
Skrifter från SMDF, Nr. 13

*Perspectives on professional
development of mathematics
teachers*

Proceedings of MADIF 11
The eleventh research seminar of the
Swedish Society for Research in
Mathematics Education
Karlstad, January 23–24, 2018

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SMDF Svensk Förening för MatematikDidaktisk Forskning
Swedish Society for Research in Mathematics Education

Skrifter från Svensk Förening för MatematikDidaktisk Forskning, Nr 13

ISBN 978-91-984024-2-1

ISSN 1651-3274

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SMDF

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Swedish Society for Research in Mathematics Education

For information see web page <http://matematikdidaktik.org>

Printed by Stema Specialtryck AB, Borås, Sweden

Preface

This volume contains the proceedings of MADIF 11, the eleventh Swedish Mathematics Education Research Seminar, held in Karlstad, January 23–24, 2018. The theme for this seminar was *Perspectives on professional development of mathematics teachers*. The MADIF seminars are organised by the Swedish Society for Research in Mathematics Education (SMDF). MADIF aims to enhance the opportunities for discussion of research and exchange of perspectives, amongst junior researchers and between junior and senior researchers in the field. The first seminar took place in January 1999 at Lärarhögskolan in Stockholm and included the constitution of the SMDF. The list shows all MADIF seminars.

- MADIF 1, 1999, Stockholm
- MADIF 2, 2000, Göteborg
- MADIF 3, 2002, Norrköping
- MADIF 4, 2004, Malmö
- MADIF 5, 2006, Malmö
- MADIF 6, 2008, Stockholm
- MADIF 7, 2010, Stockholm
- MADIF 8, 2012, Umeå
- MADIF 9, 2014, Umeå
- MADIF 10, 2016, Karlstad
- MADIF 11, 2018, Karlstad

Printed proceedings of the seminars are available for all but the very first meeting. This volume and the proceedings from MADIF 9 and 10 are also available digitally.

The members of the MADIF 11 programme committee were Johan Häggström (University of Gothenburg), Yvonne Liljekvist (Karlstad University), Jonas Bergman Årleback (Linköping University), Maria Fahlgren (Karlstad University) and Oduor Olande (Linneaus University). The local organisers were Yvonne Liljekvist and Mats Brunström (Karlstad University).

The programme of MADIF 11 included two plenary lectures by invited speakers JeungSuk Pang and Peter Liljedahl. As before, MADIF works with a format of full 10 page papers and with short presentations. This year the number of full papers was eighteen, which is twice as many as in MADIF 10.

The number of short presentations were eleven. The seminar also had one symposium, where three papers around a common theme were presented and discussed. As the research seminars have sustained the idea of offering formats for presentation that enhance feedback and exchange, the paper presentations are organised as discussion sessions based on points raised by an invited reactor. The organising committee would like to express its thanks to the following colleagues for their commitment to the task of being reactors and moderators: Abdel Seidouvy, Anette Bagger, Cecilia Kilhamn, Hanna Palmér, Jan Olsson, Jöran Petersson, Kajsa Bråting, Kirsti Hemmi, Kristina Juter, Linda Marie Ahl, Lotta Wedman, Mette Susanne Andresen, Mirela Vinerean Bernhoff, Nils Buchholz, Ola Helenius, Olaf Knapp, Peter Frejd, Peter Nyström and Tomas Bergqvist.

This volume comprises summaries of the two plenary addresses, 18 research reports (papers), one symposia and abstracts for the eleven short presentations. In a rigorous two-step review process for presentation and publication, all papers were peer-reviewed by two or three researchers. Short presentation contributions were reviewed by members of the programme committee. Since 2010, the MADIF Proceedings have been designated scientific level 1 in the Norwegian list of authorised publication channels available at <http://dbh.nsd.uib.no/kanaler/>.

The editors are grateful to the following colleagues for providing reviews: Abdel Seidouvy, Allan Tarp, Andreas Ryve, Anette Bagger, Angelika Kullberg, Anna Teledahl, Anna-Lena Ekdahl, Arne Engström, Barbro Grevholm, Björn Textorius, Camilla Björklund, Cecilia Kilhamn, Cecilia Segerby, Cristina Skodras, Cristina Svensson, Djamshid Farahani, Eva Taflin, Frode Rønning, Gerd Brandell, Hanna Palmér, Hamid Asghari, Helena Roos, Håkan Lennerstad, Håkan Sollerwall, Ida Bergvall, Jan Olsson, Jannika Lindwall, Joakim Samuelsson, Jonas Dahl, Jorrryt van Bommel, Judy Sayers, Jöran Petersson, Kajsa Bråting, Karolina Muhrman, Kenneth Ruthven, Kirsti Hemmi, Kristina Juter, Lars Madej, Leslie Jiménez, Linda Marie Ahl, Lotta Wedman, Madis Lepik, Maike Schindler, Magnus Österholm, Maria Alkhede, Maria Larsson, Maria Reis, Marie Tanner, Margareta Engvall, Mathias Norqvist, Mette Susanne Andresen, Morten Blomhøj, Ola Helenius, Olaf Knapp, Paul Andrews, Per Nilsson, Peter Frejd, Peter Nyström, Reidar Mosvold, Robert Gunnarsson, Suela Kacerja, Takashi Kawakami, Thomas Lingefjärd, Tomas Bergqvist, Troels Lange and Uffe Thomas Jankvist.

The organising committee and the editors would like to express their gratitude to the organisers of Matematikbiennalen 2018 for financially supporting the seminar. Finally we would like to thank all participants of MADIF 11 for sustaining their engagement in an intense scholarly activity during the seminar with its tight timetable, and for contributing to an open, positive and friendly atmosphere.

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Development of elementary teacher classroom expertise: issues and approaches

JEONGSUK PANG

Professional development of mathematics teachers is of great significance to implement effective mathematics instruction. It is especially true for elementary school teachers who are educated to teach many subjects rather than only mathematics. This keynote speech looks back on the experience of the speaker as a mathematics educator and a teacher educator and reviews several studies of the speaker from the perspective of teacher expertise with three themes: (a) development of instructional materials in mathematics, (b) effective mathematics instruction, and (c) elementary teacher education for classroom expertise. As such, this speech is expected to be informative for Swedish and Nordic researchers to consider the perspectives on professional development of mathematics teachers.

Development of instructional materials in mathematics

For better mathematics teaching and learning, educational leaders change the national curriculum. Traditionally, the mathematics curriculum in Korea emphasizes three aspects: acquisition of mathematical knowledge and skills, enhancement of mathematical thinking ability, and cultivation of problem-solving ability and attitude (Pang, 2014a). In the 2007 curricular revision, mathematical communication and positive attitude were emphasized. In the 2009 minor revision, creativity and character-building were added as a slogan for all subjects including mathematics. Most recently, in the 2015 revision, six core competencies in mathematics are emphasized, which are problem-solving, communication, reasoning, creativity and convergence, data processing, and attitude and practice (Ministry of Education, 2015).

However, such curricular emphases are abstract and ideal so that teachers may have difficulty in understanding specific new expectations regarding their teaching practices. Rather teachers get a sense of such new expectations through the changes of mathematics textbooks, workbooks, and teacher manuals which are aligned with the national curriculum.

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More importantly, we have only one series of elementary mathematics textbooks, workbooks, and teacher manuals for Grades 1 to 6. Instructional materials, specifically, teacher manuals, are the main resources pre-service teachers study in order to pass the National Teacher Employment Test. While preparing for the test, they read every page of teacher manuals. Mathematics textbooks are also main resources for in-service teachers to teach mathematics. They usually deal with every problem in textbooks. Given these, we make every effort to develop the best instructional materials.

However, there are at least three issues in developing instructional materials as follows:

- Do the instructional materials provide key activities tailored to the mathematical topic to be taught regardless of the curriculum changes?
- Do the instructional materials provide necessary knowledge for teachers?
- Do the instructional materials help teachers be sensitive to students' varying responses to the same task?

Analyses of multiple versions of textbooks developed under the revisions of the national curriculum showed that there have been similar mathematical topics except when to teach, but that there have been different approaches, not-necessarily important knowledge for teachers, and not-enough information of student thinking for teachers.

Against this trend, while we have been developing new materials according to the 2015 revised curriculum, we have taken a research-based approach in writing textbooks (Pang, 2016). For instance, textbook writers first check the curricular changes to find major emphases. They read key introductory elementary mathematics education books to strengthen theoretical foundation. The writers look closely at many research papers regarding the topic to be written in an effort to search for detailed suggestions or implications. They also look at the previous textbooks and white papers to trace down any trends and background. In addition, the writers check comparative analysis of textbooks in other countries to look for alternative approaches.

Triangular analysis in the research-based approach is emphasized in terms of content, student, and teacher:

- What are the key instructional elements of the specific mathematical content?
- What do students understand regarding the specific content and what kinds of challenges or difficulties are there in *learning* the content?
- What do teachers understand regarding the specific content and what kinds of challenges or difficulties are there in *teaching* the content?

Note that this analysis helps textbook writers to deal with the three issues in developing instructional materials as mentioned above. Let me provide an example with the unit of "pattern and correspondence" for the fifth graders. Four key instructional elements (i.e. context, task, function rule, and variable) were extracted to be included in the textbooks (Pang, Sunwoo & Kim, 2017). Firstly, the activities in the textbook need to deal with correspondence relationships in real-life contexts. Secondly, the activities need to include various pattern tasks including number and geometry pattern as well as additive and multiplicative relationships. Thirdly, the activities need to encourage students to explore the correspondence relationship by focusing on the changes of two quantities. Finally, the activities need to encourage students to represent the correspondence relationship with symbols including variables.

To provide necessary knowledge for teachers, the specific corner "background knowledge of the unit" is emphasized in the teacher manual. For instance, regarding the unit of pattern and correspondence, the following four aspects were elaborated: (a) importance of teaching "pattern and correspondence" in school mathematics, (b) three types of thinking in exploring the relationship between two quantities (i.e. recursive pattern, covariational thinking, and correspondence relationship), (c) key instructional elements to teach "pattern and correspondence" (i.e. relation to real-life contexts, diversity of pattern tasks, exploration for a correspondence relationship, and teaching variables meaningfully), and (d) instructional strategies to teach "pattern and correspondence" such as the use of a non-sequential correspondence table and use of position number cards. Note that these were intended to provide content-specific knowledge for teachers.

To help teachers be sensitive to students' different responses to the same task, the teacher manual provides teachers with detailed examples of students' potential responses and possible feedback. This helps the teacher prepare to monitor students' varied thinking while they solve a given task and to provide timely feedback tailored to their responses. To summarize, development of effective instructional materials is the basis of developing teacher expertise.

Effective mathematics instruction

There is always a fad for new approaches in teaching and learning mathematics. For instance, there have been emphases of flipped learning, productive mathematical discussion, integration of mathematics with other subjects, and process-oriented approaches in the Korean context. As the autonomy of educational policy in each province is increased, key words spring up everywhere including "understanding-focused", "learning-focused", and "mathematical thinking". What is crucial is then what the teacher really values in their mathematics teaching.

Against this background, the following is a brief report of a study which explored Korean teachers' perspectives of effective mathematics teaching (Pang & Kwon, 2015). A questionnaire was developed with two parts. Part I asked teachers to describe any aspects they regarded as important to an effective mathematics lesson and aspects which led to poor lessons. Part II then asked teachers to check how much they agree with the 48 items related to effective mathematics teaching in terms of 5 Likert scales. The items were categorized into four main domains (i.e. curriculum and content, teaching and learning, classroom environment and atmosphere, and assessment) and seven sub-domains (construction of curriculum, selection of content, teaching and learning method, learner, instructional materials, classroom environment and atmosphere, and assessment).

The subjects for this study were selected by stratified cluster random sampling. Finally, 135 questionnaires were analyzed from elementary school teachers, 132 questionnaires were analyzed from middle school mathematics teachers, and 124 questionnaires were analyzed from high school mathematics teachers.

The results of Part I showed that all groups of teachers believe that enhancing students' self-directed learning is effective in mathematics teaching. Other aspects were differently emphasized among the groups. For instance, elementary school teachers thought that using concrete materials is important for effective mathematics teaching, whereas secondary school teachers gave their priority to communication between the teacher and students. Middle school teachers specifically mentioned students' motivation and engagement, whereas high school teachers emphasized the reconstruction of a curriculum tailored to students' various mathematical abilities.

A result of Part II showed remarkably similar trends among three groups of teachers. This reflects that teachers' perspectives are entrenched in their socio-cultural contexts. The top 5 items of effective mathematics instruction include "teaching by re-constructing the mathematics curriculum tailored to students' various levels", "teaching by interaction between the teacher and students", "teaching to improve students' self-directed learning ability", "providing students with appropriate feedback", and "teaching the essential concepts in mathematics". These results demonstrate that teachers recognize the importance of doing meaningful mathematics beyond simply teaching a mathematics topic.

However, there are challenges in implementing what teachers perceive as being important in their actual mathematics lessons. For instance, teachers perceive that effective mathematics instruction is to foster students' meaningful learning so that not only conceptual understanding of mathematics but also meaningful engagement need to be emphasized. However, incorporating these two is very challenging. For instance, using concrete materials in elementary mathematics classrooms is recommended. But teachers often tend to focus on

the completion of doing activities, rather than understanding the concept or principle behind the activities (Pang, 2002). More recently, eliciting students' different solution methods is recommended. But teachers often emphasize their pre-determined solution method, instead of orchestrating for powerful mathematical discussion on the basis of students' multiple approaches (Pang & Kim, 2013). This leads us to conduct systematic assessment of new teaching approaches and to support teachers as needed.

Elementary teacher education for classroom expertise

As a cultural background, the following is a summary of the process of becoming an elementary school teacher in Korea. A student who wants to be an elementary school teacher must enter one of 13 specific universities offering elementary teacher education programs. Due to the popularity of the teaching profession, only about the top 5% of high school graduates can enter such programs. They then need to finish teacher education programs which consist of general studies, pedagogical preparation, subject matter preparation, and fieldwork experience. The programs require approximately 140 credit hours. Then, in order to be a public school teacher, pre-service teachers need to pass the competitive National Teacher Employment Test which consists of written examinations, evaluation of impromptu teaching performance, and interviews.

While completing teacher preparation programs, pre-service teachers are asked to choose one subject matter for concentration. This requires only about 20 out of 140 total credit hours for completion. As such, we take the generalist model for elementary teacher preparation to teach many subjects. Most coursework is quite the same regardless of the concentration within the university (see Pang 2015 for the detailed information of teacher preparation programs). The issue is then that pre-service teachers may not necessarily understand effective *mathematics* instruction. There is a lack of knowledge and skills to analyze and reflect on a lesson by mathematics-specific ways.

Against this issue, the following is a summary of a *case-based pedagogy* taken to increase expertise of pre-service elementary teachers in Korea (Pang, 2011). The term case-based pedagogy is used to underline a series of pedagogical flow by which pre-service teachers first analyze others' teaching practices and then design, implement, and reflect on their own instruction both individually and collectively. Two methods were used to collect video-taped mathematics lessons. Firstly, mathematics lessons were collected as it were, taught by master teachers, in-service teachers, or pre-service teachers. Secondly, some lessons were purposefully planned and implemented to address key ideas of mathematical teaching and learning which might be difficult to observe in ordinary classrooms. Then the collected lessons were analyzed with three criteria: (a) productivity of the lesson to raise important issues of mathematics instruction,

(b) specificity of the lesson to understand what happens in the classroom, and (c) the representativeness of the lesson to cover big mathematical ideas taught across grade levels.

A comprehensive written case was then developed for each selected lesson. Each case included five elements:

- The *Overview of the case* covers mathematical topics, textbooks and workbooks, classroom situation, and overall lesson flow.
- The *Description of lesson* provides a detailed explanation of what exactly happened in the lesson with episodes, video-clips, and teacher's own reflection.
- *Questions for discussion* are lists for teachers to reflect on the case.
- *Theoretical background* is a summary of theory or issues related to the case.
- *Focused analysis* provides teachers with what to learn from that case.

There are two phases of implementation. The first phase is to analyze others' practices. Before each class, the pre-service teachers are asked to read a part of each case, specifically from "Overview of the case" to "Description of the lesson," in order to be familiar with general lesson flow. In class, the teachers watched the video-taped lesson together and wrote down whatever stood out. On the basis of their comments, the teachers discussed extensively the instruction. After class, the teachers read the rest of each case in order to be familiar with theoretical background and focused analysis.

The second phase is to apply whatever they have learned from the first phase to their own practice. They are asked to videotape their mathematics lessons during their practicum period, and then to write a report on one's lesson design, implementation, and reflection. They are asked to present the report with video clips and discuss and receive various feedbacks from the instruction as well as other pre-service teachers.

The characteristics of case-based pedagogy include that the pre-service teachers have rich opportunities to assess various cases on the basis of detailed information of the classroom events with strong theoretical background and focused analysis. The teachers in a group context are able to interpret the same event from multiple perspectives. This experience of examining critically one's own initial analytic focus and elaborating on it regarding specific mathematical features seems fundamental in using cases in teacher preparation programs. Thanks to the productive discussion of the cases and carefully designed courses, the teachers are able to develop mathematics-specific analytic ability. For instance, they focus on mathematical tasks, teaching strategies specific to

the topic and students, rather than on general strategies. They focus on students' mathematical thinking. More recently, our pre-service teachers are able to improve their lesson analysis ability in terms of topic, actor, stance, and evidence (Pang, 2014b). In addition, they came up with alternative approaches.

Another issue in teacher education in the Korean context is that if the pre-service teacher is employed, it means tenured. In-service teachers are expected to get two kinds of professional trainings throughout their teaching career: qualification training and duty training. The teachers take some duty training courses on a yearly basis. But it is optional in practice. Elementary school teachers tend to take courses about integrated subjects rather than courses targeted at the particular subject matter like mathematics. In this respect, developing teacher expertise is rather voluntary and easy to be superficial. Note that, as mentioned in the previous section, the issue of implementing effective mathematics instruction is that some changes are superficial.

Against this issue, the following is a summary of a project taken to transform teaching practices in Korean elementary mathematics classrooms (Pang, 2012). One of the participants, Ms. Y showed substantial changes, which were very noticeable not only to the researcher but also to all of the participants. Two mathematics lessons per month were videotaped and transcribed. The teacher was interviewed three times and there were monthly inquiry meetings. An analytic framework was developed to examine what had changed and what had not changed in the process of changing teaching practices.

At first, the teacher was very concerned about going through all of the activities and problems. She faithfully followed the sequence of activities in the textbook but did not necessarily recognize the inter-relations. To be sure, the teacher provided detailed guidance with praise and encouragement. The students participated in the activities and solved the problem easily, but their answers were limited to short or fixed responses.

What had changed? *Dramatic* changes happened in the early stage of teacher change in case of using manipulatives, reconstructing the activities of textbook, presenting one's own ideas, and using small-group or individual activity. *Substantial* changes happened less dramatically but considerably in the middle of stage of teacher change, for instance, emphasizing students' reasoning and communication. The teacher often used open-ended questions and provided timely feedback. She also solicited and used students' ideas. *Gradual* changes occurred over a longer period of time. For instance, it takes time to foster students' positive disposition. The teacher's demonstration gradually decreased, while peer communication among students was increased.

What had not changed? Overall characteristics of the lessons were consistent, progressive, and systematic. The teacher focused on conceptual understanding and problem solving and used instructional strategies considering content. These aspects were consistent aspects over student-centeredness.

However, two aspects, considering students and teacher role of emphasizing mathematical communications, showed some positive changes but did not reach the full expectation. These are the areas we as teacher educators need to provide more pro-active support.

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What teachers want from their professional learning opportunities

PETER LILJEDAHL

Teachers do not come to professional learning opportunities as blank slates. Instead, they come to these settings with a complex collection of wants and needs. The research presented here takes a closer look at these wants, across five different professional learning settings, distilling from the data a taxonomy of five categories of wants that teachers may approach professional learning with. The resultant taxonomy, as well as teachers behaviours vis-à-vis this taxonomy, indicate that we need to rethink our roles as facilitator within these settings.

Research on mathematics teachers and the professional development of mathematics teachers can be sorted into three main categories: content, method, and effectiveness. The first of these categories, *content*, is meant to capture all research pertaining to teachers' knowledge and beliefs including teachers' mathematical content knowledge, both as a discipline (Ball, 2002; Davis & Simmt, 2006) and as a practice (Hill, Ball & Schilling, 2008). Recently, this research has been dominated by a focus on the mathematical knowledge teachers need for teaching (Ball & Bass, 2000; Ball, Hill & Bass, 2005; Davis & Simmt, 2006; Hill, Rowan & Ball, 2005) and how this knowledge can be developed within preservice and inservice teachers. Also included in this category is research on teachers' beliefs about mathematics and the teaching and learning of mathematics and how such beliefs can be changed within the preservice and inservice setting (Liljedahl, 2007, 2010a; Liljedahl, Rolka & Rösken, 2007). Some of the conclusions from this research speaks to the observed discontinuities between teachers' knowledge/beliefs and their practice (Cooney, 1985; Karaagac & Threlfall, 2004; Skott, 2001; Wilson & Cooney, 2002) and, as a result, calls into question the robustness and authenticity of these knowledge/beliefs (Lerman & Zehetmeir, 2008).

The second category, *method*, is meant to capture the research that focuses on a specific professional development model such as action research (Jasper & Taube, 2004), lesson study (Stigler & Hiebert, 1999), communities of practice

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(Little & Horn, 2007; McClain & Cobb, 2004; Wenger, 1998), or more generally, collegial discourse about teaching (Lord, 1994). This research is "replete with the use of the term inquiry" (Kazemi, 2008, p. 213) and speaks very strongly of inquiry as one of the central contributors to teachers' professional growth. Also prominent in this research is the centrality of collaboration and collegiality in the professional development of teachers and has even led some researchers to conclude that reform is built by relationships (Middleton et al., 2002).

More accurately, reform emerges from relationships. No matter from which discipline your partners hail, no matter what financial or human resources are available, no matter what idiosyncratic barriers your project might face, it is the establishment of a structure of distributed competence, mutual respect, common activities (including deliverables), and personal commitment that puts the process of reform in the hands of the reformers and allows for the identification of transportable elements that can be brokered across partners, sites, and conditions. (ibid., p. 429)

Finally, work classified under *effectiveness* is meant to capture research that looks at changes in teachers practice as a result of their participation in some form of a professional development program. Ever present in such research, explicitly or implicitly, is the question of the robustness of any such changes (Lerman & Zehetmeir, 2008).

As powerful and effective as this aforementioned research is, however, it can no longer ignore the growing disquiet that somehow the perspective is all wrong. In fact, it is from this very research that this disquiet emerges. The questions of robustness (Lerman & Zehetmeir, 2008) come from a realization that professional growth is a long-term endeavour (Sztajn, 2003) and participation in preservice and inservice programs is brief in comparison. At the same time there is a growing realization that what is actually offered within these programs is often based on facilitators' (or administrators' or policy makers') perceptions of what teachers need as opposed to actual knowledge of what teachers really want (Ball, 2002). But not much is known about what teachers really want as they approach professional learning opportunities.

The research presented here provides some answers in this regard.

Methodology

As articulated in Liljedahl (2010b), working in a professional development setting I find it difficult to be both a researcher and a facilitator of learning at the same time. As such, I generally adopt a stance of noticing (Mason, 2002). This stance allows me to focus on the priorities of facilitating learning while at the same time allowing myself to be attuned to various phenomena that occur within the setting. It was through this methodology that I began to notice that

there was a distinct difference between the groups of teachers that came willingly to the professional development opportunities that I was leading and the teachers that were required, often by their administrators, to attend. This was an obvious observation. Nonetheless, it was as a result of this observation that, I began to attend more specifically to what these differences were. In doing so I began to notice, subtly at first, that the teachers who came willingly came with an *a priori* set of wants. With this less obvious observation I changed my methods from *noticing* to more directive research methods. I began to gather data from five different professional learning contexts over a period of two years.

Master's programs

Teachers in this context are practicing secondary school mathematics teachers who were doing their Master's Degree in Secondary Mathematics Teaching. This is a two year program culminating in either a comprehensive examination or a thesis depending on the desires of the teacher and the nature of the degree that they are seeking. From this group I collected interview data and field notes during two different courses I taught in the program.

Induction group

This group began as an initiative to support early career teachers (elementary and secondary) as they make the transition from pre-service teachers to in-service teachers. In truth, however, it also attracted more established teachers making it a vertically integrated community of practicing teachers of mathematics. Although this group now meets every second month for the duration of the study we met monthly. From this group I collected interview data, field notes, as well as two years of survey data.

Hillside middle school

Hillside (pseudonym) is the site of a longitudinal study. For the last five years I have meet with a team of three to six middle school teachers every second Wednesday for an hour prior to the start of the school day. This group began as an administration led focus on assessment of numeracy skills, but after the first year took on a self-directed tone. The teachers in this group lead the focus of the sessions and look to me to provide resources, advice, and anecdotal accounts of how I have seen things work in the many other classrooms I spend time in. For the two year period that constitutes the study presented here I collected field notes and interview data.

District learning teams

Very much like the professional learning setting at Hillside, district-based learning teams are self-directed. Teachers meet over the course of a year to

discuss their classroom-based inquiry into teaching. This inquiry runs throughout an entire school year, but the teams themselves only meet four to six times a year. The data for this study comes from three such teams that I facilitated in two different school districts. One of these teams ran during the first year of the study, the other two teams ran in the second year of the study. Like at Hillside, my primary role is to provide resources, advice, and insights into their plans and reported classroom outcomes. The data from these teams consisted of field notes, interviews, and survey results.

Workshops

During the two years that I collected data for this study I was also asked to do several *one-shot* workshops. These were workshops designed around a variety of different topics, either decided by myself or the people asking me to deliver the workshop. They ranged in time from 1.5 hours to 6 hours with no follow-up sessions. Data, consisting of field notes, comes from six such workshops. Data from two additional workshops consists of field notes and survey results.

The data

Field notes in the aforementioned settings consisted primarily of records of conversations I had with individual teachers during breaks as well as before and after the scheduled sessions. I used these times to probe more specifically about the origins of questions asked, motivations for attending, querying about what they are getting out of the session, and if there is something else they need or want. This sounds very formal and intentional, but in reality, this was all part of natural interactions. In all, I collected notes on over 70 such conversations.

More directed than these natural conversations were the interviews. These were much more formal in nature and provided an opportunity for me to probe further about the conversations we had previously had or the things I had observed during our sessions together. Each interview lasted between 30 and 60 minutes. In all, 36 interviews were conducted over the course of the two years, resulting in 26 hours of audio recordings. These recordings were listened to as soon as possible after the interviews and relevant aspects of the recording were flagged for transcription.

The survey used with the Induction group, The District learning teams, and two of the Workshops consisted of an online survey instrument that was sent to the teachers prior to a professional learning session. The survey contained five questions:

1. Name?
2. Where are you in your teaching career? Are you in a teacher education program (please specify semester), a substitute teacher (how many

years), on a long term temporary placement (for how long), or do you have your own classroom (for how long)?

3. If relevant – what grades/subjects are you teaching right now?
4. What do you hope to get out of our next session together? You can ask for understanding of mathematical concepts, teaching strategies, resources, lesson ideas, ideas about classroom management, networking opportunities, specific lesson plans, etc. In essence, you can ask for anything that will help you in your teaching or future teaching. List as many as you want but please be specific.
5. Please list something from a past session that you found particularly useful.

The last two of these were of obvious relevance to the study. However, as will be seen later on, question two contributed data that became relevant to the analysis.

Analysis

The field notes, interview transcripts, and survey data were coded and analysed using the principles of analytic induction (Patton, 2002). "[A]nalytic induction, in contrast to grounded theory, begins with an analyst's deduced propositions or theory-derived hypotheses and is a procedure for verifying theories and propositions based on qualitative data" (Taylor & Bogdan, 1984, p. 127, cited in Patton, 2002, p. 454). In this case, the *a priori* proposition was that teachers come to professional learning settings with their own wants in mind (Ball, 2002) and that these wants are accessible through the methods described above. With a focus on teachers' wants the data was coded using a constant comparative method (Creswell, 2008). What emerged out of this analysis were a set of themes specifically about the wants expressed by teachers as well as a broader set of themes that cut across these wants. In what follows I present these themes in two distinct sections. The first section is a taxonomy of five types of wants. The second section are the themes that cut across this taxonomy.

Results – wants

As mentioned, one of the things that emerged out of the aforementioned analysis was a taxonomy of five distinct categories of wants that teachers come to professional learning settings with. To these I add a sixth category. Although not a want *per se* this sixth theme deals with the resistance with which some teachers engaged in some of the professional developing opportunities. In what follows I present each of these categories in turn, beginning with resistance and following it up with each of the five categories of wants.

Resistance

In the course of the two years of the study I collected data on a number of teachers who were flatly opposed to being part of the professional development setting I was working in. All of this data consisted of observation and conversations and came solely from the workshops and learning team settings. To a person, these teachers were participating in these settings at the bequest of an administrator or a department head. As a result, their want was to *not* be there.

First, these resistant teachers were present and they did participate in the sessions. They engaged in the activities, they asked questions, and they collaborated with others in the room. But this participation was guided by their reluctance at being there. As such, their contribution to the group was often negative, pessimistic, defensive, or challenging in nature. They would say things like "that will never work" and "I already do that". This is not to say that these teachers were the only ones to utter these types of statement, but rather that they only uttered these types of statements. Their questions to me were always challenging in nature with greater demands for evidence, justification, and pragmatism. These challenges were welcomed as they often provided others with an opportunity to engage in the content more critically. The call for pragmatism, in particular, was not unique to resistant teachers, but the goals for making that call were clearly different. When they challenged ideas based on their infeasibility the goal seemed to be to detract from the value of what was being offered; to invalidate it. When non-resistant teachers made the same call it seemed to be motivated by a goal to try to navigate the space between the ideal and the feasible; to find a way to make it happen.

The second reason I include this theme is that these teachers did not always remain resistant. There were several cases in my data where teachers, who initially approached the setting with resistance, softened their stance over time. In the workshop settings this was marked by a shift in the types and ways in which they asked questions, the ways in which they engaged in activities and interacted with their peers, and in the parting comments and conversations I recorded. In the learning team settings, this was marked by the fact that between meetings, these initially resistant teachers, reported back at subsequent sessions about efforts made, and results seen, in their own classrooms.

The third reason for including this theme here is because I want to differentiate between the resistance a teacher may have to an idea in a professional learning setting and the *a priori* resistance a teacher may approach that setting with. In the former case I am talking about a healthy form of scepticism that, as mentioned, allows teachers to negotiate the space between the ideal and the real, between the theoretical and the practical. The later, however, is a stance that can prevent the uptake of good ideas and helpful suggestions and can act as a barrier to learning and professional growth.

In all, out of the 70 conversations that I made notes on, 10 were with teachers who were, at least originally, resistant to being in the setting. Of these, four changed their stance over the course of the setting. However, my field notes record observations of many more such *a priori* resistant teachers as well as observed changes in some of them.

Do not disturb

This category of wants characterizes those instances where a teacher engages in professional learning because they want to improve their practice, but is reluctant to adopt anything that will require too much change. Ideally, what they want are small self-contained strategies, lessons, activities, or resources that they can either use as a replacement of something they already do and cleanly insert into their teaching without affecting other aspects of their practice. Such wants were rarely stated outright. Instead, they manifest themselves as overly specific statements of what it is they seek.

I was hoping to learn a new way to introduce integers.

I want something to do for the first 10 minutes of class.

A different way to do review.

All of these are indicative of situations where the teacher is looking to improve one thing about their teaching. The *don't-let-it-affect-anything-else-around-it* is implicit in the specificity of the statement. In conversations or in interviews, however, this can sometimes come out more explicitly.

I'm happy with the rest of my fractions unit. It's just division of fractions that messes me up. I was hoping that you could show me a better way to explain it.

Delving deeper it became clear that in many of the instances, where concern over disturbance and tight control over impact was important, there was an underlying anxiety, most often around the possibility of deconstructing what they have worked hard to build up.

I've been teaching for seven years now, and I'm really happy with the way things are going. After the last curriculum revision and with us getting a new textbook I have worked really hard to organize all of my lessons and worksheets in math. I don't want to mess with that. So, please don't tell me anything that is going to mess me up. I really just want to know if there is a lesson that I can do using computers that will be fun and that I can just sort of insert into my area unit.

Less often this anxiety is around what they have worked hard to understand.

When I started teaching I was fine with math. But when I was given a grade seven class this year I sort of panicked about math. Especially the unit on integers. I had never understood it when I was in school and it took me a long time to teach it to myself. So, I don't really want to learn anything new that will rock the boat for me.

In other instances there didn't seem to be an underlying anxiety, but just a pragmatic disposition that small change is good. The teachers with this disposition came to the professional learning settings with a want to learn new things and a willingness to make changes, so long as these were small changes. Although only one teacher spoke directly about this "less is more" disposition there was lots of evidence of it in the way teachers spoke about what they got out of the sessions. For example, in an interview after a session on formative assessment, one teacher told me that she had learned "I am not going to give out zeros anymore". Although important, in relation to the larger conversation of the difference between formative and summative assessment, this is a by-product of a shifting assessment philosophy, not a change unto itself. However, when probing further it revealed that for this teacher "no zero's is something I can start doing on Monday". This was something that she could cleanly insert into her practice.

Regardless of the motivation, the teachers who wanted to make only small changes know what they don't know, or don't do well, and want to learn new things to help change them.

Willing to reorganize

A slight nuance on the previous theme is when teachers want very specific improvements and they are willing to significantly reorganize their teaching and resources to accommodate the necessary changes. Although specific in nature, these wants do not come with limitations. They are stated with an implicit openness to the consequences that the desired improvements may necessitate.

I am looking to redo my unit on trigonometry. I have been following the text up until now, but I think it is time to build a new unit.

So, yeah. I'm looking for an improved way to have my students learn how to do problem solving. Right now I do it as a unit in February, but it's not working. I've heard that other teachers do it throughout the whole year and I'm hoping to get some ideas around that.

Further probing of these teachers, as well as the others who made similar statements, revealed that they are not hampered by anxiety around invalidating existing resources or undoing things learned. Like their counterparts in the previous category, however, they know what they don't know or what they don't do well and they want to make changes to improve these things. The difference is the scale at which they are willing to make these changes.

Willing to rethink

Unlike the previous two categories, the wants that fit into this are much broader in scope and often welcome a complete rethinking of significant portions of a teaching practice.

I'm pretty open to anything. I mean with respect to differentiated learning.

From the interviews it became clear that for this teacher, as well as for those who expressed similar wants, there exists something in their practice that they want to bolster. In many cases these teachers are wanting collections of resources that they could then organize and integrate into their teaching.

Anything to do with numeracy is good for me.

I'm looking for new ideas about assessment for differentiated learners.

In some cases, however, these teachers are branching out into new territories and are looking for a comprehensive package of what to do.

I'm hoping to introduce the use of rubrics into my teaching and want to get the rubrics I should use as well as instruction how to do it.

Either way, these teachers have a rough idea of what it is they want and are willing to rethink their teaching in order to accommodate new ideas. They do not have the anxieties of disrupting already held knowledge or resources that the teachers in the first category did and their wants are broader in scope than the second.

Inquiry

This category consists of those wants which align with the ideas and ideals of inquiry (Kazemi, 2008). As such, these wants consist, most often, of a general desire to acquire new knowledge and ideas about teaching. The teachers who express these wants are open to any new ideas and often come to professional learning settings without an agenda.

I'm not really looking for anything in particular. But, I'm eager to hear about some new ideas on assessment.

I was at your numeracy workshop last year and I liked it, so I thought I would come and see what else you have to say.

This is not to say that these wants are flighty and unrefined. The teachers whose wants fall into this category are often methodical in their change, pausing to ask exactly "what is it I am doing" and "if it's working". And if it is working they question "what is it that is telling me it is working".

I'm piloting a new textbook this year. So, far I'm not that impressed, but it has really opened my eyes to different ways to think about fractions.

They want evidence of success, but they want it to come from their own classroom.

Out with the old

The previous categories of wants are characterized by the willingness, to varying degrees, to add new ideas into current teaching practices. The *Out with the old* category is not at all about what teachers want to add, but rather about what they want to take away. Teachers with this category of wants come to professional learning settings unhappy with something in their practice and goes well beyond the awareness that something needs to be improved. For these teachers there is nothing to be integrated, there isn't a replacement of some aspect of their teaching to be made. They have already rejected the current paradigm and are now looking for something to fill the void that is left behind.

My kids can't think for themselves in problem solving. I don't know what I'm doing wrong, but it doesn't matter. I just need to start over with a new plan.

I can't stand the way group work has been working in my classroom. Or not working is a better description. I have given up with what I've been doing and am looking to learn something completely different.

This is not to say that these wants are coupled with blind acceptance of anything that fits the bill. The teachers who express these wants are often hypercritical of new ideas, usually as a result of their dissatisfaction with something that they have previously changed in their practice.

I spent a whole year trying to teach and assess each of the processes [communication, connections, mental mathematics and estimation, problem solving, reasoning, technology, and visualization] that are in the curriculum. In the end my students are no better at estimating or communicating, for example, than they were at the beginning of the year. My approach didn't work. I need a new way to think about this.

This is not to say that they are closed minded, but rather that they exert a greater demand on me, as the facilitator, to bridge the theoretical with the pragmatic.

Results – cutting across the taxonomy

As mentioned earlier, aside from the taxonomy of wants, there were also a set of themes that emerged out of the analysis which can be characterized as cutting across the taxonomy presented above. In what follows I present each of these themes.

Pseudo-hierarchy

Four of the aforementioned wants – *do not disturb*, *willing to reorganize*, *willing to rethink* and *inquiry* – seem to form a hierarchy in the way each category requires a slightly greater openness to change than the previous one (see figure 1). Although the teachers in the more longitudinal aspects of the study tended to have wants that became more and more open as the study went on, there were still days when they would come into the session wanting something as overly specific as a problem to do with the students the next day. There was also evidence in the field notes of individual teachers changing the scope of their wants within a single session. Sometimes this was a broadening of wants to ones that were more open to changes in teaching practice. Other times they regressed to wanting easily insertable resources, especially when the discussions shifted to tricks and best practices.

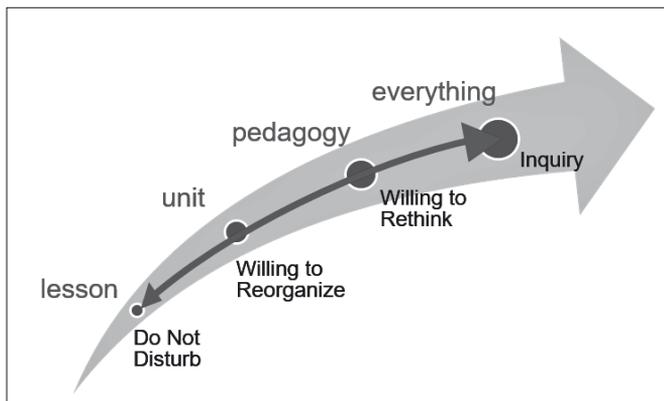


Figure 1. *Pseudo-hierarchical organization of teacher wants*

Also evident in this pseudo-hierarchy is that the scope of change associated with each want seems to increase in magnitude as the level of openness to change increases (see figure 1). That is, a teacher with wants in the *do not disturb* category is willing to entertain changes at the level of the lesson while a teacher with wants in the *willing to reorganize* category is looking at changes at the unit level. A teacher who is *willing to rethink*, on the other hand, is not focused on content as much as they are on pedagogy – differentiation, assessment, etc. Finally, the *inquiry* teacher is open to anything – from a lesson to a unit, to pedagogy.

This association between the scope of change and the pseudo-hierarchy of wants is useful in placing the teachers who express resistance as well as those who have rejected parts of their practice (*out with the old*). Resistant teachers are not wanting to change anything in their practice – no matter how small. As such, their wants do not sit on the pseudo-hierarchy (see figure 2).

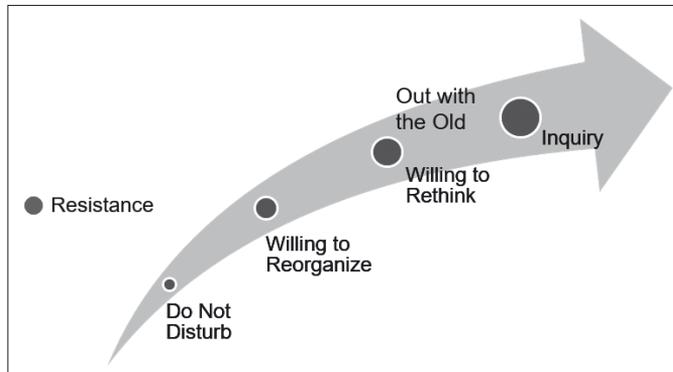


Figure 2. Expanded pseudo-hierarchical organization of teacher wants

Meanwhile, the wants of the teachers in the *out with the old* category are, almost to a person, wanting to make changes at the level of pedagogy. As such, the *out with the old* category sits at the same place in the pseudo-hierarchy as *willing to rethink* (see figure 2).

Three further nuances of this theme are worth noting. The first one has to do with novice teachers. Almost without exception, these teachers came to professional learning opportunities with wants that fit into the *willing to rethink* category (see figure 3). Deeper probing revealed a very good reason for this – these teachers do not have deeply seated practices to disrupt, they have not reified their resources, they have not yet found things about teaching that they wish to reject, and they have not yet routinized aspects of their teaching to the point where they feel comfortable engaging in inquiry. What this leaves is the category of rethinking practice. Except, with their newness to teaching this often became more of a *willingness to think* about their practice than *rethink* their practice. Given that I met many teachers whose wants were in the first two categories this means that time in the field can cause a regression regarding openness to change. This was not surprising, but troubling nonetheless.

The second nuance has to do with resistant teachers. Although teachers who were resistant about change largely did not change, there were a few instances where they did. In each of these instances the change was at the level of pedagogy (see figure 3). This revealed that their resistance to change was not rooted in a desire to hold onto reified resources, but rather in their beliefs that their pedagogy was sound. So, when change did happen it happened at the level of these beliefs.

The third point worth noting is the fact that as a facilitator I was constantly trying to upgrade the teachers' wants. That is, I was always trying to create more openness and broaden the scope of what it is they wanted out of their work with me. This was especially true of the teachers who were either resistant or came

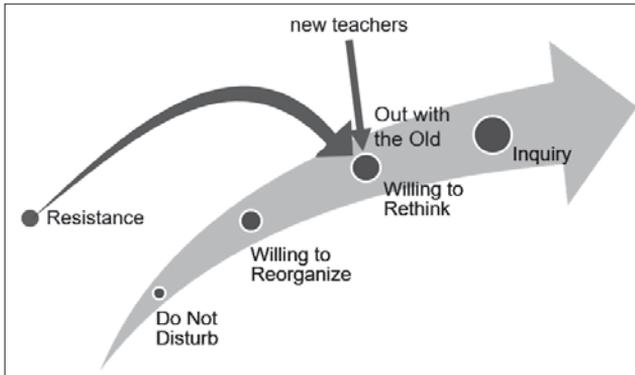


Figure 3. *Entry into the pseudo-hierarchy by new teachers and resistant teachers*

with wants in the first two categories (see figure 1). And, many teachers did expand their wants because of these efforts. There was even evidence in the data of my efforts to, and success at, shifting the wants of resistant teachers; although to a much lesser extent than those teachers who came willingly. Although not the focus of this article, this is an important point in that it shows the potential effectiveness of a facilitator in fostering changes. But it also speaks to the fact that teachers who come willingly to professional learning settings are already engaged in thinking about change and, as such, are predisposed to changing.

Engagement

Something that emerged very clearly from the data was that the wants that teachers had coming into a professional learning setting affected the way in which they engaged in the session. These types of engagements can be seen as fitting into three categories.

First, the teachers who wanted to make minimal change tended extract things from single sessions that spoke of small change. An example of this was presented above in the way one teacher took away from a wide sweeping session on the differences between formative assessment and summative assessment only the one small, and easy to implement, strategy of not "giving out zeros". Other such examples include "having students tell the story of how they solved a problem" as the only tangible thing that came out of a session on improving students' communication skills in mathematics, or "not telling students if their answer is correct" out of a session on problem solving. These examples, almost all coming out workshop settings, are evidence that a teacher who is committed to making small change will find ways to make small change, even in the face of complex and broad topics. However, as mentioned above, in the more longitudinal settings of the District learning teams or among the teachers at Hillside there was a general trend towards more openness.

The second category pertains to those teachers who approached these professional learning settings already open to change. Unhampered by the need to restrict their changes these teachers were more willing to take on ideas that went beyond the scope of the wants that they came into the professional learning settings with.

So, I wanted to understand why our district is saying that we can't use zeros anymore. I was willing to make changes around this in both testing and homework if I could figure out what to do instead. Now I realize that what I really need to do is change the way I collect information about my students' performance. I need to get away from the collection of points and focus more on the collection of data.

They were also more willing to walk away from professional learning settings with commitments to make change in areas other than what they came in with.

I originally wanted to work on numeracy skills, but now I realize that I also need to work on my students' group work and communication abilities.

This was true irrespective of the nature of the professional learning setting. This willingness to take on broader or different ideas than they initially came in with was seen even across very short single workshops. Unlike the teachers who wanted small change, these teachers' openness to change is not limited to what they know they don't know, but also extends to what they didn't know they didn't know.

The final category pertains to those teachers who were resistant to participating in the first place. Although there are a few rare exceptions, for the most part these teachers were unaffected by the ideas presented in workshops. Their resistance to being present extended to their resistance to new ideas. But as mentioned, they were still present and they did participate. However, their contribution to the group was often negative, pessimistic, defensive, or challenging in nature. Having said that, the two teachers who were required to be part of a District learning team did change over time and both started coming to the sessions with expressed wants that broadened in openness with time.

Autonomy

A final theme that emerged from the analysis pertains to the autonomy of teachers. As mentioned earlier, the impetus for the research presented here grew out of the obvious difference between teachers who want to be present and those who do not. This speaks greatly to the autonomy I saw exercised not only in the participation in professional learning settings, but also in the way in which the teachers participated. The teachers were free to take up new ideas, or not. They were free to broaden their thinking on new ideas, or not. What drove this freedom was their autonomy as teachers and as learners. Although I presented

new things to them they got to decide what they would do with them. They could reject them, they could think about them, or they could act on them.

Among the teachers who I had repeated interactions with, this autonomy extended beyond the professional learning settings and into their teaching. They were free to implement change, or not. They were free to try out new ideas, or not. And again, they exercised this autonomy.

This autonomy is obvious and it didn't take reams of data and deep analysis to see it. What the data and the analysis showed, however, was that the teachers exercised their autonomy in ways that redefined my role as a facilitator of professional learning. Although I was behaving as though I was driving the agenda of professional learning the reality is that at every stage the teachers had their own agenda, that they pursued this agenda, and that they used me as a resource in this pursuit. This is not to say that I did not have influence or that I was not able to change agendas, but rather than at every stage the teachers exercised the ultimate control; they could chose to learn or they could choose not to, they could choose to agree, or they could choose not to. The strongest evidence of this is what brought these teachers to the sessions, sometimes repeatedly. Each time they had a goal for attending – a want they needed satisfied – and they saw me as a resource likely to satisfy this need.

Conclusions

Much can be taken from the results presented above. The most obvious is that teachers come to professional learning settings with a variety of wants and needs. The results indicate that these wants can be organized into a taxonomy with pseudo-hierarchical properties. More importantly, however, is what the results say about teacher autonomy and the role that workshops play in the professional growth of teachers.

It is a long-held belief that single workshops are an ineffective means of creating professional growth (Ball, 2002). Although the data indicates that this was generally true for teachers who are either resistant to change or are only willing to make small changes, the results also indicate that this was not at all true for teachers whose wants coming into the session were broader in scope. In settings where participation was voluntary this accounted for the large majority of teachers. These teachers were quite willing to not only broaden their thinking on what they wanted out of the session, but were also willing to take up entirely new ideas. These results nuance the way we should view the effectiveness of workshops in facilitating teacher change.

Teacher autonomy, too, is something that needs to be taken into greater consideration. Coupled with the taxonomy of wants the results of this study feeds into a new paradigm in which the professional growth of teachers is seen as natural (Leikin, 2006; Liljedahl, 2010b; Perrin-Glorian, DeBlois & Robert,

2008; Sztajn, 2003) and teachers are seen as agents in their own professional learning (Ball, 2002). Teachers do not approach their professional learning as blank slates. They come to it with a complex collection of wants and needs and use professional development opportunities as resources to satisfy those wants and needs. Recognition of this has an impact on how we view our role as facilitator in these settings. Working from the perspective of a resource we need to be much more attuned to what it is that teachers want, while allowing the taxonomy to inform us of what they could want.

Acknowledgement

This article was originally published in *The Mathematics Enthusiast*, 11 (1), 109–122 (Liljedahl, 2014). This is an updated version of that article.

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Teoretiska och praktiska perspektiv på generaliserad aritmetik

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Generaliserad aritmetik är en av de så kallade stora idéerna inom algebraiskt tänkande. Ändå verkar generaliserad aritmetik vara svagt framskrivet både i det svenska nationella styrdokumentet i matematik och i läromedel för tidiga åren. I denna artikel fördjupas begreppet utifrån hur olika forskare har definierat det samt utifrån uppgiftsexempel. Artikeln bidrar således till forskningsfältet genom att skapa djupare förståelse av begreppet generaliserad aritmetik. Detta är av speciell vikt för den svenska skolvärlden eftersom generaliserad aritmetik är starkt kopplat till utvecklandet av datalogiskt tänkande, ett nytt inslag i det svenska styrdokumentet.

Goda kunskaper i algebra är viktiga för att klara studier i matematik efter grundskolan och gymnasiet. Flera studier visar att studenter med bristande algebrakunskaper får stora svårigheter att klara av universitetsmatematiken (Brandell, Hemmi & Thunström, 2008; Hiebert m fl, 2005). Traditionellt sett har algebra introducerats relativt sent i grundskolan eftersom man ansett att elever i yngre åldrar inte är tillräckligt mogna för att ta till sig algebra. Under senare tid har dock matematikdidaktisk forskning visat att man med fördel kan börja undervisa algebra redan i de tidigare skolåren (Cai m fl, 2005; Carraher & Schliemann, 2007; Kieran, Pang, Schifter & Ng, 2016). Detta har lett till att många länder, däribland Sverige, har ändrat sina kursplaner och börjat integrera algebran i skolmatematiken redan på ett tidigt stadium (NCTM, 2006). Vikten av algebran har också lyfts inom lärarutbildningen i Sverige sedan 1990-talet (se Bergsten, Häggström & Lindberg, 1997, en bok som har använts som läromedel i flera svenska lärarutbildningar). Forskningen visar dock att det är svårt att implementera nya sätt att undervisa om man inte beaktar den rådande undervisningskulturen (Hiebert m fl, 2005). Därför har vi riktat in oss på att studera den undervisningskultur som är typisk för svensk skolalgebra.

I Sverige har algebra under en längre tid varit en del av matematiken som skolelever haft stora svårigheter med. I den internationella mätningen TIMSS (Trends in International Mathematics and Science Study) har svenska elevers

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resultat i algebra legat under det internationella genomsnittet ända sedan början av 1960-talet¹. Även i TIMSS test från 1995, då svenska elever presterade som allra bäst, låg resultatet i algebra under det internationella genomsnittet. I TIMSS tester från 2007 och 2011 då svenska elevers resultat i matematik försämrades kraftigt var algebra och geometri de områden som drog ned resultatet allra mest (Skolverket, 2011). I den senaste TIMSS-mätningen från 2015 förbättrades de svenska elevernas resultat jämfört med 2007 och 2011 men ligger alltså fortfarande under det internationella genomsnittet. Fortfarande är det algebra och geometri som de svenska eleverna presterar sämst i (Skolverket, 2016).

