships of numbers in arithmetic tasks. This will eventually allow them to identify more general mathematical relationships and properties that enables more advanced arithmetic strategies such as decomposition and commutativity to be used (Venkat et al., 2019).

Gray and Tall (1994) argue that individual differences in arithmetic skills development and enactment are related to preferences of counting procedures, or ability to derive from what is already known. Most children have both alternatives available and can choose the most convenient way to handle a task. Nevertheless, Gray and Tall show that children who primarily rely on procedural strategies, such as counting single units, do not relate a task to earlier solved tasks as known facts, which in turn is not provoking a need to remember facts, since the counting procedure provides a security. Methods based on deriving from what is already known, on the other hand, enhance the ability to remember facts and support children in using those facts; ’I know 4 plus 4 is 8, then 4 plus 3 must be one less, 7’.

In a long-term perspective the strategy preference in solving arithmetic tasks becomes critical when the number range increases and more advanced arithmetic is introduced. According to Ostad (1998), a strategy learnt in isolation from its conceptual foundation induces more errors (see also Geary et al. 2004) and is hard to transfer to novel problems. Furthermore, if a single unit counting strategy (such as double counting) has been established as the preferred one, it is not easily abandoned (Cheng, 2012). Thus, it seems that children’s use of strategies is more complex than learnt ways to solve problems. Consequently, for children to advance their arithmetic skills they need to develop a conceptual understanding of numbers, or in other words, to see numbers’ part-whole relations in ways that allow them to act in accordance with the structural relationship identified in an arithmetic task.

Methods

A structural approach in teaching elementary arithmetic to preschool children was implemented in the FASETT intervention program with the intention to afford a conceptually solid basis for arithmetic skills development. The structural approach meant directing attention to numbers’ part-whole relations as an outset, rather than counting single units. This was enacted in designed activities conducted by preschool teachers in five preschools with children attending their last preschool year (as 5-yearolds). To evaluate the outcomes of the intervention we investigated children’s ways of solving arithmetic tasks and interpreted their actions and solutions as expressions of ways of experiencing numbers and number relations, in line with the theoretical framework Variation theory of learning (Marton, 2015). Assessments were done before, right after and delayed one year after the intervention. All assessments were video-recorded, with the children’s legal representatives’ informed consent. The third (delayed) assessment is the object of inquiry in this paper, in which 51 children participated. Video-documentation was a criterion for inclusion in the analysis, since children’s ways of using their fingers were considered important data, which could not be collected in sufficient details by any other method. The assessments were done individually by members of the research team in the children's own school settings.

Items in the assessment were designed to evoke the possibility to enact different strategies both in the number range 1-10 and bridging through 10. The tasks were given as oral word problems and the children were asked to explain verbally or in other ways show how they had arrived at their answer. No manipulatives were available but the children were encouraged to use their fingers if they thought it would be helpful. The target tasks for this particular study are:

A) Your friend has 2 shells and you have 5. How many do you have together?

B) On Saturday you get 10 candies. You eat 6 of them. How many are left?

C) Today you are going to set the table. There are 3 glasses already on the table but there will be 8 persons eating. How many more glasses do you need to get?

D) On your birthday, you are blowing up balloons. After the party, 3 are broken and 6 are whole. How many balloons did you blow up from the beginning?

E) You have 8 marbles and your friend 5. How many do you have together?

F) If you have 15 stickers and give 7 to your friend. How many are left?

To answer the question *how is the awareness of numbers' part-whole relations reflected in the children's ways of handling arithmetic tasks* a total of 306 observations across the tasks A–F (51 children x 6 tasks), were analysed. Each answer was coded as correct/incorrect and according to the children's different ways of handling the tasks, first considering the strategies observed among all children and then regarding the relation to tasks below or bridging through 10.

The analysis of children's expressed awareness of part-whole relations was done using Variation theory as theoretical framework. According to Variation theory (Marton, 2015), children can only act in accordance with their way of experiencing a phenomenon, that is, in this study the children's actions and explanations are expressions of their way of experiencing the task and the numbers given in the task. Some aspects of numbers are prominent when enacting counting single units strategies and others are discerned when re-grouping or decomposing number sets, determining different ways of experiencing (and thus acting) the same task. If the child is only able to see some, but not other necessary aspects, it is assumed to limit what strategies the child is able to enact. Based on these theoretical principles, an analysis of the strategies enacted by the children can reveal which aspects they have discerned and which are yet undiscerned, in particular when some tasks are provoking difficulties and the observations show children encountering such tasks in different ways. This is further discussed in terms of the children's awareness of numbers' part-whole relations, as this theoretically driven approach allows us to describe what aspects in particular the children need to 'see' in order to make use of powerful strategies. The following excerpts are answers given to the task C (3+_=8) and will illustrate the difference between how a child who is interpreted as experiencing numbers as part-whole relation acts and thus enacts a structuring strategy, and a child who sees numbers as single units, thus enacting a counting strategy, acts:

> *Counting*: The child starts with three fingers unfolded on the right hand (thumb, index and long finger), unfolds the two other fingers and two more in consecutive order on the left hand, moving the lips silently. Then moving each finger from the right thumb in same order and unfolds the long finger on the left hand as well. Starts moving each finger on the right hand again, then says "I think I need eight, no", counts by pointing at each finger on the right hand, saying "I think I need six more".
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> *Structuring*: The child unfolds three fingers on the left hand and three more on the right hand, saying "five, there were three glasses and it should be eight, then I have to get five more to make eight" raises the two folded fingers on the right hand showing eight unfolded fingers.

