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Nordic Research in Mathematics Education

Papers of NORMA 17

The Eighth Nordic Conference on Mathematics Education
Stockholm, May 30 - June 2, 2017

Editors:
Eva Norén, Hanna Palmér and Audrey Cooke

SMDF Svensk Förening för MatematikDidaktisk Forskning
Swedish Society for Research in Mathematics Education

Eva Norén, Hanna Palmér and Audrey Cooke (Eds.)

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Preface

This volume contains a collection of papers from the Eighth Nordic Conference on Mathematics Education, NORMA 17, which took place in Stockholm, Sweden, from the 30th May to 2nd June 2017. The conference was hosted by the Department of Mathematics and Science Education, at Stockholm University.

The first NORMA Conference on mathematics education NORMA 94, was held in Lahti, Finland, in 1994. Four years later, it was held in Kristiansand, Norway, and since then it has taken place every third year. After each conference, selected papers have been published in a proceeding.

The NORMA conferences are always organized in collaboration with NoRME – the *Nordic Society for Research in Mathematics Education*. NoRME is open for membership from national societies for research in mathematics education in the Nordic and Baltic countries.

The scientific committee of NORMA 17 represented all Nordic countries and one representative from the Baltic countries. There was also a mix of junior and senior researchers. The members of the committee were:

- Eva Norén, Stockholm University (chair),
- Paul Andrews, Stockholm University,
- Hanna Palmér, Linnaeus University, Växjö,
- Johan Prytz, Uppsala University,
- Martin Carlsen, University of Agder,
- Janne Fauskanger, University of Stavanger,
- Morten Misfeldt, Aalborg University,
- Lena Lindenskov, Århus University,
- Markus Hähkioniemi, University of Jyväskylä,
- Tomi Kärki, University of Turku
- Freyja Hreinsdottir, University of Island,
- Madis Lepik, Tallinn University.

The theme for the NORMA 17 conference was Nordic research in mathematics education. Nordic and Baltic researchers in mathematics education were given opportunities to introduce their research by regular papers, short communications, working groups and symposia. At total 44 regular papers, 39 short communications, three working groups, and three symposia were presented during the three days. There were also three plenary speakers. Thus, the conference offered a comprehensive forum for the discussions and constructive meetings of researchers, teachers, teacher educators, graduate students, and others interested in research on mathematics education in the Nordic context.

The collection of papers presented in this book are a selection of the papers presented at the conference. The collection contains mostly regular papers but also includes several papers from the symposiums. The papers have been selected based on the reviews, one before the conference and one after the conference. Some participants at the conference chose to publish their papers elsewhere.

Based on this selection the papers in this book cover the areas of:

- Early years mathematics
- Primary mathematics
- Secondary mathematics
- Upper secondary mathematics
- University mathematics
- Communication, language and texts in mathematics education
- Mathematics teacher education
- Continuing professional development
- Curricular aspects of mathematics education
- Mathematics Education in general

Although teaching and learning of mathematics is the common interest for all participants, the papers make visible a great diversity in how this is considered. They include a variety of mathematical topics as well as a currency from preschool to university mathematics. Furthermore, various methodologies and theoretical perspectives are used in the research presented. This variation shows that the Nordic research in mathematics education is a broad field and that the field was well represented at the conference.

Stockholm July 2018
Eva Norén, Hanna Palmér and Audrey Cooke

Contents

Preface i

Content iii

Early years mathematics

Mathematics in Swedish and Australian Early Childhood Curricula 1
Audrey Cooke

Paper ~~or~~ and digital: a study of combinatorics in preschool class 11
Jorryt van Bommel, Hanna Palmér

"I find that pleasurable and play-oriented mathematical activities create wondering and curiosity" Norwegian kindergarten teachers' views on mathematics 21
Trude Fosse, Magni Hope Lossius

Primary mathematics

Collaborative tool-mediated talk – an example from third graders 31
Heidi Dahl, Torunn Klemp, Vivi Nilssen

Narratives constructed in the discourse on early fractions 41
Ole Enge, Anita Valenta

Second graders' reflections about the number 24 51
Marianne Maugesten, Reidar Mosvold, Janne Fauskanger

Secondary mathematics

Tablet computers and Finnish primary and lower secondary students' motivation in mathematics 59
Timo Tossavainen, Laura Hirsto

Supporting students' mathematical problem solving: The key role of different forms of checking as part of a self-scaffolding mechanism 69
Joana Villalonga Pons, Paul Andrews

Negotiating mathematical meaning with **on***eself – snapshots from imaginary dialogues on recurring decimals* 79
Eva Müller-Hill, Annika M. Wille

Upper secondary mathematics

Mixed notation and mathematical writing in Danish upper secondary school 89
Morten Misfeldt, Uffe Thomas Jankvist, Steffen Møllegaard Iversen

University mathematics

Proof by induction – the role of the induction basis 99
Niclas Larson, Kerstin Pettersson

Interpreting teaching for conceptual and for procedural knowledge in a *teaching video about linear algebra* 109
Ragnhild Johanne Rensaa, Pauline Vos

Research study about Estonian and Finnish mathematics students' views about proof 119
Antti Viholainen, Madis Lepik, Kirsti Hemmi, Mervi Asikainen, Pekka E. Hirvonen

Communication, language and texts in mathematics education

The national validation of Finnish mathematics teachers' Lexicon 129
Markku Hannula

A correlation study of mathematics proficiency VS reading and spelling proficiency 139
Arne Kåre Topphol

Students with low reading abilities and word problems in mathematics 149
Hilde Opsal, Odd Helge Mjellem Tonheim

Attending to and fostering argumentation in whole class discussion 159
Markus Hähkiöniemi, Pasi Nieminen, Sami Lehesvuori, John Francisco, Jenna Hiltunen, Kaisa Jokiranta, Jouni Viiri

The roles of mathematical symbols in teacher instruction 169
Marit Hvalsøe Schou

Second language students' achievement in linear expressions and time since immigration 179
Jöran Petersson

Mathematics teacher education

Prospective class teachers' attitude profiles towards learning and Teaching mathematics 189
Tomi Kärki, Harry Silfverberg

An initial analysis of post-teaching conversations in mathematics practicum: researching our own practice 199
Suela Kacerja, Beate Lode

Opportunities and challenges of using the MDI framework for research in Norwegian teacher education 209
Reidar Mosvold, Janne Fauskanger

Negotiating mathematical meaning with oneself – snapshots from imaginary dialogues on recurring decimals 219
Arne Jakobsen, Mercy Kazima, Dun Nkhoma Kasoka

Continuing professional development

Towards an organizing frame for mapping teachers' learning in professional development 229
Daniel Brehmer, Andreas Ryve

Good mathematics teaching as constructed in Norwegian teachers' discourses 239
Janne Fauskanger, Reidar Mosvold, Anita Valenta, Raymond Bjuland

Teachers' mathematical discussions of the Body Mass Index formula 249
Ragnhild Hansen, Rune Herheim, Inger Elin Lilland

Teachers' attention to student thinking, mathematical content and teachers' role in a professional learning community 259
Odd Tore Kaufmann

Teacher learning in Lesson Study: Identifying characteristics in teachers' discourse on teaching 269
Anita Tyskerud

Adopting the developmental research cycle in working with teachers 279
Jónína Vala Kristinsdóttir

In-service teachers' positioning when discussing the body mass index 289
Toril Eskeland Rangnes, Rune Herheim, Suela Kacerja

Curricular aspects of mathematics education

Characterizing Swedish school algebra –initial findings from analyses of steering documents, textbooks and teachers' discourses 299
Kirsti Hemmi, Kajsa Bråting, Yvonne Liljekvist, Johan Prytz, Lars Madej, Johanna Pejlare, Kristina Palm Kaplan

A cross-cultural study of teachers' relation to curriculum materials 309
Leila Pehkonen, Kirsti Hemmi, Heidi Krzywacki, Anu Laine

Mathematics education in general

Estonian and Finnish teachers' views about the textbooks in mathematics teaching 319
Leila Pehkonen, Sirje Piht, Käthlin Pakkas, Anu Laine, Heidi Krzywacki

Inquiry-based Learning in Mathematics Education: Important Themes in the Literature 329
Jonas Dreyøe, Dorte Moeskær Larsen, Mette Dreier Hjelmborg, Claus Michelsen, Morten Misfeldt

E-mail Addresses to the Contributors 343

Mathematics in Swedish and Australian Early Childhood Curricula

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Opportunities for young children to engage in activities that develop their mathematical skills, understandings, and disposition are impacted by early childhood education curricula through the ways early childhood educators interpret the curricula. Investigating how mathematics is incorporated in early childhood curricula can provide insight into these impacts. An investigation of the Swedish Curriculum for the Preschool Lpfö 98 and the Australian Early Years Learning Framework was conducted to identify the use of terms indicating mathematics. The results for the two curricula are compared and discussed in terms of their impact on the mathematical skills, understandings, and disposition of young children.

Introduction

In the past, young children were viewed as incapable of engaging with mathematics and thinking mathematically (Hachey, 2013). It is now believed that "in their everyday interactions with the social and physical world, young children engage in diverse types of mathematical thinking" (Hachey, 2013, p. 420). The early childhood educator is responsible for creating experiences that enable the child to use and develop mathematical skills and knowledge. Early childhood curricula provide an orientation within which the educator can create these experiences (Gasteiger, 2014).

Mathematics in early childhood

In contrast to previous beliefs, Baroody, Lai, and Mix (2006) claim that mathematical understandings develop from early ages and pre-school children can engage with mathematics. They describe this as informal mathematical knowledge that comes from children's everyday lives and underpins the successful development of formal mathematics. The capacity for children to both bring mathematical ideas and learn new mathematical ideas should be recognized in experiences and activities that are provided in early childhood education settings. This consideration reflects aspects of Lembrér and Meaney's (2014) examination of 'being' and 'becoming' in early childhood. They proposed that positioning the child as 'being' acknowledges the mathematical understandings the child has, whereas 'becoming' highlights the mathematical understandings to be developed.

The early childhood educator's positioning of the child may impact on the activities created and the mathematics enabled within those activities (Hachey, 2013).

Mathematics in early childhood curricula

Curricula

The Working Group on Early Childhood Education Care [WGECEC] (2014) proposed that the curriculum is one of the five elements that can be evaluated to help determine the quality of the care provided in early childhood. They described curriculum as providing both content and pedagogy to enable children to engage and learn. Although the Australian Early Years Learning Framework [EYLF] (Australian Government Department of Education, Employment and Workplace Relations [DEEWR], 2009) is called a framework, Arlemalm-Hagser and Davis (2014, p. 5) considered the EYLF (DEEWR, 2009) and the Swedish Curriculum for the Preschool Lpfö 98 [SCP] (Skolverket, 2011) as both steering documents and curricula in their comparison of sustainability and agency in the two documents. Following the lead of Arlemalm-Hagser and Davis (2014), this paper will also use the term curricula for these documents.

Organisation of the curricula

The SCP (Skolverket, 2011) is organized into two parts - *Fundamental values and tasks of the preschool* and *Goals and guidelines*, with the *Goals and guideline*s separated into *Sections* then *Goals* (for children) and *Guidelines* (for educators and team members). The EYLF (DEEWR, 2009) has six parts - *Introduction*, *A vision for children's learning*, *Early childhood pedagogy*, *Principles*, *Practice*, and *Learning outcomes for children birth to 5 years*. The last part is divided into five *Outcomes* and each of these has *Key components* with points for children and for educators.

Domains of empowerment

Curricula learning outcomes and guidelines that incorporate mathematics encourage the educator to view young children as maths-able (Hachey, 2013). However, how the learning outcomes and guidelines address mathematics can influence the experiences created by educators. One way of interpreting how these address mathematics is via Ernest's (2002) domains within mathematics. His domains focus on the empowerment of the individual based on the sphere within which mathematics could be engaged with. Specifically, mathematical empowerment enables power over "language, skills and practices of using and applying mathematics" (p. 1) within narrow settings (such as school); social empowerment enables power over the use of mathematics in social settings; and epistemological empowerment enables power over "the creation and validation of knowledge" (p. 2) and incorporates the individual's identity. In terms of early childhood education, the domains could be construed as focusing on children developing specific mathematical language and processes (mathematical empowerment); using mathematical ideas effectively in social situations, including outside of the pre-school setting (social empowerment); and confidently using mathematics and creating solutions through mathematics (epistemological empowerment).

Connections between curricula, the educator, and domains of empowerment

The inclusion of mathematics within curricula may orient the educator, but the educator still has choice in the mathematical activities that are developed, and this choice can depend on the educator's perception of mathematics (Ernest, 1989). Ernest (1989) described three philosophical views of mathematics - instrumentalist, where mathematics involves unrelated and unbending rules and facts; Platonist, where mathematics is an external, static, and unified knowledge; and problem-solving, where mathematics is a human, cultural creation that is dynamic and expanding. Likewise, Grigutsch, Raatz, and Törner (1998) considered a static or dynamic view of mathematics. The static view incorporated the aspects of formalism or schema and the dynamic view incorporated the aspect of process. Benz (2012) described the aspects within the Grigutsch et al. (1988) framework as comprising terminology that enables logical and exact application (that is, formalism), concerned with calculations following rules (that is, schema), a process involving problem-solving (process), and the practical or direct use (application). Ernest's (1989) problem-solving view or Grigutsch et al.'s (1988) problem-solving (process) or practical or direct use (application) are most similar to Ernest's (2002) description of activities likely to result in epistemological empowerment.

The incorporation of mathematical ideas in early childhood curricula may be difficult for educators to act upon due to their past experiences with mathematics (Anders & Rossbach, 2015). Some educators fear or hate mathematics or dislike the idea of teaching mathematics (Bates, Latham, & Kim, 2013), and this can lead to an avoidance of mathematical activities (Chinn, 2012). However, the inclusion of mathematics in early childhood curricula reiterates the importance of young children engaging with mathematical ideas in early childhood settings. Educators must engage with mathematics themselves to improve the learning opportunities for their children (Benz, 2012). The educators' actions, when informed by the curriculum, will impact on the activities created for children (Ernest, 1989), which will flow into the types of engagement children will have with mathematics and the domain of empowerment enabled within mathematics (Ernest, 2002).

The inclusion of mathematics in early childhood curricula will prompt educators to see young children as maths-able (Hachey, 2013). This influences the activities educators plan and implement (Baroody et al., 2006) and how the educator observes and interprets what young children do in terms of mathematical understandings (Anders & Rossbach, 2015). Educators with mathematical understandings will ‘look’ for mathematics in their children’s play (Lee, 2014) and will provide resources for play that enable children to bring their existing mathematical understandings into the classroom and develop them further (Mixon, 2015). These perspectives can be influenced by whether the child is positioned as ‘being’ or ‘becoming’ in relation to mathematical understandings (Lembrér & Meaney, 2014). The experiences that result from the educator seeing young children as being maths-able and becoming maths-able, such as recognizing that children create solutions using mathematics, are more likely to lead towards epistemological empowerment (Ernest, 2002).

Research questions

An interpretive approach (Merriam, 2009) is used to investigate how the curricula might orient mathematics for the educator. The focus is on how the terms mathematics, math, maths, mathematical, mathematically (that is, the targeted terms) are used within the curricula and how they might be interpreted within the three domains of Ernest’s (2002) empowerment framework. Variations of the word ‘mathematics’ were used as this is the term Ernest (2002) used. ‘Numeracy’ was not used as it includes confidence, initiative and risk taking (Geiger, Goos, & Dole, 2014), which reflects Ernest’s (2002) epistemological empowerment. The targeted terms were searched for within the SCP (Skolverket, 2011) and the EYLF (DEEWR, 2009) to determine:

1. Which sections or outcomes contain goals or points incorporating the targeted terms?
2. How do the goals or points address mathematics in terms of Ernest’s (2002) empowerment domains?

Method

The research focused on how the targeted terms (variations of the word ‘mathematics’) were incorporated within the SCP (Skolverket, 2011) and the EYLF (DEEWR, 2009). As the researcher’s language was English, the official English translation of the SCP (Skolverket, 2011) was used. Occurrences of the targeted terms within the *sections* and *goals* of the SCP (Skolverket, 2011) and within the *key components* and *points* of the EYLF (DEEWR, 2009) were noted.

Each *goal* and *point* were analyzed in terms of Ernest’s (2002) empowerment domains. The author and a highly experienced early childhood educator colleague used their understandings and experiences within early childhood education and mathematics education to interpret how the two curricula incorporated the targeted terms and how the *goals* and *points* could be met. This process reflected the purpose of the interpretive approach in several ways, through describing and interpreting what was found and acknowledging that these descriptions and interpretations were determined by the experiences and understandings of the author and her colleague (Merriam 2009). Codes were developed to describe what the analysis found:

Explicit (E) - the goal or point can only be met within the empowerment domain.

Potential (P) - the goal or point can be met both within and without the empowerment domain.

Not needed (N) - the goal or point can be met without the empowerment domain.

Results

The targeted terms (variations of 'mathematics') were found in both curriculum documents. In the SCP (Skolverket, 2011), the targeted terms were found within three *goals* for children and two *guidelines* (one for educators and one for the team) in one section, *Developing and Learning* (p. 10), of the SCP (Skolverket, 2011). The targeted terms were found in two *outcomes* of the EYLF (DEEWR, 2009) and in one *key component* within each of these. In *Outcome 4Children are confident learners*, three points for children and two for educators in the key component *Children develop a range of skills and processes such as problem solving, enquiry, experimentation, hypothesising, researching, and investigating* (DEEWR, p. 35) contained the targeted terms. In *Outcome 5Children are effective communicators*, one point for children and one for educators within the key component *Children interact verbally and non-verbally with others for a range of purposes* (p. 40) contained the targeted terms. The description for Outcome 5 included a discussion of numeracy that used the targeted terms seven times. The targeted terms were also found within two definitions for numeracy. This research focused on the goals for children within the SCP (Skolverket, 2011) and points for children within the EYLF (DEEWR, 2009) as these provided orientation (Skolverket, 2011) and observable evidence (DEEWR, 2009) for children's engagement with mathematics.

The location of the goals and points were within sections and outcomes addressing learning, *Developing and Learning* of the SCP (Skolverket, 2011, p. 10) and *Outcome 4Children are confident learners* of the EYLF (DEEWR, 2009, p. 35) and communication, *Outcome 5Children are effective communicators* (DEEWR, p. 40). When considered in terms of Ernest's (2002) three empowerment domains, all of the three goals of SCP (Skolverket, 2011) were coded *E* (considered to have been explicit) for all empowerment domains. All goals from the SCP (Skolverket, 2011) and all points from the two EYLF outcomes were coded *E* for Ernest's (2002) mathematical domain.

Curricula Section or Outcome Goal or Point	Ernest's (2002) Empowerment Domains		
	Mathematical	Social	Epistemological
Swedish Curriculum for the Preschool Lpfö 98 (Skolveket, 2011)			
Section 2: Development and Learning			
Develop their ability to use mathematics to investigate, reflect over and test different solutions to problems raised by themselves and others (p. 10)	E	E	E
Develop their ability to distinguish, express, examine and use mathematical concepts and their interrelationships. (p. 10)	E	E	E
Develop their mathematical skill in putting forward and following reasoning. (p. 10)	E	E	E
Australian Early Years Learning Framework (DEEWR, 2009)			
Outcome 4: Children are confident and involved learners, Key component 2 - Children develop a range of skills and processes such as problem solving, enquiry, experimentation, hypothesising, researching, and investigating.			
Create and use representation to organise, record and communicate mathematical ideas and concepts. (p. 35)	E	P	P
Make predictions and generalisations about their daily activities, aspects of the natural world and environments, using patterns they generate or identify and communicate these using mathematical language and symbols. (p. 35)	E	E	E
Contribute constructively to mathematical discussions and arguments. (p. 35)	E	E	P
Outcome 5: Children are effective communicators, Key component 1- Children interact verbally and non-verbally with others for a range of purposes:			
Use language to communicate thinking about quantities to describe attributes of objects and collections, and to explain mathematical ideas. (p. 40)	E	E	P

Table 1: Analysis of curriculum goals and points

Discussion and conclusion

Both curricula refer to 'develop' or 'development' in the *Section* and *Key Components*. The word *develop* is also included in each of the three goals, which may focus more on becoming than being (Lembrér & Meaney, 2014). The EYLF (DEEWR, 2009) focuses on *create*, *make*, and *contribute*, phrases that indicate 'being' more than 'becoming' (Lembrér & Meaney, 2014). The use of these terms reflects concerns that early childhood education is moving towards a schoolification of young children (Lembrér & Meaney, 2014). However, the results do not necessarily indicate this as epistemological empowerment occurs when the child has ownership of their skills and is empowered in their knowledge (Ernest, 2002). The goals of the SCP (Skolverket, 2011) were explicitly linked to the three domains of empowerment outlined by Ernest (2002). When considering the points from the EYLF *Outcome 4* and *Outcome 5*, only one of the points was considered to explicitly link to all of Ernest's (2002) domains of empowerment, compared to all the three goals for the SCP (Skolverket, 2011). Mathematical empowerment (Ernest, 2002) was evident in all goals identified from the SCP (Skolverket, 2011) and all points from the identified key components from *Outcome 4* and *Outcome 5* of the EYLF (DEEWR, 2009), reflecting the role of language children's mathematical experiences (Hachey, 2013).

Of the three points from the EYLF *Outcome 4*and the point from the EYLF *Outcome 5*, three were coded as potentially incorporating Ernest's (2002) epistemological empowerment domain. This represents a possible disconnect of mathematics from the context of the child's everyday life. When compared to the goals of the SCP (Skolverket, 2011), the points from the EYLF (DEWWR, 2009) could produce a narrower focus of the educators' perceptions of the children's capabilities in terms of mathematical understandings and their application (Anders & Rossbach, 2015). This is evident in the point under *Outcome 5*, as the outcome focuses on communication, which requires mathematical language (mathematical empowerment) within social situations (social empowerment), but not necessarily creation of ideas (epistemological empowerment).

The curricula provide an orientation for the educator but the educator chooses how to enact it in learning experiences (Gasteiger, 2014; Geiger, Goos & Dole, 2014). The educators' past experiences with mathematics, such as a lack of engagement (Chinn, 2012) or a dislike of teaching mathematics (Bates et al., 2013), will contribute to this. The educator's philosophical views - instrumentalist, Platonist, and problem-solving (Ernest, 1989) - or static and dynamic perceptions of mathematics (Grigutsch et al., 1998), may also impact. Specifically, holding an instrumentalist philosophy (Ernest, 1989) or a static view (Grigutsch et al., 1998) may result in a focus on skills and practice within a formal environment leading to mathematical empowerment (Ernest, 2002). In addition, the educator may only look for or identify mathematics in these more formal situations (Lee, 2014) and create fewer opportunities for children to engage mathematically (Hachey, 2013).

The inclusion of the targeted terms in early childhood curricula reiterates the idea that young children are capable of engaging with mathematical ideas (Hachey, 2013) and encourages educators to provide opportunities for children to show their mathematical understandings and participate in discussions (Mixon, 2015), and to have confident mathematical dispositions (Baroody et al., 2006). Stating the mathematical requirements assists the educator in determining how mathematical understandings and skills can be addressed with children in early childhood in ways commensurate with epistemological empowerment (Ernest, 2002). If this occurs, the child is positioned as maths-able (Hachey, 2013) and concurrently ‘being’ and ‘becoming’ (Lembrér & Meaney, 2014).

Limitations

Although official translations are acceptable to use (Lembrér & Meaney, 2014), use of the original text for the SCP may have added to the authenticity of the method. In addition, although the search was for the targeted terms (all of which were iterations of the term ‘mathematics’), it was noted that the term ‘numeracy’ occurred frequently in the EYLF (DEEWR, 2009) in the text providing the overall description of *Outcome 5*. Finally, it is inherent in an interpretivist approach that the perceptions of individuals are constructed versions of reality (Merriam, 2009). Although much discussion was generated in the process involved in allocating codes, this was dependent on the experiences the two educators brought to the discussion. This was a clinical interpretation of the curricula that did not consider human and environmental factors or their impact on the interpretation of the curriculum in live settings. As a result, other educators may have alternative interpretations. This final limitation highlights the impact of the educator, as it is their own interpretation of curricula, developed from their experiences, that they use when creating experiences.

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Paper ~~or~~ and digital: a study of combinatorics in preschool class

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In a design research study on problem solving conducted in Swedish preschool class (six-year-olds) children were given the task "in how many ways can three toy bears sit in a sofa?". The focus of this paper is on how the children's' explorations and solutions of this task developed as they, in addition to the analogue version, were exposed to a digital version of it. We compare the documentation made by children who have used, respective not have used the digital application. The results indicate that working with the digital application led to more systematic documentation with fewer duplications. Further, the children who worked with the digital application created more complete solutions. The findings indicate that the digital version of the task enhanced children's understanding of what a combinatorial problem encompasses.

Introduction

Appropriately designed and implemented activities enable young children to develop mathematical competencies that were earlier considered only attainable by older children (English & Mulligan, 2013). The results in this paper derive from an educational design research study of the implementation of problem solving in mathematics. The focus in the paper is, however, not on the full study but on the representations and systematisations young children spontaneously use when they are solving a (for them) challenging combinatorial task and how both of these are influenced by the use of a digital version of the task. The task given to the children concerned how many different ways three toy bears could be arranged in a row on a sofa. To make the task meaningful for the children, it was presented as a conflict between the toy bears, where the bears cannot agree on who should sit at which place on the sofa. One toy bear then suggests changing places every day. The task for the children was to find out how many days in a row the bears could sit in different ways on the sofa.

In a first design cycle, we noticed that children who used an iconic representation when working on the task produced more duplicate combinations than those using pictographic representations (Palmér & van Bommel, 2016). This was quite surprising as iconic representations are considered to be connected to a higher level of abstract thinking than pictographic representations (Hughes, 1986; Heddens, 1986). We also noticed that children's documentation lacked systematisation. Based on these findings, in the second design cycle we developed and introduced a digital version of the task that the children were to explore before they worked on the paper and pencil task similar to the first design cycle. The main aim of the digital application was to make the children notice duplications.

The focus of this paper is *if* and *how* the use of the digital application influenced the systematization and representation the children spontaneously used when working on the combinatorial task. The paper is organised as follows: It starts with a presentation of the study's theoretical foundation, followed by the study itself with the two design cycles and their results. Finally, several implications for further research are given.

Theoretical foundation

To be able to work successfully with combinatorial tasks, you need to have understanding about four important principles: systematic variation, constancy, exhaustion and completion (English, 1996). The principle of systematic variation means that a different combination will occur if at least one item is varied systematically. The principle of constancy means that a different combination will occur if at least one item is kept constant while at least one other is varied systematically. The third principle, the principle of exhaustion, means that a constant item is exhausted when it no longer generates new combinations when the other items are varied. Finally, the principle of completion means that when all constant items have been exhausted all possible combinations have been found. English (1991, 2003) has showed that young children can develop understanding of the four aforementioned principles and that a proper and meaningful context makes it possible for young children to work effectively on finding permutations in combinatorial situations.

Listing items systematically has been shown to be difficult for young children when solving combinatorial tasks (English, 2005). A variety of graphic representations can be used when solving combinatorics task (for example lists, diagrams, sketches and tables), all of which can be made systematic or not. English (1996) identified three stages of systematization when young children solve combinatorial tasks; the random stage, the transitional stage and the odometer stage. At the random stage, children use trial-and-error which is why constant checking becomes important to succeed with a task. At the transition stage children start to adopt a pattern in their documentations but the pattern is not kept throughout the task, instead the children often revert to the trial-and error approach. At the odometer stage, the children use an organized pattern for the selection of combinations where one item is held constant while the others are varied systematically.

When the children in this study were to work on the combinatorial task, they were offered to work with paper and pencils in different colours and when documenting possible permutations, they were free to choose their own representations. Historically, most studies on children's representations have been connected to quantity, with few studies on young children's use of representations when solving tasks within other mathematical areas. In relation to quantity, Hughes (1986) distinguished between idiosyncratic, pictographic, iconic and symbolic representations. Idiosyncratic representations are irregular and not related to the number of objects represented. Pictographic representations are pictures of the represented item. Iconic representations are based on one mark for each item. Symbolic representations are the standard forms like numerals or equal signs. Also, in relation to quantity, Heddens (1986) focused on the connection between the concrete and abstract when analysing children's representations. He defined two levels, semi-concrete and semi-abstract, to describe representations used in between the concrete (objects) and the abstract (symbolic). At the semi-concrete level, pictures of real items, as a representation of the real situation, were considered. The semi-abstract level concerned a symbolic representation of the concrete items, with a constraint that the symbols would not look like the objects they represented. Thus, what Hughes (1986) named pictographic representations are semi-concrete in the wordings of Heddens (1986), whereas iconic representations are semi-abstract.

When analysing children's documentation produced when solving the *combinatorial tasks in this study,* we used English's (1996) notions *trial and error*, *transition* and *odometer* combined with Hughes' (1986) notions *pictographic* and *iconic* representations.

The study

As mentioned previously, the results in this paper derive from an educational design research study of the implementation of problem solving in mathematics in Swedish preschool class (six-years-olds). In Sweden, the compulsory school starts at age 7. Prior to that, children can attend a year in the optional preschool class (will become obligatory in August 2018). Preschool class serves to make the transition from preschool to school smooth since the traditions of play in preschool and the focus on learning in school otherwise can become problematic (Pramling & Pramling Samuelsson, 2008). Before 2016 there were no specific goals for preschool class in the curriculum, which is why the mathematics content and the design of the teaching differed a lot between preschool classes (National Agency for Education, 2014, 2016).

The study has been ongoing for five years and is conducted through several design cycles with the stages of defining, testing and adjusting interventions (McKenney & Reeves, 2012). In this paper we focus on one of the tasks – the combinatorial task described above – starting with the results from the initial design cycle in which we found that children who used iconic representation when working on the task made more duplicate combinations than children using pictographic representations (Palmér & van Bommel, 2016). We will compare these results from the initial design cycle with results from a later design cycle when a digital application, specifically designed for this study, was added to the intervention (van Bommel & Palmér, 2017). The initial design cycle involved 123 children from ten preschool classes, the later design cycle involved 61 children from eight preschool classes. The children's guardians were given written information about the study and approved their children's participation in line with the ethical guidelines provided by the Swedish Research Council (2011).

The initial design cycle

When introducing the task, the children were verbally told the task and shown three small plastic bears, one red, one yellow and one green. After this introduction the children worked individually. They were given white paper and pencils in different colours but no instructions regarding what or how to do any documentation on the paper. It is these documentations that we have analysed using English's (1996) notions of *trial and error*, *transition* and *odometer* together with Hughes' (1986) notions *pictographic* and *iconic* representations. When analysing systematization, we looked at the order of the drawn permutations, for example, to see if one item had been kept invariant, if one item had been varied or if the permutations seem to occur randomly (see example figure 1).

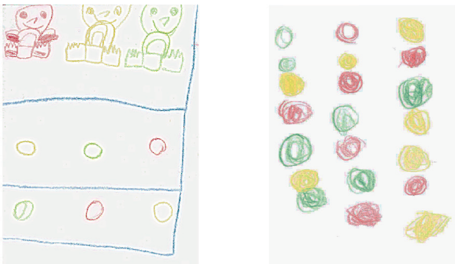

Figure 1: Two examples of children's documentation
Left: Pictographic & Iconic; Some permutations – no duplications (3 unique permutations)
Right: Iconic; Some permutations – with duplications

Of course, the analysis on systematization is made from an observer perspective and it is possible that children had systematizations not visible to us. Table 1 below shows the categorization of the 123 documentations. Nine documentations were not possible to categorize regarding systematization as they included only one permutation or a picture of more than three bears, thus the table shows the categorization of the remaining 114 documentations.

	Pictographic	Pic/Icon	Iconic	Total	Total
Trial and error – with duplications	7	3	21	31	68
Trial and errors – no duplications	19	2	16	37	
Transition – with duplications	2		9	11	20
Transition – no duplications			9	9	
Odometer - not all solutions	7	3	14	23	26
Odometer – all solutions			2	2	
Total	35	8	71	114	114

Table 1 Categorization of children's documentation in the initial design cycle

A total of 35 children used pictographic representations, 71 children used iconic representations and 8 children used both pictographic and iconic representations. Thus, the majority of the children spontaneously used an iconic representation. Four of the 114 children found six unique permutations when they worked individually with the task. These four children used iconic representations; two with a trial and error approach and two with an odometer approach. Using a trial and error approach implies that these two children had to check each of the new permutations with all the previous permutations to figure out if each drawn permutation was new or not. As shown in Table 1 the children made quite a lot of duplications. Of the documentations using a trial and error approach or a transition approach, 30 of the 55 iconic documentations, three of the five combined documentations and nine of the 28 pictographic documentations included duplications. In contrast, 19 of the 28 documentations using a trial and error approach or a transition approach together with pictographic representation consisted only unique combinations. Thus, there was less duplication in documentations with pictographic representations. While at a first glance, it looked as if iconic representations did not generate a higher level of solution of the combinatorial task; quite the opposite occurred, as pictographic representations resulted in less duplication. As long as a trial-and-error approach was used, pictographic representations seem to work best. However, a transition approach was visible more often in iconic (18) than in pictographic (2) documentations and there were more iconic (16) than combined (3) or pictographic (7) representations on the odometer level. Hence, the majority of children who showed systematization in their documentations used iconic representations. The development of representations and systematizations seemed to be somehow synchronized however, an early use of iconic representations did not seem to support the development of systematizations. This result led to the development of a digital application to be added to the intervention in a new design cycle.

The digital application

To further investigate possible connections between representations and systematization, we developed a digital version of the task. This digital application offers a semi-concrete pictographic representation (Hughes, 1986; Heddens, 1986) together with a systematic way of documenting each permutation (van Bommel&Palmér, 2017). The issue of duplications is included in the application to the extent that if a previous documented permutation is selected again, the application indicates this with a red frame (see third image figure 2). The images in figure 2 show the semi-concrete representation within the digital application (an image of bears on a sofa), as well as the documentation of the permutations in the frames on the right hand side. In the first image, the child has only placed one bear on the sofa, in the second image, the child has completed one permutation which is visible in the little frame on the right hand side of the image. In the third image, the child has accomplished three permutations and the fourth attempt resulted in a previously obtained permutation which is made visible in the application through the red frame to the right.

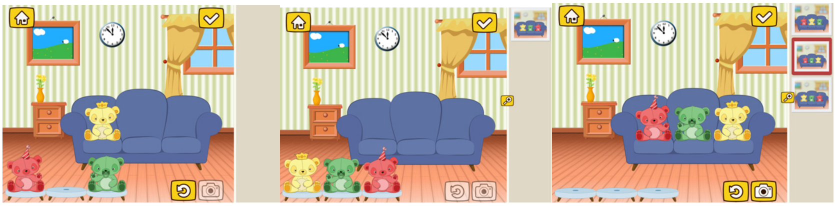

Figure 2: Sequence of images of the digital application

Results - the later design cycle

In the next design cycle, we let the children work with the digital application before introducing the paper and pencil version of the task. By doing this, we could investigate if and how the use of the digital application influenced the systematization and representation the young children spontaneously used when they work on the paper and pencil version of the task. In total, 61 children from eight preschool classes were involved in this design cycle. Table 2 below shows the categorization of these children’s paper and pencil documentation of the task (after using the digital application). The table is organized based on the children making duplications or not.

	Pictographic	*Pictographic & Iconic*	*Iconic*	*Total*	
Trial and error – with duplications	1	1	3	5	*12*
Transition – with duplications		2	5	7	
Trial and error – some permutations – no duplications	2	1	2	5	
Trial and error – all permutations – no duplications		1	1	2	
Transition – no duplications	5		8	13	*49*
Odometer – some permutations – no duplications	9	1	6	16	
Odometer – all permutations – no duplications	3	1	9	13	
Total	*20*	*7*	*34*	*61*	

Table 2: Categorization of children's paper and pencil documentation

After using the digital application, one could imagine that the children would not find the paper and pencil part of the task interesting or challenging. However, still few children "solved" the task (15 of 61) and even those who found all permutations had to work quite a time to find the permutations. Unlike the first design cycle, few children made duplications, only 12 of 61. Still, there were fewer duplications among pictographic representations where only 1 of the 20 documentations included duplications. The 16 documentations categorized as *odometer – some permutations – no duplications* consisted of exactly three combinations, each bear sitting one time at each place. Notable is that 2 of the documentations categorized as *trial and error – some permutations – no duplications* as well as 6 documentations categorized as *transition – no duplications* included five unique permutations.

Discussion – comparing the two design cycles

At this stage, it is interesting to compare the results from the initial design cycle with the results from the later design cycle. The digital application was designed to offer a semi-concrete pictographic representation together with a systematic way of documenting each permutation, which seems to have created a different understanding of the combinatorial problem. In the initial design cycle,4 of 114 children found the six unique combinations, in the later design cycle 15 of 61 children found the six unique combinations (two on a trial and error level and 13 on an odometer level). In the initial design cycle 42 of 114 children made duplications (37%) while in the later design cycle only 13 of 61 children made duplications (19%). In the later design cycle, 49 of the documentations showed a transition or odometer level indicating the application promoting systematization in the children's paper and pencil work. 2 of the documentations from the later design cycle categorized as *trial and error – some permutations – no duplications* as well as 6 documentations categorized as *transition – no duplications* included five unique permutations. Documentations with that many permutations without any duplications was unusual in the initial design cycle. In the initial design cycle 23 of the 114 documentations consisted of exactly three combinations, each bear sitting one time at each place. in the later design cycle, such documentations with exactly three combinations were found in 16 of the 61 documentations. According to English (1996), this solution is common for young children working on combinatorial tasks since the repeated selection in systematic combinatory goes against the wording “different combinations”. Especially young children often interpret “different” as different in all aspects. They do not think that keeping one item constant and change the others ends up as a “different combination”. Instead, when each bear has been sitting one time at each place they think of the problem as solved.

Implications for further research

The digital application was developed to offer a semi-concrete pictographic representation together with a systematic way of documenting each permutation. Thus, the children who began with using the digital application started to work at the semi-concrete level and had possibility to explore systematization. Based on our analysis, we cannot claim that the digital application influenced children’s paper and pencil documentation, but at the same time, nothing in the results speaks against the use of the digital application influencing the systematization and representation the young children spontaneously used when they worked on a combinatorial task. One thing that was interesting with classes of children who had worked with the digital application was that all but one of the children from two of the classes used iconic representation in their paper and pencil documentations, and in contrast, almost all of the children from a third class used pictographic representation. This diversity is something that we intend to explore further by interviewing children about their choice of representation, in close connection to working on the task. Finally, we want to emphasize that we do not understand these preliminary results as a choice between paper and pencil *or* digital application but as the results indicate; paper, pencil *and* digital application. Based on this, we consider it to be justifiable to proceed with a larger study, both to elaborate on how the analogue and digital version of the task can be combined in teaching to contribute to children’s understanding and to further explore the rationale for children’s choice of representation

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"I find that pleasurable and play-oriented mathematical activities create wondering and curiosity" Norwegian Kindergarten Teachers' Views on Mathematics

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This paper investigates the results of a questionnaire given to kindergarten teachers in Norway. The focus is on the mathematical topics the kindergarten teachers found important to work with and their arguments for doing so. The Norwegian kindergarten tradition is play-oriented, with mathematics learning during daily activities as a central part of this tradition. We analyze the quantitative and qualitative data according to how the kindergarten teachers positioned themselves with respect to play-oriented and school-oriented mathematics. The findings demonstrate how different kindergarten teachers view and rationalize potential learning opportunities in mathematics.

Introduction

Today in Norway, nearly all children attend kindergarten between the ages of one and six years of age. The guidelines in the Framework Plan for the Content and Tasks of Kindergartens (The Ministry of Education and Research, 2011) regulate the rules, content and tasks that should be undertaken in Norwegian kindergartens. However, the guidelines are not explicit about what teachers or kindergartens should do of activities, resources, scheduling and so on. Therefore, interpretation and implementation might differ from kindergarten to kindergarten. According to Olsen (2011), the reason for this diversity might be tensions between what official documents, including the Framework Plan, prescribe, and kindergarten teachers' own perceptions, meanings and practices.

In Sweden, Lembrér and Meaney (2014) used the concepts of being and becoming to examine how children were positioned in the newly-revised Swedish curriculum in regard to their mathematics learning in preschool. From their perspective, the concept "being" might be discussed in terms of democracy: children's right to express their views and children's right to influence their daily life in kindergarten. This positions the child as an active learner with his or her own initiative, imagination and sense of wonderment. They consider the concept of "becoming" as describing the situation in which the child seems to be incomplete and lack knowledge. The kindergarten teacher's role is then to fill the child with knowledge for the future. Lembrér and Meaney's analysis suggested that although the curriculum situates the children as both "being" and "becoming", the aims for mathematics are likely to suggest to kindergarten teachers that their focus should be on children's becoming. They considered this to be in alignment with the strong schoolification forces operating on kindergarten (Lembrér& Meaney, 2014). As this is in contrast to the Nordic tradition of kindergarten being play-oriented, this may lead to teachers experiencing conflict about their planning.

Benz (2012) conducted a questionnaire survey among kindergarten teachers and assistants in Germany. She analyzed educators' statements about mathematical domains or topics and views on teaching mathematics in kindergarten. The educators´ agreed mostly to statements related to scheme and formalism competences instead of process and problem solving activities. Findings from the study indicated that how the kindergarten educators view mathematics seems to influence their beliefs concerning children's learning of mathematics.

Østrem et al. (2009) completed a national evaluation of the implementation of the Norwegian Framework Plan. In the report, kindergarten leaders answered a questionnaire survey on the implementation, use and their experience with the Framework Plan. The findings indicated that the kindergarten leaders emphasized activities concerning counting and shapes rather than mathematical activities related to for example spatial thinking.

The aforementioned studies suggest that the implementation of mathematical learning goals may be difficult for kindergarten teachers if they are perceived to be in conflict with their own beliefs about the position of mathematics in kindergarten. The following study investigates this issue within the Norwegian context, exploring Norwegian kindergarten teachers' thoughts on mathematics in terms of their work with children and in relation to the curriculum (The Ministry of Education and Research, 2011). At the time the data were collected, the Framework Plan had been in place for nine years since the implementation in 2006 and had a revision in 2011. If we find differences between what the guidelines provides, and the kindergarten teachers reports of what they do, then there may be some inherent problems for the kindergarten system. Awareness of and knowledge about the kindergarten teachers' choices and reasons for working with mathematics is important as it can help strengthen the kindergarten teaching profession. According to Biesta (2011), it is essential "to understand what forms and ways of learning are made possible through a particular learning culture and what forms of learning are made difficult or even impossible" (p. 202). Consequently, our research question focuses on this: What do Norwegian kindergarten teachers consider to be important in the implementation of potential learning opportunities about mathematics?

Theory

To better understand mismatches that might occur between the curriculum and teachers' views about mathematics in kindergarten, we have drawn on theories about socialization (Biesta, 2007; Giddens, 1979). Socialization might be considered a part of kindergarten teachers' preparation for children's mathematical learning. Investigating kindergarten teachers' socialization and their views on children's learning of mathematics can provide a nuanced interpretation in terms of what influences these kindergarten teachers. Socialization has been considered in a variety of different ways. Biesta (2010) distinguishes between three functions of education: qualification, socialization and subjectification. A major function of educational institutions, such as kindergartens, lies in the qualification of children through the development of knowledge, skills and understandings. In contrast, Biesta (2007) considered socialization to be the "insertion of 'newcomers' into existing cultural and socio-political settings" (p. 26). Thus, much of what occurs in institutional settings, such as kindergartens, can be considered socialization, as it is an institution in which young children come into contact with valued understandings of how to participate in the society. From this perspective, socialization is about making children become like 'existing members', usually in the sense of becoming appropriate adults for the society in which they are situated. Biesta (2007) points out that one of the dangers of socialization is that it also reproduces, consciously or unconsciously, less desirable aspects of the culture. In our case, for example, traditions about valued knowledge might be preserved even though new policy documents indicate a change in the mathematical knowledge that is valued. Kindergarten teachers are cultural agents, who, in working with young children, socialize them in regard to the knowledge seen as valuable, including understandings about mathematics.

However, teachers are not the only contributors to the socialization process. Giddens (1979) stated that children need to be considered as active agents who have relevant knowledge and skills for structuring their own participation. This is in alignment with a "being" perspective of young children (Lembrér & Meaney, 2014). Children's play, therefore, has an important role in the continuation of the culture and of the kindergarten tradition as it enables children to control the knowledge that is raised, and which is examined within an interaction (Biesta, 2010). The guidelines in the Norwegian Framework Plan (2011) emphasize the importance of working with mathematics in children's daily life experiences. As socialization is an active process, participants in the culture have possibilities to not just reproduce valued cultural knowledge but to also influence what becomes valuable. For Biesta (2010), the possibilities of producing valuable cultural knowledge is no longer consist with socialization but with subjectification. "The subjectification function might be understood as the opposite of the socialization function. It is not about the insertion of 'newcomers' into existing orders, but about ways of being that hint at independence from such orders" (Biesta, 2010, p. 21). Subjectification is necessary if education is to lead to democracy, because in subjectification children's participation is given weight. The Norwegian Framework Plan (2011) encourage these subjectification processes. Children's views shall be heard and influence the daily activities.

Method

This project investigates the views of Norwegian kindergarten teachers and how these views might be affected by different societal influences, such as the Nordic tradition for kindergarten education, kindergarten curriculum, social and cultural settings. By studying the kindergarten teachers' argumentation for their views about the kind of mathematics that should be introduced in kindergartens, we anticipate determining how they position children's learning. For instance, do they use arguments from the Framework Plan or do they use other arguments to justify their practices regarding mathematics?

In order to answer the research question, 160 kindergarten teachers completed a survey about their views on the mathematics that should be introduced to children in kindergartens. The survey was conducted in 2014–2015 and given to 16 males and 144 females from the western part of Norway. As the number of males is low, we have combined the results of males and females and chosen not to analyze the data with respect to gender. The survey contained questions that provided both quantitative and qualitative data.

This paper discusses data from two of the nine questions in the questionnaire. The first survey question, "Which topics do you find important to work with related to the learning area 'Number, space and shape'?", was a multiple-answer question where the recipients had to indicate one or more relevant answers from the following set: patterns; locating; measuring; abstract thinking; sets; shapes; concepts; classification; and counting. The potential answers reflect different topics from the learning area "Number, space and shape". In addition, a follow-up open-ended question asked the teachers to indicate reasons for their choice. We analyze the written responses concerning how the teachers position themselves with respect to play-oriented and school-oriented mathematics. From the written justifications, we discussed the answers and identified four categories; 1) no written argument was provided, 2) arguing based on children's interests, 3) arguing based on school preparation or 4) a mix of arguments mention in categories 2) and 3). Three written justifications representative of categories (2), (3) and (4) are discussed later in this paper.

Results and discussion

All respondents answered the question "Which topics do you find important to work with related to the learning area 'Number, space and shape'?" Our findings show that counting, classification, concepts and shapes were the topics identified by most kindergarten teachers as important (see Table 1). 94 % of the kindergarten teachers found counting to be important, whereas 88 % indicated that shapes were important. In contrast, only 60 % of recipients found it to be important to work with patterns, 63 % identified localization and 65 % considered measuring important for working with mathematics (see Table 1). In the middle of the table we find sets and abstract thinking with respectively 87% and 77%. These are relatively high scored, and the majority of the kindergarten teachers say they facilitate activities that support these topics.

Counting	94 %
Classification	92 %
Concepts	91 %
Shapes	88 %
Sets	87 %
Abstract thinking	77 %
Measuring	65
Localization	63 %
Pattern	60 %

Table 1: "Which topics do you find important to work with related to the learning area: Number, space and shape?"

These results are comparable with studies by Østrem et al. (2009). In their report, kindergarten leaders also indicated that many counting and shape activities were provided in the kindergarten, and there was less focus on localization. Østrem et al. (2009) did not ask about classification and concepts, yet they are mention in the guidelines. These topics make a high score in our survey, and it may because they are close to daily activities like sorting toys and mathematical conversations, for example related to constructions activities (Fosse, 2016). Given that the Framework Plan (The Ministry of Education and Research, 2011) emphasizes space, it is possible that kindergarten teachers would identify localization as an important part of mathematics. Similarly, the Framework Plan emphasizes the use of everyday activities, yet activities such as measuring, which could be considered as being more related to everyday activities than counting or shapes, are considered important by fewer kindergarten teachers. This is in alignment with findings from Benz (2012) where the German kindergarten educators mention counting and sets as central content in mathematics in kindergarten and rarely mentioned activities related to measuring.

The results from this question made us want to explore the teachers' reasoning for their choices to see what might be influencing their views on what valuable mathematics for kindergarten children was, and potential learning opportunities about mathematics. Therefore, we had a follow-up question about their reasons for identifying working with specific mathematical topics. The question: "I think __ (one or more) topics are important to work with because…" had an 80% response rate. This is in contrast to the 100% response rate to the multiple-answer question regarding working with specific mathematical topics. The difference in the response rate might indicate that kindergarten teachers are more willing to identify what they are doing than their reasons for why they were doing it. Research on doing surveys indicate that people are more likely to complete multiple-answer questions than open-ended questions (Zhou, Wang, Zhang & Guo, 2017).

The first response is typical of an answer from kindergarten teachers' which highlights the importance of children's interests (Category 2). Maria's (pseudonym) response (translated by the authors): "It is important to work with numbers and shapes, because children's interests are often there." To stimulate the mathematical development of children related to the children's interests is in alignment with the Framework Plan (The Ministry of Education and Research, 2011) and it could be this part of the Framework Plan that teachers draw on with this justification. According to Lembrér and Meaney (2014), Maria's utterance is in alignment with a "being" perspective, since her arguing is based on the children's interest that may also involve play-oriented activities. Nevertheless, if this valuing of numbers and shapes as important mathematical knowledge is restricted to being because it is what interest children, it may be problematic in that it limits children's possibilities to learn to only the ideas they themselves raise.

Maria's responses to the multiple-answer question were in alignment with the results shown in Table 1, in that she did not mark localization and measurement as important areas of mathematics. This might influence her daily practice related to mathematics and the children's mathematical learning. As Biesta (2007) emphasized, one of the dangers of socialization is that you could reproduce the culture even if it is not what you intended. By following the children's interests, Maria may deprive the children of potential learning opportunities about mathematics that can occur in daily life situations, for example, related to measuring as described by Helenius, Johansson, Lange, Meaney, Riesbeck and Wernberg (2014). In this way, she may limit the children in reproducing valuable mathematical knowledge. Maria's response could be seen as both subjectification and qualification (Biesta, 2007): subjectification in that it reinforces children's interests as being important, and qualification in the way she encourages learning about number and shapes, which are mathematical knowledge both in daily life and for the future.

Other respondents gave reasons linked to the children's perceived mathematical needs for school readiness. An example from the category school preparation (Category 3), was offered by kindergarten teacher Helen (pseudonym): "Counting, sets and concept, measuring. It is important for children's school start that this is automatized." This statement indicates the importance of some mathematical topics due to them being needed by children when they start school. The teacher does not relate her work to expectations in the Framework Plan but to wider societal expectations. The focus on children's needs for school is interpreted as an example of Biesta's (2010) qualification because the kindergarten teachers argued with respect to an outcome related to school.

This way of arguing is related to the concept "becoming," described by Lembrér and Meaney (2014). Helen focuses on children becoming mathematicians, or at least school mathematicians, and in this statement, she is not referring to the skills and knowledge that the children already had. Such a focus might contribute to some teachers not recognizing and making use of children's current knowledge and skills. The many responses which connect specific mathematical knowledge with preparation for school may be due to politicians such as the Norwegian Minister of Education (Isaksen, 2014) suggesting that children should focus on mathematics in kindergarten in order to prepare for school. Kindergarten teachers' perceptions of what mathematics children are likely to meet when they begin school suggests that some areas are getting too much focus. This means that other areas of mathematics, for example location and patterns, may be ignored or only feature as a minor focus, even if they might provide better connections to children's existing knowledge and skills, a point highlighted as important by the Framework Plan (The Ministry of Education and Research, 2011).

The results also showed that there was another common type of response that indicated that the kindergarten teachers valued many different topics as being valuable mathematical knowledge. Ann's (pseudonym) comment exemplifies this type of mixed argument (Category 4), demonstrating children's interests, play-oriented activities and learning as part of being in a democracy.

> I think that all the mentioned topics are relevant to work with in the kindergarten. I find that pleasurable and play-oriented mathematical activities create wondering and curiosity. We discover things together; the pleasure of discovering is great. It conduces good communication between children and adults and provides an arena for mastery and desire to learn – motivation. I think purposeful, systematic, pleasurable and play-oriented mathematics activities might help to reduce social inequalities and give children a sense of safety and curiosity that will be useful for them later. The activity is meaningful in itself.

Ann indicated that she saw the child as an active agent with whom she worked together to discover and wonder about different experiences. In doing so, she seemed to draw on statements about mathematics from the Norwegian Framework Plan. This can be seen in how close her statements are to the description in the Framework Plan that: "in order to work towards these goals, staff must listen and pay attention to the mathematical ideas that children express through play, conversation and everyday activities" (The Ministry of Education and Research, 2011, p. 42). We interpret Ann's response as aligning with the "being" perspective (Lembrér & Meaney, 2014), as she is consistently arguing for the child's participation in everyday activities and situations.

In the second last sentence where Ann emphasizes how mathematics might be used to reduce social inequalities, she indicated that she was aware of the power in social and cultural settings of learning. We interpret Ann's response is an example of all of Biesta's (2010) three functions of education: qualification, socialization and subjectification: Qualification by mentioning that all the topics are important to work with and by arguing that "play-oriented mathematical activities … will get useful for them later". Her arguments might be seen as a qualification as they are about long-term need for mathematical competence. Socialization in that Ann argued for a learning environment where the children experience the social and culture setting. Subjectification in the way she argues for children as active agents "We discover things together and to reduce the social inequalities and give children a sense of safety – curiosity will get useful for them later". Qualification, socialization and subjectification are not seen as three separate functions of education, but they are overlapping (Biesta, 2010). In our research some teachers' views seem to be drawn from different influences, but they are able to blend them into a cohesive whole, rather than seeing them as being in conflict.

Conclusions

The findings demonstrate how different kindergarten teachers argue about potential learning opportunities in mathematics. Some kindergarten teachers did not provide a response, others argued based on children's interests, a third group based their arguments on school preparations and a fourth group had mixed arguments related to children's interests, play-oriented activities, school preparation and children's possibilities to participate actively in a democratic society. The data provided examples of kindergarten teachers' justifications about learning mathematics and these are related in different ways to Biesta's (2010) three functions for education: socialization, subjectification and qualification.

The diversity in the responses shows the tension between what official documents prescribe and kindergarten teachers' own perceptions, meanings and practices that are affected by a range of different influences, some of which are noted in the results. This has considerable influence in relation to the daily work with children and mathematics in the kindergarten. This is in alignment with

Biesta's (2010) dimension of socialization as the kindergarten teachers' views on the implementation of potential learning opportunities about mathematics are influenced by the culture. Biesta (2011) highlights how important it is to discuss what forms and ways of learning opportunities are made possible through a learning environment. The play-oriented guidelines in the Framework Plan (The Ministry of Education and Research, 2011) give many opportunities for different mathematical practices and supports the subjectification dimensions focusing on child-initiated activities, participation and democracy. There is less focus on the qualification functions, such as assessment and measurement. In our findings, some of the kindergarten teachers argue for mathematical activities based on children's interests related to the subjectification dimension.