Den här artikeln är en del av ett större forskningsprojekt vars övergripande syfte är att hitta möjliga orsaker till misslyckandet med att implementera algebra i den svenska skolmatematiken på ett produktivt sätt (Hemmi m fl, 2017). Mera specifikt handlar det om att undersöka hur algebra traditionellt har behandlats i svenska läroplaner och läroböcker för årskurs 1 till 9 (ett diakront perspektiv). I projektet undersöks också dagens situation (ett synkront perspektiv) genom att analysera hur algebran behandlas i aktuella styrdokument och läroböcker samt genom att intervjua aktiva lärare. I föreliggande artikel kommer vi att utgå från ett av de första resultaten i projektet där aktuella kursplaner och läroböcker undersökts för att ta reda på vilken sorts algebra som karakteriserar den svenska skolalgebran i årskurs 1–6 (Bråting, Hemmi, Madej & Røj-Lindberg, 2016).

Bakgrund och syfte

Ett ämne som intresserat forskare inom algebradidaktik är att försöka identifiera och kategorisera algebrans olika delområden (Blanton m fl, 2015; Kaput, 2008). När vi i vårt projekt analyserat grundskolans kursplaner och matematikläroböcker för årskurs 1–6 har vi i vårt analytiska verktyg utgått från Blantons m fl (2015) så kallade ”stora idéer” (big ideas): ekvivalenser, uttryck, ekvationer och olikheter; funktionslära; variabler; proportionalitet; generaliserad aritmetik (se Bråting m fl, 2016). Dessa något överlappande delområden bygger på resultat från det internationella forskningsfältet inom algebra. Därför valde vi att utgå från dem i våra första analyser. Det första delområdet inkluderar bland annat likhetstecknets betydelse, att förstå matematiska relationer och att kunna resonera kring uttryck och ekvationer. Det andra och tredje delområdet handlar om funktioner och variabler och inkluderar bland annat att kunna konstruera och läsa av tabeller, identifiera såväl mönster som funktionsregler och kunna beskriva dessa med ord samt förstå vilken roll variabler kan ha i olika matematiska kontexter. Det fjärde delområdet handlar om att kunna se när två kvantiteter är proportionella mot varandra och kunna ge exempel på och resonera kring proportionella samband. Slutligen i det femte delområdet, generaliserad aritmetik, fokuseras det på strukturer som uppkommer inom aritmetiken (Blanton m fl, 2015). Det femte delområdet är alltså i fokus i denna artikel och vi återkommer i mera detalj till det nedan.

Resultatet från våra tidigare analyser visar att de fyra första delområdena mer eller mindre alltid är och har varit välrepresenterade i såväl svenska kursplaner som läroböcker. Betoningen på funktioner och variabler har stärkts med tiden vilket man kan utläsa i den senaste kursplanen i matematik från 2011 där ”Samband och förändring”, som tidigare varit utspritt över olika områden av matematiken, lyfts fram som en egen kategori (Skolverket, 2011). Detta är inte någon tillfällighet utan följer en internationell trend där ”study of change” har identifierats som ett nyckelområde i PISAs ramverk över skolmatematik (OECD, 2010).

När det gäller det femte delområdet ovan, generaliserad aritmetik, verkar det vara underrepresenterat inom den svenska skolalgebran, åtminstone under de tidigare skolåren (Bråting m fl, 2016). Vad detta beror på är svårt att säga men förmodligen kan det vara så att det helt enkelt inte tillhör den utbildningstradition inom skolalgebra som vi är vana vid att hålla oss till i Sverige. Exempelvis förekommer inte termerna generalisera eller generalisering i den nuvarande kursplanen i matematik för grundskolan (Lgr11). Generaliserad aritmetik har en stark koppling till strukturer vilket är speciellt viktigt att fokusera på inom svensk skolmatematik med tanke på att programmering och datalogiska tänkande (Wing, 2008) nyligen implementerats som en del av den svenska skolmatematiken.

Generaliserad aritmetik är ett relativt nytt delområde av skolalgebra jämfört med de övriga delområdena och forskare har ännu inte enats om en unik definition. Syftet med denna artikel är att fördjupa förståelsen av begreppet utifrån den internationella forskningslitteraturen samt ta reda på variationen i litteraturen. Vi kommer att använda flera exempel på uppgifter inom generaliserad aritmetik för att konkretisera innebörden av begreppet. Uppgifterna exemplifierar även det som vi anser saknas inom den svenska skolalgebran enligt våra tidigare analyser. Vår avsikt i denna artikel är dock inte att genomföra en uttömmande analys av allt som har skrivits om generaliserad aritmetik utan denna studie kan ses som ett första steg i att fördjupa förståelsen av begreppet.

Olika perspektiv på generaliserad aritmetik

Under de senaste decennierna har ett omdebatterat ämne inom algebradidaktiken varit när algebra ska introduceras i skolan och vilken svårighetsgrad algebran bör ha. Vissa forskare menar att begreppsutvecklingen inom algebra hos enskilda individer avspeglas i den historiska utvecklingen (Katz & Barton, 2007) vilket bland annat skulle medföra att i enskilda individers begreppsutveckling föregår alltid aritmetik och retorisk algebra den ”riktiga” algebran. Enligt den teorin bör man inte syssla med algebra i tidiga grundskolan utan lära sig aritmetiken ordentligt först. Denna rekapitulationsteori har kritiserats starkt av flera forskare (se tex Bråting & Pejlar, 2015). Grundtanken bakom begreppet generaliserad aritmetik är istället att aritmetik och algebra inte ska

ses som två separata områden där aritmetiken föregår algebran utan att dessa två områden hänger samman och kan läras in parallellt (Kieran mfl, 2016). Även om det finns en gemensam grundtanke bakom begreppet generaliserad aritmetik finns i nuläget inte någon entydig definition.

Vi kommer härnäst att jämföra och laborera med tre olika beskrivningar av generaliserad aritmetik; 1) Blantons mfl (2015) beskrivning av generaliserad aritmetik som en av fem stora idéer (big ideas), 2) Fujii & Stephens (2001; 2008) beskrivning av generaliserad aritmetik utgående från kvasi-variabler, 3) Carraher, Schliemann, Brizuela & Earnests (2006) beskrivning av generaliserad aritmetik baserat på funktionsbegreppet. Vi stöder oss på konkreta uppgifter i beskrivningarna och avslutar med ett exempel på hur en uppgift inom generaliserad aritmetik kan konstrueras. Vår utgångspunkt är Blantons mfl (2015) beskrivning vilken vi använde i vår egen undersökning av svenska kursplaner och läroböcker i matematik för årskurs 1–6 (se ovan).

Generaliserad aritmetik som en av de ”stora idéerna”

Ur Blantons mfl (2015) perspektiv består generaliserad aritmetik huvudsakligen av två delar, 1) *Resonemang kring strukturer hos aritmetiska uttryck*, samt 2) *Generaliseringar av aritmetiska samband*. Vi kommer här att titta närmare på innebörden av dessa två delar.

”Resonemang kring strukturer hos aritmetiska uttryck” handlar om att flytta fokus i algebraundervisningen från enskilda beräkningar till att resonera kring olika egenskaper hos tal och operationer. Istället för att fokusera på till exempel resultatet av enskilda additioner kan elever redan i de tidiga skolåren börja tänka på olika egenskaper hos operationen addition och de hela talen. Exempelvis skulle en uppgift kunna handla om vilka utfall som kan uppkomma när man adderar olika kombinationer av udda och jämna tal. Det strukturella tänkandet innebär då att eleverna uppmuntras att upptäcka mönstret $udda + udda = jämn$, $udda + jämn = udda$ och så vidare.

Enligt Blanton mfl (2015) kan en orsak till elevers svårigheter i algebra vara att man inte tar fasta på strukturella egenskaper hos tal och operationer. Vad som menas med ”strukturella egenskaper” illustrerar Blanton mfl med hjälp av att skilja mellan de två lösningsstrategierna *Beräkning* respektive *Struktur* till uppgiften ”Är utsagan $46+23=47+22$ sann eller falsk?” (se tabell 1). I tabell 1 kan vi utläsa att beräkningsstrategin innebär att vänster- respektive högerled räknas ut var för sig och därefter konstateras att resultaten är lika. Den strukturella strategin innebär istället att man, utan att utföra någon beräkning, resonerar sig fram till rätt svar med hjälp av den aritmetiska strukturen bakom talen och additionerna. Om man i en given addition ökar den ena termen med 1 och minskar den andra termen med 1 kommer svaret fortfarande vara detsamma.

Tabell 1. *Strategierna "Beräkning" respektive "Struktur"*

Är utsagan $46+23=47+22$ sann eller falsk?		
Strategi	Tillvägagångssätt	Motivering
Beräkning	Vänster- resp högerledet räknas ut var för sig	Likheten är sann eftersom $46+23=69$ och $47+22=69$
Struktur	Strukturen i likheten beaktas, inga beräkningar utförs	Likheten är sann eftersom man adderar 1 till 46 och subtraherar 1 från 23

Detta går enkelt att generalisera; vad händer om man istället ökar den ena termen med 2 (3, 4 osv) och minskar den andra termen med 2 (3, 4 osv)? Vi övergår då till den andra delen av Blantons mfl (2015) beskrivning av generaliserad aritmetik, "Generaliseringar av aritmetiska samband" som innefattar grundläggande egenskaper hos tal och operationer. Ett konkret exempel på detta är att man först konstaterar att $2+3=3+2$ gäller genom ett resonemang om att ordningen inte spelar någon roll, ett resonemang om struktur. Därefter kan man generalisera detta till alla tal, vilket såklart är regeln om att addition är kommutativ.

Generaliserad aritmetik utgående från kvasi-variabler

Fujii & Stephens (2001; 2008) perspektiv utgår från att beskriva generaliserad aritmetik som en bro mellan aritmetik och algebra. Deras idé är att introducera algebraiskt tänkande i de tidiga skolåren med hjälp av så kallade kvasi-variabler. Detta innebär att elever resonerar om likheter inom aritmetiken är sanna eller inte med hjälp av att betrakta de ingående talen som utbytbara eller med andra ord som kvasi-variabler. Ett exempel på detta är att låta elever resonera sig fram till om likheten $78-49+49=78$ är sann eller inte.

Tanken med uppgiften är att eleverna ska upptäcka att likheten är sann oavsett vilket tal man drar bort och sedan adderar tillbaka, det vill säga 49 är en kvasi-variabel. Vidare är likheten sann oavsett vilket det första talet är under förutsättning att det är samma tal som dras bort och sedan adderas tillbaka. Därmed kan även talet 78 ses som en kvasi-variabel (Fujii & Stephens, 2001)².

Fujii & Stephens (2008) påpekar att syftet med kvasi-variabler är att eleverna ska introduceras till ett algebraiskt tänkande utan att behöva använda bokstavs-symboler. Med andra ord är syftet inte att eleverna ska komma fram till det generella sambandet $a-b+b=a$ utan istället diskutera om likheten är sann eller falsk genom att fokusera på sambandet mellan de olika talen och operationerna. Om en elev säger "det är sant eftersom man tog bort 49 och sen tog man tillbaka det" menar Fujii & Stephens (2008) att eleven har påvisat ett kvasi-variabelt tänkande. Detta svar kan jämföras med att en elev räknar ut vänster- och högerled var för sig och därefter konstaterar att uttrycket är sant, vilket är ett korrekt sätt att lösa uppgiften men inte ett exempel på ett kvasi-variabelt tänkande.

Notera likheten med det Blanton m fl (2015) benämner ”resonemang kring strukturer hos aritmetiska uttryck till skillnad från att fokusera på enskilda beräkningar” (se ovan). I Blantons m fl (2015) uppgift ”Är utsagan $46 + 23 = 47 + 22$ sann eller falsk?” (se tabell 1 ovan) gjordes en åtskillnad mellan att resonera sig fram till att likheten är sann med hjälp av *strukturen* bakom talen och additionerna respektive att *beräkna* höger- och vänsterled var för sig. Det som Fujii & Stephens (2001; 2008) kallar ett kvasi-variabelt tänkande är således ett exempel på det som Blanton m fl (2015) benämner ”strukturell lösningsstrategi”. Dessutom är kvasi-variabler ett exempel på det som Blanton m fl (2015) kallar ”generaliseringar av aritmetiska samband” då det visar hur man tar sig från aritmetiska uttryck till motsvarande generella algebraiska uttryck $a - b + b = a$, dock utan att introducera variabler. Notera att det är detta som Fujii & Stephens (2008) kallar en bro mellan aritmetik och algebra.

Till skillnad från de två perspektiv som vi hittills har behandlat kommer vi nu att beskriva ett perspektiv på generaliserad aritmetik som inkluderar användandet av bokstavssymboler i de tidigare skolåren.

Generaliserad aritmetik utgående från funktion

I Carrahers m fl (2006) perspektiv på generaliserad aritmetik spelar funktionsbegreppet en central roll. Utgångspunkten är att generaliseringar utgör själva kärnan i den tidiga algebran och att aritmetiska operationer kan ses som funktioner. Vidare betraktas hela den tidiga algebran som en generaliserad aritmetik av tal och kvantiteter, vilket skiljer sig från Blantons m fl (2015) syn på generaliserad aritmetik som en specifik del av den tidiga algebran. Införandet av algebra i de tidigare skolåren ses som en förflyttning från specifika tal till samband mellan tal, där de samband som avses framförallt är funktioner. Ett exempel på hur Carraher m fl (2006) tänker sig detta är att betrakta addition med 3, det vill säga ”+3”, som funktionen $f(x) = x + 3$.

Till skillnad från de ovan beskrivna perspektiven framhålls här att bokstavssymboler kan vara användbara även i de tidigare skolåren. I en klassrumsstudie i årskurs 3 introducerar Carraher m fl (2006) bland annat en ”N-tallinje” där origo kan variera (variable origin N). Positionerna på N-tallinjen, där N motsvarar origo, är från vänster till höger $N - 2$, $N - 1$, N , $N + 1$, $N + 2$ och så vidare. Med hjälp av N-tallinjen påvisades i studien att elever i årskurs 3 kan börja prata om operationer mellan obekanta tal som är skrivna med bokstavssymboler. Exempelvis kunde eleverna behandla uttrycket $N - 4$ som resultatet av förskjutningen ”fyra steg åt vänster från N”, oavsett vilket tal N stod för. I praktiska kontexter kunde uttrycket $N - 4$ uppfattas som ”4 mindre än vad John hade från början”, oavsett hur mycket John hade från början. Detta funktionstänkande kan enligt Carraher m fl (2006) introduceras hos elever redan i 8-årsåldern.

Uppgiftskonstruktion inom generaliserad aritmetik

Som vi sett ovan betonas det inom forskningslitteraturen att elever redan i de tidigare skolåren kan börja resonera kring aritmetiska strukturer och generaliseringar av aritmetiska uttryck. Men det betonas också att elever behöver hjälp med detta (Fujii & Stephens, 2008). Vi kommer därför att presentera en uppgift inom generaliserad aritmetik som är konstruerad så att lösaren leds in mot ett mera algebraiskt tänkande. Detta görs med hjälp av deluppgifterna a–c (se figur 1). Vidare har vi valt ut uppgiften utifrån resultaten av våra egna empiriska studier där denna uppgiftstyp var mycket svagt representerad i svenska läroböcker i matematik. Uppgiften utgår från Mason, Stephens och Watsons (2009) studie där elever i åldersspannet 10–13 år löser problem som involverar algebraiskt tänkande. Vi kommenterar uppgiften inom ramen för de perspektiv på generaliserad aritmetik som vi behandlat ovan.

<p>Fundera över följande likhet: $18 + \square = 20 + \square$</p> <p>Kan du sätta in tal i den blå och röda rutan * så att likheten blir sann? Kan du komma på ytterligare två exempel på tal som gör att likheten blir sann?</p> <p>När du satt in tal i rutorna så att likheten blir sann, vilket samband är det då mellan talen i den blå och röda rutan?</p> <p>Vad blir sambandet mellan talen i den blå och röda rutan om talet 18 byts ut mot 226 och talet 20 byts ut mot 231?</p>
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Figur 1. Uppgift inom området generaliserad aritmetik

* I originalet är den vänstra rutan blå och den högra röd

Syftet med uppgiften är att eleverna ska fokusera på sambandet mellan talen i likheten och inte bara på specifika beräkningar som gör likheten sann. Denna tanke är central i alla tre beskrivningar av generaliserad aritmetik som vi behandlat ovan (Blanton m fl, 2015; Carraher m fl, 2006; Fujii & Stephens, 2001).

I deluppgift a) ska eleverna ge tre exempel på olika par av tal som gör att likheten är sann. I deluppgift b) leds eleverna in mot att börja fundera över sambandet mellan talen i rutorna utifrån exemplen de valt i deluppgift a). I Masons m fl (2009) studie varierade elevernas svar här från ”den blå rutan är större än den röda” till ”talet i den blå rutan är *alltid* 2 mer än talet i den röda rutan”. Det senare illustrerar ett funktionellt tänkande liknande det i Carrahers m fl (2006) studie ovan där eleverna till exempel tolkade uttrycket $N-4$ som ”4 mindre än John hade från början *oavsett* hur många han hade från början”. De kursiverade orden ”alltid” respektive ”oavsett” är av central betydelse eftersom de indikerar att eleverna börjat generalisera. Notera dock att Carrahers m fl (2006) uttryck $N-4$ endast består av en obekant medan denna uppgift innehåller två obekanta

eftersom sambandet är implicit i likheten. En ytterligare skillnad mellan Carrahers mfl (2006) exempel och denna uppgift är att i den senare används inte bokstavssymboler. Slutligen, i deluppgift c) ska eleverna upptäcka att om de givna talen i likheten ändras så påverkas sambandet mellan talen i rutorna. Detta motsvarar ytterligare en nivå av generalisering där de fixa talen betraktas som det Fujii & Stephens (2001; 2008) kallar kvasi-variabler.

Diskussion

Vi har i denna artikel belyst olika sätt att begreppsliggöra generaliserad aritmetik med hjälp av aktuell forskning och uppgifter som illustrerar olika aspekter av generaliserad aritmetik. Således bidrar denna artikel till en ökad förståelse av begreppet både från teoretisk och praktisk synvinkel. Mot bakgrund av våra tidigare studier är exempel på uppgifter i generaliserad aritmetik baserade på forskares definitioner något som vi anser vara värdefullt för utvecklingen av den svenska skolalgebran. Det är inte möjligt att inom ramen av detta konferensbidrag genomföra en uttömmande analys av begreppet. Vi har därför valt att fokusera på de perspektiv som vi bekantat oss med i samband med våra empiriska analyser. Bland annat har vi i vårt empiriska arbete avgränsat oss till forskningslitteratur som inte sträcker sig mer än ca 20 år bakåt i tiden.

När det gäller begreppet generaliserad aritmetik kan vi utifrån denna artikel dra slutsatsen att Blantons mfl (2015) beskrivning i form av *resonemang kring strukturer hos aritmetiska uttryck* (till skillnad från att fokusera på enskilda värden eller resultat av specifika beräkningar) samt *generaliseringar av aritmetiska samband* utgör kärnan i alla tre perspektiv som vi behandlat här. Vidare har vi sett hur kvasi-variabelt tänkande (Fujii & Stephens, 2001; 2008) respektive funktionellt tänkande (Carraher mfl, 2006) är två olika angreppssätt som kan användas som stöd vid konstruktion av uppgifter inom generaliserad aritmetik för de tidiga skolåren. Dock är det viktigt att ta hänsyn till den rådande undervisningskulturen innan man introducerar nya undervisningsmetoder (Hiebert mfl, 2005). Utifrån våra tidigare studier framstår framförallt Carrahers mfl (2006) introduktion av addition med hjälp av funktioner långt ifrån den svenska undervisningskulturen i skolalgebra.

Ett nästa steg i vårt projekt är att djupare undersöka relationen mellan den stora idén generaliserad aritmetik i förhållande till de andra stora idéerna hos Blanton mfl (2015). Vi kommer även att undersöka aktiva lärares syn på generaliserad aritmetik som en del av algebrainläring. Förutom för forskningsfältet är vårt arbete att reda ut och konkretisera stora idéer inom utvecklande av elevers algebraiska tänkande betydelsefullt för författare av kursplaner och läromedel samt för lärarutbildare och lärare på fältet. Slutligen är vårt arbete med att fördjupa förståelsen av generaliserad aritmetik betydelsefullt för hur strukturellt tänkande kan användas i de tidigare skolåren i samband med införandet av programmering i den svenska skolmatematiken.

Acknowledgment

Det forskningsprojekt som denna artikel baseras på är finansierat av Vetenskapsrådet.

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Noter

- 1 TIMSS föregicks av SIMS (The Second International Mathematics Study) från 1980 och FIMS (The First International Mathematics Study) från 1964. Sveriges resultat hamnade klart under genomsnittet i båda dessa test (Murray & Liljefors, 1983).
- 2 Idén med att låta elever resonera om likheter är sanna eller falska kommer ursprungligen från Carpenter & Levis (1999) arbeten inom algebraiskt tänkande.

Large-scale professional development and teacher change

The case of Boost for mathematics

JANNIKA LINDVALL

This study contributes to the ongoing discussion of conceptualizations and measures of teacher professional development (PD). It does so by using data from the TIMSS 2015 teacher questionnaires to study the impact of *Boost for mathematics* (BfM), a large-scale teacher PD program, on Teacher confidence, Teacher collaboration, and Support from school leadership. The results indicate that BfM has had no sustained impact on Teacher confidence and Support from school leadership. A positive impact on Teacher collaboration was found for the 8th grade teachers, but not for the 4th grade teachers. These results are somewhat contradictory to the ones presented in national evaluations of BfM and potential explanations for the differences are discussed.

In educational reforms worldwide, a large amount of resources are invested in teacher professional development (PD), not just at single schools, but also at larger scales, such as districts (e.g. Cobb & Jackson, 2011) or whole countries (Boesen, Helenius & Johansson, 2015). In the light of growing research recommendations regarding critical feature of teacher PD (cf. Desimone, 2009; Sowder, 2007), the employed PD programs are also becoming more and more sophisticated. They are thus moving away from one-day workshops to programs lasting over longer periods of time, involving collaboration among teachers, and learning activities connected to teachers' classroom practices. Still, even programs incorporating the recommendations given in research literature have shown to have difficulties in improving student achievement (cf. Kennedy, 2016).

One example of such a large-scale PD program is the national state-coordinated project *Boost for mathematics* (BfM). BfM includes a year-long teacher PD program in which, according to the Swedish National Agency of Education (Skolverket, 2016), 76% of all mathematics elementary school teachers have participated in somewhere between 2012 and 2016. The program corresponds well with critical features of PD (Lindvall, Helenius & Wiberg, 2018) identified in the research literature (Desimone, 2009). Despite this, studies indicate that it, up to this point, has shown no (Lindvall, Helenius, Eriksson & Ryve,

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2018) or only limited (Lindvall, Helenius & Wiberg, 2018), impacts on student achievement.

The reasons for the discouraging results can be several. It may have methodological reasons, such as that the tests used to measure students' results have not been designed for the particular program (Bryk, Gomez, Grunow & LeMahieu, 2015), or that data have been gathered before changes in teaching have yielded any impact (Kennedy, 2016). These explanations do not, however, seem to hold for BfM. First, by using the Swedish data from TIMSS 2015, the results from Lindvall, Helenius & Wiberg (2018) point to that BfM has had no effect on student achievement. This finding holds for students whose teacher participated in BfM both one and two years ago. Second, BfM focuses on the mathematical content and abilities mentioned in the national syllabus (Skolverket, 2011), and the results from a recent publication (Skolverket, 2017) demonstrates that students performing well on TIMSS 2015 also perform well on measures designed to assess their knowledge of the mathematics in the national syllabus (e.g. grades and national exams).

That a PD program does not demonstrate an impact on student results may also depend on the design and implementation of the program itself. A teacher program aiming at improving student achievement rests on two theories (cf. Desimone, 2009). The first one, the *theory of teacher change*, concerns the program's underlying assumptions about what causes increases in teachers' knowledge and/or practices and how this comes to happen. The second one, the *theory of instruction*, concerns how and what in the changed practices that leads to increases in student achievement. To be able to determine the reasons for PD programs' impacts it is important to adhere to both theories, otherwise it is impossible to know where the model broke down. Was, for example, the PD ineffective in changing teachers' knowledge and practices, or did the changes teachers made fail in improving student achievement? Studies of PD programs that adhere to both theories are, however, few (Desimone, 2009). Considering that previous studies (Lindvall, Helenius & Wiberg, 2018; Lindvall et al., 2018) have emphasized BfM's impact on student achievement, this study focuses on its impact on teachers and factors that can support teacher change. The aim is to contribute to the ongoing discussion (cf. Desimone, 2009) about measures and conceptualizations of teacher PD by investigating BfM's impacts on teachers' reports regarding (1) their confidence in carrying out mathematics instruction, (2) their collaboration with colleagues, and (3) the support they receive from school leadership. Below, I elaborate on the reasons for why these three constructs connected to teacher change were chosen.

Aspects of teacher change and factors that support it

Different models, both linear and cyclic, have been used to describe the relationships between PD programs, teachers, instruction, and student outcomes

(e.g. Clarke & Hollingsworth, Desimone, 2009). Common for all the models, however, are that they recognize teacher change as an important component of PD programs aiming at improving student achievement. It is therefore important, not only to adhere to PD programs' impact on student results, but also to their impact on teachers and factors which may support teacher change. The aspects included in teacher change (e.g. changes in knowledge, beliefs) and factors that may support it (e.g. policy environment, curriculum materials) are several. In this study, I focus on three aspects that are emphasized in the literature.

First of all, the central focus of current PD most closely aligns teacher change with growth and a process that involves learning (Clarke & Hollingsworth, 2002; Sowder, 2007). What is to be learned, on the other hand, is more diffuse and it is argued that teacher learning may be the most difficult aspect to measure in PD initiatives (Desimone, 2009). In recent years, *Teacher confidence* has gotten more and more attention as an important part of teacher learning in and from PD. For example, studies suggest that teachers' feelings of self-efficacy greatly influence their implementation of new practices (Gabriele & Joram, 2007; Guskey, 1988) and have a positive impact on student results (Goddard, Hoy & Hoy, 2000).

A second aspect of teacher change is *Teacher collaboration*. Several scholars (Borko, 2004; Sowder, 2007) argue that an important goal for teacher PD should be to develop a shared professional culture and a common vision of teaching, which in turn can help sustain changes in practice over time. Moreover, teacher collaboration is not only seen as an aspect of teacher change, but also as a factor that support it. The collective participation of teachers may function as a tool that supports increased teacher change, and interaction among colleagues is claimed to be a powerful form of teacher learning (Borko, 2004; Desimone, 2009).

Thirdly, an additional factor that may contribute to teacher change is *Support from school leadership*, such as organizers and principals (Bryk et al., 2015). For example, the study by Youngs and King (2002) suggests that principals play a central role in sustaining high-quality PD by establishing trust and creating structures that promote teacher learning.

The Boost for mathematics

BfM's overarching aim is to improve student achievement in mathematics by strengthening the mathematics teaching (Skolverket, 2016). In order to reach this aim, the goals are to develop the mathematics teaching culture and the PD culture at the schools. Österholm et al. (2016) point to that the two cultures are strongly dependent on each other and that they have three common central points. These point may all be compared to the previously described aspects of teacher change and the factors that support it. The first point, teachers'

pedagogical content knowledge, stresses that teachers' development and use of this knowledge should be based on research and proven experience. Although, at a first glance, this point may be primarily associated with change in teacher knowledge, it can also be connected to change in Teacher confidence. Previous studies (Gabriele & Joram, 2007; Guskey, 1988) have, for example, shown that teachers' feelings of self-efficacy greatly influence their use of new practices. The second point concerns the instructional design and stresses that the planning and reflection on teaching is a central part of teachers' collaborative learning. It can, hence, be connected to the previous description of Teacher collaboration. The third and final point, cooperation between organizers, principals and teachers regarding instruction and PD, is closely connected to the previous arguments regarding the importance of Support from school leadership. In other words, the aspects of teacher change and the factors that may support it that are stressed in this study are not only emphasized in the PD literature, but are also closely connected to BfM's goals.

Evaluations and studies of Boost for mathematics

Both national evaluations (Ramböll, 2016; Österholm et al., 2016) and research studies (e.g. Boesen et al., 2015; Lindvall et al., 2018; van Steenbrugge, Larsson, Insulander & Ryve, 2017) of BfM have been conducted and focused on different aspects of BfM's impact on teachers, instruction and students. Concerning teacher change, results from the national evaluations (Ramböll 2016; Österholm et al., 2016) point to that teachers, after their participation in BfM, reflect more on the instructional practices emphasized in the program. Impacts on teacher collaboration do, however, seem to have been less positive. Based on the results, the evaluators give a common recommendation for future PD initiatives: to develop them at a more comprehensive school level and support the collaboration between teachers and principals to ensure sustainability of the positive impacts.

Regarding changes in instruction and student achievement the results between the different evaluations and studies are, however, sometimes contradictory. For example, by drawing on data from lesson observations, Österholm et al. (2016) conclude that BfM has affected teachers' instructional practices and that they teach more in line with the didactical perspectives emphasized in the program. A study using data from TIMSS 2015 (Lindvall et al., 2018), on the other hand, indicates that BfM has had no lasting impact on teachers' classroom activities. Moreover, the results presented in Lindvall et al. (2018) suggest that BfM does not predict student achievement on the TIMSS 2015 test. At the same time, using a test emphasizing students' number sense (McIntosh, 2008), another study (Lindvall, Helenius & Wiberg, 2018) suggests that BfM has affected student results in some grades.

In the light of the contradictory results, additional studies are needed to clarify BfM's impact and factors that may support it. In this paper, focus is put on BfM's impact on two aspects of teacher change (Teacher confidence and Teacher collaboration), and two factors that may support this change (Teacher collaboration and Support from school leadership). Although some of these aspects and factors have been touched upon in the national evaluations (Ramböll, 2016; Österholm et al., 2016), multiple research methods are beneficial when studying the complex processes of school reforms (Borko, 2004; Desimone, 2009).

By using the Swedish data from the TIMSS 2015 teacher questionnaires, this study contributes with important complementary data to the final national evaluations of BfM (Ramböll 2016; Österholm et al., 2016). First, as opposed to the questionnaires used in the national evaluations, the TIMSS questionnaires do not state that their purpose are to evaluate BfM. As teachers tend to over-report their implementation of practices promoted in PD (e.g. Ross, McDougall, Hogaboam-Gray & LeSage, 2003), studies using measurements which do not state an evaluation of the PD as its main purpose are also required. Second, the TIMSS questionnaires were not specifically designed to measure the impacts of BfM. Though instruments designed to evaluate particular programs have advantages (cf. Bryk et al., 2015), they may also miss hidden outcomes of interventions (Guskey, 2002a). Third, the results from Lindvall, Helenius & Wiberg (2018) suggest that BfM has had different impacts on student achievement for different grade levels. However, variations in results for teachers working in different grades of elementary school are not addressed in the final national evaluations of BfM.

Method

The TIMSS 2015 4th and 8th grade mathematics teacher questionnaires ask teachers about a range of topics. In this paper, focus is put on items related to aspects of teacher change and factors that could support it. Indexes consisting of multiple items from the questionnaires (see Appendix A), were created to measure three constructs related to Teacher confidence, Teacher collaboration, and Support from school leadership. The first index included nine items (Cronbach's alpha 4th grade = .89, and 8th grade = .89) in which teachers, on a four point Likert scale, were to characterize their confidence in doing certain things in relation to their mathematics teaching. The second index included seven items (Cronbach's alpha 4th grade = .87, and 8th grade = .86) in which teachers, on a four point Likert scale, were to assess how often they collaborate and work together with other teachers. The third index included three items (Cronbach's alpha 4th grade = .93, and 8th grade = .87) in which teachers, on a five point Likert scale, were to assess their collaboration with and support from the school leadership.

To be able to determine BfM's impact on the three constructs, two groups of teachers, participants of BfM and non-participants, were created for the 4th and the 8th grade sample. Teachers who's answers regarding participation in BfM were missing ($N_{4^{\text{th}} \text{ grade}} = 13$, $N_{8^{\text{th}} \text{ grade}} = 9$), and teachers who participated in the program during the spring term 2015 ($N_{4^{\text{th}} \text{ grade}} = 82$, $N_{8^{\text{th}} \text{ grade}} = 74$) were excluded from the sample. The reasons for this were twofold. First, these teachers had only gone through about half of the program which may had influence the results. Second, as BfM demands that teachers meet weekly to plan instruction, teachers' current participation in the program would probably have affected their answers regarding Teacher collaboration, and the aim of this study was to investigate the influence of BfM on the continuing PD culture, i.e. the culture that remained when the PD program had ended. The final sample consisted of a group of participants ($N_{4^{\text{th}} \text{ gr.}} = 43$, $N_{8^{\text{th}} \text{ gr.}} = 46$) and non-participants ($N_{4^{\text{th}} \text{ gr.}} = 96$, $N_{8^{\text{th}} \text{ gr.}} = 77$). For both the 4th and 8th grade sample, the groups were comparable in respect to age (median in the span 40–49 years for all groups), and years of experiences (4th gr. $M = 13.8$, $SD = 8.2$ vs. $M = 14.7$ $SD = 13.0$; 8th gr. $M = 14.8$, $SD = 8.4$ vs. $M = 13.4$ $SD = 9.4$).

Independent sample *t*-tests for the 4th respectively 8th grade sample were employed to assess if there were any statistical significant differences between participants and non-participants in respect to the three constructs. One could, however, argue that the Likert scales should be seen as ordinal instead of continuous and that the non-parametric alternative (Mann-Whitney U test) should therefore has been employed instead. However, De Winter and Dodou (2010) have shown that, for five-point Likert items, the *t*-test and the Mann-Whitney U test have similar power with the exception of heavily peaked or skewed distributions. The Shapiro Wilk test and visual inspections of the data sets in this study showed that most sets were normally distributed, with the exception of the 4th grade non-participants in relation to Teacher confidence. Hence, a Mann Whitney U test was also run for this data set, but it showed the same results as the *t*-test.

Finally, to assess the strength of the differences between participants and non-participants, the effect sizes were calculated in terms of Cohen's *d*. For interpreting the results, the guidelines proposed by (Cohen, 1988) were used with .2 representing small, .5 representing moderate and .8 representing large effect.

Results

The analysis of the data indicates that BfM has had a sustained impact on only one of the three constructs. The results are summarized in table 1.

For both the 4th and the 8th grade sample, no significant differences between participants and non-participants regarding Teacher confidence and Support from school leadership were found. In other words, the results indicate that

Table 1. Teachers' reports regarding Teacher confidence, Teacher collaboration and Support from school leadership

Construct	Participants			Non-participants		<i>t</i>	<i>p</i>
	Grade	M	SD	M	SD		
Teacher confidence	4 th	2.89	.58	3.00	.59	- .88	.38
	8 th	2.93	.48	2.95	.51	-.23	.82
Teacher collaboration	4 th	2.28	.56	2.50	.64	-1.95	.05
	8 th	2.49	.60	2.26	.58	2.09*	.04
Support from school leadership	4 th	2.98	.66	3.21	.76	-1.69	.09
	8 th	3.13	.88	3.05	.89	.47	.64

Note. M = Mean, SD = Standard Deviation, * difference between participants and non-participants is significant at the .05 level.

BfM has not had any sustained impact on teachers' confidence in teaching mathematics to students and their experiences of school leadership support.

Regarding teachers' reported collaboration with other teachers, a significant difference between participants and non-participants was, however, detected for the 8th grade teachers. An examination of the descriptive statistics shows a more positive value for participants in comparison to non-participants. These results suggest that BfM has had a positive and sustained impact on Teacher collaboration for the 8th grade teachers, though the effect size ($d = .39$) is considered to be small. For the 4th grade teachers, no significant difference between participants and non-participants was established, though the numbers indicate that it was close with a p-value just above .05 ($p = .054$) and a small effect size ($d = -.36$). An examination of the descriptive statistics points to that non-participants in the 4th grade reported on more collaboration with colleagues in comparison to teachers who participated in BfM one or two years ago.

Discussion

The results from this study raise several questions. First they are, to some extent, contradictory to the national evaluations (Ramböll, 2016; Österholm et al., 2016). Second, the contrasting results for the 4th and 8th grade teachers concerning Teacher collaboration are interesting given that a previous study (Lindvall, Helenius & Wiberg, 2018) have shown differences in BfM's impact on student achievement for the lower and upper grades of elementary school. The two issues are discussed below.

To begin with, the results presented in this paper indicate that BfM has had no impact on Teacher confidence. The first national evaluation (Ramböll, 2016), on the other hand, concludes that BfM has impacted teachers'

confidence in conducting mathematics instruction. This conclusion is, however, based on questionnaire items where teachers were asked to indicate to what extent BfM has increased their knowledge and skills, and increased knowledge and skills do not necessarily contribute to increased confidence. The conclusion drawn in Ramböll (2016) about teachers' confidence may therefore be built on insufficient grounds.

The results from the second national evaluation (Österholm et al., 2016), on the other hand, suggest that teachers, after their participation in BfM, reflect more on their mathematics teaching. The results from this study cannot be used to comment on BfM's impact on teacher reflection and it is therefore possible that BfM has affected teachers' reflection about their mathematics instruction, but that this reflection has not, up to this point, resulted in increased teacher confidence in carrying out mathematics instruction. Considering that teachers' sense of self-efficacy greatly influence their implementation of new practices (Gabriele & Joram, 2007; Guskey, 1988), this is, however, an important result to take into account in trying to explain other outcomes related to BfM's theory of teacher change, for example results which indicate that BfM have had no impact on teachers' and students' reported instructional practices (Lindvall et al., 2018). Are, for example, the seemingly differing results depending on methodological issues such as the time factor (e.g. teacher reflection has not yet had an impact on teacher confidence), the use of teacher interviews versus questionnaires, or that, when asked explicitly about a PD programs' impact, teachers tend to over-report on items related to practices promoted in the program (e.g. Ross et al., 2003)? Or are they depending on the design of the PD itself (i.e. the PD has not affected teacher confidence in carrying out mathematics instruction)?

Also in relation to change in Teacher collaboration, the results presented in this paper are contradictory to the conclusions drawn in the national evaluations. Both Österholm et al. (2016) and Ramböll (2016) state that BfM has had no, or very limited, impact on the collaboration between teachers, and between teachers and school leadership. This paper, on the other hand, point to that while BfM has had no impact on Support from school leadership, it has affected Teacher collaboration, especially in grade 8. An explanation for the contradicting results may be that, in this study, the analysis of data from the 4th respectively 8th grade teachers were done separately and point at opposite directions. If only a single composite analysis had been conducted, these results would not have been visible. Even though the descriptive statistics presented in one of the intermediate evaluations (Ramböll, 2014) point to differences between teachers in grades 4–6 and 7–9, none of the final national evaluations (Ramböll 2016; Österholm et al., 2016) have attended to such differences and may thus explain their conclusions. In line with Bryk et al. (2015), the results from this study thereby point to the importance of attending to variations in the sample when studying impacts of PD.

The variations found between the 4th and 8th grade teachers are, in point of fact, what makes the results from this study especially interesting. These results indicate that BfM has had a positive impact on Teacher collaboration for the 8th grade teachers, but not for the 4th grade teachers. Considering that a previous study (Lindvall, Helenius & Wiberg, 2018) indicates that BfM has had no impact on student achievement for the lower grades in elementary school, and a positive impact for the upper grades, these results are noteworthy. Possible explanations for the differences found between the two groups of teachers may be several. One could be related to that 8th grade teachers in Sweden are usually subject specialists. They may therefore not be as occupied by other subjects and initiatives in comparison to 4th grade teachers, who are usually class teachers. An imminent risk may therefore be that when the 4th grade teachers have participated in BfM, PD in other subjects have been put aside. However, once the teachers are done with BfM, mathematics might have been given lower priority in favor of other subjects. This explanation is further supported by the descriptive statistics in Ramböll (2014), which suggest that teachers in grades 4–6 experienced less support (e.g. disposal of time) in going through with the PD program as compared to the teachers in grades 7–9. Further studies are needed, however, in order to clearly establish the explanations for the differences found.

In summary, the results from this study indicate that BfM has had no impact on Teacher confidence and Support from school leadership, a positive impact on Teacher collaboration for the 8th grade teachers, and no impact on Teacher collaboration for the 4th grade teachers. These results are somewhat contradictory to the national evaluations (Ramböll 2016; Österholm et al., 2016) and may depend on methodological issues, such as methods for collecting data, or items in the teacher questionnaires. A probable factor in relation to Teacher collaboration is that this study, as opposed to the national evaluations, attended to variations in the sample by conducting separate analysis for the 4th and 8th grade teachers. Without such separate analyses, the differences found in Teacher collaboration would not have been visible. Future studies of PD that adhere to variations between teachers and contexts are therefore needed in order to bring further light on questions not only related to “what works”, but also for who, when and how.

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Appendix A

Items in the TIMSS 2015 teacher questionnaire connected to Teacher confidence, Teacher collaboration and Support from school leadership.

Construct	Items	Scale
Teacher confidence	In teaching mathematics to this class, how would you characterize your confidence in doing the following?	1. Very high
	a) Inspiring students to learn mathematics	2. High
	b) Showing students a variety of problem solving strategies	3. Medium
	c) Providing challenging tasks for the highest achieving students	4. Low
	d) Adapting my teaching to engage students' interest	
	e) Helping students appreciate the value of learning mathematics	
	f) Assessing student comprehension of mathematics	
	g) Improving the understanding of struggling students	
	h) Making mathematics relevant to students	
	i) Developing students' higher-order thinking skills	
Teacher collaboration	How often do you have the following types of interactions with other teachers?	1. Very often
	a) Discuss how to teach a particular topic	2. Often
	b) Collaborate in planning and preparing instructional materials	3. Sometimes
	c) Share what I have learned about my teaching experiences	4. Never or almost never
	d) Visit another classroom to learn more about teaching	
	e) Work together to try out new ideas	
	f) Work as a group on implementing the curriculum	
g) Work with teachers from other grades to ensure continuity in learning		
School leadership support	How would you characterize each of the following within your school?	1. Very high
	a) Collaboration between school leadership and teachers to plan instruction	2. High
	b) Amount of instructional support provided to teachers by school leadership	3. Medium
	c) School leadership's support for teachers' professional development	4. Low
		5. Very low

The role of language representation for triggering students' schemes

LINDA MARIE AHL AND OLA HELENIOUS

Schemes were Piaget's most important concept. Through work of Vergnaud, schemes were connected to representations and theoretical models from Piaget were connected to principal insights from Vygotsky. We suggest that the scheme concept can be elaborated further by detailing the relationship between schemes and semiotics. We consider a case of an adult student's work on a situation involving average speed. Linguistic representations in the problem formulation triggers two separate schemes for the student, one associated to the speed concept and one to the arithmetic average. By identifying exemplary phenomena in the presented case, we show how previous theory connecting schemes and representations can be extended to allow alternative explanations for a well-known class of students' errors.

Scheme may be the most important concept in both Kant's theory of knowledge and in Piaget's genetic epistemology. Yet, schemes are sparsely described in their respective writings. In Kant's extensive production, it covers only 11 pages and Piaget does not even describe his scheme theory formally (Heidegger, 1997, von Glasersfeld, 1995). For both Kant and Piaget, the purpose of the scheme concept is to account for the relationship between individuals' mental activity and empirical objects in the world. Before Kant, the empiricist tradition of Locke and Hume saw knowledge as originating in impressions from our senses while the rationalist tradition of Descartes and Leibniz saw knowledge as originating in the inner mental activity of reason. The scheme is the instrumental construct in Kant's theory that combines these two traditions. When Piaget, inspired by Kant, built his own scheme theory, it was rather to counter the, at the time, dominant behaviorist tradition. For Piaget, mental or physical action was not just a result of experiential stimuli. Mental constructs shape our interpretation of the world and shape our actions on the world. At the same time mental constructs are shaped by the experiences and actions in a dynamic equilibrium process. The scheme is the construct that Piaget gives the role as a mediator in this process. The principal characteristics of schemes are that they must be rigid enough to organize behavior associated to classes of known situations, flexible enough to allow association to new situations where the behavior is still viable, and adaptable enough for allowing re-organization

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when new situations so require. This is what Piaget describes by the concepts accommodation and assimilation.

In mathematics education Piaget's theories have had the largest influence through constructivism as an epistemological perspective, as for example formulated by von Glasersfeld. Analyses, principles, theories and debates have built on the constructivist grounding principle, i.e. knowledge can not be transmitted in a direct way to students, instead students construct knowledge themselves, based on their encountered experiences. Von Glasersfeld (1995) himself relates very strongly to scheme theory, but interestingly, in much of the mathematics education research that builds on constructivism, schemes are invisible. Schemes are also absent from much of the socio-constructivist tradition that, inspired by Vygotsky's focus on language and social settings as the core of learning and knowing, develops theories aiming to bridge the divide between cognitive- and sociocultural theories (Cobb, 1994, Mason, 2007). Such theories are epistemologically interesting and important (Noorloos, Taylor, Bakker & Derry, 2017) perhaps in particular since they offer interesting ways of connecting the classroom culture with individuals' conceptualization of mathematical matters (e.g. Yackel & Cobb, 1996).

In the present theoretical article, however, these epistemological and philosophical perspectives are backgrounded. Instead, by using work from in particular Vergnaud, we use an exemplary case to show how an increased attention to how language representations relate to schemes provides additional explanatory power for interpreting students' actions. We conclude the paper by arguing for an extension of Vergnaud's scheme theory, and thereby further connect core concepts from Piagetian and Vygotskian thinking.

Schemes and language

Mathematics itself is organized in concepts, which makes it appealing to think about students' knowledge in terms of their conceptual understanding. However, the concept "understanding" is both very difficult to use analytically for researchers and the mind does not seem to be organized by understanding, neither do behaviors seem dependent on understanding. Schemes on the other hand provide a theoretical connection between classes of situations in which we expect the student to act, and the ensuing actions. If you accept that schemes decide behavior, you can create hypotheses about students' actual schemes, by analyzing their observable behaviors. This is the reason why schemes are both analytically and didactically more useful than for example the idea of conceptual understanding.

Humans' intuitive knowledge of space, quantities and order relations is the foundations of mathematics but every analysis of such an intuitive framework must be performed in mathematical terms since, "there is no way to reduce mathematical knowledge to any other conceptual framework" (Vergnaud, 1998, p. 167). Vergnaud emphasizes two psychological constructs to mimic the roles

that concepts and theorems have in mathematical theorization: *concepts-in-action* and *theorems-in-action*. These notations are important for the definition of a scheme. A scheme is the invariant organization of behavior for a certain class of situations. For such a class of situations, certain propositions are (explicitly or implicitly) held to be true. Those are termed theorems-in action. Similarly, some objects, predicatives or categories are held to be relevant in a class of situations and these are the concepts-in-action (Vergnaud, 1998). This part of the theorization largely builds on Piagetian ideas. But Piaget’s scheme theory lacks too many central components to be applicable to educational research, perhaps since he was never interested in didactical matters (Vergnaud, 1996). In particular, Piaget did not pay enough attention to the importance of language. Vygotsky, on the other hand, did. Of all Vygotsky’s work, the systematic treatment of how the use of signs, language and other artifacts organizes the mind is his most seminal contribution to our understanding of knowing and learning. Vergnaud combined Vygotskian insights with semiotics and developed a theory for the role of representation in scheme theory.

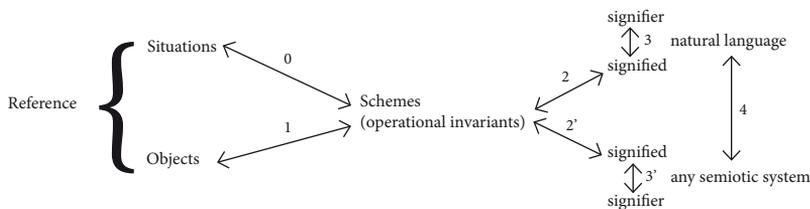


Figure 1. Vergnaud’s development of the classical semiotic triangle. Adapted from Vergnaud (1998)

In figure 1, arrow 0 symbolizes the relationship between situations and schemes, while arrow 1 show the main function of operational invariants, namely to identify objects, their properties, relationships and transformations. Arrows 3, 3’ denotes the relationship between signifiers (words and other signs) and the signified, the operational invariants, as indicated by 2 and 2’. Arrow 4 denotes the relationship between language systems. It should be noted that in general there is no one to one correspondence between signifier and signified or between natural language and a semiotic system, because the same word or sign can signify different objects and ideas, and the same idea can be signified by different words or signs. We refer to Vergnaud (1998) for further elaboration.

In this paper we will develop how Vergnaud’s theory, structuring knowledge within individuals (cf. figure 1), can be complemented with Vygotskian theory of language and other symbol systems, structuring interpersonal knowledge, in order to obtain a didactically useful extended theory (symbolized in figure 3) where semiotics and language representations have a more decisive role than in Vergnaud’s theory.

Schemes in mathematics education research

Schemes have played an important role in educational research for its explanatory potential of students' behaviour. There are many examples of studies that use the with a Piagetian interpretation of the concept, without reference to Piaget, like the great number of studies on *schema-based instruction* (e.g. Fuchs et al., 2004; Jitendra et al., 2009). Among research that closely follow Piaget's track, some use scheme theory to try to classify development, as exemplified by Noeling (1980). Others use schemes as a language for describing individual cases. Lo and Watanabe (1997), for example use the scheme concept to frame a fifth grader's process of schematizing his informal strategies for proportional reasoning over a period of six months. With reference to von Glasersfeld's (1989) they defined schemes as consisting of three parts: "(a) the child's recognition of an experiential situation as one that has been experienced before; (b) the specific activity the child has come to associate with the situation; and (c) the result that the child has come to expect of the activity in a given situation" (p. 219).

Vergnaud's variant of scheme theory has also been used by others. One prominent example is Schliemann and Nunes' (1980) investigation of fishermen with low schooling that apply schemes for mental calculations in their everyday practice. Schliemann and Nunes, conclude that applicable schemes of proportionality can develop without formal schooling, on the basis of everyday experience.

Highly relevant for the case to be presented to support this article's theoretical construction is the work by Thompson (1994), where he uses scheme theory to examine a student's development of the concept of speed and their relationship to concepts of rate. We will return to this paper in our analysis and discussion.

It is however through the work of Vergnaud that scheme theory has been most clearly elaborated. In particular, and in addition to complementing schemes with semiotic perspectives, Vergnaud stresses the role of situations. For didactical purposes it is through creation of different situations that the teacher can plan for expansion and development of students schemes. We will illustrate this by a student solution to a problem involving the concept of average speed.

A situation about average speed

Problems where students have to consider two different speeds during the same journey appear in many settings and are classical in school mathematics. The problem below is adapted from a screening test by Niss and Jankvist (2013).

Steep hill: There is a path up a quite steep hill in Athens. Rickard, who is in good shape, is going up the hill in an average speed of 3 km per hour. He goes down in double speed. What is Richard's average speed for the whole walk?

Rate concepts such as speed, density and unit prices are widely applied in school mathematics and everyday life. In contrast to ratios dealing with the relations between dimensionless quantities, such as the ratio between base and height in a triangle, the interpretation of the relations between quantities with dimensions is more challenging (Thompson, 1994). Central to the problem above is the concept of speed. Let's ponder the speed concept for a while in the simplest situations where we only deal with constant speeds. Then, it is enough to divide a distance by time, to obtain the average speed of the journey, since speed is proportional to the distance, when time is kept constant. Distance is a bi-linear function of time and speed, $d(v, t) = vt$, where the distance is proportional to both time and speed, why in situations where either time or speed is kept constant, students get away with relying on theorems-in-action involving linearity. However, speed is inversely proportional to time when the distance is kept constant. Therefore, when the situation involves the distance being constant, a theorem in action based on linearity does no longer apply. The situation above involves two average speeds, over the same distance. Therefore, simplistic manipulation of the $d = vt$ formula, fails as a scheme for calculating the answer. Another hurdle is that no distance is specified. Thus, the problem includes an element of modeling. Those who have an appropriate scheme for the situation can make an easy mathematization and assume that the hill is 3 km. The journey will then take one hour up and one half-hour down. Then Rickard has walked 6 km in 1.5 hours, so by elementary calculation, his average speed is $6 \text{ km} / 1.5 \text{ h} = 4 \text{ km/h}$.

