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Components of knowledge in solving linear equations

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This article identifies knowledge components needed for successfully solving linear equations. Data for this purpose is 359 Swedish year 9 students' written responses to the test task "solve the equation 2x+3=11". The following set of knowledge components were identified; arithmetic knowledge, parsing knowledge, balancing equations, giving a value to the unknown, not ignoring terms and the habit of verifying the solution. This paper discusses for which of these knowledge components, students could discover and correct their own errors if both solving an equation and verifying its solution.

Introduction

In mathematics textbooks, a standard method for teaching how to solve linear equations is the *canonical method* (Buchbinder, Chazan, & Fleming, 2015). This method includes the steps of first simplifying on each side of the equals sign arriving at the form $ax + b = cx + d$. After this follows inverse operations by making appropriate addition and/or subtraction operations arriving at the form $ex = f$ thus isolating the unknown on one side and finally multiplying and/or dividing in order to identify the value of 1 (one) unit of the unknown. Moreover, it seems that many students use the canonical method as a mechanical procedure (Huntley, Marcus, Kahan, & Miller, 2007).

Though solving linear equations has been taught and learnt since Babylonian time (Friberg, 2005), and there now is a large body of research on teachers teaching and students learning how to solve them, Otten, Van den Heuvel-Panhuizen, and Veldhuis (2019) yet made a call for further research in this area. The reason for their call is that they found the often-taught balance model (Andrews & Sayers, 2012; Marschall & Andrews, 2015), be it physical, digital or drawn, to be complex as a didactical tool in the presence of negative numbers (Vlassis, 2002). Nevertheless, even when only positive numbers are present, students make errors that seem difficult for the student to identify as errors if simply applying the canonical method as a procedure (Petersson, 2018b). The aim of the present study is to explore the role of verifying a solution in relation to students' responses to the task of solving a linear equation.

Components of prerequisites for solving linear equations

As stated above, the standard tool for solving linear equations is the canonical method (Buchbinder, Chazan, & Fleming, 2015) and that students seem too often use this as a mechanichal procedure (Huntley, Marcus, Kahan, & Miller, 2007). Now, one component in the canonical method is that it rests heavily on that the students view the equals sign as relational (in contrast to operational), which Knuth, Stephens, McNeil and Alibali (2006) found to be crucial for success in linear equation solving tasks. Another way to say this is that the equality must be kept balanced. A second component in the canonical method is arithmetic knowledge. For example Hall (2002) and Petersson (2018a) found that in subtractions, students may sometimes use counting down strategies where they include or exclude both the starting and ending number thus getting, for example, the difference $11 - 3$ one unit too small or too large. A third component in the canonical method is parsing algebraic expressions correctly, for example $2x$, as a multiplication and not as an addition (Humberstone & Reeve, 2008; MacGregor & Stacey, 1997; Petersson, 2018b). A fourth component in the canonical method is the concept of the unknown. Asquith, Stephens, Knuth and Alibali (2007) described perceptions of the unknown as a hierarchy of seeing unknowns as a multiple number, a specific number and an unknown digit. This includes giving a value to one unit of the unknown; that is setting 'x=…'.

Research question

A lot of research on solving linear equations explores some single component, such as those mentioned above. Less research seems to have explored several components simultaneously. Moreover, less research seems to have explored verifying a solution. Hence, the research question in this study is to explore how often students use verifying and what role verifying a solution hypothetically could play in helping students identifying and correcting their own errors made in some component of the canonical method.

Methods

To answer the research statement, the author collected 359 Swedish year 9 students' written responses to one task on linear equations given in a mathematics test. The students' responses were analysed with respect to the knowledge components described above.

The test task

Vlassis (2002) separated between what he called arithmetic linear equations, having the unknown on only one side, and non-arithmetic equations, having the unknown on both sides. The explored test task was "Solve the equation 2x+3=11". There are two reasons for choosing this arithmetic equation as test task. The first reason is that this task is simple enough to get many responses, since a difficult equation might result in many blank responses not contributing to the result. Still, it is possible to, in this task, make errors corresponding to all the components mentioned earlier. The second reason for choosing this task is that it is similar to linear equation tasks in released Swedish year 9 national tests in that the unknown occurs on only one place. See table 1, with source < su.se/primgruppen/matematik/årskurs-9/tidigare-prov >. Moreover, table 1 shows that in all years except 2006 and 2012, two operations with different priority are present and the unknown has a positive integer solution and the coefficients are integers since it is natural to view for example x/2 as dividing by an integer rather than multiplying by the decimal number 0.5. The exceptions are 2006 and 2012, where the equation task contained only addition. On the other hand, in 2006 the solution was a negative integer and 2012 the solution was a decimal number.