It seems that the necessity for mathematics in kindergartens as qualifying is an argument for some kindergarten teachers in our research, even though this is not reflected in the curriculum. Several respondents state that they will work with mathematics because it is a way to prepare children for school. Others argue for qualification as mathematics will become useful for children later, seen as qualification for the future. Further studies might investigate kindergarten teachers' actual practice and how that is in accordance with their reasoning for doing mathematics.

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Collaborative tool-mediated talk – an example from third graders

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The importance of language and social interaction in learning mathematics has been widely emphasized the last decades. In this paper, we present a dialog between two eight-year-old girls, working on a multiplication task. The study is video-based and carried out within a sociocultural framework. The analysis shows that the girls' communication skills and their competence to use drawings and other written representations are intricately interlaced. On one hand, the mathematical progress is dependant of the girls' ability to accompany their written work with verbal explanations and gestures, on the other hand, the written representations act as means to elicit the girls' thinking. Our study thus adds to the field throwing light on how representations like drawings, are necessary mediational means in young learner's collaborative talk.

Introduction

The base for this study is part of a larger research and development project called *Language Use and Development in the Mathematics classroom* (LaUDiM). The main objective of the project is to develop deeper knowledge of the learning environment's significance for developing young learners' mathematical thinking and understanding, as well as to develop their ability to express mathematical concepts and ideas. Amongst the other aims, one is to understand more about how young pupils collaborate on solving mathematical tasks.

Theoretically (Vygotsky, 1987) and research-based (Mercer & Sams, 2006), the importance of language and social interaction for learning mathematics has been emphasized. This is also a claim in the Norwegian national curriculum for primary school (LK06). There are, however, some precautions from researchers arguing that just putting pupils together will not always work. The talk is then often uncooperative, off-task, inequitable and ultimately unproductive (Mercer & Sams, 2006). Sfard and Kieran (2001) concluded that "interaction with others, with the numerous demands on one's attention, can often be counterproductive. Indeed, it is very difficult to keep a well-focused conversation going when also trying to solve problems and be creative about them" (p. 70). They argue that strong motivation is necessary to engage in mathematical conversations and make it work, and a prerequisite for a mathematical discourse to be productive is the effectiveness of the communication among partners. Research claims that there is a need to find out more about what productive dialogs that support mathematical thinking and learning entail (van Oers, 2013).

In this paper we present, analyse and discuss a dialog between two Norwegian eight-year old girls, here named Kate and Beth, solving a multiplication task. The dialog ended with the exclamation "Yes, we did it" which we took as a preliminary evidence of a successful collaboration. Thus, the research question for this paper is: What features of talk and communication stimulates mathematical progress in the collaborative process of solving a task?

Theoretical framework

Two important features of sociocultural theory are relevant for our study (Vygotsky, 1987). First, the claim that higher mental functioning, like reasoning and problem solving in the individual derives from social life. Second, that higher mental functioning and human actions in general are mediated by tools and signs. Vygotsky's accounts of mediation provide the bridge that connects the external with the internal and thus the social with the individual. Vygotsky viewed language to be the most important tool, both for the development and sharing of knowledge among people and also for structuring the process and content of individual thought. From a sociocultural perspective, it is particularly interesting to study talk in educational settings and identify in what ways humans learn to handle and use cultural tools effectively to solve problems.

Exploratory talk is a typification of a way of using language effectively for joint, explicit, collaborative reasoning (Barnes & Todd, 1977, Littleton & Mercer, 2010). In exploratory talk knowledge is made publicly accountable and reasoning is visible. It represents a form of co-reasoning where speakers share knowledge, challenge ideas, evaluate evidence and consider options in a reasoned way. Explanations are compared, and joint decisions reached. "It is a speech situation in which everyone is free to express their views and in which the most reasonable views gain acceptance" (Littleton & Mercer, 2010, p. 279). According to Barnes and Todd (1977) exploratory talk depends on learners who share the same idea of what is relevant to the discussion and have a joint conception of what they are trying to achieve. Two other kinds of talk are presented by Littleton and Mercer (2010). In *cumulative talk*, speakers build positively but uncritically on what the others have said. It is characterized by shared information, joint decisions, repetitions, confirmations and elaborations, but there are no critical considerations of ideas. *Disputational talk* is characterized by disagreement and individualized decision making with few attempts to combine resources, offer constructive criticism or make suggestions.

Duval (2006) claims that all mathematical activity involves the use and change of semiotic representations. He introduces a classification of semiotic representation into four different registers; natural language, symbolic systems, iconic and non-iconic drawings, and diagram and graphs, based on the possibilities for performing mathematical processes. Natural language has a special position amongst the registers, as it can be used also for communication, awareness, imagination etc. Duval denotes transformations between representations within the same system as treatments, and transformations between different registers as conversions. He claims that conversions are more complex than treatments, "because any change of register first requires recognition of the same represented object between two representations whose content have very often nothing in common" (p. 112). Hence, the ability to change from one representation register to another is often a critical threshold for progress in problem solving.

The dialog presented in this paper is taken from a teaching sequence where the mathematical aim was to give the pupils experiences with different multiplicative situations. A multiplicative situation is characterized as one where "it is necessary to at least coordinate two composite units in such a way that one of the composite units is distributed over the elements of the other composite unit" (Steffe, 1994, p. 19). Depending of the situation, four different multiplicative structures can be distinguished; equal groups, multiplicative comparison, rectangular area, and Cartesian product (Greer, 1992). The task involved in this paper concerns the first structure. In an equal group situation, the multiplier counts the number of groups, while the multiplicand tells the number of objects in each group.

Methodology

LaUDiM is an intervention project where two teachers from different schools and researchers from the field of mathematics education and pedagogy plan and set goals for the teaching of mathematics, which subsequently is carried out by the teachers. In the classroom, whole class discussions and dialogs between selected groups of pupils are video recorded. Parts of these video recordings, together with pupils' written work, are discussed by researchers and teachers. This represents the first step in analysing data as interesting sequences are identified. The presented dialog is chosen from video-recordings of six collaborating pairs working on the same task. By carefully viewing all the recordings we chose this dialog due to the task-focused content, and to the engagement and passion we could see between the two girls. Moreover, the session ended as already told with the exclamation "Yes, we did it" which we took as a preliminary evidence of a successful collaboration.

The video-recorded and transcribed session is 7 minutes long, the two girls are working on the task:

> The 3rd grade will have a party at school. The day before the party, they are baking muffins for the party. Anne is going to the store to buy eggs for the muffins. In the recipe, it says that they need four eggs in one portion. The children have decided

that they are going to bake twelve portions of muffins. How many eggs does Anne need to buy?

The girls' discussion is a collaborative effort to solve the mathematical problem. According to Blum and Niss (1991), a mathematical problem is a situation that challenges somebody intellectually who is not in immediate possession of direct procedures sufficient to answer the question.

To address the research question, we started the analysis by looking for keywords described by Littleton and Mercer (2010) as characteristics of the three different types of talk. Further we asked questions to the material, e.g. how do the girls respond to each other, how do they give reason, and how do they share ideas. Due to the video-based design of the study, we were able to identify not only their oral talk, but also use of gestures and other mediational tools. The second step was to identify shifts of focus in the dialog. This helped us to divide the dialog into sequences, which were analysed further with respect to the mathematical content. In this process, uses and shifts of representations became visible. This turned our attention to Duval's (2006) work on this issue. In the third step, we analysed and interpreted each sequence more thoroughly by combining these two analytical perspectives. We have decided to present and analyse the dialog as it unfolds, just leaving out a few utterances we found unnecessary.

Analysis of the dialog

The dialog starts by Kate reading the word-problem aloud, Beth interrupts her.

1	B:	I'll draw four eggs?
2	K:	Wait, wait (continues to read the task aloud). (…)
7	B:	I'll just draw some circles (starts to draw a row of small circles).
8	K:	Draw four circles. There you are. Good. And then we should..., and then we have twelve…, just write twelve, no, forget it.

While Kate is still reading the word problem, Beth suggests a conversion from the problem stated in natural language to an iconic representation (1, 7). Kate supports this transformation, by monitoring and evaluating Beth's action (8). She wants to build on Beth's drawing, but she does not know how to represent the twelve. It is not likely that the girls recognize the problem as multiplication at this point. Kate then goes back to the written task, and after some thinking time, the conversation continues.

13	B:	This is an addition problem.
14	K:	No, (whispers) it is 12 times 4.
15	B:	Oh, yes.
16	K:	No, it's 4 times 12

17	B:	(Laughs) Yes, that is the same.
18	K:	It is 4 times 12, …no, it is not the same. For if we take 12 times 4, then we take 12 four times[1].
19	B:	Yes.
20	K:	And that does not work here.

At this point, it seems as if the girls have given up pursuing the iconic representation, instead they try to find a number sentence that fits the word problem. Eagerness to explain the difference between 12·4 and 4·12 (18), is taken as an account for that it is important to Kate to make her knowledge publicly accountable, so that Beth can follow her reasoning. Beth is not given the chance to explain her thinking, and she accepts Kate's way of interpreting the problem without further questions. This sequence has features of cumulative talk. Recognizing the situation as multiplicative gives Kate some new input on how the problem situation can be modelled, and so the problem-solving moves on.

22	K:	(Points at the four eggs) So that means four…, we should get to… we are going to have twelve. (Takes the paper from Beth.) If I draw twelve. (…)
25	B:	Just do it there (points right beneath the four eggs).
26	K:	I'll draw twelve muffins[2] (starts to draw bigger circles, stops to count).
27	B:	That's funny looking muffins.
28	K:	I know, but we can see, we can see what it is anyway (completes the drawing of twelve muffins; two rows with six circles in each row).
29	B:	Now you have twelve.
30	K:	Here we have twelve muffins, and then there should be four in each muffin (points at the eggs Beth has drawn at the top of the paper).
31	B:	(Points at the four eggs) Then we put these down here, these four in one, then we have to… (points from the four eggs to the twelve muffins).

Kate identifies that the muffins are the essential units to start with in an iconic representation, and she makes the crucial connection between the muffins and the eggs by pointing at Beth's drawing of four circles (30). This shows that she has grasped the multiplicative structure of the problem, one unit distributed over the other, and is thus a mathematical breakthrough. The gesture also serves as an acknowledgement of Beth's contribution. Beth is not challenging Kate's

[1]Kate is aware of the difference between 4·12 and 12·4, but her interpretation does not follow the usual convention.

[2]There is some confusion between muffins and portions, but that is not important for the solution.

reasoning, but actively monitoring Kate as she draws the muffins (29).By suggesting to “put down” the eggs (31) she lets her know that she both understands the structure of the problem and approves of her representation of it, and the girls are ready to proceed.

34	K:	Because in this, if we add them together we get eight. (Points to the first muffin in each row, writes the number 8). Because in each there is eight.
35	B:	Here, just read from here again. Slowly. (…)
39	B:	Stop. We need four eggs in a portion, right?
40	K:	Yes, because one portion, that is one muffin for us then (points at herself). So that means that in this one there is four (points at the first of the muffins).
41	B:	(Points at the four eggs) all of these circles here, just draw a line down to… (Points at the first of the muffins).
42	K:	In one there are four, and in that one there are four, so if we add them, we get eight.
43	B:	I’ll take four of them in here (draws four small circles inside the first muffin).
44	K:	No, just... I will… (takes the pencil from Beth). Eight plus four, we do it like this, four, four, four (writes the number 4 above each muffin).
45	B:	Can I do the last ones?
46	K:	Yes, you can do these four.
47	B:	Oh no (draws a negligent looking 4).
48	K:	That’s fine, that’s fine, we can see it anyway.

Having seen through the multiplicative structure of the task, Kate seems ready to use the representation of the twelve muffins to start calculating. She attempts to justify her reasoning by words and gestures (34, 40, 42). Beth interrupts her, suggesting that they make a more concrete representation of the eggs (41, 43). Kate agrees, and starts to write “4” over each muffin (44). This exchange contains several characteristics of exploratory talk. Reasoning is made visible, and the girls consider and compare different options of representations, before a joint decision is reached.

52	K:	No, look here, do you know what, wait, we have to do it again now, because…, if we take… (Points to and counts the six muffins in the first row) this is six, right (writes 4+4+4+ on a line below the drawing of the muffins). Now I have taken these three (puts a mark after the first three muffins, counts as she writes more +4’s) 1, 2, 3, 4, 5, 6, 7, 8.

53	B:	(Counts the muffins silently.) Just take twelve of those. Ok, I'll just read (takes the problem sheet, reads to herself, following the text with her finger).
54	K:	(Counts aloud, finishes to write +4) 9, 10, 11, 12. Ok, here I made a plus-problem with all these (points to the muffins). Then we have twelve fours, just that…, here we have the answer (writes =___ below the row of +4's).

Both girls are able to use the drawing of the muffins, combined with the rows of 4's, to start a process of repeated addition, but they face some challenges keeping track of the preliminary calculations. Kate takes the lead of transforming into a more structured symbolic representation (52), making her thinking visible to ensure that Beth agrees. There are no critical considerations of ideas here, hence this sequence can be characterised as cumulative talk. However, Beth is not passive in this process, she monitors Kate's work, and checks once again that the representation they have come up with is in line with the written task (53). After some negotiation on the notation, the girls are ready to perform the needed calculations.

63	B:	It is 16 (points). (…)
66	K:	Ok, ok I believe you. Plus four, 16… (Draws more vertical lines and writes 16), and here we have four.
67	B:	16
68	K/B:	(Both counting on their fingers) 17-18-19-20 (Kate writes 20).
69	B:	24 (Kate writes 24), 28 (Kate writes 28)
70	K:	(Counting on her fingers) 29-30-31-32 (writes 32)
71	B:	(Counting on her fingers) 33-34-35-36 (Kate writes 36)
72	K/B:	(Counting on their fingers) 37-38-39-40 (Kate writes 40), 41-42-43-44 (Kate writes 44)
73	K:	Oh, that one, that one we could have done right away.
74	B:	48 … I think.
75	K:	Yes, it is 48.
76	B:	Yes, it is 48. (Kate writes 48 behind =). So, we have to buy 48. Yes, we did it!

The new representation works for calculating and the girls share the same strategy, taking turns counting in fours. They trust each other's calculations, so there is no need to question or challenge ideas in this exchange. When there are only a few more fours to add, they turn into a choral count, which indicates that they are enthusiastic as they approach an answer. Beth's "Yes, we did it" shows pride of having fulfilled their common project.

Discussion

To be able to address our research question, we first identify what comprises the mathematical progress in the dialog. We then present the features of talk and communication which stimulates this progress.

The solution process is not straight forward for the girls. Anghileri (1989) claims that multiplication differs significantly from addition in complexity because there are three pieces of information to coordinate; the number of sets; the number of elements in each set; and the procedure for executing the product. The mathematical progress in the dialog can be described in two steps. First, the mathematical breakthrough happens when the girls identify the multiplicative structure of the problem situation (30, 40-43). They recognize that the group of eggs constitute a composite unit that is to be distributed over the muffins. The task can then be solved by repeated addition of 4's. The girls' actual calculation constitutes the second step of the mathematical progress. This, of course, leads them to the final answer, but identification of the multiplicative structure is crucial in order to be able to start the calculation. The analysis shows that when the girls are stuck in the process of solving the task, they use two strategies to make progress; they either re-read the task, or they perform a shift of representation (Duval, 2006). By constantly going back to the written problem the girls check that they have a joint conception of what they are trying to achieve (Barnes & Todd, 1977), while the changes of representations serve as a tool that helps them uncover the structure of the task, to perform calculations, and to structure and communicate their thoughts. The girls' need of a model of the problem situation as a tool for thinking is in line with previous research on young children's pre-instructional multiplicative strategies (Kouba, 1989).

First and foremost, the mathematical progress in the dialog is stimulated by the fact that the girls have a common goal in solving the task (Sfard & Kieran, 2001). The repeated use of "we" instead of "I" indicates that they share the responsibility for the project. There is an atmosphere of trust and acknowledgement between them, visible for instance when Kate gives positive feedback on Beth's drawing (8), when they don't mind that their drawings are not perfect (28, 48), and when Kate trusts Beth's calculation (66). Though not sufficient, mutual acceptance is a necessary condition for co-reasoning and exploratory talk, as it creates a space where the girls dare to share ideas.

Two features of the girls' communication seem especially important for stimulating mathematical progress; the girls' ability to communicate their thinking by words and gestures, and their eagerness to actively involve themselves in each other's reasoning. First, making their thinking public makes it possible to follow each other's reasoning, to evaluate it, and build upon it. These are important features of exploratory talk. An example is when Beth draws four eggs, stating aloud what she is drawing. Kate then tries to build upon Beth's work, but is unsure of the role of the number 12 (1-8). Another example is the sequence where they are considering different options of how to represent the four eggs inside each of the twelve muffins (34-48). Putting thoughts into words also enables the one sharing her idea to think it through more thoroughly, leading to a deeper insight (Vygotsky, 1987). An example of this is when Kate explains the difference between 4·12 and 12·4 (18). Almost immediately it seems like she sees the connection between the pair of numbers and an iconic representation of the problem, making her able to model the situation in a way that illustrates the multiplicative structure.

Secondly, the girls constantly involve themselves in each other's reasoning, either by monitoring each other's actions, as when Beth confirms that Kate has drawn exactly 12 muffins (29), or by actively participating in the other's construction of a new representation (45). In exploratory talk, ideas are often challenged or questioned. This does not happen often – if at all – in the dialog between Kate and Beth, giving the communication a cumulative flavour. This does not mean that they passively accept each other's ideas, and their active involvement is most important for mathematical progress. It ensures that the reasoning is supported and understood by both participants, and hence serves as a green light to continue.

The communication of ideas and reasoning in the girls' dialog seems to be especially interrelated with the use of drawings and other written representations. It is striking that whenever a change of representation is performed, the girls very carefully explain their actions. We see this when Kate makes the drawing of twelve muffins (22-30), and later when she turns the problem into a repeated addition problem (52-54). Making their thinking public in these situations is especially important because the written representations are the dominant mediational means in the solution process. On one hand, one can say that the mathematical progress is dependant of the girls' ability to accompany their written work with verbal explanations and gestures, as this may contribute to a shared understanding, crucial for keeping the solution process a common project. On the other hand, the drawings act as means to elicit the girls' thinking, giving their verbal reasoning a necessary support. In a way, the drawings and the girls' ability to communicate their thinking seems to be interdependent. Hence the girls are involved in what we will call a collaborative tool-mediated talk in order to solve the mathematical task. They are using language and drawings effectively for joint, explicit, collaborative reasoning. We claim that our study adds to the field throwing light on how drawings are necessary mediational means in young learner's collaboration.

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Narratives constructed in the discourse on early fractions

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Fractions are one of the most demanding topics for teachers to teach and for students to learn. In this paper, we examine narratives about general properties of fractions constructed in a class when they were introduced through an equal-sharing context. The students' work and discussions constitute the starting point in planning further teaching, moving from lesson to lesson. Three episodes are presented in order to illustrate and discuss our findings. We argue that the analysis of the narratives provides insights into opportunities for students to learn as well as regarding the complexity of the topic.

Introduction and theoretical framework

The aim of this paper is to contribute to the understanding of opportunities and constraints that may occur when teaching and learning fractions. A class of Norwegian 4th grade students worked on fractions over a five-week period, which was their introduction to the concept in school. In this study, we are particularly interested in statements about general properties of fractions and the relations between fractions that were discussed in the class when the topic was introduced through a context of equal sharing. We analyze video recordings of lessons and illustrate and discuss our findings through three selected episodes.

From a pedagogical point of view, fractions and rational numbers take on many "personalities". Kieren (1976) recommends that work on fractions should be conceptualized as a set of interrelated meanings, which he calls subconstructs: part-whole, ratio, operator, quotient and measure. Behr, Harel, Post and Lesh (1993) have further developed Kieren's model, connecting it to operations on fractions, equivalence and problem solving. However, Olive and Laboto (2008) argue that the model is a semantic top-down analysis, which represents the adult view on fractions, and that it is not certain that it describes children's construction of fractional knowledge. Thompson and Saldanha (2003) have also been critical of the model, but their critique arose from a mathematical point of view: the mathematical motivation for rational numbers did not emerge from meanings, but from arithmetic and calculus. They suggest that fractional reasoning is tightly connected to multiplicative reasoning, arguing that fractional reasoning develops concurrently with reasoning on measurement, multiplication and division.

Nevertheless, Kieren's model has had a great influence on developments in the area of research. Lamon (2007) suggests that it can be important to choose one of the subconstructs as a starting point in the instruction and gradually include the others. In a realistic mathematics educational tradition (see Streefland, 1993), the notion of fractions is usually introduced through the context of equal sharing, i.e. as a quotient. Streefland (1993) argues that equal sharing gives rich learning opportunities regarding different aspects of fractions and that part-whole and operators appear naturally in this context. In their approach to fractions through cognitively guided instruction, Empson and Levi (2011) also started by working on equal-sharing contexts. In the class that is the focus of this study, the instruction started with an equal-sharing context, but the direction of further teaching was not decided a priori. Rather, each lesson was designed based on the students' work and the classroom discussions that occurred in the previous lesson.

How do students learn mathematical ideas? Sfard (2008) takes the position that learning mathematics is learning to participate in a particular discourse, where discourse is a special type of communication within a particular community. A discourse is made mathematical by a community's *use of words, visual mediators, narratives* and *routines*. The *use of words* in mathematics includes the use of ordinary words that are given special meaning in mathematics, such as function and ring, and mathematical words such as fractions and trapezium. In mathematical communication, participants use *visual mediators* to identify the object of their talk. These visual mediators are often symbolic, such as mathematical symbols, graphs, illustrations (e.g. number lines) and physical artefacts (e.g. centicubes). Within discourses, any spoken or written text that discusses properties of objects or relationships between objects is called a *narrative*. Narratives can be numerical, e.g. ½ is equivalent to 2/4, or more general, e.g. addition is commutative (see Sfard, 2015). Narratives are subject to endorsement or rejection, which is labelled true or false based on specific rules defined by the community. *Routines* are well-defined practices that are regularly employed in a discourse by a given community. These include how one talks about geometrical objects, how one performs calculations, how one substantiates a calculation, how to generalize and justify as well as when to use a particular action. A routine is called an *exploration* if it produces an endorsable narrative. Examples of explorations are numerical calculations, such as $21 \cdot 19$, the generalizing of patterns and the justification of these generalizations.

Several studies have reported on initial fraction learning through equal sharing (e.g. Empson, 1999), but none have used Sfard's framework for learning. We argue that Sfard's thinking—regarding the learning of mathematics as learning to participate in a particular discourse—is suitable to describe and analyze students' learning processes as well as opportunities for learning. Students gradually start to use fraction words and develop routines and narratives about the properties of and relations between fractions as they engage in work on fractions. Constructions of narratives and their rejections or endorsements are central to mathematical discourse. We argue, therefore, that an analysis of constructed narratives can provide insights into opportunities for learning in a given discourse. Narratives about general properties and relations between objects are of particular interest here, as they can be lifted above the numerical situations and used in new situations. Aiming to gain additional insight into learning opportunities, our research question is: “What narratives about general properties of fractions and relations between fractions can be constructed in an early discourse about fractions when they are introduced through an equal-sharing context?”

Method

The study stems from a collaboration between an elementary teacher and two researchers, the authors of the paper. The teacher has been teaching the class in all subjects from their first grade. She was concerned about the students’ participation and understanding in mathematics. There were 20 9-10-year-old students in the class, who attended a conventional Norwegian school.

The class worked on fractions over a period of five weeks: two 70-minutes lessons per week. The teacher´s motivation for the collaboration was further development of her teaching practice. She suggested that the researchers sketched ideas for lessons. The ideas were then discussed with the teacher. The teacher´s comments and suggestions on the researchers’ ideas were built on the students´ prior knowledge and the way of working they were used to. The instruction on fractions began with a problem about a school trip, whereby different groups of students shared sandwiches: one group of four students shared three sandwiches, and another group of five students also shared three sandwiches, etc. (inspired by Fosnot & Dolk, 2002). The first activity set the basis for the series of lessons, as all other lessons, and are connected to the students’ work on this first problem. The researchers were present as participants observers during the lessons, videotaping, observing and sometimes talking to individual students or even leading the instruction for short periods. After the lessons, the researchers and the teacher discussed students’ work. Based on these discussions, they sought to identify areas that should subsequently be emphasized and how.

Data and data analysis

The data that is the focus of this paper is the video recordings of the class discussion. Starting the analysis together, we watched through the recordings and marked out all utterances, spoken and written, that could be considered true or false. These utterances made up the set of all narratives discussed in the class. Most of the narratives were numerical, such as “one-fourth is half of one-half” or “three children get more than four children when they share a chocolate”. As our research question is about narratives concerning general properties of fractions or relations between fractions, the next step in the data analysis was to identify these instances. We discussed the generality of each of the narratives marked out in the data and ended up with five narratives:

A. When we share equally, we can express shares as fractions
B. When we talk about a fractional part of something, it is crucial to be aware of what it is a part of
C. The more parts we divide something into, the smaller the parts that we get
D. When we want to express part of something as a fraction, the parts have to be equal
E. Fractions can be the same even though they do not look the same

Finally, in order to present our findings, we agreed on three episodes (not sequential) in which all five narratives were evident to illustrate the way they were constructed and discussed in the class. The first episode is from week two, the second from week three and the third from week four.

Episode 1 is an excerpt from a classroom discussion on sharing two chocolates among three children. Two solutions that the students worked out before the discussion are presented in Figure 1.

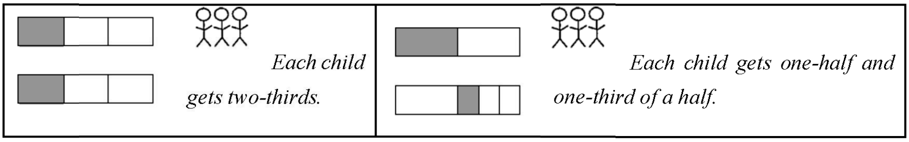

Figure 1: Students' solutions to the problem of sharing 2 chocolates among 3 children

The teacher wants to discuss with the students whether the answers in the two solutions are equal and how this is so. She also wants to discuss which fractional part of a chocolate "one-third of a half" is. The teacher begins the discussion by asking the students to talk about their solutions. Thomas suggests dividing the first chocolate into three equal parts, giving one part to each child; the same with the second chocolate. He concludes that each child gets one-third of the first chocolate and one-third of the second chocolate. The other students agree. The teacher asks how much of one chocolate one child gets in total.

John:	I think it is two-thirds.
Teacher:	Two-thirds of one chocolate? Is that what you think?
John:	Yes… Two chocolates are divided in three parts [each], and each child gets two of them.
Lena:	Yeah, but… If we take them together, then each child gets two-sixths too?

Some students agree with John, while some agree with Lena. If one considers the two chocolates as one unit (as Lena suggests when saying "take them together"), then the first picture in Figure 1 shows a unit divided in six parts, two of them are shaded – so 2/6 of the unit. In order to emphasize this change in the unit, the teacher writes both suggestions, "2/3 of 1 chocolate" and "2/6 of 1 chocolate" on the blackboard, one below the other, and asks whether both can be right. Lena suggests that it is the same. John and several other students agree.

Teacher: So, two-thirds are the same as two-sixths? Can it be? Here, we divide each chocolate in three parts (the first example in figure 1). If we divide it in six parts, then the parts are smaller, right? Remember that the question is how much of *one* chocolate each child gets. In what parts is *one* chocolate divided?

James: Thirds.

Teacher: And how many of such thirds does each child get?

James: Two.

Teacher: So, each child gets two-thirds of one chocolate. We can say that each child gets two-sixths, but then it is not of one chocolate. Two-sixths of what is it?

Lena: If you take two chocolates [as a unit], then it is two-sixths. If you take one chocolate [as a unit], it is two-thirds.

Teacher: Right. When we talk about fractional parts, then we have to say parts *of what*. It makes a difference. If you are about to get one-third of one chocolate, or one-third of a big bag full of chocolates, it is different, right [students nod and smile]? Shall we try to find out how much chocolate we actually get if we get one-third of a big bag of chocolates, 100 chocolates in the bag?

The narrative "When we share equally, we can express shares as fractions" is central in the given context. Two chocolates are to be shared equally among three children, and the students share the chocolates in one of the two ways presented in Figure 1. They have worked with similar tasks several times before in the series of the lessons in this research, and the use of "fraction words" (two-thirds, one-half, third of a half) to describe shares seems to have become part of their routine in such tasks. They emphasize that each chocolate is shared equally among the three children and that each part is one-third of the chocolate.

Thomas suggests that "each child gets one-third of the first chocolate and one-third of the second chocolate". In the written work, many students suggest that "each child gets one-half of the first chocolate and one-third of a half of the second chocolate". There is nothing in the context that makes it necessary to consider these parts together as a fractional part of "one chocolate", which is emphasized by the teacher in order to compare the two different solutions. The teacher presses on, expressing the share as a fractional part of one chocolate, and another narrative is being constructed in the process: "When we talk about a fractional part of something, it is crucial to be aware of what it is a part of". In other words, the role of the unit is emphasized by the teacher.

In order to challenge the students' claim that both "2/3 of 1 chocolate" and "2/6 of 1 chocolate" can be right answers, the teacher points out that one-sixth of a chocolate is smaller than one-third. Here, she brings in another narrative, which was discussed earlier in the context of sharing one sandwich among five and six kids: "The more parts we divide something into, the smaller the parts we get".

Episode 2 is an excerpt from a classroom discussion on a task in Figure 2. Tasks of this type were designed to discuss problems regarding type "one-third of a half", which appear in the context of equal sharing (of chocolates and such), as in Episode 1.

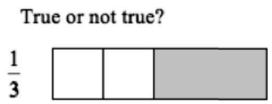

Figure 2: Task discussed in Episode 2

Nelly: Not true. Because... it is ... there are three parts, but if we are to share equally, it will not be equal ... hmm ... it will be unfair because one part is big.

Several students agree and suggest dividing the shaded part into two equal parts. Martin says that he does not understand.

Teacher: Ann, can you explain to Martin why we need to divide the big part into two parts?

Ann: Mmm. Because that part is too big if it is to be shared equally.

Teacher: That part is too big if it is to be shared equally. Do you understand it, Martin? [Martin nods.] Can you explain it in your own words?

Martin: Mmm. If it was one chocolate and three kids, then … well ... two kids would get equal parts, but the last one would get bigger than the others.

Teacher: Yes. If it was a chocolate, as Martin says, it would be rather unfair because one would get more. Exactly as you say, it would be unfair.

The problem in Episode 2 is without context, but the students independently connected it to an equal-sharing context. Nelly starts to talk about equal sharing and uses the word "unfair" – the shaded part cannot be 1/3 because it would be unfair. This is supported by Ann and expounded by Martin. This indicates that the meaning of fractions for students is strongly connected to equal-sharing situations and "fairness". The teacher supports the argumentation, and the narrative emphasized is that if we want to express part of something as a fraction, the parts have to be equal.

Episode 3 is part of a whole-class discussion on "tell me what you see". The aim was to discuss different partitioning of a figure. The tasks were designed so as to discuss solutions of the form: "each child gets one-half and one-third of a half", as in Episode 1. The students discuss Figure 3a) and find that it is "divided into two equal parts, one of which is shaded" and "we have one of two". The teacher then adds Figure 3b) so that both figures are on the blackboard.

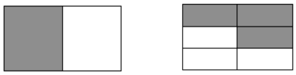

Figure 3: Figures a) and b) discussed in Episode 3

Teacher:	What does this one look like? Mary [meaning 3b]?
Mary:	It is six parts, and three are shaded.
Teacher:	Yes, what about you, Ben?
Ben:	One of two [several students: What?]
Teacher:	How do you see one of two?
Ben:	I move the one shaded from the left down to the right and then one of two.
Teacher:	Ok. Someone disagrees? Nelly?
Nelly:	I see three of six.
Teacher:	You see three of six. But can it be both one of two and three of six? Can it be? Is it the same figure? Lena?
Lena:	Yes, it is true if the parts are equally big.
Teacher:	It can be true if the parts are equally big [writes this on the blackboard]. What does that mean, Lena? If all the parts are equally big?
Lena:	If the parts are not equally big, it is not a real fraction.
Teacher:	Then I challenge you, one of two and three of six, they do not look the same. How can they be the same?
Lena:	If we move some parts, they become the same.
Teacher:	So, one-half can look different; is that what you mean [Lena nods]?

The discussion is about Figure 3b), which can be seen as *three of six*, but also *one of two* if "we move some parts" and then "we erase some lines", i.e. the idea of equivalent fractions is discussed. The constructed narrative is "the fractions can be the same even though they do not look the same".

Results and discussion

In Episode 1, two earlier-discussed narratives came up again: *When we share equally, we can express shares as fractions* and *the more parts we divide something into, the smaller the parts we get.* Both narratives emerged from the equal-sharing context that was initially used in the teaching. In addition, the narrative *When we talk about a fractional part of something, it is crucial to be aware of what it is a part of* was constructed as a consequence of an operator subconstruct appearing in the quotient context (one-third of a half chocolate). The constructed narrative emphasizes the role of the unit, which is one of the critical aspects of learning fractions (see, e.g. Lamon, 2007).

In Episode 2, the narrative *When we want to express a part of something as a fraction, the parts have to be equal* was constructed and endorsed by the students by referring to “fair sharing”. The task used in the episode was designed as it was shown to be important in emphasizing partitioning so as to illuminate a challenge that came up in a quotient context. The task can be seen as a part-whole subconstruct, but the constructed narrative was endorsed by connecting the situation to the equal-sharing, i.e. quotient, subconstruct.

In Episode 3, the narrative *Fractions can be the same even though they do not look the same* was constructed. The idea of equivalent fractions, another important aspect of fractions, were discussed in the episode. The need to discuss tasks as the one in the episode is imbedded naturally in the equal sharing context, as different ways to share two chocolates among three children. The task was given in a part-whole context, and the students endorsed the narrative by partitioning.

The teaching period started with a quotient subconstruct of fractions (Kieren, 1976). However, both the operator (as one-third of a half) and the part-whole construct (as one of three parts) appeared almost immediately in the students’ work. Their work and discussions were the starting point in teaching planning from lesson to lesson. Moreover, looking back, we see an interplay between the quotient, part-whole and operator subconstructs throughout, as illustrated in the three episodes presented in this paper. This contradicts Lemons’ (2007) recommendation that the initial instruction should concentrate on one subconstruct, indicating that focusing only on one subconstruct can be restrictive and unnatural in teaching. In our study, the context of equal sharing was shown to be a rich starting point that brought out many important aspects of fractions, as suggested by many researchers (e.g. Empson & Levi, 2011; Streefland, 1993). However, it also seemed to be highly complex for teaching and learning for the same reason, and one can say that the class worked on basically the same problem for the whole teaching period, as the teacher tried to help the students delve deeper into the emerging ideas.

It is well known that a teacher plays an important role in creating leaning opportunities for students. The three episodes illustrate, in particular, the teacher’s crucial role in the process of constructing narratives. The equal-sharing (like sharing chocolates among some kids) situations were imaginable for the students, as they constituted part of their everyday experiences, and they had no difficulty suggesting a solution. However, as everyday experiences, there is no need for students to dwell on moments as “what part of a chocolate is one-third of a half” or “is one-half the same as three-sixths”. The equal-sharing situation was moved into a new, mathematical, discourse in the teaching. It was the teacher who pressed with new questions in the situation and tried to emphasize narratives on properties and relations between fractions, making the equal-sharing a context for learning fractions.

In the process of discursive learning, the use of words, routines and narratives developed in a community are in continual flux and refinement (Sfard, 2008). We started our teaching on fractions by equal sharing, and after a while, it became a routine for the students to use fraction words to denote shares. Fractions became related to equal sharing and fairness, which constitute everyday experiences for students. This made way for several explorations and narrative constructions. Fractions are complex, both in terms of teaching and learning, and the question is how to make the concept more accessible without oversimplifying it. We hope that our paper and analyses of the general narratives constructed in the discourse can contribute to research on this question. However, our study was conducted over a short time period and further longitudinal studies on the construction of narratives are needed to gain more insights. We suggest that the episodes presented in the paper can be used in teacher education to discuss the complexity of teaching fractions with pre-service teachers.

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Second graders' reflections about the number 24

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Students' written responses to an open task were examined to identify potential indications of emerging number sense. Content analysis indicates that the number of responses given by students varied, with addition tasks being more commonly provided than tasks that involved other operations. Whereas several students refer to place value, no students mention possible applications of the number. From these findings, implications are discussed in terms of the mathematical demands that teachers are faced with when presenting such tasks in a mathematics lesson.

Introduction and theoretical background

Definitions of number sense differ, but they often refer to students' general understanding of numbers and operations, as well as ability to use their understanding in flexible ways to make mathematical judgements (McIntosh, Reys, Reys, Bana, & Farrell, 1997). Number sense is often described as a prerequisite for students' further development of mathematical knowledge (Verschaffel, Greer, & de Corte, 2007). Children's number sense has been investigated for decades (e.g., Gelman & Gallistel, 1978; Verschaffel et al., 2007), and understanding of the place value system is regarded as particularly important in students' development of number sense and eventually in their work with multi-digit numbers (Kilpatrick, Swafford, & Findell, 2001). Students' understanding of place value develops over time, and it influences understanding of multi-digit numbers, which includes a person's general understanding of numbers and operations (Jones et al., 1996). A fully developed number sense enables students to flexibly operate on numbers and develop useful strategies (McIntosh et al., 1992). This includes understanding how numbers are ordered, how different representations of numbers are connected, what effects and mathematical properties different operations have, as well as understanding how the arithmetical operations are related.

Jones et al. (1996) present four core components that constitute the process of developing multi-digit number sense: counting, partitioning, grouping and number relationships. They then distinguish between five different levels for each of the four components: pre-place value (level 1), initial place value (level 2), developing place value (level 3), expanded place value (level 4), and essential place value (level 5). With reference to the competence aims of the national curriculum, we assume that students in grade 2 are in one of the first three levels. Whereas older students develop more advanced counting strategies (Camos, 2003), students at the level of pre-place value count by ones and know how to partition a number in different quantities, for instance $8 = 6 + 2 = 1 + 7$ (Jones et al., 1996). In their work, they indicate that students at these initial levels can tell if a number is bigger or smaller than another number, but they cannot tell how big this difference is. Students with an initial understanding of place value can think in groups and they can count with tens and ones. To rationalize by counting by tens, the students realize they need to group objects. They understand that they can partition two-digit numbers, for example $24 = 15 + 9$, and in addition they understand that grouping facilitate estimation and counting. When the digits' place change, the students understand that it represents different numbers. Students developing place value (level 3) know how to count by tens and ones and are capable of applying it in operations. This level differs from the previous ones because of the ability to think part-part-whole with two-digit numbers. Within grouping, the students can estimate between which tens a sum of two two-digit numbers will be located, and they master operations and comparing simultaneously (Jones et al., 1996).

Thompson (2003) describes two sub-concepts of the place value system: quantity value and column value. One is more important in (written) mental calculation and the other in using standard algorithms. For instance, the two-digit number 24 can be decomposed into 20 and 4, which relates to the quantity value of the number. Mental calculation is mainly based on quantity value. As an example, 24 and 38 can be added as $20 + 30 = 50$ and $4 + 8 = 12$. The sum is $50 + 12 = 62$. Column value is when 24 is considered to consist of two tens and four ones. The standard algorithm for (written) addition focuses on column value by putting tens over tens and ones over ones (two-digit), and then each of the digits are added (Thompson, 2003)

In this paper, we investigate what Grade 2 students' responses to an open task about the number 24 may reveal about their emerging number sense. We consider data material from two classes of Grade 2 students, who were given the open task called "The number of today".

The study

Our examination of Grade 2 students' reflections about the number 24 is part of a larger school-based research project focusing on developing in-service teachers' knowledge. The first author of this paper has supervised the teachers in the planning of the lessons, observed their teaching, collected material from the students and discussed the teaching with the teachers in retrospect. Prior to the study presented in this paper, the teachers participated in a half-day long in-service course focusing on tasks that invite the students into discussions and different solution strategies. The task used in this study is one example.

The data material is collected from two different classes from the same school, referred to as Group A (N=17) and Group B (N=21). The two teachers who taught these groups used the task, “The number of today”, as one of four tasks that the students worked on during a 60 minutes long session. Prior to this lesson, the students have mainly been working with numbers between 0 and 20. Following the textbook (Alseth, Arnås, Kirkegaard, & Røsseland, 2011) they have first focused on the numbers 0–9. After this, they have spent time on the numbers from 10 to 20, which have been partitioned into tens and ones. They have worked with numbers that add up to 10, addition and subtraction of numbers between 0 and 20, and they have encountered the concept of numerical neighbours. According to the competence aims of the national curriculum, they are supposed to know how to “count to 100, divide and compose amounts up to 10, put together and divide groups of ten up to 100, and divide double-digit numbers into tens and ones” (Ministry of Education and Research, 2013, p. 5) by the time they finish Grade 2.

The students, who were in the first semester of 2nd grade (seven years old), worked individually for approximately 15 minutes on each of four different tasks. All four tasks had been introduced in a previous lesson, and the students could therefore start working on them without any further introduction in this lesson. The students had been told by the teachers that the task (which is the focus of this paper) was related to the question of what they know about the number 24. In each group, the students provided written responses on a worksheet. The teachers made some slightly different choices in how the worksheet was designed. In group A, the worksheet was a blank piece of paper with the number 24 on top of it (Figure 1).

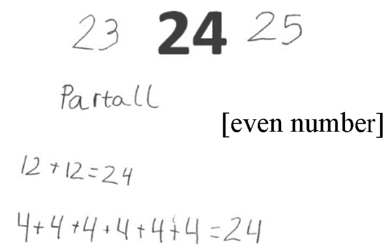
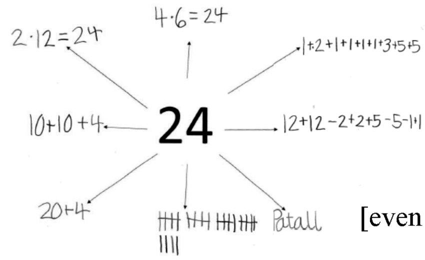


Figure 1. One example of student response from group A (A1).

In group B, the teacher had added eight arrows that were sticking out from the number 24 (Figure 2), but he did not indicate that only eight pieces of information should be provided. The first author of this paper was observing while the students were working on the task. Although various data materials were collected, only the written responses are analyzed for this paper.



[even number]

Figure 2. One example of student response from group B (B5).

The students had previously encountered similar tasks in whole-class discussions, and they were now allowed to collaborate and use manipulatives to develop their written responses. Unstructured material like milk caps and structured material like multi-base material were available for the students to use, but few students used the material. The teacher allowed them to work in groups, but most students decided to work individually on the task.

The students' responses were collected immediately after they had worked on the task for 15 minutes. To ensure anonymity, each worksheet was assigned a letter A or B to indicate what group the student was affiliated with and a number to distinguish between students in each group. For instance, A3 is student number 3 in group A. The students' written responses were analyzed using content analysis. We began by identifying how what was written related to aspects highlighted in previous research on children's understanding of number (see theoretical background), specifically. This was followed by a theory driven approach to content analysis (Fauskanger & Mosvold, 2015; Hsieh & Shannon, 2005). The theory driven analysis was based on 1) Thompson's (2003) quantity value and column value, 2) McIntosh et al.'s (1992) aspects of fully developed number sense, and 3) Jones et al.'s (1996) constructs of counting, partitioning, grouping and number relationships.

Findings

The 38 students provide a total of 161 responses. Students in group A provide 61 responses, and students in group B provide 100. Table 1 presents an overview of the different types of responses. Below we discuss these results with a focus on differences among students and groups of students. Examples of student responses are displayed to indicate the variation of responses given to the task.

Only one student (B4) does not provide any response to the task, whereas four students provide eight responses (see e.g., Figure 2). The two groups of students vary in the type of responses they give. In group A, 13 out of 17 students mention concepts or characteristics of the number 24 (e.g., even number, numerical neighbours, number of digits). The students in group B provide responses within all categories, but they have more focus on arithmetic operations than the students in group A. Five students provide examples that involve a combination of arithmetic operations. The two most advanced examples are *10 × 2 + 4* (B12) and *100 – 80 + 4* (B3). The responses contain few errors; 20 of the 38 students do not have any incorrect responses. Five students have two incorrect responses (A13, A16, B7, B16 and B21), but no students have more than two errors. Few responses from a student does not necessarily indicate a lack of knowledge. For instance, B13 only provides three responses, but these responses include three different operations: *12 + 12*, *28 – 4* and *8 × 3*.

The teacher in group B added eight arrows from the number 24 on the worksheet, and this adjustment might have influenced the students' interpretation of the task. For instance, 15 of the 21 students in group B appear to believe that the arrows should point to examples involving arithmetic operations rather than referring to place value. The students have some previous experience with the place value system; seven students—from both groups—draw arrows towards the digits of the number 24 or write about the value of the digits. For instance, students B8 and A15 write about how many tens and ones the number consists of like "2 tens and 4 ones", whereas student A3 write 10 above 2 and 1 above 4 to indicate tens and ones. This corresponds with what is often referred to as column value (Thompson, 2003). There are also examples of quantity value in the students' responses. For instance, student A7 draws an arrow from 2 and wrote 20, and another arrow from 4 and wrote 4. This student also write 10 and 1 over the digits 2 and 4.

Among the responses that include addition, many of these also indicate knowledge of place value. For instance, some students partition the numbers into tens and ones, or group numbers that add up to 10. Such responses are categorized as relating to place value, although they also include addition. Several students include *20 + 4* (six responses) and *10 + 10 + 4* (e.g., A2, A9, B1, B3, B4 and B6, 17 responses). Six students only include *10 + 10 + 4*, whereas two students include *10 + 14*. The responses of these students indicate that they have developed understanding of quantity value (Thompson, 2003).

The responses that include addition also provide other examples of partitioning. Examples are *4 + 5 + 5 + 5 + 5* (B1) and *8 + 2 + 8 + 2 + 4* (B12). These responses indicate ability in partitioning as well as regrouping, which are two important elements of Jones et al.'s (1996) model of number sense. Emerging understanding of place value involves knowing that grouping in ones and tens simplify the arithmetic operations (Jones et al., 1996). Two students' (A3 and B5) responses include tally marks or small circles that are grouped in fives. These are examples of grouping without using numerals and illustrate use of different representations of number (McIntosh et al., 1992). A response like *8 + 2 + 8 + 2 + 4* (B12) indicates understanding that one representation is more useful than another—in particular a representation that involves grouping of tens (cf. McIntosh et al., 1992).

Although addition is the most frequently used arithmetic operation in the responses, there are also examples that involve subtraction, multiplication and division. Some responses also combine arithmetical operations. An interesting example is $12 + 12 - 2 + 2 + 5 - 5 - 1 + 1$ (B5). This response indicates knowledge of mathematical properties of operations, including awareness that adding and subtracting the same number does not change the answer. By providing the responses of both $10 \times 2 + 4$ and $10 + 10 + 4$, B12 indicates understanding of relationships between operations, and this might also be interpreted as indicating emerging understanding of how multiplication can facilitate addition (cf. McIntosh et al., 1992).

In group A, two students wrote down the numerical neighbours 23 and 25, either by writing that 24 is "numerical neighbour of 23 and 25" (student A6), or by writing 23 to the left of 24 and 25 to the right of 24 on the worksheet (A1). No students in group B mention numerical neighbours, and this may be due to the adjustment of the worksheet for group B that may not invite to mentioning numerical neighbours.

Among the students' responses, only occasional errors occur. For instance, student B16 writes $10 + 10$ above the 2 in 24. This is correct, but then the student writes 8 and $4 + 4$ above the 4. This might indicate an understanding that two tens automatically mean that there must also be two ones.

Mathematical focus	Gr. A	Gr. B	Examples (students)	Incorrect examples (students)
Place value	3	1	2 tens and 4 ones (B8 and A15) Arrows under the number with 20 and 4, and arrows over the number with 10 and 1 (A7)	Arrows from the digit 2 with 10 +10 and 20, arrows from the digit 4 with 4 + 4 and 8 (B16)
Concepts and the number's characteristics	21	4	Even numbers (B6) Numerical neighbors: 23 and 25 (A6) Two digits (A4)	
Writing digits, reversing	1	2	24, 42 (B1) Wrote that the numbers were reverse. Reversing the number 4 and the number 2 (B2)	
Addition Two different addends Some similar and some different addends Only similar addends	 5 8 12	 13 30 22	3 + 21 (B1), 9 + 15 = 24 (B19), 24 = 2 + 22 (B1) 8 + 2 + 8 + 2 + 4 (B12), 24 = 4 + 5 + 5 + 5 + 5 (B1), 10 + 10 + 4 = 24 (A2) 12 + 12 = 24 (A9), 4 + 4 + 4 + 4 + 4 + 4 = 24 (B6),	24 = 5 + 15 (B2), 12 + 2 (A16) 1 + 2 + 1 + 1 + 1 + 1 + 3 + 5 + 5 (B5), 10 + 3 + 4 + 1 = 24 (A2), 10 + 4 + 10 + 4 = 24 (A4) 4 + 4 + 4 + 4 (B1)
Subtraction	1	5	24 – 0 (B11), 28 – 4 = 24 (A8), 34 – 10 (B17), 25 – 1 (B21)	17 – 4 = 24 (B19), 100 – 86 = 24 (A13)
Multiplication	0	5	4 × 6 = 24 (B5), 24 × 1 (B12), 8 × 3 = 24 (B13)	
Division	0	4	24 ÷ 6 = 4 and 24 ÷ 4 = 6 (B3), We divide by four (B17)	
Combined arithmetical operations	0	5	100 – 80 + 4 = 24 (B3), 10 + 10 + 6 – 2 (B10), 12 + 12 – 2 + 2 + 5 – 5 – 1 + 1 (B5), 10 × 2 + 4 (B12)	
Different representations	2	1	B5	A3
Wrong answers	8	8		
Total	*61*	*100*		

Table 1: Overview of responses to the task, "The number of today is 24"

Concluding discussion

Analysis of students' responses to this open task about the number 24 provide indications of emerging understanding of place value. Many students are able to group and partition the number 24, but we cannot conclude from this study that the other students are lacking understanding in this respect. The students' responses might have been influenced by the way the task was presented, and it is important to consider the possibilities and limitations of a task like this. We will highlight five issues. First, arranging the worksheet like a blank piece of paper with the number 24 on top (group A, Figure 1) or as eight arrows sticking out from the number 24 (group B, Figure 2) might affect the students' responses. With students who fill in responses at the end of each of the eight arrows, the arrows may have restricted them from providing more responses to the task. Second, there is an issue related to the responses students give and if the responses are at a more advanced level than recommended by the curriculum at the actual grade level. For instance, when student B12 responds $10 \times 2 + 4$ and student B3 responds $24 \div 6 = 4$ and $24 \div 4 = 6$, they include multiplication and division in their responses—concepts that are in focus on a later grade level (Ministry of Education Research, 2013). Third, there is an issue of how to interpret the lack of responses from some participants. Some students do not provide any response or one response only, but there is not necessarily a correlation between number of responses to an open task like this and students' knowledge and understanding of place value. Fourth, one might wonder why so few students use the concrete materials that were available or work in groups. Finally, one can ask why no students mentioned anything about applications of the number 24, e.g. that 24th of December is Christmas Eve. The reason can be that this was a written task, and the students may have interpreted it as a task where they were supposed to make arithmetic problems. Following up on the students' responses by adding cognitive interviews might have provided additional information about their number sense. An interview with the teachers about their teaching in advance could also have given answers to some of these questions.

Our focus in this study has been strictly on the students' responses, but the results of our study may also have implications for teachers. Investigations of Grade 2 students' mathematical reflections about the number 24 may indicate some mathematical demands teachers are faced with when facilitating such an open-ended activity. For instance, teachers must interpret students' responses on tasks like these and act upon them—often quickly. A teacher must also figure out what students know and are able to do from looking at their responses to open-ended questions like this. These are some examples of the mathematical demands that are embedded in the work of teaching early number sense. To skilfully carry out the work of teaching, teachers need a professional knowledge that includes—but is not restricted to—knowledge of quantity value and column value (e.g., Thompson, 2003), knowing models for examining important components of number sense like counting, partitioning, grouping and number relationships (e.g., Jones et al., 1996; McIntosh et al., 1992). Such knowledge is required to analyze students' responses and draw out their thinking through carefully selected questions and tasks and to consider and check alternative interpretations of the students' ideas as visible in their written responses.

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Tablet computers and Finnish primary and lower secondary students' motivation in mathematics

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In this paper, we report in terms of the expectancy–value theory and self-efficacy from the experiences of utilizing tablet computers for the learning of mathematics among primary and lower secondary students (N=256) *in one school in Finland. Our main findings are as follows. Using tablet computers seems to increase especially boys' intrinsic values in studying mathematics, yet both boys and girls preferably disagree than agree with the claim that tablet computers have made it easier for them to learn mathematics. Girls clearly prefer to study mathematics with paper and pencil. The utility value of using tablet computers in studying mathematics does not depend on the students' beliefs about their competence in mathematics.*

Introduction

The latest national guidelines for curricula in the Finnish primary and lower secondary schools, which have been implemented since August 2016, emphasize the versatile usage of technology in teaching and learning. However, due to limited financial resources, to which degree the schools have taken technology in use, varies a lot. In 2013, the investigated school – among the very first ones in Finland – provided an iPad for every student. Since then, tablet computers have been used daily in the teaching of most subjects. In some subjects, iPads have replaced printed textbooks completely, but in mathematics, students have used both iPad applications and printed textbook side by side. Consequently, iPad has been a primary medium for younger students and, for the lower secondary students, a printed textbook has been their foremost learning material, yet they have used iPads as a secondary medium for three or more years.

In this paper, we report from our survey on the students' experiences from using tablet computers. Experiences were surveyed both at a general level and concerning the teaching and learning of mathematics and mother tongue. We focus on students' beliefs about how tablet computers have affected their motivation and learning in mathematics and their views of themselves as learners of mathematics. The participants (N=256) are from the grades 1–8.

There are many theories on learners' motivation and some studies on how bringing technology in school affects students' motivation. In the next two sections, we review some earlier studies that are relevant to ours, and then discuss our theoretical framework. The research questions and method will be given after that, and the results are represented and discussed in the last two sections.

Review of earlier research

Earlier research has shown mixed results on the effects of the use of tablet computers in mathematics education. For example, Henderson and Yeow (2012) report from a school which was one of the first primary schools in the whole world to adopt the use of iPads. They conclude that the main strengths that tablet computers can provide are a quick and easy access to information and support for collaboration. Attard and Curry (2012) also explored the use of iPads in engaging young students with mathematics. After a six-month trial, students' engagement in mathematics seemed to have improved. However, for example, Carr (2012) reports from an experiment with a control group where fifth graders studied mathematics with iPads and game-based learning approaches. The result was that no significant differences in learning achievements occurred. A possible partial explanation may be provided by Ravizza, Uitvlugt and Fenn (2016) who, in the context of psychology education, found out that non-academic use of Internet during lessons is common even among adult learners. In their study, the students' class performance was even inversely related to the use of technology.