The different ways of handling the same task reveal differences in the children's ways of experiencing numbers: the first child creates sets by adding ones but fail to coordinate different sets since the counted units are not seen as separated from the whole. This makes it difficult to find out what constitutes the missing part. The second example shows on the other hand a child handling the part and the whole simultaneously, he sees 'the five in the eight' and his actions reflects an awareness of the part-whole relations of the numbers in the task. In the results section below, these differences in awareness of numbers are discussed in terms of discerned aspects of numbers.

Results

The observations of the strategies enacted by the children reveals a variation of ways to solve the tasks. The strategies found in the tasks below 10 are:

- Known facts. The child knows the answer, retrieved from memory or as a result from mental arithmetic.
- Structuring. Transforming a problem into a simpler one, attending to number relations, by decomposing numbers or structuring the task with finger patterns.
- Counting. Using a strategy that includes counting single units, such as Counting-all, Counting-on or Counting-down.
- Guessing. Answering without apparent counting or reasonable explanations.

In the tasks bridging through 10 the same strategies were observed, with an addition of:

- Structuring towards 10. Transforming a problem as above but in addition using base-10 properties of the number system.

• Double-counting. The number of units taken away or added are kept track on by (mostly) fingers raised for each counting word and then identifying the counted units (fingers) to get the answer.

Among the children in the study many know number facts in the number range 1–10 and are able to retrieve from facts to find an answer. These children are considered having discerned the number relations and can attend to numbers' part-whole relations in solving the tasks. Number facts are seldom observed when bridging through 10, which means the children have to enact some structuring or counting strategy to find an answer. The dominating strategy when bridging through 10 is counting (64 of 102 observations = 63%). However, the tasks are difficult to handle by double-counting (41 of 64 = 64% correct answers) while the structuring strategy in the tasks bridging through 10 mostly ends up with the correct answer (17 of 19 = 89% correct).

When further analysing the children's ways of solving arithmetic tasks an inconsistency in strategy use was discovered. Some children who were found to successfully make use of a structuring approach in the lower number range failed to use this approach when bridging through 10 and were then observed to enact a double-counting strategy in the higher number range. In the following sections a pattern of enacted strategies are presented and discussed in terms of children's awareness of the part-whole relations, based on individual children's enactment in tasks C, D, E and F (the first two tasks, 2+5 and 10–6 are excluded since answers were most often given as known facts). Three children were not categorized since they were only giving random answers without explanations to the tasks.

Counting as primary strategy. Fourteen children were using counting and double counting as the *only* strategy for solving the arithmetic tasks. The cumbersome double counting strategy becomes their only option (other than guessing) because the children do not experience the tasks constituting a structured part-whole relation. Consequently, these children may not be able to use any decomposing strategies or retrieving facts from memory – it does not make sense to them.

Child: I have 8. And then it's 5 more. Then I count 9, 10, 11, 12, 13.

Interviewer: How do you know when to stop counting?

Child: I count at the same time as I count. Like this, 8 and 1 (pointing at the table when holding up his index finger), 9 is 1, 10 is 2 (holding up two fingers), 11 is 3 (holding up three fingers), 12 is 4 (holding up four fingers) and 13 is 5 (holding up five finger).

This child keeps track of the counted number words both verbally "9 is 1, 10 is 2…" and by raising one finger for each added unit and succeeds in solving the task. The strategy is nevertheless time-consuming and the child has to keep simultaneous attention to two parallel sequences. Enacting these kind of strategies may be due to the children experiencing numbers as single units to be added to the larger set but do not identify the relation between the given parts and the whole.

Nevertheless, in straight forward additions like task E they quite often succeed in finding out the answer. However, in the task F, this strategy becomes an obstacle when having to keep track of units on the number sequence *backwards*.

> Fifteen, then I count down. Fifteen, fourteen (moving the thumb and index finger). Fifteen, fourteen. Thirteen (holding the ring finger). I don't know what comes next (counts silently). Twelve. Is it seven then?

Experiencing numbers as single units induces these counting strategies that may solve simple arithmetic task, but when the parts exceed the number of object that the child is able to perceive or when the fingers cannot represent each unit necessary for operating on the task, the absence of structure based on number relations becomes critical.