Schemes triggered by the representation of the problem

The following excerpt illustrates the student Emilé's solution to the average speed problem, Steep hill, collected during a problem solving session in the Swedish prison education program.

As seen in the upper central part of the solution (figure 2), Emilé provides the answer 4.5 obtained by adding 3 and 6 and dividing by 2, that is by averaging the two average speeds (we do not deal with the handling of units here), something which is very common (Ahl, submitted). When asked about his solution afterwards, Emilé says:

Emilé: I added 3 km with 6 km and divided by 2.

Linda: Why?

Emilé: Well, the average.

We will come back to the rest of the solution after discussing the analysis by Thompson (1994), in his well-cited book chapter. Thompson carries out a long teaching experiment with the fifth grader JJ, and analyses her responses and

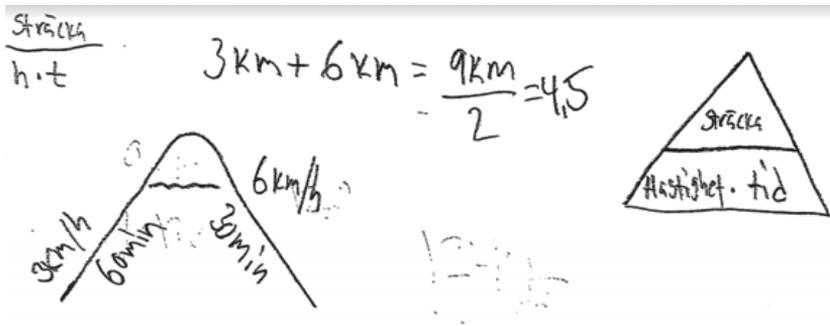


Figure 2. *Emilé's solution.*

Note. Sträcka means distance, hastighet means speed, tid means time

progress by means of scheme theory. In a similar manner to Emilé, JJ also responds to the same type of average speed problem by calculating the arithmetic average. Later in the teaching experiment, JJ learns to handle the situation appropriately. Thompson's explanation of her behavior is conceptual, and to some extent developmental, coinciding with observations by Piaget (1977). Intuitions of speed precede those of time. It is only through conceiving time as a unit and then later speed as a ratio and, later still, a rate that make it possible for JJ to develop a scheme that lets her handle the problem above. Hence, Thompson explains JJ's error by seeing it as a consequence of her undeveloped speed scheme.

Let us now return to Emilé's notes. Emilé uses several concepts-in-action related to speed. The triangle top right is a well-known schematic representation of a theorem-in-action involving the concepts of speed, time and distance and their arithmetic relationship. A different representation of the same theorem in action can be found in the upper left corner. The drawing of the top of the hill, bottom left, shows that the student also makes an appropriate real-world interpretation of the situation described in the problem. Emilé also makes an appropriate mathematization of the situation by assuming the distance to be 3 km, which leads to the uphill walk taking 60 minutes and the faster downhill walk taking 30 minutes. Emilé is hence very close to a solution, which would involve dividing the sum of the up- and downhill distances with the total time. It is known from Emilé's previous work that he knows how to solve the standard situation, where distance and time are given and speed is asked for. Here however, this theorem in action is not invoked. The added complexity of first adding the distances to get $3 + 3 = 6$ km and then dividing by the time $1 + 0.5$ hours = 1.5 hours might be outside of what Emilé associated with this scheme-situation pair. Instead, the student invoked the arithmetic mean value, and then swiftly calculated the mean.

The explanation Thompson provides for JJ's actions does not hold here. Being a grown up, Emilé has certainly conceptualized time as a unit and in his

notes, he illustrates a rather well developed scheme concerning speed, including a concept in action of speed as a ratio. Before providing an alternative explanation, we will return to Vergnaud’s theory that indicates how schemes relate to linguistic representations. In terms of Vergnaud’s schematic image presented in figure 1 above, Thompson’s analysis concerns the situation-scheme-object triangle. His assertion amounts to saying that the student has not objectified speed in an appropriate manner. We claim that in particular for students with more school experience, but perhaps also for the fifth grade student JJ in Thompson’s study, the semiotic representation of the problem also has an important influence. In Vergnaud’s schematic representation (figure 1), the semiotic perspectives, whether natural language or other semiotic systems are connected to the schemes. It is operational invariants residing in situation-scheme pairs that are signified by symbols and then projected back into the situations as objects. However, a fundamental aspect of scheme theory is the interpretation of situations. Schemes are invoked when situations associated to the schemes are encountered. Since a mathematical problem is presented by semiotic systems, the wordings, terms and other symbolic expressions used may themselves trigger different interpretations of the situations. This is particularly apparent in the case of average speed where one single expression (in the Swedish language even one single word ”medelhastighet”) signifies both the idea of average and the idea of speed. Emilé’s solution shows how he invokes a scheme related to speed first, but then moves to using a scheme related to arithmetic average. It will in fact be typical that situations are already symbol-laden when encountered. Some symbols might be unknown to the student and some may be known and associated to particular signifiers, and particular schemes, and hence be instrumental for how situations are interpreted and acted on. We therefore propose that Vergnaud’s schematic description of how scheme theory is related to semiotics should be complemented according to figure 3.

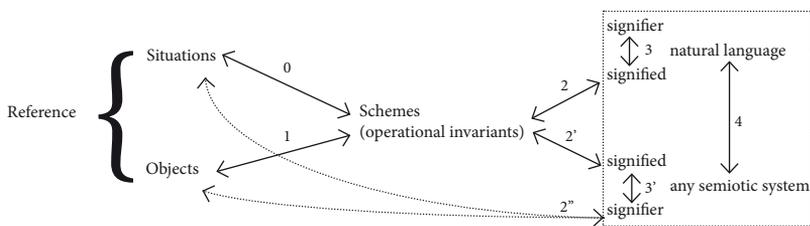


Figure 3. *Extended relation of scheme theory and semiotics*

As seen, we add paths between the semiotic block and the reference side, consisting of situations and objects. The motivation is the type of phenomena seen in Emilé’s case, but also general sociocultural theory that gives language a stronger role for forming thoughts. In Vergnaud’s interpretation, the signified

part of the semiotic systems relates to mental constructs. This is the only way they can signify something. But still, symbols (and here we include written or spoken words) are distinctively present in situations too. They are put there by someone with some intent, but left there for others to interpret. This represents that while semiotic systems are products of minds in society or history, the signifier part of the systems are made available outside of those minds. Arrow 2'' denotes, that the imprints of semiotic systems can be found in situations, in the form of words and other symbols used to describe the problem. Once there they are potential carriers of meaning for someone encountering the situation, like Emilé. The symbols trigger some scheme (arrow 0), which acts as a signifier for the signified concepts (arrows 2 and 2'), which are mental constructs associated to an individual's scheme. Our case with Emilé's solution is illustrational because of how the same word triggers two schemes. Depending on which scheme that is invoked, the same word (in Swedish) triggers different *concepts-in-action* and *theorems-in-action* that also produce different answers.

We can compare this with JJ's statements, commented in two footnotes in Thompson (1994). Footnote 11 reveals that JJ already knew about the arithmetic average after having been taught it earlier by her teacher Mrs. T, causing great confusion, which Thompson tried to remedy by telling JJ that this interpretation of average should from now on be termed *Mrs. T's average*. But later, as revealed in footnote 12, JJ's sister had again told her that averages were dealt with by adding up and dividing. "It took two full sessions to deal with this confusion; those sessions could be the subject of another paper" (Thompson 1994, p. 41). Thompson deals with this as a "confusion" more or less unrelated to what JJ has to do to develop her average speed scheme. But by our extension of scheme theory it can be described in detail. We propose that the phenomenon is the one encountered in the description of Emilé's actions, and the mechanism involved is very general. Both JJ and Emilé have strong schemes related to arithmetic averages. The arithmetic average scheme is very strong since it is effective and produces an answer. Effective schemes are dominant because efficacy is one of the drivers of schemes. The signifier of the signified arithmetic average scheme is the term *average*, and when this term is encountered in a new situation; it is interpreted semiotically (arrow 2'') and triggers the arithmetic average scheme (arrow 2). As observed in both the JJ and the Emilé case, this happens even when another scheme, a speed scheme, has already been invoked.

Discussion

We have extended Vergnaud's theoretical framing of schemes with an added attention to the relationship between language representations of situations. Mathematics education research has for many years been hugely influenced by socio-constructivist perspectives, joining ideas from both Piaget and Vygotsky.

But most of this has built on Piaget's epistemological contributions, ignoring the scheme concept. It is, however, the scheme concept from Piaget's theory that has the potential to account for individuals' mathematical conceptual development. Vergnaud has developed the scheme concept by acknowledging the importance of language and representations. This can be understood as a development in sociocultural direction and Vergnaud clearly acknowledges that the importance of language and symbol systems are best described in Vygotsky's contributions. We argue that semiotics has a more decisive role for children's mathematical actions than suggested by previous scheme based theories. If we want to understand students' formation and use of concepts, there is a need to consider the relationship between language representation, situations and schemes.

We displayed a situation where one composite word average speed triggered two different schemes. "The scheme always applies to a type of situation in which the subject can identify a possible target for his/her activity, and sometimes intermediary sub-targets too" (Vergnaud, 1996, p. 189). The possible target in the displayed situation above appears to be either "average" or "speed". They are both examples of operational invariants, concepts-in-action, that can be efficient for dealing with the problem at hand. But they are not compatible. The average speed is not the average of the speeds.

The special case illustrated made the importance of words visible. But it is reasonable to assume that words regularly have this effect on how scheme-situation pairs are chosen. Therefore we suggest that Vergnaud's schematic illustration of the relationship between schemes, situations and representations should be modified to also include a distinct connection between the semiotic representations and the situation. Once schemes are connected to symbols and language, the scheme itself can be said to contain such representations. Therefore, even though the situations themselves are not symbolic, natural language representations in the description of the situation in the problem will influence what schemes that might get activated. This development of Vergnaud's theory of schemes and representations corresponds to further incorporation of Vygotskian theories.

With this model (figure 3), it can be explained how the same word can trigger different schemes. But classic scheme theory also gives a hint on why both Emilé and JJ do not readily abandon their average scheme. After all, Emilé had all the needed concepts and theorems-in-action ready for a proper handling of the speed concept but still abandoned the relation between distance, speed and time and invoked the arithmetic average scheme. One wonders why? We suggest that although Emilé's speed scheme is effective in simple situations, involving one distance and one time, he does not have the competency to handle a situation involving two different average speeds on the same journey. His speed scheme is efficient also for such situations in the sense that it allows him to produce relevant and correct reasoning. But it is not effective in the sense

that it readily allows him to produce a final answer. His scheme of arithmetic average is on the other hand very effective. It is simple and produces an answer that is reasonable. However, in Emilé's solution we can see that he is close to building a conceptual understanding of average speed, since he elaborates on all the necessary facts for making an efficient reasoning on the situation. We argue that confronting Emilé with situations where the arithmetic mean value scheme makes no sense, by for example producing an obviously unrealistic answer, may make him de-associate his arithmetic average scheme from the average speed scheme. But as we can see from Thompson's description of JJ's actions, this may take a lot of time and energy. It may involve both expanding the speed scheme by connecting it to more complex situations and by introducing new theorems and concepts-in-action. Thompson's article describe such a process. It may also involve invoking a "catch mechanism" in the speed scheme, corresponding to a theorem in action stating, "arithmetic average is not relevant here". Thompson made this observation.

From the cases we described, and the theory we have developed, we can generate the following hypothesis: When an individual associates a linguistic representation with some scheme, that contains theorems-in-action that produce answers in a simple way, it will be very difficult to make the individual abandon this initial scheme in favor of a more complex one, or a different one. The initial scheme will act like an irresistible escape route any time the second scheme gets cumbersome and does not produce swift answers.

Mathematics education is in fact full of cases when linguistic representations are associated to new or expanded concepts. When we clarify the effects of interaction of schemes and semiotics, we can identify and understand such phenomena better and produce didactical suggestions for dealing with them.

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Evaluating 3D-DGS under the perspective of didactic usage in schools

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This paper gives a short overview over two selected empirical studies as part of a research project. The main goal of the project is to find the most powerful ("best") 3D-Dynamic Geometry System (3D-DGS). GeoGebra 5 is the most popular DGS with a module for the third dimension, which enlarges it from a (2D-)DGS into a 3D-DGS. It enables spatial constructions in schools as other 3D-DGS do. A comparative theoretical analysis amongst different 3D-DGS showed, that GeoGebra 5 has by far the largest number of implemented features. In contrast, the results of two quantitative empirical studies showed that GeoGebra 5 is only one of many 3D-DGS. At the moment, we have to recommend at least Cabri 3D and GEUP 3D as more appropriate 3D-DGSs regarding spatial constructions in school.

We can find spatial geometry contents in almost all mathematics curricula all over the world. Additionally, learning with the help of digital media is becoming increasingly important in all areas of education. In the field of didactics of mathematics the NCTM states: "Technology is essential in teaching and learning mathematics, it influences the mathematics that is taught and enhances students' learning" (NCTM, 2015).

In 2016, Stacey gave an overview of technological implications for mathematics teaching in her plenary talk at MADIF 10. There are numerous suggestions on how appropriate technologies can be used in school mathematics lessons in the field of computer algebra, dynamic geometry, spread sheets or function plotters (Stacey, 2016).

To support and enrich spatial geometry, technology can be part of this learning process. If we follow these arguments, computer tools ought to get a bigger and more essential role in mathematical education's media literacy. As part of the "mathematical literacy" within the seven "fundamental mathematical capabilities" digital tools are even essential for learning mathematics (Stacey, 2012). Following these ideas, teachers and students in school may benefit in many ways by using digital tools as part of mathematics. But how can we foster students learning spatial contents within school's practical math?

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Research results suggest that 3-Dimensional Dynamic Geometry Systems (3D-DGS) can contribute as didactical digital tools enhancing and supporting teachers and students in learning and instruction of spatial geometry (e.g. Kasten & Sinclair, 2010; Martín-Gutiérrez et al., 2013). That means within a school context you can achieve this contribution via e.g. interactive constructing, calculating, mapping, modelling, generating and measuring, spatialization, analogization, extension of planar geometric construction methods without methods of descriptive geometry in virtual space (Schumann, 2007). For designing these 3D-DGS, we refer to Fahlgren (2015) or Mackrell (2011).

The existing research literature mostly refers only to theoretical analysis of different 3D-DGS. The existence of more than three 3D-DGS has not been realized by many mathematical didactic researchers and therefore is not often investigated. The existing (theoretical) research desiderata recommend GeoGebra 5, as it is free of charge, easily available, exists with more than 50 language versions and has become widely used in schools due to its elaborate, freely available media pool. There are also numerous practical examples of the use of GeoGebra 5 in schools (GeoGebra, 2018).

There is a serious need for quantitative empirical studies with more than two 3D-DGS, with secondary schools' students, in general education schools with at least two different complex tasks. In the research field of 3D-DGS, however, there was a considerable lack of corresponding quantitative empirical studies with 3D-DGS, especially with GeoGebra 5 in 2015. Under the perspective of didactic usage in schools, we want to evaluate 3D-DGS to find the most powerful on the market.

Theoretical framework

Knapp (2015) defines 3D-DGS by the four paradigms *Virtual space-*, *Drag-modi-*, *Construction-* and *Usability-*paradigm as a minimal catalogue.

Following ISO (2017), 3D-DGS can be classified as not embedded, standardized user software. They support students in school focusing on synthetic 3D geometry and solid geometry to deal with geometry. Limitations to analytic geometry and its systems are inherent due to the computer system (Knapp, 2015).

But with Archimedes Geo3D, Cabri 3D, GeoGebra 5, Geoplan-Geospace and GEUP 3D, there are at least five 3D-DGS with an English user interface available. As GeoGebra 5 is the most popular 3D-DGS, it is necessary to investigate if it is also the "best" available 3D-DGS. This is one of the main questions within the research project "Comparative analysis of 3D-DGS for schools". After a multistage analysis process, the theoretical framework with more than 500 criteria shows an advantage of GeoGebra 5 concerning the implemented features. This 3D-DGS offers the most elaborated tool spectrum compared to the other four above mentioned 3D-DGS (Knapp, 2015). But do the students in

school also benefit from this large spectrum compared to other 3D-DGS? If we take a look at the international research publications (e.g. Hattermann, 2011; Hugot, 2005; Kovárová, 2011; Mackrell, 2011), there are no empirical results to answer the aforementioned question.

There is a lack of comparative analysis studies with more than two 3D-DGS. There is another lack of studies with students of secondary schools. Furthermore there is a lack of quantitative empirical studies and different complex tasks.

A spatial construction task is simple if it consists of a maximum of three design steps, which only requires the reproduction of a given expert solution. A more complex spatial construction task is a task which requires more than three construction steps based on each other and a higher degree of intelligence (like attention, mental rotation) of the subjects. Both kinds of tasks should be investigated.

However, the question if a specific 3D-DGS is really didactically more comprehensive ("better?") compared to others is a largely unresolved field of research. Furthermore, we tried to find out if the results of the different theoretical frameworks can also be found in quantitative empirical studies. In two empirical studies it should be explored which 3D-DGS is the "best" one.

If we follow the numerous publications with teaching proposals and research results on GeoGebra 5 it would first of all be assumed that GeoGebra 5 should be the "best". In this context, "best" means that the subjects in the studies would have to show the highest reproductive results of all 3D-DGS examined.

In order to be as close to the school practice as possible, studies with quasi-experimental design were therefore obvious (Howell, 2009; Rost, 2005). More precise details as well as a summary of research into these initiatives are reported in Knapp (2015).

Purpose and research questions of the studies

Starting the empirical research based on the theoretical framework with the five analysed 3D-DGS, these systems were compared to each other in a quantitative way. Given the preliminary choice of the computer as medium, the establishment of a control group didn't make sense. The purpose of the studies was to investigate which one out of the 3D-DGS-pool showed the highest effects in different complex spatial construction tasks.

1. Which 3D-DGS shows the highest results regarding a simple spatial construction?
2. Which 3D-DGS shows the highest results regarding a more complex spatial construction?
3. Are these effects and differences significant?

Method and material

The participants were taken as an entire group in a random sampling of school classes. To compare their spatial construction results, potentially intervenient variables have to be controlled. Methodological empirical field studies with quasi-experimental study plans with control variables (Keppel, 1991; Kirk, 1982; Montgomery, 2012) were chosen. According to Knapp (2010) the most effective of these variables concerning spatial constructions with 3D-DGS are "mental rotation" (Shepard & Metzler, 1971; Vandenberg & Kuse, 1978), "spatial visualization" (Linn & Petersen, 1986; Thurstone, 1950), "attention" (Anderson, 2004), "reasoning" (Thurstone, 1938) and "flexibility of closure" (Seel, 2003; Wertheimer, 1959). These variables were controlled via corresponding tests together with a questionnaire.

After data collection, data were tested descriptive and in an analysis of covariance. By means of MANCOVA covariance analyses the variable "construction" was adjusted by the partialization of potentially intervening variables. First, in a one-way analysis amongst the means of the five 3D-DGS, amongst the sexes, then within the 3D-DGS and, if the prerequisites are given in a two-ways analyse of covariance "3D-DGS-sex" (Howell, 2009). KMSS 8 (Kleiter, 1988 & 1990) was used as a statistical software package to calculate descriptive and inferential statistics.

To deal appropriately with the " α - β -error problem" (Rost, 2005) of hypothesis-testing statistical methods, the significance level was set to 0.05. Following the "good-enough-principle" of Serlin and Lapsey, the effect sizes η^2 of "power analyses" (Rost, 2005, p. 174–176) were also given.

In comparative empirical analysis study 1, participants had to create a simple spatial configuration by constructing a cube model by its vertices and edges. In study 2, participants had to design a more complex spatial figure by constructing and adjusting an octahedron in a multistep construction process using basic construction tools, mapping tools and dragmodi.

Participants

The studies took place from December 2014 until April 2015 in the south of Germany at secondary schools with 14-years old students. They were recruited as entire school classes. Each of these classes got one of the five 3D-DGS and one of two tasks randomly. In study 1, 113 students participated, in study 2, 121 participated. German versions of tests, questionnaire and user interfaces were used.

Instruments

In both studies, the aforementioned variables were controlled. Therefore, the corresponding tests in their German versions ("Mental rotation test" (Quaiser-Pohl, in press), "D2-Aufmerksamkeits-Belastungstest" (Brickenkamp, 2002), "Prüfsystem für Schul- und Bildungsberatung" (Horn, 1969) were used. The

construction tasks cube and octahedron and their construction process were chosen in a way that for every investigated 3D-DGS it was possible to construct the product in a similar and comparable way.

In study 1 the participants were instructed by the author with a worked-out example (Atkinson, 2000) for each 3D-DGS expert's solution. The example was shown via a direct live instruction presented via a data projector. Figure 1 illustrates the solutions of this simple construction of a cube.

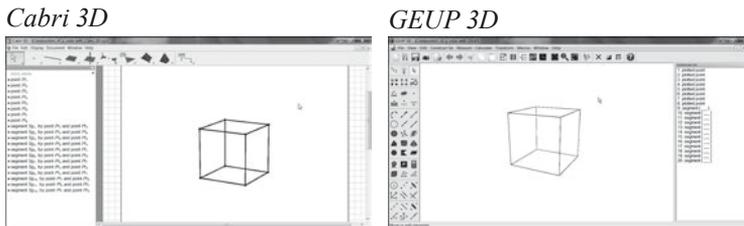


Figure 1. Solutions of the construction of a cube

In study 2 the participants interact individually with an *interactive on-screen video* (Knapp, 2010) which shows a worked-out example for each 3D-DGS procedure. Methodical access via these videos was chosen to ensure a more objective presentation of information. This minimizes any form of implicit or explicit manipulation by the instructor. Figure 2 illustrates this with GeoGebra 5 as an example.

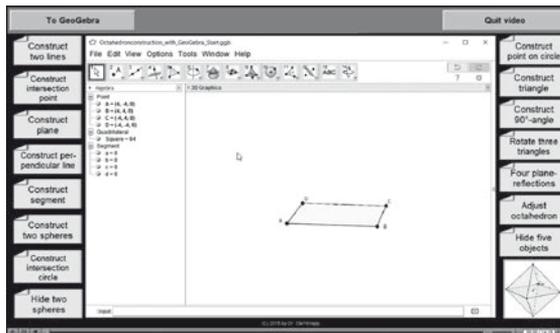


Figure 2. Interactive on-screen video of an octahedron construction (translation by the author)

Procedure

Both studies were designed as a quasi-experimental study with control variables. These variables were controlled via corresponding tests together with

a questionnaire (Knapp, 2015). The participants had 90 minutes in a specific order, according to the manuals, to work on the tests and to answer questions.

The next day, they were instructed by the worked-out example in study 1 and could work with the interactive on-screen video in study 2. At the end of each study, the participants had to save their constructions. These results were coded with points for each construction step. The maximum points for the cube construction were 28, for the octahedron construction were 52.

Results

Since the scope of the paper is limited, we focus on elementary results. More precise details are reported in Knapp (2015).

Study 1

Minimum values (girls): Archimedes Geo3D (16 points), Cabri 3D (20 points), Geogebra 5 (10 points), Geoplan-Geospace (10 points), GEUP 3D (16 points).

Minimum values (boys): Archimedes Geo3D (16 points), Cabri 3D (20 points), Geogebra 5 (10 points), Geoplan-Geospace (10 points), GEUP 3D (16 points). There are no sex differences.

Table 1. *Descriptive parameters*

3D-DGS (sex)	Means		
	Arithmetic means	Medians	Modes
Archimedes Geo3D (Boys)	24.6	27	28
Archimedes Geo3D (Girls)	23.8	24	28
Cabri 3D (Boys)	26.2	28	28
Cabri 3D (Girls)	25.3	26	28
GeoGebra 5 (Boys)	21.8	22	28
GeoGebra 5 (Girls)	21.0	20	28
Geoplan-Geospace (Boys)	22.4	28	28
Geoplan-Geospace (Girls)	22.8	24	28
GEUP 3D (Boys)	25.3	28	28
GEUP 3D (Girls)	24.6	27	28

No student achieved less than 10 out of 28 points $\approx 36\%$ of the maximal points. In all investigated 3D-DGS (adjusted) means showed, that students achieved at least 71 % of the maximal points. No significant sex differences could be found. The modes of all 3D-DGS results are 28, which is equal to 100 % of the maximal points. There are differences in arithmetic means and medians amongst the investigated 3D-DGS by solving a simple spatial construction task. Concerning these descriptive parameters, the groups Cabri 3D (boys) and GEUP 3D (boys) showed the highest, the group Geogebra 5 (girls) the lowest results. A one-way

analysis of covariance amongst the 3D-DGS showed significant differences at least on a low level (Howell 2009; Rost 2005).

Table 2. *Adjusted means of an one-way analysis of covariance (3D-DGS)*

3D-DGS	Adjusted Means				
	Mental rotation	Spatial visualization	Attention	Reasoning	Flexibility of closure
Archimedes Geo3D	25.0	24.0	23.8	23.8	23.7
Cabri 3D	25.9	26.1	25.7	26.1	26.4
GeoGebra 5	20.0	21.1	20.8	20.9	20.9
Geoplan-Geospace	21.9	22.3	22.7	22.5	22.2
GEUP 3D	25.8	25.7	25.7	25.4	25.5

Note. Effects: $p_{\max} < 0.05$; $\eta^2_{\min} > 0.05$

There are also differences between the participating groups concerning 3D-DGS. These differences are not significant. Concerning all 3D-DGS, Cabri 3D and GEUP 3D showed the highest absolute and adjusted average results.

Study 2

Minimum values (girls): Archimedes Geo3D (20 points), Cabri 3D (20 points), Geogebra 5 (20 points), Geoplan-Geospace (9 points), GEUP 3D (29 points).

Minimum values (boys): Archimedes Geo3D (22 points), Cabri 3D (38 points), Geogebra 5 (29 points), Geoplan-Geospace (13 points), GEUP 3D (36 points).

Table 3. *Descriptive parameters*

3D-DGS (sex)	Means		
	Arithmetic means	Medians	Modes
Archimedes Geo3D (Boys)	28.0	29	29
Archimedes Geo3D (Girls)	22.8	22	20
Cabri 3D (Boys)	46.2	47	47
Cabri 3D (Girls)	35.4	35	35
GeoGebra 5 (Boys)	37.4	36	37
GeoGebra 5 (Girls)	26.9	26	26
Geoplan-Geospace (Boys)	18.7	20	20
Geoplan-Geospace (Girls)	15.0	15	15
GEUP 3D (Boys)	44.5	46	47
GEUP 3D (Girls)	32.9	33	33

No student achieved less than 9 out of 52 points $\approx 17\%$ of the maximal points. In all investigated 3D-DGS (adjusted) means showed, that students achieved

at least 29% of the maximal points. Significant sex differences could be found. There are differences in arithmetic means, medians and modes amongst the investigated 3D-DGS given the task of solving a more complex spatial construction task. Concerning these descriptive parameters, the group Cabri 3D (boys) showed the highest, the group Geoplan-Geospace (girls) the lowest results. A two-ways analysis of covariance between the 3D-DGS and sex showed significant differences on a low level (Howell 2009; Rost 2005). Concerning these inferential parameters, the group Cabri 3D (boys) showed again the highest, the group Geoplan-Geospace (girls) the lowest results.

Table 4. *Adjusted means of a two-ways analysis of covariance (3D-DGS-sex)*

3D-DGS (sex)	Adjusted Means				
	Mental rotation	Spatial visualization	Attention	Reasoning	Flexibility of closure
Archimedes Geo3D (Boys)	29.5	14.5	24.2	29.6	30.1
Archimedes Geo3D (Girls)	20.6	9.1	16.1	20.4	20.9
Cabri 3D (Boys)	54.2	36.9	48.3	54.3	55.0
Cabri 3D (Girls)	29.6	18.4	25.9	29.5	30.2
GeoGebra 5 (Boys)	39.6	24.6	34.3	40.0	40.3
GeoGebra 5 (Girls)	22.3	11.7	18.4	22.3	22.8
Geoplan-Geospace (Boys)	19.1	6.5	14.0	19.2	19.9
Geoplan-Geospace (Girls)	14.1	1.6	10.0	14.2	14.7
GEUP 3D (Boys)	47.4	34.9	41.9	47.5	48.2
GEUP 3D (Girls)	27.7	17.1	23.8	27.5	28.1

Note. Effects: $p_{\max} < 0.03$; $\eta^2_{\min} > 0.01$

Over 3D-DGS or sex or combination of 3D-DGS-sex, Cabri 3D and GEUP 3D showed the highest absolute and adjusted means. More details are documented in Knapp (2015).

Discussion and conclusion

The research design "quasi-experimental study plans with control variables" is also suitable for 3D-DGS in school practice and could be successfully analogized. Its reproducibility is guaranteed. The design can be used for further empirical comparisons. Learning to construct in virtual space with 3D-DGS can lead quickly to experiences of success. 3D-DGS can support and enhance teaching and learning of spatial constructions. There are (significant) mean differences amongst the investigated 3D-DGS in a descriptive and inferential statistical way. *At least* concerning the two investigated constructions, Archimedes Geo3D, GeoGebra 5 and Geoplan-Geospace showed not the highest

construction results. Cabri 3D or GEUP 3D showed higher learning impacts in constructing a simple as well as a more complex construction. Teachers and didactics of mathematics have to keep in mind, that the popular GeoGebra 5 enables spatial constructions, but is at least not the best in all construction fields. In the future, we will continue our project with more complex tasks and practical educational suggestions. We also consider the usage of different 3D-DGS for exam tasks and their implementation in various curricula.

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Differences in pre-school teachers' ways of handling a part-part-whole activity

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The data in this paper draws from an eight-month intervention study based on the conjecture that children need to discern the first ten natural numbers as relations of parts and whole to develop their arithmetic skills. A group of nine Swedish pre-school teachers worked closely with our research team, planning, enacting and analyzing activities in order to test the conjecture. In this paper I describe how the pre-school teachers, across 67 video-recorded films, handled one of these activities, called the *snake-game* with their groups of five-year-old children. Results from the analysis based on variation theory of learning, point to differences in the enactment of the *snake-game* in terms of if and how the teachers foregrounded the part-part- whole relations of numbers embedded in the activity.

The FASETT-project (The ability to discern the first ten numbers as a necessary ground for arithmetic skills¹) includes an intervention study that is put into practice in Swedish pre-school. The project builds on Davydov's curriculum (Schmittau, 2003) and Neuman's (1987) research, who both emphasize the structured relations of numbers' parts and whole as a necessary ground for arithmetic problem solving. This structured approach stands in contrasts to descriptions of young children's development of arithmetic skills as based on operational counting (e.g. Carpenter & Moser, 1984; Fuson, 1992; Resnick, 1983). A key distinction between the two approaches is that in structured relations of parts and whole, the part and whole of quantities are handled simultaneously, compared to operational counting-based approaches, where the quantities are handled in sequence.

The FASETT-intervention is based on the idea that children need to discern the first ten natural numbers as relations of parts and whole to develop their arithmetic skills. Also, Neuman's (1987) findings of the benefit of using fingers for structuring number relations is an important part of the conjecture as well as the variation theory of learning (Marton, 2015; Marton & Booth, 1997), described below. Educational activities and games were in accordance with this conjecture designed, implemented and refined in collaboration with a group of pre-school teachers during an eight-month period of time.

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Several international studies have identified positive effects of interventions in early childhood education (e.g. Jordan, Kaplan, Ramineni & Locuniak, 2009; Sarama, Clements, Wolfe & Spitler, 2012). By studying Swedish pre-school teachers' ways of handling the theoretical designed activities, this study offers complementary knowledge about how to create opportunities for children to learn specifically about numbers and number relationship and foregrounding fingers for representing structured number relations. In this paper one of these activities, called the *snake-game* (a string with five or ten beads) is in focus.

The aim of this paper is to examine how nine teachers participating in the intervention study enacted the snake-game with their pre-school groups, and to discuss how ideas of structural relations of parts and whole were manifested in the enactment.

Theoretical background

Based on a study of 7-year-old students, Neuman (1987, 1989) argued that students who successfully solved arithmetic problems, not knowing the number facts, used structured fingers patterns, in which the relations of parts and whole were discernible simultaneously. Children in her study who did not spontaneously structure their fingers efficiently struggled with solving the problems. As an illustration of those differences, imagine two children that are solving the problem: $9 - 7 = _$. The first child opens nine fingers simultaneously, looks at her fingers and answers immediately: "two". She discerns the "part 7 within the whole 9" and does not have to count single units. In contrast to the other child who solves the task by counting backwards in ones, from 9 (9, 8, 7, ...) folding a single finger at a time.

The ability to discern sets of units (3–4 items) in an instance is called subitizing (Wynn, 1992). When larger sets of units are to be determined, other strategies are needed. However, when children structure finger patterns, they can use the "five" (one hand) or "ten" (two hands) as a reference for structuring numbers near five or ten, for example seven as a whole hand and two more. Thus, they do not have to count single units.

When children experience part-part-whole relations of a number, they are able to use the combinations of the three numbers solving various addition and subtraction problems (Baroody, 2016). From one specific part-part-whole relation, for instance $7/5/2$, you might bring out different combinations of that relationship. Additive relation problems with one missing value (e.g. $5 + _ = 7$; $_ + 5 = 7$; $7 - _ = 2$; $7 = _ + 2$) can be solved using the fact of the relation of the whole and its parts (Neuman, 2013).

From a variation theory perspective (Marton, 2015), to learn something specific, certain aspects need to be discerned. Marton has theorized Neuman's empirical findings and argues that cardinality (each number refers to a certain group of items), ordinality (each number refers to a place in an order) and that

numbers can be represented in different ways (e.g. fingers, beads) are aspects that have to be discerned simultaneously. Moreover, numbers as relationship of parts and whole is a necessary aspect that children also need to discern for being able to develop their arithmetic skills.

According to variation theory, the experience of differences (Gibson & Gibson, 1955) is a necessary condition for learning something specific. Learners' attention should be drawn to what is intended to be learned, by foregrounding aspects of the specific learning object and opening them up as dimensions of variation (Marton, 2015; Marton & Booth, 1997). This involves designing and enacting activities where those aspects are possible to discern, using certain patterns of variation. For instance, to learn that there are different possible parts within a whole number (e.g. 7), the whole number can be kept invariant, while various parts within that whole are presented as a sequence of examples (1 and 6; 2 and 5; 3 and 4). So, depending on which aspects are opened up as variation different things are made possible for the children to learn.

Methods

The intervention

Nine pre-school teachers and 65 five-year-old children from two outer districts of a large Swedish city participated in the study. Teachers and children's legal guardians had given their written consent according to the ethical guidelines from the Swedish Research Council (2011). The intervention study is a kind of design based research (Cobb et al., 2003) where teachers and researchers work collaboratively in an iterative process to develop and test the underlying ideas of the conjecture in natural settings (pre-school). A group of researchers regularly met the pre-school teachers across two semesters. In those meetings, the conjecture mentioned above and how activities could support children's learning of part-part-whole structure of the first ten numbers, were discussed. The activities were most often introduced by the research team and developed in collaboration with the teachers. The program mainly focused on four activities which the pre-school teachers enacted in their practices. The teachers were asked to video-record and upload at least one enactment of each of these activities to a server. In preparation for the next meeting, the research team watched these videos. Episodes from the video clips formed the basis for the pre-school teachers' reflection and collaborative discussions about how the enactment of the activity could be improved. Most often the teachers agreed on refining the activity. The snake-game was the second activity introduced in the intervention-program.

Snake-game

The snake-game constitutes a string of five or ten beads. The "5-snake" has five beads in the same colour and the "10-snake" has five beads in one colour and

five in another colour, grouped together (figure 1). The teacher was supposed to emphasize the number of beads on the whole snake to begin the activity and encourage children to represent this whole number with their fingers. Then she would hide some beads in her hand or under a cloth, and children were asked to show the visible number of beads on their fingers. Then, by looking at their opened and folded fingers simultaneously they were asked to identify the hidden part (figure 2).



Figure 1. "5 snake" and "10 snake"



Figure 2. Six beads visible, the child shows the same quantity with the fingers

Data and analysis

In order to examine how the pre-school teachers enacted the snake-game, 67 video-recorded situations were analysed. All nine participating teachers uploaded videos to a server during a three-month period. After having watched the films to get an overview of how the snake-game was handled, sequences of examples (hidden/visible beads) within each film were noted. Thereafter, teachers' utterances and gestures related to the mathematical ideas discussed in the planning meetings were transcribed. Principles from variation theory were used for analysing the video-recorded situations (Marton & Booth, 1997; Marton, 2015). First, an analysis was made of which dimensions of variation were opened up and how those dimensions were emphasized in the teachers' enactment of the snake-game. Then, notations were made of what was kept invariant and what varied within each dimension. Teacher's linking actions, by talk and gesture, in line with the coding framework in Ekdahl, Venkat and Runesson (2016), were also analyzed. Finally, based on all 67 films, I focused on differences and similarities in the enactment of episodes where a certain dimension of variation or several dimensions of variation were opened.

Counting activity

When the snake-game was handled as a *counting activity*, counting in ones was used to determine the number of beads or fingers. Instructions to count could be seen when the children were asked to find out how many beads they could see on the whole snake (5 or 10 beads), on the visible part of the snake or after having identified the hidden part. For instance, after a child had counted the beads in ones (10), the teacher said: "Neo counted the ten beads, do you also want to count them?". The other child counted the ten beads in the same way. This points to an enactment of encouraging counting in ones for determining a well-known quantity. In other situations, for instance after the children had identified nine hidden beads on the snake, a teacher asked the children to "check" if it really was nine beads on the string, saying: "Is it 9? Let's check [pointing to each bead] one, two, three, four, five, six, seven, eight and nine". Even if the teacher did not explicitly ask the children to count in ones, children spontaneously used "single unit counting" and were not offered any alternative by the teacher. That sort of enactment was seen for example in a situation when a teacher introduced the "10-snake".

Teacher: How many on this snake?

Children: One, two, three, four, five, six, seven, eight, nine, ten [pointing to every bead, one at a time]

Teacher: It's ten, now how many do you see? [hides five beads in her hand]

Peter: One, two, three, four, five [pointing to the beads one by one]

Anna: It's so easy, because we have five and five fingers [holds up her two hands].

From this excerpt, we can see that children were given one alternative to identify the number of beads on the whole snake (10) and the visible part (5). By asking the children: "How many?" the teacher did not offer any alternative than counting for determining different quantities. Even if Anna said: "It's so easy because we have five and five fingers", initiating the idea of the semi-decimal structure of five fingers on each hand and the connecting to five beads in two colours, that idea was not picked up by the teacher. Also, the excerpt illustrates how only one representation of numbers (bead string) were focused, without connection to fingers. The use of counting strategies to identify the number of beads is similar across these instances. The varying aspect is the number of countable items within each collection, and the activity therefore opens up possibilities for the children to count different units within different collections (a whole or a part) separately.

Structured part-part-whole activity

Dimensions of variation related to the part-part-whole structure of 5 and 10 and representing numbers with beads and fingers were opened up in the teachers' enactment of the activity by emphasizing different part-part-whole relations

and connecting different representations. Besides these dimensions, teachers also focused on structured finger-patterns (first sub-category) and extending the basic subitizing range (second sub-category).

Focusing on structured finger patterns

The teachers opened up a dimension of variation using the same whole number ("5-snake" or the "10-snake") presenting different visible/ hidden parts in a sequence. The enactment was related to starting with all five/ ten fingers, then showing the same numbers of fingers as the number of visible beads, and then discerning the hidden number of beads by seeing closed fingers on hand/s. The following illustrates this enactment.



Figure 3. *Illustration of the structured finger-pattern of 10/7/3*

- Teacher: All ten fingers. Show me on your fingers how many you see?
[Tove shows five fingers on the left hand and two on the right hand, three fingers are folded]
- Teacher: Can you look at your fingers and see how many beads are hidden?
- Tove: Three [...]
- Teacher: Seven and three, all together ...?

Here, initially, the teacher emphasized the whole number (10) and made sure that the children put their ten fingers in front of them. The children were asked to look at the snake, and structure their fingers into the visible part (7). Thereafter, the teacher did not ask how many beads are hidden. Instead she encouraged them to look at their folded fingers and see the relation of seven open fingers and three folded fingers. The child (figure 3) turned her right hand around and looked at the three folded fingers on the hand and said "three". Finally, by asking "Seven and three together ...?" teacher paid attention to the fact that having identified the hidden part, you can compose the two parts into the same whole number. In this sequence, the parts and the whole were emphasized as simultaneously possible to discern.

In situations where the children counted the beads or fingers in ones or did not model the fingers structurally, the teacher opened up possibilities for structured discernment herself: "Look at mine [shows her finger-pattern, opened/

closed fingers, in a structure way].” Alternatively, attention was directed to a child who had structured his fingers and identified the hidden part: ”Look at Steven, he *saw* the answer when he looked at his hand”. Another example was observed in a situation where the ”5-snake” was handled. A child was asked to represent the number of beads (4) with fingers. He put up the index and middle finger on each hand (total 4). However, it became difficult for that child to find out the missing number. The teacher noticed that and said: ”You see, how many did we have from the beginning?” The child said: ”Five”. The teacher said: ”Five okay, then you just need to use one hand.” In wanting the children to discern the relation of 5/4/1, the teacher recognized here that two fingers on each hand did not easily support discernment of the part-whole relation of 5. Therefore, she instead emphasized the whole hand (5). Thus, once again, the parts and the whole were made possible for the learners to discern simultaneously.

Extending the subitizing range

In this structured way of handling the snake-game different part-part-whole relations of 5/10 were emphasized, and beads and finger representations were focused on. Further, in this category teachers opened up the dimension of seeing quantities without counting for *extending the subitizing range*. One way of extending the subitizing range is to pay attention to the semi-decimal structure (5 and 5) and/or to use ”5 as a benchmark”, illustrated in figure 4.



Figure 4. *Structuring 7 as 5 and 2*

Teacher: You do not have to count. All purple and two another colour. Whole hand five and two more fingers [points to the children’s seven fingers].

In figure 4, the bead string and children’s finger representations, in conjunction with teacher talk, contrasted counting with ”seeing” to identify a quantity, opening up seeing ”5 as a benchmark” by focusing on the five purple beads. She made connections to the ”undivided 5”, pointing to the children’s structured finger pattern, with seven opened and three folded fingers.

Elaboration of the structured part-part-whole ideas

Another way of handling the activity was seen in teachers’ enactment of the snake-game, where several dimensions of variation were opened up. These were:

- a whole number (5 or 10) constitutes a part-part-whole relation,
- numbers can be represented with beads and fingers,
- structured finger-patterns make the part-part-whole discernible, and
- seeing numbers without counting.

These dimensions of variation were opened up in the previous category as well. However, in this category several dimensions were opened up simultaneously (e.g. within one example) and in a systematic way (e.g. connecting different examples). Thus, teachers made a further elaboration of the basic instruction related to the activity by "interweaving" seeing numbers as part-part-whole relations and extending the subitizing range for basic numbers. An example of this is seen in the following excerpt, where the children just have found out the hidden part of 10, having discerned the visible part 7.

Teacher: You have five fingers on each hand and five beads in two colours here [points to fingers and beads]. What do we know ...? Always ten fingers on our hands, even if we move the beads on the string [separates the beads, seven to the left and three to the right].

In this short excerpt, the teacher opened up for the semi-decimal structure, both hands and colours of the snake simultaneously. She emphasized the whole number 10 (two hands) and then focused on different possible parts within 10, by comparing 5 and 5, and 7 and 3.

Apart from several dimensions of variation were opened up simultaneously, connections between examples (numbers of hidden beads on the snake) were identified in teachers' enactment within this category. For instance, in a sequence of examples where first 6, then 7 and finally 8 beads were visible.

Teacher: [6 beads visible] Remember all those [points to the five beads and simultaneously hold up her left hand] and six ... five pink and one purple [holds up one finger on the other hand], also six.

Teacher: [7 beads visible] Remember, those are five [makes a circle with her hand around the five purple beads of the 10-snake], you do not have to count those ... and six and seven.

Teacher: [8 beads visible] Do you remember the "tip", not needing to count all. How can you find out?

This sort of enactment shows a systematic sequencing of examples where the teacher continuously focused on "5 as a benchmark". She opened up for a dimension of variation separating the "undivided 5" from 6, 7 and 8 in slightly different ways in the sequence of examples. Also, she paid attention to both fingers and beads simultaneously.

Discussion and didactical implications

The results of this study show that different ways of enacting the snake-game, in terms of which dimension of variation were opened up give implications for what is made possible for the children to learn (Marton & Booth, 1997). When the snake-game was handled as a *counting activity* children may learn to count units within different collections. When the number of items exceed the subitizing range, the children were not offered any alternative ways of finding out the visible/ hidden part. By just focusing on counting whole and parts, separately, it becomes difficult for the children to see the three collections as a part-part-whole relation and that a whole quantity consists of different plausible parts. In contrast, when the activity was handled as a *structured part-part-whole activity*, and several dimensions of variation were opened up, the children were given opportunities to learn that a whole quantity can be decomposed in different parts and those parts could be composed to the same whole, fingers can be structured as patterns and by looking at the folded fingers a hidden quantity can be identified. When the teachers, by variation opened up for extending the subitizing range, the children could learn that grouping objects, using 5 as a benchmark are alternatives to counting in ones. In the third category (*elaboration of the structured part-part-whole ideas*), the way of enacting the activity, opened up for several dimensions of variation simultaneously. Teachers being able to connect and compare different hidden/ visible parts in the presented examples and open up for several dimensions of variations simultaneously might enable the children to make generalisations to other relations of numbers within the number range of 10. The way in which the snake-game was enacted, in this category, may indicate that the conjecture (Neuman, 1987; Marton, 2015) was manifested in practice.

The design of the intervention, the iterative process and regular meetings might have had an impact on the way the teachers enacted the activity. In this study, the development of individual teachers' enactment over time was not studied and compared. However, handling the activity as an *elaboration of the structured part-part-whole ideas* was more common in films uploaded in the third month. The snake-game may be a well-known activity in Swedish pre-school. The question is, however, if the potential of bringing out important mathematical ideas embedded in the game, are also known. The results indicate that if the snake-game will become a structured part-part-whole activity it is not enough to present different visible/ hidden parts of the snake. Instead, the teacher needs to bring fore the ideas that structured finger-patterns can make the part-part-whole relation of numbers visible and "seeing" quantities without counting in ones. Therefore, this research has a pedagogical importance for teacher education as well as professional development in early childhood mathematics.

Acknowledgements

This work was supported by the Swedish National Research Council.

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Notes

- 1 In Swedish: Förmågan Att Sinnligt Erfara de Tio första Talen som nödvändig grund för aritmetiska färdigheter

In the case of jeopardising the learning outcomes – could we have known better?

METTE SUSANNE ANDRESEN

The aim of the study in progress reported in this paper was to establish the framework for a forward-looking learning perspective on teachers' professional development in larger projects. The paper presents the first outline of an instrument for analysis of learning potentials in the form of a matrix which mutually combines a theoretical perspective, a social perspective and a psychological perspective on the implementation of a professional development project in its actual context. Pieces of data from an earlier evaluation completed by NAVIMAT in 2010 of Skolverket's national initiative "The mathematics project" were re-analysed by means of the instrument. The re-analysis revealed discrepancies which might obstacle the teachers' learning.

This paper reports from a theoretical study in progress of mathematics teachers' professional development seen from a learning perspective. The aim of the study was to establish a learning perspective on teachers' professional development by building a theoretical framework for analysis and interpretation of mathematics teachers' learning from participation in professional development activities. The framework supports analysis taking social and psychological perspectives on teachers' learning into account, and it was inspired by the framework developed by Paul Cobb et al. (eg. see Yackel, Gravemeijer & Sfard, 2011), which has shown to be fruitful for analysis and interpretation of students' learning and interaction in mathematics classrooms. The study deals with issues of resonance, synergy, coherence and tensions between the three perspectives and its results may, hopefully, contribute to the understandings of how to interpret and explain discrepancies between teachers' beliefs and their enacted teaching. This paper focuses on the first part of the study demonstrating the process of re-analysing data from an already completed evaluation by the means of an instrument in the form of a matrix, based on the framework.

A framework for studies of teachers' learning

Retrospect studies of teachers' learning from projects imply methodological complications when it comes to tracing the influence or effect of particular

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project events or course modules on the participating teachers' later teaching. This study's basic idea was to set focus on forward-looking studies and to take projects' potentials for teachers' learning as the unit of analysis.

It took as the starting point that learning potentials for participating teachers of a project when it is implemented in a specific context basically relay on the interplay between:

- a The basic idea of the project (*theoretical perspective*): principles, basic to the projects content and organization,
- b The professional context (*social perspective*): norms and rules of the community of teachers/participants in the project, and
- c The teacher (*psychological perspective*): the participating teachers' individual views on their professional life in a broad sense.

The analysis and interpretation of potentials for teachers' learning from a project then must include study of synergy, coherence and consistency, tensions and discrepancies between theoretical, social and psychological perspectives on the project. The content dimension of being a mathematics teacher was included in the design of the framework and divided into three not mutually exclusive

Table 1. *The instrument for analysing teachers' learning potentials*

Teacher's role	Perspective		
	<i>A. Theoretical perspective:</i> Content and organisation of the project, underlying principles and decisions	<i>B. Social perspective</i> Professional context for the teacher's work/ the project	<i>C. Psychological perspective</i> Ideas and values, emotions, beliefs and visions
1. Mathematician	1.A Mathematical content, for example early algebra, mathematical modelling, geometry or mathematical problem solving	1.B Norms about teaching mathematical content, shared interest in mathematical subjects, collaboration opportunities in mathematics	1.C Visions about value and relevance of mathematical subjects, personal preferences and beliefs about own competence and mastering of mathematics
2. Mathematics teacher	2.A Mathematics as a school subject, the role of mathematics in school and in society, teacher training/education in mathematics	2.B Students' attitudes towards mathematics, input and reactions from parents and colleagues in mathematics	2.C Professional knowledge about and experiences from teaching mathematics, ideas about mathematical teaching competence, experiences from earlier projects
3. Professional teacher in general	3.A Visions about teachers' roles and professional learning, ideas about goals and means for development	3.B Headmaster's support, interest amongst colleagues, working habits in the particular school, parents' attitudes, reactions from colleagues	3.C Ideas and beliefs about teachers roles, individual working conditions, experiences from earlier projects

levels: The teacher in the role of i) Mathematician, ii) Mathematics teacher and iii) Professional teacher in general. The combination of perspectives and professional content expressed by the teachers' roles resulted in a matrix format. The idea was not to establish or organize mutually exclusive categories but rather to suggest a structure for analysis of implementation of a project with the aim to capture teachers' learning from the project. Table 1 shows the matrix provided with examples of topics and objects of study in its entries.

The study in progress

Initially, the idea of setting up the new interpretative framework in this study arose in retrospect from experiences with evaluations of two corpuses of projects carried out by NAVIMAT (The National Knowledge Centre for Mathematics Education in Denmark) (Andresen & Henriksen, 2010; Andresen, 2011; Ejersbo, 2010). All materials from these two evaluations serve as dataset 1 and dataset 2, respectively. This paper only deals with parts of the study concerning dataset 1.