The above-mentioned studies do not discuss gender issues. Another typical feature of previous research on the use of tablet computers in mathematics education is that they focus on short-term teaching experiments; studies on the enduring effects on motivation in mathematics are hard to find. All in all, previous research suggest that tablet computers have potential to increase students' interest in studying mathematics, but this effect may, at least, partly be explained by the novelty value involved in introducing new technology in classroom. Our study aims at proving a farer-reaching view of the situation since tablet computers have been in use in our research context for several years, and at giving some information whether boys and girls consider the value of tablet computers in mathematics education in a similar or different way.

Theoretical framework

The theoretical perspectives in this paper base on two motivational theories: We use the expectancy–value theory (Eccles et al., 1983; Wigfield & Eccles, 2000) to discuss the participating students' motivation in mathematics. Further, we discuss their perceptions of themselves as learners of mathematics and as users of tablet computers in terms of self-efficacy (e.g. Bandura 2012).

According to the expectancy–value theory, an individual's choices and performance in studying a subject can be explained by her/his beliefs about the possible success she/he can reach and the extent to which she/he values the subject. A part of this model of individual's motivation are the subjective task values. These values are usually divided into four components: attainment value (the importance of activity), intrinsic value (interest in the activity or the liking of it), utility value (the usefulness of the activity), and cost (how much effort an individual is ready to pay for succeeding in the activity). Due to limited space, we focus in this study only on the participants' intrinsic and utility values of studying mathematics and using tablet computers. However, these two values depict students' motivation in mathematics quite well also in general due to the correlations between the values, cf. Tossavainen & Juvonen (2015).

Self-efficacy means the extent of an individual's beliefs in her/his own ability to complete a task or reach a goal. According to Bandura (2012), perceived self-efficacy varies according to different domains. Therefore, Bandura (2012) argues that self-efficacy is better to be measured in a contextualized manner as human behaviour is socially situated and richly contextualized. In this study, we use Likert type items to measure students' perceptions of their contextualized self-efficacy in mathematics and using tablet computers.

Research questions

We are interested in knowing how the utilization of tablet computers support girls' and boys' motivation and learning in mathematics, and how the use of tablet computers is related to students' view of themselves as learners of mathematics. Our research questions are as follows.

1. What kind of intrinsic and utility values related to studying mathematic with tablet computers primary and lower secondary students do have?
2. Do tablet computers support boys' and girls' learning in mathematics in a similar way?
3. How the use of tablet computers is related to students' sense of self-efficacy in mathematics?

Method

Data for this study were collected using a questionnaire which contained a few open questions and altogether 92 five-point Likert scales inquiring students' general enjoyment and motivation to going to school, their views of themselves as learners in various subjects both when tablet computers are used in education and in the traditional context of teaching and learning, and information about students' activities in knowledge acquisition and how the daily work in classroom is usually organized. Since the questionnaire became very large for young learners, we had to avoid the use of multiple questioning. Consequently, the factors of motivation in mathematics were measured only with single items. We acknowledge that this solution reduces the reliability of our findings to some degree, yet previous research has also shown that single items are sufficient to depict a general overview of learners' motivation in mathematics, cf. Tossavainen & Juvonen (2015).

The Likert scales were coded as follows. 1 = "strongly disagree/never/not at all", 2 = "disagree/only seldom/only a little", 3 = "neutral opinion/occasionally/to a certain amount", 4 = "agree/quite often/quite a lot", and 5 = "strongly agree/very often/ very much".

The students were given 45 minutes' time to answer the questionnaire through their iPads. For the younger students (1st and 2nd graders), teachers read the questions out loud and students answered through a scale of smiley faces. Items surveying enjoyment were developed as contextualized counterparts concerning the use of tablet computers. Also, the students' perceptions of self-efficacy were surveyed in the context of mathematics as well as the context of using tablet computers. The scale of task motivation and intrinsic value, in the context of mathematics, was adopted from earlier studies (Nurmi & Aunola, 2005; Aunola, Leskinen & Nurmi, 2006) and it included three items measuring the liking of mathematics in different contexts. Further, two items measuring the liking of studying mathematics with different devices were developed for this study ("How much do you like doing mathematical exercises with iPads?" and "How much do you like doing mathematical exercises with paper and pencil?").

As already said, the participants of this study are students from one school and from the grades 1–8. It is obvious that, due to the large variation in age, it would require splitting the set of participants into two or more subgroups in order to make reliable detailed conclusions. Due to the limited number of pages to use, we restrict ourselves only to producing an overview of the role of tablet computers in motivating students in mathematics and, therefore, we consider the participants as one group, yet taking carefully this limitation into account in interpreting our quantitative results.

In our data, the number of boys is 118 and that of girls is 138.Data were analysed using SPSS software. In addition to applying standard descriptive methods, Student's t-tests and Pearson correlation analysis were performed. In order to avoid confusion in reading our results, we remark that there were some younger students who did not answer all items. Therefore, the degree of freedom may vary between the single items and tables. For example, in Table 1, "N=91–118" for boys means that the number of the boys who answered the four items reported in this table varied from 91 to 118 in the set of these items. To be able to apply Student's t-test for comparing means, the most important thing is that there are, at least, twenty participants contained in each group to be compared. In all items, this condition is clearly satisfied.

Results

We answer our research questions by recording first the descriptive measures for the participating students' intrinsic values related to going to school and using tablet computers in general, and their self-efficacy in mathematics and using tablet computers (Table 1). Then we summarize their liking of studying mathematics at school vs. at home and with vs. without tablet computers (Table 2) and give the descriptive measures describing the participants' views, how useful tablet computers are for their learning of mathematics (Table 3). Lastly, we study the correlation coefficients between the included items (Table 4).

Table 1 shows that the participating students like going to school and they have positive experiences from using tablet computers in studying at school (Items 1–2). For the boys, the mean of the second item is a little higher than that of the first item, yet the difference is not statistically significant ($t(91) = 1.55, p > 0.05$). Since the order of the means of these items is opposite for the girls, and, in Item 2, the mean for the boys is significantly higher than that for the girls ($t(253) = 2.05, p < 0.05$), one may interpret that studying with tablet computers may have a positive effect on the boys' enjoyment of going to school. Yet the effect size (Cohen's d) for the difference in Item 2 is small ($d = 0.26$).

Item	**Mean boys (N=91–118)**	**Mean girls (N=107–137)**	**Total Std. dev.**
1. I like studying at school	3.78	3.86	0.85
2. We have fun at school as we study with iPads	4.08	3.77	1.20
3. I am good at using iPads in studying	4.19	3.88	0.96
4. I am good in mathematics	3.97	3.59	1.15

Table 1: Students' intrinsic values and self-efficacy related to studying, using tablet computers, and mathematics

Similar significant differences are found in the students' view of their competence in using tablet computers ($t(254) = 2.62, p < 0.01, d = 0.33$) and in mathematics ($t(252) = 2.55, p < 0.01, d = 0.33$) in favour of boys. In general, one can conclude that boys are more enthusiastic about using tablet computers in studying at school and they have a stronger sense of self-efficacy in using tablet computers than girls although, in practice, the differences are not large.

The first observation from Table 2 is that the students' liking of mathematics is quite modest. Further, the means for the boys are higher than those for the girls in every item, except Item 5. However, the difference between boys and girls is statistically significant only for Item 8 ($t(250) = 2.17, p < 0.05$). Again, the effect size for this difference is small ($d = 0.27$).

An interesting result in Table 2 is that the difference between the means of girls' answers to Items 8 and 9 is highly significant ($t(134) = 3.36, p < 0.001$). The effect size can now be considered to be moderate ($d = 0.43$). So, the conclusion is that girls clearly prefer studying mathematics with paper and pencil, and for the boys, both ways suit equally well. Another noteworthy observation is that both boys and girls like studying mathematics more at school than at home ($t(251) = 4.97, p < 0.001, d = 0.21$).

Item	**Mean boys (N=117–118)**	**Mean girls (N=135-138)**	**Total Std. dev.**
5. How much do you like studying mathematics?	3.51	3.28	1.24
6. How much do you like doing mathematical exercises at school?	3.38	3.28	1.16
7. How much do you like doing mathematical exercises at home?	3.14	3.04	1.25
8. How much do you like doing mathematical exercises with iPads?	3.32	2.94	1.39
9. How much do you like doing mathematical exercises with paper and pencil?	3.31	3.51	1.34

Table 2: Students' intrinsic values related to studying mathematics

To answer the second research question, we study the descriptive measures given in Table 3.

Item	**Mean boys (N=91–118)**	**Mean girls (N=107–137)**	**Total Std. dev.**
10. iPads help me to learn mathematics easier	2.68	2.28	1.22
11. I have got better grades in exams with help of studying with iPads	3.07	2.74	1.06
12. With iPads I am able to concentrate on school work clearly better than without iPads	3.45	2.99	1.34

Table 3: Students' utility values related to studying mathematics with tablet computers

A somewhat unexpected finding is related to Item 10 in Table 3. Both boys and girls have more negative than positive views of the help that tablet computers provide for their learning of mathematics. The views of girls are significantly more negative than those of boys ($t(172) = 2.28, p < 0.05, d = 0.33$). This result may be partly explained by the results in Item 12. Tablet computers seem to help boys to concentrate on schoolwork better than girls; the difference is significant ($t(253) = 2.78, p < 0.01, d = 0.35$). There is also a significant difference between the means in Item 11 ($t(195) = 2.19, p < 0.05, d = 0.31$), but a more important finding related to Item 11 is that, in the participants' opinion, tablet computers seem to have not helped the students to succeed better in their exams – not only in mathematics but generally in all subjects. To sum up, using tablet computers in studying seems to have increased boys' sense of self-efficacy to a certain degree but, in their experience, this has not implied an improvement in their performance in mathematics. Whether or not tablet computers have provided any support to girls is not as evident. Actually, it appears that girls think that they benefit more in mathematics from studying with paper and pencil than studying with tablet computers.

We complement our answers to the first and third research questions by reporting from the Pearson correlation analysis of Items 1–12, cf. Tables 1–3. In order to maximize readability in Table 4, we only show the significant correlation coefficients with $* = p < 0.05$, $** = p < 0.01$, and $*** = p < 0.001$.

Table 4 contains some interesting relations. First, the correlation between the liking of studying at school (Item 1) and the liking of studying mathematics with paper and pencil (Item 9) is two and half times higher than the correlation between Item 1 and the liking of studying mathematics with tablet computers (Item 8). The liking of mathematics (Item 5) and the sense of self-efficacy in mathematics (Item 4) both correlate highly significantly with Items 8 and 9, but again they are remarkably stronger related to studying mathematics with the traditional working methods than to using tablet computers.

Item	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	0.14*	0.14*	0.37***	0.46***	0.51***	0.48***	0.19**	0.45***			
2		0.25***		0.15*			0.60***	-0.13*	0.48***	0.41***	0.59***
3			0.24***	0.21**	0.26***	0.27***	0.45***		0.24**	0.38***	0.41***
4				0.52***	0.50***	0.42***	0.28***	0.40***			
5					0.82***	0.75***	0.42***	0.59***			0.13*
6						0.79***	0.41***	0.69***			
7							0.40***	0.69***			
8									0.58***	0.49***	0.64***
9									-0.16*		-0.15*
10										0.51***	0.67***
11											0.64***

Table 4: Significant Pearson correlations between Items 1–12

It is not very surprising that the experiences from having fun with tablet computers (Item 2) and the sense of self-efficacy in using tablet computers (Item 3) correlate significantly with Item 8, the experience from having got help from using tablet computers in studying mathematics (Item 10), the views of general success in studying (Item 11), and the amount of help in concentration (Item 12). However, it may be more interesting that these correlations are higher for having fun with tablet computers than for being good at using them.

The correlation coefficients between Items 1, 4, and 8–12 indicate that the better a student performs in mathematics, the more she/he likes studying mathematics both with tablet computers and with paper and pencil. Moreover, to what extent tablet computers have provided support for studying mathematics and other subjects seems to be independent of the self-efficacy in mathematics. Combined with the relative low means in Table 3, these findings suggest that success in mathematics depends more on other factors than on whether mathematics is studied with or without tablet computers, yet boys' experience from having got help in concentration and high correlations between Items 10–12 indicate that tablet computers have some potential for providing support in the engagement in learning.

Discussion and conclusions

The above results give a somewhat mixed view of the potential that tablet computers may have in improving students' motivation in mathematics. In spite of the limitations related to our data, it seems that boys may gain more motivation in mathematics if tablet computers are used (Table 2). On the other hand, it became clear that students do not agree with the claims such as tablet computers have helped them to learn mathematics easier or to succeed better in exams (Table 3).

A possible reason for the latter outcome is that the quality and usability of digital learning material in mathematics for tablet computers are not yet sufficiently high. Tossavainen (2014) surveyed this issue by exploring and analysing a hundred of the most downloaded mathematics applications for iPads in AppStore and found out that more than a half of them are games with a limited mathematical content, more than every fourth of them were tests or static tools (e.g. calculators), and only one application (GeoGebra) contained genuine, non-trivial interactive functions.

The facts that, in the participants' view, tablet computers have not helped them to learn mathematics easier and girls prefer studying mathematics with paper and pencil, may also be due to some technical or pedagogical problems in managing learning environments in which technological devices are used. Genlott's and Grönlund's (2016) study clearly shows that ICT must be integrated reasonably and functionally into the pedagogical solutions in order to benefit from the use of it. Similar observations were also made by Attard and Curry (2012) and Henderson and Yeow (2012). Since students' experiences were investigated in this study only at a general level, an important topic for future research is to examine, how a pedagogic design can support innovative use of technology and students' learning with tablet computers.

Table 2 showed also that both boys and girls like studying mathematics more at school than at home. This result is in accordance with the results in Tossavainen's and Juvonen's (2015) study, where this phenomenon was seen with even a larger effect size than in the present study. It can be interpreted as an evidence for that the availability of a teacher and peer support is important for students' motivation.

As already noted, in this paper, we have analysed primary and secondary students’ data as a whole. Having done differently, we may have got a different kind of perspective to the results. For instance, we know that motivation in mathematics remarkably varies along the grades (e.g., Tossavainen & Juvonen, 2015). Further, interest and performance in mathematics have been found to form a cumulative cycle in the early years of primary school (e.g., Aunola, Leskinen & Nurmi, 2006). We also acknowledge that, in our data, the primary level students have a more thorough experience from studying with tablet computers than the secondary students, who have started their compulsory education without using tablet computers. Concerning future research, there is an obvious need for investigating the potential of tablet computers in supporting students' motivation and learning in mathematics also across different age groups.

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Negotiating mathematical meaning with oneself – snapshots from imaginary dialogues on recurring decimals

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In an imaginary dialogue study with students in grade 6 and preservice teachers, twists in assigning meaning when passing over from proper decimal fractions to recurring decimals are observed. These twists are modeled with regard to the theory of domains of subjective experience by Bauersfeld, particularly helping to explain the phenomenon of changeable impact of having concepts of limit and sequence at disposal on the persisting perception of math learners that $0,\overline{9} < 1$. *We also introduce Wille's instrument of imaginary dialogues in mathematics education, and Tall and Vinner's "concept image/concept definition"-distinction. Furthermore, we discuss our empirical data on the basis of Bauersfeld's framework. We particularly argue as a result, that learners need more explicit instruction and guided analogy regarding issues of properly representing real numbers in different modes and ways. Finally, we draw conclusions regarding consequences of our findings for preservice teacher education.*

Introduction

A frequently reported observation is that a majority of secondary school children and first year students at universities or colleges think that $0,\bar{9}$ is less than—instead of equal to—1 (cf. e.g. Tall, 1977; Tall & Schwarzenberger 1978; Monaghan, 2001; Eisenman, 2008). Different reasons are given in the literature. For example, Tall and Schwarzenberger (1987) argue that students misinterpret the number of decimals of $0,\bar{9}$ as large but finite, that the limit concept is not sufficiently understood, or that a verbal definition of limit suggests a sequence can never reach a limit. Monaghan (2001) outlines differences between the world of mathematics and the real world. Within the latter, one cannot add up infinitely many summands and get a result. Similarly, Eisenman (2001) elaborates on difficulties of changing the perspective from the process of adding on the one hand, and conceiving the limit as an object, on the other hand.

In a study with preservice teachers for mathematics at the Alpen-Adria-University of Klagenfurt, who participated in a university course on standard analysis, and attended a course on didactics of school analysis where different alternative arguments backing the equality $0,\bar{9} = 1$ were discussed, we observed that despite their previous teaching, six out of fourteen preservice teachers argued for $0,\bar{9} < 1$, or reasoned contradictorily or changeably. In this sense, the perception $0,\bar{9} < 1$ appears to be persistent. The research question that will be pursued in this article is: How can the persistence of the perception $0,\bar{9} < 1$ be explained? To this end, we also compare the results with those of a similar study with students in grade 6.

Theoretical framework

Regarding our research question, different theoretical lenses are conceivable. For example, Tall and Vinner (1981) introduce the term concept image for the "total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes", and the term concept definition for "a form of words used to specify that concept" (p. 152). They state that "different stimuli can activate different parts of the concept image, developing in a way which need not make a coherent whole" (ibid.) and speak of potential conflict factors if a part of the concept image or concept definition conflicts "with another part of the concept image or concept definition" (p. 153). In this sense, they describe difficulties with $0,\bar{9}$ as a "typical phenomenon occurring with a strong concept image and a weak concept definition image" (p. 159).

As we are particularly interested in understanding the functioning of such potential conflict factors, and in the impact of different stimuli, we widen the scope of our theoretical framework to include Bauersfeld's theory of so called *domains of subjective experience* (short: DSE) (Bauersfeld, 1985; Fetzer & Tiedemann, 2017) in order to fine-tune and differentiate the "total cognitive structure" of the concept image. In short, a DSE contains the totality of what was experienced and processed in its generating and reactivating situations, in all its perceived complexity, including emotions and haptic and motor perceptions. A DSE is generated essentially on the basis of the *actions* an epistemic subject is conducting on and with certain *objects*, and the individual sense-making process corresponding to those actions, which is navigated by social interaction. It is important to note here that "objects" as constitutive elements of a DSE do not necessarily coincide with what one might call the "mathematical objects" in the background. In particular, a mathematical object like a decimal fraction can be represented in various ways. Different representations can be the objects of different, even isolated DSEs. We will then speak of DSEs with differing objects and actions, though they are objectively dealing with the same "mathematical objects" (whatever "mathematical object" really means).

From the perspective of Bauersfeld, the "total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" is organized into different, initially isolated DSEs.

Learning can be described as building *integrating DSEs* by grasping similarities in differing experiences and constructing analogies navigated by social interaction. An integrating DSE can take over (and may also expand) the task domains of several existing, but isolated DSEs, and allows the learning subject to encounter a wider task spectrum more flexibly.

Hence, viewing concept images with the lens of DSE theory, a learner's concept image of recurring decimals can consist of different DSEs that can be activated by certain stimuli, in particular, actions on certain objects, but also by characteristic contexts, for example, the school context or the university context. A *potential conflict factor* then means, e.g., a stimulus that may systematically trigger the activation of a strongly consolidated, but task-inadequate DSE, and hence inhibits the building or consolidation of a more task-adequate DSE. DSE theory will be used below to explain the persistence of the perception $0,\bar{9} < 1$.

Method

In the studies cited above, participants were first asked whether $0,\bar{9} < 1$ or $0,\bar{9} = 1$ holds and requested to give reasons for their decision afterwards. Regarding this procedure, it appears to be less likely that the reasons given by the participants will uncover the actual line of thought that brought them to make their decision, embracing uncertainties, and perhaps dialectical or even contradictory "inner" argumentations. In our studies, we chose a form of communication that allows for more openness, and gives participants the possibility to ask questions, respond to them, or change a viewpoint while writing. The method of imaginary dialogues meets these requirements. It is a form of mathematical writing where a single student writes a dialogue between two protagonists who discuss a mathematical task or question (Wille, 2008). This form of communication allows the author of an imaginary dialogue to perceive distance as well as closeness to the protagonists.

On the one hand, distance allows the author to write without the pressure of writing something completely correct. Therefore, different thoughts or solution processes can be tried out. Furthermore, the author can put different voices into play. On the other hand, closeness allows the author to let the protagonists express the author's own thoughts and considerations. Thus, the author's own voice can be part of the imaginary dialogue. Though an imaginary dialogue is written, it displays an imagined oral dialogue. It typically includes, e.g., qualifiers like "probable" or "actually" as in spoken dialogues (cf. Wille, 2017) which will potentially be used by the dialogue author as a means to express uncertainties.

Imaginary dialogues serve particularly well with regard to our theoretical framework. As the building and consolidation of either isolated or integrating DSEs is navigated by negotiating meaning in social interaction, imaginary dialogues function as a medium to simulate and display corresponding "inner" or intersubjective processes of negotiating and assigning meaning, and hence, make these processes observable. Assuming that preservice teachers have already developed certain DSEs on fractions and decimals during their own school education period before entering university courses, we compare our findings with the results of a second imaginary dialogue study we conducted with secondary school students at a German Gymnasium in Marburg. We analyzed each imaginary dialogue by a two-step analysis (cf. ibid.). Due to space limitations, we cannot give the concrete details of the analysis in the Findings section. In brief, the general analysis procedure was as follows: In the first step, we assumed the dialogue to be real and used interaction analysis (cf., e.g., Krummheuer & Naujok, 1999) to draw conclusions about the dialogue author's DSEs (for an interaction analysis of a fictional dialogue, namely Plato's Meno dialogue, cf. Struve & Voigt, 1988). In a second step, we additionally considered that it was a student or preservice teacher who wrote what the protagonists "say". In particular, we assume that if both protagonists agree on an issue, it is indeed the author's voice that can be heard, because otherwise the author would give his differing voice to at least one protagonist (cf. Wille 2017, p. 44).

Data sources

The imaginary dialogues that form our data base were written in 2016 from students in grade 6 of a Gymnasium in Marburg, Germany, and preservice teachers who study mathematics at the Alpen-Adria-University in Klagenfurt, Austria. All students and preservice teachers got an initial dialogue which each student or preservice teacher had to continue by him- or herself in written form. All initial dialogues concerned the question if $0,\bar{9} = 1$ holds, and how an explanation for this can be given (cf. figure 1). The imaginary dialogues were written in German and translated by the authors. The initial dialogues for the grade 6 students were created by preservice teachers in a seminar at the Philipps-University of Marburg in summer term 2016. The initial dialogue for preservice teachers at Klagenfurt was created by one of the authors (A. Wille).

Two preservice teachers S1 and S2 are talking to each other. Continue the dialogue.

S1: Can you help me to understand something?

S2: Sure. What is it?

S1: We know from school that 0,99999… is equal to 1. But now, all of a sudden, I do not understand why?

S2: This has something to do with sequences and it is not that simple as you would think at first.

S1: It is not?

S2: No. Wait, I try to explain it to you.

S1: Thanks, but I will often ask in between.

Two students S1 and S2 are talking to each other. Continue the dialogue.

S1: Someone told me recently that $0,\bar{9} < 1$, but I always thought that $0,\bar{9} = 1$. Now, what is right?

S2: Why should it be 1, actually, there is always a missing piece from $0,\bar{9}$ to 1. It would only work by rounding.

S1: But when you regard $0,\bar{9}$ and 1, no number fits in between. So it must be equal.

S2: Ok, this sounds logical, too. But maybe we can show it differently so we can both agree.

S1: Good idea! Probably something from one of the last maths lessons can help us. For example, lets consider…

Figure 1: Initial dialogues that were given to preservice teachers (left) and to the grade 6 students (right)

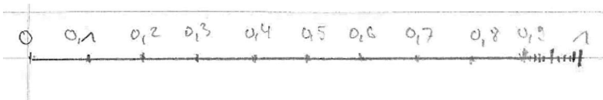
Findings

In the imaginary dialogues of the secondary school students Nick, Pia, and Melanie, and the preservice teachers Leonard, Peter, and Anna, we can observe the activation of DSEs on proper decimal fractions by certain objects, negotiation processes of possible analogies between proper and recurring decimal fraction as first attempts to build integrating DSEs, and the activation of different, isolated DSE shaping one and the same imaginary dialogue.

Different objects activate DSE of proper decimal fractions

In Nick's interchange, we can see how the action of *locating on the number line* comes into play. He writes and draws:

S1: (...) let' s draw a number line, thus:



Look, also at the number line, always one little piece is missing so it must be smaller, namely if in the decimal system one number has 0 in the front and the other 1, the number with 1 must be bigger, no matter what comes after 0.

Nick draws 0,9 and 0,99 on the number line, then stops and lets S1 conclude that "always one little piece is missing". The representation on the number line appears to activate a proper decimal fraction DSE, with regard to which of the concrete perception of the "pieces", the proper parts of the line between the 0,9- and 0,99-strokes and the 1-stroke, is generalized to: "always one little piece is missing". S1 infers that $0,\bar{9}$ has to be smaller than 1 (later in the dialogue, S2 reinterprets the "missing piece" as just a "Tick", so small that it cannot be described by a number). Then he changes from the number line representation to the symbolic one, dealing with $0,\bar{9}$ as if it was represented in the decimal system explicitly (which it is not, due to the overbar). Now he disregards the decimal places when ordering $0,\bar{9}$ and 1 and refers to a general rule which he seems to accept as valid in his proper decimal fraction DSE. It appears that the notational element "0,", which is a familiar element from the proper decimal fraction DSE, plays the role of an activating stimulus here.

Traces of negotiating possible analogies

The negotiation of possible analogies between the comparably consolidated DSE on proper decimal fractions and the recently developing DSE on recurring decimals by the students can be regarded as an attempt towards building an integrating DSE. We observe such negotiation processes in the dialogues of the students Pia and Melanie. Pia describes an iterative process of drawing number lines:

S1: How do you want to do it? It is a recurring decimal after all! Thus, infinitely long.

S2: You are right. I simply make several number lines.

S1: I do not know what you mean...

S2: Very simple: First you make a number line from 0 to 1. There you draw the 0,9. Then you make a second number line. But this time from 0,9 to 1 and you draw 0,99 into it. So it goes on and still the $0,\bar{9}$ never reaches the 1. It just infinitely goes on.

S1: Now, I understand.

Pia uses the number line to illustrate a procedural interpretation of the sequence 0,9; 0,99; … But this does not lead her to identify a limit. We can interpret this as an attempt to find an analog for the action of locating proper decimals on the number line within the developing DSE on recurring decimals.

Amanda's imaginary dialogue displays a negotiation process regarding an adequate analog for the criterion that a proper decimal fraction, *a* is greater than another *b* if and only if there is a number that can be added to *b* in order to reach *a*. Although Amanda concludes that there is no number (probably she omits "unequal 0") that can be added to $0,\bar{9}$ to yield 1, she states the inequality of $0,\bar{9}$ and 1:

S1: (...) 0,999... and 1 are indeed not equal, but still, it fits nothing in between. There is namely no number that you can add to $0,\bar{9}$ in order to reach 1.

Melanie finds and uses analogies to conclude the equality of $0,\bar{9}$ and 1 by analogizing operations with natural numbers, proper decimal fractions and ordinary fractions and the operation operating with recurring decimals. She writes:

S1: Let's take a number that fits well into 9, without rest. Like, for example, 3 ($3 \cdot 3 = 9$).

S2: Exactly, and if it is 3 for 9, it is 0,333… for 0,999...

S1: As a fraction, 0,333... would be $\frac{1}{3}$. Because 1 whole (in German: "1 Ganzes") is $\frac{3}{3}$, $\frac{2}{3}$ are missing. Thus, it would be $3 \cdot 0{,}333\ldots$ and that is 0,999...

S2: From 0,333... to 0,999..., 0,666... are missing. It is just the same with fractions.

S1: And because we have a similar calculation, probably ("wohl" in the original) $0,\bar{9}$ and 1 are equal.

The first analogy is between multiplicative decompositions of 9 and 0,999…, the second between additive decompositions of $\frac{3}{3}$ and 0,999…. Altogether, Melanie is not completely sure about her conclusion of the equality of $0,\bar{9}$ and 1, which can be read off the German word “wohl” (translated here with “probably”) that weakens S1’s statement.

Preservice teachers activate different or isolated DSEs for school and university mathematics

Peter’s imaginary dialogue employs changeable and ambiguous argumentation. First, it is argued that a little piece is missing to 1 and concluded that $0,\bar{9} = 1$ is wrong. But the conclusion is then challenged again: S1 asks: “So, $0,\bar{9} \neq 1$?”, and S2 answers: “Yes and no”, and explains how 1 is reached at infinity. We interpret the utterance “yes and no” as a sign that the dialogue author has already developed an integrating DSE for the issue of $0,\bar{9} = 1$ in school mathematics context and in university mathematics context. At the same time, we suppose that this DSE may be rather poorly developed, in the sense that it doesn’t allow Peter to resolve the contradictory conclusions his protagonists draw.

Leonard’s dialogue starts with an argumentation on why $0,\bar{9} < 1$ which is quite similar to the arguments displayed in the imaginary dialogues of the secondary students that were reported above. He concludes: “I realize that $0,\bar{9}$ can never take the value 1, but it can approximate 1 arbitrarily, because $0,\bar{9}$ consists of ∞- many 9-s.” This seems to express the idea that $0,\bar{9}$, being something that can “approximate 1 arbitrarily”, is not a fixed number at all, but rather something that can *move towards* 1. Then, Leonard appears to switch to an isolated DSE for the concept of limit and writes: “I can understand $0,\bar{9}$ as a limit and additionally 1 as a limit. Thus, it is $1 = 0,\bar{9}$.” In his dialogue, we find no traces of a negotiation process on these two contradicting conclusions. The contradiction between $0,\bar{9}$ being equal and smaller than 1 at the same time is not even mentioned. We interpret this as a sign for two isolated DSEs that are activated for the issue of $0,\bar{9} = 1$ in school mathematics context and in university mathematics context.

As well as the preservice teachers who conclude the inequality or switch like Leonard and Peter, there are also students who conclude equality. For example, Anna writes: “And therefore, we have $0,\bar{9} = 1$. And it is the same!” The additional sentence “And it is the same!” reveal that even after deducing equality by argumentation, Anna has the need to reinforce that conclusion.

Compared to the grade 6 students’ dialogues, we infer from these observations that for preservice teacher students, the perception $0,\bar{9} < 1$ tends to be *persistent*. Moreover, the persistence also affects the way they attempt to explain and give meaning to this perceived inequality, despite their university mathematics knowledge on sequences, limits and series.

Discussion

In the findings, it is exemplified how actions on certain objects and representations that occur in dealing with recurring decimals activate DSEs of proper decimal fractions, and how they are used in processes of transferring and negotiating meaning. In particular, the symbolic notation of recurring decimals resembles those of proper decimal fractions. Moreover, the action of locating on the number line, combined with a procedural reading of $0,\bar{9}$ as an approximating sequence 0,9, 0,99, 0,999, …, persistently triggers the perception of a proper segment of the number line that is always between 0,9…9 and 1, and that will never vanish completely. Referring to Bauersfeld's theoretical conception of DSEs with objects and actions as core constitutive components, these considerations may have explanatory power with regard to the perception $0,\bar{9} < 1$ itself, and also with regard to its observed persistence, even after undergoing university mathematics education. Such considerations also bring up the question in how far the emphasis on the number line representation, with nested intervals as a method to locate infinite decimal fractions, needs more explicit instruction and guided analogy to other kinds of conceptualizations and representations of real numbers to form an appropriate basis for further understanding.

It is also exemplified how imaginary dialogues can display inner negotiation processes (and corresponding obstacles) of assigning meaning to a new domain of experience by analogy, which is regarded as an initial step in building integrating DSEs due to Bauersfeld. This highlights the usefulness of the method of imaginary dialogues for diagnosis, but also as a means for planning accurate teaching interventions regarding the individual processes of constructing meaning. In the case of the preservice teachers, we observed that both ways of thinking, $0,\bar{9} < 1$ and $0,\bar{9} = 1$, can exist in parallel, indicating isolated DSEs on decimal fractions. This points to another possible explanatory pathway with regard to the observed persistence of the perception $0,\bar{9} < 1$ within the theory of DSEs: A frequently activated DSE becomes increasingly consolidated and isolated. It is reasonable to assume that preservice teachers customarily built a strongly consolidated DSE on proper decimal fractions during their own school education period. Moreover, a DSE contains the totality of what was experienced and assimilated in its generating and reactivating situations. Hence, the context of school mathematics (evoked, e.g., by the mere representation "$0,\bar{9} = 1$" itself) may intrinsically activate an isolated school mathematics DSE for dealing with the issue of $0,\bar{9} = 1$, which would hint at a lack of an appropriate integrating DSE that covers both the school mathematics and the university mathematics point of view.

What are possible consequences for preservice teacher mathematics education? Our considerations suggest that it is not sufficient, e.g., to merely teach preservice teachers university proofs of $0,\bar{9} = 1$ by means of the "toolbox" of university mathematics in addition to their school knowledge. It might happen that they are perfectly able to manage such a proof, but at the same time argue for $0,\bar{9} < 1$ by means of school mathematics without even mentioning the tension. We might suspect that this is not an appropriate basis for a sophisticated teaching on real numbers in school. Hence, university education of preservice mathematics teachers should foster the building and sufficient consolidation of appropriate integrating DSEs. This has to be addressed on the object-level of the relevant mathematics itself but can also be backed up on a meta-level. Imaginary dialogues may provide a valuable tool to both ends, by reflecting on self- or peer-written dialogues as well as on dialogues written by secondary school students.

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Supporting students' mathematical problem solving: The key role of different forms of checking as part of a self-scaffolding mechanism

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It is widely accepted that with appropriate scaffolding students are able to overcome the obstacles they face when solving mathematical problems. In this paper we describe the development and implementation of an orientation basis (OB), a device for self-scaffolding Catalan first year secondary students' mathematical problem solving. The OB comprises twelve problem solving-related actions derived from the literature and earlier classroom observations. Three unfamiliar and non-routine problems, spread over 3 months, were posed to students alongside instructions for the use of the OB. Analyses of their responses, to both the tasks and the OB, indicate that a necessary but not sufficient condition for solving tasks is the completion of seven or more OB actions. In particular, the quality of two actions connected to the checking of different parts of the problem was seen as crucial in determining a student's success.

Introduction

When children solve a problem without being explicitly conscious of the relationship between their actions and their solution their ability to transfer their solution process to new situations will be limited (Coltman, Petyaeva & Anghileri, 2002). However, appropriate adult intervention can help children become aware not only of the obtained solution but also the processes which led to it (Coltman et al., 2002). This adult, as an expert intervention is known as scaffolding and it aims to support learners complete tasks not otherwise possible. It builds on what learners already know in order to close the gap between current learner competence and task objective (Bruner, 1985; Wood, Bruner & Ross, 1976). Further, it can also be provided reciprocally by peers and, ultimately, students themselves (Holton & Clarke, 2006). In this paper we describe the implementation of a device for problem solving use in Catalan first year secondary students and discuss the outcomes of a first longitudinal analysis of its use. Called an orientation basis (OB), its role is to support the transition towards students being able to scaffold their own mathematical problem solving actions.

Problem solving

Problem solving is a dynamic and not necessarily linear activity requiring the organization and activation of multiple strategies and skills (Mason, Stacey & Burton, 1982; Pólya, 1945). Therefore, it can be considered as an example of a goal-directed human activity (Schoenfeld, 2013) that entails an appropriate mathematical knowledge, an awareness and experience of solution strategies, self-regulatory or metacognitive competence and beliefs especially regarding not only that the problem is worth solving but also that the solver can solve the problem (De Corte, Verschaffel & Op'tEynde, 2000; Schoenfeld, 2013). Evidence shows that expert solvers continuously reflect on the state of the problem solving process and spend more time understanding and analyzing the problem and solution process than calculating, behaviours typically absent with weak problem solvers (De Corte et al., 2000). This regulative competence, which includes reflecting on existing knowledge and thought processes (Sanmartí, 2007), can be learnt with appropriate support (Schoenfeld, 2013). In other words, students need scaffolded support with respect to interpreting a task, identifying its sub-objectives and planning a strategy (De Corte et al., 2000; Mason et al., 1982).

Scaffolding

Drawing on Bruner's (1975) observations with respect to how parents scaffold their infants' learning, Wood, Bruner and Ross (1976) argued that knowledgeable adults can scaffold students' problem solving activity. Here, the adult seeks to reconcile implicit theories of the task components, the necessary steps to solution, and the child's capabilities (Stone, 1998). It is a socially imitative process comprising six forms of assistance; recruiting the child's interest, reducing the degrees of freedom, maintaining goal direction, highlighting critical task features, controlling frustration and modelling preferred solutions paths (Wood et al., 1976). Through this process, whereby teacher and learner actively build a common understanding (Stone, 1998; van de Pol, Volman & Beishuizen, 2010), learners become incrementally independent (Smit et al., 2013). Indeed, the role of tutor (or scaffolding agent) can be devolved from the teacher to the learners as the learners scaffold their own learning –self-scaffolding– or other learners learning –reciprocal-scaffolding– (Holton et al., 2006). However, much remains unknown with respect to scaffolding's processes and effectiveness (van de Pol et al., 2010).

As in the construction industry, where each scaffold is unique to a specific building, learning scaffolding can be provided at different ages and in a variety of ways, addressing learners' knowledge gaps as part of an ongoing progress (Wood et al., 1976). Hence, scaffolding is not a 'technique' that can be applied in every situation in the same way (van de Pol et al., 2010). However, effective scaffolding is thought to comprise three components (van de Pol et al., 2010):

- Contingency: Support should be adapted to the student's current level of performance.
- Fading: Support is gradually withdrawn over time.
- Transfer of responsibility: Task completion is gradually transferred to the learner.

Orientation Basis for Problem Solving

One means of scaffolding students' problem solving-related self-monitoring skills is to use an orientation basis (OB) (Sanmartí, 2007). We conceive a problem solving-related OB as the necessary sequence of actions based on the problem solving behaviour of experts that leads the learner to a solution in ways that structure an emergent independence and problem solving autonomy. The OB is not a 'one size fits all' tool but tailored according to learners' requirements and achievements. Then, at every age and according to the learner's needs, the OB should be presented through different statements. In this paper we present a first longitudinal analysis of the efficacy of the OB shown in Table 1 for scaffolding first year secondary students' problem solving. By the start of secondary school, Catalan pupils are typically expected to have acquired a minimum background in problem solving. However, experience has shown that they lack regulative and problem solving competence, especially in understanding and analyzing the problem, and planning and implementing a solution process. The OB depicted in Table 1, translated from the original Catalan, was designed to be a contingent, hint-giving, feedback tool focused on facilitating both fading and transfer of responsibility (van de Pol et al., 2010). It is structured by Pólya's (1945) four principles, each addressed through three actions, to be tracked in the right hand column. Each action derived from earlier observations of the problem solving behaviours of Catalan pupils and the problem solving strategies found in the literature (e.g. De Corte et al., 2000; Mason et al., 1982).

Dimensions	Actions	Track
I understand the problem	A1. I have read the question twice, at least.	
	A2. I understand what the question wants.	
	A3. I have identified and understood the data.	
I devise a plan	A4. I have played with the data from the question.	
	A5. I have prepared a strategy.	
	A6. I have checked that my strategy fits the data.	
I apply my plan	A7. I have implemented my strategy.	
	A8. I have recorded all my actions in ways that I understand.	
	A9. I have recorded all my actions in ways others can understand.	
I review my task	A10. When I get stuck I go back to the beginning.	
	A11. When I have finished I have checked my answer(s).	
	A12. I have checked for other answers or better solutions.	

Table 1: The orientation basis (OB)

The study

Participants were 12-13 years old students in a first year of a Catalan secondary school. They solved three mathematical problems P1, P2 and P3, shown respectively in Figures 1, 2 and 3, at different points during the period March to June of the academic year 2015-2016. The problems, each comprising two parts, were posed during regular lessons and students invited to use the OB depicted in Table 1. During other lessons in this period, students did not work on other atypical mathematical problems or use the OB. During the first occasion of its use, the teacher explained the purpose of the OB and together with the class discussed and clarified the meaning and purpose of each element. This ensured, as far as is practicable, that students understood its vocabulary and overall purpose.

Problem: Multiply two two-digit numbers

With the digits 5, 6, 7 and 9 we can make two two-digit numbers with non repeated digits. For example: 67 and 59. We multiply these two numbers.

1. For which two numbers do we get the largest product? Explain how you found it out.
2. For which two numbers do we get the smallest product? Explain how you found it out.

Figure 1: Problem posed during the first session, in March 2016 (P1)

Students were each given a copy of the OB's rubric, which included a grid in which they should record their engagement with the OB as well as a paper copy of the problem on which they had to write their own solution. They were instructed to solve the problem, using the OB to guide their activity, and record the OB actions they addressed. They were also told that their teacher would not intervene in the problem solving process but check, as they worked, that they completed their OB tracking. In the following, we compare and contrast students' responses to three problems and their OB use in order to address the question; what can be inferred from students' use of the OB with respect to their development as mathematical problem solvers?

Problem: The USB price

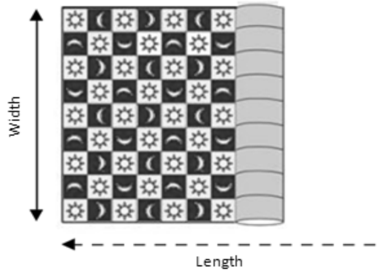
Agnes and Jan want to buy the same 32 Gb USB, but none of them have enough money to acquire it for themselves. Agnes is eleven euro short and Jan lacks one euro.
If they combine their money to buy one of these USBs and share it, they still have enough money to buy it.

1. Is it possible to know how much one of this USB is/costs? Argue why and, in affirmative case, find its price out and explain how you did it.
2. So, do you know how many euro Agnes has got? And Jan? Explain why you know it and argue why they have these quantities and no other ones.

Figure 2: Problem posed during the second session, in April 2016 (P2)

Problem: The Sun-&-Moon big carpet

Alex bought a big carpet for the meeting room of his office. It is 6 m long and 3,6 m wide. The image shows part of the enrolled carpet. As you can see, it is made by small squares containing the drawing of a sun or a moon.



1. How many squares do you see if the carpet is totally unfolded? Explain how you found it out.
2. Is it possible to know how many of these squares will contain a sun and how many a moon? Argue why it is possible to know it and, in affirmative case, explain how you found it out.

Figure 3: Problem posed during the third session, in June (P3)

Results

Table 2 shows a selection of students' use of the OB and summary data with respect to their completion of each of the three tasks respectively. Just two OB actions have been included as analyses, shown later, indicated that A6 and A11, both concerned with the checking of different aspects of the solution, proved significant in determining later success. With respect to A6 and A11, a mark of • indicates that students completed the check. The table shows the number (OB) of OB actions students completed and the number (CS) of correct solutions for each problem, 0, 1 or 2. So, for example, it can be seen that student 4 solved both parts of problem 3 correctly and undertook all 12 OB actions, including both A6 and A11. However, the same student failed to solve either part of problem 1 and undertook only 4 OB actions, none of which were A6 or A11.

The figures of Table 2 show 13 fully correct solutions, 15 part correct solutions and 44 failures. Not one student completed all three problems successfully, with the 28 full or half solutions being distributed across 17 students. Further, every student who attempted all three problems, with just two exceptions, failed completely on at least one of them. Also, with just two exceptions (see students 4 and 7), not one student's OB-related actions increased over time, with most students showing considerable fluctuation in their OB use. Finally, where students solved the two parts to a problem successfully those students always completed seven or more OB actions.

		Students																											
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
P1	A6	•	•	•		•	•	•	•		Absent	•	•	•	Absent	•	•	Absent			•	•			•			•	•
	A11	•	•	•			•	•	•	•		•	•			•	•		•	•	•	•		•	•	•		•	
	OB	9	10	9	4	5	8	7	8	9		8	10	8		8	11		8	9	9	10	3	5	11	6	4	11	8
	CS	1	0	0	0	0	0	2	0	1		1	2	2		0	2		0	0	1	1	0	1	1	1	0	1	0
P2	A6	•	•		•		•	•						•	Absent		•			•	Absent	•			•			•	
	A11	•	•		•		•	•	•	•			•	•			•	•		•		•		•	•			•	
	OB	8	9	4	10	3	7	8	6	5	8	0	9	10		3	10	9	6	11		10	4	7	11	4	2	11	0
	CS	0	0	0	0	0	0	0	0	0	2	0	0	1		0	2	0	0	2		2	0	2	2	0	0	0	0
P3	A6	•	•	•	•	Absent	Absent	•	Absent				Absent	Absent	•		•				•	•					Absent	Absent	
	A11	•	•	•	•			•							•	•	•		•	•	•	•		•					•
	OB	8	12	7	12			9		4	7	5			8	9	12	6	6	9	10	10	0	6	0	3			7
	CS	0	0	0	2			2		0	1	0			0	0	1	0	1	0	1	2	0	0	0	1			0

Table 2: Students' problem engagement and OB-related data for each problem

As indicated earlier, informal analyses suggested that the two OB checking actions, A6 and A11, appeared to be indicators of problem solving success. In this respect, the figures of Table 3 show the relationship between the number of correct solutions and students' use of these two actions for each problem individually and summatively. A Fisher exact probability test run on the summative data suggests that the frequency distribution is unlikely to be due to chance. Indeed, it can be seen quite clearly that students who complete the OB checking actions are more likely to solve the problems than students who do not. That being said, there remain 15 occasions where students completed both checking actions and still failed completely.

		P1			P2			P3			**All problems**		
		Number of A6/A11 actions											
		0	1	2	0	1	2	0	1	2	**0**	**1**	**2**
Number of correct solutions	0	3	4	5	9	4	6	5	4	4	**17**	**12**	**15**
	1	0	3	6	0	0	1	2	1	2	**2**	**4**	**9**
	2	0	1	3	1	1	4	0	0	3	**1**	**2**	**10**
Fisher Exact Probability											**p = 0.045**		

Table 3: Students' OB-checking actions (A6 and A11) and achievement

However, as shown in Table 2, the majority of problem attempts, 44, ended in failure, a result that led us to investigate in more detail their characteristics. The figures of Table 4 show a comparison between the number of OB actions completed and whether or not the students concerned had completed, albeit incorrectly, a solution attempt. The figures show that students who completed six of fewer OB actions were nine times as likely as those who completed seven or more OB actions not to complete a solution attempt. The figures also show that students who completed seven or more OB actions almost always completed their attempted solution. In short, even when their solutions were incorrect, students who completed seven or more OB actions were considerably more likely to complete a solution attempt than students who did not.

		P1		P2		P3		**All**	
		F (Finished attempt)/U (Unfinished attempt)							
		F	U	F	U	F	U	**F**	**U**
OB-actions	0–6	4	0	6	5	2	4	**12**	**9**
	7–12	8	0	8	0	6	1	**22**	**1**
							Fisher Exact Probability	**p = 0.003**	

Table 4: Failing students' problem completeness

The figures of Table 5 show, also for the 44 failures, a comparison between students' OB actions and the extent to which they engaged with the checking actions, A6 and A11. While the results for the individual problems hint at a fairly strong relationship, the figures for the total show very clearly, at a level of probability that effectively presents chance as an impossibility, that even when their solutions are incorrect, students who complete the checking actions, A6 and A11 are highly likely to complete the majority of the OB's actions. Alternatively, students who do complete these two actions almost never complete more than six of the OB's actions. In other words, the two checking actions of the OB, A6 and A11, seem strong predictors of a student's broader engagement with the orientation basis, even for those students who fail on a problem.

		P1			P2			P3			**All**		
		Number of A6/A11 actions											
		0	1	2	0	1	2	0	1	2	**0**	**1**	**2**
OB-actions	0–6	3	1	0	9	2	0	5	1	0	**17**	**4**	**0**
	7–12	0	3	5	0	2	6	0	3	4	**0**	**8**	**15**
								Fisher Exact Probability				**$p < 10^{-9}$**	

Table 5: Failing students' OB-checking actions (A6 and A11) engagement

Importantly, a qualitative analysis of the responses of those students in the second row of Table 5, those who failed to provide correct solutions to a problem but completed seven or more OB actions, revealed that their engagement with the OB checking actions, A6 and A11, was shallow. For example, Figures 4 and Figure 5 show the answers of student 3 to the two parts of P1. For the first part he wrote, "9×7=63, because 9 and 7 are the biggest" and for the second, "5×6=30, because 5 and 6 are the smallest". While both multiplications and his reasoning are correct, he has clearly misunderstood the task, which requires him to produce a multiplication involving two two-digit numbers. Our argument is that had he undertaken either A6 or A11 more deeply, he may have realised the extent to which he had misinterpreted the tasks given him.

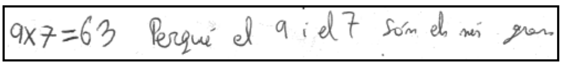
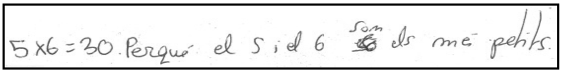


Figure 4: Student 3 solution to the first P1 question



Figures 5: Student 3 solution to the second P1 question

Discussion

In this paper we have reported on the trial of an orientation basis, designed to support first year secondary students' problem solving-related self-scaffolding. Our initial goal was focused on longitudinal changes in students' behaviour, both in terms of their problem solving competence and their use of the OB. However, the results were not simple to interpret. Most students' OB use fluctuated from one problem to another with only two students showing an increasing usage. All others, apart from student 21, who consistently addressed ten OB actions, appeared to exploit the OB randomly, alluding to at least four possibilities related to the impact of prior problem solving practices (Schoenfeld, 2013). Firstly, different students respond differently to different problem types, secondly, the tasks were at the very edge of students' problem solving competence and, thirdly, in a related manner, most students remained unsophisticated problem solvers throughout the intervention. This latter issue, acknowledging the typicality of the study's students, is unsurprising in light of research that high achieving students may need as long as two weeks to solve a problem in order for them to become competent, confident and independent problem solvers (Sriraman, 2003). Fourthly, three problems may be too few for students to have internalised the OB as a means of scaffolding their problem solving activity. Such matters will inform our work in the future.

The remaining analyses represented a diversion from our original goal, prompted by an emergent awareness that two of the OB's actions, 'I have checked that my strategy fits the data' and 'When I have finished I have checked my answer' appeared to have a greater predictive impact than the others. In particular, by focusing on students who failed to obtain correct solutions, the importance of checking emerged as an indicator for both the number of OB actions addressed and the likelihood that students would complete the problem, albeit incorrectly, confirming earlier research concerning the importance of taking time to read and interpret a problem before planning a solution strategy (De Corte, et al., 2000). However, students' written arguments, as exemplified in the comments of student 3, indicated a broad failure to interpret tasks correctly, findings that resonate with a Dutch study, also of high achieving students, that found that students who had checked their interpretation of task expectations continued to misinterpret them (Elia, van den Heuvel-Panhuizen & Kolovou, 2009). Consequently, future work will focus on ways of encouraging students to check more effectively both their interpretation of task expectations and their results.

So, has the orientation basis supported the development of students self-scaffolding behaviours? Well, the impact of any form of scaffolding is difficult to evaluate (van de Pol et al., 2010) and may in part be dependent on the nature of the tasks presented to learners. Actually, we are not able to give a concluding answer to this question, but we feel we can offer a tentative, yes, not least because the data show that a necessary but not sufficient condition for a fully correct solution is the completion of seven or more OB actions. More significantly, the data also show, for successful and unsuccessful students, that the chances of completing a task correctly are enhanced if students address the two OB checking actions. However, as has been discussed above, the inadequacy of many students' checking behaviours is an issue that will influence future project activity. Finally, two of the three problems posed to students were presented in context. However, unlike the findings of others' studies (Coltman et al., 2002), students' responses to these were not discernibly different from their responses to the single decontextualised problem. This, too, along with the aim of introducing the OB to younger children as a non sporadic device will inform future project work.

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Mixed notation and mathematical writing in Danish upper secondary school

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The paper points to the emergence of the phenomenon "mixed notation" as a result of the use of Computer Algebra Systems (CAS) in upper secondary education. We provide an illustrative example or "existence proof" of mixed notation consisting in a student's written answer to an exam question from the Danish national mathematics exam as an instance of mathematical writing. Based on a qualitative analysis of the student's written answer, we discuss how mixed notation might emerge as a result of the technological context and the classroom culture. We argue that mixed notation calls for awareness in regard to how the ongoing transformation of written mathematical activities, as a result of using CAS, influence students' mathematical learning and identity work.

Introduction

The increased use of digital media for students' mathematical writing does influence their mathematical work. One example is that students now hand in various printouts and computer files rather than handwritten assignments. This is also the case at the national exams in Danish upper secondary education. These changes in the materiality of mathematical writing contribute to a change in the specific notations and diagrams used by the students to signify mathematical objects and processes. However, such a change might not be as superfluous and innocent as one might think. Several theoretical and empirical contributions suggest that there is a complex interplay between notation and other representations and the cognitive processes, both in general and in relation to upper secondary mathematics (e.g. Kieran & Drijvers, 2006). One reason is that the use of different media for writing in mathematics might amplify and/or reduce the use of specific semiotic resources in students' responses, and that such semiotic resources are associated with different cognitive processes (Duval, 2006; Mariotti, 2002).

In Danish upper secondary education there is a growing proportion of students who use Computer Algebra Systems (CAS) as a medium for writing in mathematics. Since CAS have a slightly different mathematical notation, and strong interactive abilities (Lagrange, 2005), including the capability to black-box certain mathematical processes (e.g. Nabb, 2010), it is worth focusing on how CAS-related notation affects mathematics learning. Currently it seems likely that CAS eventually take over as the common mathematical medium in Danish upper secondary education, which has a number of potential problematic consequences (Artigue, 2002; Jankvist & Misfeldt, 2015; Trouche, 2015). In this paper we focus specifically on the influence on notation, discourse, learning and identity. Knowledge about such influence of CAS on notation and learning is important – and even more so, since we know from research on literacy that a change in medium and language "transmitting" knowledge will affect other dimensions of learning, education, and competence. As an example, Kolstø (2010) and Vollmer (2009) point out the importance of learning a subject's subject-specific-language as being closely related to learning the specific subjects' certain ways of thinking and doing. This point is also made in relation to mathematical discourse and learning (Darragh, 2016; O'Halloran, 2005 Sfard, 2008; Sfard & Prusak, 2005; Steentoft & Valero, 2009). Hence, a change in "notation" can have a deep influence on students' mathematical work. Of course, mathematical notation has always been in some sort of flux, and in that sense the CAS influence on notation is nothing but a natural continuation of such changes. However, the way CAS are used in education in general (and in Danish upper secondary school in particular) might have a specific influence that we find it useful to investigate.

In order to focus on this influence, we will augment the "instrumental approach" literature on CAS in mathematics education with a lens from literacy studies and focus on how CAS influence students' identity work. Our ambition with the present paper is to show how classical algebraic notation and CAS-related notation is entangled by students in upper secondary education and affect their identity work. We present this "mixed notation" as a phenomenon of relevance to us as mathematics educators and present an illustrative case of one student's mathematical writing in order to aim at a first characterization of the phenomenon. Hence, this paper is not to be viewed as a traditional empirical research study, but rather as a theoretical piece providing an "existence proof" and characterization of an observed phenomenon, and furthermore showing how the literature on CAS in mathematics education may start to consider how change in practice and discourse may affect students' identity work, which is an underdeveloped aspect of this literature.