Changing strategies when bridging through 10. Twenty-five children were found to use mixed strategies. Nine of them, who were observed to enact double-counting when bridging through 10 did seem to experience numbers' part-whole relations within the number range 1–10, as they enacted structuring strategies or known facts when solving the tasks below 10. In other words, they *did* identify number relations below 10 as a local relationship but could not generalize this idea to a larger number range. This brings forth an important insight: In tasks where children need to bridge through 10, quite many still seem not to experience some of the necessary aspects of the part-whole relations of numbers to enact a structuring approach, since they change a highly successful strategy (structuring or deriving from known facts) to a more cumbersome, error-prone and time-consuming strategy (e.g. double-counting).

Experiencing 10 as a benchmark. The analysis showed that nine children were structuring throughout all of the tasks. Based on their verbal reasoning when solving the tasks, they were experiencing 10 as a benchmark and directed their efforts to solve the task by structuring on their fingers and by decomposing numbers, as illustrated in the following excerpt from task E (8+5=_).

> Thirteen. If you take 8, and 2 from the 5 and adding those marbles, it makes 10 here, and 3 left. If you put them together, it makes 13.

In other words, the child is able to enact the associative law of arithmetic and seems to experience the parts in the arithmetic task in two ways, simultaneously: 8+5 is seen as 8+2+3 where 2 is part of a new part 10 but also a part of the decomposed 5. A similar way of reasoning about how to solve the subtraction task F (15-7=_), bridging through 10 is shown in the following:

> Eight. First I thought like take away 7 and then I started counting 7. And 8, 9, 10, then it was 3 more and then I had 5, and 5 plus 3 is 8.

This child considers the inverse relation between addition and subtraction and adds units to 7. However, there seems to be an important benchmark when reaching 10 as the child then experiences the difference between 10 and 15 to be added to the 3. The task 15–7=_ is thus experienced as a composition of 7+(3+5)=15. A closer look at these children's ways of handling the tasks reveals that they have a very clear idea of the number relations in the task, seeing them as composite sets relating to each other in a part-whole structure. What stands out is that they create units of 5 or 10 and decompose given numbers to create new units to 'fill up' 5 or 10. In our terms, they experience numbers in a quite different way than children who use counting as their only strategy and also compared to the children who change strategy when bridging through 10 – they experience numbers' part-whole relations simultaneously and take hold on ten as a benchmark for their structuring and furthermore see the task as to be mathematical rather than empirical.

Conclusion

Young children's struggle with bridging through 10 is not unknown to either the research community or teachers. Laski, Ermakova and Vasilyeva (2014) for instance showed that children's use of base-10 decomposition in arithmetic tasks was related to their knowledge of number structure and children are able to use multiple strategies depending on the task. The intervention program in FASETT emphasised numerical relations and most children learned and enacted structural strategies when solving tasks below 10. However, of interest in this particular paper is *why* children who have learned to structure and use the part-whole relations of numbers below 10 change to strategies that do not attend to the part-whole relations when tasks are bridging through 10. The results from the analysis presented here might shed light on what constitute knowledge of number structure in a general sense and consequently what teaching elementary arithmetic should consider.

Counting seems to be a safe way to find a solution among our participants and many use fingers for keeping track of counted units. Some children use (double)counting as their *only* strategy, and do in fact solve many of the tasks. However, the field of research converge on the view that structuring strategies and using retrieved facts is preferred in a longer perspective (Baroody & Purpura, 2017; Gray & Tall, 1994). Nevertheless, based on this investigation it can be suggested that this cannot be taught as a strategy alone, it is rather a question of which aspects of numbers that the child experiences in a task. When comparing the mixed strategy users with those who only use structuring strategies, one aspect emerges as to be critical to experience: structuring *towards 10*. Children are observed to attend to the part-whole relations of numbers and enact structuring strategies, but only those who also experience 10 as a benchmark seem to be able to generalize their seeing part-whole relations to numbers above 10 and decomposing parts to fit a 10-structure.

Fuson (1988) concluded, supported by Steffe (2004), that counting-the-count methods were advanced. Based on the results presented above, double-counting is however not an advanced strategy, even though it is cognitively demanding to keep track of parallel number lines. It rather becomes a necessary way of handling arithmetic tasks if the child does not see the task as composite sets with 10 being a critical benchmark. Strategies such as counting single units and double counting, do not support the child in experiencing the relation between and within numbers, or the base 10-structure to simplify an arithmetic task. If counting single units is the only strategy that the child is able to enact, more advanced arithmetic tasks in the number range above twenty and using multiplication will probably not be possible either. This is in line with Gray and Tall's (1994) conclusion that more able children appear to do a qualitatively different sort of mathematics than the less able, since they have alternatives and can choose the most convenient way to solve a problem. The study presented in this paper contributes that in this particular arithmetic context (below and bridging through 10) it is necessary that the child experiences number relations but also experiences 10 as a benchmark, in order to access the more powerful structuring strategy in the larger number range. In forthcoming interventions, it would thus be important to direct attention to this aspect as to be critical for learning arithmetic skills, in addition to facilitating the discernment of part-whole relations of numbers as an outset.

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