After completing the evaluation (dataset 1, see below), and after publishing its results, the evaluators in NAVIMAT got an impression of holding hidden treasures in their hands in the sense that data contained much wisdom that was not apparent in the evaluation report. NAVIMAT's evaluation was forward-looking since it was carried out midway in the project, and its results were acknowledged and used but never "checked" upon a measurement of teachers' change although "The mathematics project" has continued and still is ongoing¹. A wish to dig deeper into data from the evaluation arose from curiosity about how the participating teachers had perceived the project and how well the project did "hit the basket" in the sense that the participants had managed to take ownership and experience genuine, professional development and learning. The basic idea of capturing teachers' learning by the means of a study of resonance, synergy, coherence and tensions between a theoretical, social and a psychological perspective on the three roles of teachers grew out of further reflections upon the meaning of the phenomenon to "hit the basket" in the context of large developmental projects for mathematics teachers. Therefore, the advisors' learning opportunities in the Mathematics' Project was chosen to be the unit of analysis in this present study which aimed to create and examine the matrix model.

The theoretical study in hand was planned in the following steps:

- a An outline of the new interpretative framework was established on the basis of recent research (paragraph 3 below).
- b The framework was tested on dataset 1. The test served to identify and group observations, statements and results in accordance with the matrix format. The aim was to examine and refine the content elements of the matrix and to see distinctions between them.

- c The framework was refined and elaborated in accordance with the results of the test.
- d The refined framework was used for analysis of dataset 2 with the aim to obtain additional results besides the ones presented in the evaluation report.
- e The usefulness of the framework, for analysis and interpretation of teachers' learning, was evaluated.

This paper presents examples of work in progress from part b and c of the study. This work only deals with dataset 1.

Dataset 1

The first NAVIMAT evaluation (Andresen & Henriksen, 2010) inquired Skolverket's and NCM's (Nationellt Centrum för Matematikutbildning/ National Centre for Mathematics Education) national initiative "The mathematics project" in Sweden 2010. The aim of "The mathematics project" was to establish and maintain a national structure for professional development for mathematics teachers with the organisation of a network of mathematics teacher-advisors – organized and supported by NCM – as the main means. The mathematics advisors' role was to be in charge of local projects and, besides, to guide colleagues in using research and other inspirational materials. NAVIMAT's evaluation encompassed a web based questionnaire distributed to 389 mathematics advisors (response rate 50 %). Almost 20 % of the questionnaire's respondents did not work as a teacher at present. The questionnaire was followed up by 12 face-to-face or Skype interviews. The complete set of original data from the evaluation including questionnaire with answers, recordings of interviews, field notes, interview guides, memories, work sheets and notes etc. constitutes dataset 1 for the study in progress, reported here.

Dataset 2

The second NAVIMAT evaluation (Ejersbo, 2010) inquired the construct "Projectforum" which was a corpus of 17 research and development projects organized into the structure of a learning community described in (Andresen, 2011). The complete set of original data from the evaluation of "Projectforum" will constitute dataset 2 for future analysis in the study.

Theoretical background

The NAVIMAT evaluation (Andresen & Henriksen, 2010) of "The mathematics project" included an opening study which examined its underlying ideas explicated in the NCM background report *Hur kan lärare lära?* (Mouwitz, 2001) inspired by NCTM, USA. The opening study concluded that the

foundation was in accordance with international research on mathematics teachers' professional development as it was summarized and presented by Sowder (2007). NAVIMAT's evaluation report summarised the theoretical perspective on activities in "The mathematics project" in what it called the "NCM-paradigm". For this study, the theoretical basis was expanded but it still encompassed the "NCM-paradigm". Paul Cobb's (in Yackel et al., 2011) interpretative framework for understanding mathematical classroom activities was the main source of inspiration for the expansion. The teachers' learning from projects, though, had to be contextualised differently. "The project's theoretical perspective", column A in the matrix in Table 1, was added with the intension to capture an essential aspect of the learning context for participating teachers. The prerequisite that learning potentials rely on the interplay between the three perspectives was in line with ideas within the NCM-paradigm about taking ownership to projects and about developing a professional teacher identity (see for example Sowder, 2007). It was, amongst others, inspired by ideas underlying the concept of constructive alignment in higher education teaching (Biggs, 1996). This study was also inspired by the program "Educational reconstruction" (Prediger et al., 2015) in taking the teachers' perspectives into account. Though, it combines rather than chooses between the individual's and the shared perspectives. Depending on the choice of unit of analysis, discrepancies between teachers' beliefs and their enacted teaching have been interpreted and explained in various ways for example in (Hundeland, 2011) and more generally in (Skott, 2004). This study sets apart from a deficit-oriented view on teachers' enacted professional activities and knowledge.

Data analysis, examples

The aim of part b. of the study was to examine the first outline of the theoretical framework represented by the matrix (Table 1) by the means of Dataset 1. NAVIMAT's questionnaire was designed to examine the key issues for success of the mathematics advisors and it covered three headings: 1. Mathematics advisors' background and conditions, 2. Mathematics advisor as participants in the MSU's project ("The mathematics project") and 3. Progress factors in MSU project. Only short excerpts of the re-analysis were presented in this paper, for exemplification.

Starting with the framework vs starting with observations

Every single entrance in the matrix pinpointed distinctive aspects picked out from the teachers' learning seen as a whole. If the matrix was to be useful for its purposes, it should be possible to identify data that illustrate these aspects and their interaction with the other aspects of the particular case. Therefore, the first example took one of the roles, the mathematician, in the matrix as its

starting point and study the teachers view on the project's guidelines for teaching algebra. On the other hand, the matrix should also capture the main part of observations and ascribe meaning to these by contextualising and contrasting them. Therefore, the following two examples took pieces of data as their starting points.

Role: the mathematician

Taking the role as mathematician (row 1 in the matrix, Table 1) as the starting point, the observation that the book "Förstå och använd tal" (To understand and use numbers) published by NCM (McIntosh, 2008) was highlighted by some of the respondents led to letting the entry *1.A Mathematical content* in the matrix include "early algebra". In connection with that, *1.B Norms about teaching mathematical content* came to include groups of participating teachers' local interpretation of the book's view on teaching early algebra in school. The entry *1.C Visions about value and relevance of mathematical subject* should, if possible, include the individual teacher's personal view on early algebra. The book's view on teaching early algebra was in accordance with the NCM-paradigm (it referred to Reys & Reys, 1995) in characterising a pupil with number sense (McIntosh et al., 1997). Accordingly, the book's view on teaching early algebra should be designated to *2.A Mathematics as a school subject*.

The mathematics advisors' personal view

Under heading 1, "1. Mathematics advisors' background and conditions", NAVIMAT asked question 1.5.1: *Are you happy to work as a mathematics advisor?* (Yes, No, explain in free text). According to the evaluation report 95 % (of 142 respondents) of the advisors were happy with their job as mathematics advisor. The reasons for being happy with the job were, in the report, picked out among the answers, grouped and counted up (65 in total): Personal growth (12), networking (11), positive response (8), interest in mathematics (8) and a wish to improve mathematics education (6) and others: Interesting, great etc. (20).

Re-analysis

Re-analysis by means of the matrix of the answers from teacher-advisors revealed among other:

- i The teaching respondents did not mention concrete mathematical subjects (1.A) but only the teaching levels: projects or initiatives targeted kindergarten, mathematics in primary school, kindergarten + primary + secondary school. One of the respondents wrote (1.C):
It is very stimulating and interesting to have time for exploring the exciting world of mathematics.
- ii A psychological perspective on mathematics and norms about teaching (1.C and 2.C) was revealed:

I like to set focus on mathematics in primary school. Mathematics is not only about calculation; it is about much more

I want to change the way we teach mathematics. Today it is, in too many places, monotonously counting in the textbook. Discussions and practical work has all too little space. The pupils do not see connections to the real world. Mathematics is fun, why don't the pupils have that experience.

- iii From a psychological perspective (3.C), one of the respondents wrote about the projects' basic idea (column A) and the local school culture (column B):

It is incredibly fine with school development. Though, it is not fine when changes in mathematics teaching are implemented at regional level because many teachers find it frustrating and hard to change patterns and give up less good but well established algorithms.

- iv Respondents about being in the project, from a psychological perspective (3.C):

Very exciting and rewarding task.

Incredibly stimulating and funny.

The most exciting task I could dream of. A dream job!

Badly supported by school leaders and headmasters. Except some enthusiasts.

I have too little time and get no resources. I want to do a lot but lack time and money. I am in doubt whether to continue as an advisor.

Results

Maybe because mathematics was understood to be the subject, only a few of the answers mentioned mathematics explicitly. Almost none mentioned specific mathematical subjects but many advisors praised the lectures, courses and seminars included in the project. The answers in ii) revealed the view on mathematics as school subject in accordance with the NCM materials, i.e. the book on early algebra, meaning that there was coherence between 1.A and 1.C – 2.C. in the matrix in this case. These respondents either had changed to the project's view or they were agreeing with it in advance. In iii), one respondent pointed to a discrepancy between (column A) and (column B). It was unclear which of the role levels (1, 2 or 3) he or she talked about, maybe the comment was meant to be general. The respondent's viewpoint was in accordance with the NCM paradigm. In iv), the most extreme answers are quoted to demonstrate how wide the observations spanned. The first three demonstrate good conditions for learning, whereas the latter two are in line with other negative answers discussed in next paragraph.

The mathematics advisors' experiences and visions

The questionnaire contained questions about the advisors' experiences and visions. Under heading 1, "1. Mathematics advisors' background and conditions", NAVIMAT asked question 1.5.4: *Do you think there are sufficient resources for you to fulfil your current role as a mathematical advisor satisfactorily?* (Yes, No) followed by 1.5.4.a: *What do you think is required for you to fulfil your role as mathematical advisor satisfactorily?* (Free text). 65,5% (of 116) answered "No" to question 1.5.4. The report picked out resources and support which were mentioned by respondents as necessary, grouped and tallied those (201 in total). The report showed: Time (80), money (45), accept and support from leaders (38), network (12), autonomy (9), and others (11).

Re-analysis

Re-analysis by means of the matrix of the answers from teacher-advisors revealed among other:

- v Some answers included the need for a budget for the advisor function (3.C), for example:

Budget to invite lecturers and for example build up a math. lab. 100.000/ year.

Own budget so that I can make priorities of what initiatives we need.

[...] more money for laboratory material.

A budget to arrange training days.

Time (money) to be able to [...] and to arranging study groups about mathematics workshops and 'the Handbook' which we have been trained for running.

- vi A number of respondents repeatedly mention in the answers to 1.5.4.a and several other questions, that they lacked time to do their job (3.C), for example:

Time to complete the task!

The employer should offer a reduction in the work time.

MUCH more time.

I should have at least 20% for feeling that I have time to do what is expected.

- vii Other answers to 1.5.4.a. revealed the respondents' personal/psychological perspective (column C) on the local school contexts (2.B and 2.C), for example:

Engaged colleagues and headmasters.

Mandate from school management.

Support mainly from management and all relevant school leaders, teachers.

Clearer directive on what is expected of me.

Results

The high percentage who answered "No" to question 1.5.4 highlighted the importance of this part of the evaluation and re-analysis. Here, the answers in v), vi) and vii), and the last in iv), were interpreted to reveal disappointment as a result of discrepancies between what was expected of the advisors from the project's set up (2.A and 3.A) on the one hand, and their experienced interest, support and mandate on the other hand (2.B and 3.B). According to the NAVIMAT report the NCM paradigm (the theoretical perspective on "The mathematics project") implied that professional development should include time and resources for the teachers to engage in sustained efforts to improve their practices. Neither for the teachers nor for the advisors was the prerequisite of good working conditions met, in a number of cases. The re-analysis stressed this discrepancy and made it more explicit.

Conclusion and perspectives

In 4.2 the re-analysis suggested that the respondents had little focus on mathematics content compared to the teaching of mathematics and supervision of mathematics teachers, conduct study groups etc. Apparently, nobody had complaints about the level or content of mathematics but only a closer examination of the NCM programme would reveal how the content was distributed between mathematics, teaching etc. Once the re-analysis is concluded, the results may point to the need of a closer examination of the NCM programme in this aspect. The re-analyses in the examples followed a pattern of interpretation that set the matrix's theoretical perspective to be the NCM-paradigm, its psychological perspective to be the advisors' personal or individual view on their experiences with the project and its social perspective to be the local schools' contexts (working conditions, colleagues' views on the project and its activities etc.). The main results of the re-analyses in the examples suggested that theoretical perspective (Column A) resonated with psychological perspective (column C), whereas both conflicted with the social perspective (column B). Although these two examples were far from conclusive, they still point to the need of another closer examination of the NCM programme. Such an examination should aim at explaining or interpreting the observation that a high percentage of advisors who found there was a lack of support and resources still confirmed that they were happy with the job. The larger (main) study in progress will focus on elaborating on the professional level (rows 1, 2 and 3 in the matrix) based on further re-analyses of dataset 1. The main weakness of the matrix so far seems to be column B in its present unspecified form. The social perspective must be better tailored to encompass norms for teaching and for other professional activity delineated from, and contrasted with consequences of decisions about economy and politics.

This new interpretative framework with its instrument in the form of the matrix (Table 1) was meant to be used in general for analysis of the potentials

for mathematics teachers' learning when they participate in larger professional development projects. The matrix will be loaded depending on the actual project and its context, like in the case of Dataset 1 where the advisors to a large extent were enthusiastic with the NCM programme including its underlying NCM-paradigm, and therefore saw the troubles not with the content but in the local professional context. Other large-scale professional development projects will reveal other strengths and weaknesses by means of the matrix!

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Notes

- 1 <https://www.skolverket.se/kompetens-och-fortbildning/larare/matematiklyftet/matematiklyftet-1.178141> (in Swedish)

Swedish year one teachers' perspectives on homework in children's learning of number: an ongoing controversy

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This paper draws on semi-structured interviews undertaken with twenty teachers of year one children in Sweden. Interviews focused on teachers' construal of their own and their pupils' parents' roles in supporting year one children's learning of early number. Data, which were analysed by means of a constant comparison process, yielded homework as a theme that dichotomised teachers between those who set homework for learning number and those who do not. Of those who set homework, the majority construed it as a means of facilitating number-related fluency, particularly for children in danger of falling behind their peers. Of those who do not, the majority argued that differences in family backgrounds would compromise societal principles of equality of opportunity.

Much research effort in for example USA, England, China, Spain and Portugal has been expended on trying to understand the relationship between mathematics homework and student achievement (Cooper, Robinson & Patall, 2006; Farrow, Tymms & Henderson, 1999; Hong, Mason, Peng & Lee, 2015; Kitsantas, Cheema & Ware, 2011; Núñez et al., 2015; Rosário et al., 2015). Still the extent to which homework influences achievement remains unclear. Indeed, the results of such studies highlight continuous uncertainty about the efficacy of homework. This uncertainty, controversy even, underpins the narrative of this paper, which reports on an interview study of Swedish year one teachers' perceptions of the role of homework in their pupils' learning of number.

The controversy of homework

Homework is broadly construed as any task set by a teacher for students to undertake outside school or during, say, an after-school club (Corno & Xu, 2004; Muhlenbruck, Cooper, Nye & Lindsay, 1999). Its purposes have been categorised as being practice, preparation, participation, personal development, parent-child relations, parent-teacher communications, peer interactions, policy, public relations, and punishment (Epstein & Van Voorhis, 2001). With

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this follows equity aspects on homework design (Epstein, Foley & Polloway, 1995; Strandberg, 2013) where parents may have a crucial role in mathematics homework (Bryan & Burstein, 2004; Van Voorhis, 2004). Another example of the importance of homework design is Rosário et al. (2015), who in a pretest-posttest study compared the three homework designs extension (i.e. problem solving), practice and preparation and found only extension homework to improve students' mathematics achievement. Key in the definition and purposes is the role of the teacher in initiating the activity, the focus of this paper. However, while homework is ubiquitous in many countries, its use is often based more on rhetoric than warrant (Farrow et al., 1999). Indeed, with respect to the United States,

[n]o item on the nation's educational reform agenda seems more solidly grounded than the belief that students at all grade levels will benefit from more homework; indeed, the more the better.

(Gill & Schlossman, 1996, p. 28)

In this respect, drawing on earlier research in the field, Corno (1996) writes of five myths that underpin the use of homework. These state that the best teachers give homework regularly, more homework is better than less, parents want their children to have homework, homework supports what children learn in school and, finally, homework fosters self-discipline and responsibility.

Such rhetorical assertions, particularly when assumed true by politicians, create dilemmas for teachers and researchers. For example, the rhetoric of the government at the time, prompted the British Minister for Education, David Blunkett, to assert to a public meeting of British industrialists that some researchers are

so out of touch with reality that they churn out findings which no-one with the slightest common sense could take seriously [...] a report [...] recently suggested that daily homework is bad for you. If that is so, why is it such a firm part of provision in independent schools [...] ?

(Blunkett in Tymms, 1999, p. 22)

While it may be argued that the Minister's response fails to understand the class structure of the English independent schools and the disproportionate advantages they have afforded over decades (Croxford & Raffe, 2014), it seems clear that the mention of homework has the propensity to provoke heated debate between the holders of different viewpoints, some of which have empirical warrants and some not. For example, Farrow et al. (1999) explored the relationship between homework and achievement in mathematics, English and science of nearly 20,000 year six pupils in 492 English primary schools. Their analyses demonstrated unambiguously that learning gains in primary school were more closely related to less rather than more homework. Indeed, at the turn of the

twentieth century, to "most educational experts [...] the benefits of homework were anything but self-evident" (Gill & Schlossman, 1996, p. 29).

Homework in Sweden

The historical narrative of homework in Sweden has, in many ways, reflected the international. Thus, while the Swedish discourse on parental involvement in children's education has been strong, the perception of homework within that involvement has varied considerably (Wingard & Forsberg, 2009). For example, there are currently no legal expectations that teachers should set homework, with recent curriculum guidelines indicating that the responsibility for such decisions lie with individual schools and teachers (Skolverket, 2014). However, over the previous decades the official narrative has vacillated between homework being an essential element of the school experience to its being, effectively, forbidden before being rehabilitated into the mainstream of educational thinking (Hellsten, 1997). Today, official ambivalence has been highlighted by legislation allowing parents to receive tax deductions on tuition service they buy to support their children's home-based learning (Prop. 2012/13:14). That being said, earlier expectations that homework should not be set underpin current teacher perceptions, particularly with younger children, that homework is problematic. A not uncommon argument is that school is children's work and that extending it to the home creates

a stressed worker: homework blurs the boundaries between home life and school life, stealing time from children's leisure time, and puts unhealthy pressure on the ambitious students. (Forsberg, 2007, p. 213)

Consequently, homework remains a rarity for young Swedish children, as highlighted in every iteration of TIMSS in which Sweden has participated; Swedish year eight students consistently claim to receive less mathematics homework than their peers in other countries (c.f. Mullis, Martin, Foy & Arora, 2012). This leads to the following research question: What arguments do Swedish grade one teachers give for setting and not setting homework on learning number?

Methods

This paper reports on a particular set of results from an interview study of English and Swedish teachers' perspectives on number-related learning of year one pupils. Funded by the Swedish Research Council (Vetenskapsrådet), the Foundational Number Sense (FoNS) project is a comparative study of the role of parents and teachers in the support of year one children's acquisition of the number-related competences necessary for later mathematical success (Andrews & Sayers, 2015). The semi-structured interviews had two broad aims. Firstly, to yield constructs appropriate for inclusion in a later survey and, secondly,

to uncover in-depth the views of a representative selection of teachers on the teaching and learning of number to year one children.

Participants were contacted in various ways, including teacher electronic bulletin boards, emails and calls to randomly selected schools across the two countries. As a result, teachers were drawn from a range of geographical locations and represented different genders, ages, professional education and teaching experience. In each country, twenty interviews were arranged because it represents a number sufficient to ensure categorical saturation (c.f. Guest, Bunce & Johnson, 2006), whilst avoiding the ethical embarrassment of interviewing people whose data will never be used.

Interviews were conducted in teachers' schools and video-recorded directly onto laptops. Transcripts were made by the interviewers and analysed by the project team. In accordance with the aim of identifying constructs for inclusion in a later survey, a constant comparison analytical process was adopted (Strauss & Corbin, 1998). Thus, a transcript would be read and codes of response identified. When a new code was identified, previously read transcripts would be re-read to determine whether the new codes applied to them also or whether it could be refined against earlier evidence. To ensure consistency, the data for each country were analysed independently by two members of the project team before being compared. This process resulted in robust categorisations for each country. Homework emerged as an important theme from both sets of interviews and it is the results for the Swedish data that we report here.

Results

All teachers, irrespective of whether they actually set homework, were aware of at least three potential benefits homework can bring. Firstly, homework can act as a way of communicating with parents, so that, as Julia remarked "parents see what we are doing in school". Secondly, homework can provide children with opportunities to spend time with parents, as emphasised by Lena, who said, "that's why I think homework is important ... they (children) get a chance to sit down with their parents". Thirdly, homework develops good study habits, as seen in Pauline's comment that "I think homework is good because they (children) still think it's great fun. And they will have more of it when they grow older, so it's good to introduce it now". Such comments fit comfortably within the official discourse that parents have an obligation to involve themselves in the learning of their children (Wingard & Forsberg, 2009). However, acknowledging such generic benefits, teachers' views polarised between those who did set formal homework on number and those who did not. It is on these that we report.

Teachers who do not set homework

Ten teachers, exactly half the sample, spoke of not setting homework, either because they work in homework-free schools or because they have principled

objections to it. With respect to the former, teachers' responses were interestingly varied. For Jenny, her school's argument for not setting mathematics homework was pragmatic rather than principled. Having commented that her school sets reading homework, she said that "I don't know why, but somehow it feels, well, easier to involve [...] parents in reading and writing than in mathematics", before adding that during her regular development meetings, many parents "think it (mathematics) may be difficult to explain [...] that's probably what I hear, it's hard to explain". In other words, Jenny's views were that if parents were as consistent in their mathematical competence as in their reading competence, mathematics homework would be set.

For Irene, there was a different problem concerning over-enthusiastic parents trying to encourage their children to move beyond what the school is currently teaching. She commented that parents frequently ask

"Well, what can we do at home, ... what can we do?" And then I usually say, "No, you should not do anything" ... (and they ask) "Well, can we move on, can they work with multiplication, we have practiced multiplication at home?" It is so forced the belief that the more they practice at home the better they will be.

Finally, Marianne's comment indicated that while her school did not set homework (with exception for reading), it was a decision that left her uncomfortable. She asserted that "this is a homework-free school, which I do not support 100 percent", before adding that "I think that the home has a large role. Not to send home challenges, more to consolidate things like multiplication tables a bit ... homework should not be difficult".

From the perspective of the latter, teachers who opposed the setting of homework typically argued that differences in children's home environments influence learning in ways that make it difficult for schools to compensate. In this respect, Ellinor's comment was typical. She said, "It should not matter what you do at home, but it does. And we can never, within the school's context, weigh up for what children get at home". For the same reason Wilma's school decided to only set homework to be done during the leisure time activities organised by the school and Wilma motivated this as follows; "We do not send home things that need to be explained at home, it's we who teach – not the parents".

In other words, as noted by Julia, education "is supposed to be equal, we should all be given the same opportunities, and therefore we cannot put the learning responsibility on homes". Interestingly, as seen above with Jenny's comments, some teachers applied such principles of equality only when parental competence in pedagogically explaining mathematics was in doubt.

Two other teachers mentioned different but principled reasons for their rejection of homework. Firstly, Hanna, sets homework only for children with specific needs of individual support, explaining,

instead of having a lot of homework for all, I (attend) only to this girl (who struggles), and these parents, and ask them to do something at home. And focus on it. If they had more homework beyond that, then it would be very difficult.

Secondly, in contrast to the strongly-made arguments of Marianne, who regretted her school's decision not to set homework, Kerstin said, quite simply, that she knew of no convincing research showing that homework is beneficial to students' learning, commenting that "there is nothing that shows that you do better at home than you do at school". Consequently, she avoids setting homework to her students.

Teachers who set homework

Of those teachers who set homework, three reasons dominated. The first concerned homework as the consolidation of routine competences, as discussed above by Marianne. Here, homework is explicitly an opportunity to reinforce learning which has already taken place in school. Indeed, as Anders pointed out, "homework (should only) be a repetition of a lesson". In similar vein, Isabelle commented that mathematics homework should be "more practice at home only" and "nothing new", while Lena saw it as focused on practising "something that should be automated" and something that students are already confident in and can complete without parental support. In other words, while there was a reasonably strong group of teachers who set homework, their enthusiasms were underpinned by the same principles by which other teachers rejected homework; namely that it should not involve parents whose actions may compromise equality of opportunity.

The second reason, despite differences of opinion with respect to the general desirability of homework, all teachers seemed to agree that homework is an important strategy for supporting those children who are falling behind or struggling academically. In such situations, as highlighted by Lena, "when you have a student who has difficulty with something, then you have to contact the parents and give them things ... they can practise at home ... like number bonds". Hanna used the same argument to set homework only to those with special needs.

Teachers' third reason for setting homework concerned the ways in which it contributed to the communication between school and home, particularly from the perspective of parents learning about what their children are currently doing in class. In this respect, Erika's comments were not dissimilar to those of others. She said that,

they have a maths homework each week and have had it since they started in year one. And they have from Friday to Friday ... so that the parents get a little insight into what we are doing right now. [...] And I as a parent myself think it's great and fun to see what my child is working with, and I think most parents like that too.

In similar vein, Lovisa commented that "we have mathematics homework ... something we have done in school (and) they go home and show their parents and ... think it through again".

Results summary

All teachers acknowledged that differences in the home environment could compromise principles of educational equality, which influenced how they perceived the place of homework. Some teachers and schools rejected homework altogether, while others seemed to adopt a compromise position whereby homework would be a simple consolidation of number routine skills that required no parental involvement. That being said, the majority conceded that children who struggle with their learning of number should be given homework to be undertaken with their parents.

Finally, there was limited evidence that some teachers, irrespective of their articulated position with respect to the role of formal homework, believed that parental engagement in informal activities at home would help their children's learning of number. Ellinor, for example, continuing to express her concerns about the variable impact of the home environment, described how she and her colleagues advise parents to follow-up some school-based activities. She said,

Although we can never even up what children get from home, in the meantime, we usually write tips such as: ... today we worked with volume, you are very welcome to go bake at home ... please play cards, please play dice games.

Such views, although not frequently mentioned, indicated a belief that informal activities can not only support learning but also facilitate the development of both good study habits and contact between parents and school.

Discussion

In this paper, drawing on interview data from 20 Swedish year one teachers, we set out to examine colleagues' perspectives on the role of homework in children's learning of number. The analyses found a profession divided between those who do set mathematics homework and those who do not, although this simple dichotomy, as exemplified in the comments of Marianne who is compelled by her school's policy not to set homework when in principle she would prefer to do so, belies a more complex narrative. In this respect, there was much commonality of belief about homework's potential benefits and desirable practice.

From the perspective of the former, teachers were aware of at least three potential benefits homework can bring. Firstly, homework can act as a way of communicating with parents (Van Voorhis, 2004), so that, as Julia remarked "parents see what we are doing in school". Secondly, homework can provide

children with opportunities to spend time with parents (Epstein & Van Voorhis, 2001), as emphasised by Lena, who said, "that's why I think homework is important ... they (children) get a chance to sit down with their parents". Thirdly, a view that Corno (1996) regards as a myth, teachers believe homework encourages the development good study habits (Epstein & Van Voorhis, 2001), as seen in Pauline's comment that "I think homework is good because they (children) still think it's great fun. And they will have more of it when they grow older, so it's good to introduce it now". Such beliefs about homework's potential benefits fit comfortably within an official discourse in which parents have an obligation to involve themselves in the learning of their children (Wingard & Forsberg, 2009).

From the perspective of the latter, and resonant with the findings of the studies reviewed by Bryan and Burstein (2004), teachers were unanimous that homework is appropriate for those children who struggle with their learning of number. However, with the single exception of Hanna, teachers, whether they valued homework or not, believed that there was no role for parents in its completion. Arguing from an equity perspective, the completion of homework, which should be designed to be achievable without additional support, should be the sole responsibility of the child (Strandberg, 2013).

Despite such commonality of belief, actual practice polarised colleagues, with half setting homework, believing it to be both desirable and necessary, and half not, believing it to be counter-productive and divisive. Of the teachers that did set homework, the dominant argument was that it serves as a consolidation of number routine skills, practices exploited by teachers internationally (Cooper, Robinson & Patall, 2006). What is, perhaps, interesting, is that internationally, researchers have reported on a form of homework, which draws on a tacit awareness that the school day offers insufficient time for teachers to cover all curricular material and is frequently found in exploratory tasks designed to prepare pupils for forthcoming learning (Epstein & Van Voorhis, 2001). In this respect, not only was there no evidence of Swedish teachers using such tasks but some, as in Ellinor's assertion that she does "not send home things that need to be explained at home", who see such material as promoting inequality, in that not all children can get the same support from their home (Forsberg, 2007). And while absence of evidence is not evidence of absence, the fact that twenty teachers failed to discuss such homework seems to pose fairly strong evidence that it is not commonly found in Swedish year one classrooms. This principled argument takes us to those teachers who do not set homework.

For these teachers, with the exception of Marianne, homework undermines a deeply-held principle that their role was to prevent differences in family background undermining equality of learning opportunity (Epstein, Foley & Pollo-way, 1995). This sense of equity, which was strong among all teachers and not just those who did not set homework, is further reflected in official statistical

evidence showing that parental involvement in Swedish children's homework is typically limited to no more than five minutes a day (Forsberg, 2007). Interestingly, this dominant issue of equity, which has deep-seated historical roots (Hellsten, 1997), appears to be completely missing from the international literature and identifies Sweden as unique in its principled resistance, to homework. In other words, for many Swedish teachers, unlike their colleagues internationally, homework is not the job of childhood (Corno & Xu, 2004).

Acknowledgements

The authors are grateful to Anna Löwenhielm for her significant contribution to the collection, transcription and initial analysis of the Swedish data.

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Swedish teachers talk about school algebra

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In this paper, we report the first results from an explorative multiple case study on how Swedish compulsory school teachers talk about algebra in school mathematics. Focus group interview data (9 schools, 41 teachers) were studied mainly through open thematic analysis. The analysis revealed several interesting strands for further and deeper studies: teachers' understanding of algebra and its place in school mathematics, teachers' feelings towards algebra, resources and support, and teachers' openness for change.

Algebraic thinking seems to be a cornerstone in pupils' progression towards mathematical proficiency (e.g. Kieran, 2004). This means that the teaching tradition in, for instance, early algebra and algebra is important for pupils' learning outcome. Due to recent research findings, many countries have included algebra in national guidelines and curricula from the early school years (e.g. Hemmi et al., 2018). This is also the case in Sweden (Prytz, 2015). Besides curricular directives and objectives, there have been various attempts to improve algebra teaching in Sweden through in-service training projects for teachers (e.g. *Boost for mathematics*) and in teacher education for some decades (see for example Bergsten et al., 1997, a textbook included in teacher education in several Swedish teacher education departments). However, it is not possible to discern a general positive effect of these efforts on Swedish pupils' learning in algebra (Hemmi et al., 2018). In the current national mathematics curriculum, implemented in 2011, algebra is included as a central content in all school years. By combining more outspoken content with general subject competencies (e.g. mathematical reasoning and communication), the aim is to alter the emphasis in mathematics teaching.

In a four-year project funded by the Swedish Research Council, we and our colleagues investigate the complex issue of implementing algebra in school mathematics (years 1–9). We know that it is hard to change the practice without considering the prevailing traditions (Hiebert et al., 2005). We attempt to find

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possible reasons for the failure in increasing the quality of algebra teaching in Sweden (see e.g. Hemmi et al., 2018), by examining how algebra is traditionally treated in different *arenas* in the Swedish school system. In the project, the school system is regarded as stratified into the levels of 1) formulation arenas, comprising the group of people who decide content in policy documents, for example curricula, but also people producing textbooks, and 2) realisation arenas, comprising the teachers who interpret texts and design and carry out lessons (cf. Lindensjö & Lundgren, 2000). Our particular interest is to reveal how more or less tacit traditions in teaching practices are related to the curricular intentions and identify possible mismatches, thus contributing to the future development of algebra teaching at different levels of the education system.

In this paper, the focus is on the realisation arena. We explore how Swedish teachers in different schools talk about the place of algebra in school mathematics and their own relation to algebra and teaching of algebra.

Relevant literature

In many countries, algebra was traditionally postponed until adolescence partly because of the earlier assumptions concerning child cognitive development, and partly because of the parallels made between the learning trajectories of pupils and the history of mathematics (cf. Carraher, Schliemann, Brizuela & Earnest, 2006). However, recent studies show that it is possible and even beneficial to start working with algebraic ideas and generalisations in parallel with arithmetic already in early grades (e.g. Blanton et al., 2015; Cai & Knuth 2011; Carpenter, Franke & Levi, 2003; Carraher et al., 2006). Carraher et al. (2006), for example, suggest that pupils in the lower grades can make use of the algebraic ideas and representations usually absent from the early mathematics curriculum as these are considered to be beyond pupils' reach.

The idea of early algebra is to facilitate pupils' progression towards and understanding of more formal algebra. Scholars agree that algebraic thinking in early grades should reach beyond arithmetic and computational fluency in order to focus on the underlying structure of mathematics (Cai & Knuth 2011). Blanton et al. (2015) found that young pupils (i.e. 9–10 year olds) are capable of engaging successfully with an extensive and diverse set of algebraic ideas ("Big ideas") comprising Equivalence, expressions, equations and inequalities; Generalized arithmetic; Functional thinking; Proportional reasoning; and Variables (see Bråting, Hemmi, Madej & Røj-Lindberg, 2016).

In Sweden, Attorps (2005) has studied secondary school teachers' conceptions of teaching equations in compulsory school. She found that teachers stressed the procedural side of equations rather than the central ideas and concepts of algebra. Nyman and Kilhamn (2014) have investigated how teachers introduce algebra to year 6 pupils. The teachers' in their study mostly focused

on activities and social interaction in order to engage the pupils in mathematics. The algebra content itself was in the background. International studies on how teachers interpret curriculum intentions in algebra cover issues such as teacher concerns about the new algebra curriculum with the goal to teach all pupils algebra (Crawford, Chamblee & Rowlett, 1998), and teacher attitudes to changing algebra teaching towards more problem-based working manners (Agudelo-Valderrama, Clarke & Bishop, 2007).

Crawford et al. (1998) have shown that it is important to offer teachers long-term support regarding the content and the methods of implementation in order to influence their beliefs and pupil learning outcomes. Regarding teacher attitudes, most teachers in the multiple case study of Agudelo-Valderrama et al. (2007) showed negative attitudes towards change and these attitudes were connected to their poor self-efficacy and to external factors that, according to the teachers' views, hampered their attempts to change their teaching. However, there are studies showing that teachers are willing to change their teaching when they see that pupils' results improve as consequence of the change (e.g. Guskey, 2002). Moreover, recently, the role of supportive artefacts and tools have been raised as important factors for improving teaching and teacher change (e.g. Cobb & Jackson, 2012; Remillard & Bryans, 2004).

Method

Our study of teachers' relation to school algebra is based on focus group interviews, as this method is suitable for revealing meaning-making in groups of people (Gulliksen & Hjordemaal, 2014). The method provides "access to participants' own language, concepts and concerns" (Wilkinson, 1998, p. 188) and rests on the assumption that meaning-making is created in the interactions between people.

Nine groups of teachers from nine compulsory schools participated in the data collection. The interviews lasted for 1.5–2 hours. In all, 40 certified teachers, and one uncertified (i.e. MSc degree in engineering), for years 1–9 (mean 15.6 years of teaching experience, $SD=9.4$) participated in the interviews. The sample is neither stratified, nor random. However, we have controlled for diversity in school locations in different socio-economic areas, and had different levels of newcomers. School sizes differ from 150 pupils up to 600. We aimed to reach theoretical saturation, that is, collect data until no further information is revealed on the issues raised in the interview. The focus group interviews were therefore spread out in time (9 months).

An interview guide contained 14 open questions steering the conversation into two themes: 1) what is algebra/early algebra in the school, and 2) what mathematical tasks are suitable for teaching algebra (at some specific school level). In the second theme, we used tasks from Blanton et al. (2015) in order to

cover the big ideas to be developed throughout the school years (see Bråting et al., 2016). Moreover, we selected tasks from mathematics textbooks and tests to investigate how the teachers related to specific aspects of algebra to be able to characterise the typical Swedish teaching tradition in the realisation arenas. Two pilot interviews were conducted in order to refine the interview guide. The interviews were audio recorded and transcribed.

The study is conducted in accordance with the ethical guidelines for the Humanities and Social Sciences set out by the Swedish Research Council. For example, written consent was collected from all participants, and audio recordings and other data are handled in accordance with guidelines and regulations in order to maintain confidentiality.

Analytical approach

We used both deductive and inductive approaches in the data analyses. The initial thematic analyses were conducted with NVivo software using both a priori categories concerning big ideas and the specific aspects of algebra teaching and learning (Blanton et al., 2015), and an open approach to capture items that may be invisible in the documents or in previous research (Braun & Clarke, 2006). As our aim from the beginning was to explore teachers' views of expected student progression throughout the years 1–9, we analysed their responses to questions connected to it. We also used Blanton et al.'s big ideas as a-priori categories when analysing this question.

Several a-posteriori categories evolved from the inductive thematic analysis (Braun & Clarke, 2006), which was informed by the analytical questions raised by Little (2002).

- What is made visible? (topics, materials, planning, etc.)
- What parts of practice become described, demonstrated or rendered? (i.e. the face of teaching that is available as a resource for development)
- The portrayal of teaching: simple–complex; static–dynamic; certain–ambiguous.

The analysis aimed to reveal the representations of practice, and the meaning-making in the teachers' talk with a focus on how they relate their praxis to the steering documents, and how they talk about their interaction with curricular material on algebra.

Results

We start the reporting the results of how teachers talked about algebra and its place in school mathematics. Then we present the main themes from the inductive analysis. The themes are somewhat overlapping and intertwined. We exemplify the results with extracts from our data.

Teachers' understanding of algebra and its place in school

The teachers recognise the place of early algebra in compulsory school and they talk about the importance of introducing algebraic thinking in the early school years. Concerning the first big idea (connected to equations, expressions and equalities/inequalities), the teachers stressed that understanding the equals sign and simple equations is especially important in years 1–3, but also highlighted it as an important and difficult topic to continue working with in the following years.

Teacher 3: But, what I feel about algebra [teaching] is the huge importance of the equals sign. As long as they're not sure of what it stands for there's no point in starting to work with algebra. That's the tilt and balance we work a lot with in the 4th year.

[Primary teacher in school 8 talking about algebra]

Teacher 1: The equals sign is a huge obstacle.

Teacher 2: they don't know that ...

Teacher 3: ... or actually they do know, but they cheat.

Teacher 2: yes, they put in different values after the equals sign and just continue counting. "It becomes".

Teacher 3: It still is "it becomes".

[Lower secondary teachers in school 2]

Working with patterns sometimes followed by a formulation of a rule (functional thinking) is another item that the teachers in all of the focus groups mentioned when discussing algebra at different school levels.

Teacher 7: Right now, I work with the youngest, so it's not so frequent that we work with advanced stuff. But to turn what we see as a pattern into a general rule, that is, I feel, not at all easy. I don't know it properly. I'd like to have curricular material showing how one could get it into the thinking, sort of.

Teacher 4: I agree. I often think that it's the hardest ... Our sixth-year pupils, it's the thing that is most of an obstacle to them, to generalise. It's easy-peasy to build the next figure, and they could build the nth one. But to write it down, as a rule, that's hard, really hard.

[Primary teachers in school 1]

The teachers' discussions seldom focused on variables or generalized arithmetic.

Teachers' feelings towards algebra

The first major theme that emerged from the inductive analysis concerns teachers' feelings towards algebra. Especially the teachers who lacked formal training for teaching mathematics, talked about their unease working with algebra in their classes.

Teacher 2: ... I probably shouldn't teach algebra [laughs].

Teacher 1: I feel so too [laughs].

[Primary teachers in school 3]

This feeling of unease towards algebra had an impact on how the teachers talked about their pupils' algebra learning. The teachers often used terms such as "hard to learn" or "don't want to frighten them" when discussing the introduction of early algebra.

However, teachers that expressed self-confidence regarding mathematics often used terms such as "challenging", "interesting", and "fun" when talking about early algebra.

Teacher 4: ... yes, it's something that is challenging it's, somehow, mathematics, kind of, now we are talking mathematics for real. We are dealing with algebra [...] and it's often about finding something unknown, something we don't know from the beginning, and the pupils think it's fun, it becomes some kind of riddle and problem solving.

[Lower secondary teacher in school 9]

Resources and support

This theme comprises teachers' collaborative work, support from local school authorities, textbooks, curricula and national tests, which they used as resources in their teaching, and in their professional development.

Teacher 3: For me, a good teacher's guide is where I can learn what to pay attention to ... what can be difficult for the pupils, that is, [advice like] "observe if the pupils manage this" or "observe what they answer to this task". Kind of what to do more than the stuff in the textbook. [...]

Teacher 5: and perhaps an explanation: "you need to do this or that otherwise this kind of obstacle occurs". Of course, we know this but it's good to have it in front of you so as not to forget.

[Primary teachers in school 1]

The teachers discussed the need for more extensive teacher's guides: "I would like to have a curricular material that shows ...", or the need to develop their own teaching material as they feel that the textbooks do not fulfil the expectations of varied mathematical tasks.

Teachers from schools where local support from the school authorities was described as structured (i.e. guidelines for assessments and local learning outcomes), and where the teachers cooperated within and between schools (i.e. from primary to lower secondary level), talked more specifically and the offered resources seemed to have a clear role in their teaching.

Teacher 2: Yes, we also work with ... we have an assessment tool in the municipality [for all grades] ... and mathematical tasks we developed. We have extra material not only the textbooks ... and we share problem solving tasks with each other

[Lower secondary teachers in school 2]

Teachers from schools where the local support from school authorities as described as inadequate (i.e. unclear local goals, and absent plans for compulsory school as a whole, and low attendance in professional development programmes), talked more tentatively. The excerpt below shows how the teachers display a more uncertain picture of available resources and artefacts, and, hence, an uncertain feeling towards algebra teaching and their collaborative work is also displayed.

Teacher 3: Often, I think, that we lack this, what we expect of the pupils when they leave the 3rd year... and also the other way around, what I can expect when the pupils come to me in the 4th year. What have they done? What skills do they have? I can't start where the pupils haven't yet arrived. And yet, we [the teachers] are often in the same building ... what about when the pupils leave for the 7th year [lower secondary level], and they change to a new school. Then it's even more difficult to know what we as teachers can kind of expect from one another.

[Primary teacher in school 4]

Openness for change and interest in developing algebra teaching

The teachers were, overall, open for change and interested in developing their algebra teaching. We can see in their talk that different artefacts are used as support for change, namely national curricula; tools for assessment (national, local); pre-service, and in-service education; curricular material (textbooks, teacher guides, both digital and analogous). The teachers in several schools described a development of their teaching from the approach of pupils mostly working with textbooks to teaching with hands-on material.

Teacher 4: I recognise that [...] I also try to work a lot hands-on and show ...

Teacher 1: Well, you also have to recognise this hands-on material and practical maths, and all kinds of maths that we are doing ...

[Primary teachers in school 4]

However, when the teachers talked about goals this was done in general terms with a focus on how they changed (or wanted to change) the teaching without specifying what pupils should actually learn, that is, the "how question" and not the "what question". Mainly, the direction of change is towards more communicative teaching.

- Teacher 3: The textbook is designed so you could have a collective math-talk each day. Building on these ... principles. [...] and they speak a lot of the importance that the children talk with each other about mathematics [...]
- Interv.: Do you work like that often in your classroom?
- Teacher 3: No (laughs). We would like to, but aren't there yet. I think it's a goal [...] more material, math games, and working together.
- Teacher 2: I try to. The pupils I work with. A material called button-maths, algebra, that is ... you use the buttons to represent missing values. Then you always do it in pairs, so they have to talk to each other. I'm also planning to do more stuff like this. To work with mathematical tasks in groups. It's my forthcoming project. I think it's important to talk ... we need much more math-talk ...
[Primary teachers in school 5]

Discussion

The results from the initial analysis of the focus group interviews show several interesting strands to study further. We focused on how teachers relate their praxis to curricula, and how they talk about their interaction with curricular material. We were able to reveal, to some extent, how teachers meet the demands of introducing early algebra in school.

Algebra is by tradition vaguely addressed in the Swedish curriculum (see e.g. Prytz, 2015). This means that the teachers need other resources and support to do their jobs. Consequently, the teachers' talk display that textbooks, teacher guides, etc., in-service teacher education, and working together with colleagues (both in the local school and across school levels) are important sources for teaching development. This is in line with several studies pointing to the important role of supportive artefacts and tools for improvement and teacher change (e.g. Cobb & Jackson, 2012; Remillard & Bryans, 2004). The teachers described their struggle with choosing and constructing suitable tasks, and suitable classroom settings, and thereby problematized how curricular materials work as resources. This may be connected to the available resources both in terms of teacher knowledge and local support, but also the kind of textbook used. The question then is: How can we think about the resources teachers need in order to support teachers who have different understandings of mathematics and teaching mathematics?

Overall, we see an openness for change, despite the expressed unease working with school algebra. This is a somewhat different picture than previous studies show (e.g. Agudelo-Valderrama et al., 2007). Our interpretation is that the teachers trust the message advocated in curricula, in-service teacher training, and by researchers, to introduce algebraic thinking in early grades. This is clear in the teachers' talk about, for instance, the big ideas (Blanton et al. 2015). However, the need of local support regarding both methods and content is also apparent (cf. Crawford, 1998) as the teachers speak in very vague and in general

terms when discussing the goals of algebra teaching. In line with Nyman and Kilhamn's (2014) study of year 6 teachers' design of algebra lessons, our study shows that the content-related issues of teaching need more attention. This is also pointed out by Attorps (2005) concerning lower secondary teachers' conceptions of teaching equations. We know that the enacted norms of professional practice open up or close down particular considerations for teaching (Little, 2002). This implies a need to shift focus from the "how question" to the "what question" when developing algebra teaching.

Acknowledgements

This work was supported by the Swedish Research Council.

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Teaching finger patterns for arithmetic development to preschoolers

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In this paper we describe the theoretical idea behind an intervention programme that included 65 five-year-olds and teachers at seven preschool departments in Sweden. The primary aim of the programme was to investigate how finger patterns could be taught to facilitate the development of preschool children's ways of experiencing the first ten natural numbers. Informed by Variation theory of learning, we developed teaching activities and artefacts to promote children discerning necessary aspects of the first ten numbers by using their fingers to construct numbers as part-part-whole relations. In this paper we discuss the theoretical background in relation to expected learning outcomes and the educational context of mathematics learning in Swedish preschool.

This paper derives from a research project FASETT (funded by the Swedish National Research Council), which aims to study preschool children's number sense and how a pedagogical approach implemented in an intervention programme might strengthen children's developing knowledge of numbers and emerging arithmetic skills. One of the main research questions in the project is: How can the ability to experience numbers' manyness and part-whole-relations be developed using a pedagogy based on specific empirical and theoretical insights? In this paper we describe the empirical and theoretical idea behind the intervention conducted in the project and in particular how finger patterns were used as means to facilitate the intended way of experiencing numbers.

The intervention programme is founded in empirical insights made by Neuman (1987) in her studies of school-beginners' number sense and origin of arithmetic skills. Her findings indicate that children experiencing mathematics difficulties more often use inefficient strategies involving a strong dependence on finger counting strategies that do not lead to learned number facts. More recent research within the cognitive and neuro-scientific field emphasise the

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relation between using fingers in arithmetic reasoning and successful arithmetic skills (Berteletti & Booth, 2016; Moeller et al., 2011; Butterworth, Varma & Laurillard, 2011). However, Neuman claims that it is not finger use in itself that leads to deficits in arithmetic problem solving, but some ways of using fingers hinder conceptual development of numbers while others support number fact acquisition based on numbers' part-part-whole relations and other critical aspects of number. In our project we take up on the idea, that it is not finger use per se that promotes or deficits development; we investigate how finger patterns used to represent a conceptual understanding of numbers as composite sets, which cardinal value can be perceived rather than counted one by one, might promote arithmetic skill development.

Through recent advancements in pedagogical theory (Variation theory of learning, see Marton, 2015) about concept learning and teaching, we aimed to put a bold conjecture to the test. It is well known in the research literature that cardinality, ordinality, commutativity and the inverse relation of addition and subtraction are important principles to assimilate in order to solve arithmetic tasks (Fuson, 1988; Baroody, 2004). We conjecture, that the ability to experience the meaning of numbers in terms of numbers' manyness and part-part-whole-relations as primary to counting, would improve young children's readiness to encounter arithmetic problem solving situations. In line with earlier and more recent findings of the power of finger use for arithmetic problem solving, we hypothesised that learning to use finger pattern sets to represent the part-part-whole-relations of numbers would provide children a necessary strategy to discern the relations and furthermore to model novel arithmetic problems as a coherent task, constituting a part-part-whole-relation rather than a counting procedure. This conjecture was implemented in an intervention programme and conducted together with preschool teachers and 5-year-old children in preschool practice.

Using fingers in preschool arithmetic learning

There is a large field of research on early arithmetic learning and rising awareness of the early childhood learning experiences' effects on later development in mathematics (Duncan et al., 2007). Many studies have described children's strategies in arithmetic problem solving and conclude that fingers are often used to aid the counting strategies (Fuson, 1988; Berteletti & Booth, 2016). The relation between numerical reasoning and finger knowledge earn ever more evidence from the neuro-scientific field, a relationship which is observed long after a person has stopped using her fingers in arithmetic problem solving (Rusconi, Walsh & Butterworth, 2005). One could say that number sense is related to finger sense and Rusconi et al. even conclude that finger calculation is "an almost universal stage in the learning of exact arithmetic" (pp. 1610).

Fingers come in use either to count *on* or to count *with*. The conceptual difference is described for example by Sinclair and Pimm (2015), as the former

means that fingers can be counted as items and the latter as a representation of an arithmetic operation. Fuson (1988) described early on the strategies children adopt as they need to keep track of counted addends that exceed the subitizing range. Commonly, children successively extend one finger for each number word said. Fingers then represent the counted words – they can be counted *on*. Fuson and Secada (1986) have done efforts to teach children strategies, for example finger counting patterns where each added unit is represented by a specific finger pattern. Still, the arithmetical process heavily relies on counting-on and counting-up as the primary strategy for arithmetic problem solving in their intervention.

There is a distinct difference between using finger patterns in the way Fuson and Secada (1986) tried out and to use finger patterns as representations of part-part-whole relations, as our intentions are in the research project at hand. The latter allows a flexible use of fingers, enacting both cardinal and ordinal meaning of numbers and emphasizes the whole rather than the differentiated parts of a whole. Brissiaud (1992) advocated a pathway to developing number knowledge by using finger patterns to represent numbers, without counting. This approach builds on the intuitive ability to perceive a small number in an instant (subitizing) which can be developed towards recognizing patterns of sub-sets in larger sets without counting, so called conceptual subitizing (Clements, 1999). This supports the perception of number structures in arithmetic tasks (Baroody, 2004). Mulligan and Mitchelmore (2009) hold that awareness of mathematical patterns and structure is a general feature of young children's cognition that predicts their later mathematical achievement. Children's mathematical development is, thus, not only based on numerical competences like knowledge about numbers, numeration and quantities, but also on non-numerical competences like structure sense.