Theoretical framework

As an outset for looking at mixed notation we will use the distinction between *epistemic* and *pragmatic mediations* (Artigue, 2002; Lagrange 2005; Trouche, 2005), and augment it with a consideration of students' identity work (Iversen, Misfeldt & Jankvist, in review). An epistemic mediation is directed towards the user's cognitive system; the tool is used to create a different understanding or to support learning. For example, Lagrange (2005) refers to experimental uses of computers, e.g. in relation to students' mathematical concept formation. In contrast, a pragmatic mediation is directed towards something external to the user; the tool is used to create a difference in the external world. Lagrange (2005) refers to the mathematical technique of "pushing buttons". We augment this understanding of the roles of technology with the concept of identity – or what we will argue that we meaningfully can refer to as *identity directed mediation*.

Our focus on identity is based on socio-cultural perspectives of teaching-learning processes. Ivanič (2006) argues that students' learning is closely linked to processes of identification, meaning the extent to which students identify with the values, beliefs, goals, and activities that prototypical participants in the learning activities represent. The view that identification is an important factor in learning is shared by a number of scholars (e.g. Gee, 2001). In the words of Hyland (2009, p. 70), "identity is something we do; not something we have." All of us do identity all the time, and this doing has been coined as *identity work* by Gee (2003). In this way, identity can be understood as negotiated ways of participating in different social groups, cultures and institutions, and of course identity work is *mediated* by the tools, technologies and representational systems at hand. Hence, we apply a theoretical lens based in the same sociocultural outset as the instrumental approach, but we include identity work as a third type of mediation (in addition to those of epistemic and pragmatic).

The basic insight from the instrumental approach is that there is a dialectics between artefact and individual, when adopting artefacts as tools for work. In relation to identity this means that students, on the one hand, are expected to use artefacts (digital tools, forms of notation, etc.) to perform identity work in ways not foreseen by teachers and technology developers. And, on the other hand, that these artefacts change and affect the students' identity work.

Presenting an illustrative case of identity work with mixed notation

The following illustrative case is taken from a longitudinal study of students' mathematical writing in the subject of mathematics. This field study took place over a two-year period (2011-2013) and consisted of several studies from different types of Danish upper secondary education. One of the findings of the study was that CAS are increasingly used as a medium for students' mathematical writing (Iversen, 2014). We present an excerpt taken from the student Anna's reply to a written examination in mathematics (see Figure 1). Anna's answer serves as an example of key differences between classical algebraic notation and CAS notation as well as an example of their entanglement. Notice how the central formula used in the solution of the task is written up in two different ways (line 4 and line 6 in the excerpt of Figure 1).

b. I denne opgave skal jeg bestemme koordinatsættet til projektionen af $\overrightarrow{AB}$ på $\vec{a}$.

Jeg benytter følgende formel:

$$\overrightarrow{AB_a} = \frac{\mathbf{Ab} \cdot \mathbf{a}}{(|\mathbf{a}|)^2} \cdot \mathbf{a}$$

Jeg beregner vha. Nspire:

$$\frac{\text{dotP}(\mathbf{Ab},\mathbf{a})}{(\text{norm}(\mathbf{a}))^2} \cdot \mathbf{a} \blacktriangleright \begin{bmatrix} -5 \\ 10 \end{bmatrix}$$

Altså er koordinatsættet til projektionen af $\overrightarrow{AB}$ på $\vec{a}$ lig $(-5,10)$

Figure 1: Excerpt of Anna's written answer to a sub-question (7.b) in a written exam. The text reads: (line 1-2) "b. In this task I am to decide the coordinates to the projection of $\overrightarrow{AB}$ on $\vec{a}$." (line 3) "I use the following formula:" (line 5) "I calculate using Nspire:" (line 7) "Hence, the coordinates to the projection of $\overrightarrow{AB}$ on $\vec{a}$ equals (-5,10)**"**

In the first two lines Anna paraphrases the formulation of the task. In line 3 she indicates the mathematical formula that she is going to use to solve the task. The formula is firstly written with algebraic notation as it typically occurs in textbooks and lecture notes as well as in formula tables and task formulations. A deviation from this is that Anna indicates the vectors included in the formula (to the right of the equal sign) in bold (**Ab** and **a**). As seen in the first two lines, i.e. the paraphrasing of the exam text by the Danish Ministry of Education, vectors are conventionally written using a notation of small horizontal arrows above letters, $\overrightarrow{AB}$ and $\vec{a}$.

We cannot know why Anna uses the notation of putting vectors in bold. Typically, this is not taught in Danish upper secondary school, and to the best of our knowledge her teacher did not introduce this notation. Putting vectors in bold is of course used in various international sources, e.g. on the Internet, which Anna might have consulted. It does, however, seem much more likely that is is because she previously defined $\vec{a}$ in her CAS *TI-Nspire* – and *TI-Nspire* uses the notation **a** for a vector. (Notice that the problem Anna is working on is a sub-problem of a larger set of problems, where the vector $\vec{a}$ has been used previously). In that sense, one can argue that Anna is using elements of CAS notation already in the formula in line 4 of Figure 1. We do, however, consider this formula mainly as an example of algebraic notation. This makes sense if we compare the formula with the version of the same formula shown on line 6. In fact, it illustrates some key differences between algebraic notation and CAS notation. In the latter case, **Ab · a** is replaced by dotP(**Ab**, **a**), and norm(**a**) is used instead of the typical $|\vec{a}|$. In addition, Anna is using a small triangle between the formula and the calculated result, whereas in algebraic notation one would typically use an equal sign (=). The transition from line 4, where one single notational element from CAS moves into the students' writing, to more full blown CAS notation in line 6 captures what we mean by *mixed notation*. Namely, the fact that elements from CAS notational conventions are imported into students written products.

How to understand the phenomenon of mixed notation

We now analyze the case of Anna presented above and aim at characterizing some relevant dimensions in the phenomenon of *mixed notation*. In the case of Anna, the use of CAS notation is not just a meaningless markup language used to document her work with a CAS tool, which then ideally should be translated back to classical algebraic notation. Rather we see elements of CAS notation being used as a natural part of the communication around Anna's solution of the problem. It appears that the two types of notation assist each other in the construction of Anna's argument – and hence also in her *identity work* (Gee, 2003) as someone doing mathematics (see also Iversen, 2013). When Anna for example indicates the length of the vector $\vec{a}$, not by the algebraic notation $|\vec{a}|$, but by the more CAS-oriented and keyboard friendly norm(**a**), she is in part reporting on her CAS-based calculations. But at the same time, she is also transforming her communication with the teacher to include CAS notation. These types of notational transformations are performed by students and are to a large extent accepted – sometimes even endorsed – by mathematics teachers in Danish upper secondary education (Iversen, 2014).

Furthermore, elements of CAS notation and CAS use are contributing to the shaping and molding of the mathematical identities of the students (Iversen et al., in review). That mixed notation is part of students' identity work and students' learning – we argue – goes counter to a first approximation of the role of CAS notation in upper secondary school students' work, namely as a technical discourse related only to the instrument. This first approximation of mixed notation, as a superfluous byproduct of the technical means that students bring into play, would suggest that skilled students take out the CAS aspects in the theoretical parts of their communication with their teacher and only provide the teacher with a genuine algebraic translation of the CAS work. And this is not the case (Iversen et al., in review).

In fact, it is obvious that one of the affordances of the mixed notation is that the students are able to report on their CAS-based work in a direct manner. Line 6 in the example (Figure 1) does have aspects of that in it. However, we do not see students or teachers unanimously suggesting that CAS-related mathematical notation should only be used for reporting the CAS work. In Anna's case, line 6 actually provides the conclusion on her investigation, whereas line 1 and 2 is her problem statement, and 3 and 4 describes her approach. In other (empirical) cases, we see both teachers and students endorse the use of CAS-related notation in a mix with algebraic notation. From a functional perspective (O'Halloran, 2005), this means that the mixed notation potentially serves purposes related to identity work and idea development work as well as functions related to pointing to the state of things in the world. The case of Anna shows that CAS do more than just the latter, i.e. point to state of affairs; CAS potentially change notation, create identities, and influence learning.

Discussing the potential influence of mixed notation

Our previous work (Iversen, 2014), as well as the mathematics education literature (e.g. Artigue, 2002), show that CAS can play an active and constructive role in students' identity formation. The illustrative example presented in this paper confirms this by showing how notational transformation may be induced by CAS and viewed as identity work. Our analysis shows that mixed notation is not a superfluous phenomenon, if we want to understand the way that CAS shape students' conditions for leaning mathematics. Rather we see that mixed notation has a diverse and complex influence. Mixed notation can lead to misunderstandings, and to loss of skills regarding mathematical formalism (for related examples, see Jankvist & Misfeldt, 2015). But at the same time, it could be interpreted as an active part of students' identity work and cognitive apparatus, in the sense that it might open up the students' potential ways to express mathematics (Iversen et al., in review), and hence mixed notation should not a priori be considered only a problematic phenomenon. We believe that the way teachers address and evaluate students' work involving mixed notation needs to be the object for further investigation and dialogue, not least because it raises a number of important concerns for the teaching and learning of mathematics. In the following, we outline four potential points of awareness for such further investigation.

Firstly, the difficulties that students often encounter when having to handle multiple representations in mathematics is well established in the semiotic approach to mathematics education (Duval, 2006). Hence, the introduction of a new notational system to be used for working in CAS is likely to lead to further difficulties for some students, especially if this notational system is introduced in a covert manner and as a superfluous and simple translation from "mathematics" to CAS notation and then back again. There is a risk that such an approach may lead to the kind of learning difficulties for students that Duval has described, i.e. that students see one of the representational forms as being *the* mathematical object, and the other representations (for instance the CAS notations) as being signs referring not to an abstract mathematical object, but merely to the privileged representation.

Secondly, the introduction of CAS notation is in some sense redundant, which may lead to both confusion and loss of meaning for the students. But more than that, it may contribute to the creation of new "stumbling blocks" for students, who are already experiencing difficulties related to mathematical symbols and formalism (e.g. see Niss & Jankvist, 2017). It seems easy to imagine situations, where students who are mixing CAS notation with mathematical notation ends up disabling themselves in performing, say, algebraic reductions either with paper-and-pencil or in a CAS environment. Furthermore, small discrepancies in the notations may lead to misunderstandings compromising the usual mathematical rigor. As an example, $3a$, where a is a number, is usually taken to mean $3 \cdot a$, and for that reason we regard it true that $3a = a3$. However, as part of the Danish “maths counsellor” program (see Jankvist & Niss, 2015), maths counsellors have found that several upper secondary students consider this as false, because they read $a3$ to mean a_3 due to the CAS-related convention of regarding this as such.

Thirdly, and as mentioned previously, notational transformation, including those involving mixed notation, are not consistently evaluated by teachers, and the acceptance and endorsement of “CAS notation” varies widely from teacher to teacher (Iversen, 2014). Of course, this is not unproblematic, and it may potentially challenge the didactical contract (Broussau, 1997) regarding the use of CAS in the classroom. An unclear didactical contract can lead to severe obstacles for the students, as described by Jankvist, Misfeldt and Marcussen (2016). In a situation of teacher change in a second year upper secondary mathematics class, it was observed that unclear contractual relations concerning the role of CAS fostered misguided winning strategies on the students’ behalf (in relation to Brousseau’s game metaphor), either by leading to students loss of confidence in their own mathematical skills or by causing metacognitive shifts, where the students’ focus was shifted away from the mathematical object to something else, e.g. a CAS-related procedure.

Fourthly, we should not forget that digital technologies change and, in many respects increase the “mathematical muscles” of the students. This has both obvious and relatively well-described didactical potentials (e.g. Lagrange, 2005). If we want to capitalize on these potentials, it requires that students are able to report on their CAS activities, which is likely to involve some sort of reference to CAS notation in their mathematical writing. Taking seriously that CAS constitute an important part of the mathematical environment for today’s students, mixed notation is also a healthy sign of students’ leaning. When students use notational elements from CAS in their written mathematical work, it may be because they are expressing mathematics in a language that they find meaningful. In that sense, CAS notation becomes a register of mathematical representation (Duval, 2006) that has relevance, and mixed notation may become a somewhat meaningful mathematical discourse. Mixed notation may assist students in clearly describing a working process involving CAS, and it may provide students with a language for expressing mathematical meaning. This “language” is of course slightly different from the standard notation – which can lead to a number of problems as described above – but nevertheless it is a language for mathematical meaning and as such writing with mixed notation may in some respects potentially enhance students’ learning of mathematics. Finally, mixed notation allows students a broader range of ways to present themselves as mathematical writers, e.g. when answering mathematical tasks. They may also present themselves as “CAS super users” (Iversen et al., in review), since mixed notation affects students’ identity work by providing a larger range of possible mathematical identities and possibilities for self-presentation (Iversen, 2014).

Concluding remarks

We argue that it is important to consider the influence of CAS in upper secondary school and suggest that an investigation of the resulting mixed notation is indeed a relevant phenomenon to consider in future studies. Keeping in mind the growing proportion of upper secondary students who make use of ICT as a medium for writing in mathematics courses, it seems clear that the influence of CAS, and the use of mixed notation, is growing. In the current situation in Denmark mixed notation exists, but norms and rules for accepting CAS notation as part of students' written work are neither systematically negotiated among teachers nor described in learning standards or official curricular materials. As discussed above, this can give rise to a number of difficulties for the students. However, CAS notation is not a static thing and the technological development is promising to slowly close some of the gaps between CAS-related notation and standard algebraic notation, leaving mixed notation as a concept in flux. Still, since the potential impact of this notation covers both students' learning and their identity work, it appears highly relevant to follow closely the emergence and development of mixed notation.

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Proof by induction – the role of the induction basis

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Proof by mathematical induction is a conceptually difficult, but important form of proof. The proof contains three steps and this study focuses the first one, the induction basis. The aim of the study is to explore how university students treat the induction basis in a proving task. Data were collected from 38 students' solutions to a task in a written exam and were analysed using content analysis. The results reveal that the students used different cases as the induction basis, the majority chose n = 1 although n = 0 was the preferred choice for the given task. A majority of the students used one case in their verification of the induction basis, but it was also common to use more than one case, which is superfluous for this task. Among the students who chose n = 1 as the initial number, a majority included more than one case in the basis step. We discuss how students' choices were influenced by the course literature and the formulation of the current task.

Introduction

Mathematical induction is an important form of mathematical proof that university students meet in the beginning of their studies. However, *proof by mathematical induction* (PMI) is conceptually difficult and there are different kinds of misconceptions that may cause problems for the students (e.g. Ernest, 1984; Ron & Dreyfus, 2004; Stylianides, Stylianides, & Philippou, 2007). In this study we focus on university students and how they treated PMI in a first course at university. Before presenting the study, we focus the structure of PMI and what previous research has taught us according to students' ways of treating such proofs.

Proof by mathematical induction

Mathematical induction is useful when you want to prove a statement that can be connected to the set of natural numbers. We exemplify this by the task used for our data collection. The task comes from a written exam:

> The number sequence a_n is defined through the recursive formula $a_n = na_{n-1} - n + 1$ for $n \geq 1$; $a_0 = 2$.
>
> a) Compute a_1, a_2, a_3, a_4 and a_5.

b) Find an explicit formula for a_n and prove by induction that it is correct. (Compare with $n!$.)

In this task, one first has to solve problems not directly connected to PMI, namely the whole a)-task and the problem to find an explicit formula in the first part of the b)-task. Here this formula is $a_n = n! + 1$. In the second part of the b)-task, PMI shall be used to prove that formula. A proof by mathematical induction can be said to contain three steps:

i) The induction basis aims to show that the statement is true for (in the example above) $n = 0$.

ii) The induction step starts with the *induction hypothesis*, which here can be expressed as "suppose k is a number for which the statement is true". Then we aim to show that this implies that $k + 1$ is a number for which the statement also is true.

iii) If step i) and step ii) hold, the induction principle claims the statement is true for every $n \geq 0$ (where n is an integer).

That the proof in itself contains three steps does not mean that every task can be solved with these three steps only. In the example above, one had to first find a closed formula that seemed to give the same result as the given recursive formula, before using PMI to prove that this closed formula actually gives the correct result for every n. There are also variations in how the three steps are applied. In the most common tasks the basis step deals with $n = 0$ or $n = 1$, but depending on what to prove you have to adapt the starting point to an adequate number or include more than one number in the induction basis.

Previous research

Ernest (1984) pronounces a number of conceptual difficulties experienced by students, and we will here focus on two of those; difficulties related to the induction basis and to the induction step respectively, and also how these two are connected in the structure of the proof.

There are different kinds of misconceptions regarding students' understanding of the induction basis. One finding is that students fail to include or do not understand the role of the induction basis. Getting the induction started, i.e. verifying the first step, is often treated as a formality without any meaning and not seen as really essential for the proof (Dubinsky, 1986; Ernest, 1984; Palla, Potari, & Spyrou, 2012), or as a preliminary activity just checking the validity of the initial case to give confidence that the statement to prove is true (Ron & Dreyfus, 2004). However, there are many examples where the induction step can be proved, but the proof fails in the induction basis, e.g. to prove that $2n + 1$ is even. There also exist uncertainties about where to start the basis step, as the misconception that the induction basis must always contain the case $n = 1$ (Stylianides et al., 2007). Connected to this is a lack of understanding regarding how many cases you need to include in the basis step and the consequences caused by the choice of starting point. Ron and Dreyfus (2004) have shown that it is not clear for all students that one has to check only for the very first case and that other checking activities are not necessary parts of the proof – except for more complicated examples where the induction basis needs to include more than one case.

A second issue is the induction step. Students often construe PMI as a method where you assume what you have to prove and then you prove it (Ernest, 1984). However, in the induction step you neither prove the statement is true for $n = k$ nor for $n = k + 1$; in fact, the truth-values of these cases are irrelevant since it is the implication “true for $n = k$ implies true for $n = k + 1$” you need to prove.

The final step of the proof is setting the results from the induction basis and the induction step together, which connects the understandings and misunderstandings due to the induction basis and induction step. Previous studies indicate that some students appear to conduct proofs without really understanding the steps involved, and that a proof has to follow a very strict scheme. In a study, some students admitted they view the basis step as nonessential, and something they did just because it was a rule stated by the teacher (Harel, 2002). Other studies showed that some students believed the induction basis had to be verified *before* the induction step for the proof to be valid (Pang & Dindyal, 2012), or that the basis step is always verifiable and thus one only needs to worry about the inductive step (Stylianides et al., 2007).

This paper is an initial report from a study aiming to explore students’ understandings of PMI, and in forthcoming papers we intend to present results according to all steps of the proof. However, several researchers have identified the induction basis as one of the difficulties (e.g. Dubinsky, 1986; Ernest, 1984; Palla et al., 2012; Ron & Dreyfus, 2004; Stylianides et al., 2007), hence we here choose to focus exclusively on this initial part of the proof. Thus, this paper aims to explore how university students treat the induction basis in tasks where PMI is employed. This limitation made it possible in depth to uncover details in a crucial part of PMI and through that produce a richer description of students’ different ways of handling the first step in PMI.

The context of the study

In the syllabus for compulsory school in Sweden, the word *proof* is not mentioned. However, the students shall develop their ability to apply and follow mathematical reasoning, which also is a preparation for conducting proofs. In Sweden, almost all students (98 % year 2014) continue to upper secondary school and about a fourth of the students follows the natural science or technological programme, which contain up to five courses in mathematics. In the first and third course, proofs are mentioned related to other parts of the core content, e.g. to prove and use the sine theorem. In the fourth course different methods of proof in mathematics is also an explicit part of the core content, mentioning proofs with examples from arithmetic, geometry or algebra. Although course 4 has mathematical proofs as a core content, PMI is usually not a part of the topic. However, in the fifth course one part of the core content is "Mathematical induction with concrete examples from e.g. the area of number theory" (Skolverket, 2012, p. 39). Thus, PMI is explicitly treated during course 5.

To apply for *Mathematics I*, the first mathematics course at the current university for this study, a student needs a passing grade in at least course 4 from upper secondary school. Hence, not all students have met PMI before they start Mathematics I, although they repeatedly have met proofs in general.

Mathematics I is a full time one-semester course, given at the department of mathematics at a university in Sweden. The students are aiming for a general exam in mathematics or physics, or for a teacher exam. The course has two parallel halves; algebra and calculus. PMI is included in the algebra part, which is examined mainly by a written exam at the end of the semester. PMI is introduced in *one* lecture (number 17), followed up by tutoring and task solving on PMI. In addition, one or two written hand in tasks deal with PMI. However, PMI is rarely used for proving theorems in other parts of the course. Thus, in Mathematics I, the introduction of PMI is limited to learning the method for its own sake or for future use. The current semester, the task presented in the introduction of this paper was the only task dealing with PMI in the written exam.

Regarding what number to choose as starting point in the induction basis, the course literature (Bøgvad, 2014, p. 143) uses $n = 1$ when the induction principle is established. $n = 1$ is also the most common starting point in the examples, but there are also examples with other starting points, e.g. $n = 0$ and $n = 4$. However, in 10 out of 13 exercises, the induction basis should be at $n = 1$ (including one task where both $n = 1$ and $n = 2$ are needed as basis), implying this is the usual case.

Method

In order to explore how students treat the induction basis, we chose to use data collected from students' solutions to a task of the written exam in the course Mathematics I (the task was presented above in the introduction of this paper). In total, 109 students took part in this exam, of whom eight students did not solve the current task at all, and ten students' solutions were marked with 0 points. We got permission from 38 students to use their solutions in our analyses. Of these 38 students, one gave a partly correct proof, where however the induction basis was missing; one student just presented an induction hypothesis and nothing more; while three students did not start the b)-part of the task at all. Since the focus of this paper is how students treated the induction basis, these five students will be excluded from the following analyses, which then will contain solutions from 33 students.

A content analysis (Cohen, Manion, & Morrison, 2011) of the students' solutions was undertaken. Aware of findings in previous research (e.g. Ernest, 1984; Ron & Dreyfus, 2004; Stylianides et al., 2007), we read and re-read the students' solutions, striving to identify similarities and differences in their treating of the induction basis. This content analysis generated three themes, in which each of the 33 student solutions was categorised. The first theme was whether or not the student presented a statement to be proved – recall the first part of task b) was to find a closed formula, which validity then should be proved. The second theme was what number the students chose as starting point in the induction basis (e.g. n = 0), and the third theme dealt with how many cases the students included in the basis step.

Results

In this section, we elaborate on the three themes mentioned above. We exemplify the different categories by including parts of the solutions from some of the 33 students included in the analysis. The given excerpts were chosen as representative examples of solutions in the respective category.

Did students clarify what they aimed to prove?

The first part of task b) was to find a closed formula, which was likely to give the same result as the recursive formula given in the task. Remember that the students had computed the values of a_1 to a_5 in part a), which was an obvious support when they should find the closed formula. The correct formula is, as presented above, $a_n = n! + 1$. This formula was stated by 31 of the students, e.g. one student wrote

Student A: It seems like we get the following formula for a_n, $a_n = n! + 1$.

However, one student (C) started his/her proof without giving the closed formula. That is, there was no statement to be proved, when s/he started the 'proof' by writing:

Student C: We first show the statement holds for a basis case. $n = 0 \rightarrow a_0 = 2$.

A few lines down the same student however gave the explicit formula referring to part a), and then used this formula as induction hypothesis and in the induction step. Another student (B) just began to show the (obvious) validity of the recursive formula. The first two steps presented were:

Student B: 1. $a_n = na_{n-1} - n + 1$ for $n \in [1, 5]$ as shown above.
2. $a_{n+1} = (n+1)a_n - (n + 1) + 1$ is supposed to be valid for the following n.

That is just repeating what was already given and student B also continued the 'proof' by reasoning about what came out from the recursive formula.

Starting point for the induction basis

As mentioned above, 31 students gave the correct formula ($a_n = n! + 1$), which is essential before starting the proof. However, since student B and C anyway started their proofs (see above), they have been included in the following two categorisations.

Even though it is not explicitly said in the task for which n the formula for a_n should be valid, it is implicitly given that it should be for $n \geq 0$ since the given sequence in the task starts with a_0. In the solutions analysed, 14 students included $n = 0$ in the initial step, while 17 students started at $n = 1$. We here give two examples starting at $n = 0$ and two examples starting at $n = 1$.

Student D: Check whether P is true for $n = 0$. $P(0) = a_0 = 0! + 1 = 2$. P is true for $n = 0$.

Student J: Basis step: Valid for $n \in [0, 5]$. (see above) [the student wrote "see above"]

Student F: Basis case: We check for $n = 1$: $1 + 1 = 2 = a_1$ so yes, it is true.

Student A: 1. The formula is proved for the cases 1–5. [referring to the first part of the task]

Two students started at $n = 2$. One of them did not give any motivation of his/her choice of starting point. The other student starting at $n = 2$ wrote

Student G: As basis we can use any number from task a). For example, $a_2 = 3 = 2 + 1 = 2! + 1$

Student G did neither motivate his/her choice of $n = 2$ as starting point, nor include that the formula anyway is valid for all $n \geq 0$ since s/he already had shown the equality for a_0 and a_1, which in fact is necessary for his/her proof to be complete. Despite this deficiency, the proof could be seen as valid.

The number of cases included in the basis step

As induction basis, 20 of the 33 students showed the validity of the formula for one specific case ($n = 0$, $n = 1$ or $n = 2$). Two examples were student D and F above, and two other examples are:

Student H: Basis case: We show the formula is valid for $n = 1$. $1! + 1 = 2 = a_1$

Student I: 1) Basis step: the formula is true for $n = 0$. $0! + 1 = 1 + 1 = 2$

Twelve students showed the validity for all elements from part a). Several students showed that by simply computing a_0 (or a_1) to a_5. We have above seen other forms of examples by student A and J, and yet another example is:

Student L: Basis assumption: The formula is valid for a_0–a_5 (even for a_0, since $0! = 1$, which means $0! + 1 = 2$. [referring to computations in part a) for a_1–a_5]

Finally, one student showed the validity for two cases.

Student K: $k = 0$ gives $0! + 1 = 2 = a_n$ for $n = 0$. $k = 1$ gives $1! + 1 = 2 = a_n$ for $n = 1$.

Summing up the results, focussing on the second and third theme, there are some differences in the students' choices in their solutions. Almost all students used either $n = 0$ or $n = 1$ as the first case in the induction basis. A majority of the students verified the basis for one specific case, but it was also common to use more than one case as basis. In Table 1, we combine the results from these two themes. This is a cross-table where e.g. the first column shows that of the 14 students choosing $n = 0$ as induction basis, 10 included just that case, while 4 included at least one more case.

	$n = 0$	$n = 1$	*Sum*
1 case	10	8	*18*
>1 case	4	9	*13*
Sum	*14*	*17*	*31*

Table 1: Starting point and number of cases included in the induction basis (number of students)

Here, we can notice that students who gave $n = 1$ as the first number in the induction basis also to a greater extent included more than one case in the basis step. In fact, a majority of the students starting at $n = 1$ included more than one case, while less than one third of the students starting at $n = 0$ did the same.

Discussion

The study presented in this paper is the initial part of a project about teaching and learning of PMI. Since the induction basis is the initial step of a proof by induction and this step has been identified as a difficulty (Ernest, 1984; Ron & Dreyfus, 2004; Stylianides et al., 2007), we chose in this paper to focus on the induction basis only. This narrow focus offered opportunities to a deeper exploration on students' understanding of an essential part of PMI, which is known as problematic for students.

One important finding was the variation in the students' solutions, whether $n = 0$ or $n = 1$ should be the case to verify in the induction basis. Since the recursive formula had $a_0 = 2$ as its initial value, $n = 0$ is to prefer as starting point for the proof, rather than $n = 1$. There can be various explanations for why a majority of the students anyway started with $n = 1$. Due to the course literature (Bøgvad, 2014, p. 143), the basis in the definition of PMI is conducted for $n = 1$ and most exercises start at $n = 1$ too. Hence the students are used to proofs starting at $n = 1$ and some might have the misconception that the proof always starts at checking for $n = 1$ (cf. Stylianides et al., 2007). This misconception can also depend on that students have memorised the structure of PMI and hence conduct their proof mechanically (Pang & Dindyal, 2012; Ron & Dreyfus, 2004). The task formulation may also contribute to this misconception, or at least not prevent it, since $a_0 = 2$ is already given. In addition, the task did not explicitly tell from what n to verify the formula, it just

said verify for a_n. Thus, it may not be obvious that a_0 is also computable by the closed formula and hence should be verified in the proof. The misconception that $0! = 0$ could be another possible reason to skip the case $n = 0$, since the closed formula then would not give the result $a_0 = 2$. However, we did not identify any signs of this misconception, although it cannot be ruled out.
A second finding is that over one third of the 33 students involved more than one case in their basis step, although in the current task just one case (n = 0) is needed as induction basis. This can possibly be explained by the conclusion that they are not aware of the role of the induction basis. Including more than one case, when not necessary, can be a matter of seeing the basis step as a formality (Ernest, 1984), and not understanding that …

> checking the validity of the initial case is an integral part of the proof – not a preliminary activity that is intended to shed light on the statement or to give confidence that the statement to be proved is true. (Ron & Dreyfus, 2004, p. 114)

However, the current task might encourage the adoption of including more than one case in the basis step. Before even starting the proof in the b)-part of the task, the students had to find a (closed) formula which was likely to give the correct result. Hence it is necessary to first be convinced that the formula found actually seems to coincide with the given recursive formula, i.e. "to give confidence that the statement to be proved is true" (Ron & Dreyfus, 2014, p. 114). In addition, the a)-part of the task was to, by the recursive formula, compute a_1 to a_5, which automatically gave the student five cases where the closed formula $a_n = n! + 1$ easily could be verified. Thus, that students gave more than one case as induction basis could just be a matter of that the cases were already verified. Moreover, it is not incorrect to include more than one case, though it is superfluous in the current task. It would be interesting to give almost the same task, but exclude the a)-part, give the closed formula $a_n = n! + 1$, and just ask the students to by mathematical induction prove it is correct. Possibly more students would then just verify one case in the basis step, since the initial computations of a_1 to a_5 are then not requested.

Even though the design of the task possibly had an impact on the students' tendency to include more than one case in the basis step, the results arising from combining theme two and three indicate a lack of understanding of the role of the induction basis. These results show that students who chose $n = 1$ as the (first) number in the induction basis, to a greater extent also included more than one case in the basis step. Recall that $n = 0$ was to prefer as basis. Hence, students who made one less appropriate decision were also more likely to make a second less appropriate decision. The tendency to include more than one case in the basis step indicates that the students connect the verification of the basis rather to the computations in the a)-task than to the formula to be proved. This shows a lack of understanding of the essential role of the induction basis (cf. Ernest, 1984).

Through this study, it has been possible to identify some issues about PMI. What we found most interesting was that a majority of the students chose $n = 1$

rather than $n = 0$ as induction basis and that those students also to a larger extent included more than one case in the basis step. However, when analysing written solutions to a task, it is not possible to draw deeper conclusions about how the students have reasoned when solving the task. Anyway, this study has illuminated some issues to be immersed in further research, e.g. through interviews get a clearer picture of why students include more than one case in the induction basis. Another view of the same issue is in what way the task design affects the students' solutions regarding the number of cases included in the induction basis. Hence this study has provided valuable information for the research to come.

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Interpreting teaching for conceptual and for procedural knowledge in a teaching video about linear algebra

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The aim of this study is to investigate teaching videos about mathematics, seeking to uncover research-based foundations for their quality. By drawing on the notions of procedural and conceptual knowledge, the research was operationalized by asking professionals in undergraduate mathematics education (n=18) to interpret sections of a teaching video. The video dealt with a topic in linear algebra. The results indicate rather divergent interpretations of conceptual knowledge. This can hinder a reliable evaluation of teaching in terms of aiming for conceptual or procedural knowledge. It is recommended that the notions should be carefully used, defined and explained when used to evaluate the quality of teaching videos in particular, or of teacher's explanations in classrooms in general.

Introduction

On public internet platforms such as YouTube, there are many teaching videos for mathematics. In such videos a single, often invisible speaker teaches about mathematical topics in a confined environment. These videos are meant to assist students in their learning. They can also be resources for other people than learners, for example to seek inspiration for and to compare with one's own production of teaching videos, or to do research on teachers' explanations, whether it is in videos or in classrooms. We belong to the first category, producing videos ourselves. Yet, being researchers of mathematics education for engineers, we want to find research-based foundations for such work. Therefore, we were interested in finding research-based criteria for the quality of these videos.

Research on multimedia learning offers design principles that enhance learning, such as: the use of visualizations, limiting surplus information, personalization (a friendly voice, showing the teacher's face) (Mayer, 2005). However, these guidelines are not didactical, describing how mathematical topics are or can best be taught in a video. By lack of tools for analyzing and evaluating the teaching of mathematics in videos, we turned to the teaching in mathematics classrooms in general, where one can distinguish between different activities, such as activities that involve teacher-student interaction (e.g. probe, evaluate or extend students' ideas) and an activity that does not necessarily involve 2-way-interaction: explaining. Explaining of mathematical topics is a complex activity and research on it is still ongoing (Baxter & Williams, 2010). Explaining aims at supporting students, on the one hand to better understand mathematical ideas, and on the other hand to better carry out tasks. To capture this distinction, we turned to the notions of conceptual and procedural knowledge. These notions are widely used by researchers of mathematics education, based on work by, among others, Hiebert (1986). Backgrounds and definitions of these notions will be explained below. At this stage, it suffices to say that procedural and conceptual knowledge are connected to student's learning and thinking, rather than to teaching, and that "(t)he general consensus, in research on mathematical thinking and in mathematics education, is that having conceptual knowledge confers benefits above and beyond having procedural skill" (Crooks & Alibali, 2014 p. 345). In studying the quality of teaching videos, we can look for whether the teacher is *aiming* at enhancing procedural or conceptual knowledge. As an example, a teacher who aims at procedural knowledge can emphasize how tasks are to be done by demonstrating subsequent steps of the solution process. If a teacher rather aims at conceptual knowledge, he/she can focus on why a procedure works, show different representations, compare procedures or show how classes of problems have similarities.

The purpose of the present study is to support the evaluation of teaching videos, investigating whether the explanations offered in a video are aiming at procedural or conceptual knowledge, and how this can be judged. We operationalized our study by selecting from the web a video on linear algebra, in particular about bases and dimensions of vector spaces. We selected this topic, because (1) it is a topic that is part of many bachelor engineering curricula, and (2) because of the interaction between procedural methods (Gaussian elimination, finding pivots) and a connected network of concepts (vector spaces, bases and dimensions). We watched a dozen YouTube videos on this topic. The majority had an emphasis on the "how", although not one could be indicated as "purely aiming at procedural knowledge". We selected a video with a high didactical quality, clearly aiming at conceptual knowledge, for example by comparing between different solution approaches and by jumping over tedious calculations. We showed it to professionals interested in mathematics education, asking them to judge sections of the video in terms of teaching for conceptual or for procedural knowledge. Would they reach a common agreement? Would their judgment agree with our own? How would they interpret conceptual and procedural knowledge? In this paper we will report on the commonalities and divergences in participants' interpretation of teaching for conceptual and procedural knowledge in mathematics, with respect to the content presented in the chosen video. The judgment could later be useful to evaluate teaching videos on didactical qualities.

We formulated the following research question: What are, according to a group of mathematics-interested professionals, the sections in a teaching video that emphasize conceptual or procedural knowledge?

Theoretical Framework

The notions of conceptual and procedural knowledge in mathematics are widely used by researchers. Hiebert (1986) characterized conceptual knowledge as a set of connecting pieces of knowledge. Kilpatrick, Swafford, and Findell (2001) explain conceptual knowledge as "an integrated and functional grasp of mathematical ideas" (p. 118). Procedural knowledge includes familiarity with symbols and representation systems in mathematics together with knowing rules and procedures that are used to solve a class of tasks in mathematics (Hiebert, 1986).

Researchers agree on a dynamic interplay between conceptual and procedural knowledge, showing that conceptual and procedural knowledge can grow interactively (Baroody, Feil, & Johnson, 2007; Rittle-Johnson & Alibali, 1999): "Linking procedural to conceptual knowledge can make learning facts and procedures easier, provide computational shortcuts, ensure fewer errors, and reduce forgetting" (Baroody et al., 2007, p. 127). However, it is warned not to confuse or equate these notions with deep and superficial knowledge, respectively (Baroody, 2003; Star, 2005). Conceptual knowledge is a basis for procedural *fluency*, which differs from procedural *knowledge*. A superficial procedural knowledge refers to disembodied task preforming procedures, most often algorithmic computations, while procedural fluency may be of a deeper, richer nature, for instance when knowing how to generate solution processes beyond standard problem types (Kilpatrick et al., 2001; Star, 2005). A conceptual knowledge type may be of a superficial quality if the building of schemas for conceptual structures is weak and mainly related to primary level concepts. Bergsten, Engelbrecht, and Kågesten (2015) investigated engineering students' learning and they created the following working definitions: "Procedural approach: Use and manipulate mathematical skills, such as calculations, rules, formulae, algorithms and symbols. Conceptual approach: Show understanding by e.g. interpreting and applying concepts to mathematical situations, translating between verbal, visual (graphical) and formal mathematical expressions and linking relationships" (p. 932).

Crooks and Alibali (2014) offer a review of research on conceptual and procedural knowledge, explaining that this mainly focuses on students, and the most frequently used instruments are written tests. The more rare studies about deliberate teaching that aims at conceptual knowledge (e.g. Eisenhart et al., 1993; Even & Kvatinsky, 2010) show that this kind of teaching requires, amongst others, flexibility, diligence and conceptual knowledge from a teacher, and it does not necessarily lead to conceptual knowledge with students. These studies were case studies of carefully observed teachers and how they offered the students inquiry-based tasks, used different representations, made connections, asked the students to discuss alternative approaches, and so forth. These studies did not offer categories for the quality of the teaching in terms of conceptual and procedural knowledge, and they did not specify whether a higher quality was reached through student-student interaction, teacher-student interaction, or through teacher's explanations without teacher-student interaction. By studying mathematics teaching videos, we can only observe the latter. We hope that studying the didactical quality teaching videos can also contribute to research on classroom-based explanations that aim at conceptual knowledge.

Methods

Our research design entailed a survey based on a mathematical teaching video. The data collection took place at a Norwegian conference on Undergraduate Mathematics Education. The conference attracted professionals in mathematics education: mathematics education researchers, mathematicians with teaching tasks and teachers of mathematics. Within this conference we conducted a workshop on didactical approaches in teaching videos. Part of the workshop was to show a video and collect judgments by participants in terms of teaching aiming at conceptual or procedural knowledge. Because of time limitations, however, they could only evaluate one video.

The video

From the wealth of videos freely accessible on YouTube, we selected the video "Linear algebra, Basis and dimension" published by Massachusetts Institute of Technology (MIT), available at www.youtube.com/watch?v=AqXOYgpbMBM. We deliberately chose an English video as the Nordic mathematics community is rather small and we run the risk of having the teacher of the video in our workshop. Also, the MIT-video satisfied many guidelines for multimedia (Mayer, 2005): the use of space is well-planned, we see the speaker's face, the video is relatively short (8:09 min.) and the user is activated: after having explained the task (Figure 1, left), the teacher asks users to first hit the stop button and solve the task by oneself.

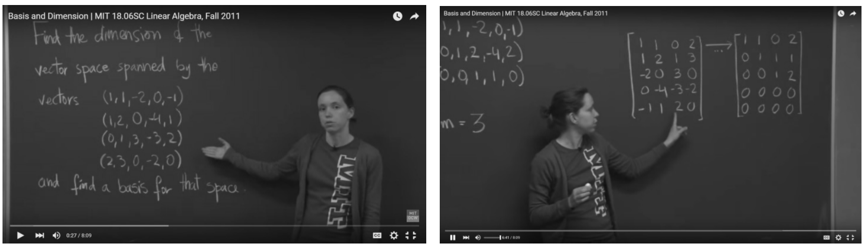

Figure 1: Stills from the video "Linear algebra, Basis and dimension" from MIT

The task in the video is to find the dimension and basis of a vector space spanned by four given 5-dimensional vectors. The solution could be demonstrated step-by-step aiming at procedural knowledge. However, there are several aspects indicating that the teacher aims at conceptual knowledge: at the beginning the teacher links to prior knowledge; before starting calculations, the teacher gives a rough outline of the approach; towards the end she presents an alternative approach for the given problem explaining how the two approaches are related. The procedural aspects, such as carrying out the Gauss operations, are accelerated and the teacher says she will go fast, because *"you must have seen eliminations a million times"*. When she explains the alternative approach, she avoids losing time on calculations and only shows the first and final matrix, indicating the calculations by an arrow and dots (see figure 1, right).

We analysed the video by splitting it into sections and describing these with cognitive steps:

1. Starts by giving the pre-knowledge (linearly independence, spanning, basis, dimension).
2. Gives a rough outline – how to work on the given problem (1st: find basis, 2nd: find dimension).
3. Talks about linear independence (until after 2:00).
4. Takes two minutes to do the elimination of rows. At 3:58: one row of zeros.
5. At 4:05: Circles the pivots and talks for a minute about the last obtained matrix.
6. At 5:04: Writes the basis on the right hand board; talks about alternative bases.
7. At 5:50: Writes down the answer to the question: dim = 3.
8. Summarizes and talks about alternative approach (vectors as columns).
9. At 6:39: Moves to the right, where she had prepared some work (the same vectors, but then as columns + the matrix after the elimination).
10. At 7:25: Stresses that she now cannot use the columns as basis.

Data collection

We created a questionnaire consisting of two pages, on which the above ten video sections were described with 4-5 cm space between, five on each page, in order to provide space for comments. During the workshop, we introduced our interest in the use of videos and gave illustrations of the variety of types of videos available on the web. Then we outlined the content of the MIT-video, defining it as "rather good" and giving the main headlines 'pre-knowledge', 'elimination of rows', 'pivots and basis' and 'another strategy' to describe its progress. The participants were asked to watch the video and indicate about each section whether it was aiming for conceptual or procedural knowledge, and additional comments could also be given. We deliberately did not offer definitions of what is meant by the notions of procedural and conceptual knowledge to avoid funnelling the participants' answers. These notions are frequently used by researchers, often without amplifying their meaning. By not giving the audience definitions, we wanted to get a grip on how the audience interpreted the conceptual and procedural notions - unaffected. Thereafter, we ran the video and the participants filled in the questionnaire. After the video was finished, we initiated a discussion, with questions: “What was good (both procedural and conceptual)?”, “What could have been done differently?” We made field notes of the comments. As the participants left, we collected 18 anonymous responses.

The data analysis process

To analyze the answers on the questionnaire we took advantage of the definition of conceptual and procedural knowledge provided by Bergsten and colleagues (2015). We first tried to organize the responses according to degrees of similarities, this resulted in quite many groups of responses, as few were to a large degree equal. Then, we discovered that most disagreements were on the first page. This made us decide to let the second page on the final five sections of the video be more important for coding. This choice could be supported by the argumentation that (1) in the final sections of the video the teacher was aiming at conceptual knowledge by explaining an alternative approach without losing time on calculations (see figure 1, right), and (2) the participants needed time to get used to the video and the questionnaire, thus the second page better represented their interpretations. This refinement made three categories crystalize: (1) participants who had interpreted most parts of the second half of the video as conceptual - the C-group; (2) participants who had interpreted most parts as procedural - the P-group; (3) participants who had answered either P-P-P-C-C or P-P-C-C-C, which we coded as the PP-□-CC-group. The remaining participants offered blank responses, or responses which were not written in terms of conceptual or procedural knowledge. This group was named “Answering something else or not answering at all”.

We are well aware of methodological limitations of our approach. The participants may have interpreted questions differently from what was intended, and we may have interpreted their answers incorrectly. The participants may not have been well enough prepared to characterize the sections in the video (some did not remember well the linear algebra). The English language in the video, in the workshop and in the questionnaire may have hindered (most participants were Norwegian), and so forth. Therefore, we take our results with caution.

Results

The participants’ responses yielded four groups. Below we will present their additional remarks in the questionnaire and their contributions to the discussion.

The C-group consisted of four participants. Their categorization of the different sections of the second half of the video was ‘conceptual’ or as one participant expressed: “conceptual about ‘what can a basis be?’”. There were also responses stating: “P→C, good: Clear about procedure, link to concepts”. In the group discussion, one of the participants explained this view. He emphasized that since there are linear algebra concepts, on which all the calculations in the video are based, his reading of the video was that most parts were aiming for conceptual knowledge. Here we observed an interpretation of conceptual knowledge as knowledge based on the presence of mathematical concepts - even when presenting only the “how?” of a procedure. Thus, because these participants recognized the underlying concepts, they judged it as aiming for conceptual knowledge.

The P-group consisted of five participants. They interpreted at least four of the five final sections in the video as procedural. One of the participants in this group interpreted nearly all ten sections as procedural writing: “Procedural, less explanation – doing aspect. Non-concept” and “Discussing strategies – not concepts”. In this group, a common view appeared to be that there was something missing: “Presents alternative strategy; - no or little discussion of the general idea behind” and “Procedural (relies on us to remember initial definition introduced)”. In the discussion, several participants stressed that in the video mathematical definitions were missing. They emphasized that definitions should have been given greater attention in the video. The participants in this group considered definitions as important constituents of teaching for conceptual knowledge.

The PP-□-CC-group consisted of four participants. They described the first two sections of the second half of the video as procedural. These sections showed the teacher concluding the first solution approach. The participants in the group did however not have a common interpretation of the ensuing section in the video (section 8), which we cannot explain. The final two sections in the video, referring to how an alternative way of solving the task can be done, was by all participants in this group interpreted as conceptual. An explanation offered was: “C: ‘What if we did something else’”. This indicates that the participant apprehends the variety in methods as a conceptual feature. The responses in this group seem to agree that the alternative solution approach aims at conceptual knowledge.

Group 4 ‘answering something else or not answering at all’ consisted of five participants. Some comments from this group were on quality of the explanations, such as: “Necessary to write how to transform one step to another in elimination process. But explanation was good”. There were also descriptive responses: “explains a little”. Another participant in this group wrote: “General comment: Linear algebra is outside my area, therefore lost focus and understanding of what was going on. Did also lose track of where we were in the video, thus there are not many fruitful comments here.” (translated). These responses could not be analyzed in terms of aiming for conceptual and/or procedural knowledge.

Discussion, conclusion and recommendations

Our research question was: 'What are, according to a group of mathematics-interested professionals, the sections in a teaching video that emphasize conceptual or procedural knowledge?' This question cannot clearly be answered because of diverging apprehensions by the participants of what they recognize as conceptual or procedural knowledge. We can discern several interpretations.

One interpretation is that teaching is judged as aiming for conceptual knowledge, if it is based on mathematical concepts. For the participants who were familiar with the concepts used in the MIT-video it was easy to relate the discussions and processes in the video to the mathematical arguments founding the processes. Thus, because these participants recognized the underlying mathematical concepts, they judged it as conceptual. However, any sequence in the video, whether aiming at procedural or conceptual knowledge, used linear algebra concepts. According to this interpretation then, as there were underlying concepts throughout, all sections were 'conceptual'. With all mathematical thinking and reasoning being based on mathematical concepts, this interpretation of conceptual knowledge will blur any distinction between procedural and conceptual knowledge.

A second interpretation is that a certain approach to teaching is judged as aiming for conceptual knowledge, if it includes formal definitions. Such definitions were lacking in the video, thus connections between concepts and their definitions are up to the viewers of the video to draw themselves. Lack of formal definitions made these respondents interpret the teaching in the video as aiming for procedural knowledge. The importance of formal definitions to mathematicians has been discussed by many researchers (o.a. Van Dormolen & Zaslavsky, 2003; Vinner, 1991), writing that the organization and presentation of mathematical content in textbooks and lectures are often based upon the assumption that concepts should be 'acquired' through definitions. However, the definitions of conceptual knowledge in the research literature do not mention formal definitions. In fact, conceptual understanding may be informal or intuitive, as long as it is rich in connections (Baroody et al., 2007; Hiebert, 1986).

Of the four groups in the study, it was only the PP-□-CC-group that made interpretations of teaching aiming at conceptual knowledge as being about offering relationships between concepts and solution approaches. One of the participants in the PP-□-CC-group put up a definition of what (s)he meant: "Procedural – talks about a method: What is going to be done first and last. How. Conceptual – short about why, (but mostly about what one has to do and the order)" (translated). This interpretation is quite in line with the definitions given in the literature on mathematics education research.

We started with a need for didactical quality descriptors for mathematical teaching videos and chose to study to what extent the explanations in videos can be judged as aiming at conceptual or at procedural knowledge. The dynamic interplay between conceptual and procedural knowledge (Baroody et al., 2007; Rittle-Johnson & Alibali, 1999) may at times make it hard for teachers to distinguish between the approaches. However, at times these are simple to observe: A teacher who just tells about the "how" is clearly procedurally oriented, and one who jumps over a calculation is clearly avoiding procedures. Our study shows that these notions do not yield reliable judgments at all when used by professionals in mathematics education, without first explaining, discussing, defining and explicating these terms. It can be assumed that a number of professionals in mathematics education aren't well aware of the definitions from the research literature. In particular, mathematicians who strongly stick to formal definitions as one of the bases of their explanations, may have misconceptions about conceptual understanding.

What is illuminated by the present project is that there are a number of typical combinations of conceptual and procedural interpretations of a mathematics lecture. The rather diverging interpretations in the first three groups – along with responses in the fourth group that mainly indicate uncertainty – illustrate that the understanding of the notions conceptual and procedural knowledge is rather diverging and, also, that these notions are 'difficult'.

The present project embraces only a small number of responses gained from a small part of the professional community. Thus, it is exploratory. Nevertheless, locating such divergences in a small group of professionals sends a signal of difficulties obtaining a unique apprehension within bigger communities. When studying a teacher explaining mathematics, whether this is within a teaching video or within a live classroom, the judgment of whether it is aiming for conceptual and procedural knowledge should be done. Asking professionals in mathematics education may yield unreliable results if the notions are not carefully defined, explained and discussed.

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Research study about Estonian and Finnish mathematics students' views about proof

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Proof belongs to university mathematics almost indisputably, but quite often it has an important role in school mathematics, as well. In literature, several functions for proof have been presented. In this study, university students' views about the importance of proof and its different functions were explored. 97 students in Finland and 215 students from Estonia participated, all in the beginning of their mathematics studies in university. These countries are interesting to compare because earlier studies show substantial differences in how proof and proof-related items are addressed in the school curricula of these countries. The results show that the students in both the countries appreciate quite highly the importance of proof both in school mathematics and in mathematics in general. Support for understanding and development of thinking skills were reasons the students considered most important for studying proof and proving.

Introduction

Proof and proving are often seen as essential elements of mathematics, especially at the advanced level. Quite often the amount and importance of proving increases considerably when a student starts mathematics studies in tertiary education (Selden, 2012). It has been reported in several studies that university students often have difficulties with proof and proving (e.g., Gueudet, 2008; Selden & Selden, 2003; Hemmi, 2006; Reid & Knipping, 2010). Learning of proving skills requires different kind of mathematical thinking than, for example, training of calculations based on algorithmic thinking. However, it is not so broadly studied how students understand the reasons for why proofs and proving are studied in mathematics. In this study the main focus is to explore the following questions:

1) Do Estonian and Finnish university students appreciate the role of proof and proving in mathematics at the beginning of their mathematics studies?
2) How important do they consider different functions of proof?
3) Which reasons do they possibly state for studying proofs and proving in mathematics?

The study adds to our general knowledge about students' relation to proof in the beginning of their university studies. Moreover, the results between the countries are also compared because the comparison helps us to interpret and explain students' views with respect to differences of proof-related approaches in respective secondary curricula. The aim is to analyse if and to what extent differences found in Finnish and Estonian curricular approaches (see Hemmi, Lepik & Viholainen, 2013) are reflected in students' views.

Hemmi et al. (2013) analysed and compared proof-related issues in the Finnish and Estonian mathematics curricula. They found that proof and proving are addressed in secondary school curriculum in both the countries but in different manner. The Estonian curriculum explicitly states goals concerning proof and proving: The primary introduction to proving issues is prescribed at the lower secondary level, and there is a heavy emphasis on these topics at the upper secondary level. The presentation style resembles the 'traditional' way of starting to work with proving within geometry and continues by presenting rigorous ready-made proofs. Students' solving of proving tasks is not stressed. The Finnish curriculum is less explicit in terms of proving, proof is not explicitly present in the Finnish upper secondary school curriculum. At the same time, the Finnish curriculum addresses the proof-related competences from grade 1. In addition, students' justifying and investigative activities are emphasised from the primary to the upper-secondary levels. The new approaches to proof-related competences that could enhance students' experience of meaning (e.g. de Villiers, 2010; Heinze & Reiss, 2004) are strongly present in the Finnish curriculum.

During the last decades several researchers and mathematics educators (e.g. de Villiers, 1990; Hanna, 2000) have presented that the most central function of proof in mathematics is not to *verify* the truth of the statements. For example, de Villiers (1990) suggests five different functions for proof. He stresses that beside verification, proof has an important role in *explaining*: providing an insight why something is true. Proof also *systematizes* various results into a well-organized deductive system. It is also possible to *discover* new results through proving by using deductive reasoning. de Villiers also proposes that proof may be seen as a tool for *communication*, which means that mathematical knowledge can be communicated via proofs. Hanna (2000) discusses various functions of proof and emphasizes that enhancing of mathematical understanding is the most important goal for the use of proofs and proving. Hemmi (2006) introduces *transfer* as an important function of proof. She suggests that proofs may introduce techniques or methods that are useful in other problems, and they may also offer understanding for something different from the original context. Also, the development of logical thinking skills can be included in this function. Researchers have also found the functions of *aesthetic experiences* and *intellectual challenges* as important aspects of proof to be considered (e.g. Hemmi, 2006).

Method

In this study, a questionnaire was applied for data collection. The questionnaire included 22 statements presented in Table 1 and one open question. The statements have been tested and developed in our earlier studies (e.g. Hemmi, 2006). The statements 1-6 focus on the role of proof in school mathematics and the statements 7-9 the role of proof in mathematics in general. The questionnaire included also statements about the following functions of proof: verification, explanation, transfer, aesthetics and intellectual challenge (statements 10-22). These functions were chosen for the study, because they were assumed as the most relevant and best known by the respondents.

Students were asked to respond how they agree or disagree with the presented statements using a six-point-scale (1 = strongly disagree, 6 = strongly agree). An even-point scale was selected, because it forces respondents to either agree or disagree by omitting a neutral option (Allen & Seaman, 2007). In the case of an odd-point scale, it might have been too easy to take a neutral view without reflecting the statement. In the questionnaire, the statements were presented in a mixed order. It was assumed that in this way the students might react to each statement without comparing them to other statements, and, thus, responses between statements might be more independent. However, mutual dependencies between the statements are not studied in this paper.

After the survey there was also the following open question:

> Please mention some reasons why students should familiarize themselves with proofs and proving in school mathematics.

The aim of the survey was to measure *how strongly* students appreciated different aspects of proof and proving. Furthermore, the open question explored *what the reasons are* (the most important ones) for studying proofs and proving according to the students. The term ‘school mathematics’ was used in this question, because it was assumed that the respondents did not yet have experience in university mathematics, and it was aimed that their responses would be based on their experiences rather than preconceptions. Naturally, the respondents could get hints from the statements presented in the survey to their responses in the open question.

97 students from one Finnish university and 215 students from three Estonian universities participated in the questionnaire. Among the Finnish students, 38 (39 %) were majoring in mathematics, 21 (22 %) chemistry and 17 (18 %) physics. 47 out of the sample (49 %) were already studying in or planning to apply to a teacher education program. 89 students (90 %) had studied the advanced syllabus in mathematics in the upper secondary school. In the Estonian sample, 50 respondents (23.3 %) were majoring in mathematics, 24 (11.2 %) in mathematical statistics, 33 (15.3 %) in gene technology, 24 in other natural sciences (11.2 %) and 78 (36.2 %) in different areas of engineering or technology. 80% of Estonian

students had taken the extensive mathematics course during their secondary education.