Year 9 national test task	**Task formulation**
2014 part B task 7	Solve 25 – 5x = 10
2013 part B task 9	Solve x/2+1 = 5
2012 part B1 task 9	Solve 2,35 = 0,5 + x
2010 part B1 task 10	Solve 13 - 3x = 7
2009 part B1 task 7	Solve 17 = 3x + 5
2008 part B1 task 8	Solve x/3+ 2 = 5
2006 part B1 task 6	Solve x + 6 = –2

Table 1: Linear equation tasks on released national tests

Analysing the students' responses

Each student's response to the task was categorised with respect to verifying or not verifying the solution. In addition, each student's solution to the equation was categorised as correct or incorrect. Now, a response with an incorrect solution could contain several simultaneous errors. For example, a single response could contain both arithmetic errors such as setting $11 - 3 = 9$ and parsing errors such as interpreting $2x$ as $2 + x$. Thus, each incorrect response was analysed with respect to each of the components of balancing, arithmetic, parsing, explicitly giving the unknown a value and a fifth category that was found while examining the responses, namely ignoring parts of the equation.

The students

359 Swedish year 9 students agreed to participate in this study and the linear equation task was given in a teacher administrated classroom test in mathematics. To check the generalisability of the sample, the students in this study were compared with a sample from whole of Sweden with respect to the achievements on the national test part B, which is the part in which linear equations of the studied

type occur, as seen in table 1. The author received achievement data from the whole Sweden sample from the National Test Team. In their sample, percentage of correct responses were 46% for the second language students and 60% for the first language students. The author got national test achievement data from five of the six schools that participated in the present study and these second language students (N=146) achieved on identical level as the national sample while the first language students (N=113) achieved 56%, which a little lower than the national sample. This latter difference is likely due to residential segregation effect (Hansson, 2010, 2012) since the students sampled were from schools with a high proportion of second language students. The similarity between the sample in this study and the national random sample with respect to achievements on the national test, suggests that the knowledge in mathematics of these two samples are similar, which in turn indicates high reliability of the student sample used in this study. It should also be noted that the small difference between the two samples is not crucial for the validity of the present study not comparing achievements quantitatively but instead qualitatively identifying knowledge components that the students need for mastering solving linear equations. In fact, a lower achievement might mean a larger proportion of students not giving an answer to the task but it likely also implies a larger proportion and thus richer mix of incorrect responses, which should increase the saturation (the actual occurrence) of the different knowledge components of the canonical method found in the literature. This should contribute to a higher validity of the results.

Results

Balancing as a component in solving linear equations

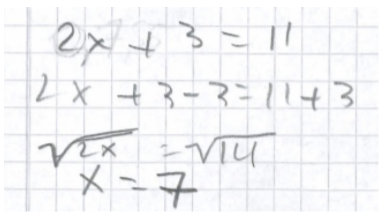

Figure 1a: Opposite operations

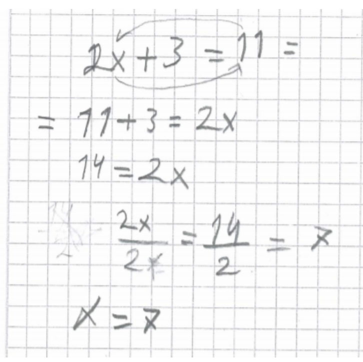

Figure 1a: Terms moved around

Figures 1a and 1b exemplify unbalancing the equation. In figure 1a the student tried to isolate 2x on the left side by subtracting the number ‘3’ on one side and adding the same number on the other side. Since the student in the next two lines did identical operations on both sides, though interpreting the square root sign as halving, it seems as if the student has confused 'doing the same operation on both sides' with 'change sign when moving a term to the other side'. In that sense, this error is neither an arithmetic error nor a parsing error but an erroneous balancing of the equation. The same holds in figure 1b where a student swapped 2x and 11 with each other in order to isolate 2x on one side. Moreover, the student in figure 1b used the equals sign to mark the transformation of an equation into another equation, which clearly indicates an operational use of the equals sign as 'becomes' instead of 'equals'.