Based on the above described theoretical and empirical findings we developed the FASETT project's intervention programme that would implement the structure framework including numbers' part-part-whole relations and meaning of "manyness" through a specific way of using finger patterns in preschool mathematics education. The FASETT project introduces a pedagogical approach that to some extent stands in contrast to the traditional pedagogical preschool practice. Our approach implements a demarcated learning object and systematic activities designed to develop certain skills and knowledge. Preschool curriculum in Sweden (Swedish Agency for Education, 2011) does have goals for mathematics learning to strive for, such as the basics of number concepts, quantity and order, but does not provide any directions for how to teach or stimulate this. The designated way of teaching is directed towards children's interests and initiatives, which are elaborated and extended in thematic and play-oriented practice. The purpose of this paper is thereby to highlight and discuss the theoretical framework of one intervention programme that is aiming to enhance preschool children's number sense and basic arithmetic skills, as

a contribution to the scientific base for early childhood education in Swedish preschool stipulated in the Education Act (SFS 2010:800).

Methodological framework

The intervention programme is informed by Variation theory of learning, a pedagogical theory that describes learning as differentiation. To learn means to become able to discern relevant features of the surrounding world, which means there are different ways of experiencing this world. In order to discern a feature, variation in regards to the feature is needed. Hence it has to be varied and other features need to remain invariant. Teaching is consequently an act where aspects of phenomena (learning objects) are brought to the foreground and related to each other in patterns of variation and invariance. The theory has been used to develop teaching in systematic investigations of how learners experience a learning object and ways of teaching that support the learner in discerning necessary aspects of that learning object (Marton & Tsui, 2004; Marton, 2015).

The theoretical framework is used both to design potential patterns of variation and to analyse the learning opportunities that emerge in the iterative process of the intervention programme. This is described in terms of what aspects of numbers that are made possible to differentiate by the children. Based on earlier research and theory (see earlier overview in this paper), some aspects of numbers are more important to discern than others, and will therefore become objects of learning in the intervention programme.

The intervention programme is characterized as field experiments due to the design in which we implement learning activities and artefacts in natural settings (Salvin, 2010), and design experiment, in that the intervention is iterative, process focused, collaborative and theory driven (Cobb et al., 2003). This allows us to try out activities and artefacts in action with preschool children and evaluate the learning opportunities and outcomes in order to further refine the acts and design and thereby our understanding of children's number sense development.

The study

Seven preschool departments with 65 five-year-olds and their preschool teachers participated in the intervention programme during eight months. Video documentations of the activities formed the basis for meetings with the research group (bi-weekly) and for the collective work of refining the activities and artefacts. The choice of activities and artefacts followed two criteria: first there had to be some elements that could facilitate the necessary aspects to be discerned, second the activities had to appeal preschool practice (such as games, play or narratives).

The pedagogical approach adopted in this project is based on Variation theory of learning, which indicates that the learning object is essential to establish and

to constantly be directed towards. A functional number concept as learning object constitutes several aspects that are necessary to discern, some of which we had anticipated due to earlier research and theory (Marton, 2015; Neuman, 1987; 2013). Subsequently, during the intervention, we found other aspects emerging to be critical to some children (see below). The teachers conducted the intervention activities in accordance with the children's previous experiences and aspects that were found to be critical for every individual child. Analyses of the critical aspects and how to develop the activities were done in close collaboration between teachers and researchers in meetings held bi-weekly.

In the following we will present the intervention programme's four main activities and discuss their learning potential in accordance with the theoretical framework. More explicitly, we direct attention to the question *what did it mean that the teachers taught finger patterns to the preschool children?* Considering the strong theoretical foundation for the designed activities, this discussion will contribute to the scientific base in early childhood mathematics education.

Activities in the programme and the theoretical contribution to learning

Four main activities were developed and executed within the project during eight months, focusing finger patterns enacted to facilitate children in experiencing the meaning of the first ten natural numbers: a) the statement game, b) the five/ten-snake, c) finger patterns, and d) context tasks.

The statement game

The statement game (inspired by Sensevy, Quillo & Mercer, 2015) is played with two or more participants using a die and fingers. The game brought for two aspects that were found to be necessary for all the children to discern: *Numbers can be represented with fingers*, and *Numbers can be composed and de-composed in different ways*. Some of the children did not in the start of the programme experience the cardinal value of numbers and consequently it did not make sense to them to use fingers as a representation of numbers. To develop a fluent use of finger patterns, it was concluded important not only to be able to illustrate numbers with specific finger patterns, but to experience numbers as compositions of parts and that the number illustrated with fingers mirrors this experience of numbers. In other words, a "six" on the die would by the majority of children be illustrated as one whole hand and one more finger raised on the other hand (preferably the thumb). The statement game challenged this, by first having the children to think out a number they thought the die would show. Second, they were encouraged to show on *both* of their hands the number they thought out (which would make them de-compose the finger pattern and create a two-part pattern). Furthermore they needed to show the number in a *different* way than the other participants. Third, they threw the die to see if they were correct in their guessing (see figure 1).



Figure 1. *The statement game enacted with finger patterns using both hands as 4+1 and 3+2*

The game directs the children’s attention towards numbers represented with fingers and how it is possible to compose the same number in different ways. The first aspect, that numbers can be represented with fingers, is necessary for the other dimension of variation to open up (to compose and de-compose numbers). Both are promoted in the statement game (see table 1).

Table 1. *Dimensions of variation in the statement game*

	Number	Finger pattern
Numbers can be represented with fingers	Varies	Invariant
Numbers can be composed and de-composed in different ways	Invariant	Varies

5-snake and 10-snake

The aim of the intervention programme was in particular to enhance children’s ways of experiencing numbers’ manyness and part-part-whole relations. These aspects were introduced in the statement game but one important dimension of variation was not emphasised: the importance of relating parts to the whole. To highlight this we introduced a second game: 5-snake and 10-snake. This game is known in Swedish preschools and can be found in different layouts, in our programme a string of five pearls in similar colors. The teacher hides some of the pearls in her hand, leaving the rest visible on the string outside her hand and the child is to find out how many are hidden. To enable the child to discern the part-part-whole relation of five, the child was encouraged to represent the whole (5) with her hand, then look at the visible pearls (for example 3) and model the same three on her hand (most likely by folding down two fingers). The child has now created a similar two-part structure as in the statement game, but in this game the parts’ internal relationship and to the whole is emphasised simultaneously (see table 2). The use of finger patterns aided the child in representing numbers as parts and their relation to the whole, which deliberately was chosen to be commensurable to one hand (or two in the larger snake).

This game furthermore allowed for another supporting dimension of variation, when introducing the 10-snake constituting five pearls in one color and five pearls in another color: the semi-decimal structure (see figure 2). The teacher was now able to point out the ”fiveness” of pearls in similar colors and the same ”fiveness”

Table 2. *Dimensions of variation opened up in the snake-game*

	Whole	Parts
Part-part-whole relation	Invariant	Varies



Figure 2. *The snake-game emphasises a semi-decimal structure and while enacted brings fore the part-part-whole relation*

on the child’s hand, which the child does not have to count, rather ”know” there are five and perceive also larger parts (through conceptual subitizing).

Finger patterns

The snake-game seemed to provide an eligible strategy to handle the relation of parts and the whole of five or ten. The general intentions were to enable children to perceive numbers’ manyness and part-part-whole relation, in that it would facilitate their developing arithmetic skills. Attention thereby had to be brought towards how to handle number relations in a fluent and flexible way concerning sums different than five and ten. This activity was influenced by Neuman’s (1987) finger patterns.

The teachers introduced finger patterns constituting a semi-decimal structure (the whole hand for 5 and adding fingers in consecutive order on the other hand to represent 6, 7, 8 and 9). The task was to name the finger pattern shown with a number word, to extend the subitizing range by enacting on the semi-decimal structure. If the teacher showed 7 (5+2 fingers), the teachers would then ask how many more fingers have to be raised to get 9, then showing the suggested addition and check whether they ended up showing nine as a finger pattern of 5+4. The teacher would then reverse the task and ask how many she would need to remove to get seven again, and so on (see table 3). This task added an arithmetic component, in that children would discern ”the 7 and 2 in the 9”. Since the finger patterns were invariant in the presented structure, it was made possible for the children to discern the relations in for example 7/2/9 and to make use of the semi-decimal structure.

Table 3. *Finger-number relations aiming towards understanding arithmetic principles that rely on part-part-whole flexibility*

	Number	Finger pattern building on semidecimal structure
Finger-number relations	Varies	Invariant

Context problems

Critical for developing arithmetic skills is the ability to model a task as a mathematical task, rather than an empirical. Most preschool arithmetic problem solving occurs in context problems where familiar situations frame an addition or subtraction. The programme aimed to develop children’s perception of numbers with the help of finger patterns as a way to model and abstract the meaning of numbers. Context problems were therefore introduced as the last activity, in which the structured finger pattern use would aid the child in discerning the mathematical aspect of the tasks. The part-part-whole relation of numbers was focused through short stories, such as ”eight bears are walking in the woods, three run away, how many are left” (see table 4). By modelling the numbers 8, and 3 goes away, 5 (a whole hand) will be left.

Table 4. *Context problems emphasising how to model the part-part-whole structure of arithmetic tasks, aided by finger pattern*

	Number	Finger pattern
Part-part-whole relationship	Varies	Invariant

Conclusions and implications for mathematics education

The presented activities have been designed in accordance with Variation theory principles, to ensure that necessary aspects are made possible for the children to discern. These aspects are not new, already Fuson (1988) and Carpenter, Moser and Romberg (1982) found it necessary to learn ordinal and cardinal meaning of numbers. Baroody (2016) also has emphasized the part-whole relation as essential for learning to solve arithmetic tasks. But the way to systematically enable young children to experience these aspects of numbers in designed activities is a novel approach in early childhood mathematics education and particularly in the Swedish theme-based preschool practice. Our approach directs attention towards children’s ways of experiencing numbers and to enable children to discern what they have not been able to experience before. This stands in contrast to most earlier research within the field of arithmetic development, who rather describe children’s acts and strategies in arithmetic problem solving but not how the development of these strategies may be facilitated. This paper describes in detail how we put our theoretical conjecture (structure before counting) into practice and thereby provides some insights to the learning opportunities that may lie in designed activities that are based in empirical investigations and a solid theoretical framework.

What then did it mean that the teachers taught finger patterns to the preschool children? The detailed programme was developed during the intervention process, which enabled us to pick up on those aspects that were found necessary for individual children in order to follow the intended progression. First, children had to experience numbers as composite sets and that these could be represented by objects, e.g. fingers. This further means, that when children are

encouraged to use their fingers in situations where numbers, sets and arithmetic tasks are focused, there is an advantage for those children who are able to represent numbers in different ways rather than a static image of fingers. The flexibility in representing numbers opens up the opportunity to experience also numbers as composed of smaller parts. The following three activities further enabled the children to experience how parts are integrated in a whole and how numbers are seen as *relations*. This is a contrasting approach to for example Fuson (1988), who suggests that arithmetic development builds on counting that emphasises number sequences rather than structure. We find that the use of finger patterns, to structure numbers in a visible way, supported the children in both experiencing numbers as compositions of parts, and in handling a part-part-whole relation of numbers in simple arithmetic tasks. Nevertheless, we are fully aware of the powerful impact teachers have on how the activities were enacted and thus the actual learning outcome for the children (Ekdahl, 2018).

The overall purpose of this paper was to contribute to early childhood mathematics education, since scientific basis in preschool practice is mandatory, but not sufficient in the Swedish context, according to national assessment of preschool quality (Swedish Schools Inspectorate, 2018). Our study provides some deeper understanding of the learning potential that finger patterns in designed activities may have, as ways to enhance children's number sense and basic arithmetic skills, but also that arithmetic skills are a complex of ways to experience numbers that has bearing on the possibilities children have to engage in reasoning about numbers.

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Views on *concept* in the framework of Three worlds of mathematics: a concept analysis

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There are examples of conceptual incoherencies within the field of mathematics education. In this paper, a concept analysis of the notion concept in Tall's framework of Three worlds of mathematics is conducted. In the first phase of the study, an analysing tool is constructed from a literature review in philosophy and cognitive science. This tool makes two distinctions. One distinction is between views considering concepts as abstract and views considering concepts as mental. The other distinction is between views considering concepts as intersubjective and views considering concepts as subjective. Preliminary results show that different views on concept are mixed in the framework of Three worlds of mathematics.

The word "concept" is used with different meanings in texts within the field of mathematics education. As one example, Sfard (1991, p. 3)¹, on the one hand, explicates the notion *concept* as a mathematical idea that is presented as a theoretical construct within formal mathematics, while Asiala et al. (1997, pp. 2, 6), on the other hand, use a theoretical perspective reconstructed from an interpretation of Piaget, who in turn explicates *concept* as a mental representation (Furth, 1969, p. 53). Without any deeper reflection, it seems as if these views on *concept* are incoherent. Further, in some research settings, the word "concept" is imprecise and undefined, and in other settings, the explication of *concept* and the usage of the notion are not coherent. The usages of terms like "concept development", "concept formation" and "concept acquisition" in Sfard (1991, pp. 10, 16–17), all seem to refer to a Piagetian view, where concepts are mental representations, which is not coherent with the explication of *concept* as a mathematical idea.

The fact that "concept" is used with different meanings may lead to unclear theoretical frameworks and makes results difficult to interpret and to compare. Several ways of tackling conceptual incoherencies may be found in the field of mathematics education. One way is to stick to a single framework and use the notions in it, without analysing the relations to notions in other frameworks. As one example, Tall (2013, p. 6) uses the notion *knowledge structure* instead

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of *concept image* (Tall & Vinner, 1981, p. 1), *conception* (Sfard, 1991, p. 1) or *schema* (Asiala et al., 1997, p. 8). These notions are similar, but lack of precise explications makes it difficult to compare them; they are formed within different frameworks. Another way is to make a concept analysis to clarify the meanings of words, both in explications and within textual contexts, while evaluating the consistency of underlying theoretical assumptions (Machado and Silva, 2007, p. 671). The procedure for finding the meaning of a certain word in a text consists primarily in searching for other words, keywords, indicating a meaning of the word that is the object for the study. This meaning could then be compared with the meanings of other words, in order to find relations and gain an understanding of the notions.

The purpose of this paper is to present some preliminary results from an ongoing concept analysis of the notion *concept* in the framework of Three worlds of mathematics (Watson, Spyrou & Tall, 2003; Tall, 2004; 2013), in short called "the Three-world framework". This framework is chosen as one of those currently used with the ambition to offer a unified theory of the development of mathematical thinking, from the newborn child to the researcher in mathematics (Tall, 2013, pp. 5, 419). The present study should be seen as a step towards the comparison of views on *concept* in different frameworks in mathematics education, with the aim of developing a coherent framework for analysing concepts in mathematical problem solving. Here it is presupposed that different views on *concept* could (and should) be compared in order to broaden the understanding coming from a certain framework. Further, one stance is that the meaning of the word "concept" in different fields should basically be the same, but that there are different kinds of concepts, with different features. There are several attempts to distinguish mathematical concepts from non-mathematical ones. Recent examples are Sjögren and Bennet (2014, p. 834) and Priss (2017, p. 201). This question, however, is not in the object for the current study.

In a first step of the study, ideas in philosophy of language and cognitive science are used, in order to build a tool for analysing texts in mathematics education. The reason for this approach is that views on *concept* often are more clearly explicated in these fields, and that this could offer a deeper understanding of the field of mathematics education. The next step consists of a concept analysis, where texts using the Three-world framework are analysed with the help of the philosophical tool. The research question here is which views on concept that could be found in the framework, from the perspective of the analysing tool.

A tool for analysing views on *concept*

The analysing tool is the result of a configurative literature review in philosophy and cognitive science. This review aimed at finding key references and the broad lines within the discussion about concept in the philosophical field. A smaller,

more systematic, review was conducted where two journals, *British Journal for the Philosophy of Science* and *Philosophia Mathematica*, were manually scanned for texts handling views on *concept*, published during the period between January 2000 and March 2014. From the review, some anthologies were found, besides some other references. The texts were then chosen depending on how clear the description of *concept* was, if they were referred to by other texts, and if they contributed to the general discussion with new perspectives.

The tool makes two distinctions. The first distinction is between concepts seen as abstract and concepts seen as mental. Views where concepts are considered abstract go back to Plato, who saw concepts as ideal forms, independent of the mind. Such views are common, e.g. within philosophies of language (Katz, 1972/1999; Peacocke, 1991; Zalta, 2001), where concepts are claimed to be senses and parts of meanings of signs. One example of an abstract view is when Sfard (1991, p. 3) explicates *concept* as a mathematical idea that is presented as a theoretical construct within formal mathematics. Note that in educational contexts, mental objects are often considered abstract. An example of this is when Tall (2013, pp. 9–10) claims that different kinds of concepts (seen as mental representations) are developed from different kinds of abstractions. In this text, however, the term abstract is used as meaning nonmental, as e.g. in a Platonic sense. Views where concepts are considered mental, on the other hand, go back to Descartes, who saw concepts as mental images of sensory impressions (Furth 1969, pp. 70–71). Such views are common in contemporary empiricism (Jenkins, 2008) and cognitive science (Carey, 2009), where concepts are claimed to be mental representations. Asiala et al. (1997), using a Piagetian view on *concept* where concepts are seen as mental, is one example from mathematics education.

The second distinction is between concepts seen as subjective and concepts seen as intersubjective, and is independent of the distinction between abstract and mental. There is a range among views claiming that concepts are abstract, where some views mean that concepts are subjective (Zalta, 2001, p. 344) and others mean that they are intersubjective (Katz, 1972/1999, p. 133) or even objective (Peacocke, 1991, p. 525). The difference between intersubjective and objective views is not decisive for educational purposes. Therefore, this distinction will be handled fairly loosely and both of these views will be called "intersubjective". Further, views claiming that concepts are mental might distinguish between "individual concept" and "concept" (Carey, 2009, p. 354) in order to handle the fact that the word "concept" has an ambiguous meaning, it might be a subjective representation or something that we share. In social representation theory (Potter & Edwards, 1999, p. 448), just to take one example of many, representations have two roles; partly they are our personal representations, which are used for our thoughts about the world, and partly they are integrated in a collective understanding about the same world, enabling us to communicate

with other people. The difference between a view on concept as *subjective and abstract*, and a view on concept as *subjective and mental* is ontological, and may not have practical consequences for education.

The different views in the distinctions should not be considered incommensurable and Jackendoff (1989/1999, pp. 305–306) argues that it is possible to have a view including both concepts seen as abstract and concepts seen as mental. This argumentation also holds for a view including concepts seen as intersubjective and concepts seen as subjective, and for educational purposes both subjective concepts and intersubjective concepts might have important roles to play. The students develop their own subjective concepts, which in turn are judged against a curricular intersubjective concept. Within a theory of learning, these roles are different and should not be mixed.

The classical view on *concept* in Frege (1892/1985, pp. 188–190), as one example, distinguishes between *concept* and *idea*. Concepts are classes of objects (called "references") which are pointed out by senses in specific ways, e.g. through definitions or descriptions, and are both abstract and intersubjective. Ideas, on the other hand, are internal images arising from memories of impressions and are as such both mental and subjective. Inserting these ideas into a matrix, the results may be presented as in figure 1.

	abstract	mental
intersubjective	concept	
subjective		idea

Figure 1. *The Fregean view on concept*

Piaget, as already mentioned, explicates *concept* as a mental representation (Furth, 1969, p. 53). Already in a sensory-motor stage, the mental schema could be seen as some kind of practical concept, a pre-concept. Piaget (1941/1969, p. 156) claims that in the beginning of childhood this pre-concept is merely a synthesis of qualities in the mind. At the representational plane, the pre-concept develops into a mental representation. When the child may imagine the object even when there are no perceptions from physical objects, she has acquired the concept. Through a communicative interaction with other people, real, social, concepts, like *number*, are developed. These are logical classes on which operations, like addition, can be performed. The view that concepts are logical classes, and hence seen as abstract, is approximately a Fregean view. There is an ambiguity here with the two views on *concept*, one where concepts are seen as mental and one where concepts are seen as abstract, with a larger focus on the view on *concept* seen as mental. Inserting these ideas into the matrix, the results may be presented as in figure 2 (Piaget, 1954/1976, pp. 84–85, 360–363, 373; Furth, 1969, pp. 36–37).

	abstract	mental
intersubjective	concept	concept schema
subjective		pre-concept schema

Figure 2. *The Piagetian views on concept*

The concept analysis in this paper is a search for the meanings of the word "concept", and related terms, in order to find the view (or views) of the notion *concept* in the Three-world framework. In case there are clear explications, the task is more straightforward. In analysing views through the usage of words, the method is to search for keywords indicating a view where concepts are seen as abstract or mental, and as subjective or intersubjective, respectively. Keywords for a view where concepts are considered abstract are terms like "word meaning" and "category", used in a way indicating that concepts are meanings of words, besides formulations like "talk about a concept", since in a view considering concepts as mental we are using concepts in our thinking – we do not think about them. Further, expressions claiming that concepts are building blocks in formal mathematics might indicate a view where concepts are seen as abstract. Keywords for a view where concepts are considered mental are terms like "conceptual representation", "internal concept" and "constructing concepts", together with expressions indicating a Piagetian view, like "acquiring concepts". Keywords for a view on concept as *subjective* are formulations like "our own way of constructing concepts" and "the students develop concepts of *triangle*" (indicating that there are different concepts for different students). Keywords for a view on *concept* as intersubjective are expressions like "the social concept", "theoretical conceptual knowledge in the culture" and "the concept *rational number*" (indicating that there is just one such concept). Besides that, formulations indicating that concepts are developed in a culture of mathematicians or teachers, like "concepts are constructed in an axiomatic system" or "the notion *vector* in the curriculum" might be seen as keywords for a view claiming that concepts are intersubjective.

Further, references to a text with a certain view on *concept* might indicate that view. As an example, references to Frege might indicate a view where concepts are seen as abstract and references to Piaget might indicate a view where concepts are mainly seen as mental. Generally, formulations associated with philosophy of language indicate a view on *concept* seen as abstract and formulations referring to discussions in cognitive science indicate a view on concept seen as mental. The keywords together with the matrix used in figure 1 and figure 2 constitute the analysing tool for the concept analysis. In the next section, this tool is used for analysing views on *concept* and similar notions in the Three-world framework.

The notion concept in the Three-world framework

The Three-world framework first appeared in Watson et al. (2003) and combines the approaches in the process-to-object frameworks (Sfard, 1991; Gray & Tall, 1994; Asiala et al., 1997) with the theories of the van Hiele and affective theories, in order to provide a broader view on conceptual development. When Tall (2004, p. 32) discusses the notion *concept*, it seems clear that "concept" in this text has two different meanings. Sometimes concepts are seen as abstract and sometimes they are seen as mental, as is seen in the following quote.

When the term "concept" is used in this context, it therefore has the meaning that mathematicians consider to be an (abstract) object. However, it is not an object in the sense of a physical thing that we can perceive in the world. [...] I freely use the term "concept" when I speak about numbers, fractions, algebraic expressions and so on, in a manner which fits with common usage [sic!] but, at the same time is in a context where the symbol refers dually to process or concept (as a mental object).

(Tall, 2004, p. 32)

On the same page, the term "mental concept" is used and Tall (2013, pp. 6, 15) uses terms like "mental number concept". Also, in the quote "the duality between concept and schema is based on the same fundamental idea in which a named concept has rich internal links that reveal it to be a schema" (Tall, 2013, p. 80), the word "concept", via the usage of "schema" in a Piagetian meaning, refers to something mental. The view on *concept* seen as abstract appears in formulations like "we think about the process of counting and the concept of number" (Tall, 2013, p. 13). Further, formulations like "concepts in the calculus" (Tall, 2013, p. 7) and "these ideas may be expressed axiomatically in the formal theory of mathematical analysis" (Tall, 2013, p. 7) points to a view considering concepts as abstract. Hence, "mental concept" in the text refers to a mental representation, but concept seems, dually, to be seen as mental or as abstract.

In Gray and Tall (2007, p. 25), the notion *thinkable concept* is introduced as referring to "some phenomenon that has been named so that we can talk and think about it." (Gray & Tall, 2007, p. 25). These phenomena are noticed in a situation. When these are verbalised and named, then we can begin to talk about them and refine the meaning of the words in an analytic way. In this formulation, similar to the one in Tall (2013, p. 21), thinkable concepts are something that are distinguished from the human brain, which is indicating an abstract view on *thinkable concept*. However, in Tall (2013, pp. 79–81) the idea that *thinkable concept* and *schema* are essentially the same notion is presented; schemas develop over time and are compressed while building thinkable concepts. Here, a Piagetian view on *thinkable concept*, where thinkable concepts are seen as mental, is indicated. "Concept" and "thinkable concept" are used interchangeably in Tall (2013), without any explicated difference. Hence, the duality of the two different meanings of "concept" affects the interpretation, with the consequence that several interpretations of the notion *thinkable concept* are possible.

The word "knowledge structure" "refers broadly to the relationships that exist in a particular context or situation, including various links between concepts, processes, properties, beliefs and so on" (Tall, 2013, p. 6). It looks like "knowledge structure" may have two meanings. The first, more apparent, meaning is a cognitive one, where "knowledge structure" is another term for "schema", or a mental representation (Tall, 2013, p. 80). In that meaning, links between neurons in the brain are seen as constituting a foundation of mathematical thinking, and "[a]s these links are alerted, they change biochemically and, over time, well-used links produce more structured thinking processes and more richly connected knowledge structures" (Tall, 2013, p. 6).

Also, the expressions "knowledge structure", "cognitive structure", "schema", "conception", "mental conception" and "concept image" are used interchangeably in the text. The second, more invisible, meaning of "knowledge structure" is a non-cognitive one, where the term refers to relations between abstract concepts, processes and properties in a (mathematical) context or situation. Hence, the ambiguity within the view on *concept* leads to an ambiguity in the view on *knowledge structure*. All of this points to the fact that there is an ambiguity in the text, concerning the views on *concept*, *thinkable concept* and *knowledge structure*, between these notions considered as mental and considered as abstract.

Besides that, Tall (2013, p. 426) uses the term "individual thinkable concept". This indicates that besides the distinction between concepts seen as mental and concepts seen as abstract, there is a distinction between concepts seen as intersubjective and concepts seen as subjective. While the view on *concept* seen as intersubjective is present in formulations like "children are introduced to counting physical objects to develop the concept of number" (Tall, 2013, p. 7), the view on *concept* seen as subjective is present in formulations like "[o]ur biological brains evoke thinkable concepts by a selective binding of neural structures" (Tall, 2013, p. 24).

Since the views on *concept* in the Three-world framework seem to be similar to the Piagetian views (figure 2), partly including a Fregean view (figure 1), it seems plausible to suppose that there are both a view on *concept* seen as mental and intersubjective, and a view on *concept* seen as abstract and intersubjective. When it comes to concepts seen as subjective, it could both be similar to what Frege calls "idea" or similar to what Piaget calls "pre-concept". In both cases, concepts seen as subjective are mental, and I have not found any signs indicating a view where concepts are subjective and abstract. Hence, mental concepts might be both intersubjective and subjective, but abstract concepts might only be intersubjective. The word "knowledge structure", as seen before, can refer both to a mental and subjective structure, and to an abstract and intersubjective structure. Inserting these ideas into the matrix, the results may be presented as in figure 3.

A general conclusion from the concept analysis is that there are several views on *concept* in the framework. As an answer to the research question,

	abstract	mental
intersubjective	concept thinkable concept knowledge structure	concept thinkable concept
subjective		concept thinkable concept knowledge structure

Figure 3. *Views on concept in the Three-world framework*

there is one view on *concept* seen as abstract and intersubjective, one view on *concept* seen as mental and intersubjective, and one view on *concept* seen as mental and subjective. The interpretation of *concept* must therefore depend on the textual context.

Discussion

The purpose of this paper has been to present some preliminary results from an ongoing concept analysis of the notion *concept* in the framework of Three worlds of mathematics. The first phase of the study presented a philosophically flavoured tool for analysing views on *concept*, based on two distinctions: between concepts seen as abstract and concepts seen as mental, and between concepts seen as intersubjective and concepts seen as subjective. The concept analysis then showed that both these distinctions were present in the Three-world framework.

Underlying this framework, there seems to be an implicit view that concepts appear on two different arenas. The first one is a cognitive arena where concepts are considered mental representations, with cognitive processes, conceptions, concept images and cognitive structures. The other one is an abstract arena where concepts are considered abstract, and where knowledge structures appear in a linguistic context. The reason for why there seems to be a need for two different views on *concept* might be that the framework is to be used both for describing, or explaining, a psychological development of the individual, and for understanding the nature of mathematics, at the same time. The duality between the two different views on *concept* is present already in Piaget’s theoretical perspective, where *concept* is explicated as a mental representation, but where real concepts are classes of objects. It is also present in Sfard (1991), where *concept* is explicated as a mathematical idea, but where the usages of some words seem to refer to a view where concepts are mental.

While concepts seen as abstract and concepts seen as mental generally might appear in different theories of learning, the distinction between subjective and intersubjective views is found in the relation between the individual and the community, or between the student and the curriculum. The student develops his or her own subjective concept, which is judged against a curricular concept. This distinction might be independent of theory of learning.

As Jackendoff (1989/1999, pp. 305–306) argues, it is possible to have a view including both concepts seen as abstract and concepts seen as mental, approximately as the one in Tall (2004; 2013). However, if different views are implicitly combined, it could be difficult to interpret what the framework actually describes, making it problematic to use as a theoretical base for understanding learning. It is my position that such a dual view has to be explicit and clear, in order not to create confusion. One way might be to consequently use "mental concept" when a concept is seen as mental and "concept" when a concept is seen as abstract. In a similar way, "individual concept" could be used when concepts are seen as subjective and "concept" could be used when concepts are seen as intersubjective. The three different views on *concept* in the Three-world framework (figure 3) might then be addressed with the terms "concept", "mental concept" and "individual mental concept" (or just "individual concept").

For the concept analysis in this paper an analysing tool, consisting of a matrix together with keywords, has been used. The advantage of such an approach is that the philosophically flavoured tool offers a preconception of the notion *concept*, which gives opportunities to a deeper understanding of the texts. However, other approaches might have resulted in other kinds of descriptions of the views on concept. Even so, the results show that a concept analysis of this kind can reveal views on a certain notion, e.g. the notion *concept*, in texts in mathematics education. What has yet to be done in the study is to use the analysing tool in a more systematic literature review and continue making concept analyses of other frameworks, to see how the views on *concept* differ and relate to each other. One result from that work will be a view on *concept* that builds a ground for a new framework for analysing concepts in mathematical problem solving. Another result will be an insight in how frameworks could be compared and combined.

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Notes

- 1 References are, when particular formulations are of importance, specified with page numbers.

The role of mathematical competencies in curriculum documents in different countries

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The inclusion of competencies in curriculum documents can be seen as an international reform movement in mathematics education. The purpose of this study is to understand which role mathematical competencies have in curriculum documents in different countries, with a focus on the relationship between competencies and content. Curriculum documents from 11 different countries were analysed. The results reveal three different themes of variation, concerning if the competencies are specific to mathematics, if competencies are described as learning goals, and if such learning goals are differentiated between grade levels.

In this study, curriculum documents from different countries are analysed, focusing on the role of mathematical competencies. The inclusion of competencies in curriculum documents can be seen in the light of an international reform movement in mathematics education (cf. Boesen et al., 2014; Niss et al., 2016). This reform is visible in several countries (as is also evident through the analysis in this paper), where different notions are used for essentially the same thing, such as competencies, proficiencies, abilities, or practices. The notion of competencies is used here unless a direct reference is made to a document where another notion is used.

The reform highlights two aspects of the learning of mathematics: knowing mathematics and doing mathematics (Niss et al., 2016). The first aspect focuses on content, what to know, such as the mathematical notions, concepts, and methods. The second aspect focuses on competencies, how to act, describing the activities of doing mathematics (i.e., the practice of mathematics). Several questions could be asked concerning these two aspects, and some of these are addressed in this paper: What relationship exists between these two aspects (competencies and content)? Are they seen as two independent dimensions of mastering mathematics or is one of them seen as primary in some way?

Different answers to these types of questions "will give rise to very different kinds of mathematics teaching and learning" (Niss et al., 2016, p. 612), making international comparisons concerning these questions very important

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in order to understand the diversity of mathematics education in the world. This paper makes a contribution to this area of research by analysing curriculum documents from many different countries.

The use of competencies in curriculum documents

Two major strands of research can be located that focus on the use of competencies in curriculum documents. First, there is research that focuses on analyses of the message portrayed in curriculum documents. Second, there is research that focuses on different types of potential effects on teaching and learning from such documents.

Research on the content of curriculum documents

Analyses of curriculum documents from several countries highlight issues of clarity in these documents. An analysis of the former Swedish curriculum documents, valid between 1994 and 2011, shows that the reform message in these documents is vague and formulated with complex wording (Bergqvist & Bergqvist, 2016). The intended reform was a focus on competencies (or abilities as labelled in the Swedish context). Similarly, analyses of curriculum documents in USA have shown that "reformers wrote words to convey a new kind of mathematics teaching and learning, yet the meaning those words could convey was imprecise, at best" (Hill, 2001, p. 302).

Furthermore, an analysis of the Australian curriculum focused on aspects of cohesiveness (Atweh, Miller & Thornton, 2012). Through their analysis, the authors question "whether only lip service [...] is given to the General Capabilities, Cross-curricular Priorities, and the high order Proficiencies" (Atweh et al., 2012, p. 2). That is, they notice a lack of cohesiveness concerning the role of proficiencies in the curriculum. In particular, they notice that proficiencies sometimes "are not presented as outcomes or aims to be developed and assessed" (Atweh et al., 2012, p. 7) but sometimes they are.

Research on potential effects of curriculum documents

There is also research about potential effects of competency-oriented curriculum documents on teaching. In general, it has been shown that educational reforms often do not give desired effects in schools (Cuban, 2013). Analyses of mathematical competencies, in the context of the NCTM Standards and in the Swedish 1994 curriculum context show similar results: The reform of including and focusing on mathematical competencies as a goal for mathematics education does not seem to have made a clear impact on school practice. For example, studies in relation to the NCTM Standards show that

- there are no clear differences in textbooks between before and after the introduction of the NCTM Standards (Jitendra et al., 2005),

- teaching is still ”more like the kind of traditional teaching reported for most of the past century than the kind of teaching promoted in Principles and Standards” (Jacobs et al., 2006), and
- ”reformers have not yet succeeded in getting district leaders to grasp [...] the meaning of their reform proposals” (Spillane, 2000, p. 169).

In Sweden, a combination of national curriculum documents and national tests are used to convey reform messages. However, the 1994 reform, which for mathematics included a focus on competencies, was not successful, for example since

- textbooks focus on procedural competence (Boesen et al., 2014),
- teacher-made tests do not focus as much on competencies as the national tests do (Boesen, 2006), and
- teachers have not modified their teaching in alignment with the reform (Boesen et al., 2014).

In line with these specific empirical results from USA and Sweden, in their survey of situations in several countries, Niss et al. (2016, p. 625) also note that ”it has been found to be challenging for teachers to come to grips with notions of mathematical competence/competencies and their relatives and, not the least, with their implementation”.

Connecting content and implementation of curriculum documents

As noted above, the implementation of the reform to focus on competencies as a central aspect of learning mathematics seems to be a difficult problem. One issue of implementation could be an unclear message in the curriculum documents, as also suggested in empirical studies of curriculum documents (see above). Perhaps the message is unclear because of the complexity of the message itself, about seeing mathematics knowledge also as competencies, and not only as knowledge about content. But the message could also be unclear because of how the message is described in curriculum documents. Therefore, more analysis is needed of curriculum documents regarding this complex message of mathematical competencies. Such analysis is done in this paper.

Purpose

The purpose of this study is to understand the role of mathematical competencies in curriculum documents in different countries, with a focus on the relationship between competencies and content.

The focus on the relationship between competencies and content is chosen since this addresses the core question of how mathematics is characterized in a school context. Thus, this focus makes it possible to contribute with a deeper

understanding of different ways to conceptualize the learning of mathematics, when including competencies in curriculum documents.

The purpose is not to compare the different documents to be able to draw conclusions about different countries, since the documents are of different types. For example, some documents are prescriptive, legal documents while other documents are guidelines or resources of some type. Instead, documents from different countries are here used to find different ways of describing the role of mathematical competencies, in particular in relation to content.

Method

The curriculum documents selected for analysis came from *The ICMI database project*¹. Only documents available in English were included, due to limited proficiency in other languages. The criterion for inclusion of a document was that it should contain *an explicit presentation of a set of competencies*. This criterion is specified through a focus on two aspects of it: the notion of *competencies* and the notion of *a set*.

First, as discussed in the introduction of this paper, competencies refer to an answer to questions around what it takes "to become 'a doer' of mathematics" (Niss et al., 2016, p. 612). That is, competencies describe the activities of doing mathematics (i.e., the practice of mathematics). Thus, competencies can be included in a curriculum document when different types of verbs are used, besides forms of to "know", to describe what it means to master mathematics. However, it is not enough if a curriculum document uses notions that could be interpreted as competencies (such as "reasoning" or "solving problems"), but there needs to be an explicit presentation of a *set* of competencies. This part of the criterion comes from a result in the survey by Niss et al., (2016, p. 621): "It is further characteristic of any competency construct that it involves a number of distinctions between different instances of the construct or between different sub-constructs or strands".

This study focuses on how the relationship between competencies and content is described in curriculum documents. Therefore, the analysis focused on explicit descriptions about such relationships. In particular, two parts from documents are included in this analysis. First, direct statements about this relationship are included, for example, if competencies are described as the focus for teaching while knowledge of content is described as the goals for learning. Second, how competencies and content are presented structurally in the document is also included in the analysis, for example, if lists of competencies are given together with lists of content strands, but only the content strands are used when describing learning goals. Finally, these statements and structures from all curriculum documents are compared to characterize different ways of describing the role of mathematical competencies, in relation to content. That is, a bottom-up type of analysis is used, to find different themes in how the relationship between competencies and content is described.

Results

Curriculum documents from 11 different countries have been analysed. First, an overarching result is here presented, through three themes of variation that have been located in the analyses of the 11 documents. Thereafter, a short description of each document is given, focusing on explicit statements about the relationship between competencies and content, and on the structural presentation of this relationship. Each document is also characterized in relation to the three themes of variation.

Themes of variation

The analysis of the curriculum documents revealed different ways of describing the relationship between competencies and content. These results are here characterized through three themes of variation. The abbreviations given within each theme are used in the characterization of each document. The A category in each theme refers to a somewhat more focused or elaborated view of mathematical competencies.

First theme: Competencies are described as specific for mathematics (1A), or as more general for all subject areas, and thereby as more separated and independent from content (1B).

Second theme: Competencies are described as learning goals, that is, there is an aim for students to develop the competencies (2A), or competencies are described as teaching methods or classroom activities, that is, as an important way of learning mathematics, but where focus is on content when describing learning goals (2B).

Third theme, which is a sub theme of 2A: When competencies are described as learning goals, this is done in different ways. In particular, competencies are described as specific learning goals for different grade levels, with variation over grade levels (3A), or they are described more generally as a focus of development over all grade levels, but with no variation when describing each grade level (3B). In the latter case, content is focused on when describing learning goals for each grade level.

Australia

The Australian curriculum for mathematics² includes content strands and proficiency strands (1A). Proficiency strands describe "how content is explored or developed" and "the actions in which students can engage when learning and using the content". That is, proficiencies are described as means in order to learn the content (2B).

For each year level, the Australian curriculum includes year level descriptions, content descriptions and achievement standards. The year level descriptions give an "overview of relationship between proficiencies and content" and relate explicitly to the proficiency strands but no explicit reference is made to the content strands nor to the content descriptions for the year level in question. The content descriptions relate explicitly to the content strands but not to the

proficiency strands. The achievement standards refer explicitly neither to the content strands nor to the proficiency strands (2B).

Canada – British Columbia

The curriculum document from British Columbia³ includes curricular competencies, which describe "what students can do with mathematics" (1A). The content, on the other hand, "reflects what students should know". Learning standards are given for each grade level, structured in a table with two columns, one with competencies and one with content. Thus, the competencies are described as central learning goals (2A), on equal terms as content, and where all standards are specific for each grade level (3A).

Czech Republic

The curriculum document in the Czech Republic⁴ describes objectives for mathematics, which focus on "the formation and development of key competencies by guiding pupils towards", which is followed by a list of competencies specific for mathematics (1A).

Content is described through different thematic areas, and the subject matter is seen as "a means to master activity-oriented expected outcomes which are gradually combined and create preconditions for an effective and comprehensive use of acquired abilities and skills at the level of key competencies" (2A). The curriculum document describes expected outcomes at two different stages and structured around the thematic areas, where no explicit connections are made to the competencies (3B).

England

The curriculum document in England⁵ describes aims for mathematics formulated through competencies (1A & 2A). However, there are no explicit descriptions of relationships between competencies and content. The detailed description of programme of study for each year level is structured around content, with specific goals that are not explicitly connected to the competencies (3B).

New Zealand

There are no competencies specifically for mathematics in the curriculum document from New Zealand⁶ (1B). However, the document includes key competencies that are common for all subject areas, and they are described as "both end and means" (2A & 2B). The achievement objectives for mathematics are presented in different content strands, without any explicit connections to competencies (3B).

Northern Ireland

For key stages 1 and 2 in the curriculum document in Northern Ireland⁷, competencies are described as processes in mathematics (1A). This section is

described on equal terms as other sections that describe content (e.g., number and measures). In each section, learning goals are described (2A), which vary between stages (3A).

Norway

The Norwegian curriculum document⁸ includes competencies (labelled as basic skills) that are common for all subject areas, but these are also explicitly interpreted within each subject area (1A). There is a framework describing progression through year levels, using the general basic skills, which serves as "a basis and point of reference for developing subject and grade relevant competence aims" (2A). However, the competence aims in the curriculum are structured only through content areas, and there are no explicit connections to the general basic skills or to the competencies that are specific for mathematics (3B).

Singapore

A central part of the curriculum document from Singapore⁹ is mathematical problem solving, which is also placed in the centre of a pentagon, with labelled sides: attitudes, metacognition, processes, concepts, and skills. Processes include different mathematical competencies (1A), while concepts are described as "content categories".

Several of the processes are described as means in the learning process, for example, that communication "helps students develop their understanding of mathematics" and "through mathematical modelling, students learn to ..." (2B). However, processes are also described as learning goals, for example: "The teaching of process skills should be deliberate" (2A).

In one section of the curriculum, a table is presented with the processes and indicators of them. In another section, a much larger table is presented with content, connected to learning experiences, structured around many sub-strands and also around several levels, which is not the case for processes (3B).

South Africa

The South African curriculum document¹⁰ includes "specific skills", which describe mathematical competencies (1A). There is one description of the relation between content and these skills: "Each content area contributes to the acquisition of specific skills", which emphasizes content as means towards learning the competencies (2A).

The competencies are only described through their names, while a table is given over "mathematics content knowledge", which is structured through content areas, general content focus and specific content focus. The curriculum document also includes grade overview tables, which are very extensive, describing aspects of teaching and assessment and showing a progression, which is structured around content (3B).

Sweden

The Swedish curriculum document¹¹ describes the development of mathematical competencies as the purpose of mathematics (1A & 2A), which is in a part of the curriculum separated from parts describing the content of mathematics. However, there are no explicit statements about the relation between content and competencies. The competencies are used for describing the knowledge requirements, that is, the grading criteria, which also vary for different grade levels (3A).

USA

The *Common core standards*¹² include competencies through standards for mathematical practice (1A). It is stated that students should develop these competencies at all educational levels, that is, they are described as learning goals (2A). The grade level standards, which "define what students should understand and be able to do", are structured around content areas. For each grade level, the mathematical practices are listed, in the same way for all grade levels (3B).

Conclusions and discussion

The analysis of the curriculum documents has revealed different ways of describing the relationship between competencies and content. These results have been characterized through three different themes of variation (see the first part of the results).

Concerning the first theme, almost all documents describe competencies specific for mathematics (1A), but one document describes only more general competencies (New Zealand). This issue, whether to derive mathematical competencies from more general competencies has been debated in research literature (Niss et al., 2016, p. 621), but the results here show that it is very common to describe competencies specific for mathematics. Concerning the second theme, most documents describe competencies as learning goals (2A), but one document primarily describes competencies as means (Australia). Some documents explicitly describe competencies both as means and ends in the learning process (New Zealand and Singapore). Concerning the third theme, most documents do not describe learning goals for competencies that are specific for different grade levels, but only as a more general focus of development (3B). Three documents specify different learning goals for competencies for different grade levels (Canada – British Columbia, Northern Ireland, and Sweden).

For all the themes of variation, there are also different levels of clarity in describing the role of competencies in relation to content. Issues of clarity have not been focused on specifically in this paper, but in many curriculum documents not much is said at all about the role of competencies in relation to content. Furthermore, even when competencies are explicitly described as learning goals, which is common in these 11 documents, it is still not common

to use competencies when describing specific learning goals, but then content is used as the main building blocks to describe progression in students' learning.

A Danish framework (Niss & Højgaard, 2011), not analysed here, gives a competence description of mathematics education, and describes competencies and content as two separate and independent dimensions of mathematics knowledge. That is, every formulation of a learning goal must include some competency and some content. No such clear description exists in the analysed documents, and the view presented in the Danish framework is not compatible with many of the documents from other countries. This is the case since the documents describe learning goals without the inclusion of competencies, or describe competencies and content at the same level, that is, learning goals are *either* formulated through competencies *or* formulated through content.

Based on these themes of variation, and the varying degree of clarity, it would be of interest to analyse the effects these variations might have on teaching, when relying on different types of descriptions of the relationship between competencies and content. Within the former Swedish national curriculum, there was no clear effects in the classrooms (Boesen et al., 2014) from curriculum documents that were unclear in their reform message (Bergqvist & Bergqvist, 2016). Aspects of unclarity in curriculum documents have been shown also in this study. In particular, unclarity concerns if and how competencies can and should be seen as learning goals, while at the same time most emphasis in the documents is on content when describing specific learning goals. Empirical studies are therefore needed to know how teachers interpret these different types of descriptions of competencies and content, and how teaching is influenced.

Furthermore, by analysing documents from 11 different countries, the results from this study can contribute with valuable contextual, and potentially explanatory, information in other international comparative studies. In particular, the differences seen in this study could relate to differences at a larger (political) context, or could help explain differences in classroom activities in different countries.

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Notes

- 1 <http://www.mathunion.org/icmi/activities/database-project/introduction/>
- 2 <http://www.australiancurriculum.edu.au>
- 3 <https://curriculum.gov.bc.ca/curriculum/mathematics/introduction>
- 4 <http://www.msmt.cz/areas-of-work/basic-education-1>
- 5 <https://www.gov.uk/government/publications/national-curriculum-in-england-mathematics-programmes-of-study>
- 6 <http://nzcurriculum.tki.org.nz/The-New-Zealand-Curriculum>
- 7 <http://ccea.org.uk/curriculum>
- 8 <http://www.udir.no/in-english/>
- 9 <https://www.moe.gov.sg/education/syllabuses/sciences/>
- 10 <http://www.education.gov.za/Curriculum/NationalCurriculumStatements-GradesR-12.aspx>
- 11 <http://www.skolverket.se/publikationer?id=2687>
- 12 <http://www.corestandards.org/Math/>

Investigating the politics of meaning(s) in Nordic research on special education mathematics: developing a methodology

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This paper aims to develop a methodology to explore the politics of meaning in special education mathematics research. Mediated meaning, directions of intentionalities and perspectives on special education have been analysed in eight reviewed articles. Results indicate that the politics of meaning in the Nordic sample are about processes of normalisation and effectiveness through methods and approaches. The teacher is emphasised as the centre for change and development also when it comes to organisational factors. Disabilities are not researched, perhaps cloaked by an overall relational approach or due to research paying attention to milder difficulties. The developed methodology seems to be fruitful and will be applied on a broader international sample.

Schools, both nationally and internationally, are struggling with an increased number of students in need of support and a decrease in mathematical knowledge (Martens, Knodel & Windzio, 2014). This implies an increased need of special educational support in mathematics. The nature of the support given – to whom it is given, by whom, why it is given and when – is understood as being constituted by its underpinnings in the politics of meaning (Skovsmose, 2016) regarding special education mathematics in policy, research and practice. By the "politics of meaning" in this article, we refer to the underlying societal motives, values and assumptions. Challenges in the process of teaching and learning mathematics is further on in this article labelled as special education mathematics (SEM).

SEM has been shown to be a diverse branch of the social sciences (Bagger & Roos, 2015; Magne, 2006). How research can support the practice of SEM has been explored by Scherer et al. (2016, p. 19) who claim: "There is need for more evidence-based research in the field of inclusive mathematics classrooms". In addition, there is no consensus regarding who the students supposed to be included are, or how the students should be supported or how this practice should be investigated (Bagger & Roos, 2015; Heyd-Metzuyanim, 2013;

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Sherer et al., 2016). Conceptualising the student is dependent on amongst other things national steering documents, approaches to learning and teaching, and how mathematics and learning difficulties are understood (Scherer et al., 2016). However, how a concept is understood and what values it is loaded with affect the approach of individuals (Llewelyn & Mendick, 2011). The concepts are in turn not always explained but exist implicitly in research reports (Bagger & Roos, 2015).

The underpinnings in research derive from the stakeholders' individual and collective politics of meaning. Furthermore, it will direct research in the drive towards improving teaching and learning in mathematics. There seems to be some obscurity of the politics of meaning regarding SEM, something that contributes to a black box of underpinnings of the direction of research. When individuals act, speak, write about and handle the world, these actions are understood as mediating meaning, meaning that is created and recreated collectively (Scollon, 2001). In this case, the concealed area of meaning concerns what SEM is, for whom it is, from whom it derives etc. This meaning might be visible in texts depicting what the research derives from, aims towards, discovers and concludes in relation to SEM. A review and an exploration of these points holds the potential to unravel the politics of meaning. It might in addition result in a resourceful evidence-based advice for further research and practice.

Aim and research questions

The aim of the study is to develop a methodology to explore the politics of meaning in SEM. To make a necessary demarcation, this article will contribute a snapshot of Nordic research in an international context of peer-reviewed journals. The study is performed through an exploration of the following three research questions:

RQ1: What is stated as the cause and intentional goal, effects and suggestions of the implications of SEM?

RQ2: What opportunities and obstacles are mediated in the texts?

RQ3: Is it possible for the developed methodology to deconstruct the meaning-making in SEM? If so, how?

Theoretical underpinnings of the study

In order to uncover the meaning-making and underlying values in research on SEM, we have explored the internal relations of texts and also the inter-related meanings between texts. The texts are together understood as collective communication and mediation of meaning. For this purpose, we draw on Fairclough (2003) and Scollon (2001) in our ontological foundations of "what is". This is complemented with Skovsmose (2016) regarding how this is to be researched from a mathematics education perspective together with a

framework of perspectives on special education which is built on Persson's (2008) and Nilholm's (2005, 2007) theories. The special educational perspectives are central parts of the operationalisation of the theory in this article (see method for analysis on p. 4). Fairclough's (2003, p. 10) notion represents a useful theory for understanding what a text represents.

Part of what is implied approaching texts as elements of social events is that we are not only concerned with texts as such, but also with interactive processes of meaning-making.

Accordingly, this study approaches the text in the articles as accounts of discourse that are constructing and reconstructing meaning on SEM as a social event. In addition, text is understood as collectively produced shared meanings in which mediated actions "produces and reproduces social identities and social structures within a nexus of practice" (Scollon, 2001, p. 5). Written language is the mediational means by which mediated actions take place. The actions are expressions of the intrinsic motives for and the underlying values and assumptions underlying the site of engagement of research in SEM.