The means and the standard deviations of the students' response distributions to the statements for both the Estonian and Finnish samples were calculated. Significances of the differences between the samples were analysed by applying the t-test. If the significance was under .06, Cohen d was also calculated. In the significance analysis, the equality of variances was tested by Levene's test. If Levene's test gave significance under .05, the equality of variances was not assumed in the t-test. Otherwise the t-test was completed by assuming equal variances.

To analyse students' responses to the open question all the proposed motives for studying proof and proving were listed and grouped by the similarity. Later the number of responses in each motive- group was counted. Each detected motive was, if possible, also related to a certain function of proof. Students' responses were first analysed by the research team member by the respective country, who read texts in the original language and created initial categories to classify the proposed statements. Then all the categories together with examples of statements were translated into English. In the following the initial categorizations were jointly discussed. After several cycles of similar analyses, the final list of categories was fixed. Later the number of responses in each category was counted. Each detected motive was, if possible, also related to a certain function of proof (see Table 2 in Results).

Results

The results concerning the survey are presented in Table 1. Students in both the countries were quite convinced that proof does not belong only to university mathematics (S4) but it should also be studied at least at the upper secondary level (S3). The Estonian students were a little bit more critical than the Finnish students with respect to the statement about practicing proof and reasoning in the lower secondary school (S2). In regard to practicing proof and reasoning also in the primary school (S1), the variances of students' responses were quite large in both the countries. There was a significant difference between Estonian and Finnish students' responses about the idea of including problems related to proving and derivations into the final or national examinations (S5). The Estonian students tended to oppose the idea while the Finnish students were significantly more positive toward it. When proof and proving were contrasted with the practical applications of mathematical knowledge (S6), most of the students claimed that practical applications are more important to learn. Among the Estonian students this view proved to be stronger than among the Finnish students.

Students from both the countries tended to support the idea that in mathematics no claim can be considered true before it has completely been proven (S8). The

Finnish students tended to support also the claim that proof is the most central activity for the mathematicians while the Estonian students stayed neutral (S7). Students from both countries also lightly agreed (on average) that proving skills are important in applying mathematics to practical problems (S9). Thus, the importance of proof in mathematics was generally supported by the respondents.

Students considered proof to be a powerful tool in verifying mathematical statements (S10). Among the Finnish students this belief was somewhat stronger than among the Estonian students. However, students from both countries stayed neutral towards the necessity of proof in convincing about the truth of mathematical statements (S12). Instead, students were not very convinced about the necessity to feel uncertainty about the truth of the claim before proving (S11). With respect to the explanation-function, students from both countries seemed to be equally supportive. On average, they agreed that proofs help to understand mathematical connections (S13), they considered proofs to be important in presenting answers to why-questions (S14) and they also agreed (at least lightly) that proofs are needed in understanding how mathematical truths are derived (S15). However, the nature of these statements has to be taken into account – it may be difficult for students to strongly disagree with them.

The students considered proving exercises as an important tool for developing logical thinking (S16). They also believed that proofs develop critical thinking (S18). Students from both the countries tended to stay somewhat neutral towards the claim that proofs teach students techniques that are valuable in other contexts (S17). In the case of questions concerning the aesthetics of proofs (S19 and S20), students did not have a strong opinion on average and the variances of their responses were quite large. However, the results indicate that the Finnish students were more supportive toward the statements on the aesthetic elements of proof than the Estonian students. The students were also very convinced that proving tasks offer intellectual challenges (S21), and that they are suitable for students who like challenges (S22). Again, however, the Finnish students were more supportive than their Estonian colleagues.

	Estonia		**Finland**			
	Mean	**St. dev**	**Mean**	**St. dev**	**Sig. (t-test)**	**Cohen d**
Role of proof in school mathematics						
1. Pupils should somehow practice proof and reasoning in primary school (age about 6-12).	3.4	1.5	3.7	1.3		
2. Pupils should somehow practice proof and reasoning in lower secondary school (age about 13-16).	4.3	1.4	4.7	0.9	.008 (-)	0.33
3. Proof should be included in most mathematics courses in upper secondary school.	5.0	1.2	4.8	1.0		
4. Proof belongs only to university mathematics.	2.6	1.3	2.6	1.2		
5. Problems related to proving or derivations should be represented also in final examinations/national examinations.	2.6	1.5	3.9	1.3	.000 (-)	1.06

6. It is more important for the students to learn practical applications of mathematical knowledge than work with proofs and proving.	4.6	1.2	4.2	1.1	.003 (-)	-0.42
Role of proof in mathematics						
7. Proof is the most central activity of the mathematicians in their work.	3.9	1.1	4.3	1.4	.004 (+)	0.33
8. In mathematics, no claim can be considered to be true before it has been completely proven.	4.4	1.5	4.7	1.3		
9. Proving is an important skill when applying mathematics to practical problems.	4.0	1.3	4.2	1.1		
Verification						
10. Proof is a powerful tool in order to verify mathematical statements.	4.7	1.1	5.0	0.8	.033 (-)	0.27
11. Students have to feel an uncertainty about the truth of the claim; otherwise there is no use to give a proof.	3.1	1.5	3.8	1.1	.000 (-)	0.55
12. Proof is needed in order to become convinced about the truth of mathematical statements.	4.2	1.2	4.2	1.0		
Explanation						
13. Proofs and proving help students to understand mathematical connections.	4.6	1.2	4.7	0.8		
14. Proofs are important to present as answers to why-questions.	4.8	1.2	4.9	1.0		
15. Students need to always see the proofs in order to understand how mathematical truths are derived.	4.5	1.3	4.4	0.9		
Transfer						
16. Exercises in proving are important for students to develop logical thinking skills.	4.7	1.2	4.8	0.8		
17. Proofs teach students techniques that are valuable in other contexts.	4.2	1.2	4.3	1.0		
18. Proof and proving develop students' critical thinking.	4.3	1.2	4.7	0.9	.002 (-)	0.40
Aesthetics						
19. Through proofs and proving, students have a possibility of experiencing the beauty of mathematics.	3.5	1.4	4.0	1.2	.001 (-)	0.44
20. The proofs suitable for school level have not the potential of offering students an aesthetic experience.	3.8	1.3	3.4	1.1	.003 (+)	-0.34
Intellectual challenge						
21. Proving tasks offer students intellectual challenge.	4.3	1.3	5.0	0.8	.000 (-)	0.74
22. Proving tasks are suitable for students who need challenges.	4.6	1.2	5.0	1.1	.006 (-)	0.38

Table 1: The means and the standard deviations of students' responses for the statements in both the countries. Marking (+) after the significance of the t-test means that the equality of variances was assumed and marking (-) means that it was not assumed.

In their responses to the open question (see Table 2) students dominantly emphasized the explanatory power of proof. More than half of the respondents thought that proof should be treated in the secondary mathematics because it "explains the content", "supports understanding" or "answers to why-questions". The second most popular motive suggested by the students was the development of learners' thinking, reasoning or argumentation skills. Almost one fifth of the Estonian students and about one fourth of the Finnish students mentioned this in their responses. In addition to reasons mentioned in Table 2, some students pointed out the importance of proof in the developing of problem-solving skills (2.8 % of Estonian students and 2 % of Finnish students) or application skills (1.9% and 5 % respectively). These reasons refer to the transfer-function. Only few respondents indicated that the role of proof in the secondary mathematics is to establish the truth of mathematical statements (0 % and 2 %). Also, motives related to the discovery (0 % and 2 %) or intellectual challenge (0.5 % and 0 %)) were mentioned only by very few students, and reasons referring to systematization, communication or aesthetics were not mentioned at all. Not many students questioned or denied the need to study proofs in their responses.

Reason to study proofs and proving	Function	Est (%)	Fin (%)
Support for understanding, explanations to why-questions	Explain.	54.9	54
Development of thinking, reasoning or argumentation skills	Transfer	19.5	25
Support for further studies in mathematics	–	15.8	8
Learning about the nature of mathematics	–	7.0	3
Support for remembering or less things to remember	–	4.7	3
Need to study proofs denied or questioned	–	4.2	2
No reasons mentioned	–	0.0	14

Table 2: The most frequently mentioned reasons to study proofs and proving in school mathematics. Proportional distributions among the samples of Estonian (n=215) and Finnish (n=97) students.

Discussion

Students in both the countries highly appreciated the role of proof both in the school mathematics as well as in mathematics. In general, the Estonian students seemed to be more critical toward proof and proving and to the usefulness of the functions of proof than the Finnish students. A comparative analysis of curricula (Hemmi et al. 2013) revealed that the Estonian approach to proving tends to be more traditional while Finnish mathematics education has implemented so-called developmental proof approach, which means that proof-related activities are trained little by little so that word proof is necessarily not explicitly mentioned. In addition, proving tasks are more common in the school mathematics in Finland than in Estonia. These reasons may explain why the Finnish students saw proof and proving in a more positive light than their Estonian colleagues. However, it has to be noted that there were differences between the Estonian and Finnish samples at least in students' study programs and intentions with respect to studies. These may also have an effect on the observed differences.

Students in both the countries agreed that proof should be treated in upper secondary mathematics (S3). Differently from their Estonian counterparts, the Finns also tend to support the idea of practicing proof and reasoning already in lower secondary school (S2). According to the curricula the teaching practice is contradictory to this result: Proof is explicitly introduced in the lower secondary level in Estonia and not in Finland (Hemmi et al., 2013). This might be explained by the fact that students in these countries may have different views about the nature of proof. Supposedly, many students in Estonia find the formal and rigorous way, in which proof is introduced in Estonian schools, to be difficult and unsuitable for younger pupils. On the contrary, if proof is introduced by applying the developmental approach as prescribed in Finnish lower secondary curriculum, proof may appear more achievable for pupils at lower secondary level.

Based on the results, the students highly appreciated the role of proof in the understanding of mathematics and in the learning of logical thinking skills. These refer to the explanation- and transfer-functions of proof. When reasons to study proof and proving were explicitly asked, most often the students mentioned reasons that also referred to these functions. Other functions were either not mentioned at all or mentioned only in a few responses to the open question. Explanations- and transfer-functions were also emphasized in Knuth's (2002) study, where secondary school mathematics teachers' views about the role of proof in school context were examined.

It is surprising that only few students mentioned the verification-function in their responses to the open question, even though the students generally agreed in the survey that proof is a powerful tool to verify mathematical statements. On the basis of the survey, it seems that the verification-function is generally acknowledged by the students, but at the same time they remain neutral toward the claim that proof is needed to become convinced about the truth of mathematical statements (S12). It seems that the students are ready to accept the results presented in the textbooks without any proof, and the learning of ready-made proofs has some other goals than ensuring the truth of presented mathematical results. This is supported also by the result that the students were lightly critical towards the claim that feel of an uncertainty would be a prerequisite for to give a proof (S11). Therefore, they seem to accept that proof is a way of communicating mathematics independently from the need to verify the truth.

In the case of questions concerning the aesthetics of proofs (S19 and S20), students unanimously stayed neutral. The result tends to indicate that the way proof has been treated in school has not provided students with the possibilities to experience the beauty of mathematics. Also, support for further studies in mathematics was mentioned by many as the motive to study proof-related issues in school. Because the sample consisted of first-year students who all were studying mathematics at tertiary level, probably the respondents felt personally the difference in approaches between secondary and university mathematics, especially that in university mathematics proof has a more central role.

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The national validation of Finnish mathematics teachers' lexicon

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This report describes the national validation of Finnish mathematics teachers' professional lexicon for describing events in the mathematics classroom. As part of an international Lexicon project, we had created a lexicon of 104 terms and their more extensive descriptions. This was then validated through the responses of 72 Finnish mathematics teachers. Overall, the terms were very familiar to the respondents, although some terms were somewhat less frequently in use. Some terms were clearly problematic and require modifications. Overall, the teacher responses suggest that the Finnish mathematics teachers' terminology is more focused on teacher-student interaction and lesson organization rather than mathematics specific aspects of teaching.

Introduction

Our language often enables and limits our thinking in ways we are not fully aware of (Lakoff & Johnson, 1980). Therefore, examining the professional language of teachers is one method to examine the pedagogical thinking of teachers. A study on the metaphors used by Finnish mathematics teachers (Oksanen, Portaankorva-Koivisto & Hannula, 2014) reported that most of them saw themselves primarily as experts in mathematics teaching (51%) while some saw themselves as experts in pedagogy (14%), and only a few used metaphors highlighting their role as experts in mathematics (6%).

In our current study we look at the Finnish mathematics teachers' language on a more fundamental level. What teachers see in a classroom situation, and even more strongly, what they can think and discuss about is mediated by what they can name. The richer and more nuanced the teachers' professional language is, the more elaborate reflections and discussions are possible (Mesiti et al., 2016). Clarke (e.g. 2013, see also Mesiti et al., 2016) has pointed out that the language differences have implications for international comparative research.

The international Lexicon project aims to identify and compare the lexica used by mathematics teachers for describing mathematics lesson events in Australia, Chile, China, The Czech Republic, Finland, France, Germany, Japan, and The USA (Mesiti et al. 2016). The purpose of the research project is to identify how mathematics teachers in different countries see the teaching-learning process and the terminology used by professional educators. The national lexica will improve possibilities for international comparative research.

Our research question is: Which terms for describing educational events in a mathematics classroom do the Finnish mathematics teachers recognize and use frequently?

This presentation outlines the process for generating the first draft version of the Finnish Lexicon and its national validation. At this stage the reporting is mostly descriptive. However, there are some tentative conclusions at the end.

Method

The generation and development of the national Lexicon

For the generation of the lexical terms, the Lexicon project wanted to avoid too strong influence of the academic research terminology. Therefore, experienced practicing teachers had a key role in the process. In Finland, the lexicon was first generated by a team consisting of the author and three experienced mathematics teachers, who alternated between viewing and annotating video events and discussing to find consensus on the relevance of each term. The team used lesson videos from grade eight mathematics lessons from the nine participating countries as a stimulus to identify activities they have a name for.

In Finland, this process of naming events led to a realization that many of the important things that teachers name in the lessons are not activities, as suggested by the original protocol. For example, the term “Revision” is not used primarily as a name for an event, but rather as a qualifier for several different things, such as “A revision lesson” or “A revision task”. Other terms that did not refer to activities were “Realization” (The moment the student ‘gets it’), “Lesson plan”, and “Use of humour”.

The first draft version was discussed at a Lexicon project meeting which inspired generation of some additional terms that were later approved of by the expert teachers. Moreover, we clarified our definition regarding the scope of the lexicon. For example, we decided to exclude terminology that is specifically mathematical. After these amendments, the Finnish Lexicon included 104 terms. Each term was accompanied with a verbal description and two examples as well as a non-example that was almost within the meaning of the term, but not quite. The 104 terms were categorized under five categories: “Kasvatus” (upbringing/education/fostering; in Swedish “Uppfostran”); Organizing; Evaluation; Teaching methods; and Mathematical content

Procedure for validation

In Lexicon project each participating country was responsible to design and implement their own national validation. In order to validate the Finnish lexicon an electronic survey was conducted in November-December 2016. The validation study aimed at recognizing how familiar the terms were for the Finnish mathematics teachers, how frequently they use these terms, and how well they recognized the terms from the descriptions and examples. Moreover, they were asked to suggest new lexical terms to be included and improvements for the names and descriptions given by us.

The survey was influenced by the Australian survey for their national validation, but it was made shorter by removing some sections. The Finnish survey consisted of six sections: 1) Demographics, 2) Giving the term only and asking four different questions about that term: a) How familiar is the term?; b) How often do you use the term?; c) How often do your colleagues use the term?; and d) How often does the phenomenon referred to by the term happen?, 3) After being presented with the verbal description, examples, and the non-example, the respondent was asked to suggest a lexical term matching the description, 4) After being given a full description including the term, the respondent was asked the familiarity of the term and to suggest improvements for the term or its description. 5) After presenting a list of all terms (including synonyms, alphabetically arranged) the respondent was asked to suggest additions to the list, and finally 6) a Thank you -page asking for contact information for future surveys and with information about reward lottery. For sections 2 and 4 five point response scales were used. Four parallel versions of the survey were developed, rotating all lexical terms through sections 2 to 4. In each version each of the sections included 26 terms. Because we were worried about the length of the survey, we encouraged the participants to skip the open response items and respond to the multiple choice items, if in a hurry.

We first piloted one version of the survey to identify possible glitches with the form and confirming that the survey is not too exhaustive for the respondents. We got 6 responses in the pilot study, most responding only to the multiple choice items. The careful completion of all items had taken one respondent 45 minutes while those who responded to multiple choice items only were able to complete the survey in ten minutes. Based on the pilot study we corrected a couple of minor errors and these six responses are included in the pool of responses.

Data

The main validation survey and one reminder letter were distributed in November –December 2016 through MAOL (mathematics teachers' union) weekly newsletter that has 4400 recipients. The four different versions of the survey were randomized by asking the respondent to select one of the four possible links based on the month of their birthday. The survey was also sent through the mailing list of Finnish Mathematics and Science Education Researchers' Association with about 200 recipients. Moreover, I used my personal contacts to ask about 20 teachers to fill in the survey.

Altogether we received 77 responses to the survey. Out of these responses 72 were by mathematics teachers and only these are analyzed for validation. The four different versions of the survey received 24 /17 / 11/ 25 responses. Based on uneven numbers and the geographical bias of different survey's responses we assume that the randomization was not always followed, and we suspect that some participants have shared with their colleagues the links to a specific survey rather than the randomization starting page. However, for our purposes this is not a significant problem as the respondents in all four versions still represent a broad variation of geographic regions and ages.

For the open response items, the number of responses was smaller. For naming the Lexical terms based on the long description, we got fewer responses towards the end of the survey. The number of suggested term names varied for the different survey versions between 13-20/ 4-8 / 7-8 / 9-15. In addition, we received 140 suggestions to improve descriptions. Moreover, 17 persons made altogether 78 suggestions for adding in total 49 new terms to the lexicon. Out of these we have selected 40 new terms that we will include in the next round of validation.

Analysis

Our data analysis consisted of three stages. First, we ran some descriptive statistics on the respondent populations to confirm that there are no significant biases towards certain types of respondents. Next, we computed the mean values and standard deviations for each survey item type to get an overall feeling of the data set. Finally, our main validation analysis was based on identifying the most familiar and unfamiliar lexical terms based on the following criteria.

For familiarity, the validation results had to meet at least two of the following four criteria:

- Rather or very familiar to over 90% of respondents
- Used frequently (2 highest options) by most (>50%) respondent or colleagues
- The respective event occurs frequently (2 highest options) in most respondents (>50%) classes
- Most respondents (> 50%) are able to produce the correct term or its synonym based on the description

For unfamiliar terms the validation results had to meet at least one of the following four criteria:

- Very familiar for less than half of the respondents
- Most respondents (>50%) use the term seldom or never.
- Most respondents (>50%) identify the event happening seldom or never.
- Less than one third of the respondents (< 33%) are able to produce the correct term or its synonym based on the description

In addition, we identified lexical terms that fulfilled at least one criteria for both familiar and unfamiliar terms. We call these contradictory terms.

Results

The results section includes a description of respondent demographics, an overall summary of the responses, an analysis identifying the most familiar terms, an analysis identifying unfamiliar terms and an analysis of contradictory terms that were identified as both familiar and unfamiliar.

All respondents have studied mathematics either to master level (51 respondents) or to bachelor level (20 respondents). They all have the formal mathematics teacher qualifications and teach mathematics. Nine of them teach only mathematics, 50 also science (physics and/or chemistry), 22 also computer science, and 1 teaches another subject. Most of them (45) teach at lower secondary level, 26 at upper secondary level, 2 at elementary level, 6 at vocational education, and 3 at tertiary education. Fifteen of these responses include teaching at more than one level.

The overall outcome of the survey was that the teachers were familiar with the given terms, but not all terminology was in frequent use (Table 1). We see that the overall familiarity of the items (and the variation of responses) did not depend on whether the respondent was given the term only or a longer description. Therefore, we decided to combine the two survey item types for familiarity for further analysis. Similarly, the frequency of usage of terms by the respondent and their colleagues was rather similar and we decided to combine also these data in our future analysis.

Survey item type		$\bar{x}$	SD
Term only	How familiar?	4.5	0.89
	How often you use?	3.1	1.23
	How often your colleagues use?	3.0	1.14
	How often this thing happens?	3.7	1.14
Full description	How familiar?	4.5	0.85

Table 1. The mean values and standard deviations for different survey item types

Most familiar lexical terms

Altogether close to half of the terms (47) fulfilled at least two criteria for being familiar and no criteria for being unfamiliar.

In the area of “Kasvatus” (Upbringing/education/fostering; In Swedish “Uppfostran”) eight of the 15 terms were identified as familiar: Use of humour, Classroom climate, Good working climate, Maintaining good working climate, Encouragement and pep, Caring, Bullying, and School rules.

In the area of Organizing 13 of the 23 terms were identified as familiar: Communication between school and home, Distribution of material, Differentiation, Opening the lesson, Lesson plan, Take Attendance, Material, Use of technology, Use of material from the web, Student collaboration, Group work, Scheduling, Seating order, Giving instructions, and Notebook work.

In the area of Evaluation 9 of the 16 terms were identified as familiar: Giving homework, Checking homework, Explaining and discussing assessment, Setting assessment goals, Self-evaluation, Giving feedback, Providing positive feedback, Test, and Returning assessed tests.

In the area of Teaching methods, 10 of the 38 terms were identified as familiar: Orienting, Independent work, Student raises their hand, Student response, Student question, Request for justification, Summary, Revision, Worked-out example, Guidance, Realization.

Finally, in the area of Mathematical content, 4 of the 12 terms were identified as familiar: Word problem, Exact mathematical language, Mental calculation, and Application task.

Although a large part of the terminology was confirmed to be well recognized, some of these terms or their descriptions still need to be reconsidered, because the names generated by the respondents did not always match the name we had chosen. For example, most suggested the names “Class spirit” or “Group spirit” for our description of “Classroom climate”. As another example, 91% of the respondents recognized the term “Exact mathematical language” as familiar, yet none of the respondents was able to produce the same exact term based on the description.

The unfamiliar lexical terms

Altogether 17 terms were identified as unfamiliar. Their validation results are presented in Table 2. The results indicate that even the unfamiliar terms are familiar to and used by quite many of the respondents, highlighting the important difference between teachers’ active and passive vocabulary. Moreover, even the least familiar terms are very familiar to a significant share of the respondents. Therefore, the question is not so much whether these terms are not part of the Finnish mathematics teachers' lexicon, but, rather, whether they are more essential than many others that we have not thought of. For example, the poor validation results for the specific terms of teacher response to student answers (confirming, amending, or rejecting) suggest some kind of bias present in the process of generating the lexicon.

After a closer examination, we consider removing only four of the items (Orienting students for a work mode, Brainstorming, Confirming a response, Rejecting a response). However, we plan to rename some terms and revise some description and then include all these unfamiliar items in the second round of validation to get more data for deciding which terms to leave in. Moreover, we plan to combine the term “Hurrying a student” with a very similar term “Prompting a student to work” that appears as one of the contradictory terms.

Lexical term	Very familiar (%)	Not used (%)	Not occurring (%)	Naming success*/attempts
Education of good manners	56	43	24	2/6
Orienting students for a work mode	27	95	90	2/15
Student assisting the teacher	42	67	57	2/4
Getting the attention of the class	73	62	40	2/15
Encouragement by peers	55	59	60	5/14
Teacher lecturing	65	45	24	1/7
Hurrying the students	48	68	21	3/10
Reading the textbook	78	52	7	6/12
Debate	59	74	64	3/14
Checking the result from a “solution book”	81	55	21	3/14
Going through the group work outcomes	45	73	60	7/15
Brainstorming	45	74	71	2/6
Confirming a response	27	91	30	5/14
Amending a response	55	59	10	6/16
Rejecting a response	27	95	60	3/13
Routine exercise	59	53	14	3/13

Table 2. Validation results for unfamiliar terms. *A naming is considered successful, if the respondent produces the correct term or a synonym of it.

Contradictory lexical terms

The validation indicated 22 contradictory terms that met at least one of the criteria for both being familiar and for being unfamiliar. Of course, the contradiction is only apparent, as the criteria for familiarity address different dimensions. A term may fall into this category, for example, if it is well known by teachers, but the event happens very seldom.

These include four terms that describe an undesirable event in the classroom: Mocking a student, Teacher’s pet, Cheating in test, and Cramming. These terms are not used, and these do not occur (except for cramming). However, teachers are very familiar with these terms and especially they were surprisingly successful in naming these events. We believe that it is important that the lexicon includes also terminology for undesired events and we intend to keep these terms in the lexicon. However, we use the validation information to make some modifications (e.g. replace "Mocking a student" with "Embarrassing a student").

Another five of these terms related to events that occur very frequently and the term is recognized, but the event is perhaps so uninteresting that the term is not used: Repeating a response, Closing a lesson, Mathematical content, Connecting to earlier, Paralleling ideas.

Two of the contradictory terms were specific pedagogical practices that seem to be unevenly distributed among teachers: Flipped learning, Personal feedback discussion.

There are also terms, where we have chosen a rather extraordinary concept rather than a better known alternative. For example, the respondents recognize the term "Lesson structure" but they prefer using the term "Lesson plan". In a comparable vein, we suggested "Teacher question" while the respondents preferred a broader term "Teaching discussion". Moreover, we suggested terms "Evaluating student solution" and "Checking lesson task", while our respondents seemed to prefer a more general term "Going through solutions". Lastly, we had used a specific term "A hint", but our respondents suggested the more general term "Guidance".

Finally, there were four terms, where we had either chosen an unfamiliar name for a familiar concept, or our description of the term was misleading. As the problems with these terms were specific to Finnish language, we cannot describe them here in detail. Nevertheless, we intend to revise term names and descriptions and revalidate the terms.

Conclusion

The national validation of the Finnish mathematics teachers' lexicon was quite successful. Teachers were very familiar with most terms and we have identified more than 50 key terms for a national lexicon for Mathematics Teachers. There was a slight tendency for the terminology for organizing and evaluation to be more familiar to the teachers than the terminology for teaching methods and mathematical content.

Among the most familiar terms there were many and specific concepts relating to the good teacher-student relationship, including "Use of humour", "Classroom climate", "Maintaining good working climate", and "Caring".

There are also specific terms related to how the lesson can be organized. For example, the following terms more or less define a typical Finnish mathematics lesson: "Checking homework", "Orienting", "Worked-out example", "Giving instructions", "Distribution of materials", "Independent work", "Guidance", "Differentiation", "Summary", "Giving homework". Of course, there is some variation, as the terms “Student collaboration”, “Group work”, “Use of technology”, “Revision”, and “Notebook work” indicate.

With respect to teacher-student interaction during guidance, we see here some interesting specific terminology: “Student question”, “Request for justification”, and “Realization”. It is also worth noting, that the Finnish word for Guidance, “Ohjaus”, means “To steer”. As the Finnish term relates to movement rather than building, we have decided to not use “Scaffolding” as the English translation.

On the other hand, few words specific to mathematics teaching met the criteria of familiar terminology. There were three terms for specific types of mathematical task, and the term “Exact mathematical language”. Furthermore, the terms “Mathematical content”, and “Paralleling ideas” we recognized quite well, but used very little.

Taken together, this all suggests that the Finnish mathematics teachers conceptualize their teaching primarily through their relationship and interaction with their students, rather than through the teaching of mathematical content. One might argue that the extent of terminology related to a topic is not necessarily an indication of the perceived importance of that topic. However, if there is significant and continued attention and discussion on a topic, would that not inevitably lead to a more detailed vocabulary to foster such discussions?

We realize that the number of responding teachers was not high, especially regarding the least popular version of the survey (11 respondents). Therefore, it is important to get additional validation data to make more informative judgement regarding unfamiliar terms.

When comparing these results with the earlier metaphor study (Oksanen et al., 2014), we can see that the results of both studies suggest a primary focus on the expertise in organizing and orchestrating mathematics teaching, while some attention is given to general pedagogy (“Kasvatus”), and rather little attention is placed on the content knowledge. Taken together, these studies indicate that the main focus of Finnish mathematics teachers – at least in their language – is on teaching. They do pay some attention on educating the child, but quite little on the content itself.

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A correlation study of mathematics proficiency VS reading and spelling proficiency

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This study tested the connections between different elements of language competence and mathematics competence. We tested the proficiency in mathematics, reading and writing/spelling for 2376 Norwegian students in grades 5, 6, 8 and 9. We found a correlation between proficiency in mathematics and reading comprehension as expected from previous studies. More interesting is that spelling, tested with a dictation, seems to correlate stronger with mathematics on grades 5 and 6 than comprehension does. This supports an assumption that the correlation between proficiency in language and mathematics is not simply a matter of the ability to read and understand the mathematical task.

Linguistic skills, reading, writing and mathematics.

It is important for a student to have proficiency in reading comprehension when solving mathematical tasks (Adelson, Dickinson, & Cunningham, 2015; Fuchs, Fuchs, Compton, Hamlett, & Wang, 2015; Nortvedt, Gustafsson, & Lehre, 2016; Pearce, Bruun, Skinner, & Lopez-Mohler, 2013). The student has to read and comprehend the task in order to solve it. The effect of language and linguistic skills are prominent when students solve mathematics word problems (Abedi & Lord, 2001; Nortvedt, 2010; Vilenius-Tuohimaa, Aunola, & Nurmi, 2008), but are also related to solving other types of mathematics tasks (Vukovic & Lesaux, 2013). In the present paper, the correlation between reading comprehension and results on a mathematics test is studied.

Studies of mathematics and reading have mostly dealt with comprehension. Comprehension means to understand both what the words and the sentences mean. The process of reading is more than comprehension. We can roughly split it in two parts, decoding and comprehension. Decoding means to be able to identify the characters and combine them into words that are pronounced. Reading speed can be used as a measure of decoding. Gough and Tunmer (1986) and Hoover and Gough (1990) suggested *The simple view of reading (SVR)* as a measure of a person's ability in reading. They claimed that reading could best be understood as a combination of decoding and comprehension, and proposed the SVR-formula for reading, reading = decoding x comprehension (R = D x C). Later studies give support to the SVR model (Adlof, Catts, & Little, 2006; Kendeou, Savage, & van den Broek, 2009). In this paper, we will also compare the SVR scores with proficiency in mathematics.

In Norway, we often test students' proficiency in spelling with dictations. The teacher reads a text, sentence by sentence, and the students write down the sentence from memory. This rather complex process relies on listening comprehension, short-term and working memory and writing. It has been shown that short-term or working memory (Baddeley & Hitch, 1974) influence language comprehension (Daneman & Merikle, 1996) and comprehension of oral messages and ability to follow directions (Engle, Carullo, & Collins, 1991). Solving a mathematical task is in some ways a similar process. It relies also on comprehension, this time reading, on memory, and on writing. Studies show that the capacity of short-term or working memory (Baddeley & Hitch, 1974) has an influence on mathematics achievement (De Smedt et al., 2009; Gersten, Jordan, & Flojo, 2005; Raghubar, Barnes, & Hecht, 2010; Siegel & Linder, 1984). It has also been shown that measures of short-term memory at the age of 4 is a good predictor of later proficiency in mathematics (Bull, Espy, & Wiebe, 2008). These factors predict a possible correlation between scores on mathematics test and scores based on dictations.

The research questions are:

How do results on a mathematics test correlate with results from reading and writing tests?

How do these correlations compare to each other?

Method and data sources

The analyses in this paper are based on data collected by the Norwegian SPEED project (The Function of Special Education) (Haug, 2017). The project's principal aim was to study special education, not as an isolated subject, but as an integrated part of the overall education. We studied both special- and ordinary education, and both students with and without special needs, with a variety of instruments. The SPEED-project is a rather large study with a sample of more than 2500 students and their teachers and parents. This large sample is one of the strengths of the study reported in this paper. The students covered a wide range of both mathematics and language skills, and were tested with general mathematics and language tests, making it possible to study relations between these two over a wide range of skills.

This paper use data from a mathematics test and a language test comprised of both a reading and a spelling test. Results from all students was included in the analysis, regardless of their level in mathematics or language, in order to bring to the fore results that are valid for the entire proficiency span.

For a more comprehensive account of the whole project see Haug (2017) and of the methods used see Topphol, Haug, and Nordahl (2017). Only the parts relevant for this paper will be explained here.

The sample

The SPEED-project collected data from 29 schools in two medium sized municipalities. The two municipalities were from different parts of Norway, representing a variety of cultural, social and other backgrounds. We invited all students in grades 5, 6, 8 and 9 to participate during winter and spring 2013. A total of 2376 students completed both the mathematics and the language test, 70 % of all the students, and 86 % of those who had consented to participation. Although the sample did not meet the requirements of randomness needed for statistical generalization, we will argue that the broad coverage in background makes the results valid for a larger population than the two municipalities only. Analysis also showed that our data conformed to national statistics on important factors (Topphol et al., 2017).

The mathematics test

All students were given 40 multiple-choice tasks. Students in grades 8 and 9 were given 12 additional multiple-choice tasks. Every task had seven response alternatives including “do not know”. One of the alternatives was the correct answer, and the rest were so-called distractors. The assignments were paper-based with check boxes making digitizing through optical scanning possible. Researchers in the SPEED-project developed the tests. The tasks covered mathematical topics, and had a level of difficulty, that was in accordance with the Norwegian curriculum. The majority of the tasks were based on situations the students could meet outside the classroom, in their daily life, such as understanding the clock, bus schedules, fractions, decimal numbers, geometry, arithmetic and statistics. There were a mixture of word problems and non-word problems. The construction of the mathematics test is discussed in more detail in Opsvik and Skorpen (2017).

For each student we used his or her percentage of correct answers as the mathematics score. In order to eliminate the effect of grade, and of the two tests being slightly different, the scores were normalized to have mean value equal to zero and standard deviation equal to one for each grade separately, z-scores.

The language test

We used *Norwegian spelling and reading test for compulsory primary and secondary school*[3] (the Carlsten-test) (Carlsten, 2002) to measure the students’ proficiency in reading and writing. Carlsten developed this test primarily as a screening test to identify students struggling with vital areas of the Norwegian language, and not as a research tool. Nevertheless, we chose to use this test mainly by two reasons. First, the test has been widely used in Norwegian schools for many years. The teachers know the test well and can easily relate our results to their

[3] The author’s translation of “Norsk rettskrivings- og leseprøve for grunnskolen”.

classroom situation. Secondly, the test's aim fitted well with the SPEED-project's goal and with the aim of the mathematics test.

The test estimated reading proficiency as both reading speed and reading comprehension. The students read a narrative text, and the unit of speed was words per minute. We measured comprehension through multiple-choice questions. In several places in the text, the reader was supposed to pick the correct word from three alternatives. An example from the 6th grade test: "Hard as (stone–wool–tree)"[4]. We used percentage of correctly chosen words as a measure of comprehension.

The reading proficiency in this context was related to verbal text. The test provided information about parts of the students' literacy but gave us no information about the students' reading skills related to interpretation, evaluation and reflection.

We tested the writing proficiency with a dictation. The teacher read a text, sentence by sentence, and the students wrote down each sentence from memory. The test was thus more than a pure writing test. It relied on both listening comprehension, and on the ability to remember the sentence. We calculated two dictation based scores from the number of errors the student made. The number of spelling errors made a *spelling score*. The total number of errors, both spelling errors and missing words or sentences, made what is called the *dictation score*.

As with the mathematics test, scores were normalized to z-scores for each grade separately. Reading score understood as *simple view of reading* was calculated as the product of the speed and comprehension scores and normalizes as above.

Analysis

Ordinary Pearson's product moment correlation coefficients between the mathematics score and the different language scores were calculated. Linearity was tested with simple scatterplots, which revealed no indications of non-linearity.

The distributions of the reading comprehension and the two dictation based scores were rather skew, with an accumulation towards the high values. They were negatively skew and in addition rather narrow. This was a result of the Carlsten test's aim towards the less proficient students. A substantial part of the students reached the maximal score. This could give smaller correlation coefficients than a test that also challenged the best students would do.

I used statistical tests to compare correlation coefficients. Since all the coefficients were calculated with the mathematics score as one of the variables, the null hypotheses were of the form $\rho_{xz} = \rho_{yz}$. This means tests of equality of dependent correlations. William's (1959) formula was used to calculate p-values,

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[4] The author's translation

in accordance with Steiger (1980) and Chen and Popovich (2002) suggesting to use this formula for such tests.

Results

Table 1 contains Pearson's correlation coefficients between the students' mathematics results and their results on the language tests; the reading scores: *decoding*, *comprehension* and *simple view of reading*, and the two dictation based scores: *spelling score*, based solely on spelling errors, and *dictation score*, based on all errors. Results are presented for all the students together and split by grade; 5, 6, 8, and 9.

Grade	N	Reading decoding	Reading compre-hension	Reading (Simple View of Reading)	Dictation score	Spelling score
All	2376	.317**	.397**	.388**	.453**	.428**
5	560	.403**	.354**	.459**	.491**	.478**
6	626	.279**	.346**	.317**	.443**	.437**
8	619	.292**	.469**	.395**	.491**	.446**
9	571	.303**	.415**	.388**	.387**	.352**
**. Correlation is significant at the .01 level (2-tailed).						

Table 1: Correlation coefficients (Pearson) between mathematic results and five different language scores

All the correlation coefficients were significantly higher than zero with all p-values smaller than 10^{-11}.

From Table 1 we can see that, except for 5th grade, comprehension correlated more strongly with the mathematics score than reading speed did. This was as expected. When a student faces a mathematical task, there is no use in speed if she does not understand what she reads. The difference between speed and comprehension correlation was statistically significant for all students ($p = .0005$), for grade 8 ($p < .0001$) and for grade 9 ($p = .015$).

The correlation coefficients between mathematics and simple view of reading fell in between those with speed and with comprehension. This was not surprising, since it was constructed as the product of them. This will not be followed up any further in this paper.

We found interesting results when comparing the correlations between the mathematics score and the two dictation based scores, with the correlation between mathematics and the different reading scores. Except for the 9th grade, the two results from the dictation scores seemed to correlate at least as strong with the mathematics score as reading did. For the 5th and 6th grades, the *dictation score* correlated significantly more strongly with mathematics than reading comprehension did (p-values < .001). For the 8th grade, the difference was too small to be statistically significant. There seemed to be an age dependent effect, strongest with the youngest students in our sample. The same effect was present if we restrict the dictation data to the *spelling score* only, but now the difference was statistically significant only for the 5th grade (p = .0009). The *dictation score* correlated slightly more strongly with mathematics than the *spelling score* did.

Discussion

This paper examines the correlations between proficiency in mathematics and proficiency in reading and dictation. The correlation between mathematics and reading was found to be as expected from previous studies (Adelson et al., 2015; Fuchs et al., 2015; Nortvedt et al., 2016; Pearce et al., 2013). This agreement with earlier research serves primarily as a validation of the study and will not be further discussed.

In this study student competency to write sentences from dictation correlated as strong as, and even stronger for grades 5 and 6, with mathematics score than reading comprehension did. Correlation with reading can partly be explained by the necessity of reading and understanding a mathematics task before solving it. Correlation with the dictation scores, a writing test, cannot be explained in a similar way, by the necessity of writing to solve these tasks. The mathematics tests required just a small amount of writing. The tasks were multiple choice and the writing was thus limited only to some drafting on a separate paper. The relation between mathematical task solving and dictation must therefore be of a more complex nature. I will bring to the fore one possible explanation based on similarities in the process of solving mathematical tasks and in taking down dictations, similarities involving memory and memory effects.

Solving mathematics tasks, taking down a dictation and reading are all processes that are affected by the student's memory, through how working memory influences comprehension (Daneman & Merikle, 1996; Engle et al., 1991) and proficiency in mathematics (De Smedt et al., 2009; Gersten et al., 2005; Raghubar et al., 2010; Siegel & Linder, 1984). Working memory capacity and memory function will thus contribute to the correlation between all three of them. I will now argue that memory is a more crucial factor in mathematics task solving and dictation than in reading.

Solving mathematical tasks and taking down a dictation starts with an element of comprehension, one with reading and the other with listening to oral messages or instructions, both previously shown to be affected by memory capacity (Daneman & Merikle, 1996; Engle et al., 1991). This contributes indirectly to the correlation between them. The next step in both processes involves a more direct use of memory. When taking down a dictation the student has to remember the sentence, with the exact wording, long enough to be able to write it down, and when writing the words with correct spelling, she has to remember the words not written down yet. The mathematics student must remember the task, both the structure and the pieces of information, during the solving process. The last thing she does is to "write" down the answer by placing a mark in the correct check box. Of course, short-time memory is also involved in the reading process: the entire sentence has to be "remembered" to be understood, but this memory use is not to the same extent competing with other mental processes. The processes of mathematics task solving and dictation make in this way a more direct use of memory, and relies more heavily on it, than reading does. Memory will thus contribute more to the correlation with dictation than with reading. This can explain why the mathematics score correlated more strongly with the dictation scores than with reading. The result that the *dictation score*, with missing words and sentences, correlated more strongly with mathematics than the pure *spelling score* did, supports the assumption that memory plays a part, since missing words and sentences can be related to memory.

In Norway, one can often hear teachers complain about too extensive use of word problems in mathematics. They claim that students with reading difficulties get extra difficulties with mathematics because they struggle reading the tasks, especially tasks with a lot of text. This could of course be part of the explanation, but there are probably more to it than that. The results in this paper show that the relation between a student's proficiency in language and mathematics is more complex than the student's ability to read and understand the mathematical task. It involves also factors that influence the students' proficiency in taking down dictation. Memory can play a substantial part. If teachers do not take the influence of memory into consideration, the may miss an important factor.

Concluding remarks

Based on a large sample of students, covering a wide range of mathematics and language skills, this study has revealed that students' scores on a dictation correlated more strongly with mathematics than did their scores on a reading test. Based on similarities between the two activities, one possible explaining factor, working memory, has been discussed. More dedicated studies should be done to investigate this further.

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Students with low reading abilities and word problems in mathematics

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In this paper we explored whether text in word problems is extra challenging for students with low reading abilities in 5th and 6th grade. We analysed data from four tasks, which were part of a larger survey sample, and compared the results of students with low and students satisfactory reading abilities. Our findings indicate that text might be a barrier for students, but the context can also be a possible help for students in solving word problems.

Introduction

Word problems (WPs) in mathematics are not a recent notion. Some of the earliest example of human writing take the form of WPs (Swetz, 2009). "The term *word problem* is used to refer to any math exercise where significant background information on the problem is presented as text rather than in mathematical notation" (Boonen, Van der Schoot, Van Wesel, De Vries, & Jolles, 2013, p. 271).

Since most students face mathematics-related problems in written form in an out-of-school setting, it is natural that they should be taught and evaluated on their ability to solve WPs (Helwig, Rozek-Tedesco, Tindal, Heath, & Almond, 1999). When students solve a WP, they first have to read the text and then solve the problem. Students draw on both mathematical competence and general reading strategies when they solve WPs (Nortvedt, 2013). There has been a discussion about whether or not the extensive use of WPs makes it more difficult for students who have problem to read to learn mathematics. The purpose of this paper is to see if students with low reading abilities (LRA) struggle more with WPs in mathematics than students with satisfactory reading abilities (SRA).

Theoretical background

Students confronted with WPs in school "are engaged in a peculiar kind of activity wherein they typically solve these problems in a stereotyped and artificial way without relating them to any real-life situation" (Verschaffel, Greer, & De Corte, 2000, p. 12). According to Boaler (2009), students should not be involved in solving WPs that are in a context that requires them to engage partly in the real world while at the same time ignoring everything they know about the real world. Also Greer, Verschaffel, and Mukhopadhyay (2007) claims that student in mathematics learn to play what they call the "Word Problem Game" where one of the rules is "violations of your knowledge about the everyday world may be ignored" (Greer et al., 2007, p. 92). We agree with Boaler, that real world context is important. Still, students will meet WPs which are not in a real world context in school and in assessments, therefore it is important to study if students are able to solve these WPs and, if not, find explanations as to why.

In the Norwegian curriculum for the common core subject of mathematics (LK06), reading is one of five basic skills. The basic skill *reading* in mathematics is defined as:

> … understanding and using symbolic language and forms of expression to create meaning from texts in day-to-day life, working life and from mathematics texts. (…) Reading in Mathematics involves sorting through information, analysing and evaluating form and content, and summarising information from different elements in the texts. (Utdanningsdirektoratet, 2013)

According to Nortvedt (2010), there is a strong positive correlation between numeracy and reading comprehension. She has studied how 8th grade students in Norway are responding on multistep arithmetic WPs on the national test in numeracy and compared this result with students' responses on the national test in reading comprehension. "Student's reading levels explained 44 % of the variability in their scores on the multistep arithmetic word problem scale" (Nortvedt, 2010, p. 33).

Normally, a mathematical problem is defined as a task where no standard procedure is known to the students (English & Gainsburg, 2016). With this definition, not all WPs are a mathematical problem (Björkqvist, 2003). We can have WPs which are/are not a problem solving task and problem solving tasks which are/are not WPs. There are several ways of defining level of difficulty in WPs. When solving a WP, the students first have to translate the text into an internally represented model of the problem. "The translation phase is related to linguistic and factual knowledge and requires the skill of number selection to solve word problem" (Kingsdorf & Krawec, 2014, p. 66). Students who create a visual-schematic representation of the situation to be solved seem to benefit from it, while a production of a pictorial representation is negatively related to WP solving performance (Boonen et al., 2013). In a study of 128 6th grade students in the Netherlands, "the production of visual-schematic representations explains 21 % of the relation between spatial ability and word problem solving performance" (Boonen et al., 2013, p. 276). This can explain why some students can solve WPs and other cannot, but we do not have data to investigate this further. However, this is still relevant, since many students can solve common arithmetical tasks and they show good text comprehension skills, and yet they fail to solve WPs correctly, indicating other factors must be involved (Daroczy, Wolska, Meurers, & Nuerk, 2015).

What makes WPs challenging for students? One can separate problems by looking at the number of steps required to solve them, one step is normally easier to solve than multistep WPs (Nortvedt, 2012). But since two-steps tasks are often more difficult linguistically, we cannot conclude that the reason for them being more difficult is arithmetical complexity (Daroczy et al., 2015). Another way of distinguishing easy problems from more complicated ones is to look at the actual text in the WPs. For example, by counting the number of words, whether difficult of easy language is used in the text, if there is unnecessary information, or if there are words that point to a particular arithmetic operation (Kingsdorf & Krawec, 2014). A WP in a familiar context or in a context the students have a relationship to can also be crucial if students manage or fail to solve the problem. Daroczy et al. (2015) conclude that difficulties in solving WPs are influenced by the complexity of linguistic and numerical factors, and their interrelation. In this paper, we will discuss some of these factors.

Methods

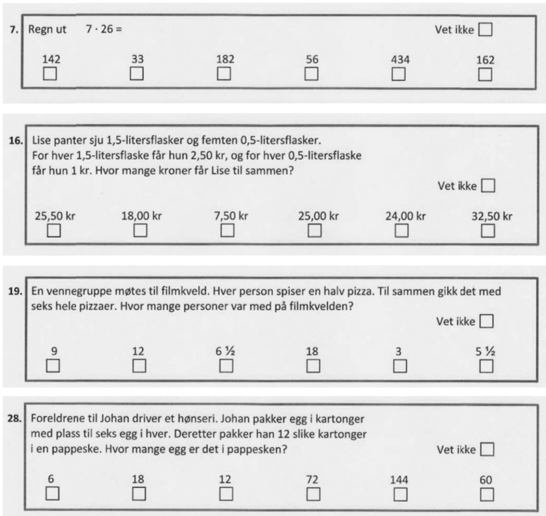
In this study, we used data taken from a survey sample of mathematics from the project: *The Function of Special Education* (SPEED) (Haug, 2017), a joint research project between Hedmark University College and Volda University College. The mathematical survey in the SPEED-project had 40 multiple-choice items, with 7 possible answers including the possibility to answer, "I do not know the answer". Some of the wrong answers on these items are related to well known misconceptions. In this paper, we have chosen four tasks from the survey that relate to each other in form of multiplication (See Figure 1). In the SPEED-project, the students also responded to the Carlsten reading test (Carlsten, 2002) as a measure of whether the student has LRA or SRA. In this test, a student is classified as having a functional literacy if he could read more than 80 words per minute with less than 15 % error on a reading test. On the 5th grade reading test there were 25 possible correct answers making more than 22 acceptable, for the 6th grade the test had 27 correct answer making more than 23 acceptable. This mean that a student with a SRA have both a satisfactory reading speed and is able to de-code satisfactory. A student with LRA fail in both of them or only one of them.

In our study, 593 students from 5th and 660 students from 6th grade participated. For each of the students, their teacher provided an assessment of their academic achievement on a scale from 1 to 6, where 1 stands for very low skills and 6 for extraordinary skills in mathematics. Since we compared students according to their reading ability, we removed students rated at an academic achievement level equal to one or two in mathematics. In addition, we have also removed any students not assessed by their teacher. This left us with 475 students in 5th grade and 552 students in 6th grade.

In our study, 348 students in 5th and 475 students in 6th grade was classified as having SRA, leaving 127 and 77 with LRA in 5th and 6th grade, respectively. According to the curriculum in Norway, students after 4th grade are supposed to be able to do multiplication in practical situations. The focus is on the standard multiplication table, but also on using different methods for multiplication. After 7th grade students should be able to “reckon with positive (…) whole numbers, decimals, …” (Utdanningsdirektoratet, 2013). Since Norway has not divided the competencies between the 5th, 6th and 7th grade, it is difficult to tell whether the students have learned about multiplication with decimal numbers yet. We argue that if the students have not learned about it, then both the students with LRA and the students with SRA still have equal conditions.

Analysis and discussion

In our analysis we present results from task 7 first (see Figure 1). It is a standard multiplication task with single and two-digit numbers, used here as a control task. Since this task has no context and only two words, the reading ability should not be a decisive factor. Both in 5th and 6th grade the students with SRA scored notable better than the students with LRA (Table 1). There is a significant difference[5] between the two groups for 6th grade ($\chi^2 = 5.38$, $p=0.02$), but not for the 5th grade ($\chi^2 = 2.76$, $p=0.1$).



7. Regn ut 7 · 26 = Vet ikke ☐
142 ☐ 33 ☐ 182 ☐ 56 ☐ 434 ☐ 162 ☐

16. Lise panter sju 1,5-litersflasker og femten 0,5-litersflasker. For hver 1,5-litersflaske får hun 2,50 kr, og for hver 0,5-litersflaske får hun 1 kr. Hvor mange kroner får Lise til sammen? Vet ikke ☐
25,50 kr ☐ 18,00 kr ☐ 7,50 kr ☐ 25,00 kr ☐ 24,00 kr ☐ 32,50 kr ☐

19. En vennegruppe møtes til filmkveld. Hver person spiser en halv pizza. Til sammen gikk det med seks hele pizzaer. Hvor mange personer var med på filmkvelden? Vet ikke ☐
9 ☐ 12 ☐ 6 ½ ☐ 18 ☐ 3 ☐ 5 ½ ☐

28. Foreldrene til Johan driver et hønseri. Johan pakker egg i kartonger med plass til seks egg i hver. Deretter pakker han 12 slike kartonger i en pappeske. Hvor mange egg er det i pappesken? Vet ikke ☐
6 ☐ 18 ☐ 12 ☐ 72 ☐ 144 ☐ 60 ☐

Task 7: Calculate 7x26 =

Task 16: Lise returned seven 1.5-liter bottles and fifteen 0.5-liter bottles. For each 1.5-liter bottle, she gets NOK 2.50 and for every 0.5-liter bottle, she gets NOK 1. How many NOK does Lise get for all the bottles?

Task 19: A group of friends meets for a movie night. Each person eats half a pizza. In total, they ate six whole pizzas. How many people were involved in the movie night?

Task 28: The parents of Johan run a chicken farm. Johan is packing eggs in cartons that holds six eggs each. Then he packs 12 such cartons in a cardboard box. How many eggs are there in a cardboard box?

Figure 1: The four different tasks. The number of the tasks indicates the tasks number in the test

[5]Chi-Square test (χ^2-test)

Task 16 is a multistep WP with decimal numbers. The situation in the task is familiar for most students and it is a money-problem, which is often seen to be easier than other contexts. There are almost 30 words in the text, some of them "unnecessary" decimal numbers, which makes the WP more complicated. Of students in the SRA group (both 5th and 6th grade), only 57 % answered correctly on this task, making this the problem where students most frequently respond incorrectly. For students with LRA, 38.5 % and 44 % of the students in 5th and 6th grade, respectively, answered correctly on this item. There is a significant difference between the SRA and LRA students in 5th grade (χ^2 =12.54, p<0.001) and 6th grade (χ^2 =4.71, p=0.03). Like Daroczy et al. (2015), we cannot conclude that the reason for students answering this task incorrectly is that the linguistics are more difficult or the task more arithmetical complex. This task might also be outside the curriculum for 5th and 6th graders. Whatever the reason, students with SRA scored better than students with LRA.

	5th grade				6th grade			
	SRA		LRA		SRA		LRA	
Task number	N	%	N	%	N	%	N	%
7 (Calculation)	343	61,5	121	52,5	464	77,8	72	65,3
16 (Bottles)	341	57,2	122	38,5	465	57,4	75	44,0
19 (Pizza)	339	77,0	119	63,9	460	83,0	75	66,7
28 (Eggs)	341	86,2	120	65,0	462	91,8	71	64,8

Table 1: Percentage (%) of students in our population (N) that answered the different tasks correctly, split by SRA and LRA

Both task 19 and 28 are single step arithmetic WPs with no extra numbers in the text. This makes these two WPs easier for the students to solve correctly. Task 19 comes from a familiar situation for students, but it has a twist. It is more common to know how many people there are, and then find out how much you need to buy. Here it is the other way around. The question wording in itself can contribute to making this task more difficult. They also have to relate to non-integers. Task 28 is probably derived from an unknown situation for most of our students. Although this item is from an unknown context, it is still the easiest because it only consists of multiplication with known numbers and integers. On both of these two tasks, there was a significant difference between students in the SRA and LRA group both for 5th and 6th grade students. (Task 19, 5th grade: χ^2 =7.81, p=0.005; 6th grade: χ^2 =11.13, p<0.001; Task 28, 5th grade: χ^2 =25.65, p<0.001, 6th grade: χ^2 =43.00, p<0.001).

Furthermore, there are more students with SRA answering correctly on the last two of the WPs (task 19 and 28) than on the control task (7). The tasks 19 and 28 contains lower numbers which makes the multiplications easier. For the students in the SRA group, it looks like reading text does not need to be an obstacle. It can also be a help for the students to get the right answer. We find the same pattern for the LRA students in 5th grade, but not as strong. For the students in 6th grade with LRA, approximately 65 % responded correctly on all these three tasks. By looking at the size of the numbers in the multiplications, we would expect more students answering correctly on task 19 and 28 than on task 7. This is not the case for this group (LRA 6th grade), therefore indicating that the text is a barrier for these students.

On all three WPs, there are a significant difference between students with LRA and SRA. This result indicates that there is a connection between students' reading abilities and ability to do mathematics, just as Nordtvedt (2010) stated.