Ignoring of forgetting terms when solving linear equations

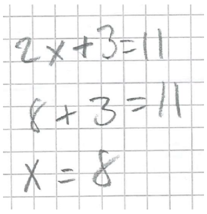

Figure 2a: Forgot to divide by 2

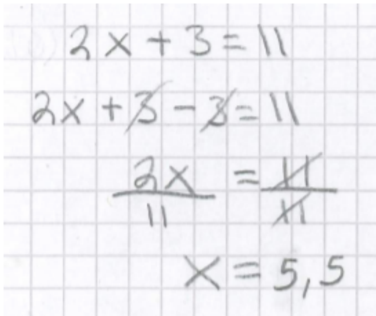

Figure 2b: Forgot to subtract 3

In figure 2a, the student verified that 8+3=11 but concluded that x=8 instead of 2x=8 thus seeming to forget the factor 2. In figure 2b, the student subtracted 3 on only one side but apart from that did arithmetically correct operations (though in an odd way by dividing by 11 instead of by 2), parsed the symbols correctly and correctly balanced the equation through the rest of the solving process. Together this indicates that the student may simply have forgotten to subtract 3 from the right hand side.

The arithmetic component in solving linear equations

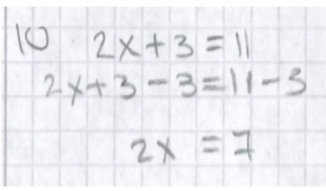

Figure 3a: Calculates 11-3 to 7

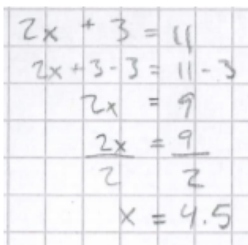

Figure 3b: Calculates 11-3 to 9

In the responses in figures 3a and 3b, it seems as if the students viewed the equals sign as relational since they consequently did the same operations on both sides. Despite the correct balancing of the equations, the two responses contain arithmetic errors. Indirectly we can also assume that their calculations of $11 - 3$

leading to wrong differences 7 and 9 instead of 8 indicate that they did not use number facts when calculating the difference. Instead, they likely used counting down strategies where the student in figure 3a excluded the starting and ending number when counting down from 11 to 7 while the student in figure 3b instead included the starting and ending number when counting down from 11 to 9.

The component of giving a value to the unknown

In figure 3a, the student responded with "2x=7" but did not proceed to determine a value of one single x. In this category of responses there were also a few cases of responses "2x=8" and those that stated "8+3=11" without explicitly giving a value to the unknown.

The parsing convention component in solving linear equations

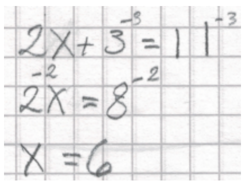

Figure 4a: 2x parsed as 2+x

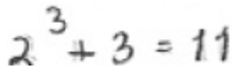

Figure 4b: 2x parsed as 2^x

The only error in the calculations in figures 4a and 4b are that these students parsed the original equation $2x + 3 = 11$ in ways that differ from what is endorsed in mathematics, namely seeing $2x$ not as a multiplication, but as an addition in figure 4a and as a power in figure 4b. Else, the arithmetic calculations in both figures 4a and 4b are correct with respect to the parsing error that each student made. When it comes to the students' view on balancing the equation, the calculations in figure 4a shows that this student consequently did the same operation on both sides until getting some solution. From this, we conclude that this student views the equals sign as relational and knows balancing as a way to solve equations. From the arithmetic statement in figure 4b we can see that the left hand side evaluates to the right hand side of the equality thus indicating at least an operational view of the equals sign while figure 4b does not give any information about if the student mastered the equals sign as relational though an operational use is evident. Finally, these two students treated the unknown as a variable, whose value should be determined, which they did explicitly in figure 4a and implicitly in figure 4b.

The component of verifying a solution when solving linear equations

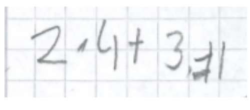

Figure 5: Verify a solution

In figures 4b and 5, the students verified that a specified value of the unknown satisfies the equation.