An action is understood as taking place within a site of engagement which is the real-time window opened through an intersection of social practices and mediational means. (Scollon, 2001, p. 5)

This real-time window allows us to get a snapshot of the policy of meaning in the reviewed research. In order to situate the analysis of the mediated discourse of meaning in the social practice of mathematics educational research, we draw on Skovsmose's theories (2016). Skovsmose emphasises learning as action and social structuring as a core. He stresses that theories of meaning are conceptual in nature and can be studied through the social structuring of intentionalities. Intentionalities are understood as being expressed through and in the intrinsic motives for, as well as the underlying values and assumptions of research. Skovsmose highlights the urgency of studying the intentional interpretation of meaning in mathematics education. This urgency is due to the politics of meaning that are underlying the researcher's interpretations of what is considered as desirable and good. This harmonises with Scollon's theory on mediated action and social practices. Meaning is constructed by the direction of intentionalities that are formed through historical, political, cultural, economic and discursive factors. Therefore, an exploration of the direction of intentionalities has the potential to unravel the collective mediated policy of meaning in research concerning SEM.

Thus, the mediated policy of meaning is emphasised as the dialogical and collective processes of meaning-making which takes place in the act of writing about SEM. The contribution of this article is a search for a way of unravelling the politics of meaning in SEM. This is done by studying the direction of intentionality in the mediated actions in written research reports and specifically in research taking place within the Nordic school context.

Method of selection

The first criterion in the selection was to find studies that dealt with SEM. The keywords in the search were therefore "special" and "mathematic". These keywords limited the search to English written research. The second selection criterion of this study was to find recent research articles of quality. Thus, a time limit was applied lasting from 2015 until as late as possible in 2017. For the quality requirement, publications were sought coming solely from peer-reviewed journals. This provided a snapshot of what research has been up to. Since special education is a multidisciplinary field, it was important to search a wide range of databases to be able to capture a variation of fields, disciplines and methodologies and allow for crossing boundaries. The databases Scopus, Eric and MathEduc were chosen since they depict special education and/or mathematics education in general. Scopus covers all sorts of research and consequently the selection was confined to the social sciences. Eric is a database that granted the study considerable breadth within the field of educational research. The MathEduc database led to the deeper aspects of mathematics education and therefore we did not restrict the range since the selected keywords already provided a limitation. Initially, 82 articles were identified. After reading the abstracts and applying the criteria of conducting research in the Nordic context, six suitable articles were found spread over a breadth of fields, disciplines and methodologies. However, since the journal *Nordic Studies in Mathematics Education* (NOMAD) is not included in the databases consulted, a specific search was made in this journal ending up with 62 additional articles, whereof two were within scope of the criteria. This led to a selection of eight articles in total.

Method of analysis

The texts contribute to an account of the policy of meaning across articles rather than an account of how common or valid the results are in the articles. "Intentionalities can be formed by possibilities as well as by obstructions" (Skovsmose, 2016, p. 9). In research on SEM, as initially described, motives, values, and assumptions made within the text are mediating meaning regarding the support, the learner and the subject. In other words, the direction of expressed intentionalities reveals socially constructed possibilities and obstacles for students in need of support. Consequently, the method of analysis needed to allow us to identify, display and scrutinise both the possibilities and the obstacles in the direction of intentionalities in SEM. This has led us to advocate the special educational perspectives – both relational and categorical – developed by Nilholm (2005, 2007) and Persson (2008) and thereafter adopted into the area of SEM by Bagger and Roos (2015). In the relational perspective, the heritage of the problem and the solution is described as located in a sociocultural setting. In the categorical perspective, the problem and solution are placed within the student and can be described as a deviation from what is "normal" (Nilholm,

Table 1. *Analytical scheme with relational and categorical perspective in relation to the research questions of the study*

Perspective	Step 1: Main <i>cause</i> for SEM	Step 2: Main <i>effects and implications</i> of SEM	Step 3: Main <i>goal</i> of SEM
Relational	Describes the environment and relations between student's properties and context		Changing the learning environment and relations between student and context
Categorical	Describes the student's prerequisites or properties		Strengthening the student or compensating for deficits

2007; Persson, 2008). These perspectives allow for a deconstruction of the policy of meaning expressed through the direction of intentionalities inhibited in concepts and expressions.

The research questions guided the analysis. First, an analytical scheme was constructed in order to capture RQ1 (table 1) – What is stated as the cause and intentional goal, effects and suggestions of the implications of SEM? This scheme is a development of a framework used by Bagger & Roos (2015). Sections of texts were compiled into a table for analysis in a document – namely the sections related to cause, intentional goal, intentional effects and suggestions of implications. A search for intrinsic motives and underlying values and assumptions was made in order to answer RQ2: What opportunities and obstacles are mediated in the texts? This question was answered through a summary and interpretation of the overall direction of intentionalities, a procedure leading to a demarcation of the mediated politics of meaning(s).

The coding of the selected texts into a categorical or relational perspective was performed by a search for central statements or clear conceptualisations in the articles in all the steps of the research reports. Here is an example of a relational perspective in the text about the cause, effect, implication or goal: "students that are externally differentiated to schools with low prior knowledge are, in general, exposed to further differentiation through the higher level of ability grouping at these schools" (Ramberg, 2016, p. 706). This is an example of a categorical perspective in the text about the cause, effect, implication or goal: "overlapping difficulties in mathematics and reading measured at the end of comprehensive school are clearly a risk for lower academic self-concept and [...] those difficulties have negative effects on school achievement across several school years" (Holopainen, Taipale & Savolainen, 2017, p. 99).

Results and analysis

The eight articles chosen for study provided a snapshot of the politics of meaning in Nordic SEM. In the following section, RQ1 and RQ2 are answered by presenting the content of the texts. The result is presented according to themes of the direction of meaning found in the analysis.

The direction of intentionalities in the Nordic sample

A long-term goal to improve achievement of students in need is seen in several texts that place the problem and solution within the student, although the methods are at times relational oriented. Special teaching methods or programmes are used in such cases. Johansen and Somby (2016) aimed at improving performance for students in need of support through a special programme in which 1,880 pupils in the 10th grade in Norway participated. The organisation and content of the programme, the teacher acting as facilitator and the students' activities were the centrepieces for heightening student achievement in mathematics. A Finnish study (Hotulainen, Mononen & Aunio, 2016) took achievement and parental educational background into consideration in a pre-test of an intervention to enhance thinking skills. Although this initiative indeed improved student thinking skills, no improvement could be seen in mathematics. In the end, the authors did not discuss the background factors. On the other hand, the presence of these factors, can be interpreted as a direction of intentionalities to contribute to levelling the opportunities to learn and promoting inclusion in the subject.

The overall implicit direction of intentionalities in the articles about methods or interventions is that it is important to find a method which enhances learning. We have interpreted the comparisons to normal achieving students as an intentionality to normalise the curve of development. The direction of intentionalities further implies the idea of a quest for a method or approach that is successful and indirectly that there might be approaches and methods less favourable for including all students in the teaching and learning of mathematics.

Organisational factors have been investigated in a study by Ramberg (2016) of ability grouping amongst 950 upper secondary school students. He concludes that although this is a method known not to assist students in need of support, it is still highly advocated; he emphasises that a discussion on equity in educational settings is needed. Ability grouping is also criticised and described as common by Ekstam, Linnanmäki and Aunio (2015). In their report, models for support after a change in legislation in Finland 2011 are investigated. Although the three-tier support is supposed to enhance inclusion, much of the practice of giving support is the same as before the legislation and "pull out of the classroom" (p. 85) is the most common form of support. The directions of intentionalities in these last two articles suggest that students in need of support should be included in mainstream learning and that the practice continues of choosing models that do not benefit student learning and self-efficiency according to the research.

In the texts, there are expressions of intentionalities that imply that the teachers' self-awareness and beliefs will affect the teaching and learning. Schmidt (2015) relates inclusion in mathematics to students encountering difficulties in the subject, coming to the conclusion that it is important as a teacher to work

with a subject such as mathematics from a socially inclusive perspective. In the text, the direction of intentionalities points towards a need to ensure that students in need of support are included in the subject. One way to do that is to extend the use of peer collaboration in the classroom. Hence, expressions of intentionalities also imply teachers' awareness and work with didactical peer collaboration. The teachers' role is also stressed by Ekstam et al. (2017), who found that pre-school teachers' interest in the subject affects their beliefs in teaching more than their actual subject knowledge.

The learners' prerequisites, perspectives and experiences are also emphasised in this selection of Nordic research. Holopainen, Taipale and Savolainen (2017) show that adolescents' difficulties in mathematics predict and influence their academic self-concept. Bagger (2016), who has explored students' experiences of pressure and thoughts about themselves and the national test in the third grade, stresses that pressure and perceived threats could be tempered by pedagogical methods if the teacher knew what kind of position of need the student is in. These eight articles express intentionalities regarding enhancing the possibilities of good teaching in mathematics which is supposed to prevent secondary difficulties. Enough of the right support might thereby affect students' feelings of being able, their self-concepts and their future academic prospects.

In conclusion, a demarcation of the politics of meaning in the above mediated actions regarding SEM is made. The overall cause, goal, effects and implications of research performed on organisational levels are mainly relational, whilst the studies looking into methods for heightening achievement are mainly categorical. Yet the categorical research also enhances the importance of the teacher and then takes a relational stance. The research in these eight articles emphasises SEM as being about relations between processes of normalisation of achievement and teacher readiness to educate these students, including organisational factors. The politics of meaning can be summarised as embodying the following points:

- 1 Improving educational effectiveness is supposed to lead to a normalisation of achievement and/or agency in mathematics. This is achieved through a search for successful methods or approaches.
- 2 Inclusion in mathematics is desirable and connected to issues of equity, teachers' and students' self-knowledge and psychosocial aspects of teaching and learning.
- 3 Teachers are at the centre of change and development, both for the student and the organisation.
- 4 Practice is unchangeable as proven experience has precedence over research, although the reasons for not changing might be complex and hard to identify.

Discussion of the results and methodology

The result of this snapshot of SEM in Nordic research shows four directions in the politics of meaning that could be summarized as follows: There is a need for improving educational effectiveness and thereby students' achievement through successful methods and approaches. In addition, these methods should promote inclusion in mathematics. The teachers are at the centre of change or stagnation because proven experience has precedence over research, although the reasons are not always known to the researchers. Taking into account this diversity of directions of the politics of meaning, it is essential to recognise and communicate the different views on mathematics teaching and learning in various social contexts. These views will influence the direction of research and practice regarding SEM. One way of doing so could be that teachers, principals, policymakers and researchers work more closely together and initiate research and development projects with a common understanding or at least an exchange of the direction of intentionalities. This goes beyond forming research aims, questions and methodology together, having more to do with the underpinnings of actions and choices through societal motives, values and assumptions. To be more explicit regarding these notions could lead to an opportunity to be aware of the possibilities and obstacles that are socially constructed. Since special education is a multidisciplinary field, it is essential that research pays attention to the diversity of research results, fields and methodologies.

It is noteworthy that in the reviewed articles there is a lack of discussion or even a mention of students with disabilities. Looking at the results through the words of Lambert (2015, p. 15):

[...] how can we create enabling rather than disabling mathematics classrooms for a broader range of learners? Mathematics education must include disability in calls for equity, as well as include learners with disabilities in research.

A reflection on this is that the students in greater need and with disabilities might be lost in the general relational approach towards learning difficulties. A question that rises is whether the research pays attention primarily towards students with more moderate needs in mathematics.

Finally, RQ 3 still remains to be addressed. The question asked was if it was possible for the developed methodology to deconstruct the meaning-making in SEM. In this study's exploration of the politics of meaning in SEM, three theoretical approaches have been employed: (1) mediated meaning-making in texts (Fairclough, 2003; Scollon, 2001), (2) the directions of intentionalities (Skovsmose, 2016) and (3) special educational perspectives (Nilholm, 2005, 2007; Persson, 2008). These frameworks were useful because they allowed the deconstruction of meaning-making while still contextualising the text within both mathematics and special education simultaneously. This contributed to the problem area: Special Education Mathematics (SEM).

The special educational perspectives are in the background in the depiction of the result. The reason for this is that the categorisation itself was an analytical step that provided the means to identify obstacles and opportunities. These obstacles and opportunities reveal the intrinsic motives for research, as well as its underlying values and assumptions. This made the direction of intentionalities explicit, and thereby made it possible to capture the politics of meaning across texts. However, there is still a need to be more explicit about the connection between perspectives in further studies. Our use of Fairclough and Scollon explicitly contributed to a way of approaching the reviewed research texts – the mediating means – as a collective meaning-making and at the same time displaying their contribution to the problem area and SEM. In this way, we could conclusively take stock of the politics of meaning in the selected Nordic publications. Hence, this methodology would be applicable to advocate for a larger international sample.

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Feedback to encourage creative reasoning

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This paper presents the way pilot studies underpin a design for a future project investigating how formative feedback can be designed in order to support students' creative reasoning when constructing solutions to mathematical tasks. It builds on the idea that creative reasoning is beneficial to students' mathematical learning. Four pilot studies have been performed with the purpose of creating an empirical base for the preparation of formative feedback to students in mathematics classrooms. The results represent a development of general theoretical guidelines for formative feedback. Our specific and empirically based guidelines will act as a starting point for further intervention studies investigating the design of formative feedback aimed at supporting students' creative reasoning.

Students, who are encouraged to construct their own solutions and create arguments when solving mathematical tasks, tend to, if they are successful, learn or remember more from such activities than students who are being guided by templates and prepared examples (Hiebert, 2003; Jonsson, Norqvist, Liljekvist & Lithner, 2014; Olsson, 2017). Despite the disadvantages, described in numerous research reports, of teaching mathematics by providing solution methods to tasks, such teaching is still prevalent in many classrooms, in Sweden, as well as around the world (Blomhøj, 2016; Boesen, Lithner & Palm, 2010; Hiebert & Grouws, 2007). Teaching where students create and justify their own solutions (i.e. engaging in creative reasoning) require other teacher-student interactions than the traditional teaching. Rather than explaining which method to use as well as how and why it works the teacher must encourage students, not only to construct own solutions, but also to challenge them to justify their choice of method (Hmelo-Silver, Duncan & Chinn, 2007).

A teacher-student interaction aimed at supporting students' construction of solutions can be compared to feedback aimed at supporting the students' learning processes and hence relies on the active involvement of the student (Hattie & Timperley, 2007; Nicol & Macfarlane-Dick, 2006). Teaching that incorporates

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feedback on process level requires the teacher to provide such feedback in very short cycles of interaction where they may have only seconds to decide on an appropriate response to a question (William & Thompson, 2008). Research on formative assessment and feedback often reports general guidelines on how to incorporate such practices in teaching but few studies present empirical results detailing how feedback can be prepared and designed in classroom situations (Hattie, 2012; Palm, Andersson, Boström & Vingsle, 2017). Hence there is need for a deeper understanding of how formative assessment/feedback is implemented on classroom level (Hirsh & Lindberg, 2015). This study is one in a series of four pilot studies with the aim of identifying what characterizes the kind of feedback that leads students to reason creatively. We plan to use the result of this study to support the design of a set of future interventions in an iterative cycle of design research with the ultimate goal of developing guidelines for formative feedback that will lead to creative reasoning.

Background

Teaching that encourages students to construct and justify solutions to mathematical tasks entails a focus on reasoning. Furthermore, it is reasonable to assume that teachers' feedback may guide the character of students' reasoning. The following paragraphs will outline distinctions between different types of reasoning and feedback relevant for the didactic design addressed in this paper.

Imitative and creative reasoning

If the teacher explains a definition of a mathematical concept and then demonstrates how to solve tasks associated with this particular concept, it is possible for students to solve similar tasks by remembering the procedure without understanding the definition. Lithner (2000, 2003) found that students trying to apply memorized procedures often had difficulties when solving tasks for which there had been no recent teaching. For example, calculating $2^3 \times 2^4$ using a memorized process could mean mixing up whether the numbers should be added or multiplied. The reasoning associated with such an approach is defined as imitative (IR) (Lithner, 2008). A variant of IR is AR, algorithmic reasoning which is relevant for this paper. AR entails recalling a memorized, stepwise, procedure or following procedural instructions from a teacher or textbook, that are supposed to solve a task (Lithner, 2008). AR is algorithmic in the sense that it solves the associated task, but it does not require an understanding of the mathematics on which the procedure is based.

An alternative approach to the example above, $2^3 \times 2^4$, may be to consider what the mathematical meaning behind the expression is, i.e. 2^3 means $2 \times 2 \times 2$, and 2^4 means $2 \times 2 \times 2 \times 2$. After realizing this, the next step is to put the two together, $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$, which is 2^7 . If the student can express

mathematical arguments for the solution she is engaged in creative mathematical reasoning (CMR). CMR is characterized by the constructing or reconstructing of a solution method and the expressing of arguments for the solution method and the solution (Lithner, 2008).

Formative feedback

As stated above it is possible for students to reach correct answers to tasks without understanding the mathematical concepts involved (Brousseau, 2002). If a student should fail in his or her attempts to solve a mathematical task, the most obvious feedback from the teacher may be an explanation regarding how to solve the task, not to explain the mathematics it is based on. Should the student, however, be responsible for the construction of the solution method, she is helped by understanding the mathematics required by the task. In such cases, if the student encounter difficulties, it is appropriate for the teacher to inquire into the student's thinking. By knowing something about the student's thinking the teacher has a base on which formative feedback that supports the student's solving process, can be built (Hattie & Timperley, 2007).

Formative feedback can address several different dimensions of learning. Hattie and Timperley (2007) defines four levels of feedback, *task level*, *process level*, *self-regulation level* and *self-level*. In this paper we are interested in feedback on task level and process level. Feedback at task-level is about how well a task is being accomplished or performed and 90% of feedback in classroom is on task-level (Airasian, 1997). Such feedback typically distinguishes correct answers from incorrect ones and suggests methods for solving a task, which often entails a focus on surface understanding (Hattie & Timperley, 2007). Feedback on process-level turns the focus to the underlying process that will solve a task. This involves relations between e.g. the mathematical content and students' perceptions of a task. Such feedback supports a deeper understanding and construction of meaning, which are cognitive processes (ibid). Feedback on process-level is typically given in dialogue.

Method

In this study we focus on teachers' feedback to students in situations where they need help with their problem solving and/or their explaining or presenting of solution methods or solutions. Feedback will be considered as both a response to students' actions and guiding their continued reasoning. The chain *students' action – teacher's feedback – students' continued reasoning* will be the unit of analysis. That chain is possible to capture through voice-recordings which may be transcribed into written text. The feedback and reasoning of interest for this framework may be described as informal and as the first visible (audible) result of human thinking. If such data would be captured later in the process,

for example through interviews, it would be more or less modified and possibly more different from the thinking processes that created it. Therefore, to be as close as possible to the feedback and reasoning, we choose to capture them in direct relation to the thinking processes that create them.

Sample

The method has been developed in 4 pilot-studies with 11–12-year-old students in classroom activities. However, it is reasonable to use it in other situations with students of different ages.

Didactic situation

The teaching environment is set up in line with ideas of Brousseau (2002), i.e. students learn mathematics when they construct solutions to mathematical problems prepared by the teacher. Learning takes place when students create meaning of the mathematics included in the problem, in the sense that they understand the way it contributes to the solution. The role of the teacher is to support students' construction of solutions and the main principles for the didactic situation are:

- The task must constitute a problem in the sense that students must not know a solution method in advance.
- Students are responsible to construct solution methods.
- Students have responsibility to justify solution methods and solutions.
- The teacher's interaction aims to support students' construction of solutions without revealing a solution method.

Tasks

The tasks are designed in line with Lithner's (2017) guidelines for tasks requiring CMR. That (1) no complete solution method is available from the start to a particular student, and (2) it is reasonable for students to justify the construction and implementation of a solution. Figure 1 shows an example of one of the tasks used in the pilot studies.

To the right is a picture of flowers and tiles.

- a. How many tiles would you need if there were 5 flowers in the same arrangement?
- b. How many tiles would you need for 10 flowers?
- c. How many tiles would you need for 100 flowers?
- d. How would you calculate the number of tiles for n flowers?

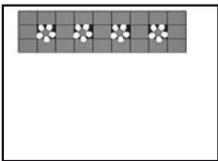


Figure 1. *The task Flowers and tiles*

Prepared feedback

The idea of prepared feedback used in our pilot studies builds on three main principles: asking students to explain their thinking, challenging students to justify why their solution method will work and challenging students to justify why their solution is correct (Olsson, 2017).

To prepare feedback a hypothetical path to solve a specific task is established. In relation to the mathematical ability of the students who will work on the task reasonable solution methods are foreseen and expected difficulties noted. Furthermore, acceptable justifications for both solution methods and solutions are formulated. For each of these items specific feedback in line with the main principles is prepared. Table 1 shows an example of prepared feedback on an expected difficulty when solving the task flowers and tiles (figure 1). The student has solved a) by counting tiles. When solving b) she doubles the result from a).

Table 1. *Example of prepared feedback*

Difficulty	Feedback
The student has not identified the "unit" through which the pattern is extended in order to add a new flower	Ask the student to verify her answer (if she is not able to verify the answer, ask her to draw and count), if (when) the student realizes that the answer is not correct, ask her why her solution method does not work (if needed, ask her to draw the extra tiles necessary to be able to include another flower). If the student manages to reach a correct answer, ask her if and why she is certain she's right, ask her what happens when one more flower is added (5 tiles must be added).

Procedure

The students work in pairs. The task is presented in written text and the students are asked to present written solutions. The students are encouraged to collaborate and ask questions if they don't understand the instructions.

During the lesson the teacher interacts with the students and provides the prepared feedback when appropriate. The recordings are transcribed into written text with a focus on spoken language.

Analysis method

The analysis will focus on the chain students' action – teacher's feedback – students' continued reasoning and will be performed through the following steps:

- 1 Parts of transcripts where the teacher interacts with students will be identified.
- 2 The relation between teachers' feedback and students' reasoning is described.

- 3 Determine how feedback supports or does not support students' CMR.
- 4 Guidelines for feedback are revised.

The revised guidelines for feedback to support students' CMR are used to design specific feedback for the next intervention, in which it will be tested and subject to further revision.

Results from pilot studies

The results from the four pilot studies underpin a didactical design that is planned to be tested and developed in a future series of intervention studies. The ambition is to develop the general principles for feedback (see method) into more specific guidelines. When analysing transcripts from teacher interactions in the pilot studies some examples have been perceived as interesting. We have observed that students' oral and written solutions are often fragmented and many of the elements are understood implicitly. In those situations, the teacher should not participate in the implicit shared understanding but should encourage students to be explicit and structured in their written and oral presentation of solutions. Feedback in this situation, when the students were engaged in understanding that 5 tiles must be added to every flower (see figure 1), was thus designed with the guideline "encourage students to explicitly articulate their understanding". The following transcript is an example when the teacher does not accept implicit understanding:

- Teacher: What did you do to figure out this for example? The first one?
Student: We counted
Teacher: You counted. OK. And what did you say that ...?
Student: That ... that I cannot ... because ... if ... well this goes to 2
Teacher: Uhu
Student: So I added five more
Teacher: Uhu
Student: ... and then there were five flowers ...
Teacher: OK and if you had added another flower, how many tiles would you have added then?
Student: Well then I would have added five more
Teacher: Five more. And if you had added one more flower, how many would you have added then?
Student: Five. Five, five, five, five ...
Teacher: OK. So here you have concluded something. What did you conclude?
Student: ... that there are five

- Teacher: ... [five] what?
 Student: ... five tiles
 Teacher: For what?
 Student: For each flower
 Teacher: OK. Five tiles for each flower.

Another example of a designed feedback was considered after the observation that a great number of students did not take advantage of the easier parts of tasks like *Flowers and tiles*, when solving the more difficult parts of the task. The teacher should thus encourage the student/s to revisit an earlier part of the task and reflect on the similarities between the earlier and present parts in order to verify solutions or discover faulty conclusions. The following extract from a transcript where students claimed that $5 \times 100 + 28$ would calculate the number of tiles for 100 flowers (see figure 1):

- Teacher: Uhu ... could you try your theory on this one [pointing at the subtask with 5 flowers]
 Student: Uhu ... then we count 5 times 5 ... which makes ... but this isn't correct ...
 Teacher: At what point did you go wrong ... or do you have to rethink this ...

This interaction proceeded and in the end the students examined if their theory worked on one flower surrounded by tiles. Then they could formulate a general calculation:

- Student: Now I got it ... $5 \times$ number of flowers $+ 3 =$ numbers of tiles
 Teacher: Why is that working ...
 Student: Because every flower means 5 tiles ... except for the last one ... or the first one ... you must add 3

Summary of results

The results presented above are examples of how general guidelines on how to provide formative feedback on process level to students are played out in actual classroom situations. Both examples show students whose reasoning is CMR before interacting with the teacher and that feedback guided to continuing CMR. The two examples of feedback; encouraging students to formulate all their justifications and conclusions explicitly through questions and encouraging students to look back at previous steps in their process, are based on empirical observations, which are used to connect what is happening in actual mathematical teaching to general guidelines of feedback. Furthermore, the expected parts of students' path to solutions could be identified and the prepared feedback to support students CMR could be delivered.

Discussion

Creative reasoning is a powerful tool in the learning of mathematics. In mathematical problem solving creative reasoning, where conjecturing and justifying are viewed as important parts, leads students to construct their own solution to mathematical task, something that previous studies have found beneficial for their learning (Jonsson et al., 2014; Olsson, 2017). Justifying and conjecturing are cognitive challenges that support understanding of the mathematical content. If students are provided with the solution method and are informed about the correctness of their answer, such challenges are removed. For example, the task Flowers and tiles (see figure 1) could be introduced by the teacher showing the class how to solve a similar task, explaining how to look for the change depending on numbers and how to identify the constant-term. This kind of teaching has the potential to guide students towards trying to remember and apply the strategies rather than understanding when and why they are appropriate. If teaching however is focused on guiding the students to reason creatively, they will have to formulate mathematical support for their solutions. If students' justifications are indeed based on the mathematics of the particular task, it is reasonable to assume that this entails a deeper and more sustainable learning compared to solutions based on remembering and applying strategies correctly.

Teaching where students are constructing solutions while engaging in CMR does not entail a passive teacher (Hmelo-Silver et al., 2007). To foster CMR a teacher must consider how to provide feedback that supports students' problem solving without revealing solution methods. The first example presented in the results shows how a teacher instead of explaining how to solve a task poses questions addressing the process of solving the task rather than the numbers or answers involved. Furthermore, the teacher challenges students to justify both the solution method and the correctness of the solution. The teacher also refrains from formulating or interpreting implicit ideas and continuously encourages the student to articulate her thoughts. The teacher refuses to finish any of her student's sentences or to provide clues as to whether the student is right or wrong. To ask students about their thinking, and to challenge them to justify their conclusions, may be seen as part of general guidelines on feedback. The contribution of these pilot studies is that they originate in empirical observations, e.g. that students are not using their previous experience of similar tasks or that they are often not explicit when they present their solutions. Based on these observations, appropriate formative feedback could be prepared. In literature, there are numerous reviews and effect studies reporting the great effectiveness of formative feedback for learning (Hattie & Timpeley, 2007; Hattie, 2012; Palm et al., 2017). Furthermore, they often provide general guidelines for how to design such feedback. In our opinion, research seldom offers concrete, and empirically supported, examples of how feedback specifically designed for mathematical teaching should be prepared and provided.

In the classroom, the teacher deals with the complexity of interacting with 20+ individuals. Providing feedback on process level or self-regulation level, which is typically given in dialogues, requires the teacher to, within seconds, decide on appropriate feedback, in the particular situation, with this particular student and with this particular mathematical task (William & Thompson, 2008). In such situations it may be easy to give direct feedback to task level and explain how to solve the task at hand, rather than challenging the student. The results from the pilot studies indicate it is possible to prepare for interactions with students through the design of feedback on process-level associated with expected paths through solutions of tasks. The two examples of feedback presented in results; using questions to encourage students to formulate all their justifications and conclusions explicitly and encouraging students to look back at previous steps in their process, are possibly parts of many teachers' natural feedback repertoire, but taken together they form part of a set of concrete examples of feedback that have the potential to support students' creative reasoning, something that we believe will lead to learning.

The formative feedback we suggest as supporting CMR builds on principles such as asking students to explain their thinking, challenging students to justify why their solution method will work and challenging students to justify why their solution is correct. To these general guidelines we have added two specific strategies: a) refraining from interpreting implicit ideas in order to give the student opportunities to not only think but also to articulate her thinking and b) asking the students to revisit earlier stages of the task. These guidelines for formative feedback are based on empirical evidence and our intention is to develop these guidelines over time. Our aim is to organise a series of interventions in order to further investigate the issue of how formative feedback can be designed to support students' CMR in mathematical problem solving.

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Elevers erfarenheter kring ett projekt om matematik med yrkesinriktning

KAROLINA MUHRMAN OCH PETER FREJD

Forskning visar att elever har svårigheter att rekontextualisera sina matematikkunskaper från skolkontexten till yrkeslivskontexten om matematikundervisningen saknar koppling till yrkeslivet. I artikeln analyseras, utifrån Bernsteins teorier, intervjuer med elever på tre yrkesprogram som deltagit i en interventionsstudie med målet att knyta matematiken närmare elevernas yrkesinriktning. Resultaten tyder på matematikämnet har en stark klassifikation och inramning som är svår att förändra, men om klassifikation och inramning förändras mot ett mer ämnesintegrerat arbetssätt verkar elevernas förmåga att rekontextualisera sina matematikkunskaper förbättras. Undervisningsmiljö och användandet av autentiska problem diskuteras som centrala faktorer för att detta ska ske.

En oundgänglig framgångsfaktor i en kunskapsbaserad global ekonomi sägs vara en välutbildad arbetsstyrka (OECD, 2010). Många yrken ställer i dag stora krav på goda matematikkunskaper. Det medför att en välfungerande och relevant matematikundervisning är av betydelse för att svara mot yrkeslivets behov av kompetent arbetskraft och den är därför en väsentlig förutsättning för en god samhällsutveckling. Med gymnasiereformen 2011 (Gy11) infördes en ämnesplan i matematik med yrkeskoppling på gymnasieskolans yrkesprogram, vilket har stött på svårigheter. Det finns risk för att yrkes eleverna, tvärt mot reformens intentioner, blir dåligt förberedda för yrkeslivets beräkningar och att de därmed går miste om framtida anställningar (se tex Muhrman, 2016). Rapporter från Skolinspektionen (tex 2014) tyder på att många elever på yrkesprogram fortfarande inte förstår syftet med att läsa matematik och andra gymnasiegemensamma ämnena, eftersom matematikundervisningen inte kopplas till yrkesämnena, trots ämnesplanens utformning i Gy11.

Matematiklärares dilemma på yrkesprogram är att utveckla, planera och organisera en matematikundervisning som tillgodoser såväl yrkeslivets krav på yrkesrelevanta matematikkunskaper, som allsidiga matematikkunskaper för eventuella fortsatta akademiska studier. Dessutom skiljer det sig mellan olika yrken vilka matematikkunskaper som efterfrågas i yrkeslivet (se tex Hoylse

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m fl, 2002; Asghari, 2014). Denna artikel bygger på elevers erfarenheter från deltagandet i en interventionsstudie där matematiklärare på en gymnasieskola har designat och implementerat yrkesrelaterade uppgifter på matematiklektionerna under ett år tid. Syftet är att identifiera hur matematikämnet förändras när man inför ämnesintegrerade arbetssätt och hur det påverkar elevernas möjligheter att använda matematik i olika sammanhang. Utifrån Bernsteins (1990, 2000) teorier avser vi att svara på följande forskningsfrågor:

- 1 Hur påverkas matematikämnets klassifikation och inramning vid ett yrkesintegrerat arbetssätt?
- 2 Hur påverkar matematikämnets klassifikation och inramning elevernas möjligheter att rekontextualisera sina matematikkunskaper?

Forskningsbakgrund

Studier visar att anknytningen mellan skolans matematik och yrkeslivets matematik är svag vilket kan leda till problem, när elever ska överföra sina matematikkunskaper från en skolkontext till en yrkeskontext (se tex De Corte, 1987; Muhrman, 2016). Ofta beskrivs matematiken som dold i yrkeslivet jämfört inom skolan (Williams & Wake, 2007). Flera forskningsstudier pekar också på att om matematik undervisas skild från sitt sammanhang riskerar man att det utvecklas två olika ”världar”, där matematikundervisningen är en värld och matematikanvändningen i yrkeslivet är en annan (se tex Dalby & Noyes, 2015; FitzSimons & Boistrup, 2017; Hoyles m fl, 2010; Pozzy m fl, 1998). Det går inte att förutsätta att alla matematikkunskaper för yrkeslivet kan utvecklas enbart genom praktik på en arbetsplats, då arbetarna själva kanske inte har så stora matematikkunskaper eller pedagogisk erfarenhet av att undervisa i matematik (Muhrman, 2016). Eleverna får en mer positiv inställning till matematik när lärarna kopplar undervisningen till elevernas kommande yrke så att de ser ett tydligt syfte med att lära sig ämnet (se tex Högberg, 2011; Lindberg, 2010; Muhrman, 2016). Skolinspektionen (2014) efterlyser därför ämnesintegrerade arbetssätt i större utsträckning än vad som ofta används, där delar av programmets yrkesämnen integreras i programmets kärnämnen eller tvärtom. I en studie av Frejd & Muhrman (inskickad) syns positiva effekter av ett ämnesintegrerat arbetssätt men också stora skillnader i utfallet beroende på hur ämnesintegreringen organiseras.

Lindberg (2010) påvisar i en studie av ett utvecklingsprojekt avseende ämnesintegrerat arbetssätt mellan matematik och yrkesämnen såväl möjligheter som svårigheter. Eleverna fick ökad motivation för matematik och visade något bättre resultat i ämnet när man arbetade ämnesintegrerat. Det fanns dock svårigheter med samarbetet mellan kärnämneslärare och karaktärsämneslärare, som kunde relateras till dels organisatoriska hinder och lärarnas utbildningsbakgrund (ibidem).

Teoretisk ansats

Den teoretiska basen utgår från Bernsteins (1990, 2000) teori om pedagogiska koder. Bernstein använder sig av pedagogiska koder för att förklara hur olika faktorer påverkar utformningen av aktiviteter i skolsystem. Med hjälp av Bernsteins pedagogiska koder kan vi analysera och diskutera matematikämnets relation till andra ämnen i skolan och till yrkeslivet, samt hur denna relation kan påverka elevernas lärande och inställning till ämnet. I denna artikel kommer vi främst att använda oss av begreppen *klassifikation*, *inramning*, *synlig- och osynlig pedagogik* samt *rekontextualisering*.

Klassifikation och inramning

För att förklara vad det finns för gränser för en kategori och hur denna förhåller sig till andra kategorier använder sig Bernstein (1990, 2000) av begreppen *klassifikation* och *inramning*. En kategori kan till exempel vara ett ämne i skolan, i detta fall matematikämnet, men det kan också handla om en hel skolkontext. Beroende på i vilken grad en kategori är avskild från andra kategorier pratar Bernstein om svag eller stark klassifikation. Vissa skolämnen är tydligare avgränsade från andra ämnen (stark klassifikation), genom att de har en stark kunskapsbas där det är tydligt uttalat vad som anses vara ”giltig” kunskap, vilket gör att undervisningens utformning blir relativt likartad i ämnet, oavsett lärare. Detta gäller bland annat för ämnet matematik, som till exempel kan jämföras med samhällskunskap som är ett ämne med svagare klassifikation och där undervisningens innehåll och upplägg har en större påverkan av lärarens egna erfarenheter och tolkningar.

Medan klassifikation handlar om ämnets yttre avgränsningar mot andra ämnen, handlar inramning om vilket lärostoff som görs tillgängligt i ämnet och hur det görs tillgängligt. Vid en stark inramning har eleverna ingen möjlighet att påverka innehållet i undervisningen eller undervisningens utformning. Vid en svagare inramning ges eleverna möjlighet att vara med och påverka val av innehåll, tidfördelning eller organisation. Katederundervisning kan till exempel ses ha en starkare inramning än ett projektarbete där eleverna själva får välja innehåll och styra över sin tid (Bernstein, 1990). Ju starkare klassifikation ett ämne har desto större blir möjligheterna till kontroll, eller inramning av innehållet. Matematikämnets inramning styrs till stor del av yttre faktorer, som innehållet i läroplanen, matematikboken, nationella prov, timplan osv, men även inre faktorer som lärarnas didaktiska val när det gäller undervisningens upplägg, har stor inverkan på inramningen. Med begreppet inramning kan vi analysera hur de yttre faktorerna, tillsammans med lärarnas didaktiska val när det gäller undervisningens upplägg kan påverka elevernas inställning till matematik.

Osynlig och synlig pedagogik

Bernstein (1990) använder sig av begreppen klassifikation och inramning för att diskutera det han kallar för *osynlig* och *synlig* pedagogik. Den synliga

pedagogiken har en starkare inramning med en mer uttalad inriktning på individuellt lärande och på faktakunskaper. Arbetsdelningen mellan elever och lärare har tydliga gränser, innehållet är till stor del styrt av en lärobok och elevernas kunskap bedöms vanligen med prov som har tydliga kunskapskriterier. Den osynliga pedagogiken har en svagare inramning och är mer inriktad på kompetens än faktakunskaper. Kunskapsinhämtningen sker ofta på gruppnivå genom sociala relationer. Innehållet är inte så styrt av läroböcker eller timplaner och kan väljas mer fritt efter den rådande situationen. Mål och kriterier är mer flytande.

Att förändra inramningen hos ett ämne med stark klassifikation kan enligt Bernstein (1990) vara svårt. Om ett nytt arbetssätt införs som innebär att ämnets klassifikation mot andra ämnen försvagas, till exempel vid ett ämnesintegrerat arbetssätt, uppstår enligt Bernstein en komplexitet som kan leda till konflikter både mellan lärare-lärare och lärare-elever, när lärare tvingas samarbeta och nya strukturer utformas. Studiestarka elever som lärt sig en sorts ”skolkod” som innebär att de har en klar strategi för hur de ska lyckas i ämnet, kan protestera om ämnets inramning försvagas. Dessa elever har enligt Bernstein ofta lärt sig ett rationellt sätt att producera faktakunskap som ofta leder till bra resultat på prov, vilket gör att de blir stressade om andra arbetssätt eller bedömningsformer införs.

Rekontextualisering

I relation till ett ämnets klassifikation och inramning använder Bernstein (2000) begreppet rekontextualisering när han diskuterar hur kunskaper från tex skolämnet matematik kan omarbetas för att användas i andra sammanhang. I denna artikel studerar vi om eleverna tycks kunna rekontextualisera sina matematikkunskaper, genom att de får berätta hur matematiken används inom det yrke de utbildas för. FitzSimons (2014) beskriver svårigheter när skolmatematik ska rekontextualisera till att användas i en yrkeskontext. För att det ska vara möjligt krävs det att eleverna kan se samband mellan skolmatematiken och matematikanvändningen i yrkeslivet.

Metod

I denna artikel beskriver vi en del av resultatet från en interventionsstudie. Målet med interventionsstudien var att analysera och bidra med djupare kunskap om matematiklärares förutsättningar att utveckla sin matematikundervisning på ett sätt som gör eleverna väl förberedda för en anställning inom det yrke de utbildas för och eventuella vidare studier. Studien som helhet har gjorts under läsåret 2016/2017 och bestod av att författarna följde matematiklärarna på ett yrkesgymnasium och deltog vid planeringsmöten, klassrumsobservationer och före- och efterintervjuer, med matematiklärare och yrkeselever. Denna artikel baseras på två intervjuomgångar som har gjorts med yrkeselever i åk 1 *före* och *efter* att de har jobbat med olika yrkesintegrerade matematikprojekt, framtagna av matematiklärarna i samband med interventionsstudien. Exempel på matematikprojekt som ingick i interventionerna var att arbeta med

kostnadsberäkningar för schamponering med frisörelever, omvandling av recept med restaurang och livsmedelselever och bestämning av cylindervolymer i motorblock med fordonslever. Eleverna som berörs kommer från hantverksprogrammet – frisör (4st), restaurang och livsmedelsprogrammet (3st), samt fordonsprogrammet (3st). Två intervjuguides utvecklades och användes (en till varje omgång). Den första intervjuguiden innehöll 8 frågor inom två teman (*Använder yrkeslever matematik i sitt arbete, i sådant fall vad och inom vilka områden. och Hur utvecklar man matematikkunskaper för sitt yrke?*) behandlade elevers förväntningar på sin kommande matematikutbildning på gymnasiet.

Intervjuguide 2, innehöll 12 frågor i syfte att utvärdera elevernas erfarenheter av termin 1, inom teman relaterade till *arbetssätt, inställning till matematik och kunskaper om matematik i yrkeslivet*. Några frågor syftar även till att undersöka elevernas *önskemål* om hur man kan förbättra undervisningen. Frågor om vikten att kunna matematik och om eleverna kunde beskriva några områden där man behöver matematikkunskaper fanns med i båda intervjuguiderna.

Empirin som denna artikel bygger på är i huvudsak hämtad från intervjuomgång 2, men en liten jämförelse gjordes huruvida eleverna gav fler exempel på situationer där de behöver matematik i sitt kommande yrkesliv efter att de deltagit i studien. Intervjuerna i transkriberades och analyserades utifrån Bernsteins (2000) begrepp om klassifikation, inramning och rekontextualisering. Nedan följer en först en sammanfattning av intervjuerna som sedan analyseras och diskuteras.

Sammanfattning av intervjuer

Frisörelever

Frisörerna tycker att ”De behöver ju matte till allt” och ger flera exempel från sitt yrke tex vid klippning för att mäta längder och vid färgblandning för att beräkna volymer och förhållanden. De uppskattade mycket när de fick ha ämnesintegrerade lektioner mellan matte och frisörämnen: ”Det är roligare, det känns bättre att lära sig det man utbildas för. Det känns som vi behöver mer av det”. Bland annat ansåg de att dessa lektioner leder till att de numera tänker mer på hur mycket matte det finns i frisöryrket. Framför allt miljön i frisörsalen gör, enligt eleverna att de förstår lättare vad de behöver kunna. De upplevde inte att ämnesintegrering i klassrummet var lika givande som i frisörsalen, en ämnesintegrerad lektion mellan matematik och frisör med kalkylberäkningar i matematikklassrummet upplevdes som ganska svår och krånglig. Helst vill de arbeta med matematik i frisörsalen en gång per vecka. De vanliga matematiklektionerna har sällan någon koppling till deras yrke och har ganska lika upplägg med genomgång och enskilt arbete i boken – ibland någon gruppdiskussion, berättade eleverna. De märkte inte att matematikläraren och frisörlärarna hade börjat samarbeta ordentligt och ansåg att lärarna är ganska oklara och inte ger så tydliga instruktioner. Integreringsprojekten, tycker de inte har haft någon stor

påverkan på deras inställning till matematik, men de anser att det påverkar deras kunskaper i både yrket och matematiken. De har blivit bättre på att förstå kostnadsberäkningar som tex att beräkna schampoåtgång för att veta hur mycket betalt man måste ta. Eleverna är klart besvikna över att matematiken inte har haft en större koppling till deras yrkesinriktning.

Fordonselever

Fordonseleverna som intervjuas säger att de tycker att de under sin första termin på utbildningen har fått större insikter i att det behövs mycket matematikkunskaper i fordonsyrket. Jämfört med den första intervjun kan de ge betydligt fler exempel på beräkningar som är kopplade till deras yrke, bland annat att det är mycket beräkningar relaterat till ellära. De är klart besvikna för att de inte har fått arbeta mer med yrkesrelaterad matematik, vilket de trodde att de skulle få göra nu i gymnasiet: ”Vi har ju inte tjatat till oss detta, utan han lovade ju det”. De tycker att det skulle vara till hjälp för dem om de fick ha mer matte i fordonssalen: ”Vi skulle kunna vara mer i fordonssalen, för att förstå mer, det gav mycket, man förstod varför, meningen alltså”. På matematiklektionerna arbetar de mest med boken enligt upplägget, genomgång – enskilt räknande. De har alltid arbetat på samma sätt i matematiken under sin skolgång, så de säger att de är vana vid detta arbetssätt. Men om eleverna själva fick bestämma så skulle hälften eller åtminstone en tredjedel av matematiklektionerna vara förlagda i fordonssalen.

Restaurang och livsmedelselever

Dessa elever tycker att det kan vara viktigt att kunna matematik för sitt vardagliga liv snarare än för sitt yrke. De säger att matematik behövs överallt, men inte så mycket i deras yrke. Beräkningar som de gör i sitt yrke är enhetsomvandlingar tex vid receptomvandlingar och överslagsberäkningar när de står i kassan. På matematiklektionerna används ”flipped classroom” med filmer som de ska se innan varje lektion, det tycker de fungerar bra. Ibland har de några diskussioner och grupparbeten, men mest arbetar de tyst enskilt i sin bok. De ämnesintegrerade uppgifterna i klassrummet tycker de inte har varit så givande. De är tre olika inriktningar i klassen och alla har fått arbeta med alla inriktningars ämnesintegrerade uppgifter. De ämnesintegrerade uppgifterna i klassrummet kändes inte alls realistiska och hade ingen tydlig koppling till deras yrke. Det var en lite ”rolig” aktivitet, men inte så väsentlig för yrket. De har inte heller några direkta önskemål om att få en mer yrkesanknytning eftersom de anser att det inte behövs så mycket matematik i deras yrke. De säger att det känns mer ”effektivt” att arbeta i boken och då kan alla arbeta på sin egen nivå och de slipper då också genomgångar på sådant de redan förstår. Målet med att läsa matte 1a är för dessa elever i första hand att få ett betyg så att de kan få en examen. De säger att ”Mycket av det här kommer jag aldrig använda” men att de ändå måste göra det för att få sitt mattebetyg.

Analys och diskussion

Resultaten från före- och efterintervjuerna har analyserats i relation till Bernsteins teorier om klassifikation och inramning för att skapa förståelse för hur matematikämnet förändras när man inför ämnesintegrerade arbetssätt och hur det påverkar elevernas möjligheter att använda matematik i olika sammanhang.

I analysen av elevsvaren framkom, precis som Bernstein (2000) beskriver, att det inte är lätt att förändra ett ämnes klassifikation eller inramning. Trots att lärarna har haft goda intentioner att integrera matematiken med yrkesämnena och har gjort försök att upprätta samarbete mellan matematiklärare och yrkeslärare, tyder elevsvaren på att matematikämnet fortfarande har en stark klassifikation gentemot yrkesämnena. Lärarnas försök att samverka har inte fått något tydlig effekt på undervisningens inramning vilket visar på komplexiteten i yrkesprogrammets matematikundervisning, som styrs av såväl krav från Skolverket, med bland annat nationella prov, som krav från branschorganisationer om anställningsbarhet (se tex Muhrman, 2016; Tsagalidis, 2008). Alla eleverna som har intervjuats berättar att lärarna målade upp en bild av att de skulle få en ”annorlunda” matematikundervisning med tydlig yrkesanknytning när de började gymnasiet. Eleverna anser dock att detta inte har infriats i tillräcklig stor utsträckning. Det har visserligen förekommit några enstaka tillfällen med yrkesanknuten matematikundervisning, men till allra största delen har undervisningen skett på ”traditionellt sätt” med i huvudsakligt arbete i matematikboken. Enligt Bernstein (2000) leder en stark inramning till att eleverna inte ges utrymme att påverka vare sig innehållet eller utformning av undervisningen. Både fordonseleverna och frisöreleverna har tydliga idéer om hur de anser att deras matematikundervisning borde vara utformad för att ge dem möjligheter att utveckla de matematikkunskaper de behöver för sitt framtida yrke. Deras önskan är att en del av matematikundervisningen regelbundet ska förläggas i yrkessalarna, eftersom de anser att de då får lättare att koppla matematiken till yrkesinriktningen, vilket gör dem mer motiverade att läras sig. Dessa resultat kan relateras till den tidigare forskningen som visar att eleverna kan se ett tydligare syfte med att lära sig matematik och därmed blir mer motiverade, när ämnet på olika sätt kopplas till deras yrkesinriktning (se tex Dalby & Noyes, 2015; Lindberg, 2010; Muhrman, 2016).

Även om eleverna i denna studie var positiva till att arbeta ämnesintegrerat i större utsträckning gick det också att se att den svagare inramningen av matematiken i vissa fall ledde till det Bernstein (2000) beskriver som en osynligare pedagogik. Eleverna efterlyste tydligare information från deras lärare eftersom de ibland kände sig osäkra på syftet med det ämnesintegrerade arbetet.

Även när det gäller frågan om att rekontextualisera sina matematikkunskaper till yrkeskontexten, visade frisöreleverna och fordonseleverna upp liknande resultat. Båda dessa elevgrupper visade en betydligt större insikt i värdet av att lära sig matematik för deras kommande yrkeskarriär vid den andra intervjun. I den första intervjun som vi gjorde med dessa elevgrupper precis i början av

deras gymnasieutbildning, hade de en bild av att det inte var så stort behov av matematikkunskaper i deras yrkesinriktningar och de kunde inte ge så många exempel från sina respektive blivande yrken där matematik används. Vid den andra intervjun, som beskrivs i denna artikel, kunde eleverna ge ett flertal exempel från sina yrkesinriktningar där de använder sig av matematik. Eleverna beskriver inte bara exempel från de ämnesintegreringsprojekt som de har arbetat med, utan de visar en djupare förmåga att generellt kunna rekontextualisera matematiken i flera av sina arbetsuppgifter. Till exempel beskrev fordons- eleverna en mängd arbetsuppgifter rörande bilars elektronik där de utför olika beräkningar. Dessa skillnader kan givetvis ha flera förklaringar, det ämnesintegrerade arbetet med att koppla matematiken till yrkesämnena kan ha viss betydelse, men en del av förklaringen kan också vara att eleverna totalt sett har större kunskaper om sitt blivande yrke nu än de hade vid den första intervjun.

Av de tre elevgrupperna utskilde sig eleverna från Restaurang- och livsmedelsprogrammet tydligt. Dessa elever visade inte någon ökad kunskap när det gäller att rekontextualisera sina matematikkunskaper till yrket, snarare tvärtom. Vid den första intervjun i början av utbildningen gav eleverna några exempel på när matematik används i deras yrkesinriktning, vilket var i stort sett samma exempel som de gav vid den andra intervjun. I den första intervjun hade eleverna däremot en tanke om att det kan behövas en hel del matematik i deras yrke, vilket de mer eller mindre i den andra intervjun ansåg att det inte behövdes. För dessa elever hade matematiken närmast fått en starkare klassifikation mot yrkesämnena än den tidigare hade haft. Restaurang- och livsmedelseleverna var inte heller missnöjda med matematikämnet starka inramning, med en traditionellt utformade undervisning som till stor del bestod av enskilt räknande i matematikboken. Eleverna visade tydliga drag av de studiestarka elever som har ”knäckt skolkoden” som beskrivs av Bernstein (1990, 2000). Dessa elever var i första hand intresserade av att reproducera kunskap för att klarar proven och därmed få ett godkänt betyg i ämnet. De upprepade flera gånger vid intervjun att de inte kunde se någon användning av den matematik de jobbade med under matematiklektionerna, men verkade anse att det inte hade någon större betydelse. Precis som Bernstein pratade de också om ”rationella” eller mest effektiva sätt att lära sig det de var tvungna till för att klara kursen. Dessa elever ansåg att det var effektivast att räkna tyst för sig själv i boken, gärna med hörlurar på för att avskärma sig från andra elever.

Anledningen till att restaurang- och livsmedelsprogrammets elevsvar avvek så tydligt från de andra elevernas svar kan vara flera. Att det förekommer beräkningar även inom denna yrkeskategori är det ingen tvekan om. Författarna som deltog i detta forskningsprojekt har sett många exempel på matematikanvändning inom både restaurang och bageriyrket när vi har observerat yrkeslektioner, men i många fall är kanske matematiken mer dold inom dessa yrken för eleverna än vad den är inom frisör och fordon. Detta gör att det i sådant fall kan krävas mer arbete av matematik- och yrkeslärarna att

synliggöra den dolda matematiken (jfr Williams & Wake, 2007). Restaurang- och livsmedelselevernas svar tyder på att matematiken kan vara alltför dekontextualiserad från yrkesinriktningen för att eleverna ska förstå dess användning och klara av att rekontextualisera sina kunskaper (se t ex Dalby & Noyes, 2015; FitzSimons, 2014; FitzSimons & Boistrup, 2017; Hoyles m fl, 2010). Även om matematikläraren har gjort försök med att koppla matematiken till yrkesämnen tyder elevernas svar på att uppgifterna i flera fall inte har varit yrkesautentiska. För eleverna kan det uppfattas som att det inte finns några autentiska matematikproblem att lösa inom deras yrke, vilket kan vara en anledning till att eleverna i efterintervjun tycks se matematiken som mer avskild från sitt yrke än vad den gjorde i den förintervjun.