So far, we have looked at the result for 5th and 6th grade separately. Is there progress from 5th to 6th grade for students with SRA and students with LRA? If we take a closer look at task 7, we find that 61.5 % of the students in the SRA group in 5th grade and 77.8 % in 6th grade answered it correctly, which is a significantly better result (χ^2 =8.26, p=0.004). Also on the task 19 and 28 there is a significant difference between 5th and 6th grade for this group of students (Task 19: χ^2 =4.55, p=0.03; Task 28: χ^2 =6.40, p=0.01), but not for task 16 (χ^2 =0.004, p=0.95). For students in the LRA group, there are no significant differences between 5th and 6th grade on any of the task (p between 0.09 and 0.98).

As noticed, the 5th graders perform better in two of three WPs than in the calculation task. This is another factor that indicates that it is not necessary the text in the WPs that are the difficulty. Actually, it looks like students with LRA can have a good informal mathematical understanding and have difficulties with doing the calculation/algorithms. By taking a correct answer on task 7, as an indication that students can multiply, is there then a difference between students with SRA and LRA when it comes to solving WPs? By picking out only those students who have a correct answer to task 7 (Table 2), there are a significant difference between the SRA and LRA students in 6th grade on task 19 and 28 (Task 19: χ^2 =9.58, p=0.002; task 28: χ^2 =10.47, p=0.001). On task 16 the difference is not significant (χ^2 =1.05, p=0.3). For students in 5th grade there are no significant difference between the SRA and LRA groups (p between 0.14 and 0.49).

As indicated before, it might be that the text actually helps the students with LRA, just like it helps the students with SRA. Another interesting finding, is that we cannot find the same difference in 6th grade. The student in 6th grade with SRA have an improvement from task 7 to task 19 and 28, but not the students with LRA. This difference can be explained by either easier numbers in the calculations or that the context (reading) in the tasks help them. According to the curriculum, the algorithm for multiplication is introduced during 5th or 6th grade. The question is then why do not students in 6th grade with LRA have the same pattern as the other group of students? This might be explained by reading being a larger barrier than first realised.

Our result shows a difference in percentage of students with SRA and LRA in solving task 7 correct although this is not a WP (significant difference for student in 6th grade, but not for students in 5th). This points towards students with SRA performing better than students with LRA on tasks where reading is not a primary part.

	5th grade				6th grade			
	SRA		LRA		SRA		LRA	
	N	%	N	%	N	%	N	%
16 (Bottles)	192	68	56	57	330	66	43	58
19 (Pizza)	207	80	63	76	352	87	46	70
28 (Eggs)	206	90	61	85	351	97	44	86

Table 2: List of how many of our population that answered task 7 (calculation) correctly (N), and how many percentages of these that answered the corresponding task correctly (%). Divided in SRA and LRA

On one-step WPs, there are a significant difference between students with SRA compared with those with LRA. The reason for this might be that students with LRA have difficulties making a visual-schematic representation (Boonen et al., 2013). Both task 19 and 28 should be possible to make such a representation, for instance in task 19 by drawing circles divided into two parts. Our results indicate that it does not seem that unknown context is as important as calculation with decimals or integers for students with SRA since the calculations in task 28 are easier than those in task 19. For students with LRA, there are roughly equal numbers of students who answer correctly to both of these two tasks. By looking only at students that are solving task 7 correctly (Table 2), we find that there are more students solving the task with whole numbers (28) correctly, than the task with decimal numbers (19).

Closing remarks

In our study, students with better reading skills were better at answering both word problems and purely symbolic computations than students who do not read well. Another interesting result is that among students in 5th grade there are more students answering correct on two word problems than there are students answering correct on the calculation task. This might imply that the text, also for those students with low reading ability, can be a help in solving multiplication problems when students are not completely competent in multiplication. Another explanation can be that the calculations in these WPs are easier. When students are more competent in multiplication (6th grade), this assistance in the text and easier calculations are not as prominent. The result of the student in 6th grade indicate that the text might be a barrier.

Nordtvedt (2010) concludes that there is a strong positive correlation between numeracy and literacy, and our data supports Nordtvedt's findings. Like Daroczy et al. (2015), we conclude that students who do not read well have both linguistic and numerical difficulties with word problems. Our data implies that the students who have problem reading also have bigger difficulties with mathematics all over, and not only word problems.

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Attending to and fostering argumentation in whole class discussion

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Prior studies highlighted the importance of whole class discussion after student activities and have focused, for example, on teacher moves in supporting discussion. We characterize two processes in teacher-students interaction in argumentation discussions: attending to and fostering. These processes describe how student argumentation feeds teacher talk that in turn feeds student argumentation. We analysed video recordings of one whole class 7th grade lesson when students made geometric constructions and engaged in argumentation discussion. We elaborated on four themes in how the teacher talk attended to and fostered student argumentation. We argue that the concepts of attending to and fostering help to understand how teachers can orchestrate argumentation discussions.

Introduction

This study focuses on the relationship between teacher talk and student argumentation in mathematics learning in lower secondary school. Often argumentation in mathematics education is considered from the cognitive point of view of what students consider as a proof or how they construct proofs or justifications (e.g., Harel & Sowder, 2007). Another line of research focuses on collective argumentation and consider argumentation as a social phenomenon in which students and the teacher together present rationale for their actions (Krummheuer, 1995). Often Toulmin's model is used to recognize argumentation components such as claim, data and warrant (e.g., Conner, Singletary, Smith, Wagner, & Francisco, 2014; Berland & McNeill, 2010). Some studies have also identified teacher moves that support students' work related to argumentation components (Conner et al., 2014). In this study, we continue to focus on collective argumentation and study argumentation as discussion in which students and the teacher pose claims, defend claims and criticize others' arguments.

Teacher talk is one of the key elements in facilitating argumentation discussion as teachers orchestrate classroom work by harnessing and interweaving students' contributions and making shifts between what is foregrounded and what is backgrounded in pursuit of overall pedagogical goals (Littleton & Kerawalla, 2012). Orchestrating productive classroom discussion after student activities has been recognized as an important but challenging phase in mathematics teaching (Stein, Engle, Smith, & Hughes, 2008). We think that this phase of a lesson may be even more important when the aim of the lesson is to engage students in argumentation discussion.

Previous studies have identified different types of teacher talk on the basis of how different points of view are discussed and who participates in the discussion (e.g., Lehesvuori, Viiri, Rasku-Puttonen, Moate, & Helaakoski, 2013). Some studies have characterized specific teacher moves that stimulate student thinking. For example, Temple and Doerr (2012) created categories for mathematics teachers' initiation (comparing, defining, describing, evaluating, hypothesizing, recounting, representing) and feedback (clarification requests, elicitation, evaluation request, expansion, explicit correction, justification request, metalinguistic feedback, recast, reinforcement, repetition) moves. Similarly, Chin (2007) identified questioning approaches (Socratic questioning, verbal jigsaw, semantic tapestry, and framing) with subcategories that stimulated productive thinking. Both Temple and Doerr (2012) and Chin (2007) found that the teacher moves depended on the purpose of the episode. However, relatively few studies (Conner et al., 2014) have explored the relationship between teacher talk and student argumentation in mathematics. As a result, research providing a more thorough understanding of the relationship between teacher talk and student argumentation is still needed.

To study the relationship between teacher talk and student argumentation, we draw on the concepts of attending to and fostering by building on previous research (Lobato, Clarke, & Ellis, 2005; Sherin, Jacobs, & Philipp, 2011). Teacher talk which *attends to* student argumentation is sensitive to the students' arguments, for example, by drawing out the students' ideas in a dialogue or by reviewing the arguments in a lecture. Attending to refers to a process in which student argumentation influences teacher talk. *Fostering* means that the teacher intends to move student argumentation forward, for example, through questioning or using examples. In fostering, the teacher's talk influences the students' argumentation. The same teacher utterance can indicate both the processes of attending to and fostering. For example, when a teacher rephrases students' argument using formal mathematical notation, this indicates that the teacher talk attends to the students' argumentation and fosters argumentation by introducing new notations.

The aim of this study is to elaborate on the concepts of attending to and fostering and to examine how these concepts may enrich the analysis of teacher orchestrated whole class argumentation discussion. One mathematics lesson was selected for this study. The following research question guided the data analysis: How does the teacher's talk attend to and foster students' argumentation?

Methods

The reported study is part of a two-year research project investigating how the participating classes develop in argumentation discussion when using argumentation tasks regularly in mathematics and in physics. For this study, we selected one 7^{th} grade mathematics lesson from the project database for a more detailed analysis. The criteria for selecting the lessons was that it included relatively high level whole class discussion in which students talked mathematics to each other. The participants were informed about the study and gave research consents. The results are reported using pseudonyms.

Data collection

The lesson was the fifth lesson of the teacher and the students in the project. The students (n = 25) were seventh grade students. The teacher was an experienced mathematics and science teacher. The topic of the 45 minutes long lesson was geometric constructions. Students were working in six groups (A, B, C, D, E and F) to construct a quadrangle that has four equal sides (a rhombus) and prepared to explain why their construction was valid. During the group work, the teacher circulated in groups. The students produced posters of their constructions. Then, the students observed other groups' posters and prepared to comment on them in the forthcoming whole class discussion. Finally, several posters were discussed during a whole class discussion.

The lesson was video recorded with a handheld video camera which followed the teacher from the back of the classroom. The camera was connected to a wireless microphone on the teacher. In addition, each student group had a small wide angle GoPro-camera attached to their desk. Students' verbal comments were recorded on the video's audio. Students' posters were collected.

Data analysis

The analysis started as two researchers observed the lesson live. Afterwards, the lesson video and particularly the whole class discussion were watched several times. In the data driven analysis, data was reduced into segments around each teacher utterances, the segments were then divided into groups and the groups were elaborated (Miles & Huberman, 1994). In detail, for every teacher utterance, it was considered how the utterance related to the ongoing student argumentation that preceded the teacher utterance. In addition, each input the teacher utterances gave for student argumentation was analysed. By comparing these instances with each other similarities and differences were noted and the episodes were divided into groups. The episodes in each group were compared to each other and common features were characterized. Through this, we composed four themes in how teacher talk attended to and fostered student argumentation.

Results

In the following, we elaborate on the four themes adressing how the teacher's talk attended to and fostered student argumentation. The given exerptcs are from the whole class discussion.

Supporting students to direct their talk to other students

In the beginning of the whole class discussion, when discussing the solution of Group A (Fig. 1), the teacher tried to get the students to talk to each other instead of talking just to the teacher.

1	Alex	At least it looks like a pretty good square.
2	Teacher	It looks like a good square. What is it Joe?
3	Joe	Why there is a circle? (…)
4	Teacher	Why there is a circle? (Directs the question to Rebecca.)
5	Rebecca	Because it had to be done by compass. Then we started do the square with the help of the circle. It had to be done geometrically, and so, we did the circle and then it was easier to do it.
6	Teacher	Did that answer you Joe?
7	Joe	Yeah.
8	Teacher	Anything else?
9	Robert	How did you do those that go up there and to the side. Those lines in the middle. How did you get them exactly in 90 degrees angle?
10	Rebecca	Well, we turned the ruler?
11	Robert	So you cannot prove in any way that.
12	Rebecca	We estimated it by eye.

The teacher talk attended to Alex's statement by repeating it (turn 2). This indicated that the statement had been heard. The teacher did not evaluate the statement, which fostered the discussion to continue. In turn 4, the teacher directed Joe's question to Rebecca. Here teacher talk attended to the fact that students were talking to the teacher instead of talking to each other. The same teacher utterance also fostered the students talking to each other. In turn 6, the teacher reinforced that the purpose is for students talk to each other by asking Joe to comment on Rebecca's response. After this, the discussion continued, and the teacher highlighted that estimating by eye is not accepted method in geometric construction. Thus, the discussion in this episode included important elements of argumentation as ideas were critically analysed and weaknesses were found.

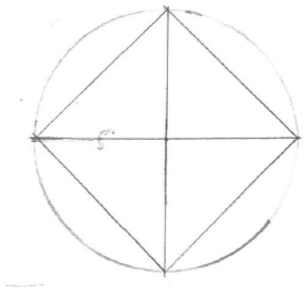

Figure 1: The solution of Group A

After the above episode, the teacher fostered student-to-student discussion by talking about discussion rules:

> Wait a minute. I have one thing that I would like to say. The first thing is that if you come up with an idea, you don't have to ask my permission. Clearly, Alex had something in his mind. So discuss, and others will listen, what one has to say. Alex.

In the excerpt above, teacher talk again attended to the need to get the students to talk to each other and fostered this by explicitly pointing this out. Later, when discussing the solution of Group B (Fig. 2), the teacher again supported student–student discussion:

14	Teacher	Carl, tell us.
15	Carl	Why there are two circles there?
16	Oliver	I can come to explain.
17	Teacher	You don't have to come to explain. Just answer Carl's question. Why-
18	Oliver	Carl, well, first we draw the outer circle and then the inner circle is just because of the angle bisectors because we did not want to draw all the small arcs separately, but we draw the full circle. It was easier. (Turning toward Carl and talking to him. Carl is nodding.)

Carl, who was not part of Group B, asked a question about the work of Group B. Oliver from Group B offered to answer the question. In turn 17, the teacher forbade Oliver to come in the front of the class to explain but instead wanted Oliver to answer to Carl from his own seat. The teacher attended to the potential of student explaining an idea to another student and fostered this by requesting Oliver to response directly to Carl. In this case, Oliver turned toward Carl, mentioned his name and explained to him. Thus, the teacher move was successful in promoting student-student discussion.

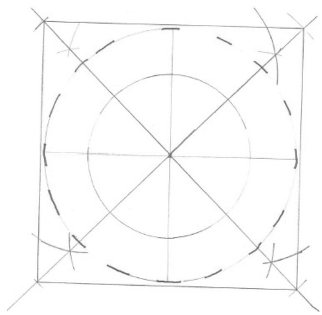

Figure 2: The solution of Group B

Seizing the potentially fruitful student utterances and using these to feed the argumentation

When Oliver continued to answer other students' questions there was a point in which Robert challenged the need to draw a certain circle:

19	Oliver	Yes you need to have those circles
20	Robert	I challenge that.
21	Teacher	Why do you challenge that?

The teacher talk attended to Robert's expression that he did not agree with Oliver and also fostered Robert to explain reasons for why he did not agree. In other words, the teacher talk attended to a disagreement and fostered counterargument. In addition, the teacher highlighted that when challenging ideas, reasons have to be explained. After the teacher's question, Robert explained how he would have modified Oliver's drawing. When Oliver responded, it became clear that his group had thought differently than Robert.

After the above discussion, Rebecca said that she did not understand anything:

22	Teacher	Do the others have something to comment on?
23	Rebecca	I don't understand anything of that.
24	Teacher	You don't understand anything. Good. Great. What do you not understand?
25	Rebecca	I don't understand anything.
26	Teacher	You don't understand anything.
27	Rebecca	I don't get the logic. (…)
28	Teacher	Do you know what? That is a brilliant answer. That is a brilliant answer. Do you know Oliver, you have a small problem.
29	Oliver	I know.
30	Teacher	Rebecca did not understand anything, and you should explain so that Rebecca and I too will

		understand, because I too have not understood anything yet.

In the turns 24, 26, 28 and 30, teacher talk attended to Rebecca's difficulties in understanding and to the need to explain in more detail. The teacher seized on the Rebecca's genuine expression of not understanding and used this as a springboard to foster Oliver to explain their line of reasoning in more detail. Furthermore, the teacher again attended to and fostered discussion rules by expressing that it is good to say when something is not understood and that others can be asked to explain in more detail.

Guiding the discussion to focus on the mathematical content of the argument

After the above episode, Oliver continued to explain their construction method, and there were more questions from the students.

31	Mike	Why did you bisect those angles?
32	Oliver	To get, u-hum. We bisected them to get like exactly 90 degrees here. So if this had been here and this here, then it would not have produced a square.
33	Robert	Is the angle in those radius 90 degrees?
34	Oliver	I'm not sure.
35	Teacher	Argh. Argh.
36	Oliver	Let's agree that it is. (Teacher laughs friendly.)
37	Robert	Oliver, Oliver, if it is 90 degrees, then how did you do it?
38	Oliver	Estimating by eye (with laughing voice).
39	Teacher	Argh.
40	Oliver	I know. We should have done it differently.

In turns 35 and 39, the teacher made sounds that signalled that something went wrong. The teacher did this in friendly manner. He attended to the insufficient justification and fostered students paying attention to this relevant issue of geometric constructions.

The teacher talk attended to and fostered the mathematical content of the argument in other points of discussion too. For example, he asked why-questions to get the students to discuss reasons, asked about specific steps to help students to describe what they did in their construction and asked to think about the construction instead of how the result looks like.

Not attending to a potentially relevant issue

Besides attending to several relevant and evidently productive issues in student argumentation, the teacher did not attend to all potentially relevant issues. One such episode happened when discussing the already mentioned solution of Group A.

41	Mike	After you draw the first line to the circle, if you had done a perpendicular line-
42	Robert	No, perpendicular bi-
43	Mike	perpendicular bisector to that line in the middle, then you would have got 90 degrees angle there, and you would have been able to connect the vertices as a square.
44	Olive	What is a perpendicular line?
45	Rebecca	What is a perpendicular bisector?

In this case, the teacher did not say anything about Mike's idea of correcting the construction of the other group. With Mike's correction, the construction would have been exact and there would have been a potential to construct other rhombuses than squares with the same technique. The attention of the teacher was potentially directed to the fact that some students did not know what a perpendicular line is even though that had been studied. The teacher also mentioned that the students should know this by now.

Discussion

In this study, we have analysed one lesson that included whole class argumentation discussion in which students talked mathematics to each other. The teacher talk played an important role in the discussion. The teacher used talk to attend to relevant points in the discussion and foster students to direct their talk to other students. In addition, the teacher spotted potentially fruitful student utterances and used these to feed the argumentation. He also guided the discussion to focus on the mathematical content of the argument. These three themes illustrate three dimensions of teacher orchestration: student–student dialog, argumentation components and content of argumentation. Previous research has studied classroom dialogue (e.g., Lehesvuori et al., 2013), components of argumentation based on the elements in Toulmin's model (1958/2003) and content of argument by examining if the argument is based on deductive or other forms of reasoning (e.g., Harel & Sowder, 2007). This study points to the need to include all these dimensions in the analysis of argumentation discussions. As shown in the results, the teacher in this study orchestrated the discussion in all these aspects. If focusing only on one dimension, we may miss important contribution of teacher talk.

In orchestrating the discussion, the teacher talk attended to the ongoing student argumentation and fostered it. Attending to meant that the teacher picked up ideas in students' argumentation and used these in his talk. When fostering, the teacher gave input to the students' argumentation. This relationship between teacher talk and student argumentation resembles to the concepts of uploading and downloading by Tabach, Hershkowitz, Rasmussen and Dreyfus (2014). According to Tabach et al., the ideas that students have developed during group work can be uploaded to the whole class discussion. Students can also download ideas from whole class discussion to their group work. Similarly, when attending to, the teacher downloads something from students into his talk. When fostering, then teacher is uploading something into students work. The difference between attending to/fostering and downloading/uploading is that the same teacher talk can be attending to and fostering. Thus, attending to and fostering are like two sides of the same coin. This also differentiates the concepts from eliciting and initiating, as proposed by Lobato et al. (2005). Another difference is that when a teacher is attending to, he or she does not necessarily try draw out students' ideas. There are also some similarities to the framework by Conner et al. (2014) who consider teacher moves that are related to different components of Toulmin's model. A difference is that attending to and fostering do not focus only on argumentation components but also to student–student dialog.

We found the concepts of attending to and fostering helpful in examining how the teacher orchestrated the whole class discussion. In particular, through attending to and fostering, we recognized the bi-directional flow of ideas from students to the teacher and from the teacher to the students. However, this study focused only on one lesson. Thus, the concepts of attending to and fostering are still preliminary concepts which need to be further elaborated in other lessons and in different contexts. In the ongoing project, we continue to study teachers' practices and investigate subtle differences in attending to and fostering.

Acknowledgment
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The roles of mathematical symbols in teacher instruction

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Mathematical symbols are essential in communicating, employing, and generalising mathematical knowledge. In this study, we develop a method for examining teacher instruction; specifically, how mathematical knowledge is presented to students in secondary school by means of symbols. Four fundamentally different roles of symbols are identified as: a label, taking part in a role-play, setting up contentual expressions and enabling transformations. By combining these with the building blocks of teacher instruction it is possible to detect certain patterns in the instruction with respect to how mathematical symbols occur and thereby obtain information about the character of the mathematical knowledge students meet in instruction. Looking at the transition from lower to upper secondary school as a case study, the method is used to recognize some of the transition problems experienced by students caused by a change in how mathematical symbols are employed.

Introduction

In mathematics classrooms, signs are used in many different roles such as communicating and operating mathematical knowledge (Steinbring, 2006). Research into symbolizing in mathematics classrooms is comprehensive and often addresses the learning process, e.g. how students interpret mathematical signs (Cobb, Yackel, & McClain, 2012; Radford, 2013), and how knowledge is constructed in classroom interaction (Steinbring, 2005). But how do mathematical signs appear in mathematical teacher instruction? In what ways do teachers employ mathematical symbols and expect students to read them? The understanding of symbols depends on what the student "is prepared to notice and able to perceive" (Sfard & Linchevski, 1994, p. 88), but what opportunities are students given to notice and to perceive? In this study we will develop a framework for studying teacher instruction with respect to the different roles of symbols. As a case study, we will look at instruction at both sides of the transition from lower to upper secondary education in Denmark. This transition is interesting in a Scandinavian context as it appears late (when students are 16–17 years old), and because the teacher education in lower and upper secondary education differs significantly in particular in Denmark.

Theoretical framework

In this paper, a mathematical sign designates 1) a numerical sign, a number symbol which can refer to a context, e.g. the length of a side or an angle in a triangle, or it can be context free; 2) an algebraic sign, often a letter that stands for either a number or an object e.g. a line or a plane; and 3) an operation such as +, – and $\sqrt{\ }$.

In a school context, a mathematical sign is often called a (mathematical) symbol. This notation will be used henceforward except when referring to references that are using the word ‘sign’.

All mathematical knowledge needs a system of signs, which carry no meaning of their own but acquire meaning through the relation with the object the sign refers to (Steinbring 1999). Steinbring states that signs are a source of information about the conditions for constructing new mathematical knowledge, simultaneously carrying this knowledge and being the means of communication about it. According to Steinbring (2006), signs have two major functions: 1) a semiotic function as “something that stands for something else” – a mathematical object/reference context; and 2) an epistemological function as vehicle for knowing the object of knowledge. This is illustrated in the epistemological triangle on figure 1. The horizontal arrow shows the semiotic relationship stressing the sign’s representational character. The epistemological characteristics of the underlying basic mathematical concept shape the resulting relation between the sign and the object.

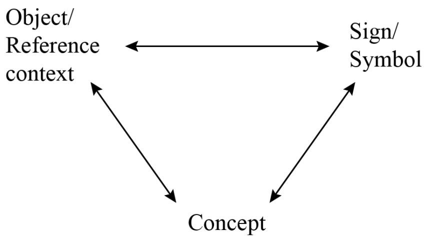

Figure 1: The epistemological triangle (Steinbring, 2006, p. 135)

Steinbring’s (2006) epistemological triangle provides a framework for modelling how mathematical knowledge is developed by means of signs/symbols. The learning process is influenced by the way in which the relationship between object and sign is mediated, which includes teacher instruction. In this study we will look further into how symbols are employed in classrooms - that is, which roles they play in teacher instruction. The more explicit consequences for student learning are beyond the scope of this paper. The presented categories are identified from literature inspired by observations of the problems experienced by upper secondary students when working with symbols that was completed by Mogens Niss (personal communication November 29, 2016), and they are related to the different ways Janvier (1996) interprets mathematical symbols.

Classifying the roles played by mathematical symbols

Four roles have been identified. The first is derived directly from the epistemological triangle as the semiotic function of the symbol. Steinbring (1999, p. 116) notes: "Mathematical concepts are constructed as symbolic relational structures and are *coded by means of signs and symbols* that can be combined logically in mathematical operations" (italics added). Symbols act as codes or labels for objects, which is what Peirce (1965) calls *an index*. Some labels always mean the same thing e.g. π, e or '+' whereas others change their meaning depending on the context. When working with triangles, a is a side or the length of a side, whereas it stands for the slope of the straight line in $y = ax + b$. In Arcavi's (1994) notion of *symbol sense,* which is described by various qualities, "sensing the different roles symbols can play in different contexts" (Arcavi, 1994, p. 31) is mentioned as one such quality. In this first category symbols merely act as 'a label' (**L**).

When more symbols are combined in expressions according to the 'manuscript' or conventions of formal mathematical language, the semantics of the expressions is of less importance. The meaning of the symbols cannot be deduced from reasoning but are defined by notation. $f(x) = a \cdot e^x + b$ says nothing about a, b, f or x and although $a(b) = f \cdot e^b + x$ means exactly the same, the symbols play completely different roles. The category treating how symbols are used in formal mathematical language we call 'the role-play' (**R**).

In the earlier mentioned quote, Steinbring (1999, p. 116) states: "Mathematical concepts are constructed as symbolic relational structures". These structures can be propositional formulas like equations, or propositions and theorems. In Pythagoras's Theorem $a^2 + b^2 = c^2$, where c is the hypotenuse and a and b are the other two sides in a right-angled triangle, the verity of the theorem does not depend on either the symbols themselves or the role-play in which they participate but rather on the mathematical substance they refer to. Activities concerning the symbol sense: "how and when symbols can and should be used in order to display relationships, generalisations, and proof" (Arcavi, 1994, p. 31), can be recognized in the generational activities in Kieran's (1992) GTG-model for algebraic activity and further this function. When symbols are playing this role, they belong to the category 'contentual expressions' (**C**).

The final category concerns how symbols take part in the manipulation of expressions following a set of rules, which make some transformations valid while others are not. Steinbring refers to this when he says, "symbols that can be combined logically in mathematical operations" (1999, p. 116). A transformation results in a new expression that is identical to the previous one. This is what Duval (2006) calls 'denotations'. While performing the transformation, the mathematical content referred to by the symbols need not be visible but can be detached from the context, if any, where it appears. From the result of a transformation, new knowledge can evolve, such as where the revising of an expression provides information about an unknown quantity. Teacher instruction comprising symbols in this category are connected to the symbol sense "An ability to manipulate and to 'read' symbolic expressions" (Arcavi, 1994, p. 31). This fourth and last category we call 'transformations' (**T**).

Aim of study

Symbols and the roles they take on are essential to mathematical knowledge (Steinbring, 2006). The aim of this study therefore, is to *establish a framework for exploring the use of symbols in teacher instruction*. By teacher instruction, we mean any activity planned by the teacher and carried out in the classroom such as presenting theory or examples on the blackboard, going over problems, class discussions, students solving problems, reading a mathematical text or performing inquiry activities, etc. The framework was developed by combining theoretical considerations and classroom observations. The connection between teacher instruction and mathematical knowledge is mediated by symbols through the symbol categories defined above. As identifying which category is employed directly from observations can be ambiguous, we make use of the (building) blocks of instruction that are easily observable. Each block is dominated by one main symbol category, and thereby the teaching can be examined through the relation between instruction and symbols as illustrated on figure 2.

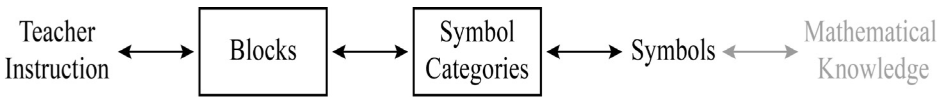

Figure 2: The developed framework

Method

As a case study, the framework was set up and used to look at the transition from lower to upper secondary education. Four classes in lower secondary school were observed during the *last* three months before the final exam for a total of 15 lessons comprising 80 minutes each. Three classes were followed during their *first* three months in upper secondary school for a total of 14 lessons comprising 100 minutes each. The observed instruction did not necessarily provide an indication of how teacher instruction was carried out generally but instead reflected the shift in instruction at the transition.

Three of the lower secondary classes came from the same city school but were different year groups (9th and 10th grades); the last class came from a rural school. The upper secondary classes came from three different schools in the same city; one from the city centre, one from a suburban area, and one from a technical high school. To make the observations as representative as possible, the seven participating teachers were chosen to provide a range in terms of experience (length of time teaching), qualifications (level of teacher education), and gender.

The empirical data consisted of both observations and artefacts. Field notes from classroom observation addressed a) the type of representation employed verbally or in writing, b) a description of the activity, and c) notes from the blackboard and further observations. Artefacts included textbooks and worksheets from the lessons. These were used to qualify the coding of the field notes. This coding was carried by the author and, in part, by a second researcher. A subsequent discussion resulted in a revision of codes and a re-coding.

Identifying blocks for the framework

Using a qualitative content analysis (Hsieh & Shannon, 2005) of classroom observations, a number of blocks in the instruction were identified.

Block	Description of Block	Symbol category
A	Attaching a numerical or algebraic symbol to another representation (sketch, diagram, table, text, etc.	L
I	Inserting a numerical or an algebraic symbol in an expression	
D	Defining a mathematical object by the style of writing, e.g. "a quadratic polynomial is defined by $p(x)=ax^2+bx+c$"	R
N	Introducing mathematical notation; "$a\cdot a\cdot a=a^3$"	
G	Obtaining knowledge about the meaning of a symbol by using a graphical tool, such as when seeing b as the y-intercept by using the slider tool in Geogebra	
E	Setting up a mathematical proposition or expression, which can and sometimes are proven by mathematical arguments. The expression can be algebraic or numerical as in "Amount in SEK $= 50\ \text{DKK}\cdot\frac{79{,}07}{100}\cdot\frac{\text{SEK}}{\text{DKK}}$"	C
F	Writing up a formula from the book that could have or has been proven earlier	
M	Manipulation, reduction, factorising, calculating, etc.	T
K	Obtaining new knowledge from a manipulation or calculation	
P	Using a computer program like Geogebra, Exel or Nspire for construction, manipulations, etc. by "clicking".	All

Table 1: The blocks and their connection to the symbol categories

Examples of teacher instruction are 1) "sketching a triangle stating the lengths of the sides", and 2) "attaching symbols x_1, y_1, x_2 and y_2 for the coordinates to a drawing of two points". Both belong to the same block, denoted A, that is defined by stating a relation between a symbol and a second representation. The ten identified blocks is explained in table 1. In the right column the dominating symbol category corresponding to each block is stated. This correspondence is deduced from the definitions of the symbol categories. Notice that the block-category P differs from the others by containing a mix of symbol-categories. P describes the act where technology is used with a mathematical program for multiple purposes more or less simultaneously and without addressing the underlying mathematical content. This makes the interpretation of the instruction based on classroom observations too complex and therefore the P-blocks have been omitted in this study.

Coding data

Each of the 29 lessons (15 in lower and 14 in upper secondary school) was divided into scripts consisting of a consecutive sequence of one or more blocks forming a coherent whole. When coding, the blocks of each script were written in a row, e.g. **A E I M**, showing the sequential order in which they occurred during the observation. When different blocks happened simultaneously in the same script, such as when students could choose from different activities or problems, the blocks were separated by a semicolon, e.g. **A**; **E I M**. In the examples from the previous section the symbols are numerical in 1) and algebraic in 2). To distinguish between numerical (n) and algebraic (a) symbols, the noted category was added an index, e.g. $\mathbf{A}_n$ or $\mathbf{A}_a$.

The example below shows how a script consisting of more blocks was inferred from observations (figure 3): A class from upper secondary school was shown how to find the distance between two points *A* and *B*. The teacher began by drawing a sketch of two points (x_1, y_1) and (x_2, y_2) attaching algebraic symbols to a second representation ($\mathbf{A}_a$). The formula was then re-deduced (the students had seen it done before) using Pythagoras's Theorem ($\mathbf{E}_a$). Finally, the numerical coordinates of the points *A* and *B* were inserted in the formula ($\mathbf{I}_n$) and the distance calculated ($\mathbf{M}_n$). The resulting sequence is $\mathbf{A}_a$ $\mathbf{E}_a$ $\mathbf{I}_n$ $\mathbf{M}_n$.

Oct. 22, 2015 Upper secondary school
The straight line

Representation	Activity	Observations
Symbolic	The teacher shows an example	Teacher draws
	Finding the distance	The formula:
	between two point	$\lvert AB \rvert = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$
	$A(-1,3)$ and $B(2,-1)$	is deduced from drawing. Coordinates are inserted
		Students calculate distance

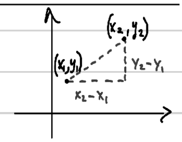

Figure 3: An example of field notes

This concludes the development of a framework for describing the use of symbols, and thereby the character of mathematical knowledge, seen in teacher instruction. The connection between the roles taken by symbols and how they enter into the building blocks of instruction is shown in table 1. By applying the framework on data from the case study, possible patterns and differences in patterns with respect to the use of symbols in the observed teacher instruction can be revealed.

Results

The occurrence of blocks and the roles of symbols

In table 2, the number of each type of block is summarised. In lower secondary school, 88 blocks were observed. The corresponding number for upper secondary school was 159. A main finding is that students meet all four roles of symbols on both levels although with different frequencies. More important, though, is that the context in which they meet them changes immensely at the transition. Where labels are concrete numbers in lower secondary school they become algebraic symbols in upper secondary. Equivalently mathematical propositions are set up as particulars using numbers in lower secondary but as generals with symbols in upper secondary school. The pattern reiterates for transformations where algebraic manipulations are much more frequent in upper secondary as opposed to the calculations at the lower level.

Block	A_n	A_a	I_n	I_a	D_n	D_a	N_n	N_a	G_n	G_a	E_n	E_a	F_n	F_a	M_n	M_a	K_n
S. C.	L				R						C				T		
Low.	10.2%	4.5%	1.1%	0.0%	2.3%	10.2%	0.0%	4.5%	2.3%	3.4%	20.5%	1.1%	3.4%	4.5%	25.0%	6.8%	0.0%
	15.8%				**22.7%**						**29.4%**				**31.8%**		
Upp.	4.4%	8.8%	9.4%	0.6%	0.6%	4.4%	6.3%	3.1%	0.0%	0.0%	3.1%	7.5%	3.1%	11.9%	18.9%	15.7%	1.9%
	23.2%				**14.4%**						**25.6%**				**36.5%**		

Table 2: The frequency of the appearing blocks in lower and upper secondary school

Numerical and algebraic symbols

The result above suggests that the use of numerical and algebraic symbols change considerably at the transition. Adding up the number of blocks reveals that the frequency of blocks employing numerical symbols is $\approx 65\%$ in the observed lower secondary classes and $\approx 45\%$ in upper secondary school. More interesting, though, is the order in which the two kinds of symbols appear. In table 3 and 4 all the identified scripts are listed: 45 in lower secondary and 74 in upper secondary classes.

Lower secondary school			Upper secondary school		
D_a A_a F_a	E_n M_n	P D_a	A_a	N_n	M_n
G_a	A_n M_n	P	N_a E_a M_a	M_a	N_n
F_n P	N_a E_n M_n	M_n	N_n E_n M_n	M_a	M_n
P	E_n	M_a	M_a; M_n	M_a	A_a
D_a N_a P	A_n N_a	D_a A_a M_a	M_n M_a	N_n	F_n F_a
E_n M_n	E_n	M_a	M_a; M_n	N_n M_n	M_n F_n
D_n; M_n	E_n M_n	M_n	F_a	A_a	M_n
D_a D_n G_a	M_n;F_a; A_a	E_n M_n;M_a	M_a	A_a A_a	F_n M_n
D_a F_a	E_n M_n	A_n F_n	E_a M_a	M_a M_a	F_a I_n M_n
D_a G_a	E_n M_n	A_n E_n M_n	M_a	M_a M_a	D_a M_a
D_a	M_n	A_n;G_n;E_n;M_a	A_a D_a I_n M_n	P	M_a M_n
	E_n M_n	E_n M_n; A_n	A_a E_a I_n M_n	F_n M_n	M_n
	A_n		A_a E_a M_a	A_a F_a	N_n N_a
E_n M_n	E_n M_n		A_n F_a I_n M_n	A_a E_a I_n M_n	F_a F_n
P; G_n A_n	N_a D_a E_a		P;A_n F_a I_n M_n;K_n	M_a	N_n
P; M_a	E_n M_n		E_a	D_a	F_a I_n M_n
A_a F_a I_n M_n			E_a M_a	A_n E_n E_a	D_n M_n D_a M_a
F_n E_n M_n			F_a M_a	D_a I_n M_n	E_n E_a
			F_a I_n M_n; P	A_a A_n E_n E_a	E_a M_a
			A_a F_a N_a N_n	D_a	N_a
			A_n F_a I_n M_n	M_n	F_a I_n M_n; M_a
			A_a F_a I_a M_a	N_n P	F_a
			A_a E_a M_a	P	F_a I_n M_n
			A_n F_a I_n M_n K_n	E_n D_a	N_n N_a
			A_n F_a I_n M_n; K_n		F_a I_n M_n

Table 3: The use of algebraic (yellow) and numerical (green) symbols in blocks

Lower secondary school			Upper secondary school		
D_a A_a F_a	E_n M_n	P D_a	A_a	N_n	M_n
G_a	A_n M_n	P	N_a E_a M_a	M_a	N_n
F_n P	N_a E_n M_n	M_n	N_n E_n M_n	M_a	M_n
P	E_n	M_a	M_a; M_n	M_a	A_a
D_a N_a P	A_n N_a	D_a A_a M_a	M_n M_a	N_n	F_n F_a
E_n M_n	E_n	M_a	M_a; M_n	N_n M_n	M_n F_n
D_n; M_n	E_n M_n	M_n	F_a	A_a	M_n
D_a D_n G_a	M_n; F_a; A_a	E_n M_n; M_a	M_a	A_a A_a	F_n M_n
D_a F_a	E_n M_n	A_n F_n	E_a M_a	M_a M_a	F_a I_n M_n
D_a G_a	E_n M_n	A_n E_n M_n	M_a	M_a M_a	D_a M_a
D_a	M_n	A_n; G_n; E_n;M_a	$A_aD_aI_nM_n$	P	M_a M_n
	E_n M_n	E_n M_n; A_n	$A_aE_aI_nM_n$	F_n M_n	M_n
	A_n		A_a E_a M_a	A_a F_a	N_n N_a
E_n M_n	E_n M_n		$A_nF_aI_nM_n$	$A_aE_aI_nM_n$	F_a F_n
P; G_n A_n	N_a D_a E_a		P;$A_nF_aI_nM_n$;K_n	M_a	N_n
P; M_a	E_n M_n		E_a	D_a	F_a I_n M_n
A_a F_a I_n M_n			E_a M_a	A_n E_n E_a	D_n M_n D_a M_a
F_n E_n M_n			F_a M_a	$D_aI_nM_n$	E_n E_a
			$F_aI_nM_n$; P	A_a A_n E_n E_a	E_a M_a
			A_a F_a N_a N_n	D_a	N_a
			$A_nF_aI_nM_n$	M_n	F_a I_n M_n; M_a
			$A_aF_aI_aM_a$	N_n P	F_a
			$A_aE_aM_a$	P	F_a I_n M_n
			$A_nF_aI_nM_n$ K_n	E_n D_a	N_n N_a
			$A_nF_aI_nM_n$; K_n		F_a I_n M_n

Table 4: Scripts containing certain characteristic scripts (blue and grey)

The three cells in the two left columns show the scripts for each of the three 9th grades, the third column presents the 10th grade, and the three columns to the right show the scripts from each of the three upper secondary school classes. As stated earlier, the order of the scripts represents the temporal presentation of the blocks. Table 3 indicates that in lower secondary school, the two kinds of symbols are usually *not* connected. The scripts are characterized by the use of either numbers or algebraic symbols. In upper secondary school the opposite is the case; most scripts contain *both* types of symbols, and often with the same relation between algebraic and numerical symbols: first the letter and then the number. Only a few times do the numbers act as a starting point for the setting up of a general expression stated by algebraic symbols.

The structure of the scripts

Looking at the order of blocks in a script (table 4), we see that especially one combination occurs frequently in upper secondary school: $\mathbf{F}_a$ or $\mathbf{E}_a$ followed by $\mathbf{I}_n$ and in most cases $\mathbf{M}_n$. Sometimes an **A** or **D** is added in the beginning and sometimes the symbols inserted and manipulated with are algebraic. But the conception of mathematics underlying the script is the same: a (general) expression, e.g. a propositional formula or a functional expression exists or can be found, numbers related to a specific problem can be inserted and the result calculated. The appearance of this kind of script is marked with blue in table 4. It is worth noticing that the script becomes more and more common at upper secondary level after the first few weeks of introduction. At the lower level, a different script is seen frequently: $\mathbf{E}_n$ $\mathbf{M}_n$. The script, which is marked with grey, indicates that expressions are set up in a specific context with known numbers and argued for in this context instead of leaning on previous work. Most problem solving was completed this way at the lower level.

Conclusion and discussion

This study show that a classification of the different roles played by symbols in mathematics combined with an identification of the building blocks of teacher instruction makes a useful framework for looking at how school mathematics is presented to students, and thereby which opportunities the mediation between sign and object is given in the learning process.

Applying the framework to teacher instruction on either side of the transition from lower to upper secondary school reveals patterns in the use of symbols that signifies some potential problems students might meet when moving from one educational level to another. One main finding obtained is in how symbols take part in contentual expressions: In lower secondary school, numbers are widely used in setting up expressions and argumentation emerges from the context of a concrete problem. At upper secondary level, algebraic symbols are used for proving or proposing general relations in a context free setting and applied again and again in specific situations afterwards. The analysis emphasizes a challenge: in lower secondary school, students are taught how to solve particular problems but not how to ask the general questions. In upper secondary school general questions are asked but the linking to meaningful contexts seems deficient. Teachers at both levels can make use of this finding in their instruction. However, at this case study care must be taken not to generalise results excessively. The limited amount of data makes the results vulnerable to for instance atypical teacher instruction.

The developed framework only looks into teacher instruction and only around the transition. When it comes to how students learn, mathematics knowledge from other research areas should be taken into account: the social aspect, the use of everyday language, the role of technologies, classroom activities, the teaching design, etc. Many of these also influence instruction but have been omitted here, which could give rise to somewhat simplified results. The outcome of the case study confirms that the framework has the capacity to point out important issues concerning teacher instruction and how mathematical symbols appear in instruction. This could be taken further by carrying out classroom observations during a longer period at one or both levels and identifying ideal types of teacher instruction (Bikner-Ahsbahs, 2015), enabling a deeper understanding of the opportunities students are given to notice and to perceive as noted in the introduction of this paper.

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Second language students' achievement in linear expressions and time since immigration

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This study investigated how 259 grade 9 students solved two test items in algebra involving linear expressions. Some students were early or newly immigrated second language students in Sweden. The findings are based on a categorization of students' written responses. The results show that for the more advanced test item on linear expressions and unknowns, early arrived second language students achieved worse than newly arrived and other second language students, while there was a minor achievement difference when solving an elementary linear equation. The interpretation of the results is that the early arrived immigrants suffer from having larger parts of their mathematics education as second language students and thus struggle with advanced mathematics

Introduction

In research, second language immigrants are often described in the two perspectives of being second language learners and of their present socioeconomic status (Ufer, Reiss, & Mehringer, 2013; Hansson, 2012). Following Cummins (2008), the present study acknowledges that early arrived immigrants have been second language students, large parts of their schooling while newly arrived immigrants likely have been first language students most of their schooling. Here their knowledge in linear expressions is explored.

First and second language students' achievements in mathematics

In large scale studies such as TIMSS, second language students in many countries are reported to, on average, achieve below first language students (Mullis, Martin, Foy, & Arora, 2012). Petersson (2017) observed that in national tests in mathematics in Sweden, there was a smaller achievement gap between first and second language students in algebra than in other mathematical content areas. Petersson hypothesized that students immigrating in late school years have contributed to that result. In TIMSS students are defined as second language test takers depending on the test takers' self-reported estimation of how frequently they speak the language of test at home (Mullis et al., 2012). In Swedish school, the students are assigned to follow one of the courses 'Swedish' and 'Swedish as second language' based on regulations stated in the school act (Skolförordning, 2011). Based on empirical and linguistic arguments, Cummins (2008) distinguished between conversational and academic proficiency in the language of instruction and found the academic proficiency to take much longer time to develop than conversational proficiency. Cummins indicated that an approximate time span for reaching conversational proficiency in a second language is about within two years and academic proficiency in about five to seven years. This means that second language students that have immigrated early in compulsory school, have experienced most of their mathematics education without full access to academic school language. This may have negative effect on their success in learning mathematics. For example, Ufer et al. (2013) found small achievement differences between first and second language students for algorithmic tasks, but large achievement differences for conceptually demanding mathematical tasks. Another explaining factor is a positive correlation between having high proportions of second language students in the school and having a larger proportion of individual school work in the mathematics classroom, which is known to correlate with lower achievement (Hansson, 2012).

Mathematical background

Linear expressions are a part of algebra. It occurs frequently in various problems in school mathematics. Good knowledge in working with linear expressions and unknowns is a gateway for the individual student to continued studies in mathematics since much of upper secondary school mathematics and mathematical modelling builds on linear expressions. The mathematical area of linear expressions, cover several mathematical ideas. There is the concept of unknown. The perception of unknowns has been described as a hierarchy of seeing unknowns as a multiple number, a specific number, an unknown digit etc. (Asquith, Stephens, Knuth, & Alibali, 2007). There is the concept of algebraic syntax. Students' difficulty in parsing algebraic expressions have been explained as a difficulty in making productive use of the information and relationships carried in algebraic expressions (Humberstone & Reeve, 2008; MacGregor & Stacey, 1997). For example, some students might confuse the implicitly given multiplication in '2x' with an explicit addition '2+x' or with a power 2^x. MacGregor and Stacey (1997) explained the confusing with a power with a combination of the following three arguments. First it is necessary that the students have been taught about powers. Secondly, some students may not have learnt to clearly distinguish between repeated addition and repeated multiplication. Thirdly, some students may think that the test items is too easy for them and expect to use more advanced mathematics.

Research question

Combining the results from Cummins (2008), Petersson (2017) and Ufer et al. (2013), the present study suggests comparing second language students' achievement in algorithmic versus demanding test items in algebra. Since there might be a large span in experience of Swedish language and schooling among second language students, the students were divided in the following categories, where ‘2L student’ denotes those following the course ‘Swedish as a second language’ in school.

Newly2L: Second language student, who due to immigration entered the Swedish school system during school years 8–9 and thus is newly immigrated.

Early2L: Second language student, who due to immigration entered the Swedish school system during school years 1–7 and thus is early immigrated.

Other2L: Second language student that have immigrated before school start age or have not immigrated at all.

Swe1L: Student following the course ‘Swedish (as a first language)’ in school.

Formal linear expressions with unknowns are introduced late in compulsory school mathematics in Sweden. This implies that second language students, with the possible exception of some newly immigrated students, have experienced probably most of their teaching in this area in their second language. Moreover, before immigration the newly immigrated students may have experienced most of their mathematics education in their first language. Given that second language students may have different length of experiences of the language of instruction in the country of immigration, the present study asks the following question: When tested in both algorithmic and demanding test items involving linear expressions, what differences, if any, are there in achievement between second language students with different length of experiences as second language mathematics students in Sweden?

Method

Test responses from 259 students were analyzed together with a survey used for categorizing the students as Newly2L, Early2L, Other2L and Swe1L.

Test instrument

To answer the research question, the author composed a test using old national test items since these have been piloted by the Swedish national test group. The following two test items involved working with linear expressions.

Item A: Solve the equation $2x+3=11$. (Original formulation in 2009 mathematics national test in school year 9, item B7 was “Solve the equation $17=3x+5$”).

Item B: $4x+5y=11$. What is $12x+15y$? (Original formulation in 2009 mathematics national test in school year 9, item B15 was “How much is $4x+6y$ if $2x+3y=12$?).

Test item A can be solved by direct substitution $2\cdot 4+3=11$ or by inverse operations $x=(11-3)/2$. For test item B, the context of a mathematics test implies a proportional relation between the two expressions. This relation can be used explicitly by solving as $3\cdot 11=33$ or implicitly by finding values of the unknowns that solves the equation (e.g. $y=1$ & $x=1.5$) and substitute these into the linear expression. The students' written responses to the test items were compiled into a database and were categorized by solution strategy. Test items A and B have the same mathematical structure as the source formulations in the 2009 national test, but the coefficients have been altered. Though language gives meaning to e.g. mathematical tasks, it might also be a source of added challenges for second language students and could obscure the mathematical meaning of the test problems (e.g. Campbell, Davis, & Adams, 2007). With the aim to minimize context and language obstacles, the test items were chosen to have problem formulations with low text intensity. Calculators were not allowed on the test. Test time was about 40 minutes.

Student sample

The participants in the present study were chosen from the last year of the compulsory school since this school year can be expected to have the largest span of experiences from Swedish language among newly arrived and early arrived immigrants. One reason for separating between Newly2L, Early2L and Other2L is their different proficiency in Swedish language, see table 1. Another reason is that while Early2L probably have had all their algebra lessons in their second language, Newly2L are likely to have met some algebra lessons in their first language before immigration. Other2L and Swe1L have experienced all their schooling in Sweden.

In the study a total of twelve entire classes of a possible thirteen, in five schools with a high proportion of immigrant students, agreed to participate in the study. Information about the students' migration background was collected in a written survey, to which the students gave their written consent. Among Swe1L no students had immigrated during school years 1–9. The sample was purposive in choosing schools with an above average proportion of immigrant students. When making a purposive sampling, there is a risk of losing external validity since the purposive sample may have other properties than a random sample (Kruuse, 1998). To control for this, the students in the present study were compared to a national random sample with respect to achievement on the written part B1 on the national test. While the purpose of test items A and B is to answer the research question, the purpose of measuring their national test achievement is different. It is to make it possible to discuss the generality of the results in this study – to compare the students in the present study with other students. The national random sample was collected by the Swedish National Agency for Education and is a part of the annual evaluation of the national test (Skolverket, 2013). The author received data from the national random sample from the National Test Team. The random sample only categorizes students as first or second language students and has no information about the students' school year of immigration. In the national random sample, the second language students achieved 46% correct responses. This is identical to the average of all second language students in the present study, whose results are given in Table 1. The first language students in the national random sample achieved 60% correct responses, which is similar to the achievement of the first language students in the present study.

Students' background	Newly2L	Early2L	Other2L	Swe1L
Number of students	23	67	56	113
Proportion of students with leaving grade in Swedish language ≥ passed	52%	78%	86%	97%
Proportion of correct responses in national test in mathematics	49%	43%	48%	56%

Table 1. Participating students' achievement in Swedish language and in national mathematics tests

The research question was implemented as comparing the achievement of the Early2L students with that of Newly2L and Other 2L students using Cliff's *d* for measuring the effect size and a Mann-Whitney test, corrected for the occurrences of equal ranks, for measuring the statistical significance.

Results

Responses to item A

The responses to item A are summarized in Table 2. A majority of the students in each student category gave correct response to item A by giving a series of inverse operations or substituted the solution into the original equation or just gave the solution. Only three students gave both a series of inverse operations to find the solution and substituted the solution into the original equation. Of students giving an incomplete or erroneous solution, most used a series of inverse operations. One group of responses was various incomplete responses similar to that in Figure 1a. These students had responded $2x=8$ or $x=8$ or made a calculation error corresponding to $2x=9$ or $2x=7$.

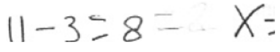


Figure 1a: Incomplete / erroneous solution

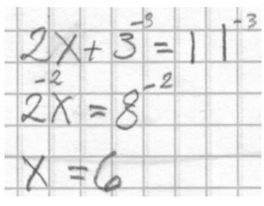


Figure 1b: Confusing *2x* with *2+x*

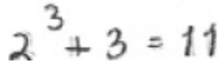


Figure 1c: Confusing *2x* with 2^x

In the students' attempts to make a series of inverse operations during the solving process, many responses contained algebraic errors. One kind of error was to confuse the implicitly given multiplication of coefficient and unknown with an addition instead as in Figure 1b. The student in Figure 1b started with a correct subtraction of the constant term from both sides of the equality. In the second row the student erroneously confused $2x$ with $2+x$ with a consequential error. Some students made multiple errors. For example, one student first stated that $2x=11+3=14$ and then continued with setting $x=14–2$, that is, confusing $2x$ with $2+x$ as in Figure 1b. Moreover, there were three responses of confusing the multiplication with a power as in Figure 1c.

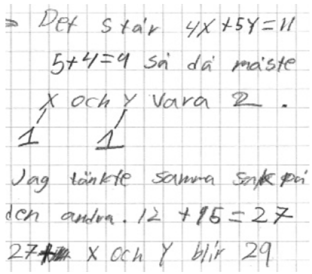
Response category	Newly2L	Early2L	Other2L	Swe1L
Correct	14 (61%)	40 (60%)	39 (70%)	82 (73%)
Incomplete or only calculation error (2x=11–3; x=11–3; 11–3=9 or 11–3=7)	2 (9%)	3 (4%)	2 (4%)	3 (3%)
Algebraic error	3 (13%)	9 (13%)	4 (7%)	10 (9%)
Unclassified	0 (0%)	1 (1%)	2 (4%)	2 (2%)
No response	4 (17%)	14 (21%)	9 (16%)	16 (14%)

Table 2. Response categories and proportions per student category for test item A

A Mann-Whitney test, found the achievement differences for test item A to not be statistically significant neither between Early2L and Newly2L ($p = 0.46$) nor between Early2L and Other2L ($p = 0.13$).

Responses to item B

The responses to item B are summarized in Table 3. Less than 40% of the students in each student category gave correct response to item B. The students who responded correctly to item B gave three kinds of responses. One correct response was to substitute for example y=1 into the first expression and then solve this equation for the other unknown, followed by substituting the values for the unknowns into the linear expression $12x+15y$ and evaluating it. Some students made calculation errors during this substitution procedure. Another correct strategy was to explicitly calculate $11·3=33$, where 3 is the ratio between the coefficients in the linear equation and the linear expression in test item B. A third alternative was to only give the answer 33. Just as for test item A, some students confused the multiplication of the coefficient and its unknown with an addition. The main idea in the response in Figure 2a is to sum the coefficients $4+5=9$ in the linear equation in item B and from this suggest that $x+y=2$. The next step is to sum the coefficients $12+15=27$ in the expression and add the number 2 from $x+y$ and get 29. This algebraic error led to several different responses such as 27, $27xy$, 30 and setting $3(12x+15y)=81xy$. Figure 2b gives an example of the response 30, where a student added three, which is the ratio relating the coefficients in the linear equation and the expression in item B.



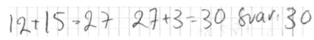


Figure 2a: Confusing 4x with 4+x etc. **Figure 2b: Erroneous use of factor 3**

Response category	Newly2L	Early2L	Other2L	Swe1L
Correct answer only	1 (4%)	4 (6%)	7 (13%)	21 (19%)
Correct substitution	1 (4%)	1 (1%)	3 (5%)	7 (6%)
Correct factor	7 (30%)	8 (12%)	11 (20%)	14 (12%)
Incomplete or only calculation error	1 (4%)	2 (3%)	2 (4%)	3 (3%)
Algebraic parsing error	5 (22%)	21 (31%)	9 (16%)	18 (16%)
Unclassified	0 (0%)	4 (6%)	2 (4%)	4 (4%)
No response	8 (35%)	27 (40%)	22 (39%)	46 (41%)

Table 3. Response categories and proportions per student category for item B

Table 3 shows that the students in Early2L achieved less well than the other student categories and made more of especially algebraic errors similar those in Figures 2a and 2b. A Mann-Whitney test found the achievement differences for test item B to be statistically significant with low effect size between both Early2L and Newly2L ($p = 0.03$ Cliff's $d = 0.28$) and Early2L and Other2L ($p = 0.013$, Cliff's $d = 0.22$).