Frequency of various solution components

There were 255 (71%) correct responses. Of the correct responses there were 121 responses using the canonical method, 83 just responded $x = 4$ while 50 wrote the verified solution as $2 \cdot 4 + 3 = 11$ and only 1 (one) both gave a solution via the canonical method *and* verified the solution. There were 56 students, who did not answer the task. Among the 48 (13%) erroneous responses, there were three non-classified responses. These were the responses "8x", "18" and "24". Of the remaining 45 responses given in table 2, 30 made one single type of error. There were 10 responses coded in two error categories and among these, it was most common to combine unbalancing the equation with some other error. In 5 cases, there were responses coded in three error categories. A clarifying note is that counting the number of errors in each category instead of the number of responses explains why the sum 65 of the content in table 2 exceeds the 45 responses categorised.

Component	Frequency (relative frequency)
Arithmetic error	6 (2%)
Parsing error	19 (5%)
Balancing error	12 (3%)
Forgets or ignores part	13 (4%)
Unknown gets no value	15 (4%)

Table 2: Frequency of components

Discussion

One focus in this study is students' use of verifying solutions when working with equation tasks. One striking observation in the data was that only one single student out of 359 responded with both solving the equation and explicitly verifying the value of the unknown by inserting it into the original equation. Else, it was common to *either* use the canonical method for solving *or* simply insert the solution into the original equation. For the students who came up with an incorrect solution to the equation, a combination of the canonical method and verifying the solution should help at least some students to discover their own errors and give them a chance to self-correct their responses.

Another focus in this study is errors in components of the canonical method. Students that respond as in figures 3a, 3b and 4a demonstrate knowledge about the canonical method (Buchbinder, Chazan, & Fleming, 2015) and thus uses the equals sign as relational (Knuth, Stephens, McNeil, & Alibali, 2006). However, the two students responding as in figures 3a and 3b, made the same kind of arithmetic errors as described in Hall (2002) and Petersson (2018a). Their responses show that also arithmetic knowledge is an indispensable component for correctly solving linear equations. Moreover, the arithmetic component in responses 4a and 4b and the balancing component in response 4a are correct with respect to the erroneous parsing described in research literature (Humberstone & Reeve, 2008; MacGregor & Stacey, 1997; Petersson, 2018b). This means that also correct parsing is an indispensable component when solving equations.

Of the balancing errors in table 2, more than half of them were similar to those in figures 1a and 1b and could be related to a non-relational view of the equals sign as described in Asquith et al. (2007) or due to a diffuse conception about the canonical method. One assumption is that students might discover errors due to both arithmetic mistakes and incorrect balancing if a task on equations asks for *both* a solution procedure *and* a verification of that solution as in figure 5. This might also hold for students that forget or ignore parts of the equation during the solving procedure as in figures 2a and 2b.

However, students making paring errors as in figures 4a and 4b will not discover their errors by verifying their solution. For example, the response in figure 4b is an arithmetically correct verification. Instead, they need instruction on the sanctioned parsing rules. Neither might asking for verification help students that do not give a value to the unknown as in figure 3a. On the other hand, a hypothesis is that learning the habit of verifying should raise the awareness of actually assigning a value to a single unit of the unknown, which might make them elaborate their incomplete solutions into complete solutions.

A third focus in this study is how verifying solutions hypothetically could help students identifying and correcting their own errors when solving equations. Since only 1 (one) student of 359 explicitly both solved and verified the solution while several students hypothetically should have discovered their own errors if they had verified their solution, one conclusion for the teaching and testing of solving equations is to encourage and remind students to *also* verify their solutions. This would help, in particular, students that else make frequent errors when solving equations. Moreover, it is likely that these students often are low-achievers and thus would benefit from this. One way to promote students' habit of verifying is to, in tests, explicitly ask for both solving and verifying, since it is commonly known that what is examined to a higher extent also is learnt.

A suggestion for further research is to, through an intervention study, explore if teaching and examining both solving and verifying would both help students discovering their own errors themselves and; if this at the same time would support them in building a relational view of the equals sign and learning correct balancing of equations. Such a study should, of course, include non-arithmetic equations, and if applicable also quadratic equations and systems of linear equations.

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