Slutsats

Vår analys av datamaterialet tyder på att det kan vara svårt att påverka matematikämnets klassifikation och inramning vid ett yrkesintegrerat arbetssätt. Eleverna uppfattar det som ”en rolig grej” och i vissa fall lärorik, men att matematikundervisningen sedan återgår till en traditionell, väl definierad struktur, med stark klassifikation och inramning. Samtidigt tyder vår analys på att förändringen av matematikämnets klassifikation och inramning ger elevernas möjligheter att rekontextualisera sina matematikkunskaper lättare. Detta kan även bero på ökade kunskaper inom yrkesinriktningen, men det kräver att eleverna ser en tydlig koppling mellan sin yrkesinriktning och matematiken. Studien påvisar att matematiken är olika dold inom olika yrken, vilket gör att eleverna inte nödvändigtvis ser någon koppling mellan sitt yrke och matematiken och därmed inte får lättare att rekontextualisera sina matematikkunskaper, trots ökade kunskaper inom yrkesinriktningen. För att eleverna ska kunna se kopplingen mellan matematikämnet och yrkesinriktningen kan uppgifterna behöva vara yrkesautentiska. Att arbeta ämnesintegrerat med uppgifter som inte är yrkesautentiska kan utifrån vår analys förstärka gapet mellan skolan och arbetslivet, snarare än att få dem att närma sig varandra.

För att förändra ett arbetssätt med än mer anknytning till yrkeslivet krävs, utifrån vår studie, att uppgifterna som eleverna ska göra känns autentiska och att de utförs i en sådan miljö där de hör hemma snarare än i matematikklassrummet.

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On the use of representations and teaching principles when teaching general vector spaces

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This pilot study aims to find principles, applied by teachers for organising their teaching of general vector spaces. Based on an analysis of transcripts from semi-structured interviews with two experienced teachers, one result is that both teachers claim that they design and use teaching sequences in order to make their students aware of the advantage of using the formal theory and of switching representations. Another result is that they do so in different ways, both characterised by treatments and conversions in and between semiotic registers. The differences and their possible consequences for the development of students' abilities are discussed.

The use of different representations is fundamental for teaching and learning mathematics (Duval, 2006), in particular in linear algebra (Hillel, 2000; Larson & Zandieh, 2013). It is often in linear algebra where students encounter the role of "formalism" in mathematics, with new concepts being introduced by axiomatic definitions such as vector spaces, subspaces, linear transformations, and represented in different ways. The introduction of formal theory in mathematics often presents a challenge for students (Dorier, 1998; Dorier, Robert, Robinet & Rogalski 2000; Hillel, 2000). Dorier and Sierpinska (2001) identified three factors accounting for students' difficulties to learn linear algebra, all related to formal theory and representations. Firstly, the axiomatic theory is by many students considered useless to learn, as many linear problems can be solved by other methods than formal theory of vector spaces; secondly, the "abstract" language and the different representations used by teachers and textbooks do not appear as natural to students; and thirdly, in order to develop "theoretical thinking", students must learn to switch between representations of concepts.

Arguments for building teaching sequences that can develop students' understanding of the advantages of using theory of vector spaces as a tool to unify problems, methods, and concepts in a general approach have been put forward (Dorier, 1998). In this context it is crucial to develop their ability to "translate" between modes of representation (Hillel, 2000).

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The aim of this explorative study is to contribute to research on the role of representations in the teaching of linear algebra in first year undergraduate mathematics education. Specifically, we look at teachers and focus on the topic of general vector spaces. Its rationale is the following: According to Dorier and Sierpiska (2002, p. 271) "the teacher 'formats'" the use of the textbook and impacts on students' learning. An interview study by Bergsten (2011) showed that lectures have impact on students at a diversity of aspects and levels, and that students see it as more important that a lecturer explains concepts and results in an intuitive way rather than by using a formal approach. Therefore the teachers' *conceptions*, about their teaching practices are essential to explore. In line with Philipp (2007, p. 259), we see conceptions as a general notion or mental structure encompassing beliefs, meanings, concepts, propositions, rules, mental images and preferences. Our study is framed by the following research questions.

- 1 What conceptions do teachers express about their teaching of general vector space?
- 2 How do they express their use of representations of different kinds and transitions between them in their teaching?

Semiotic representations in linear algebra and teaching

All mathematical objects must be semiotically represented as means for external communication and for cognitive activity (Duval, 2006). A semiotic representation stands for something to be used for mediating something about a particular object to an interpreter (Duval, 2006). There is a distinction between the production or comprehension of a representation by a sign (*semiosis*) and the conceptual activity of understanding the object (*noesis*). In Duval's (2006) terms a semiotic system that permits transformations of representations is a representation *register*. A transformation of representations within a register is termed a *treatment* and between registers a *conversion*. For example, to transform a system of linear equations to its matrix form is a treatment, but to interpret it as linear transformation is a conversion to another register (see Larson & Zandieh, 2013).

Duval (2006) conjectured that comprehension in mathematics involves the coordination of at least two registers. Students' varying ability to make conversions therefore limits their capacity to use previously acquired knowledge and to acquire new knowledge. This conjecture was confirmed by e.g. Pavlopoulou (1993) in a linear algebra context, involving semiotic representation of vectors in \mathbb{R}^3 : in the graphical register by arrows, in the tabular register by columns and in the symbolic register by elements of an abstract vector space. These three registers are related to Hillel's (2000) three modes of representations of linear algebra: the *geometric mode*, which uses the language and concepts of

2- and 3-space, the *algebraic mode*, which uses the language and concepts of the more specific theory of \mathbb{R}^n and the *abstract mode*, which uses the language and concepts of the general formalized theory.

According to Treffert-Thomas (2015) there has not been much empirical research about teaching strategies in linear algebra. Harel (2000) describes three principles for teaching linear algebra (see also Dorier and Sierpinska, 2002). The *Concreteness principle* is defined as follows: "*For students to abstract a mathematical structure from a given model of that structure, the elements of that model must be conceptual entities in the student's eyes*" (Harel, 2000, p. 180; italics in original).

Following this principle, to teach the general concept of vector space as a generalization from structures such as \mathbb{R}^n , \mathbb{P}^n or spaces of matrices, requires that students already have constructed these structures as mental entities. The second principle is the *Necessity principle*: "*For students to learn, they must see a need for what they are intended to be taught. By 'need' it is meant an intellectual need, as opposed to a social or economic need*" (Harel, 2000, p. 185; italics in original).

This means that it is a violation of this principle to derive the axioms of vector space from a presentation of the properties of \mathbb{R}^n for a student unfamiliar with the economy of axiomatic approach for mathematical theory-building. Finally, the *Generalizability principle* is defined: "*When instruction is concerned with a [...] model that satisfies the Concreteness principle, the instructional activities within this model should allow and encourage the generalizability of concepts*" (Harel, 2000, p. 187; italics in original). Following this principle the teacher should avoid to introduce linear dependence in a geometric context by collinearity or co-planarity.

Methodology

In this explorative interview study the participants were two university lecturers at a mathematics department (L1 and L2), both well known to the authors. They were selected based on their extensive work experiences (more than 10 years of teaching) and the fact that they currently teach on different programs using different textbooks with an anticipation that the teachers would express different conceptions. A few days before the interview the participants were given a set of interview questions based on research papers on students' difficulties and teacher interventions in linear algebra. (Dorier, 1998; Hillel, 2000), paying particular attention to representations and the role of the theory of abstract vector spaces. The questions were organised around three themes and the main theme of this paper was: Teaching linear algebra with a focus on general vector spaces.

Research question 1 (RQ1) concerns teachers' conceptions for organising the teaching of general vector space. In this study we have limited the scope and focus on key factors for the organisation of teaching. *Textbooks* have a

major influence on the organisation of mathematics teaching at university level (Rensaa & Grevholm, 2015); the content assessed in *examination tasks* is often emphasized in teaching (Niss, 1993); and the amount of *time* and the type of *teaching sequences* are fundamental in developing students' abilities regarding general vector space (Dorier, 1998).

The semi-structured interviews lasted 80 and 98 minutes, respectively, and were audio recorded and transcribed. The transcripts were categorised using the four teaching organisation factors above and analysed through Harel's (2000) principles for teaching strategies and through Duval's (2006) theoretical constructs around semiotic representations.

Results

Overview of the courses

Lecturer L1

For several years L1 has been in charge of the present course in linear algebra for second year engineering students in the programs Chemistry and Technical Biology. L1 uses a traditional form of teaching, i.e. presenting at the whiteboard. The *textbook* used in the course was chosen because its emphasis on geometry. As it treats general vector spaces at the end in a sparse manner, it is supplemented by a complete set of lecture notes, including notes on dynamical linear systems and exams with solutions. L1 introduces general vector spaces early (in the fourth lecture in a series of 12), allowing the use of theoretical concepts of vector spaces for a considerable *time* during the course: "When I discuss the set of solutions I can talk about vector spaces and subspaces. Otherwise I must keep quiet about the fact that the set of solutions has a certain structure that we can explain".

The course has two exams, a mid-course exam and a final, more comprehensive exam. The *examination tasks* in both exams include general vector spaces, however, "general vector space is not in focus in this course since our students are chemists and biologists ..., but at the same time I think it is useful with general vector space as generalisation [of the theory]". L1's design and use of *teaching sequences* for general vector space mainly refers to this fourth lecture and its exercises and later on to the introduction of other topics within the course.

Lecturer L2

Over the years L2 has been in charge of many courses in linear algebra for first year engineering students, using different textbooks. He is now in charge of the course in the program Industrial engineering and management, using a textbook, authored by himself, and a separate problem collection (also available online). The *textbook* is used by L2 also for the overall structuring of the

course. The lectures mainly focus on theory, with applications such as conic sections and linear dynamic systems as examples of the use of the spectral theorem. Furthermore, two "real-world" applications are presented; firstly, the problem of identifying handwritten numbers, and secondly, in the last lecture, the principles for modern internet search engines.

Chapter 5 in the textbook called "General vector space", draws on the previous chapters dealing with linear equation systems, vector geometry, determinants and matrices. The chapter begins with the defining list of axioms for a general vector space. The introduction of such spaces in the course is placed in lecture 7 (out of 21 lectures) and thus leaving *time* for students to work on this theoretical notion. L2 points out: "You take the general definition [of vector spaces] and then swiftly turn to subspaces", thereby reducing the problem of checking all axioms to the axioms that characterise subspaces of a given space.

The *examination* is divided into two written exams, a credit-giving mid-course exam without general vector space problems and a final exam. "I usually give some problem on polynomial vector spaces in the [final] exams ... [which] did cause problems in the beginning, but not anymore, since such tasks are included in the collection of exams online".

The quote above indicates that these types of problems have become standard problems for the students. L2 also emphasises that he uses the standard inner product in \mathbb{R}^n in his exams and, for example, sometimes asks for the distance of a vector in \mathbb{R}^5 to a given three-dimensional subspace.

Analysis

RQ1

L1 expresses his use of a *teaching sequence* for introducing general vector space as follows. He defines \mathbb{R}^n and then "write[s] the axioms as a theorem" and discusses the standard basis and other bases in relation to the geometric representations of two non-parallel vectors in the plane. At this stage of the course he still lacks the algebraic tools of systems of equation, determinants, matrices, etc. for explaining the concepts of linear dependence and independence, and he tells the students: "I will not explain right now what vector can determine a basis, because we will need matrices and determinates ... For now it is sufficient to discuss if a set with two operations is a general vector space or not".

L1 hence defines general vector space on the basis of the properties of \mathbb{R}^n and then focuses on the question whether a given set with two operations is a vector space. To some extent this violates all the three teaching principles in Harel (2000). The *Concreteness principle* is violated due to the fact that \mathbb{R}^n probably has not yet settled as a conceptual entity in the students' eyes as a ground for abstraction. However, this new concept reappears later on in the course. A similar analysis refers to the *Necessity principle*, since students' intellectual need for this generalization probably is non-existent at that occasion. However, L1's intention is not yet to show the benefits of using general vector space on the

abstract level. At the beginning of the course L1 uses the geometric language of linear algebra to introduce linear dependence in the plane by collinearity, which violates the *Generalizability principle*. Later on in the course, though, when the algebraic language of \mathbb{R}^n is available, he returns to this concept, and emphasizes the importance of translations between representations throughout the course.

L2 first treats linear equations, determinants, and matrices and argues that "a gradual approximation towards more formalism can be reasonable" since he noticed that "difficulties don't add up linearly; one difficulty plus another one can result in a mental wall." This approach of building up formalism by broadening the theory and adding complexity is also illustrated by his use of *teaching sequences* throughout the course, that facilitate the learning of general vector space as illustrated below: "Regarding *teaching sequences*, I build the whole course in this way ... the same system of equation[s] has three different interpretations ... I constantly and consistently extend and broaden the scope ... and thus find a unification of seemingly separate problems".

Since L2 introduces the algebraic language of vector spaces early in the course, the students have had time to develop \mathbb{R}^n as a mental entity before he introduces general vector space, thus fulfilling the *Concreteness principle*. It is questionable whether he fulfils the *Necessity principle*. Our analysis does not indicate that L2 exploits the potential of abstract vector space methods for creating an intellectual need for this abstraction, since almost all problems given to students may be solved by more elementary approaches. Compared to L1, however, L2 has more "concrete" models at his disposal when he introduces general vector space, and thus has better opportunities than L1 to fulfil the *Generalizability principle*.

RQ2

L1 emphasises the importance of being able to switch between different representations of objects in linear algebra. An example of *conversion* between registers expressed by L1 is the switching from the set of all geometric vectors in the plane or space to ordered pairs or triplets of coordinates in \mathbb{R}^2 or \mathbb{R}^3 : "Forget that you think of directed vectors, think of pairs or ordered triplets". Another example of *conversion* refers to switching between the geometric and the algebraic mode to represent the system matrix of a linear dynamical system, when the matrix is defined as being an orthogonal projection onto a given plane in a given basis: "Sometimes I formulate it as a geometric problem, but they solve it as an algebraic one. But if you think of it as a geometric problem, you solve it at once. ... It is a very successful combination task".

The quote above indicates that L1 includes problems in the course with potential to show students the advantages of cognitive activities connected to conversions. L1 also exemplifies switching representations within a register, *treatment*. For example, a linear mapping between vector spaces can be represented in the abstract mode and in the algebraic mode by different matrices in different bases:

We introduce mappings by mean of matrices ... mostly standard mappings ... projections onto a plane or line are important. But in my lecture I ... use general vector spaces ... then introduce [different] bases and the transition [between the matrices of the mapping in different bases] in a general way.

Similar to L1, L2 has included problems in the exams intended to be "easy" geometric problems, but the students solved them algebraically. This indicates that students have difficulties with *conversions* between registers. Such difficulties appear also in higher dimensions: "You must understand that you can think geometrically and draw an ordinary figure, but still understand that we work in an arbitrary dimension. You can look at the figure and use it as a support of your procedure" (L2).

Such problems involve treatment, conversion and generalisation. For example, a projection problem in \mathbb{R}^3 given in the geometric register is first solved there, a *treatment*, and the solution is *converted* to the algebraic register by using the coordinate form of the scalar product. A projection problem in \mathbb{R}^n is first illustrated by an analogous geometric problem in \mathbb{R}^3 , yielding a solution formula as above. This formula is generalised to \mathbb{R}^n , thereby yielding the solution in the algebraic register to the original problem.

L2 also consistently uses the vector notation $\mathbf{x} = \mathbf{u}X$, where \mathbf{u} is a basis matrix and X is the corresponding coordinate n-tuple of the vector.

I avoid the problem to identify a vector and its coordinate representation, by always writing out a basis matrix and a coordinate matrix. It does not matter in the beginning ... it is of no use ... until we arrive at finite dimensional vector spaces ... since in this notation everything becomes calculation in \mathbb{R}^n .

In this way, the change of bases for a vector space (treatment), the matrix representation of a linear mapping (conversion) and the connection between its matrix representations in different bases (treatment) are easy to handle. At one occasion, however, the notation may cause some confusion: "When you talk about points and vectors. I resolve this by saying that we always use the position vector of the point, not the point itself" (L2).

Summary

RQ1

Both lecturers presented similar conceptions regarding their course organization strategies. They expressed that they start with simple concepts and subsequently increase the level of abstraction. Thus, general vector spaces gradually appear as a unifying concept for previous specialised concepts and methods. They both argued for introducing general vector spaces at an early stage in the course. In order to accomplish this L2 followed the organisation of the textbook, whereas L1 had to treat two chapters in the book in reverse order and produce extra material (lecture notes).

The overview of the transcripts may seem to indicate more similarities than differences. However, the analysis (RQ1) shows that the two teachers expressed quite different conceptions regarding teaching strategies. L1's way of introducing vector spaces does not fulfil the three teaching principles. The students at the time of introduction have not had enough possibilities to form the mental entities needed for abstraction (*Concreteness principle*), to find the need for the concepts of general vector space (*Necessity principle*), and for doing generalisations (*Generalizability principle*). The *Concreteness principle* is fulfilled by L2, since he expresses that he applies the algebraic language of vector spaces early to facilitate students' development of \mathbb{R}^n as a mental entity before he introduces general vector space. L2's approach of working in the algebraic mode instead of the geometric mode at the beginning of the course (in contrast to L1's approach) fulfils the *Generalizability principle*. The fulfilment of the *Necessity principle* for introducing general vector space is questionable for both lecturers.

RQ2

Not surprisingly, both lecturers expressed that they repeatedly apply different modes of representation (the *geometric mode*, the *algebraic mode*, and the *abstract mode*) and that they often use transition within (*treatment*) and between (*conversion*) registers in their teaching. While lecturer L2 avoids many problems for himself and for students to handle treatments and conversions by always using the vector notation $\mathbf{x} = \mathbf{u}X$, lecturer L1 (who does not use this notation) makes treatments and conversions in and between semiotic registers informally, e.g. by switching the notation between \mathbf{x} and X in the case above. The analysis shows that both lecturers regard treatments and conversions in and between registers as a fundamental part of their teaching. As was pointed out above, transformations of semiotic representations may involve several conversions, treatments and also generalisation, generating complications in making them explicit to the students.

Discussion

Students attend lectures to hear and view the lecturer present and organise the content in a pedagogic manner, stressing what is essential for them to learn (Bergsten, 2011). Students' difficulties in linear algebra often refer to their view of the axiomatic theory as meaningless to learn, to the "abstract" language used by teachers and textbooks and to the frequent transitions between semiotic representations of concepts (Dorier & Sierpinska, 2001). Teachers therefore need to find teaching strategies that facilitate for students to overcome these difficulties. The two lecturers in this study expressed that they use different teaching principles for introducing general vector space. There may be several reasons for this. The textbooks used as the underlying source for their teaching are different and the organisation of content in linear algebra textbooks varies (Rensaa

& Grevholm, 2015). The lecturers have different aims for their teaching in the two engineering programs. Whilst the program taught by L2 contains advanced courses (e.g. Optimization and Mathematical statistics) that draw on techniques of abstract vector spaces, the subsequent courses in the programs taught by L1 mainly assume a pre-knowledge of linear algebra in \mathbb{R}^n . Therefore L1's discourse about the significance of general vector spaces on introducing this concept early in the course, to be returned to later, is a meta level activity in the sense of (Dorier et al., 2000). It could be seen as an "outlook" to more advanced mathematics even though it does not fulfil the three teaching principles.

However, the necessity of learning about general vector space may be questioned (Dorier & Sierpinska, 2001) if students only see mathematics as a service subject for their future occupation (Rensaa & Grevholm, 2015). This requires the lecturers to be aware of how they fulfill the *Necessity principle*. An implication of the results in this pilot study is that lecturers should be aware of this principle and arrange their teaching according to it. One strategy that forces students to focus more on general vector space is to emphasise this subject in the exams (Niss, 1983). The interviewed lecturers expressed that a vast majority of their exam tasks can be solved in more elementary ways, without using the formalism of abstract vector spaces. It is, however, questionable to increase the role of abstract vector spaces in the exams as long as students do not have enough opportunities to see the advantages of this formalism and just learn it for the exam. This could be remedied by including examples, possibly from applied courses, where the use of this formalism is important and necessary. It should also be noted that different notations in mathematics and applied courses, in particular for the concepts of vector spaces, can be a major obstacle to students. It is therefore important for lecturers in linear algebra and lecturers in charge of applied courses to cooperate in order to achieve a consistent use of terminology.

The use of terminology for representations is related both to the two other teaching principles and to the switch between and within registers. L2 expressed a consistent use of the vector notation $\mathbf{x} = \mathbf{u}X$, which according to him made conversions and treatments of representations formally easy to accomplish. However, a potential danger inherent in the consistent use of this instead of the informal switching between \mathbf{x} and X in different registers is that it may obscure students' understanding of the role of different representations, and instead make it a mechanical procedure.

As described in the analysis, teaching about general vector space is complex and includes the need to explain conversions, treatments and generalisations. The lecturers in this study argued that they consistently construct teaching sequences aiming at familiarising their students with such changes and making an increasing level of abstraction meaningful. To develop these strategies further may include simple examples, in line with Treffert-Thomas (2015), or by being explicit with the changes of representations and semiotic registers, similar to the suggestions by Larson and Zandieh (2013).

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Questioning questions – revisiting teacher questioning practices

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This paper revisits the art of questioning in mathematics classrooms through a comparative analysis of three mathematical whole-class discussions using number strings. Data was collected from two expert teachers, and one teacher engaged in professional development learning to use number strings. The teachers' questions were analysed to find out what constitutes a mathematically fruitful questioning practice. Findings show that learning about affordances of different question types is a good start, but needs to be accompanied by developing listening skills and changing one's belief about the aim of questions in the mathematics classroom.

It is through the patterns of interaction and discourse created in the classroom that students develop a mathematical disposition

(Walshaw & Anthony, 2008, p. 522).

One of the oldest and most powerful didactic tools in a teacher's toolbox is the use of good questions to initiate and enhance mathematical reasoning. Although we know a lot about what types of questions promote mathematical thinking (Franke et al., 2007; McAninch, 2015; Shahrill, 2013) we know less about what skills teachers need to develop in order to orchestrate powerful discussions. Is it enough to simply learn specific question types? This paper draws on a comparative analysis of three classroom discussions to answer the question: what constitutes a mathematically fruitful questioning practice?

As a result of the social turn in mathematics education research at the end of the 20th century, communication and mathematical reasoning were identified as important mathematical competencies in various frameworks (e.g. Kilpatrick, Swafford & Findell, 2001; Niss, 2003), initiating a shift in research towards interaction as an essential aspect in the teaching and learning of mathematics (Radford, 2011). Following the call to develop students' communication and reasoning competency, whole-class mathematical discussions have become a prominent feature of mathematics classrooms. In many reform curricula mathematical discussion, i.e. sharing, explaining, conjecturing and justifying in the

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public space, have become a defining features of a quality mathematical experience (Walshaw & Anthony, 2008). Students are expected to learn to communicate mathematics, but also to learn mathematics through communicating (NCTM, 2000). An early debate on this issue concludes that teaching mathematics through conversation is potentially powerful, but not all conversation are (Sfard et al., 1998). So, the big issue is *how* to promote effective conversation in the classroom. Walshaw and Anthony (2008) argue that students' opportunities for learning are greatly influenced by what they are helped to coproduce through dialogue, and that effective use of discourse makes students' mathematical reasoning visible. Thus, important features of the role of the teacher are to initiate, orchestrate and facilitate effective mathematical discourse in the classroom. In order to promote mathematical discussions, teachers need to take an active role in the conversation. A teacher who knows where she wants the discussion to be heading can use questions to facilitate and focus students' participation.

However, if there is a large amount of communication going on in mathematics classrooms, but the level of cognitive challenge and mathematical depth is low, students may learn to communicate in mathematics although they do not learn mathematics through communication. This paper revisits the art of questioning in mathematics classrooms to see in what ways communication about mathematics can be turned into mathematics learning.

Using questions as a didactical tool

Whether a question is judged as generically good or bad depends on the aim of the question. Ellis (1993) describes five different reasons for teachers to use questions in relation to mathematical learning: i) to check on learning, ii) to probe thought processes, iii) to pose problems, iv) to seek out alternative solutions, v) to challenge students to think critically and reflectively. A good question is one which fulfils its aim. If the aim is to challenge students to reflect on mathematical ideas, a good question is one which brings about such reflection.

Teacher questioning, as well as listening and responding to students, has been researched since the early 1980's (e.g. Boaler & Brodie, 2004; Davis, 1997; Ellis, 1993; McAninch, 2015; Ni et al., 2014; Shahrill, 2013; Wilen, 1987). Although researchers may have different foci when studying questions many build on the similar types of categorization. The categories used in the analysis for this paper build on Cunningham (1987), Ellis (1993) and Hiebert and Wearne (1993). Questions are seen as *factual*, *conceptual* or *evaluative*. Factual questions relate to something the students are expected to know. Often such questions are met with short answers. Conceptual questions are of two kinds; *convergent*, i.e. closed question directing the response towards a single answer, or *divergent*, i.e. open questions involving more creative thinking. Both convergent and divergent questions can be posed on a *low* or *high cognitive level*. High-level convergent questions are explanatory and analytic rather than descriptive. High-level divergent questions generate new problems or generalisations rather

than simply providing an alternative strategy. Evaluative questions probe students to make comparisons and consider justifications, proof, clarity, effectiveness or elegance. According to Martino and Maher (1999) and others, posing high-level cognitive or evaluative questions creates opportunities for students to engage in mathematical activities that promote mathematical learning.

Lately the interactive aspect of teacher questioning has been highlighted by for example Ni et al. (2014). When a question is seen as part of an interaction, the student's response feeds into the dialogue revealing how the question is understood, and providing input for further questions. Take a question like "What do you think?" A student who replies by reading from the board or giving an answer he knows the teacher is expecting is answering a simple factual recall question, whereas a student who replies by describing a pattern he has noticed is answering a conceptual question. Since norms and expectations influence how students respond, we have categorised the teacher's questions taking the student's actual response into consideration.

The study

In traditional teaching, new ideas are often introduced by the teacher, who takes on the role of an active *instructor* and *explainer* (Ernest, 1989). This type of interaction is in Sweden commonly referred to as the teacher's "go-through" (*genomgång*) where new ideas are introduced followed by application of those ideas in a variety of activities. In such a setting, the teacher has authority over the mathematical knowledge, carefully explaining it to the class. In contrast, a reform-based curriculum promotes a student-centred interaction where the teacher takes the role of *facilitator* (Ernest, 1989), relying on student input and adjusting the path of instruction to align with and build on student contributions. As a facilitator, the teacher presents the tasks and poses questions that aim to generate ideas, compare, clarify or justify.

The focus of this study is teacher-orchestrated whole-class interactions that are decidedly mathematical. Whole-class interaction is when all the students in the room are expected to engage in the interaction going on. Normally such interactions are orchestrated or facilitated by the teacher, and can serve different purposes such as sharing and comparing or targeting specific mathematical goals (Hiebert & Wearne, 1993; Stein, Engle, Smith & Hughes, 2008). Since orchestrated whole-class discussions are an integrated part of lessons using numbers strings, three such lessons were chosen to form the data for this study.

Number strings

A detailed analysis was conducted of three whole-class mathematical discussions around carefully crafted number strings (DiBrienza & Shevell, 1998; Lambert, Imm & Williams, 2017). A number string is a set of related math problems, with or without context, designed to support students in constructing big ideas about mathematics (Fosnot & Dolk, 2001). For example, in one string

the following three tasks: 146 – 12; 272 – 14; 283 – 275 are chosen to highlight the relationship between subtraction and addition by drawing out two solution strategies. The first two tasks are "easy" and will be solved by counting down on the number line, and then a challenge is presented where the more efficient strategy of "adding on" will hopefully emerge. Typically, a teacher presents the problems one at a time. Students first solve the problem mentally, then share and discuss their strategies. The purpose of the discussion is to support students' construction of mathematical ideas by helping them represent their thinking as well as reflecting on different ways of thinking and representing.

Data collection

Number strings and whole-class conversations around them were introduced in a Swedish primary school through an in-service development project (ROMB) using a reform-based curriculum¹ (Fosnot, 2000; Fosnot & Dolk, 2001), building on ideas from Realistic Mathematics Education (Freudenthal, 1991; Van den Heuvel-Panhuizen & Drijvers, 2014). Special attention was paid to the use of number strings and different types of questions. The teachers in the project took part in an initial two-day workshop followed by regular meetings to discuss their work with the researchers throughout the year. They were also given textbooks and other curricular documents including videos to study and use, but while these were in English the teachers struggled with the reading. During the project, several whole-class number string discussions were video recorded and analysed. For this paper one such lesson (teacher C, see table 1) was analysed in comparison to two similar lessons with teachers who were experienced in the use of number strings (teacher A and B, see table 1).

Table 1. *Three lessons using number strings*

	Teacher A	Teacher B	Teacher C
Length video (minutes)	18	28	33
Student's age	10	10	8
Teacher's experience with number strings	Yes	Yes	No
Student's experience with number strings	Yes	No	No

All teachers are experienced mathematics teachers at the relevant level. Teacher B and C were video recorded in authentic classrooms as part of the ROMB project, with a camera focussing the teacher. Permission to use the video for research purposes was collected. Teacher B was a visiting teacher educator from New York² conducting a lesson in a Swedish year 4. Teacher B is an expert on the use of number strings. She held the lesson in English with the ordinary mathematics teacher present throughout, ready to translate whenever needed. The students were very attentive and surprisingly good at using English. Teacher C was a novice in relation to the use of number strings, teaching her own students in year 2. For the sake of comparison, we also analysed a video in the commercially

produced in-service teacher training material (Dolk & Fosnot, 2006) used in the project, showing an expert conducting a number string lesson (teacher A). There is no information about the recording context or if the classroom is in any way authentic.

Method of analysis

After the three lesson videos were transcribed verbatim, the analysis was conducted in three steps. First, all teacher questions were identified and categorized according to a framework based on Cunningham (1987), Ellis (1993) and Hiebert & Wearne (1993), as described in table 2. Only questions clearly related to the mathematical content and taken up as questions by the students were included. Secondly, each question was revisited in the flow of interaction and re-categorised in accordance with the response that followed (see notes in table 2). If a short answer was given, i.e. one word or one number, the question was always categorised as factual (F). This analytical step altered the categories for some questions.

Table 2. *Framework for analyzing teacher questions*

Types of questions	Explanation
Factual (F) <i>Memory recall and Confirmation</i>	Name, identify, repeat, recall known facts or procedures, "correct" answers wanted. (often short answers, including yes/no)
Low-level Conceptual (LC) <i>Procedural descriptive questions</i>	Describe mathematical thinking and students' ideas. (What do you see? What do you think?) <i>Note:</i> LC if the student's own thinking is described, but changed to F if the student describes what "you are supposed to do"
High-level Conceptual (HC) <i>Explanatory questions</i>	Explain, analyze (Why? and How do you know?). Generalize and make connections. <i>Note:</i> HC if there is an attempt to give a mathematical explanation, but changed to LC if the answer is descriptive.
Evaluative (E) <i>Analytic and comparative questions</i>	Evaluates effective or elegant strategies or representations with valid arguments.

Finally, the questions were bundled together and treated as episodes (Franke et al., 2017). Each episode starts with a question from the teacher, continues with follow-up-questions directed to the same student, and ends when the teacher poses a new question to the whole class or to another student. An episode was categorised in accordance with the question of the highest level in the hierarchical order from the lowest (Factual) to the highest (Evaluative).

Results and discussion

The results are presented in table 3 as an overview of all the analysed episodes, indicating categories F, LC, HC and E, with a few comments about the specifics

Table 3. *Overview of all questioning episodes in the three lessons*

Number string	Episodes	Comments
Teacher A Big idea: relation between subtraction and addition. Model: number line		
146 – 12	F LC	
272 – 14	LC LC LC LC LC	
283 – 275	LC LC F F F LC HC E E E (HC) HC HC LC LC LC LC HC F LC	20 episodes discussing one task, engaging many students. (HC): student turn-and-talk
Teacher B Big idea: relation between multiplication and division. Model: array		
2 · 3	F LC	
2 · 30	F LC	Several LC questions answered as F
4 · 4	F LC LC (HC) F	(HC): many HC questions were asked here but nobody answered.
4 · 40	F (LC) LC LC LC LC F	(LC): student turn-and-talk
4 · 39	LC F F F F F	
120/12	F	
132/12	LC LC LC LC F HC	
108/12	LC F F	
Teacher C Big idea: space value and "making tens". Model: number line		
59 + 1	F LC LC	
59 + 11	F LC LC LC	
48 + 12	LC LC	Parts of the conversation not publically audible
58 + 22	LC E	Large parts of the conversation not publically audible. Many E and HC questions posed but not answered. More than 10 questions in the second episode.
38 + 6	LC	

of some episodes. Many episodes included several questions in a lower category. Specifically, F questions appeared in many episodes.

Types of questions

When comparing the teachers' questioning practice in the three whole-class discussions we notice some striking differences. Although factual recall questions (F) are on the lowest cognitive level, they are common in all three classrooms. Teacher A (number string expert) poses F questions frequently in the beginning of the lesson, and quite often as the first and/or last question in a low cognitive (LC) episode, where they serve to make sure there is agreement in the community about specific points of departure or closure. As a contrast, teacher C (the number string novice teacher) uses a lot of F question to scaffold her students in long conversations about their thinking. For example, the task $48 + 12$ has two LC episodes. Both of these consist of a bundle of around six F questions together with one LC question. As a result, the students start adjusting to the teacher's way of thinking rather than expanding their own. Teacher B (the expert American teacher in the Swedish classroom) also poses a lot of F questions, and students repeatedly answer high-level questions with short

answers turning them into F questions. Although the use of English is a restraining factor, it is obvious that the students are unfamiliar with the higher-level questions teacher B asks and they do not quite know how they were expected to answer. Both teacher B and C pose high cognitive (HC) and evaluative (E) questions that are answered on a lower level or not answered at all.

Our conclusion is that F questions can be used in a meaningful way, but they can also hinder students' from developing higher level mathematical reasoning. Furthermore, students need to be taught how to answer high-level questions, it does not simply happen when such questions are posed. This conclusion relates to the idea of sociomathematical norms, indicating that one aspect of establishing productive norms is to be explicit about what kind of answers are expected and honoured.

Engaging students in the discussion

The number of tasks and episodes differ between the lessons. While teacher A discusses three tasks in 27 episodes, and teacher B discusses eight tasks in 31 episodes, teacher C discusses seven tasks in only 14 episodes. This means that teacher A and B engage more students in each task, orchestrating the conversation using input from the students. It is visible in table 3 that many of the tasks presented by teacher A and B engage five students or more in the conversation. The most engaging conversation in teacher A's lesson involves 20 episodes, many of them categorised as HC or E (task 283 – 275), while in teacher C's lesson no task is discussed with more than four students, the last three tasks are discussed at length only with one student each.

Another difference between these conversations is the number of questions within an episode. In the second step of the analysis we noticed that teacher C several times engaged in an almost private conversation with a single student, with backs turned to the class making it impossible for the peers to hear and take part in the conversation. The questions seemed to be more aimed at scaffolding or assessing than orchestrating a discussion. In contrast to this, teacher A often created a new question based on a student response, but directed the question towards a new student, thus initiating a new episode.

In all three lessons, at least one question seemed to cause uncertainty among the students. The three teachers dealt with these situations in different ways. Teacher A and B used the didactical move "turn-and-talk" so that the students were given a chance to reflect on the question together with a peer. In teacher A's classroom, the question to discuss in pairs was why an add-on strategy and a take-away strategy both could be used to solve the same subtraction problem. After a few minutes peer-talk it was possible to continue the conversation with some interesting student ideas to build on. In a similar situation, when teacher C asked which of two proposed addition strategies was the easiest, she kept repeating the same question to the student who failed to come up with an argument, even though the student at one point begged her to stop asking. Being a novice to number string whole-class discussions, teacher C has learned that good

questions can facilitate learning, and she knows that asking students to evaluate and justify are good question types to probe thinking and reflection. However, to her, the aim of the questions is to guide students towards [predetermined] answers or solutions. What she does not yet embrace is the belief that teaching is about initiating students into a community of practice, where questions can also be used to create puzzlement and promote disequilibrium (Imm et al., 2012, p. 30). The aim of a question is not the answer but the new ideas it generates. Imm et al. (ibid) describe this as one of the landmark changes that teachers go through when learning about questioning and conferring. Moving from taking the role of explainer to that of facilitator (Ernest, 1989) does not entail handing over the responsibility for engaging in high-level mathematics. The teacher's active role in the conversation highlighted by Walshaw and Anthony (2008) incorporates much more than knowing what questions to pose. It also includes the ability to make good use of student's responses in collaboratively constructing new mathematical understanding.

Concluding remarks

A major difference between the experts and the novice in this study is the nature of the teachers' response, i.e. how the interaction continues after the initial question. When student input is not explored and further discussed the high-level questions do not flourish. Although the question types are similar, the teachers' response may be different, as shown also by Li and Ni (2009) who compared novice and expert elementary school teachers. Ni et al. (2014) describe four ways in which teachers respond to students' ideas: ignoring; acknowledging; repeating; or examining and utilizing. How students' contributions are treated in the classroom is further explored by Maunula (2018) using the categories: disregarded; selected; considered; or explored, where explored indicates a fruitful elaboration of an idea. However, exploring student ideas implies knowing where the discussion might end up and where you want it to go. Furthermore, it requires that students are encouraged to build on each other's ideas, to make connections and to be willing to revise and refine their mathematical ideas.

We conclude that posing high-level questions is only the beginning of establishing a good questioning practice. Although posing high-level questions create opportunities for learning, as Martine and Maher (1999) point out, students may not engage in them. It is the teacher's responsibility to orchestrate a fruitful conversation, but for this to happen the teacher needs to be a very active listener, who can detect, draw out and elaborate on small grains of interesting mathematics from student that can be examined, explored and made use of. Learning about affordances of different question types is a good start, but needs to be accompanied by developing listening skills and changing one's belief about the aim of the questions in the mathematics classroom.

Acknowledgements

We wish to thank all the teachers engaged in the ROMB project for welcoming us to their classrooms. A special thank you to Britt Holmberg for video recording the lessons conducted by teacher B and teacher C.

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Notes

- 1 <http://idpp.gu.se/forskning/utvecklingsprojekt/romb>
- 2 https://www.cuny.cuny.edu/education/mathematics_in_the_city

Data generation in statistics – both procedural and conceptual. An inferentialist analysis

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Data generation in statistics education is often conducted by the students themselves; however, the question of what learning opportunities the data generation process offers has only been studied to a small extent. This paper investigates to what extent data generation is an observational and procedural vs. a conceptual activity. We inquire into this question based on an empirical study where eleven year old students measured the jump lengths of paper frogs. Our analysis draws on students' discussions in group work, and it uses inferentialism as a background theory. Our results indicate that students' discussions are conceptual to a certain extent and provide various learning opportunities for the students.

Research in statistics education is organized around three major axes: data generation, data analysis, and statistical inferences (Garfield & Ben-Zvi, 2008). While the importance of data and data generation is widely acknowledged, it has received little attention in research (Garfield & Ben-Zvi, 2004), and there is a tendency in statistics education research to reduce data generation to a means to introducing and investigating other statistical ideas, or to a series of actions and procedures to get the right data (Cobb & McClain, 2004).

In recent decades, these views on data generation began to change. Garfield and Ben-Zvi (2008) argued that how data is obtained is important information about any statistical study. In a similar manner, researchers began to stress the importance of data generation for the conceptual development of statistical ideas (e.g. Cobb & Moore, 1997). Based on empirical findings on how data creation can enhance students' statistical reasoning, Cobb and McClain (2004) recommend that students should produce their own data. Other researchers recommend instead that the data does not necessarily have to be produced by the students themselves as long as it is given authentic context (e.g. Hancock, Kaput & Goldsmith, 1992). These differing views reflect the fact that there is no agreement on the significance of students' involvement in data generation processes for learning about statistical concepts. It is an open question whether

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data generation offers worthwhile learning opportunities for students; whether it is – despite certain opportunities for conceptual development – mainly procedural; and what the conceptual aspects of learning in data generation processes might be.

This paper aims to investigate the data generation process. Based on an empirical study where eleven year old students produced data collaboratively in groups, we investigate the extent to which students' discussions in data generation focus on observation and its procedures or on conceptual aspects of statistics learning. To analyze the conceptual and procedural sides of data generation, we use the semantic theory of inferentialism (Brandom, 1994; 2000). Inferentialism offers us a tool to analyze students' data generation as well as its conceptual and procedural aspects.

Previous research

Concepts such as data collection, data production, data generation, and data fabrication are used interchangeably in statistics education research, and sometimes the terms are used to describe different processes. For instance, Cobb and McClain (2004) used the term "data generation" to make a clear demarcation from simple data collection. According to these authors, data generation is a process that precedes data collection: "these preceding phases involve clarifying the significance of the phenomenon under investigation, delineating relevant aspects of the phenomenon that should be measured, and considering how they should be measured" (Cobb & McClain, 2004, p. 386).

Reviewing research on the process of data production in teaching/learning situations, we note two major approaches. In the first approach, data is handed to the students via the teacher or textbook, and the students' work with data generation consists of making sense of and contextualizing this data. In most of these studies, the researchers' target is the learning of a specific statistical concept rather than the experience of a statistical investigation (Heaton & Mickelson, 2002; Lehrer & Schauble, 2002). Singer and Willet (1990) argued that teacher-produced data or textbook data should be authentic data and that real data can connect to real context and, as such, enhance interest, relevance, and substantive learning. The underlying conjecture is that authentic contexts will trigger students' engagement, and pave the way for fruitful discussions with data. Cobb and McClain (2004) proposed that when students are handed data, they should be told the story of the data and how the data has been produced. In contrast, Noss, Pozzi and Hoyles (1999) warned that focus on real data can overshadow statistical ideas and give priority to deterministic reasoning. They suggested that, for educational purposes, focus should be put on how concepts in statistics are used in practice. The second approach, student-produced data, has been variously conceptualized in research depending on the researchers'

agenda. In general, the students produce the data either through computer-simulation (Ainley, Pratt & Hansen, 2006; Pratt 1995) or manually in experimentation (Nilsson, 2013). Lehrer and Romberg (1996) take a modeling perspective on data. From their perspective, data construction involves "posing question, collecting response, transforming response into data" (p. 76). The data collected by students may be "treated as objects independent of the existence of that which they represent" (p. 70), and may serve as objects of manipulation for further inferences. Lehrer and Romberg's study indicates that the construction of data enhances students' ability to draw sound inferences beyond the data (Makar & Ben-Zvi, 2011). Nilsson's (2013) study shows that students' involvement in data generation does not automatically improve their prediction ability. In Pratt's (2000) study, the students used a data-simulation (Chance-maker microworld) to produce data.

A trait common to these two approaches is that how data is gathered is often understood basically procedurally (Garfield & Ben-Zvi, 2008). Studies that focus on conceptual aspects and involve student data generation typically discuss how the data generation can be supportive of the learning of statistical concepts; they do not focus on learning opportunities of the data generation process itself. The conceptual learning opportunities in data generation are, therefore, an under-researched topic.

Theory

In order to understand the conceptual and procedural in data generation practices, we use the semantic theory of inferentialism, as primarily developed by Brandom (1994; 2000). By using concepts from inferentialism, we look in detail at students' practice of measuring and how it relates to the students' social practice in their group work, as well as considering the existing and emerging conceptualization of the situation that comes into play in collaborative work. This is made possible because inferentialism provides detailed accounts of how the social practice of communicating underpins the intellectual practice of being a concept user.

Inferentialism has gained popularity in mathematics and statistics education research in recent years, and it has been used in different ways in order to conceptualize students' understanding and reasoning (Bakker & Derry, 2011; Schindler et al., 2017). A recent special issue on inferentialism in the Mathematics Education Research Journal (Bakker & Hußmann, 2017), including various empirical studies (e.g. Mackrell & Pratt, 2017), gives a glimpse of the potential that inferentialism offers for understanding students' activities in statistics. However, most of the work focuses on students' individual work. Group work has not been analyzed to a great extent using inferentialism (exceptions are Schindler & Joklitschke, 2016; Schindler & Seidouvy, in press).

Inferentialism is a theory that explains how shared conceptual content can come out of interpersonal communication (and asserts that all conceptual content is of this kind); concepts only have meaning in relation to other concepts in inferential networks. In inferentialist theory, knowing a concept is knowing how to use it in communication.

Making explicit how this is possible means explaining how communication can have the necessary interpersonal precision for concepts to develop stable relationships to other concepts. Following the later Wittgenstein (1958), Brandom sees the discursive practice as a language game governed by a set of rules, but looks at this game in a different way by specifying that it is a game of giving and asking for reasons. When using a concept, one commits to its relation to other concepts. Any claim made can both serve as a reason and stand in need of other claims as justification. In this way, interlocutors can give and ask for reasons concerning some content or concept and can develop a shared way of using concepts. Interlocutors can acknowledge another person's claim explicitly, or entitle it by implicitly using it as a premise in further communication.

However, inferentialism is also compatible with the general theorization of communication and all its imprecise messiness, as discussed, for example, by Harvey Sacks (1995). As Münchhausen's trilemma exemplifies, we cannot expect to define anything fully without ending up in infinite regressions, in circular arguments, or relying on unexplained axioms. In this sense, communication would seem to lack the necessary precision to ever allow agreement about any concept's use. Why would interlocutors ever stop asking for further justification? The resolution of this situation comes from acknowledging that language use is not only a logical inferential practice but also a social inferential practice. As social, conscious, and self-conscious beings, humans can decide when it is not appropriate or practical to ask for further reasons or justifications – even in cases where our judgment tells us that it is logically called for. In the game of giving and asking for reasons, in Brandom's formulation, making an assertion means undertaking responsibility to justify the claim if appropriately challenged. If an interlocutor grants another person's claim *authority*, the claim can stand without further challenge. It is in this way that infinite regressions of justification are avoided.

Brandom (1994) discusses three types of authority: person-based authority, content-based authority, and observational authority. Content-based authority is "invoked by justifying the claim through assertion of other sentences from which the claim to be vindicated can appropriately be inferred" (p. 175). For person-based authority, the speaker does not provide content-related reasons herself, but defers to the claim of another person. The speaker reasserts the claim of another person. In this turn, entitlement to the claim is "inherited" from the original speaker who uttered the claim in the first place. The last of Brandom's three authority types, observational authority, is the most interesting to us.

Observational reports are empirical claims. As such, they are a particular type of noninferential reports, which means that they are contentful, but with no explicit inferences to other claims. In contrast to claims that refer a term to a category, such as "five is a number," empirical claims such as "there is a dog outside" require two particular sets of authorities to be in place. First, for anyone to call you a reliable reporter of an observation, they need to hold the belief that you have appropriate conceptual knowledge of the content that you are reporting, for example that you can distinguish dogs from other animals. We will call the authority associated with having conceptual knowledge of the empirical matter to be assessed *empirical content authority*. Second, for anyone to call you a reliable reporter of an observation, they need to hold the belief that you are in the appropriate circumstances to evaluate the empirical matter, for example that you are in fact looking outside. Such appropriate circumstances might involve other matters than physical position and we will here call the authority obtained by being in the right circumstances for a particular observation *circumstantial authority*. Hence, *empirical content authority* and *circumstantial authority* together make up *observational authority*.

From this line of reasoning, we can draw a theoretical conclusion. If the topic is fixed and other interlocutors entitle a person empirical content authority, gaining observational authority will be a matter only of other interlocutors also seeing that the person is in the appropriate circumstance to make the observation. And by extension, if there is a shared understanding that all people in a group have empirical content authority, anyone who gains circumstantial authority will also gain observational authority. Based on the inferentialist account of authority in students' work, we ask the following research questions: 1. What distribution of authority types can we identify in students' work? 2. What, if any, conceptual opportunities to learn can be detected in the students data generation process?

Method

The data material for the present study consists of video-taped group work sessions with two groups (A and B) of, for each group, four 11 year old students from a grade five class in a Swedish school. According to their teacher, the students were familiar with group work and with sharing the work among themselves, but were not used to conducting experiments themselves. As part of a bigger design research project, the groups were given a task of our design by their teacher. In this task, the students were asked to experiment with paper frog models of three different sizes. By pushing down on the frog with a finger and releasing, the frogs made a small jump. The task was set in a narrative where the students were called in as experts to assist a company in finding "the best-selling frog." This was explained to mean "the frog that jumps the furthest." In

the instructions, the students were asked to collaboratively test and record five jumps for each of the three frogs, and then to come up with a recommendation for the company. The students had measuring tapes and a pre-prepared protocol paper where they could record their measurements.

While the whole lesson was video-taped, here we only deal with the sequences that concern data generation (in the sense of Cobb & McCain, 2004). By using inferentialist theory, we analyze the conceptual and social content that underpins different instances of data generation. In this analysis, inferentialist theory is operationalized as follows. Deciding on which number to write down is for each measurement considered as authorizing an observational report. For each such case, we analyzed the section of the video and corresponding transcripts from the point where the frog was positioned on the starting point to the production of the written note. By analyzing the communicationally relevant acts (speech acts, positions, movements, and gestures), we categorized the claims that were acknowledged in the discussion according to which type of dominant authorization underpinned this acknowledgment, as described in the theory section. Thus, students' interactions leading up to the production of a written number are considered the primary units of analysis.

Results

The interactions in the two groups followed different patterns. Most of the results that we want to share concern group B, because their work turned out to exemplify particularly interesting phenomena in relation to our research question. We will briefly deal with group A at the end of the results section. Before presenting our general results, we exemplify our analysis by describing group B's collaboration related to some of the jumps.

The first measuring session (jump 1) starts after an initial discussion of the instructions given to the students. The students first discuss the initial positioning of the frog at the starting point, agreeing that the front foot of the frog should be at 0. After the jump, all four students lean forward to look at where the frog landed. One student says, "We measure from here" and points with the pen at a front foot of the frog. Others acknowledge. Several students state "40." Just when one student leans down to write it down, someone says "40 millimeters." This initiates a discussion on whether the unit should indeed be millimeters or centimeters. Here 40 is already acknowledged as a number representing the jump length. Through inferring that 40 mm would mean 4 cm and that the actual jump is rather close to half a meter, they eventually agree that centimeter is the correct unit. In the end, in our theorization, the claim that gets authority is 40 cm. However, leading up to this claim, distinct inferences about the purpose, content, and theory, or the production of the data point, are made salient. Through gestures, actions and speech, an agreement about the starting position of the frog in relation to the tape is made. Similarly, an agreement is reached on

the meaning and choice of the units. Since these discussions are all related to the content of the measurement (a frog’s jump that is to be quantified as a number with a unit by means of a measuring tape), we classify this as content-based authority. The discussion around jump 1 produces a measurement, but also some rules concerning the measurement. Some of them are implicit, that is, not communicated as verbal claims. However, these rules are later acknowledged in action when the students follow them in subsequent jumps.

In the next jump, the students follow the same procedure for setting up the frog, thereby acknowledging the claim about initial positioning. The rules that the students negotiate do not only concern measurement but also other aspects of the data generation. In jump 2, one student, Daniella, claims she is ”a bad jumper” indicating that her frog’s jumps are shorter. This is acknowledged by the others. Later, in jump 6, this becomes important. Now a new frog should be used. Students discuss jump order and then decide that two of the students should do two jumps each, but Daniella only one, because here jumps are shorter. This is followed in the jumps with the last frog (jumps 11–15). We interpret this negotiated turn taking rule as the students’ possible understanding that if one person systematically produces shorter jumps, letting this student do a different number of jumps for different frogs would skew the results.