Discussion and conclusion

The research question in the present study was to explore relations between achievement in linear expressions and having different length of experiences being second language students, here exemplified with on the one hand Early2L students and on the other hand Newly2L and Other 2L. The main pattern in the results is the following: For the more demanding test item B the Early2L achieved significantly below Newly2L and Other2L, while the achievement difference was small for the elementary test item A, as seen in tables 2 and 3. Newly2L and Other2L achieved as Swe1L on the more demanding test item B. While in tables 2 and 3, the proportions of incomplete or only calculation errors and of unclassified errors were about the same for both test items in all student categories, the proportions of algebraic errors and 'no response' were larger for test item B. Especially Early2L students had large proportions of algebraic errors on test item B. The algebraic errors were essentially the same as have been observed in earlier research (MacGregor & Stacey, 1997). Test item B was a challenge for students in all categories and many used the algebraic information in a pragmatic way to reach some erroneous answer (Humberstone & Reeve, 2008; MacGregor & Stacey, 1997). Choosing substitution as solution strategy in item B corresponds to seeing unknowns as carrying specific values while the strategy of identifying the factor 3 disregards any specific value of the unknowns (see Asquith et al., 2007). In all student categories similar proportions chose a correct substitution strategy, though calculation errors were common. However, Early2L had a smaller proportion of the factor 3 strategy than Newly2L. If 'Correct answer only' in table 3 was interpreted as a factor 3 strategy, Early2L had smaller proportions of factor 3 strategies than Other2L and Swe1L as well. Under this assumption, a smaller proportion of Early2L students reached the high level in the hierarchy of variable perception of Asquith et al. (2007).

Now, Petersson (2017) observed that second language students had an achievement profile emphasizing algebra when compared with Swe1L and hypothesized that newly arrived students may have contributed to that result. The present study followed up this hypothesis by separating between Newly2L, Early2L and Other2L for the case of linear expressions. Despite Early2L on average achieved higher than Newly2L in Swedish language as seen in table 1, Early2L students seem to face added challenges in advanced algebra learning. One interpretation is that this might be related to Early2L having received large parts of their mathematics education as beginner second language students, while Newly2L may have received a major part of their mathematics education and some of their algebra education as first language students before immigration. The results might also be interpreted as related to organization of the teaching for second language students due to socio-economic segregation (Hansson, 2012). With the first interpretation, the results are in line with Cummins (2008) and also with Ufer et al. (2013), who saw achievement differences between first and second language students for conceptually demanding mathematical tasks. The author suggests that the results, despite the use of only two test items, have some degree of generality at least in Sweden, since the test takers in the present study achieved similarly in the national test to a national random sample. However, there is a need of a larger study to confirm this suggestion.

Acknowledgment

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Prospective class teachers' attitude profiles towards learning and teaching mathematics

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We have measured prospective class teachers' attitudes, towards learning mathematics and, towards teaching mathematics. We used a previously validated questionnaire called ALM (Attitudes toward Learning Mathematics), and we constructed another questionnaire called ATM (Attitudes toward Teaching Mathematics). In general, the observed attitudes were quite positive and the attitudes towards teaching were more positive than towards learning. Component-wise differences between ALM and ATM were found. We also compared attitude profiles between two class teacher programs. Prospective teachers' attitudes were more positive in the more mathematically focused program, which had a test of mathematics skills in the entrance examination.

Introduction

In Finland, prospective class teachers (PCTs) are educated in master's degree programs in eight universities, all of which have their own teacher education strategies and curricula that makes the best use of the local university's resources. These teacher education programs must contain 60 European Credit Transfer and Accumulation System (ECTS) credits of minor studies in subject didactics with a focus on the teaching and learning of subjects and themes in basic education (Sahlberg, 2010). The degree qualifies class teachers to teach all school subjects in the Finnish primary schools (grades 1–6). Since mathematics in primary schools is usually taught by a class teacher who does not have a major or minor in mathematics, we find it relevant to explore PCT's attitudes towards learning and teaching mathematics (see also, e.g., Boyer & Mailloux, 2015; Hourigan, Leavy, & Carroll, 2016).

At the University of Turku, the Department of Teacher Education operates in two units in two cities, Turku and Rauma. Both units have their own class teacher programs with different curricula and different entrance examinations reflecting the profiles of the units. In Turku, the emphasis is on mathematics and natural sciences, and the students have to pass an entrance examination containing a test of mathematics and natural sciences skills. In the Rauma unit, the student admission does not take mathematical skills into consideration, and the unit specializes in arts, crafts, and physical education. Hence, we are interested in studying whether there are differences in PCTs' attitudes concerning mathematics between the two degree programs. The selection process for class teacher education is under constant review in Finland, and therefore scientific information concerning the effects of the entrance examination and different degree profiles is valuable as a basis for discussion.

There is no overall agreement concerning the definitions of the concepts like attitudes, believes and conceptions in the domain of mathematics-related affects (e.g. Goldin et al, 2016; Hannula, 2012). In this paper, attitudes are “manners of acting, feeling, or thinking that show one’s disposition or opinion” (Philipp, 2007, p. 259). We interpret that this definition covers the components of Wong and Chen’s (2012) attitude scale, which will be used in the empirical part of this study. The *Checking solutions* component is related to the looking back feature of Polya’s problem solving framework. The *Confidence* scale measures respondents’ self-conception about their ability to learn mathematics. *Enjoyment* deals with the degree to which students enjoy mathematics, and the *Use of IT* component enquires how much respondents believe that information technology supports their learning of mathematics. The *Multiple solutions* component measures students’ tendency to look for multiple solutions for mathematical problems. *Usefulness* of mathematics is related to respondents’ beliefs about the usefulness and relevance of mathematics to their daily life.

The affective domain plays an essential role both in teaching and learning mathematics. The research suggests a reciprocal causality between the learners’ achievement and affect (Hannula, 2012). Hence, promoting the students’ positive attitudes should be reflected in their improved mathematical performance (Ignacio, Nieto, & Barona, 2006). According to Atnafu (2014), students’ academic achievements in mathematics are associated with their teachers’ attitude to teaching mathematics. Namely, teachers’ pedagogical practices are aligned with their attitudes regarding mathematics, teaching, and learning (Boyer & Mailloux, 2015; Hourigan et al., 2016). Self-confident teachers who see mathematics interesting, pleasant and useful are likely to improve students’ positive attitudes towards mathematics (Boyer & Mailloux, 2015). Prospective teachers’ attitudes towards mathematics may originate from their early schooling years, and these attitudes, together with prospective teachers’ conceptions about teaching mathematics, seem particularly difficult to change (Boyer & Mailloux, 2015; Philipp, 2007). Unfortunately, PCTs have been reported as having rather negative mathematics-related attitudes, but current research also indicates some positive indications. PCTs are found to hold positive beliefs about mathematics, find it interesting and enjoyable and value its role in the sciences and in the society (Hourigan et al., 2016).

Many aspects of the affective dimension have been examined also in the Finnish mathematics education context (see, e.g., Hannula, Bofah, Tuohilampi, & Metsämuuronen, 2014; Holm, Hannula & Björn, 2017; Kaasila, Hannula, Laine,

& Pehkonen, 2008; Sorvo et al., 2017). The studies are mostly focused on the affective factors of the students. More research on teachers' attitudes, especially prospective primary teachers' attitudes, are required (Philipp, 2007; Hourigan et al., 2016).

Research questions and methodology

We formulated the following research questions for this study: 1) What kind of attitudes do the prospective class teachers have towards learning mathematics and towards teaching mathematics? 2) How do the prospective class teachers' attitudes towards learning of mathematics relate to their attitudes towards teaching of mathematics? 3) Are there differences in the prospective class teachers' attitude profiles between the two class teacher programs with different emphasis on mathematics?

In this study, we measured PCTs' attitudes towards learning mathematics using the *Attitudes toward Learning Mathematics* (ALM) scale developed and validated by Wong and Chen (2012). The scale was originally designed in the Singapore Mathematics Assessment and Pedagogy Project to be used with lower secondary school students. We chose to use this scale for several reasons. First, the test has a very concise form, and therefore it is plausible that the respondents will consider and answer the items carefully. Second, the components seem to have a practical orientation relevant to the Finnish context. Third, the test has been carefully validated by its developers, and we did not find any cultural features which would prevent us from using it. In our questionnaire, t*he variables were measured on a 5-point Likert scale* ranging from strongly disagree = 1 to strongly agree = 5.

According to Wong and Chen (2012), the psychometric properties of attitude scales towards learning may be culture dependent. Therefore, the translated scale ALM from English into Finnish was validated by confirmatory factor analysis ($\chi^2(194) = 446.36$, $p < .001$, CFI = 0.92, TLI = 0.91, RMSEA = 0.054, SRMR = 0.067). In order to examine the interconnections between PCTs' attitudes towards teaching and learning mathematics, we constructed a new questionnaire, the *Attitudes toward Teaching Mathematics* (ATM) scale, with the aim of measuring the same attitude components as in ALM. For this purpose, we rephrased the items of ALM to focus on teaching instead of learning. Examples of the rephrased items are exhibited in Table 1.

Component	Example item of ALM	Example item of ATM
Checking solutions	When I know I have made a mistake in solving a problem, I will try to find out why.	I encourage my pupils to find mistakes from their incorrect solutions by themselves.
Usefulness	I think mathematics is useful in solving real world problems.	In my teaching, I regularly emphasize the usefulness of mathematics for solving real world problems.

Enjoyment	Solving mathematics problems is fun to me.	Most of my (future) students think that solving mathematics problems is fun.
Use of IT	IT has been helpful to my mathematics learning.	IT is helpful to my (future) students' mathematics learning.
Multiple solutions	I often figure out different ways to solve mathematics problems.	During my lessons, I often emphasize that there may be several ways to solve a mathematics problem.
Confidence	I am confident in solving mathematics problems.	I am confident in teaching mathematics well.

Table 1: Example items for the components of the questionnaires ALM and ATM

Based on exploratory factor analysis, the new ATM scale was divided into six components. The rephrased items did not entirely fit into the components for which they were originally designed. Some of the rephrased items were omitted, and some were regrouped with other components. However, the components could still be interpreted to measure similar aspects as the components of ALM. The components *Checking solutions* and *Multiple solutions* measure teachers' attitudes towards guiding their pupils to check the correctness of their solutions and to figure out alternative solutions, respectively. *Usefulness* is related to teachers' attitudes towards highlighting the relevance of mathematics for solving practical problems in their teaching. *Enjoyment* measures teachers' impressions of whether or not his or her pupils enjoy learning mathematics. *Use of IT* is related to respondents' believes that information technology can be used to help pupils to learn mathematics. *Confidence* deals with teachers' self-reliance on teaching mathematics. Furthermore, The ATM scale was validated by confirmatory factor analysis ($\chi^2(120) = 162.40$, $p = .006$, CFI = 0.94, TLI = 0.92, RMSEA = 0.045, SRMR = 0.060). The Cronbach's alpha values for the six components of both questionnaires and the number of items related to each component are given in Table 2. Some of the alpha values are rather low indicating weak internal consistency of the corresponding component. In particular, in order to increase the internal consistency of the *Usefulness* component, we abandoned one item of the original ALM scale leaving only two items remaining for that component.

Components of the questionnaires	ALM		ATM	
	No. of items	Cronbach's α	No. of items	Cronbach's α
Checking solutions	4	0.616	5	0.663
Usefulness	2	0.651	2	0.522
Enjoyment	4	0.889	4	0.735
Use of IT	4	0.670	3	0.731
Multiple solutions	4	0.740	2	0.752
Confidence	4	0.830	2	0.759
Whole test	22	0.863	18	0.785

Table 2: Internal consistency of the items for the components of ALM and ATM

The data was collected using Webropol-questionnaires from the two units of the Department of Teacher Education at the University of Turku during the years 2015 and 2016. In both units, all PCTs take 6 ECTS credits compulsory mathematics education courses during their first and second year of studies. In the more mathematically focused program in Turku, the response rate was around 63% (n = 70) and in the less mathematically focused program in Rauma around 61% (n = 105).

The data was analysed using IBM SPSS Statistics 22 software. The negatively worded items were reverse-coded, and the composite variables *Checking solutions, Usefulness, Enjoyment, Use of IT, Multiple solutions* and *Confidence* were formed by calculating the mean values of the items of the corresponding components for both questionnaires separately. Moreover, the composite variables *ALM* and *ATM* were formed by calculating the mean value of all items of the corresponding questionnaire. These variables were used to describe the students' overall attitudes towards learning and teaching mathematics. Each variable was measured on a 5-point Likert scale where 1 corresponds to strongly negative attitude and 5 corresponds to strongly positive attitude.

Results

Considering our first research question, we observed that prospective class teachers' attitudes towards learning (*ALM:* M = 3.47, SD = 0.546) and teaching (*ATM:* M = 3.77, SD = 0.411) of mathematics were, in general, positive. The values of *ALM* ranged from 2.18 to 4.86 and the values of *ATM* from 2.39 to 4.89. Based on Shapiro-Wilk normality test, both variables *ALM* and *ATM* were normally distributed. By one-sample t-test, the mean values of *ALM* ($t(174) = 11.394$, $p < .001$) and *ATM* ($t(174) = 24.832$, $p < .001$) were statistically significantly greater than the neutral value 3. Furthermore, the mean values of the six components of both questionnaires differed statistically significantly from the neutral value 3, except for the ALM components *Confidence* and *Multiple solutions*. The mean values of the components are depicted in Figure 1.

The composite variables of the ALM and ATM components *Checking solutions, Usefulness, Enjoyment, Use of IT, Multiple solutions,* and *Confidence* were not normally distributed. Friedman test revealed statistically significant differences among the components of ALM ($\chi^2(5) = 411.87$, $p < .001$) and ATM ($\chi^2(5) = 250.92$, $p < .001$). Post hoc analysis with Wilcoxon signed-rank tests was conducted with a Bonferroni correction applied, resulting in a significance level set at $p < 0.0033$. Statistically significant differences were found between all the components of ALM except for the pairs *Confidence–IT* and *Confidence–Multiple solutions*. Similarly, almost all components of ATM differed statistically significantly from each other. Only between the components *Confidence*, *Use of IT* and *Multiple solutions* were no statistically significant differences found.

Hence, PCTs responded differently towards the different components of the scales. The highest mean values were obtained by the components *Usefulness* and *Checking solutions* in both questionnaires. The only component with the mean value below the neutral value was the ALM component *Multiple solutions*.

In order to examine our second research question, we compared the data from the two questionnaires. First of all, the attitudes of the respondents towards teaching (*ATM*) were more positive than towards learning mathematics (*ALM*), $t(174) = 8.646$, $p < .001$. The Spearman's rank-order correlation was used to examine the relationship between the components of learning and teaching. In the component-wise comparison, the r_s-values ranged from 0.321 to 0.631 at significance level $p < .001$. A strong positive correlation was found between the *Use of IT* components of ALM and ATM ($r_s(173) = .631$, $p < .001$), as well as between the *Confidence* components of the two questionnaires ($r_s(173) = .619$, $p < .001$). There was also a strong, positive correlation between the ALM components *Confidence* and *Enjoyment* ($r_s(173) = .694$, $p < .001$). Moreover, the scatterplot of the variable *Confidence* of ALM and ATM revealed that a PCT who is not confident in learning mathematics may nevertheless be confident in teaching it. However, confidence in learning mathematics seemed to imply confidence in teaching mathematics. Even though there were correlations between the attitude components of teaching and learning, statistically significant differences between the distributions of each of the components of ALM and the corresponding components of ATM were found by Wilcoxon signed ranks test. The components *Checking solutions*, *Confidence, Multiple solutions* and *Use of IT* obtained higher mean values in teaching than in learning, whereas the attitudes towards *Usefulness* and *Enjoyment* were more positive in ALM than in ATM (see Figure 1).

When comparing the prospective class teachers' overall attitudes towards learning mathematics between the two class teacher programs, we noticed that the students in the more mathematically focused program had higher scores of *ALM* than the students in the less mathematically focused program, $t(173) = 4.626$, $p < .001$. However, no statistically significant differences were found between these programs when considering attitudes towards teaching mathematics (*ATM*). Descriptive statistics of the components of both questionnaires for the class teacher programs are given in Table 3. We noticed that the distributions of the components *Enjoyment* (ALM: $U = 2030.5$, $p < .001$, $r = .38$; ATM: $U = 2946.0$, $p = .025$, $r = .19$) and *Confidence* (ALM: $U = 1960.5$, $p < .001$, $r = .40$; ATM: $U = 2571.5$, $p = .001$, $r = .26$) differed in both scales between the two programs, the attitudes in the more mathematically focused program being more positive than in the less mathematically focused program. There was also a statistically significant difference in the attitudes of using IT in mathematics teaching ($U = 2798.5$, $p = .007$, $r = .20$), this time in the favour of the less mathematically focused program. Moreover, the differences in the distributions of *Checking solutions* ($U = 2982.5$, *p* = .033, *r* = .16) and *Multiple solutions* (*U* = 2761.0, *p* = .005, *r* =.21) in ALM were statistically significant between the programs.

Program		*ALM*						*ATM*					
		Useful	*Check*	*Enjoy*	*IT*	*Conf.*	*Multi*	*Useful*	*Check*	*Enjoy*	*IT*	*Conf.*	*Multi*
Turku	M	4.54	4.16	4.05	3.18	3.52	3.11	4.10	4.32	3.30	3.46	4.04	3.61
	SD	0.423	0.686	0.815	0.872	0.890	0.877	0.611	0.447	0.627	0.734	0.788	0.826
Rauma	M	4.41	3.97	3.34	3.25	2.75	2.75	4.11	4.26	3.09	3.76	3.55	3.56
	SD	0.661	0.695	0.953	0.741	0.897	0.741	0.676	0.584	0.685	0.723	0.970	0.833

Table 3: Descriptive statistic of the components of ALM and ATM for the mathematically-focused program (Turku) and the non-mathematically-focused program (Rauma)

We used K-means cluster analysis to group the students into classes with different attitude profiles. The clustering was based on the twelve composite variables of ALM and ATM. After exploratory analysis, the number of clusters was fixed on three. According to the attitude profiles of the clusters depicted in Figure 1, Cluster 1 contains the students with the most positive attitudes. The prospective class teachers in Cluster 2 have the lowest confidence both in learning and in teaching mathematics. Interestingly, the use of IT for teaching and for learning mathematics was higher in this group than in the other clusters. Cluster 3 contains the students with the lowest mean values of the components, except for the ALM components *Confidence* and *Enjoyment* and the ATM component *Confidence*. Cross tabulation of the clusters revealed that the class teacher programs differed from each other ($\chi^2(2) = 19.43$, $p < .001$). For the more mathematically focused program, 71% of the PCTs belonged to Cluster 1, 10% belonged to Cluster 2 and 19% to Cluster 3. For the less mathematically focused program, 39% belonged to Cluster 1, 33% to Cluster 2 and 28% to Cluster 3.

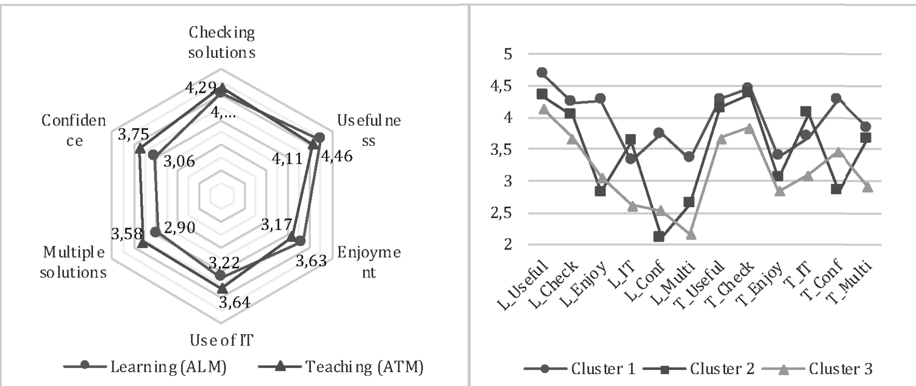

Figure 1: Mean values of the components of ALM and ATM in the whole sample and in the clusters

Discussion and conclusions

Based on our analysis, the PCTs seemed to have quite positive attitudes towards teaching and learning mathematics. We interpret that PCTs consider mathematics useful and emphasize its usefulness in their teaching. When learning mathematics, they want to check the correctness of their solutions and also guide their pupils to do so. They enjoy learning mathematics, and they believe that their pupils enjoy learning mathematics as well. To some extent, they have a positive attitude towards using IT for learning mathematics. In particular, they see the potential of using IT in order to foster their mathematics teaching and their pupils' learning. They also emphasize the importance of finding out multiple solutions when they are teaching, but do not so much report doing so when learning mathematics themselves. Although PCTs have formerly been reported as having negative mathematics-related attitudes (Philippou & Christou, 1998), our findings are in line with recent positive indications (Hourigan et al., 2016).

There are recent results showing that mathematics education programs may have positive effects on PCT's attitudes (Hourigan et al., 2016). Based on this study, we cannot draw any conclusions whether the PCTs' attitudes have changed during our programs. However, we noticed that PCTs' attitudes were more positive in the program which has a mathematics and natural sciences test as a part of the entrance examination. It seems reasonable to think that by selecting students with good skills in mathematics and natural sciences, we also select students with positive attitudes towards learning and teaching mathematics. Indeed, studies suggest a reciprocal causality between achievement and affect (Hannula 2012). Moreover, students with less skills and less positive attitudes towards mathematics will probably apply to teacher education programs which have no entrance test in mathematics. Hence, as a contribution to the reforms of the student selection processes of the Finnish class teacher education, we may say that from the perspective of PCTs' attitudes towards mathematics our findings support using a mathematics and natural sciences test in the entrance examination for the mathematically focused programs.

Furthermore, we found out that attitudes towards teaching mathematics were more positive than towards learning mathematics. It seems that the PCTs are more willing to emphasize the use of information technology, checking the correctness of solutions and finding multiple solutions in teacher's work than in solving mathematical problems personally. The PCTs had also higher confidence in teaching than in learning mathematics. In addition to earlier studies (see, e.g., Ünlü & Ertekin 2013) showing a positive correlation between mathematics teaching self-efficacy and mathematics self-efficacy, the similarity between the structures of the instruments ALM and ATM has enabled us here to compare the different components of the learning and teaching scales. We consider this comparison the most important theoretical aspect of our study.

We acknowledge the limitations of the new ATM scale, which should be further developed to better correspond to the components of ALM. Although the scales were validated by confirmatory factor analysis, some original items had to be omitted from the composite variables and the Cronbach's alpha values were still quite low. In addition, the low response rate may compromise the external validity of our results. Finally, we note that in the cluster with low mean values of confidence and enjoyment, the PCTs' attitudes towards the use of IT for both learning and teaching mathematics were higher than in the other clusters. More investigations about teacher's attitudes towards the integration of IT and mathematics teaching are needed (Goldin et al., 2016) and this interesting finding should be further examined.

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An initial analysis of post-teaching conversations in mathematics practicum: researching our own practice

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An initial analysis of post-teaching conversations about mathematics in a school-based practicum setting is presented. Using data from the practice of mathematics educators, insight is sought on ways for directing reflections to be more mathematics-based. A scheme for planning teaching was introduced to student teachers beforehand to bring into attention different aspects of mathematics teaching. It was possible to detect some features in the conversations that have a potential to develop into reflections about the mathematics. These features make it possible to identify a potentiality zone, in addition to the evaluation-based and subject-based discussions as suggested from earlier research. This paper fills a gap in the research about student teachers' reflections in practicum.

Introduction

The school-based practicum has an important part in training Student Teachers (STs) which goes beyond the role of the university based courses. Grootenboer (2005/2006) pointed out the need for it to include and induce critical reflections on STs in order to make their experiences meaningful and helpful. Despite the accepted importance of a good practicum, Haugan (2011) in his systematic review of research about Norwegian General Teacher Education found out that there is a dearth of research about STs' reflections in their practicum period. Østrem (2008) as quoted in Haugan (2011) concluded in her study that STs look at the practicum and their lectures at university as two separated practices, and that it is the teacher education programme's task to connect them both so that experiences from the practicum can be used in teacher education. Zeichner (2010) pinpointed some problems regarding the missing connections between teacher education and the practicum, such as the STs' missing feedback about their teaching practice as learned in teacher education courses, and the missing connections between content in the teacher education program and the teaching in practicum.

The field of practicum further triumphs over the field of teacher education because its rules are familiar and comfortable for the STs (Nolan, 2012). Similar results are described in a literature review (Haugan, Moen & Karlsdottir, 2013) where surviving in the classroom and managing it are primary, while pupils' learning and its facilitation, and STs' reflections about own learning become secondary or are often missing (Bjuland, Jakobsen & Munthe, 2014; Brown & McNamara, 2011; Rowland, Thwates & Huckstep, 2005).

We have noticed that STs tend to focus on the general pedagogical aspects of teaching during the post-teaching conversations in school-based practicum. Such aspects are what Shulman (1987) defined as ''broad principles and strategies of classroom management and organization that appear to transcend subject matter'' (p. 8), but also learners and learning, assessment, educational contexts and purposes (Blomeke et al, 2008). This is an important aspect of the teaching practice and important knowledge for STs. But we as teacher educators are more concerned about the STs' developing thoughts about important aspects in preparing and teaching a mathematics lesson such as the mathematical content, representations used, and choice of examples.

We aim therefore with our research to find ways to integrate the course content with the practicum and help STs to reflect on their own mathematical learning by setting the scene for such reflections. One place where such reflections can develop are the post-teaching conversations in practicum which we focus upon in this paper. In order to search insight into features of the conversations that have a potential to empower STs in their reflections about mathematics, we present and analyze empirical data from the observation of our STs' teaching and the post-teaching conversations afterwards. In the next section we present a scheme for planning mathematics teaching we have developed, that the students bring with them in practicum.

The scheme for planning mathematics teaching and the Knowledge Quartet

The aim for developing the scheme was to make students aware of the important aspects a teacher of mathematics should take into account when planning teaching. For this, we looked at the Knowledge Quartet (KQ), a framework that presents four dimensions that can be used to observe STs' mathematics-related knowledge in practicum (Rowland, Thwates & Huckstep, 2005). The four dimensions are employed in our study to inform the scheme that students use in their practicum.

The scheme starts with the mathematical theme to be taught, the aim stated in the curriculum and translated in concrete mathematical ideas/points the teacher wants to achieve with the current teaching. One of the dimensions from KQ is transformation, and it is about preparing the mathematics content for teaching in ways that help pupils understand it. Elements of this dimension are the different representations such as oral explanations, written symbols, manipulatives, real-world events or pictures (Lehr, Post & Behr, 1987) that are appropriate to achieve the aims. Representations, and how they can give mathematics learners more experiences with different sides of mathematical concepts, are an ongoing theme in our own teaching with STs. The choice of examples to bring about a mathematical point, and their connection to the ideas the teacher wants the pupils to work with, are included with the aim of making students aware of the importance of intentionally and carefully choosing examples to use in mathematics teaching.

The connection dimension from KQ in our scheme is related to: questions about what pupils have worked with earlier that can influence the present teaching, and thoughts about the future teaching topics to which the current theme is connected, different ways to solve the problems, and mathematical challenges pupils can have with the concept at hand. The two latter are also connected to the contingency dimension in KQ as a way to prepare STs about unexpected situations in the classroom. Other elements connected to contingency (KQ) are: the different questions to ask for developing good discussions in the classroom, for helping pupils if they are stuck, for evaluating them and for planning further teaching. The foundation dimension, students' knowledge and beliefs about mathematics and its teaching, is not explicitly placed in the scheme, but we encounter it in our reflections.

Mathematics-based reflections and evaluative reflections

Johnsen-Høines and Alrø (2010) identified two approaches of communication in the classroom: the evaluative and the subject-based (here mathematics-based) approach. Earlier, Johnsen-Høines and Lode (2007) used these approaches to describe and analyze the post-teaching conversations that they as MEs conducted with STs and Mentor Teachers (MTs). We use the two concepts similarly in our study. In addition, we have integrated the scheme for planning mathematics teaching in our lectures before the practicum period.

In our study, an evaluative approach is connected to those parts of post-teaching conversations where participants point to what did go well or wrong in the observed lecture, why choices were made, and how it could be done better. Johnsen-Høines and Lode (2007) call this a retrospective perspective. While Høines and Alrø (2010) point to this approach as positive when used to learn from mistakes, they also argue about obstacles it can bring for STs' empowerment.

On the other hand, we aim to create possibilities for more mathematics-based reflections with our students, reflections about the teaching and learning of mathematics, as a way to connect the course content to the practicum. A subject-based reflection is an "educational approach that aims to explore how the situation might generate discussions for further development" (Johnsen-Hoines & Lode, 2007, p.321-322). These are the kind of reflections that take a future-oriented perspective (Johnsen-Høines & Alrø, 2010), where critique toward the knowledge is not seen as critique against the person itself. In terms of our study, such an approach would focus more on what didactical possibilities offer the examples chosen for teaching, what implications do the representations have for learning the current mathematical concepts, as well as other elements from our planning scheme and KQ.

Using the two approaches, which we see as being a continuum where the discussions flow, as tools for analyzing the post-teaching conversations, helps us in identifying features that indicate potentials for empowering students in mathematics-based discussions.

Background for the data collection

The aim with our research is to find ways to make the post-teaching conversations in practicum more mathematics oriented and connect them to the teaching at our institution. For this, we discuss in our own teaching about the important points in planning a teaching session in mathematics and the planning scheme developed. We use the scheme as one way to make STs aware of those points, but also as a tool for supporting them in reflecting upon the mathematical content in the post-teaching conversations. We might focus on one or more points from the scheme, without aiming to go through all of them, and without allowing it to be limiting our reflections. In the meetings before the practicum period, we shortly presented the scheme for the MTs and expressed our expectations for the post-teaching conversations to be mathematics-based.

The course we teach is a compulsory one in mathematics (30 ECTS) for all elementary school STs in their second year. In our teaching, we look at areas of mathematics for elementary schools from a perspective of teaching and learning, with a focus on relational understanding (Skemp, 1976). Two school-based practicum periods of 3 weeks each are organized in two semesters. STs are placed in groups of 3-4 in different schools under the supervision of a MT from the school. Both authors of this paper teach the course and are in charge of visiting them in their practicum period in schools. We observe the STs while they teach mathematics, we get the plan for the teaching beforehand, and after the observation, we have a conversation with the whole group of STs and the MT if available. In our observations, we focus on episodes that can be fruitful to discuss with the students.

Data for our project were collected during practicum periods in primary schools. In this paper we use data from one group of four STs. This is the first practicum period for the year, a few weeks after the course start. STs have therefore had little teaching in mathematics education at our institution beforehand. In our observations they taught about word problems in two different fourth grades. Then we audiotaped the post-teaching conversations with the STs and two MTs. We focus here on the analysis of the conversation and use observations where needed to inform the analysis.

Analyzing post-teaching conversations

Both authors analyzed the data starting from the two approaches, the evaluative and the mathematics-based conversations. In addition, we wanted to look for features of the post-teaching conversation that can help us as Mathematics Educators (ME) to direct the conversation towards reflections about the mathematics, its teaching and learning. We differ therefore between the concrete elements from KQ being discussed. The chosen excerpts represent such elements.

Different ways to solve a problem- a potentiality zone in a continuum between an evaluative and a mathematics-based discussion

In our analysis of the post-teaching conversations, it was sometimes difficult to categorize an excerpt as being evaluative or subject-based. Such a categorization would not be our final aim either, as we want to find out more about potential features to foster mathematics-based reflections. In order to identify such features, looking for elements from our scheme that were adapted from the KQ dimensions, helped us find the focus and lead the conversation towards the desired direction. We discuss such an example here, bringing our perspective in how we use the scheme with dimensions from KQ to spot such potential and help the STs to use that potential.

Often the conversation starts by pointing at what happened in the classroom. In the excerpt, a student (ST1) is talking about different ways of solving a word problem.

ST1:	They solve it in different ways...They solve it in different ways. They think in different ways. Some do it in their head.
ST2:	Yes.
ST1:	Some line up the numbers. Some do it mentally…in (laughs) some special ways.

In the first sentence, ST1 points to the different ways pupils can choose to solve a problem. This can connect to several dimensions from the KQ, as used in our scheme. It is related to *connection*: thinking about different solutions can be a way to precede the cognitive demands of the task; *transformation*: it can foster thinking about the choice and the potential of the example; *contingency*: it can to some degree help students to be prepared for unexpected solutions; *foundation*: it tells about students' knowledge, understanding and beliefs. This is not a mathematics-based discussion as it does not discuss the mathematical and didactical details, but it remains on a general level. It is not an evaluative discussion either; the student is not giving any evaluation of the situation. We identify here a zone between an evaluative and a mathematics-based discussion, where the excerpt can temporarily be placed, a potentiality zone. Such a zone is recognized by the possibilities it offers for the participants to inquiry the topic at hand and discuss in depth the didactical aspects of the concept or task. The sentence's potential could be developed by discussing for instance the chosen task and how it responds to the goals that STs had with their teaching as connected to the *transformation* dimension (KQ), thus becoming mathematics-based. Instead, the ST in the other sentences further explains what she means by the different ways: the ways of setting up the numbers to get the answer to the problem, focusing on the calculations, the technicalities of the task, not on its structure. It can be interpreted as if the ST thinks of some special pupil when she laughs about "special ways" they solve problems. This can be one kind of evaluative discussion, and the first sentence's potential is not fully used. Thus, one ME tries to direct attention into being more specific, in order to realize that potential, and asks about the text of the word problem students were working with: A marching band is lined up in four rows. There are nine children in each row. How many children are there altogether?

The discussion continues with STs telling what they did in the classroom when they drew on the smart board different solutions that pupils presented. These are elements of an evaluative discussion, helping to make the conversation more concrete. The ME asks again, in order to conduct a mathematics-based discussion, about the different solutions STs had thought of, and answers are:

ST1: In the drawing we didn't have…there we had these four…but when you actually calculate it, then you can take…Ok there are four, right? And they know…they know how to multiply by 4, so four times nine. But you can also do it 9+9+9+9, right, if you want to do it. Or 9+9 is 18, and 9+9 is 18, and so 18+18, right? And they had…

ST2: Yes. Or multiply by 2.

At first, ST1 starts by pointing back to the drawing they had on the smart board, where four rectangles represented the four rows. This can serve as a concrete example that later can be taken up to further reflect about mathematics. She then pulls out some information about the pupils: they have learned to multiply by four, so they could use that as one way to solve the problem. Here the ST moves back and forth between a retrospective and a future-oriented perspective. Next, she thinks of multiplication as repeated addition as a possible approach. She then adds the repeated addition but using the associative rule, by grouping two and two nines, and then adding up the two eighteens. ST2 mentions multiplication by two as an alternative to adding the two eighteens. Now the discussion is based on concrete examples of mathematics, the student is not into evaluating pupils' responses. It is a mathematics-based discussion to a certain degree. In more depth, the ST could e.g. reflect upon pupils' mathematical knowledge by looking at the different solutions.

Pupils coming with a new solution- two levels of conversation as experienced by participants

During the observations, one ME heard one pupil use 40-4 as a solution to the marching band problem, but the STs missed the pupil's suggestion, and they are stuck as if trying to excuse themselves for this. It is not what the ME wants, as the aim is to reflect upon potentially fruitful situations, in this case discuss how the ST should make sense of pupils' answers. Thus, the ME asks again in an attempt to direct the discussion towards the suggestion itself, and how STs would follow this up with the rest of the pupils:

ME:	… Is that a way to solve it?
ST1:	Yes, because he can round it up. He can take 10 times, or 4 times 10.
ST2:	Mmmh
ST1:	Is that you were thinking about? He gets 40 and then…
ME:	… yes, for example…
ST1:	… takes minus 4. Because then he must subtract one?

After approvals from the MT and other STs, ST1 adds in a questioning tone:

ST1:	So … but that is also a way to solve it. Right?

ST1 is now trying to answer the question, but she is often asking for confirmation. It seems as if she feels like she and her reasoning are being evaluated. On the other hand, the ME is trying to use the situation where a pupil presents an unexpected solution as *contingency* (KQ), in order to invite STs into a mathematics-based discussion. There are clearly two levels and two different ways of experiencing the conversation by the participants, the ST and the ME. However, some reflection can be found in the conversation, and thus a potential for mathematics-based discussions, as ST1 tries to make sense of pupil's reasoning. A feeling of safety can be a condition for the student(s) to fully participate. The knowledge is not set free, the ST experiences it as connected to herself. The ME chooses not to continue with the topic. Maybe inviting the other STs into the conversation from the beginning would have helped ST1 to feel safer in reflecting upon the example.

Pupils' solutions of word problems and equations- premises to participate in a discussion

The example presented below is about highlighting the connections between different concepts in mathematics (related to the *connection* dimension in KQ). In the excerpt, pupils have worked with another word problem: Alex payed 1400 NOK for a pair of trousers, one T-shirt and a sweater. The trousers cost 620 NOK, the sweater 590 NOK. How much did the T-shirt cost?

The ME and the MT have already discussed together the connection of this word problem to equations during the observations. Now the ME tries to invite

STs in the post-teaching conversations. After re-reading the task, the ME tries to connect it to the solution drawn on the smart board.

ME: It was drawn very nice on the table … (STs are nodding, saying yes …)

ME: Because it was… It was one of the pupils that drew three boxes first, 620 in the first one, 590 in the next one, and a question mark in the third. Right? And then a new box that goes under all three, that will then be as much as (those). What is missing here, I mean under the question mark? I mean it is…it is almost a written equation, right?

620	590	?
1400		

Figure 1: The representation of the word problem on the smart board

This is an example of *connection* (KQ), where MEs try to link the word problem and pupils' work to equations, which they will meet later. Our aim as MEs was to direct the STs' attention to the connections between mathematics concepts and structures. The ME starts with an evaluation about the nice drawing of the problem's solution on smart board. This evaluation is a positive reinforcing of ST's work to encourage them to participate in the conversation. They participate by nodding, saying yes, or that they will try it next week. A more active participation could have occurred, but, in order to participate in mathematics-based discussions, both parts need to have the right premises, such as the knowledge needed. Algebraic thinking had yet not been addressed in the course. In terms of KQ, this is an example of *foundation*; it is likely that STs did not see the connection the ME is pointing at, as they do not participate in elaborating it. Still, this conversation is valuable for MEs as it informs about what MEs need to focus at in the lectures. This excerpt can be placed in a potentiality zone and be used to talk about connections in mathematics, but its potentiality cannot be fully exploited now. This will be used further to reflect upon algebraic thinking as a part of our lectures in mathematics.

Conclusions

From the analysis of the post-teaching conversations, we conclude that the questions we ask during discussions are influencing the reflections and their direction into being mathematics-based. In addition to the evaluative and subject-based discussions used by Johnsen-Høines and Lode (2007), we identified a potentiality zone, as a crucial moment to ask appropriate questions. The zone is possible to spot when looking for different elements from the planning scheme. These zones cannot always be fully used in the moment they occur. As identified in the analysis, necessary conditions for fully participating in a mathematics-based reflection are related to feelings of safety for the STs as in the third excerpt, and knowledge of mathematics and its teaching, as in the last one. We can invite students to discuss about mathematics, but it is up to them to accept the invitation to reflect.

Other potentiality zones could have been used to conduct reflections. Such an example are the representations chosen for the problem solved, which belong to the *transformation* dimension (KQ). Drawing was constantly used in the classroom. A potential for mathematics-based discussions would be to address the influence drawing and visualization have on pupils' problem solving and learning. This is also connected to the *foundation* dimension, as it is in the content of our mathematics course.

The use of the planning scheme inspired by KQ resulted helpful in both holding the focus on the mathematics and defining the potentiality zones, and to further analyze the data in addition to the two conversation approaches (Johnsen-Høines & Alrø, 2010). Similarly, Turner (2012) found that using KQ to support beginning teachers in focusing their reflections on the mathematical content of their teaching, in collaboration with others, brings about improvement of their mathematical knowledge for teaching. The potentiality zone, combined with the scheme for planning teaching, helps us to get more insight into when and how to ask questions to foster STs' reflections about mathematics.

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Opportunities and challenges of using the MDI framework for research in Norwegian teacher education

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The mathematics discourse in instruction (MDI) framework was developed in South Africa, but it may be useful also in other contexts. This framework was developed to provide nuanced interpretations of the mathematics made available for students to learn. In this paper, we discuss possible opportunities and challenges of using the MDI framework for research in a Norwegian teacher education context. Overall, we suggest that the framework provides opportunities for highlighting the complexities of mathematics teaching. However, two challenges emerged from using the MDI framework for data analysis. First, the MDI framework does not clearly define what should count as formal mathematical language, and this makes it difficult to distinguish between subcategories. Second, the MDI framework does not distinguish between mathematically valid and invalid responses in teachers' classroom discourse, and this might limit its usefulness in some situations.

Introduction

Among the numerous frameworks for investigating mathematics teaching, this paper has a focus on the mathematics discourse in instruction (MDI) framework. This framework was developed to provide nuanced interpretations of the mathematics made available for students to learn (Adler & Ronda, 2015), and it represents a shift of focus from investigating knowledge for teaching mathematics (e.g., Hoover, Mosvold, Ball, & Lai, 2016) to analyzing mathematics embedded in the work of teaching (e.g., Ball & Forzani, 2009). Simultaneously, it illustrates a turn towards grounding teacher education in practice (e.g., Zeichner, 2012). The framework has previously been used to examine connections between examples, tasks and their accompanying explanations (Adler & Venkat, 2014; Venkat & Adler, 2012). It has also been used to investigate whether and how exemplification and explanatory talk enable or constrain students' opportunities to learn (Adler & Ronda, 2017a). Furthermore, an adapted version has been used to analyze textbooks (Ronda & Adler, 2016). Recently, Adler and Ronda (2017b) discussed how and why the framework might influence professional development of mathematics teachers.

In their presentation of the MDI framework, Adler and Ronda (2015) recognize that the framework was developed in a specific context—the South African educational context—and they query its potential beyond this context. With this as a point of departure, we recently discussed a possible use of the MDI framework in a Norwegian teacher education context (Fauskanger & Mosvold, 2017). Focusing on exemplification and student participation, we suggested that the MDI framework could support mentoring conversations in the field placement of pre-service mathematics teachers and help pre-service teachers as well as mentor teachers focus on the mathematical content. In this paper, we extend these results and investigate opportunities and challenges of using the MDI framework as analytic framework in a Norwegian teacher education context, focusing on explanatory talk (shaded in Figure 1). We approach the following research question: What opportunities and challenges emerge when using the MDI framework to analyze data from a Norwegian teacher education context?

Theoretical framework

In recent years, mathematics education research has been influenced by sociocultural theories that investigate learning and development in terms of communication and discourse (e.g., Sfard, 2008). The MDI framework is situated in this tradition, as it aims at capturing the complexity of mathematics teaching by concentrating on the discourse involved in the work of teaching mathematics (Adler & Ronda, 2015). The framework characterizes mathematics teaching as a sequence of examples and tasks along with the explanatory talk that follows (Adler & Ronda, 2014). The object of learning (Figure 1) is often announced explicitly and relates to the mathematical content and/or skills that students are expected to learn in a given lesson (Adler & Ronda, 2015). An example, or a sequence of examples and related tasks (i.e. exemplification), is often used to focus students' attention towards the object of learning. Explanatory talk is another integral part of the interaction between teacher and students, and amongst students (student participation). Student participation relates to what the students are invited to say, and if they are invited to speak and reason mathematically. In the MDI framework, student participation is seen from the point of how a teacher engages with the students during whole class discussions (Adler & Ronda, 2015, 2017a). In Figure 1, the four interacting components of the MDI framework are presented: exemplification, explanatory talk, student participation ("learner participation" is used by Adler and Ronda (2015)), and the object of learning. These components characterize mathematics teaching across classroom practices and contexts.

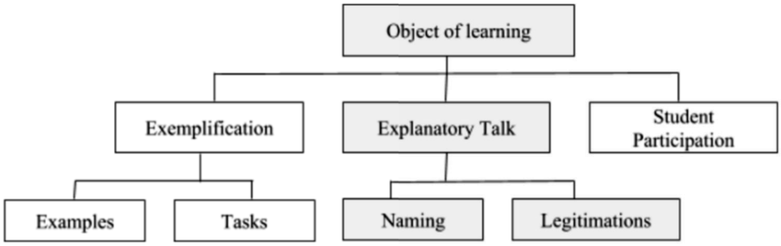

Figure 1: Constitutive elements of MDI and their interrelation (adapted from Adler & Ronda, 2015, p. 239, shading added).

Even though each element of the framework is inevitably connected with the other two, we focus only on explanatory talk in this paper (shaded in Figure 1). Explanatory talk is crucial in teachers' presentation of the mathematics that students are supposed to learn, but explanatory talk also includes analysis of the naming of objects as well as legitimation in episodes of a lesson (Adler & Ronda, 2015). In the MDI framework, the emphasis on explanatory talk draws on Bernstein's (2000) insight that continuous evaluation is at the core of pedagogic practice. The discourse in a mathematics lesson thus continuously transmits criteria as to what counts as mathematics (Bernstein, 2000; Adler & Ronda, 2015). The transmission occurs "through messages that are communicated as to what is valued with respect to the object of learning" (Adler & Ronda, 2017a, p. 69). The purpose of explanatory talk is naming and legitimating the mathematical issues discussed in examples or tasks (Adler & Ronda, 2015).

The MDI framework distinguishes between different domains of legitimation. For instance, legitimation can be reached within the mathematical domain, but also in non-mathematical domains. Legitimation can be based on the authority of the teacher, or it may refer to the curriculum. In the mathematical domain, Adler and Ronda (2015) distinguish between what counts as mathematics in a particular or local instance (L), and mathematics that has some generality. In the latter, a distinction is made between partial (GP) and full generality (GF). Non-mathematical criteria (NM) are also identified, as everyday knowledge or experience (E), non-mathematical visual cues (V) and if what counts is stated assigning authority to the position (P) of the teacher. The criteria for what counts as mathematics across episodes in a lesson relate to "the opportunities they open and close for learning" (Adler & Ronda, 2015, p. 244).

Naming of mathematical objects focuses students' attention in certain ways. Based on the assertion that a critical element of talk in the mathematics classroom is how objects focused on are named (Sfard, 2008), Adler and Ronda (2015, p. 244) define naming as "the use of words to refer to other words, symbols, images, procedures or relationships." Within episodes across a lesson, naming is categorized as either colloquial (NM, non-mathematical) or mathematical (see Table 2). In relation to mathematical naming, mathematical words used as labels or to read a string of symbols (Ms) are distinguished from formal mathematical language used appropriately (Ma). The summative judgment related to the level of naming "depends on movement across colloquial and formal mathematical naming in the lesson" (Adler & Ronda, 2015, p. 244).

Methods

In the present paper, our focus is on the process of using a particular analytic framework rather than on the results that emerge from analyzing data with this framework. The empirical data are thus used for the purpose of highlighting some *possible* opportunities and challenges of using the MDI framework in a certain context. We draw upon data from a cross-disciplinary project in Norwegian teacher education: Teachers as Students (TasS). The TasS project has a focus on teacher learning in field practice and involves student teachers from four subject areas — including mathematics. We draw upon classroom observations of one of the student teachers from the TasS project. At the time of data collection, Martin (pseudonym) was in the fourth semester of his teacher education program. In the previous semester, he had completed all required courses in mathematics. In the Norwegian education system, there are differentiated primary and lower secondary teacher education programs for years 1–7 and years 5–10. Martin attended the program for years 5–10. Based on analyses of data from group interviews, Martin stood out as a special case (Yin, 2003). He was one of two student teachers in the project who selected mathematics because they liked it and were good at it.

To describe the mathematics made available to learn during student teachers' instruction, the unit of study is a lesson (Adler & Ronda, 2014). We focus on a lesson taught by Martin in field practice. The object of learning in this lesson is multiplication of fractions, constituted by an algorithm for multiplying fractions, and the students are expected to learn how to multiply two fractions. To focus the students' attention towards the object of learning, Martin presented a sequence of examples and accompanying tasks. Based on shifts in content focus in these examples, we have divided the introductory part of the lesson into three mathematical episodes (see Table 1).

Episodes and codes	Examples
Episode 1. Content from a previous lesson.	Example 1: $3 \times \frac{5}{20}$ Example 2: $3 \times 2\frac{1}{4}$
Episode 2. Multiplying fractions and representing fraction multiplication using the area model.	Example 1: $\frac{1}{3} \times \frac{3}{4}$ Example 2: $\frac{1}{2} \times \frac{1}{3}$
Episode 3. Multiplying fractions.	Example 1: $\frac{3}{7} \times \frac{10}{2}$ Example 2: $4 \times \frac{1}{5}$

Table 1: Examples used in Martin's lesson (see also Fauskanger & Mosvold, 2017, p. 47).

These three episodes were then analyzed by use of the MDI framework. For this paper, we selected an excerpt from episode 1 as empirical example. The excerpt was selected because it illustrates the variation of opportunities and challenges that emerged when we used the MDI framework to analyze the data—focusing in particular on explanatory talk (Table 2).

Explanatory talk	
Naming	Legitimating criteria
Within and across episodes word use is: *colloquial* (**NM**), e.g. everyday language and/or ambiguous pronouns such as this, that, thing, to refer to objects in focus; *math words used as name only* (**Ms**), e.g. to read string of symbols; *mathematical language used appropriately* (**Ma**) to refer to other words, symbols, images, procedures, etc. Use of colloquial and mathematical words: Level 1—**NM**, there is no focused math talk, all colloquial/everyday; Level 2—movement between **NM** and **Ms**, some **Ma**; Level 3—movement between colloquial **NM** and formal math talk **Ma**	Legitimating criteria: *non-mathematical* (**NM**) *visual* (**V**), e.g. cues are iconic or mnemonic; *positional* (**P**), e.g. a statement or assertion, typically by the teacher, as if 'fact'; *everyday* (**E**). *Mathematical criteria*: *local* (**L**), e.g. a specific or single case (real-life or math), established shortcut, or convention; *general* (**G**) equivalent representation, definition, previously established generalization, principles, structures, properties, which can be partial (**GP**) or 'full' (**GF**). Criteria for what counts as mathematics that emerge over time in a lesson and provide opportunity for learning geared towards scientific concepts. Level 0—all criteria are **NM**, i.e. **V**, **P**, **E**; Level 1—criteria include **L**, e.g. single case; Level 2—criteria extend beyond **NM** and **L** to include generality, but this is partial **GP**; Level 3—**GF** math legitimation of a concept or procedure is principled and/or derived/proved

Table 2: Short version of the analytic framework (adapted from Adler & Ronda, 2015, pp. 242–243).

Analysis and discussion

Martin introduces the lesson by reminding the students of how they multiplied whole numbers with fractions in the previous lesson. He repeats that they can write any whole number as a fraction. As an example, he presents $3 \times \frac{5}{20}$ and explains that 3 can be written as the (improper) fraction $\frac{3}{1}$. Upon request, one of the students explains how to continue by multiplying numerators and denominators, and the episode continues as displayed in Table 3.

Line, code given to Martin's talk	Transcript*
7–14 Criteria: P, P_{inv}, V, L, L_{inv}, GP_{inv} Naming: NM, Ms, Ma	7. M: (repeats what a boy said that 3 can be written like the (improper) fraction $\frac{3}{1}$) ***We take three times five and one times twenty*** **(NM/Ms/L)**, or **like we did it yesterday (P)**, no, last week. Thursday, yes, Thursday. (writes $3 \times \frac{5}{20}$) **Three *times* five *over* twenty (NM/Ms/P)**. (Turns towards the students, and then walks back towards the blackboard and wipes out the example)
	8. M: *Another thing* that is also important, and that some of you don't always remember, is if we had (starts writing on the blackboard while looking down at his notes. Writes $3 \times 2\frac{1}{4}$, i.e. a whole number times a mixed number). *Like that one*. If we have three whole *times* two whole and a quarter. Then I observed that many of you *took* three times two and three with one. **(Ms)**. **Then we get six and three quarters (Ms/L). That is not completely correct (P_{inv}/GP_{inv})**. What you must remember, is *to convert it* (points at the number 2), *so that we get* (first writes down a number 3) (GP_{inv}). Can anyone tell me what fraction we get **(Ms)**? (points at the mixed number)
	9. S3: Nine fourths **(Ms)**.
	10. M: Yes, nine fourths **(Ms)**. **Four fourths one time, *plus* four (L_{inv}) *more, plus* one fourth, is nine fourths (Ms/L)** (writes $3 \times \frac{9}{4}$). *And then we get?* (asks a boy)
	11. S4: Eh?
	12. M: Nine *times* three **(Ms)**?
	13. S4: Twenty-seven **(Ms)**.
	14. M: Twenty-seven **(Ms)**. (writes, $\frac{27}{4}$) And ***then we can convert it back again*** **(NM/P), and then we see that it is the same as (L)** ... (walks over to write it on the blackboard, but then he hesitates a little bit. Writes $6\frac{3}{4}$ **(Ms)**, which is correct, but he just said that they could not calculate this directly) Eh, yes. (thinks a little bit) **That is not completely right (P_{inv})**, but, anyway, ***you must remember to convert*** **(NM/P_{inv}) (circling in the mixed number with his hand (V))**, ***because if we had something else here now*** **(NM/P_{inv}) (points at $2\frac{1}{4}$ (V))** (wipes out what he wrote on the blackboard to take a different example). I know that some of you fail at this too. That is, *if we had* (wipes out the number 4 in the denominator of the mixed number and writes 2 instead) two here now, then many of you manage to write the fractions as a whole number **(Ms)** (writes 1 on the other side of the blackboard) plus for instance three *over* two **(Ms/NM)** (writes, $1\frac{3}{2}$. Steps to the side and points at what he just wrote). **Then you can take away two more here** (points at the number 3 in the numerator), **and then you get the same as two and a half (writes $1\frac{3}{2} = 2\frac{1}{2}$) ($L_{inv}$)**.

*Key: M, Martin; S, student; *italics* for colloquial, underlining for formal language and **bold type** for legitimating criteria. In addition, bold types are used for MDI codes.

Table 3: Analysis and coding of explanatory talk from excerpt of episode 1.

When focusing on Martin's use of colloquial and mathematical talk (naming), we notice from the codes that Martin's non-mathematical talk (NM) involves everyday language like "convert" (line 14)—in Norwegian, he said "gjøre om"—and other utterances includes use of pronouns (e.g., "like that one"). The non-mathematical talk is frequent throughout the lesson, but there is also some mathematical talk that involves use of mathematical words as labels (Ms, e.g., line 7), reading strings of symbols (Ms, e.g., line 12), and naming fractions (e.g., line 10). From this, we suggest that Martin's use of colloquial and mathematical talk in this excerpt appears to be on level 2 (see Table 2), since there is a certain movement between non-mathematical talk and use of mathematical words as labels (Ms). Formal mathematical language is rare in Martin's lesson.