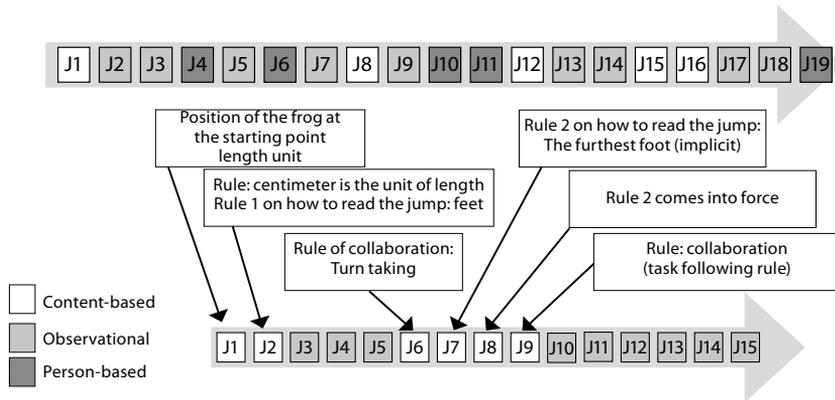


Figure 1. *Sequence of authority types for groups A (above) and B*

The summary of the analysis of group B is shown in figure 1. In their work, only content-based and observational authority appears. Their work is systematic in the sense that through collaborative work, they make agreements about how to produce the data, in a similar way to that explained for jump 1. Whenever a new jump introduces a new uncertainty of how to measure, a content-based claim that is later acknowledged produces a new rule. As seen, this happens in jumps 1, 2, 6 to 9. Whenever no new uncertainties appear, observational authority takes over. From jump 10 on the students seem to have reached a shared agreement

on how to produce the data. Each jump is dealt with through observational authority. Whichever person is handling the measurement gains observational authority due to circumstantial authority, which is then easy to obtain.

We will now briefly mention also group A. As seen in figure 1, the sequence of authorities plays out very differently here. Group A initially has a rather long discussion about whether millimeters or centimeters are the best units to report. They end up choosing millimeters. This is a discussion concerning the empirical content of the data generation. Group A never agrees on a clear set of rules for the measurement. Lack of such rules means that it is harder to obtain observational authority since there is no shared agreement about what would constitute empirical content authority. And the requirement of measuring millimeters adds to the difficulty of obtaining observational authority. Instead, group A makes choices on what data to report based on content-related claims, which, unlike in group B, do not produce converging agreement about how the measurement should be done. Several of their observational reports are also a product of person-based authority.

Discussion and conclusion

The purpose of this paper was to investigate to what extent students' discussions in data generation focus on observation and its procedures, or on conceptual aspects of data generation. Based on inferentialist theory, we analyzed students' collaborative work by means of investigating the claims that gained authority in their conversation. Our study revealed that the work of group B provides an example that collaborative work with data production can involve conceptual content (Cobb & McClain, 2004). Defining rules for measuring is in any data generation situation an aspect of giving the numbers context and creating statistical content (Moore, 2006). In group B, instances of content-based authority are followed by observational authority, which is gained by obtaining circumstantial authority. This indicates that the need for rules regarding several aspects of measuring was identified, negotiated, and shared. The fact that rules are both made explicit and integrated as shared assumptions in the subsequent work are an indication of a learning opportunity of this conceptual content.

Group B's collaboration illustrates the possible added value of data generation tasks and activities. Didactically, the task given to the students created a situation that started without clear rules about how the measurements should be done. Such absence of rules seems to be a prerequisite for discussions on the nature of the data generation to take place: If the rules are 100% described, then there is no reason to discuss their conceptual aspects. Yet, group A had few such discussions despite being given the same instructions, and choosing what measurement to write down was often based on person-based authority, frequently reflecting a social negotiation rather than producing and following data generation rules. It is subject to further research to investigate ways of refining task formulations to promote the group B type of experience.

From a theoretical and methodological perspective, viewing our data through the lens of inferentialism enabled us to analyze a negotiation that partly consisted of silent agreements. What enabled us to still draw our conclusions was the role that authority plays in reasoning. With cognitivist theories like constructivism, making the claim we do would mean having to produce ideas about what is in the collaborators minds, because that is by definition where knowledge resides in constructivist theories. Conversation analysis (Sacks, 1995) or other linguistic theories like Toulmin's (1958) model of argumentation could, in similar ways to inferentialism, be used to draw out the type of implicit agreements we deal with here. However, such theories lack the ontological depth of inferentialism where "all questions of ontology are questions on the authority structure in the social practice of claiming" (Wanderer, 2014). In collaborative work the content is, in typical communicational manner, often dealt with implicitly – and not with the explicitness of formal mathematical presentation. We have shown that inferentialism provides a basis for analyzing mathematics conceptualization in such complex communicational settings.

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Using the Mathematical working space model as a lens on geometry in the Swedish mathematics upper secondary curriculum

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This paper uses the model of Mathematical working spaces to analyse the area of geometry in the Swedish upper secondary mathematics curriculum. By applying this framework, we describe how one can understand the mathematical work advocated in the curriculum in terms three geometrical paradigms as well as in terms of different emphasis on a set of three ways of working connecting epistemological and cognitive aspects of geometrical work.

Since the beginning of the 1960s research into the learning and teaching of mathematics has been centred around and related to different aspects of the core activity of *mathematical work* (Ernest, 1999; Boaler, 2002; Gila & Villiers, 2012). Building on and adding to this line of research, the model (or framework) of *Mathematical working spaces* (MWS) has been introduced and studied collaboratively by researchers from various countries for more than 10 years. A central motivation behind the framework is that by "articulating epistemological and cognitive aspects, the MWS model is aimed at providing a tool for the specific study of mathematical work in which students and teacher are effectively engaged during mathematics sessions" (Kuzniak, Tanguay & Elia, 2016, p. 721). However, MWS are also used to analyse and characterise conditions for mathematical work at different curricula levels (Kuzniak, Tanguay & Elia, 2016).

Focusing on the area of geometry, several different frameworks have been proposed and used in research, such as for example the Van Hiele model of thinking in geometry; the theory of figural concepts; Duval's cognitive model of geometrical reasoning, and the three frameworks for the learning of geometrical reasoning discussed by Jones (1998). Indeed, the MWS model was originally developed in the domain of geometry based on the work by Duval under the name *Geometrical working space* (GWS) model, which together with the notion of geometrical paradigms (Houdement & Kuzniak, 1999, 2003) has

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been used to clarify the different meanings of the term and conceptions of geometry. The idea of paradigm is inspired by the corresponding notion introduced by Kuhn (1996), where a *paradigm* stands for a combination of beliefs, convictions, techniques, methods and values that are shared by the members of a scientific group or field.

The MWS model has been used by various researchers from different countries (mostly by French and Spanish speaking groups from France, Spain, Chile and Canada (e.g. Tanguay, 2015; Kuzniak, Montoya Delgado & Vivier, 2016)) to study for example different tasks, teaching and learning situations, and sets of activities. Like de la Torre Fernández and Blanco (2008) study in the Spanish context, we in this paper analyse the governing written curriculum. Our main aim with this paper is to try and apply the MWS framework to the Swedish context as part of a comparative study of geometry at the upper secondary mathematics levels in Chile and Sweden. The paper constitutes the first step of a larger project where we will study and compare the GWS of the curriculum with the GWS of textbooks and the GWS of the teachers' and students' work in Sweden and Chile, as well as possible differences and gaps among these three instances of the framework.

Geometry in the Swedish mathematics curriculum

In this section we present the data analysed in the paper, which consist of extracts related to geometry from the Swedish national mathematics curriculum for the upper secondary level (Skolverket, 2011). First, there are the general goals formulated in terms of ability goals that should permeate all teaching and are applicable to all educational programmes at the upper secondary level:

Teaching in mathematics should give students the opportunity to develop their ability to:

- 1 Use and describe the meaning of mathematical concepts and their inter-relationships.
- 2 Manage procedures and solve tasks of a standard nature with and without tools.
- 3 Formulate, analyse and solve mathematical problems, and assess selected strategies, methods and results.
- 4 Interpret a realistic situation and design a mathematical model, as well as use and assess a model's properties and limitations.
- 5 Follow, apply and assess mathematical reasoning.
- 6 Communicate mathematical thinking orally, in writing, and in action.
- 7 Relate mathematics to its importance and use in other subjects, in a professional, social and historical context.

Table 1. *Core content in geometry divided according to the Swedish upper secondary mathematics curriculum (Skolverket, 2011)*

Courses	a	b	c
Mathematics 1	Properties and representations of geometric objects. Methods of measuring and calculating quantities crucial in subjects typical of the programs.	The concept of symmetry and different types of symmetry transformations of figures in a plane. Mathematical reasoning using basic logic, including how to reason in everyday contexts and in different subject areas. Illustration of the concepts of definition, theorem and proof.	The concepts of sine, cosine and tangent, and methods of calculating angles and lengths of right angled triangles. The concept of vector and its representations. Mathematical reasoning using basic logic, including how to argue in everyday contexts and in science subjects. Illustration of the concepts of definition, theorem and proof.
Mathematics 2	In-depth study of geometrical concepts such as sine, cosine, tangents, vectors and symmetries. Mathematical reasoning using basic logic, including how to reason in everyday and professional contexts	Use of fundamental classical theorems in geometry concerning similarity, congruence, and angles	The concept of curves, straight lines and parabolic equations and how analytic geometry relates geometric and algebraic concepts. Use of classical theorems in geometry (similarity, congruence, angles).
Mathematics 3	The program does not include Geometry.	The program does not include Geometry.	Properties of the equation of a circle and unit circle in defining trigonometric concepts. Proof and use of cosine, sine and area theorems for an arbitrary triangle
Mathematics 4	Handling trigonometric expressions, and proof and use of trigonometric formulae including the Pythagorean trigonometric identity and the addition formulae. Algebraic and graphical methods for solving trigonometric equations. Different methods of proof in mathematics, with examples from the areas of arithmetic, algebra or geometry.		

Table 1 summarises the geometry content as the listed core content in the curriculum for the different a-, b-, and c-track of the first three mathematics courses and course 4 (geometry is not part of the content in course 5). The a-track courses are included in all vocational programmes, the b-track are taken by students in the Business management and economics-, Arts-, Humanities-, and the Social science programmes; and the c-track are the mathematics courses in the Natural science- and the Technology programmes. The courses are taken in consecutive order within each track with the exception that in addition Mathematics 2a gives access to Mathematics 3b, and Mathematics 2b gives access to Mathematics 3c. Mathematics 3b or 3c is prerequisite for Mathematics 4.

Theoretical framework

In this paper we use the framework of MWS (Kuzniak, 2011) in the domain of geometry, which was the domain in which the framework originated in terms of GWS (Houdement & Kuzniak, 1999; 2003). Nowadays, a MWS focusing on work in geometry is abbreviated *MWSG*, a notion we adapt in this paper.

A MWS provides a tool for interpreting students' and teachers' problem solving activities by mapping their mathematical work onto an epistemological and a cognitive plane. The two planes are connected along three dimensions by which the mathematical work develops and meaning is created. These dimensions, called *semiotic*, *instrumental* and *discursive*, give pairwise rise to three planes of *genesis* called [Sem-Ins], [Ins-Dis] and [Sem-Dis] which in a more nuanced way capture the complexity in navigating between and coordinating work within the epistemological and the cognitive plane.

In the MWS framework one in addition distinguishes between three different types of spaces: *reference*, *suitable* and *personal*. On one hand, the reference MWS is defined with respect to the relation to disciplinary (scientific) knowledge, ideally using mathematical concepts, notions and nomenclature. The suitable MWS on the other hand depends on the institution involved, and is defined according to the way the knowledge are perceived in relation with its specific place and function within the institution (typically a learning institution). Finally, the personal MWS is related to the individual, and is defined by the way that she or he handles a mathematical problem given her/his own knowledge and cognitive capacities (Kuzniak, 2011). Here one can note the similarities to other theoretical frameworks and notions such as the transposition of knowledge in the Theory of didactical situations (TDS) and the Anthropological theory of didactics (ATD), as well as the International association for the evaluation of educational achievement (IEA) curricula framework.

A MWS is manifested only through its user, actual or potential, and its constituents will vary with the educational system (e.g. the reference MWS as specified in curricula documents for example), with the schooling contexts (e.g. the suitable MWS such as the curriculum adapted in the teaching and learning of mathematics within a school), with the practitioner (e.g. the personal MWS capturing the mathematical work carried out by students in a classroom). The reference MWS is naturally defined according to mathematical criteria.

In this paper we focus on the content area of geometry and the ability goals as described in the curriculum of the Swedish upper secondary school to identify the corresponding reference MWS, with students and teachers as its potential users.

Geometrical paradigms

The conception of geometrical paradigms was originally discussed by Houdement and Kuzniak (2003) in connection with MWSG, in terms of (1) an epistemological component; (2) a cognitive component based on Gonseth's (1945–1955) forms of knowledge (intuition, experiment, and deduction); and (3) a philosophical component given by Kuhn's (1966) ideas. The three geometrical paradigms are used by Houdement and Kuzniak (2003) to organise the work within, as well as the interplay between, intuitions focusing on how deduction and reasoning are conceived. The three geometrical paradigms are:

Geometry I (*Natural geometry*), where validation is sense-based and intimately related to real objects. Intuition is often immediate perception, experiment

and deduction act on material objects by means of the perception and instruments. The back- and forward transition between the model and real objects is what allows the proof of assertions, and dynamic proofs are accepted as methods of proof.

Geometry II (*Natural axiomatic geometry*), were validation bases itself on the hypothetical deductive laws of an axiomatic system. A system of axioms is necessary but the axioms are close to the intuition of the physical object in Geometry I. The system can be incomplete but its use is necessary for making valid assertions and proving these.

Geometry III (*Formalist axiomatic geometry*) cuts the umbilical cord between the realistic and axiomatic base of geometry: axioms need not anymore be based on the senses and real objects. The type of reasoning is the same as in Geometry II, but the system of axioms is complete and independent of its possible applications to the world. The only criterion of truth is internal consistency.

The Mathematical working space model for geometry

The main purpose of the MWSG is to capture and describe the geometric work made by students in schools. As the name suggests, the geometric work is at the centre of the model and the model encourages and motivates reflection on the teaching and learning of geometry. In this model, the main and crucial function of educational institutions and teachers is to develop rich learning environments that will facilitate and enable students to solve geometric problems using appropriate strategies and tools. To describe the specific activity of students solving problems in geometry, the MWSG maps the students' activity onto two planes or levels; the so-called *epistemological*- and *cognitive* plane respectively.

The epistemological plane a priori defines the expectations about the activity according to the requirements of the given mathematical domain in question, in our case geometry. More particular regarding geometry, the epistemological plane has three interacting characteristics related to this mathematical dimension (see figure 1a): (a) a real and local space as material support with a set of concrete and tangible objects such as figures or drawings (*space and figures*); (b) a set of artefacts such as drawing instruments or software (*artifacts*); (c) a theoretical reference system based on definitions and properties (*reference*). See Kuzniak, Tanguay and Elia (2016) for more details and references.

Since geometry as it is taught and learned in schools not is a disembodied set of properties and objects reduced to symbols and signs which can be manipulated by formal systems, it should first and foremost be considered as a human activity. Hence, it is essential to understand how communities of individuals, but also specific individuals, use and internalize their knowledge of geometry in their practice of the discipline. The main objective of the cognitive level in the MWSG model focuses on the subject(s) viewed as a cognitive subject(s) solving problems. Considering geometric activity, the cognitive level focuses on three processes (see figure 1a): (a) a process of visualization related to the representation of both space and the material supporting validation (*visualisation*);

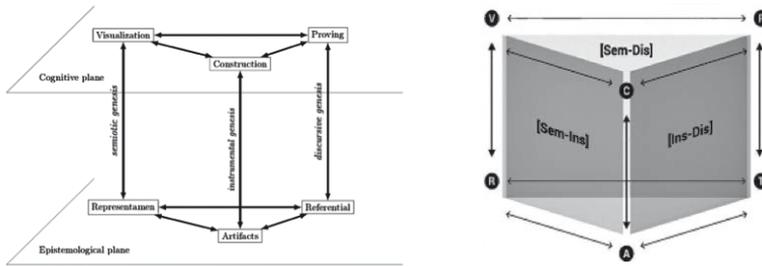


Figure 1. (a) *The MWGS model*; (b) *the vertical planes (Kuzniak, Tanguay & Elia, 2016, p. 725; 727).*

(b) a process of construction and the function of the available and used instruments (e.g. rulers, compass) and the respective geometrical configurations (*constructions*); (c) a discursive process producing arguments and proofs (*proof*).

Besides illustrating the three characteristics in each plane, Figure 1a in addition shows the relationships connecting the epistemological- and cognitive plane in terms of three different dimensions or *geneses*: *semiotic*, *instrumental* and *discursive*. Figure 1a shows three specific dimensions of work connecting the epistemological- and cognitive planes each which requires three specific genetic developments named *genesis*: a figural and semiotic genesis that gives the tangible objects their status of operating mathematical objects; an instrumental genesis that transforms artifacts into tools within the construction process, which is crucial in the case of geometry; and, a discursive genesis of proof that gives a meaning to the properties used within mathematical reasoning. Figure 1b shows three vertical planes ([*Sem-Ins*], [*Ins-Dis*] and [*Sem-Dis*]) spanned by these geneses that match the connections between the dimensions and that in addition can help to specify the precise geometric work carried out in the MWSG when students solve tasks given by their teachers. Coutat and Richard (2011) refer to these planes as inherently focusing on *discovering*, *validation* and *modelling* respectively.

Research context, methodology and research questions

In this paper we study the Swedish upper secondary school mathematics curriculum focusing in the area of geometry as specified in the national written governing curriculum document (Skolverket, 2011) summarised in Section 2. We do this by applying the MWSG model introduced in the Section 3 with the goal to gain insight into the referential MWSG for this institutional level. Hence, it is natural to separate the analysis and discussion between a-, b- and c-tracks of mathematics courses, and to identify what paradigms of geometry are conveyed for these in the governing curriculum document.

Based on the assumptions of the theoretical framework, we posit that geometric work can characterised by the three semiotic-, instrumental-, and

discursive dimensions of a MWSG. In the analysis of the general goals and specific geometrical core content in the curriculum, we connect and map the formulations used in the curriculum onto the three dimensions. The questions we address are:

- Which geometrical paradigms are portrayed and prescribed to be used in upper secondary school in Sweden? What are the similarities and differences with respect to the three different tracks (a, b, c)?
- How can geometry in the curriculum be understood and described in terms of the MWSG vertical planes [Sem-Ins], [Ins-Dis] and [Sem-Dis]?

Results

Table 2 summarise the analysis about the question what geometrical paradigm are portrayed in the different upper secondary mathematics courses.

Table 2. *Geometrical paradigm in the Swedish upper secondary school*

Courses	a	b	c
Mathematics 1	Geometry I	Geometry II	Geometry II
Mathematics 2	Geometry II	Geometry II	Geometry II
Mathematics 3	No	No	Geometry II and Geometry III
Mathematics 4	Geometry II and Geometry III		

For example, our analysis shows that the course Mathematics 1a advocates work in paradigm Geometry I as can be illustrated by formulations like "Methods of measuring and calculating quantities that are crucial in subjects typical of program". In Mathematics 1a, in general nothing is mentioned about the use of axioms, rather connections with reality and practical relevance is supposed to permeate all mathematics teaching and learning. In contrast, the course Mathematics 3c mostly advocate work in the Geometry II paradigm, but also offers a possibility to engage in Geometry III work. Partly this progression from Geometry I towards Geometry III is to be expected given the sequencing and the structure of the courses. In the case of the c-track courses, students are advocated to work in the Geometry II paradigm in Mathematics 1c ("Mathematical reasoning using basic logic, including how to argue in everyday contexts and in science subjects") and Mathematics 2c ("Use of classical theorems in geometry (similarity, congruence, angles)"), whereas the formulation "Proof and use of cosine, sine and area theorems for an arbitrary triangle" for the Mathematics 3c content rather suggest a shift towards Geometry III paradigm work. The latter indicates that systematic use and knowledge of axioms as well as a higher level of abstraction are expected of geometry work in this course. However, the type of work students are invited to participate in is naturally dependent on how the

curriculum is realised and put into practice, but at least the curriculum render it possible to work in an more abstract space (at least in some levels). On the other hand, the rest of the content in Mathematics 3c concurs to work in Geometry II. Similarly, in Mathematics 4 ”handling trigonometric expressions, and proof and use of trigonometric formulae, including the Pythagorean trigonometric identity and the addition formulae” and using ”different methods of proof in mathematics, with examples from the areas of arithmetic, algebra or geometry” suggest work in the Geometry II and Geometry III paradigms.

Generally, the main goal for many of the courses is learning how to work with ”Mathematical reasoning using basic logic” using different contexts related with the programs the students are studying. However, looking across the a-, b- and c-track, there is gradually more emphasis and promotion on using logic and theorems to do (basic) proofs and not to apply theorems as tools only, which in other words aims at geometrical work involving a (no necessarily complete) system of axioms.

Turning to the question of how geometry in the curriculum can be understood and described in terms of the vertical planes [Sem-Ins], [Ins-Dis] and [Sem-Dis] we found it generally challenging to see any major differences between the courses; see table 3 for an overview.

Table 3. *Dimensions and vertical planes promoted in the mathematics curriculum*

Courses	a	b	c
Mathematics 1	[Sem-Ins]	[Sem-Dis], [Ins-Dis]	[Sem-Dis], [Ins-Dis]
Mathematics 2	[Sem-Dis], [Ins-Dis]	[Sem-Dis], [Ins-Dis]	[Sem-Dis], [Ins-Dis]
Mathematics 3	No	No	[Sem-Dis], [Ins-Dis]
Mathematics 4	[Sem-Dis], [Ins-Dis]		

As noted previously, the course Mathematics 1 is advocating work in the paradigm Geometry I and mostly is the [Sem-Ins]-plane, because even though learning some theorems (like the theorems of Pythagoras and Thales for example) the content in the curriculum suggests the usage of these as artefacts rather than as referential objects. Nevertheless, considering the general goals 1–7 in the Swedish curriculum, we notice that the use of artefacts is promoted (like *tools* and *models*) in the goals ”manage procedures and solve tasks of a standard nature with and without tools” (2) and ”interpret a realistic situation and design a mathematical model, as well as use and assess a model’s properties and limitations” (4). Using a referential is also stressed in the ability goals: ”follow, apply and assess mathematical reasoning” (5) and ”communicate mathematical thinking orally, in writing, and in action” (6) respectively. In addition, the goals ”use and describe the meaning of mathematical concepts and their

inter-relationships” (1) and ”relate mathematics to its importance and historical context” (7) advocate the use of representamen as well.

Summarising the last paragraph above, we conclude that, primarily due to the abilities 1–7, the planes that the curriculum promote students to work in and to use the most are [Sem-Dis] and [Ins-Dis]. According to Coutat and Richard (2011) (see also Kuzniak et al., 2016), this means that the processes the curriculum stresses to be used are modelling and validation. The process of discovering is not given priority to the same extent.

Conclusions

Using the framework of Mathematical working spaces our analysis shows that the geometrical work promoted in the Swedish upper secondary mathematics curriculum by and large is work within the Geometry II paradigm. The reason for this is that the curriculum stresses the students’ constructing of geometrical referential to have as the basis for their mathematical work.

Regarding the vertical planes, it is generally not clear from the curriculum what kind of work is intentioned for the students to engage in (except in the c-track and the Mathematics 4 course). However, the analysis indicates that the process of discovering, related to the plane [Sem-Ins], is sparsely promoted in the curriculum. Rather, almost all of the courses advocate students working in the [Sem-Dis] and [Ins-Dis] planes, meaning focusing on validation and modelling.

Discussion

Since we know that MWS are manifested through its users, actual or potential, we considered the curriculum as potentially manifested foremost by students and teachers. In this context, using the MWS model makes sense for us. However, the application of the framework exhibits some limitations such as for example that the curriculum did not have enough geometric-specific goals, which made the analysis harder, especially with respect to the interplay between the vertical planes ([Sem-Ins], [Ins-Dis] and [Sem-Dis]). In addition, since we did not find many references of other work analyzing curricula using MWS, we did not have much research to compare and contrast our results with. Nevertheless, having done the analysis, we are convinced that studying the geometry MWSG in the Swedish context, starting with the curricula, is promising for continuing to analyze textbooks and students’ and teachers’ work in a coherent and way. Next step to further explore and go deeper into this analysis, is to investigate how these ambitions in the curriculum (referential MWSG) are mediated through the schools (suitable MWSG) to the students actually enacted mathematical work (personal MWSG). It is our hope that the research initiated and presented in this paper productively can support and guide us in this endeavor.

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Methodological challenges when scaling up research on instructional quality in mathematics

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This article is based on the symposium on researching the quality of mathematics instruction, presented at Madif11. Instructional quality is one of the main contributors to student achievement and motivation in mathematics. However, much research addressing instructional quality or the relationship between teaching and learning are small-scale studies. When scaling up this research to allow generalisations from the research findings, issues can be seen with regard to cultural patterns of instruction and subject specificity. Experiences from large-scale international comparative surveys and a Nordic video study are used to discuss methodological challenges connected to generalising from large-scale studies when researching instructional quality in mathematics.

While a main aim of mathematics education research is to improve teaching and learning of mathematics, scholars, policymakers and educators have spent decades debating what counts for student learning in mathematics classrooms (Learning mathematics for teaching project, 2011). Especially instructional quality is seen as one of the main factors influencing student achievement and motivation (Hiebert & Grouws, 2007; Kiemer, Gröschner, Pehmer & Seidel, 2015) and as a central mediator of teaching competence (Baumert et al., 2010). In the past decade, instructional quality has become a major research interest in mathematics education research (Charalambous & Hill, 2012; Hill et al., 2008; Kersting et al., 2012). Instructional quality has traditionally been studied using small-scale designs, thus generalising research outcomes from such studies is challenging. However, the issue of generalising results might be addressed by scaling up the number of classrooms and teachers involved in the study. Comparative large-scale studies, such as the Programme of international student assessment (PISA) and Trends in international mathematics and science study (TIMSS), offer another way to scale up research on instructional quality in mathematics. From questionnaire data representing students', teachers' and

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principals' perceptions of mathematics teaching and learning, inferences can be made about the instructional quality and its relationship to student achievement, attitudes and beliefs (Niss, Emanuelsson & Nyström, 2013). However, these and other large-scale assessments, such as national tests, only give indirect information about instructional quality, as they rely on self-reported data or test achievement. Video-based studies offer a complementary method to study instructional quality through classroom observation and are suitable for describing complex educational settings and the situated, improvised act of teaching (Fischer & Neumann, 2012). Video-based studies including a large number of classrooms therefore offer another possibility for scaling up, as they can give more generalisable outcomes (Blömeke, Kaiser & Clarke, 2015).

Both large-scale surveys and video-based classroom studies face methodological challenges regarding conceptualisation, operationalisation and measurement of instructional quality in mathematics, because most of the studies involve rating processes or are bound to specific cultural contexts. For example, regarding measurement, there is controversy about how reliably and validly teacher and student ratings describe instructional quality (Schlesinger & Jentsch, 2016). Concerning operationalisation, the stability of instructional quality across different lessons or the effects of rater biases on the reliability and validity of analysis have been questioned (Praetorius, Lenske & Helmke, 2012; Praetorius et al., 2014). As an example of conceptualisation challenges, instructional quality is often researched across different subjects, and there is ongoing discussion about which features of instructional quality are subject-specific. For instance, there is an unsettled debate about which features of "instructional quality" are unique to mathematics instruction (Hiebert & Grouws, 2007; Schlesinger & Jentsch, 2016). Another challenge related to conceptualisation is the validity and transferability of results and instruments to other contexts (Clarke & Xu, 2018). For international studies, Blömeke, Olsen and Suhl (2016) bemoaned the lack of comparative research that could extend findings from national studies to other educational cultures and systems.

The aim of this article is to discuss some of the methodological challenges connected to generalising from large-scale surveys and video-based studies when researching instructional quality in mathematics. We draw on discussions from the symposium on researching instructional quality of mathematics held at Madif11 in Karlstad. At the symposium, methodological issues were exemplified by challenges encountered in two projects, TIMSS and LISA (Linking instruction and student achievement). In this article, we will also include experiences from the PISA study. PISA and TIMSS are international, comparative, large-scale surveys aimed at monitoring trends in mathematics achievement in participating countries, but which also include items about instructional quality (Niss et al., 2013). LISA is a video-based study conducted within the Nordic context, including 3–4 consecutive mathematics lessons from 47 classrooms

from Norway, 16 classrooms from Sweden and eight from Finland (Klette, Blikstad-Balas & Roe, 2017). In the following, we will describe how the studies presented in the symposium faced specific methodological challenges. Subsequently, we will discuss the specific challenges more broadly in the discussion section.

Using large-scale surveys to research instructional quality

International comparative studies, such as TIMSS and PISA, collect data on instructional quality through several survey-based measures: Students and teachers respond to questions describing teaching methods, content and beliefs about mathematics teaching and learning. In addition, school principals provide information about school characteristics and challenges. This is later connected to student achievement data to investigate relationships between teaching approaches and student learning (Martin, Mullis & Hooper, 2016; OECD, 2014). The aim of these projects is to generate data to allow researchers to perform analyses of educational systems rather than single classrooms. Olsen (2013), for instance, utilising data from PISA2012, connected data on general teaching quality (e.g. student-teacher relationship), frequency of classroom activities (e.g. teacher feedback) and the taught content (e.g. problem solving) with student achievement to highlight strengths and weaknesses of mathematics teaching in Norway. The outcome of this analysis was that Norwegian teachers focused less on structuring activities (e.g. checking homework, summarising lessons) than teachers in OECD in general, and that this in part might explain Norwegian achievement outcomes, as more extensive use of structuring activities is associated with higher student achievement.

The use of questionnaire data to research instructional quality is a main issue when applying large-scale surveys across a large number of countries. The data are self-reported, and as such represent beliefs held by students, teachers, head-teachers and parents about what goes on in classrooms. Students from an educational system where procedural fluency is the educational goal might view statements about instructional quality very differently from students attending school in an educational system where problem solving is emphasised. In addition, how we express ourselves and how we respond to questionnaires are also affected by culture, and previous research has revealed cultural differences in response style to Likert questions among students from different countries (Buckley, 2009). Without accounting for these differences, comparison across cultures and contexts might lead to misinterpretations of the outcome. In the PISA study, for instance, this has been observed when Danish students saw themselves as very good mathematics students, while scoring significantly lower than their peers in higher-achieving countries. Furthermore, students in low-achieving classrooms and countries might value their teachers as highly

or even more highly than students in high-achieving classrooms and countries, weakening the relationship between instructional quality and achievement (Nortvedt, Gustafsson & Lehre, 2016; Shen & Tam, 2008).

Several methods designed to consider and include cultural-based response patterns to questionnaires in the analysis and interpretation of data from large-scale international surveys exist. The PISA study, for instance, used anchoring vignettes to address issues related to the cross-cultural comparability of responses to questionnaires (OECD, 2014). An anchoring vignette consists of a scenario and some statements that respondents are asked to respond to. However, this method rests on the assumption that different respondents interpret the vignette scenarios in the same way and respond consistently to the vignette statements.

Measuring instructional quality across subjects

Standardised observation instruments have been proposed as analytical tools suitable for providing transparent, valid and reliable overviews of instructional quality (Kane & Staiger, 2012) that may be suitable for analyses across contexts (Cohen & Grossman, 2016; Klette & Blikstad-Balas, 2018) and academic subjects (Cohen, 2018). Several classroom studies examined instructional quality in different subjects using the same observation instrument across subjects (Kane & Staiger, 2012; Klette et al., 2017). The reasons for using the same manual across subjects may be pragmatic (e.g. it is expensive to train raters in several manuals) or more substantial (the researchers may be interested in comparing instructional quality across subjects).

We may very well question the validity of using the same observation instrument to rate instructional quality in different subjects, because instructional quality might differ between subjects (Brophy, 2007). A general observation instrument might not be capable of measuring mathematics-specific aspects of instructional quality; thus, general manuals might give a limited (and maybe non-valid) view of instructional quality. In the LISA study, a preliminary analysis of two aspects of instructional quality was conducted to determine whether a general and mathematics-specific observation manual would rate instruction differently.

We chose the manuals *Mathematical quality of instruction* (MQI) and *Protocol for language arts observation manual* (PLATO) for our comparison because the manuals are widely used, have been thoroughly validated (e.g. Grossman, Loeb, Cohen & Wyckoff, 2013; Hill et al., 2008) and share a similar structure. Both manuals measure several aspects of instructional quality and rate each aspect on a four-point scale after set time intervals; PLATO every 15 minutes and MQI every 7.5 minutes. They differ in the number of aspects they rate: PLATO rates 12 aspects of instructional quality, whereas MQI rates 21. We chose to compare two of these aspects, *feedback* and *intellectual challenge*

(which MQI calls *remediation of students' errors and misunderstandings* and *task cognitive demand*), because they represent aspects which are present in all subjects' instruction and are closely tied to the subject content (unlike e.g. behaviour management).

To study the differences between *intellectual challenge* and *feedback* in the two manuals, we first conducted a document comparison of the manuals' rubrics to determine whether there was a match between what the manuals considered to be instruction of high and low quality. Afterwards, we purposefully sampled and rated 15-minute segments from 32 different teachers to analyse what kind of instruction the manuals rated similarly and differently. This investigation is presented more thoroughly elsewhere (Stovner, 2018), and only a summary of the results is given here.

The comparison of the *feedback* rubrics revealed that the manuals mostly agreed on what constitutes high- and low-quality feedback, and 124 out of 144 feedback instances identified in the segments were rated similarly. Both PLATO and MQI describe low-quality feedback as vague or containing only procedural suggestions, and all instances of such feedback were rated similarly ("Good work", "Use a common denominator here"). Both manuals describe high-quality feedback as addressing students' understanding, being substantial and pointing to underlying meaning or overarching skills. All examples of high-quality feedback in the segments (e.g. "you seem to divide by hundred since the exercise mentions percentages, but you've only got 25 % of the shares, so how much would 1 % be?") were rated the same.

The only difference found between the manuals' *feedback* rubrics is that MQI did not count "help with mathematical tools" as high-quality feedback. This made a difference for one of the teachers in the sample who helped students with budgeting in Excel, giving feedback such as "use relative cell-references here [pointing] so the cells automatically update when you later change the interest rate". This was rated high by PLATO but did not count as feedback in MQI.

The results for the *intellectual challenge* rubric were somewhat similar: It made no difference whether a task was rated with PLATO or MQI; all instruction rated as low-quality in one manual was rated as low-quality in the other manual, and similarly on the high end. There was only one exception: MQI has a special provision for students' unsystematic exploration when solving a task where the teacher does not support students in making sustained progress. This made a difference for a lesson in which students were asked to estimate how many people attended a local rock concert based on a birds-eye picture of the event. This segment received PLATO's highest rating, since the students explained and justified their ideas; but, conversely, the segment received MQI's lowest rating, since these ideas ("calling the ticket broker" or "using image recognition software") were not mathematical in nature.

To sum up, 30 out of 32 segments rated in this analysis were given the exact same ratings by the two manuals. But for the two segments rated differently, the discrepancy was huge, and the mathematics-specific manual awarded its lowest rating to what the general manual awarded its highest. The message for researchers wanting to compare instructional quality across subjects by using a general observation manual is clear: One should investigate in depth how using a general manual affects the measurement of instructional quality.

Measuring instructional quality across contexts

In addition to investigating possible challenges with generalising PLATO elements across subjects, the LISA study has taken a closer look at challenges encountered when using the manual across cultural contexts. PLATO was developed for rating instruction in the United States and has a socio-constructivist approach to learning (Bell, Dobbelaer, Klette & Visscher, 2018), by which we mean that oral contributions of students and teachers in whole-class situations and during group work are valued and rated highly. This occasionally causes dilemmas when analysing Nordic classrooms, where students' work is often individualised (Carlgren et al., 2006). One example became evident when coding the aspect *classroom discourse* (CD) during Norwegian and Finnish mathematics lessons. CD consists of two sub-aspects, *uptake* and *opportunity for student talk*. A high score on *uptake* involves evidence of teachers' uptake of student ideas in the form of restating in academic language, asking for clarification, elaboration or evidence. A high score on *opportunity for student talk* involves at least five minutes of opportunity to discuss content and requires that some of the questions guiding the conversation are open-ended. Students in these contexts often discuss mathematical content, but in many LISA classrooms this occurs individually with the teacher during seatwork, especially in the Finnish classrooms. Since CD requires several students to participate, the quality of these teacher-student conversations is not captured with CD in PLATO. Thus, when reporting on CD, the criteria for what is considered classroom discourse need to be clearly explained in order to make valid conclusions.

With this example, we want to demonstrate how an observation instrument may face methodological challenges when generalising instructional quality across contexts, indicating possible biases. Since PLATO is built on a socio-constructivist view on learning, content-related talk is only recognised when "in public" and when several students participate. Findings from the LISA study indicate that this theoretical stance may not reflect common discourse patterns in Nordic mathematics classrooms. Therefore, if researchers bring a standardised observation manual to another context, they are advised to critically assess the different criteria for instructional quality and how it is captured when applied in a new context. If not, the researchers might draw

contextually insensitive conclusions about instructional quality – for example, when analysing classroom discourse.

Discussion

The methodological challenges of the studies discussed in the symposium point, from different perspectives, to the issue of validity. When using self-reported survey data from students and teachers, as in international large-scale studies like PISA and TIMSS, or when using observation instruments designed to capture instructional quality across contexts or subjects, as in video-based studies like LISA, we must be careful: Is instructional quality really measurable and observable in the same manner across contexts, subjects or individuals? If so, then this would presuppose that instructional quality is a universal, cross-contextual phenomenon. While some studies assume that instructional quality is a stable phenomenon across subjects and even contexts and measures, other studies, which might conceptualise or operationalise it differently, argue in favour of the subject- or context-specificity of instructional quality. For instance, a review of mathematics-specific aspects of instructional quality in available instruments yielded several aspects distinct to mathematics instruction, e.g. the use of representations and mathematical language, teaching of mathematical content and topics (e.g. problem solving), making connections, abstractions and generalisations (i.e. mathematical richness), the implementation and role of the task in mathematics, and the use of materials and manipulatives (Schlesinger & Jentsch, 2016).

Whichever way instructional quality is conceptualised (universal or specific), in order to interpret results from research on instructional quality, it is necessary to specifically address validity issues and to validate surveys and observation instruments in the contexts and for the subjects for which they are meant to be used. Examples of qualitative approaches to validate video-based studies with in-depth analyses of PLATO ratings were presented in the symposium and are further discussed here. The results indicate that video data rated with standardised observation instruments may not be sufficient to evaluate certain elements of mathematics instruction in Nordic classrooms. In the Nordic setting, measurements of classroom discourse may need another dimension capturing teacher–student talk during individual seatwork. The results of a validation approach using a mathematics-specific and a general instrument to measure instructional quality, on the other hand, partially supported the validity of a general instrument for use across subjects, yet also revealed detailed but important differences. A validation by closer qualitative observation, as suggested above, is not the only way to respond to the issues of instructional quality affecting students differently in different subjects and contexts. Another approach to validate results when researching instructional quality would be

to use student data, as was done by some large-scale studies presented in the symposium. Blömeke et al. (2016) proposed reflecting on both students' and teachers' experiences in the classroom and including students' performance, motivation and interest as dependent variables in a single model of student outcomes. Combining or integrating different self-reported and observational data might thus enhance the complementarity and validity of findings concerning instructional quality.

Whatever approach is taken to validate instruments or outcomes in the various studies, there is always a trade-off associated with the research of instructional quality. If we always make our own measurements with our own conceptualisations of instructional quality to obtain a perfect fit for the subject or context in which we are interested, we lose both comparability between studies and a "common language" to talk about aspects of teaching (Grossman & McDonald, 2008). Which trade-off is the most constructive for a particular research project might arguably depend on the goals of the project, whether the aim is to find comparable patterns or to generate a more in-depth understanding. This is not a dichotomy, however, as we argue that there will always be the need for an in-depth understanding of context in order to nuance and challenge the results of large-scale studies.

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Short presentations

Winning team – collaborative teaching of university mathematics

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Teaching university mathematics involves many challenges. Discussing a topic and posing appropriate questions is key to learning it, but determining which questions are worth asking is difficult. Similarly, many students require examples of how mathematics connects to their main subject in order to appreciate its relevance, but seeing these connections, as well as connections between different mathematical topics, is difficult. We outline a planned project on collaborative teaching aimed at addressing these challenges. Two instructors with different mathematical backgrounds (analysis and algebra/geometry) will collaborate on teaching a first-year calculus course. Some lectures will be taught jointly, the two teachers presenting and discussing content together. Central concepts of Calculus are viewed differently in different branches of mathematics, and highlighting these differences can help students grasp these concepts. Furthermore, having instructors with different mathematical expertise discussing the topics and posing questions can provide students with alternative ways of viewing content and approaching problems, helping them gain insight into the processes of doing mathematics. We are particularly interested in how collaborative teaching might support students' participation in and development of mathematical discourse. Furthermore, collaborative teaching can contribute to teacher development, with participating teachers learning from one another and jointly developing innovative teaching practices.

The space between pre-service primary teachers' first year status and their goals

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Students' mathematics teacher identity is formed in various settings. A study with 45 pre-service students in their first year of education was conducted as part of a longitudinal study of year 4–6 mathematics teachers' identity formation, to study the development during their education in terms of mathematical knowledge, pupils' learning and the teacher role. Questionnaires and interviews were used. The results show that many students were reluctant, or even unable, to explicitly use mathematics and had mathematics conceptions that may mislead pupils. The students had stronger emphasis on the teacher role and content knowledge than on pupils' learning when they wrote about what is important to learn to become a good mathematics teacher. A large part of the students' goal descriptions of a mathematics teacher did not contain any mathematics content knowledge, which also was less important in the descriptions of what internships may provide for their learning, particularly in a more general context. Mathematics content knowledge is hence not regarded as one of the most important aspects of the teacher identity to develop, even though data implies a need for improvement in that area.

Upper secondary physics teachers' views of mathematics

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Physics teachers at upper secondary school implicitly teach mathematics in their physics classes through their teaching strategies and preferred ways of using mathematics. Their views of physics and mathematics are important for the way they depict mathematics to the students. A web-questionnaire about teaching, learning and attitudes to physics and mathematics was administered to 845 Swedish physics teachers, of which 379 teachers responded (a 45% answering rate). The data was analysed in factors and profiles with SPSS. Part of the

questions investigated views of mathematics, i.e. as a means for application, as a schema, as a formal construct or as processes. Mathematics as a means for application was the dominant opinion. Students' lack of knowledge in mathematics, both when it comes to proficiencies and mathematics topics, was regarded as a problem by many of the teachers. Particularly problem solving and modeling were regarded as big or crucial problems, whereas conceptual proficiency, communication and relevance were considered no or small problems. Algebra and equations were problematic, but not arithmetic and statistics. The results will be used to select teachers with differing profiles for classroom studies.

To set an example: shifts in awareness when working with Venn diagrams

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Venn diagrams and basic set theory can serve as a tool for pre-service teachers to grasp complex networks of mathematical concepts, such as different quadrilaterals (Moyer & Bolyard, 2003). Basic set theoretic concepts carry intuitive meanings (Bagni, 2006), and Venn diagrams where the interior of closed plane curves represent sets, are simple and non-controversial, even for children (Freudenthal, 1969). Previous research is inconclusive regarding the benefit of Venn diagrams (Moyer and Bolyard, 2003).

As a reoccurring part of a mathematics course in the primary teacher education, two lectures and associated workshops have introduced basic set theoretical concepts and Venn diagrams to the pre-service teacher. The discovery of objects belonging to several sets seemed unproblematic in the first workshop, the understanding of intersection became an obstacle when working with numerical problems.

We conclude that working with properties of objects is not enough to understand set intersection. In order to support their future students' mathematical development, pre-service teachers need to be aware of their awareness when working with problems (awareness-in-action), but also of this awareness in turn (awareness-in-discipline) (Mason, 2011).

Kompetens att leda matematiksamtal

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Inför min masteruppsats har jag som deltagande aktionsforskare följt ett utvecklingsprojekt i mitt arbetslag på grundlärarprogrammets matematikkurser. Utvecklingsprojektets syfte är att utveckla ett ramverk som ska vara användbart som verktyg för lärare i planering, utförande och analys av matematiska klassrumsdiskussioner. Arbetslaget består av forskare och lärarutbildare inom intressegruppen för matematikdidaktik inom institutionen.

I studien följs hur utvecklingsprojektet utformas och genomförs och vad det är som möjliggör och begränsar utvecklingen av vårt samarbete och vår undervisning som lärarutbildare under projektet. Genom att följa ett utvecklingsprojekt som deltagande aktionsforskare inriktas intresset på två fält samtidigt, forskning och den vardagliga praktiken. Praktiken ställer frågorna och sätter igång handlingar som studeras och diskuteras. Forskningen är det systematiska arbetet i relation till teorier som ger ny kunskap men det är praktiken som är utgångspunkten. Här handlar praktiken om att i arbetslaget utveckla undervisningen om kompetens att leda matematiksamtal. Med stöd av teorin om praktikarkitekturer och fokus på *sayings*, *doings*, och *relationships* utforskas den kollegiala samverkan i vår praktik. Samspelet och interaktionen i praktiken påverkas av dessa faktorer och möjliggör och begränsar utvecklingsprojektets utfall.

Clash of cultures? Teachers' and students' perceptions of differences between secondary and tertiary mathematics education

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The transition problem between secondary and tertiary mathematics education has been addressed from various perspectives, e.g. concerning gap in mathematical content, and metacognitive awareness. In a literature review of studies covering a range of aspects of the transition problem, five dimensions of research that highlight mismatches in criteria for what counts as

”mathematics” at upper secondary and tertiary levels were identified. Since many studies focus on some specific aspect of the transition problem, it is valuable to use all dimensions to approach the problem from a more general perspective. The purpose of this on-going study is to analyse if any of the dimensions influences the transition more than any of the others, both from a student and a teacher perspective. We use an on-line questionnaire, where answers are given either on a five-point Likert scale or a five-point scale capturing the direction of a possible difference between secondary and tertiary levels. In the first step, nine mathematics teachers teaching first year mathematics courses at engineering programs have answered the questionnaire. The analysis shows that especially *Changes in level of formalisation and abstraction* seemed to be regarded as a gap (0.70 on the normalised scale), which can be compared with the corresponding measures for the other four dimensions: between 0.41 and 0.59.

Learning models enhancing Number sense

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Taking in account that knowledge in mathematics can partly be explained as theoretical knowledge, students have to take part in theoretical work to enhance mathematical knowing. Further, taking in account that theoretical work depends on mediated actions, questions in relation to the youngest students use of mediated tools can be raised. In my presentation I discuss students work in grade 1, 2 and 3 in Sweden in relation to the perspectives of learning activity (Davydov, 2008) and early algebraization. Therefore, I discuss learning models and algebra as tools to enhance mathematical knowledge. The empirical data for the presentation is produced in a project; ”Education in mathematics together with young students using problem solving and algebra”. A task used by students in the project and in three reference schools shows that the project students handled problem solving in quite another way than the reference students. The project students used algebraic symbols and a length model to solve the task. Earlier research shows similar results (see e.g. Zuckerman, 2004; Schmittau, 2003). In the presentation I will ask the audience about learning models in relation to mathematics education and mathematics education research.

Linguistic features as possible sources for inequivalence of mathematics PISA tasks

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When mathematics tasks are translated to different languages, there is a risk that the different language versions are not equivalent and display differential item functioning (DIF). In this study, we aimed to identify possible sources of DIF. We investigated whether differences in some linguistic features are related to DIF between the English (USA), German, and Swedish versions of mathematics tasks of the PISA 2012 assessment. The linguistic features chosen in this study are grammatical person, voice (active/passive), and sentence structure. We analyzed the three different language versions of 83 mathematics PISA tasks in three steps. First, we calculated the amount of differences in the three linguistic features between the language versions. Then, we calculated DIF, using the Mantel-Haenszel procedure pairwise for two language versions at a time. Finally, we searched for correlations between the amount of linguistic differences and DIF between the versions. The analysis showed that differences in linguistic features occurred between the language versions – differences in voice were most common – and that several items displayed intermediate or large level of DIF. Still, there were no statistical significant correlations between differences in linguistic features and DIF between the language versions, that is, there must be other sources of DIF.

Elevers meningsskapande i mötet med matematikläroböcker

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Studien syftar till att studera den *meningspotential*, eller erbjudande om mening, som läroboken designats för att eleven ska upptäcka, samt den *handlingspotential* som eleven uppfattar i mötet med läroboken (Selander & Kress, 2010). Detta görs utifrån en multimodal ansats där olika resurser för lärande beaktas såsom: skrift, symboler, bild, rörlig bild, ljud gester o.s.v. Studien behandlar elevers möte med läroböcker i grundskolans tidigaste år och svarar upp mot

forskningsfrågan: *Vilken meningspotential finns i matematikläroböcker och vilken handlingspotential upptäcker elever i mötet med läroböcker?* Fokus riktas mot relationen mellan det matematikuppgiften designats till att erbjuda eleven, samt det eleven faktiskt upptäcker i mötet med läroboken. Datamaterialet består av videoinspelningar, samt dokument i form av elevlösningar från 18 elever i årskurs 1. Förväntade resultat av studien är att det matematikinnehåll som uppgifterna designats till att erbjuda eleverna, inte alltid är vad eleverna upptäcker i mötet med läroboken. Detta är av stor vikt att lärare och läroboksförfattare har kunskaper om. Med god kännedom om matematikläroboken som multimodal, ges ökade möjligheter till att goda lärsituationer i matematik kan iscensättas.

Argumentation in university textbooks

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Argumentation is central within both mathematics and science education. Since mathematics frequently is described as based on deductive reasoning and science as based on inductive reasoning, this might contribute to differences between the subjects regarding the type of argumentation that is present in textbooks. The purpose of this study is to further the understanding of the role of argumentation in science and mathematics texts. The research question is: What are the similarities and differences, concerning the *amount of explicit argumentation*, between biology, chemistry, and mathematics textbooks? We use the concept *argumentative structure*, which focuses on the key components in an argumentation: a *conclusion*, a *premise* and an *argumentation marker* (e.g. "since"). Data consists of 20 pages from a university textbook in each subject. We calculated the *number of argumentative structures per declarative sentence*. Preliminary analyses using independent samples *t*-tests were performed with textbook page as unit of analysis. The results show that the mathematics textbook contains significantly ($p < 0.01$) more argumentative structures per statement (0.45) than the science textbooks (chemistry 0.26; biology 0.15). These results indicate that argumentation might play different roles in different subjects.

Mathematical classroom discussions – developing a framework focussing on the teacher’s role

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Orchestrating a mathematical discussion is a difficult task for many mathematics teachers. We report on the process of developing a framework describing the teacher’s role in promoting mathematical discussions. A short intervention was carried out to test it in mathematics education classes for pre-service teachers. Data was collected concerning the understanding, use and effectiveness of the proposed framework. Previous research stresses the importance of mathematical discussions. The framework contributes to the development of teacher education by making pre-service teachers aware of basic components of a mathematical discussion. The framework includes two main aspects of mathematical discussions: mathematical objectives and possible talk moves. Special attention was given to cultural contexts and local terminology to make it relevant to Swedish classrooms. The framework was implemented and revised in an iterative process. Data gathering targeted how the framework was understood and used by the pre-service teachers.