Although the MDI framework provides some interesting opportunities to observe and evaluate the mathematical talk in a lesson, some challenges are also involved. For instance, we find it difficult to distinguish between colloquial language (NM), mathematical words used for naming (Ms), and appropriate mathematical language (Ma). For instance, in line 7, Martin explains: “We take three times five and one times twenty,” and later in the same utterance he says, “Three times five over twenty”. Reading a string of symbols by using the number words “three”, “five” and “twenty” can be interpreted as mathematical words used for naming (Ms). When Martin uses the word “times” instead of “multiplied by” however, it can be argued that this is more of a non-mathematical, or colloquial, language. On the other hand, this word use is quite common, and even mathematicians might say it like this. The challenge then is to decide if this is non-mathematical word use (NM), use of mathematical words for naming (Ms), or perhaps even appropriate mathematical language (Ma). This might not be a major issue, but the MDI framework does not define what counts as formal mathematical language, and coders are left to make a decision based on their interpretation. Analysis of naturally occurring talk in mathematics classrooms involves frequent challenges in making such analytic decisions. A further development of the definitions of codes in the MDI framework might be necessary. On the other hand, the discussion of what counts as formal mathematical language is an ongoing discussion that might yield different results across historical and cultural contexts.

When considering the next element of explanatory talk in the MDI framework, we observe that Martin uses a variation of mathematical and non-mathematical legitimating criteria. The non-mathematical legitimating criteria are visual (V, line 14) and positional (P, lines 7 and 14). Applying the MDI framework for analyzing Martin’s lesson clearly indicates that Martin’s way of legitimating mathematics for his students is positional. We also observe that visual legitimating criteria are frequently used, and these observations illustrates some of the opportunities of using the MDI framework.

However, our analysis of legitimating criteria in this excerpt from the transcripts also displays some challenges. We observe that parts of Martin’s explanatory talk contain invalid responses and statements. For instance, in line 8, Martin suggests that multiplying 3 with $2\frac{1}{4}$ by first multiplying 3 with 2 and then with the fraction $(\frac{1}{4})$ is incorrect. It appears to the observer that Martin himself has misunderstood this, and the explanation he presents to the students is thus invalid. In line 14, Martin repeats this invalid explanation (P_{inv}), and he argues that the students must always remember to convert a mixed number into an improper fraction before multiplying. The mathematical legitimating criteria used by Martin are both local (L, lines 7, 10 and 14) and general (level 2). Full generalization (GF) was visible in other parts of the lesson, but the extract presented in this paper only includes partial generalization (GP). In line 8, his partial generalization is incorrect

(GP_{inv}) when he contends that the students must always remember to convert the whole number into an improper fraction before multiplying. We have introduced the use of subscript in our codes (e.g., GP_{inv}) to indicate incorrect or invalid responses. The MDI framework does not include this option.

Conclusion

The MDI framework was developed in a South African context, but Adler and Ronda (2015) called for further explorations of its use outside this context. We have previously argued that the MDI framework may be useful for supporting the development of ambitious mathematics teaching in the Norwegian teacher education context—possibly as a tool for mentoring student teachers (Fauskanger & Mosvold, 2017). In the present paper, we have investigated opportunities and challenges of using the framework to analyze data in the Norwegian teacher education context. Overall, we found the application of the MDI framework to be successful. Most of the codes seem to work well across contexts, and we suggest that the framework might be useful for highlighting the complexities of mathematics teaching also in a Norwegian teacher education context. The coding allows for evaluating the level of the mathematical content made available for students to learn, and this is also a benefit of the framework. Our analysis also indicates some challenges of using the MDI framework for data analysis, however, and we highlight two of these in the following: 1) lack of definition and difficulty in making distinctions between subcategories, and 2) dealing with invalid responses.

First, the MDI framework does not clearly define what should count as formal mathematical language, and this makes it difficult to distinguish between subcategories—in particular related to naming. The question of what should count as formal mathematical language is complex, but it should be faced when the degree of formal mathematical language is used for the coding of levels. Second, the MDI framework does not involve a way of dealing with invalid responses in teachers' classroom discourse. For instance, Martin's legitimating criteria involved both local and partial generalization, which indicates level 2. However, his partial generalization was incorrect, and we suggest that the framework should include a way of dealing with this in the coding.

The results from our investigations of using the MDI framework in analysis of data from a Norwegian teacher education context might provide some relevant information regarding future development and use of the MDI framework. Some of the observed challenges may be local, and we are aware of the limitations of the study and the example presented here, but we suggest that the observations made in the context of this study are relevant also beyond the Norwegian teacher education context. In their initial presentation of the framework, Adler and Ronda (2015) indicated that it may be difficult to distinguish between some of the categories in the MDI framework, and our experiences confirm these challenges in a different context and provide some possible explanations and implications for extending the framework to avoid such challenges.

Finally, it might be argued that the episode analyzed in this paper represents a kind of teaching where the MDI framework is less relevant to use. From our experience, invalid responses and misleading statements frequently occur in the mathematical discourse of pre-service, and we have reason to believe that this is not only so in the Norwegian teacher education context. The MDI framework has a potential to evaluate the mathematical content made available for students to learn and highlight areas of problems and possibilities for improvement (e.g., Fauskanger & Mosvold, 2017), and we therefore suggest that it is relevant to use also for analysis of classroom discourse that is problematic—like the empirical example that we discuss in this paper.

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Assessing prospective teachers' development of MKT through their teacher education: a Malawian case

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This paper reports on a study that measured development of Mathematical Knowledge for Teaching (MKT) in preservice teachers by using a pre-post design before and after a mathematics course in initial teacher education. The sample comprised all pre-service teachers from 8 teacher education colleges in Malawi and were tested using adapted measures from the Learning Mathematics for Teaching project in the United States. A paired sample t-test, using $N = 1{,}223$ of the pre-service teachers' pre-test ($M = -0.069$, $SD = 0.950$) and post-test ($M = 0.070$, $SD = 1.044$) MKT scores, showed a significant improvement in the pre-service teachers MKT ($t(1{,}222) = -4.476$, $p < 0.001$). There is also a significant correlation between pre- and post-test scores ($r = .419$, $p < .001$).

Introduction

In Malawi, students' achievement in Mathematics is low at both primary and secondary school levels and has been an issue of concern for the past decade (Kazima & Mussa, 2011). The low achievement has been demonstrated in national examinations and in the Southern and Eastern African Consortium for Monitoring Educational Quality (SACMEQ) tests for grade six. For example, in the SACMEQ III tests, Malawi was one of the two lowest performing countries (SACMEQ, 2010). Looking at the details of the performance by Malawi grade six students, it was found that more than 90% of the students were operating at basic numeracy level, which is three grade levels below their expected level of achievement. Many factors contribute to Malawi's low achievement including large class sizes, limited teaching and learning resources and quality of teachers (Kazima, 2014). Although these factors are connected and need to be addressed together in order to improve the quality of education in Malawi, the quality of teachers is most important because a well qualified teacher will be able to cope and teach better within the limited circumstance of Malawi context than unqualified teacher (Kazima, 2014). Furthermore, the other factors also affect countries like Lesotho and Zimbabwe that scored higher than Malawi on the SACMEQ III test, hence teacher quality seems to be the main factor for Malawi's low achievement (Kazima, 2014; SACMEQ, 2010).

The quality of teachers has been a challenge for Malawi since the introduction of free primary education in 1994 as this resulted in more than 65% increase in enrolment in primary school within one year; from 1.9 million in 1994 to 2.9 million in 1995 (Kazima, 2014). The enrolment has continued to increase over the years such that it rose to 3.6 million in 2008 and currently is estimated at 4.5 million (Ministry of Education, Science and Technology, 2016). Consequently, there was a huge shortage of qualified teachers resulting in the employment of many unqualified teachers. Furthermore, the Malawi initial primary teacher education (IPTE) program was changed in both duration and structure, reducing from three to two years and with the first year full-time at the teacher education colleges and the second year full-time at a school completing teaching practice. There are eight public teacher education colleges in Malawi, and they are all boarding schools.

We are interested in this IPTE program and how well it prepares teachers to teach mathematics in primary schools in Malawi. Our interest is informed by research that long established that teachers' knowledge is crucial in teaching (Shulman, 1986; Ball, Hill & Bass, 2005) and that students' achievement is positively correlated to teacher's mathematical knowledge for teaching (MKT) (Hill, Rowan, & Ball, 2005). Therefore, in this study, we investigated whether or not the Malawian IPTE program develop pre-service teachers' MKT through the first year of mathematics course in the program.

Background and related literature

It is now well known that teachers need various forms of knowledge for teaching effectively. Since the seminal work of Shulman and his introduction of the term pedagogical content knowledge (Shulman 1986), many researchers all over the world have studied teaching in an attempt to understand what this knowledge entails. The earlier works of Ball and colleagues at the University of Michigan in the United States (e.g., Ball, Hill & Bass, 2005) and of Adler's QUANTUM project at University of the Witwatersrand in South Africa (e.g., Adler, 2005) focused on examining what this knowledge is in Mathematics. They described in detail Mathematical Knowledge for Teaching (MKT), what it is, why it is important, and what it looks like in specific mathematics concepts such as fractions, multiplication, probability and functions (e.g., Kazima, Pillay, & Adler, 2008). What was not clear at the time and what is still debated, is how teachers would acquire such knowledge. Ball, Thames and Phelps (2008) further categorized Shulman's subject matter knowledge into three; Common Content Knowledge (CCK), Specialized Content Knowledge (SCK) and Horizon Content Knowledge (HCK). This study focuses on only CCK and SCK. CCK is described as mathematical knowledge that is used in teaching, for example how to perform some algorithms, which is also common to other professions that use mathematics.

While SCK is described as the mathematical knowledge which is unique to the work of teaching and not needed in other professions. An example of SCK is knowing why one can 'invert and multiply' when dividing by fractions. A teacher should be able to explain this and demonstrate with examples to students. As Ball et al. (2008) demonstrated, CCK and SCK are two of the six sub domains of MKT.

The work of Ball and colleagues extended to the Learning Mathematics for Teaching (LMT) project where they developed measures of MKT—called the LMT-measures. These measures have been adapted and used in other countries and in different contexts, for example, Norway in Europe (Fauskanger, Jakobsen, Mosvold, & Bjuland, 2012), Indonesia in Asia (Ng, 2012) and Ghana in Africa (Cole, 2012). The LMT measures have been found to be useful in exploring teachers' mathematical knowledge and mathematical reasoning in teaching scenarios of specific concepts (Adler & Patahuddin, 2012). We adapted some of the LMT measures for use in the Malawi context to measure the development of some aspects of MKT through the first year of the pre-service teachers' IPTE program.

Later and recent work of Adler and the Wits Maths Connect project has progressed from describing MKT to using the knowledge in teaching (Adler & Ronda, 2017). They offer a conceptualization of how MKT can be used in teacher education, especially through professional development of mathematics teachers. They suggest a Mathematical Discourse in Instruction (MDI) framework that shows how teacher knowledge can be used in planning and implementing mathematics lessons (Adler & Ronda, 2017). Thus, they are making clearer how the knowledge for teaching can be acquired by teachers, which was not clear before. This is important to us because our overall aim is to improve quality of teacher education in Malawi, and we can do that if we find ways of developing the student teachers' MKT through initial teacher education. Furthermore, we learn from our own previous work (Jakobsen & Kazima, 2015) and that of other researchers that mathematics teacher education that is centered around MKT can be effective (e.g. Adler & Patahuddin, 2012). As research has shown, students taught by teachers with a high MKT score did better on tests compared to students taught by teachers with lower MKT score (Hill, Rowan, & Ball, 2005; Kane, McCaffrey, Miller, & Staiger, 2013), even when the teachers in proceeding years were randomly assigned students and the students were retested (Kane, McCaffrey, Miller, & Staiger, 2013). Hence, this motived us to measure student teachers' MKT in Malawi before and after the mathematics teacher education course to see the if it had an impact on the development of their MKT.

Design and Methodology

We applied a pre- and post-test design to measure teachers' mathematical knowledge for teaching. The measures used for this purpose were adapted LMT measures with number concepts and operations items. Number concepts and operations was chosen because it is one of the main focuses of the Malawi IPTE program as defined in the Malawi national curriculum for IPTE. Adaptation of the measures was completed in three stages. The first stage was to select the most appropriate itmes for our pupose from the available LMT instruments. This included aligning items in each available LTM form to Malawi's mathematics curriculum for IPTE. We found that the LMT form A from the 2001 instrument (NCOP-CK_2001A) had the closest and the most extensive range of items that covered the curriculum (Kazima, Jakobsen, & Kasoka, 2016). For this reason we selected this form and the corresponding form B (NCOP-CK_2001B) as a starting point for adaption. We selected a total of 88 items in two parallel forms A and B. The second stage was adapting the items to the Malawi context, for example, by changing some words and names of people to suit Malawi context. The third stage involved checking the mathematical content of the items and modifying where necessary to reflect the Malawi curriculum. This involved what Delaney, Ball, Hill, Schilling and Zopf (2008) call "changes related to school cultural context" and "changes related to mathematical substance". Thereafter, the two forms were piloted on 351 pre-service primary school teachers from one of the teacher training colleges in Malawi. The forms were distributed on papers. After the pilot further modifications were done following the pilot findings. In particular, we analysed difficulty level of each item and removed all items that were not around the mean ability of zero (for more details of adaptation process, see Kazima, Jakobsen, & Kasoka, 2016). The final from had a total of 67 items in two parallel forms; Form A and Form B. Form A contained 38 items and Form B 35 items, of which 6 were anchoring items. To minimize the test-retest effect, we used the split-half method and randomly gave half of the sample Form A for the pre-test, and the other half Form B. In the post-test, this was swapped (Cohen, Manion, & Morrison, 2007).

Sample

All pre-service teachers enrolled at one of the eight public teacher colleges in Malawi and not released from teaching on the test day (all colleges are boarding schools, but sick student are released from teaching) in September in 2015 constituted the sample (N = 1733). They were all students in IPTE. During the first term running from September to December of the academic year, their curriculum for this first term covered number concepts and operations, and we expected that the pre-service teachers had been introduced to tasks of teaching like 'how to teach' number concepts and operations during the first term. During term two, the pre-service teachers are mainly taught basic application of what is covered in term one. In addition, they are introduced to shapes and some basic financial mathematics.

In total 1,733 students participated at pre- and post-test. However due to some unforeseen logistical incidents at the colleges during the pre-test, we have had problems to pair pre-test and post-test scores for all participants. For this study, only 1,223 pre-service teacher students out of the 1,733 participants are paired. This is partly due to the unforeseen incidents at pre-test where some students did not provide enough identification information on the paper test. It was also partly due to the fact that some participants dropped out of college before post-test (e.g. some students were accepted at other study programs early in the term), while other students enrolled at the college after pre-test and we only had post-test data available for them. Table 1 shows the number of pre-service teachers we have been able to pair so far. We labelled the colleges as C1 to C8.

Age/Gender	**Teacher college**								**Total**
	C1	C2	C3	C4	C5	C6	C7	C8	
<21 years	68	103	56	44	42	74	55	38	480
21 - 25	97	136	84	27	63	108	44	21	580
26 - 30	20	26	26	4	13	17	6	4	116
>30 years	9	12	1	3	6	7	6	3	47
Female	102	147	36	78	37	105	48	30	583
Male	91	126	130	0	83	101	61	36	628
Gender not provided	1	4	1	0	4	0	2	0	12

Table 1: Number of paired pre-service teachers by age and college.

Data collection and scoring

The pre-test took place in the third week of term one, after the newly enrolled pre-service student teachers had just started their teacher training (September 28–October 2, 2015). In order to avoid that the pre-test directly affected the post-test, we placed the post-test the second week of term three (May 16–20, 2016). The time between the two tests also assured us that content would have been taught at all colleges, as all the colleges use the same books and curriculum material. However, we were aware that occasionally content planned for term one can end up being taught in term two due to unforeseen circumstances and delays in the colleges.

The test was distributed on paper by the researches, and it was not possible for us to test all students on the same day because of logistical reasons. The eight colleges are located across the country such that the two farthest are more than 900km apart. We therefore administered the tests to the colleges on different days but within the same week. For both pre- and post-test, each of the two forms were administered in four colleges but different colleges each time to ensure that those that wrote Form A for pre-test should write Form B for post-test, and vice versa.. The test took place during class time and without any incentive. We started by informing participants about the objective of the study, and that it was voluntary to attend, but none of theo participants withdrew.

Due to the relatively large sample, we used a two-parameter Item Response Theory (IRT) (Edwards, 2009) model to estimate the pre-service teachers MKT score—often called ability—and we used the software BILOG-MG for this estimation. An IRT scales a person's MKT (ability) and item difficulty on the same continuum. This means that a pre-service teacher with the MKT (ability) of Θ has a 50% chance to answer an item with the difficulty Θ (Edwards, 2009). MKT (ability) and parameters are scaled so that the average Θ for the whole sample is 0 and the population standard deviation is 1. The MKT scores were then entered into IBM-SPSS for analysis.

Results and discussion

Before comparing the pre-service teachers' MKT score at pre-test and post-test we tested the data for normality. Q-Q plots, estimated skewness and kurtosis confirmed the assumption of normality needed for conducting a paired sample *t*-test. We also found that pre- and post-test MKT scores were significantly correlated ($r = .419$, $p < .001$), hence pre-service teachers who scored high at pre-test were more likely to have a higher mathematical knowledge for teaching at post-test.

We then compared the pre-service teachers MKT scores from pre-test ($M = -0.069$, $SD = 0.950$) and post-test ($M = 0.070$, $SD = 1.044$) using a paired sample *t*-test. We found that the score at post-test were significantly higher than the score at pre-test ($t(1222) = -4.476$, $p < .001$).

At a first glance, this is of course promising. A significant increase of pre-service teachers' mathematical knowledge for teaching number concepts and operations during their first year in the IPTE program is important, as knowledge about number concepts and operations is critical to quality teaching of primary mathematics (Hill et al., 2005). It is critical that pre-service teachers understand and competently use basic number concepts and operations properties for them to effectively teach mathematics. The items we used in this study were all selected with the specific purpose to address aspects of number concepts and operations that can be considered prerequisite for the learning of school mathematics beyond mathematical literacy level (OECD, 2003). Both forms contained items that examined pre-service teachers' knowledge of whole number operations, subtraction of integers, representation and operations of fractions, decimal representations, prime numbers, and the order of operations.

As the results of data analysis show, when we consider the mean MKT score for the whole group of preservice teachers from the eight Malawian teacher colleges, the group's knowledge for teaching these essential aspects of mathematics improved from the beginning of term one (pre-test, week three in term one), to the beginning of term three (post-test, week two of term three). The increase is not big—M = -0.069, SD = 0.950 at pre-test and M = 0.070, SD = 1.044 at post-test—but it gives confidence to be able to confirm that the IPTE have had an impact on the pre-service students. In particular, since all the prospective teachers' colleges are boarding schools and all the prospective teachers had spent all their time at the college between pre- and post-test and were not exposed to other programs, it is likely to assume that the mathematics course was the main cause for the change in MKT that we measured.

For further research, we propose to use an ANOVA test to investigate if there are differences in development among the eight teacher colleges (Pallant, 2010). An earlier study with less participants (N = 725) indicated no significant improvement among majority of the teacher colleges (Kasoka, Jakobsen, Kazima, 2017), hence it is interesting to see if we can identify colleges who are able to improve MKT the most during their IPTE.

In conclusion, we found that after spending a year at a teacher training college and completing a mathematics course involving number concepts and operations, the prospective teachers MKT related to these concepts had significantly increased. It is likely that the change was caused by the mathematics course that all the prospective teachers were exposed to.

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Towards an organizing frame for mapping teachers' learning in professional development

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In their claim that teachers' learning is treated as a "black box" in research on professional development programs for mathematics teachers, Goldsmith, Doerr, & Lewis (2014) call for "an organizing framework that clearly distinguishes dimensions of teachers' learning and identifies catalysts of teachers' learning…" (p. 23). The aim of this study is to present initial efforts to construct a framework for categorizing descriptions of activities designed to support teachers' learning as presented in research articles within mathematics education. Based on existing literature on professional development and examination of research articles, an organizing frame is constructed.

Introduction

The field of mathematics education needs to "build a shared body of knowledge about the nature of teachers' learning and the catalysts that support it" (Goldsmith, Doerr, & Lewis, 2014, p. 25). In order to build such a shared body of knowledge, Goldsmith et al. call for "an organizing framework that clearly distinguishes dimensions of teachers' learning and identifies catalysts of teachers' learning…" (p. 23). Further, they argue that research on professional development programs (hereafter PDPs) for mathematics teachers mostly concern whether a program has affected practice or student learning, while the teachers' learning is treated as a "black box". That is, research on PDPs often lacks in explicit descriptions of, for instance, what forms of activities (e.g. reading, discussing, acting in the classroom, cooperating etc.) that promote learning and conceptualizations of teachers' learning processes.

Models for teachers' learning (e.g. Clarke & Hollingsworth, 2002; Kazemi & Franke, 2004: Vermunt & Endedijik, 2011) often center on certain *activities* that are intended to act as catalyst for teacher learning in PDP. Such activities are often described in terms of *what* to do and *how* to do it (Goldsmith et al., 2014), but not in terms of a rationale for, or description of, *how* these activities are to function as instigates for teachers' learning (Vermunt & Endedijik, 2011; Robutti et al., 2016).

In this paper, we aim to contribute to this complex research field of mathematics teachers' learning by presenting initial efforts to develop an organizing frame for categorizing PD activities as presented in research articles and discussing how they support teachers' learning. The rationale for this aim is to help "researchers share findings in a way that leads to greater accumulation of knowledge" (Goldsmith et al., 2014, pp. 23-24). Methodologically we 1) use and build upon prior models of teachers learning from PDPs in order to establish a "pilot framework" with categories for the mapping and 2) develop the pilot framework iteratively by applying it to a number of journal articles.

Constructing the frame

To build the organizing frame, we; 1) draw upon literature on different models of mathematics teachers' learning and PDPs and 2) conduct an analysis of a number of empirical articles of PDPs. We use the models of teachers' learning and PDPs as a starting point in the construction of an initial frame. Then we analyse journal articles that reports on PDPs as a means to test and iteratively evolve the set of categories from the initial organizing frame. That is, the development of the frame comprised two steps: 1) reading, choosing, and relating already established theories/models of teachers' learning in PDPs in order to establish an initial frame; and 2) iteratively developing the initial frame by applying it to a number of journal articles reporting on empirical studies of PDPs in mathematics education.

Process

The construction of the frame is based on the assumption that activities within PDPs are the catalyst for teachers' learning (Clarke & Hollingsworth, 2002; Vermunt & Endedijik, 2011). The activity is then understood in terms of the underlying *theoretical perspective of learning*, what the activity primarily aims to accomplish (its *function*), the *type of teacher knowledge* it aims to develop, and the *forms of the activity.*

In aiming at understanding PD in terms of views on how teachers develop knowledge, the theoretical perspective on learning is mapped according to the notions of Kazemi and Franke (2004). Kazemi and Franke (2004) stress that how the activity is intended to be understood as a catalyst for learning depends on what theoretical perspective one takes. We therefore start the analysis of articles by mapping this perspective.

The next step in the analysis is to study the main aim of the PDP – its function. The PDP's function and activities are analyzed using the work by Desimone (2009). That is, the question we ask is: Does the PDP aim primarily at developing teachers' knowledge and/or beliefs, improving instruction, or developing students' knowledge of mathematics? Through this, we have been able to capture whether teachers' learning is the main aim of the PD or if it serves as a means to accomplish other aims.

Whatever the main aim of the PDP, teachers' development of knowledge typically constitutes an important component of it. We aim to characterize the activities in the PDP in terms of the *type of teacher knowledge* the teachers are expected to develop. We focus on both *what* the teachers are expected to know and *how* they are expected to know this content. To capture what teachers are expected to know, we used Ball, Thames and Phelps (2008) as a starting point. To capture how they are expected to know the content, we used the framework of proficiency of mathematics teacher knowledge by Kilpatrick et al. (2001).

Finally, the form of the activity is categorized using Kwakman (2004), Kennedy (2016) and Desimone (2009). These frameworks enable us to capture how individual teachers learn through, for instance, reading, testing or reflecting, whether they develop as teachers by following prescription, practicing strategies or building a more solid knowledge base, as well as how PDP settings can be structured through, for instance, emphasis on collective participation, duration or active learning.

Thus, the process for critiquing and formulating which categories should compose the organizing frame was to: 1) depart from models of teachers' learning that suggest that the *activity* is the main catalyst for learning; 2) state and depart from the view that activities can be productively understood in terms of the underlying *theoretical perspective of learning*, what the activity primarily aims to accomplish (its *function*), the *type of teacher knowledge* the activity aims to develop, and the *forms of activity*; and 3) use research reports on PDPs for in-service mathematics teachers to iteratively determine categories that are suitable/interesting to map.

Models of professional development and teachers' learning

Learning is a concept that is difficult to define and/or describe coherently and comprehensively, but common to all theories of learning is that some *change* occurs with regard to the learner. Here we adopt the elementary definition of teachers' learning by Goldsmith et al. (2014) as "… include changes in knowledge, changes in practice, and changes in dispositions or beliefs that could plausibly influence knowledge or practice" (p. 7). To understand how research articles in mathematics understand these changes, we aim to categorize the activities and their theoretical base, function, type of teacher knowledge in focus, and how learning is intended to take place, according to the literature below.

Focusing on activities

A substantiated assumption in the model by Clarke and Hollingsworth (2002) is that, to promote change/growth, reflective participation in *activities* is preferred over the passive reception of knowledge. Similarly, Vermunt and Endedijik (2011) suggest a model for teachers' learning patterns in which an intertwining of *activities* is intended to catalyze learning. Kazemi and Franke (2004) also hold the *activity* as the primary unit of analysis, and in their model an individual is seen to develop through participation in interpersonal and cultural-historical activities. Thus, Clarke and Hollingsworth (2002), Vermunt and Endedijik (2011), and

Kazemi and Franke (2004) all emphasize the activity as the main catalyst for learning.

Theoretical perspective on learning

Kazemi and Franke (2004) emphasize that how an activity is to be understood as a catalyst for learning depends on which theoretical perspective one takes. It is well known that learning is a concept and phenomenon that is hard to define and is possible to see from many viewpoints. In mapping articles, we search for assumptions about learning by using a set of perspective on learning as our basis. The set includes perspectives such as constructivism, sociocultural theories, social practice theory, adult learning, cognitive load theory, etc. We find support in using this set of theories as a starting point, but do not restrict the analysis to only these perspectives.

Function of the activity

For studying the effects of a PDP – what has changed in its wake – Desimone (2009) suggests a model: a) increased teacher knowledge and skills and/or change in attitudes and beliefs; b) teacher change in instruction; and c) improved student learning. The suggested categories of the effectiveness of PDPs are useful in mapping the main function, aim or goal of the activities within the programs. In particular, this helps us determine whether teachers' learning is the means or the end of a PDP.

Types of teacher knowledge

Ball et al. (2008) summarize their view of mathematics teachers' knowledge in two domains: *subject matter knowledge* (SMK) and *pedagogical content knowledge* (PCK). The SMK domain is categorized into the sub-domains of *common content knowledge, specialized content knowledge* and *horizon content knowledge*, while the PCK domain is categorized into the sub-domains of *knowledge of content and students*, *knowledge of content and teaching*, and *knowledge of content and curriculum*. Even though Ball et al. (2008) proclaim that their current categorization is not a final product, it is useful in mapping the teacher knowledge a PDP emphasizes as the subject the teachers are to learn.

We are interested in understanding not only what the teachers are expected to know, but also the ways in which they are expected to know this content. We find Kilpatrick et al. (2001) useful here and adopt their five categories for proficiency in teaching mathematics: *conceptual understanding of core knowledge, instructional routine fluency, strategic competence, adaptive reasoning*, and *productive disposition*. Together, these categories constitute a basis for understanding issues that can be summarized as *mathematics teacher knowledge*, in the sense of *how* to know a certain content.

We find a combination of categories from Ball et al. (2008) and Kilpatrick et al. (2001) useful to map in order to understand *what* teachers are expected to know and *how* they are expected to know it.

Forms of activities

The form of a professional learning activity, according to Kwakman (2003), can be mapped into four categories: a) *reading* in order to collect new personal input (data, knowledge, information); b) *doing/experimenting* to gain new experiences and apply new ideas; c) *reflection* in order to recognize and change routine behaviour; and d) *collaboration*, which provides teachers with new ideas and feedback.

Desimone (2009) presents a conceptualization of features for studying effects on teachers' PDPs. Our proposed frame categorizes the core features of PDPs as: a) *content focus*, as empirical evidence points at a "link between activities that focus on subject matter content and how students learn that content with increases in teacher knowledge and skills…" (p. 184); b) *active learning*, which simply states that activities in which the teacher is active are more effective than passive activities such as lectures; c) *coherence*, which indicates whether the learning content is "consistent with teachers' knowledge and beliefs ... and state reforms and policies …" (p. 184); d) *duration*, in which both the number of hours and the spread of time count; and e) *collective participation*, which emphasizes arrangements for potential interaction between participants, based on the idea that collaboration promotes learning.

In her review on PDPs, Kennedy (2016) suggests four types of enactment that a program may facilitate, according to the extent to which the outcome gives the teacher independence in how to act upon the content of the program. A program can be *prescriptive*, with explicitly described actions the teacher should take, like following a recipe. If the "recipe" is followed by the described rationale for the suggested action this is called *strategy*, and the teacher is expected not only to act, but also to understand the strategy behind the action. When a PDP gives the teacher an "aha moment" this is called *insight*, whereby there is no suggested action to take as compared to the two previous categories. The fourth category is called *body of knowledge*; here, no particular action is suggested, and the knowledge may be communicated as a lecture or a book. This fourth category is suggested to give the teacher the most freedom and independence in how to act upon the knowledge.

Developing our initial framework by using empirical studies

To test and iteratively develop the initial organizing frame developed from the first step of this process, we applied it to journal articles in mathematics education reporting on PD initiatives. We randomly (no conceived selection was made) picked some journal articles from the literature review by Goldsmith et al. (2014). We read one article, noted plausible categories, read another article and added and deleted categories to adjust and amend the initial frame derived from the first step of the methodology. This process was repeated until no categories were added or deleted from reading a new article. The use of the list of articles from Goldsmith et al. (2014) was both timesaving (compared to searching for papers) and fulfilled the criteria for what kind of articles we were looking for (reports on mathematics teachers' learning from PDPs for practicing in-service teachers, reported in refereed journals only). We found this procedure, and the use of these articles as a means to develop our initial frame into the final organizing frame, fruitful.

Example

An example of the process is provided. The report by Anderson and Hoffmeister (2007) on a PDP for mathematics teachers addressing the procedure for examinations. The initial frame, with categories according to the frameworks of mathematics teachers' learning and PDPs, is used as a starting point. The iterative process of adding comments to improve this initial frame is undertaken. This procedure was repeated with following articles, saturating the number of categories and resulting in the finalized organizing frame.

<u>Theoretical perspective</u>

The theoretical perspective on which the paper was grounded was not explicitly reported; this lack of information was noted.

<u>Function</u>

The PDP was intended to increase teachers' own knowledge by applying the three learning strategies of a problem-solving course, examination of student thinking by interviewing students, and reading and discussing research on learning and teaching mathematics. As this program's approach aims to develop the teachers' own knowledge and does not focus on a change in their instruction or the students, it fits the category *increased knowledge* according to the frame by Desimone (2009). We conclude that the categorization according to this model is functional.

<u>Types of teacher knowledge</u>

The model by Ball et al. (2008) helps categorize the three learning strategies of the PDP according to the content it intends to mediate. The problem-solving course, which is the content of the PDP, is categorized as SMK, covering specialized content knowledge and common content knowledge. The examination of student thinking is categorized as PCK, and the sub-category knowledge of content and students. The discussion of research mostly concerns the content of one book (Ma, 1999, as cited in Anderson & Hoffmeister, 2007) and covers content specific to elementary mathematics teachers, which fits SMK with specialized content knowledge.

The three strategies in the program aim to prepare teachers for understanding certain content, how students understand this content, and what research says about the content. Thus, the category *conceptual understanding of core knowledge* by Kilpatrick et al. (2001) offers a satisfying explanation of *how* the content of the three strategies of the program is expected to be known by the teachers. We conclude that the model suggested by Kilpatrick et al. (2001) enables us to focus on *how* the teachers are expected to know the content.

Form

The three learning strategies of the PDP are performed by *reading* (Ma,1999), *doing* (studying problem-solving) and *reflecting* (discussing Ma (1999) and examining student thinking). We regard these categories, suggested by Kwakman (2004), as satisfactorily capturing the approach of how to facilitate learning activities.

Regarding the settings of the PDP, the model by Desimone (2009) works well as a template for categorization. The described PDP lasted one school year and one summer course (duration) and was in line with other activities and the school's curriculum (coherence); also, the teachers actively participated in the activities (active learning) and worked together (collective participation). Besides these categories, the examination of this article suggests the categories *number of participants, material* and *facilities* to achieve a more complete picture of the settings that may affect the learning situation for the participants. The number of participants indicates possibilities for cooperation and collective participation. Different types of material (in the example article; a book to read and discuss) may be vital for completing the program. The location in which the PDP is held may influence conditions for performing the activeties. A familiar place may facilitate finding materials and/or space for discussions and a new milieu may be perceived as uplifting. Both the alternatives could affect the outcome of the program. Thus, we regard the categories *number of participants*, *material* and *facilities* as completing the categories of the model by Desimone (2009) in order to map the settings for the PD program.

Applying Kennedy's (2016) four categories to the example article suggests that teachers should develop some conceptual knowledge of problem-solving and student thinking, but not truly focus on how to act in the classroom through engaging in strategies for teaching or following prescriptions for how to carry out teaching. Thus, the learning strategies *problem-solving* and *examining student thinking* are regarded as *body of knowledge*. Discussing Ma (1999) provides the participants with insights on learning and teaching early mathematics. This might fit the *body of knowledge* category in one sense, but as it is described in the article it better fits the *insight* category. However, the categories suggested by Kennedy (2016) provide us with information about the extent to which the outcome allows the teacher to act independently upon the content of the program.

To sum up, we created a *pilot framework* based on theoretical models for teachers learning and effective PDPs. This pilot framework was then applied to some articles in an iterative procedure until a saturation in establishing categories was reached.

The resulting frame

From the iterative procedure of analyzing the articles, a resulting organizing frame for mapping activities, intended to catalyze mathematics teachers' learning, was generated (see Figure 1).

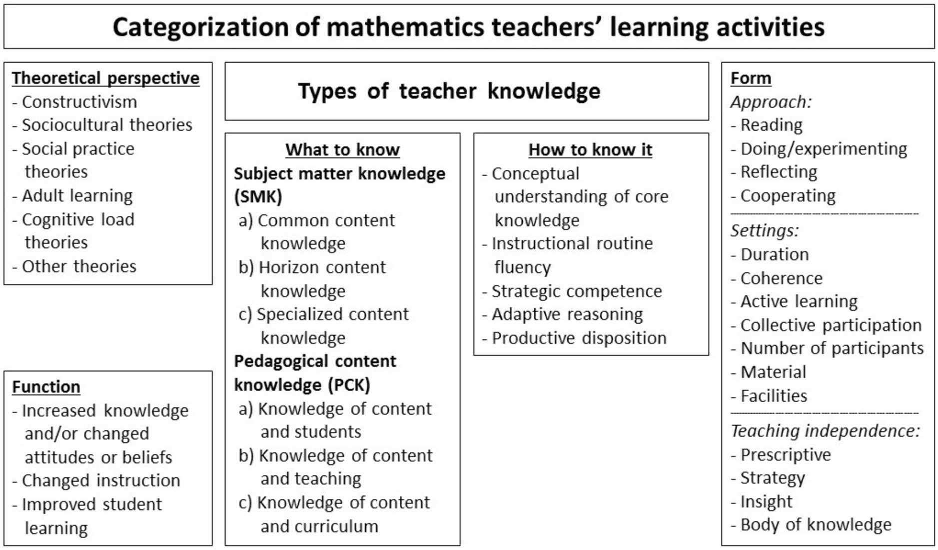

Figure 1: An organizing frame for mapping activities and settings intended to catalyze mathematics teachers' learning

This final frame contains what we find most useful to map in order to make way for understanding teachers learning in PDPs; 1) The theoretical perspective the PDP is based on, 2) What main function the PDP has, 3) What type of knowledge the PDP emphasize and 4) In what form the PDP is launched.

Concluding remarks

In this paper we have presented our initial efforts to construct an organizing frame aimed at capturing different aspects of teachers' learning in PDPs. Just as students' learning and reasoning are central in mathematical classrooms, teachers' learning, and reasoning should be central in PDPs. Nevertheless, Goldsmith et al. (2014) claim that teachers' learning is often treated as a "black box". Therefore, there is good reason to continue the work to better understand this topic. Further, in order to "… share findings in a way that leads to greater accumulation of knowledge" (Goldsmith et al., 2014, pp. 23-24), we encourage to utilize and test the suggested frame. Even if we conjecture that the suggested organizing frame will function as a tool for mapping mathematics teachers' learning, we consider future use of it as needed to establish or develop it further.

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Good mathematics teaching as constructed in Norwegian teachers' discourses

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This paper explores the notion of good mathematics teaching as constructed in the discourses of practicing Norwegian mathematics teachers. Analyses of data from group interviews show that the teachers tend to conceptualize good mathematics teaching in terms of structuring lessons, differentiating in accordance with individual students' different needs, mathematical communication between teacher and students, as well as teachers' use of tasks and resources. In addition to this, the teachers emphasize student engagement and students' learning when discussing good mathematics teaching. Possible implications for these findings are discussed.

Introduction

A continually growing body of research investigates what constitutes good mathematics teaching (Franke, Kazemi, & Battey, 2007). To better support teachers in learning to carry out ambitious teaching practices fundamental for supporting children's learning of mathematics (e.g., Lampert, Beasley, Ghousseini, Kazemi, & Franke, 2010), the focus of in-service teacher education has recently shifted from developing pre-service teachers' knowledge toward developing teaching practices (Zeichner, 2012). Although the question of what constitutes good mathematics teaching practices has been examined in numerous studies, the question of how to define good mathematics teaching continues to remain unresolved (Cai, Kaiser, Perry, & Wong, 2009; Franke et al., 2007; Krainer, 2005; Li, 2011). In their overview of research on mathematics teaching and classroom practices, Franke et al. (2007) highlight creating mathematical classroom discourse, developing norms and building relationships that support mathematical learning as three core features of good mathematics teaching, but no universal definition has been developed to this date. Attempts to define good mathematics teaching seem to depend on the views of mathematics teacher educators and mathematics teachers (Cai et al., 2009; Li, 2011). Since attempts to define good mathematics teaching can be regarded as a process of establishing norms, the views of teachers and teacher educators are arguably important

(Krainer, 2005). The views about good teaching also influence teachers' decision-making (Krainer, 2005), and they might thus influence the development and adaptation of common ideas and recommendations across countries as well as sharing of visions of effective classroom practice (Givvin, Jacobs, Hollingsworth, & Hiebert, 2009).

There are many ways to denote the way teachers talk about good mathematics teaching, but a common approach is to study teachers' views about good mathematics teaching through analysis of their discourses (Franke et al., 2007; Hemmi & Ryve, 2015). In the present study, we analyze data from Norwegian teachers' group discussions with a focus on how they construct the notion of "good mathematics teaching" in their discourses. Our approach to analyze data draws upon the study by Hemmi and Ryve (2015) of how Finnish and Swedish teacher educators conceptualize effective mathematics teaching. Where these researchers focused on teacher educators, we focus on practicing teachers. We address the following research question: What aspects of good mathematics teaching constitute a group of Norwegian mathematics teachers' discourse? To answer this question, we analyze the discussions given by 20 Norwegian teachers in focus-group interviews at the end of the first day in a professional development project. To our knowledge, few studies have examined Norwegian teachers' construction of good mathematics teaching from studying their discourses in focused discussions.

Methodology

The study presented in this paper is part of a larger project called "Mastering Ambitious Mathematics Teaching" (MAM). In this project a model for school-based professional development of in-service mathematics teachers have been developed along with resources for teachers. The model as well as the resources was originally developed to be used in pre-service teacher education. The model has repeated enactment of specifically designed instructional activities to be used in the teachers' instruction as a point of departure, and all the activities focus on numbers and operations. The activities are developed to learn in, from, and for teaching practice (see e.g., Kazemi & Wæge, 2015; Lampert et al., 2010; Valenta & Wæge, 2017).

Twenty teachers participated in the part of the MAM project that is presented in this paper. Our focus is not on investigating differences among teachers with different background, but we provide some background information to inform the readers. The participating teachers work at 10 different schools, and they teach fifth, sixth or seventh grade. Their age range vary from 23 to 59 years, their teaching experience vary from one to 30 years, and their formal education in mathematics/mathematics education vary between 15 ECTS and 120 ECTS (i.e. master's degree). The overall design and selected population makes it possible to draw conclusions in relation to these teachers only and we cannot make any claims about the Norwegian teacher population in general. The participating teachers are volunteer participants in the MAM project, which might imply that they are more concerned about mathematics teaching than many other teachers might be. In our efforts to learn more about how good mathematics teaching is constructed in these teachers' discourses, we arranged three focus-group interviews. The interviews had six or seven participants and lasted from 44 to 51 minutes. After some introductory questions, the following main questions served as point of departure for the discussions: 1) How would you characterize a good mathematics lesson? and 2) How would you characterize what for you is a "normal" mathematics lesson?

For the purpose of this paper, the focus-group discussion related to these questions were analyzed by using content analysis (Hsieh & Shannon, 2005). Two researchers (authors 1 and 3 of this paper) coded all the data material independently. Both researchers developed individual codes and grouped them into categories, in an iterative process including several cycles of analysis. The two researchers then reconciled and agreed upon categories and corresponding codes. The codes and categories were shared with two other researchers (authors 2 and 4 of this paper) who coded the data material using these codes and categories to validate the coding. Some minor adjustments to the codes were made during this process, but the categories listed remained the same:

1. Teacher's instruction/role
2. Structure in lessons
3. Differentiation
4. Communication
5. Use of tasks and resources
6. Student engagement
7. Students' learning

Categories 1–5 refer to the teachers' actions, but categories 3–5 also include the students' actions. Categories 6 and 7 refer to the students only, focusing on their engagement and learning. When seen in relation, the seven categories indicate a shared responsibility for good mathematics teaching by teachers and student. Examples of codes for the category of Teacher's instruction/role (1) are to: a) be a guide, not a lecturer, b) find a way to present the content in engaging way, c) using precise mathematical language, d) work in depth with concepts, e) predict student response, f) find a way to respond to students' thinking, g) build on students' thinking towards the learning goal, h) ask good questions, and i) use resources critically.

As can be seen from the next section, these categories are partly overlapping. An example is the category of Teacher's instruction/role (1) and the category of Communication (4): Parallel to highlighting mathematical communication as central for students' learning (4), the teacher´s role as facilitator of such discussions (1) is emphasized. Although these categories are partly overlapping, we stick to these since they all emerged in the coding process from the content analysis, and they refer to the similar aspects, but in partly different ways (Hsieh & Shannon, 2005). The seven categories illustrate the teachers' own conceptualizations of good mathematics teaching, constructed from the teachers' discourses.

Results

Our analysis reveals that the group of Norwegian teachers tend to conceptualize good mathematics teaching in terms of paying attention to their own roles as facilitators in the classroom, structuring lessons, differentiating in accordance with individual students' various needs, the mathematical communication between teacher and students, as well as teachers' use of tasks and resources. In addition to this, the teachers emphasize student engagement and student learning when discussing good mathematics teaching. In the following, examples from each of these conceptualizations will be presented.

Teacher's instruction/role

The teachers express in the interviews that it is important to present the mathematical content in an engaging way. They want to be facilitators and stimulate for mathematical discussions by responding to students' thinking, build on students' initiatives, and guide them towards the learning goals. For instance, in one of the group discussions, a teacher says that "formative assessment should be a part of our teaching all the time, to stimulate and help them [the students] further." Formative assessment is however, also described as challenging. The teachers stress the importance of working in depth with mathematical concepts like multiplicative structures and emphasize the use of a precise mathematical language in lessons. In one of the interviews, prediction of students' responses is highlighted. In another interview, teachers describe challenges of teaching. One teacher states that it is challenging to pose good questions in the classroom conversation, while another finds it challenging to summarize lessons in a constructive way due to lack of time.

Structure in lessons

The teachers dwell on the importance of having a good structure in mathematics lessons. They suggest that mathematics teachers must have clear content goals for the lessons. These goals should be made explicit in the beginning of the lesson in a way that directs the students' attention towards the content in focus and support their learning of the content, without reducing opportunities for thinking and exploration. They also make a point of varying the lessons, for instance introductions, work stations and discussions. At the end of a lesson, teachers should sum up and make connections to the learning goals. One teacher argues that this is especially important when working with inquiry-based tasks: “One can really ‘stray from the subject’ without a goal in this kind of teaching.”

Differentiation

Another aspect that pointed out by the teachers as important in good mathematics teaching is differentiation. This is exemplified by one of the teachers who states that it is important “to reach all students, find tasks that are suitable for everybody, both those students that strive in mathematics and those who are high-achieving.” The teachers find it important to allow all students to participate, either by using tasks that can be worked on in different ways or by organizing the students in groups where they can work on differentiated tasks. Differences between students are conceived as challenging, but the teachers maintain that differences can also be an asset, since different students’ ways of thinking can come up. In one of the group discussions, a teacher says that, “oftentimes, students are cleverer to explain to each other than I am as a teacher, since I often use a more difficult language in my explanations.” In one of the other group discussions, a teacher gives an example of a high-achieving student who had investigated the commutative law and made “a guest lecture” for her fifth-grade students. This teacher also expresses that other high-achieving students have been investigating “other things”, as she expressed it, and such mathematical inputs are valuable both for these students themselves and for the other students in her class.

Communication

The teachers agree that mathematical discussions are central for students’ learning and therefore important for good mathematics teaching. For instance, one of the teachers contends that, “discussions are important, no matter what type of activity. They are important for students’ learning, and they are important for the teacher to get an impression of students’ understanding.” The teachers stress that communication in a mathematics classroom must be two-ways. Students must participate actively in discussions and explain to each other, and teachers have to elicit and respond to students’ ideas.

Use of tasks and resources

In their conceptualization of good mathematics teaching, the teachers express that it is important to introduce mathematical tasks that are motivating for their students. One teacher recalls an example of a task that was motivating for his students: finding patterns to come up with a recursive formula. In addition to being motivating, the teachers suggest that tasks should be open and stimulate different approaches to reaching a solution or stimulate to find different solutions. Another teacher tells that she could present a task for her students and say, “help me to solve it!” Other teachers suggest that a good mathematical task is open for differentiation. Different uses of games with cards, dices and computers are mentioned as teaching resources in the interviews.

Student engagement

"Good mathematics teaching can be recognized when all students say 'No!' when you tell them that the lesson is finished", a teacher suggests in the group discussion. Student engagement is presented as an important characteristic of good mathematics teaching by the teachers. They describe student engagement as active participation, eagerness to solve a given problem, listening and trying to understand. The teachers stress that hard work and effort, followed by gradual mastery, is decisive for student engagement and for the quality of mathematics teaching.

Students' learning

The teachers characterize good mathematics teaching as teaching that provides students with opportunities to think, be creative, discover, use their knowledge in new problems, and develop understanding. One of the teachers declares that, "it is great to see students using strategies we have been working on before in new situations. That is a good mathematics lesson." For student learning, the teachers emphasize concentrated work on problems, explaining to others and listening to other students' explanations. Finally, the teachers suggest that good mathematics teaching supports students' learning of a way to work in mathematics, use of mathematical terminology and knowing certain facts by heart.

Discussion

Several recent studies investigate teachers' discourse of good mathematics teaching in different contexts (e.g., Krainer, 2005; Li, 2011). Our study adds to this discussion and thereby contributes to the ongoing efforts to conceptualize good mathematics teaching (e.g., Cai et al., 2009; Franke et al., 2007; Givvin et al., 2009). From our analysis of focus-group interviews of 20 Norwegian mathematics teachers, we notice that the teachers conceptualize good mathematics teaching in terms of structuring lessons, differentiating in accordance with individual students' different needs, two-way mathematical communication between teacher and students, as well as teachers' use of tasks and resources. It was also shown that the teachers emphasize student engagement and student learning when discussing good mathematics teaching, and this corresponds with results from international studies (e.g., Li, 2011).

Some conceptualizations of good mathematics teaching found in our study correspond with findings from similar studies in other Nordic contexts (e.g., Hemmi & Ryve, 2015). For instance, the teachers express that they want to be facilitators and build their teaching on individual students' thinking and initiatives. Like in the Finnish teacher education context (Hemmi & Ryve, 2015), the

Norwegian teachers emphasize the structure of the mathematics lesson and in-depth work with mathematical concepts by using a precise mathematical language in lessons. The Norwegian teachers also seem to agree with the Swedish and Finnish teacher educators about enabling individual students to participate, while simultaneously viewing differences among students as an asset. Moreover, mathematical discussions are emphasized as an important element of good mathematics teaching. This corresponds with a larger body of research that highlights creation of mathematical classroom discourse as a core feature of good mathematics teaching (Franke et al., 2007).

There are also some differences between the conceptualizations of good mathematics teaching found in the Norwegian context and previous findings in other countries. For instance, Hemmi and Ryve (2015) suggest that Swedish teacher educators emphasize constructivist thinking and student-centered teaching, and that their interpretations in this respect are extreme, but the Norwegian teachers do not have a similar emphasis in their conceptualizations as shown in the emphasis on the importance of teachers' role. In terms of differentiation, there appear to be some nuances in definitions across countries. In the Swedish teacher education discourse, differentiation is operationalized referring to letting all students work at their own pace and level, whereas in the Finnish context, keeping the group of students within the same mathematical area and at the same time support and challenge individual students are highlighted. Furthermore, the 20 Norwegian teachers contend that it is important to introduce mathematical tasks that are motivating for their students, or tasks which are open for differentiation. This aspect of these Norwegian teachers' conceptualization of good mathematics teaching appears closer to what is found in the Finnish teacher education context (Hemmi & Ryve, 2015), where problem-solving and inquiry approaches are emphasized. In the Swedish context, however, they emphasize spontaneous everyday situations and thematic work.

The conceptualization of good mathematics teaching that seems to emerge from the present study – balancing the communication between the teacher and the students – appears to be somewhere between the contexts in Finland and Sweden. Whereas the Finnish discourse described the teacher as "a very proactive agent in the classroom" (Hemmi & Ryve, 2015, p. 515), the Swedish discourse concentrated on "basing teaching on students' thinking, ideas and interests" (Hemmi & Ryve, 2015, p. 511). The Norwegian image of good mathematics teaching seems to be found in between these two. Whereas the aspects of teacher's instruction/role, the structure in lessons, differentiation, communication and use of tasks and resources in the discourse mainly refer to the teachers' actions, the aspects of student engagement and students learning refer mainly to the students. The Norwegian teachers describe student engagement as an important characteristics of good mathematics teaching. Active participation, eagerness to solve a given problem, willingness to listen and try to understand, as well as hard work, are described as important prerequisites for good mathematics teaching. In the discussions, part of the responsibility for the quality of mathematics teaching is thus given to the students. This is in line with previous studies in the Norwegian context (Fauskanger, 2016), but differs from findings in Sweden and Finland (Hemmi & Ryve, 2015). However, in line with Swedish teacher educators, the Norwegian teachers also characterize good mathematics teaching as giving opportunities for students to think, be creative and discover. The Norwegian teachers also contend that the responsibility for engagement is supposed to be shared among teachers and students (cf. Fauskanger, 2017). Such a shared responsibility is also what constitutes the Norwegian teachers' discourse about student learning. The teachers are, however, responsible for helping their students to learn mathematics, the Norwegian teachers say.

When comparing with results from international studies outside the Nordic context, it appears that the focus on student learning is always at the center. Views about the role of the teacher, however, seem to differ across countries. Whereas US mathematics teachers emphasize classroom management (e.g., Cai et al., 2009), mathematics teachers in countries like China seem to focus more on teachers' preparation, content knowledge and understanding of textbook contents (Cai et al., 2009; Li, 2011). The Norwegian mathematics teachers in our study do not emphasize classroom management, and their views appear different from those of US teachers in this respect. Unlike Chinese teachers, however, these Norwegian teachers do not emphasize teachers' knowledge, preparation and understanding of textbook content (cf. Li, 2011). Hemmi and Ryve (2015) report that Swedish and Finnish teacher educators stress the importance of teacher knowledge but practicing teachers in these countries might have different views.

Conclusion

By providing some perspectives of Norwegian teachers' views of good mathematics teaching, the results from this study add to the body of literature on views of good mathematics teaching (e.g., Cai et al., 2009; Givving et al., 2009; Hemmi & Ryve, 2015; Li, 2011). The Norwegian mathematics teachers in our study share some views of good mathematics teaching with teachers and educators from other countries, but their views also differ from findings in international studies in certain respects. We notice in particular that these Norwegian teachers emphasize a shared responsibility for engagement and learning among teachers and students, and they want to facilitate good mathematical discussions by using tasks and activities that enable differentiation among students.

Although we have described our sample as "Norwegian teachers", we do not claim that the results from this study are representative for the entire population of Norwegian teachers. The participants in this project are special, in that they are volunteer participants in the MAM project, which implies that they are more concerned about mathematics teaching than many other teachers are. Having said this, we believe that the results from this study may indicate some views of mathematics teaching that are characteristic for the Norwegian context. Like Givvin et al. (2009), we believe that variations in teachers' views about mathematics teaching across countries may relate to the cultural differences in teaching itself, and the findings from our study seem to correspond with observations of mathematics teaching in Norway.

Since attempts to define good mathematics teaching can be regarded as a process of establishing norms (Franke et al., 2007), and since views influence decision-making (Krainer, 2005), mathematics teachers' views of good mathematics teaching are arguably important. Further research may be useful to investigate if the views of good mathematics teaching reported in this study correspond with the views of a larger population of Norwegian mathematics teachers. In addition, we suggest that it may be useful to explore similarities and differences between the views of teachers and teacher educators in the Norwegian context, since there may be cross-professional differences even within countries. Researching good mathematics teaching as constructed in teachers' discourses can contribute to a better understanding of teachers' views and thus allow teacher educators to tailor their in-service education.

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Teachers' mathematical discussions of the Body Mass Index formula

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In this paper, we investigate a group of primary teachers having initial discussions about the Body Mass Index formula. Elements from Bloom's revised taxonomy and Niss' mathematical competencies are used to analyse the teachers' discussions. Their discussions show that the teachers are inexperienced with the history and certain parts of the mathematics behind the formula, but they have the ability to raise important questions about it.

Introduction

In this paper, we refer to an ongoing research project about indices as a theme of instruction in teacher education. An index can be considered as a benchmark for measuring changes in the value of a variable quantity over a time period, like alterations in the average price of an amount of commodities. Calculations of indices are based on approximations and different measurements, which often involves weighted arithmetic means, or other mean value approaches. For an overview of calculations of different price and quantity indices, see for instance Balk (2008). The output from an index formula is usually a single number, which can be tracked when calculated at different points in time. This can give information about the development of the variable quantity, a knowledge that can lead to different societal decisions. In this way, an index can be regarded as a mathematical model.

Indices are not explicitly mentioned in the Norwegian curriculum, but there are arguments to include them in teacher education. According to Skovsmose (1992), it is essential that students develop their reflective and critical awareness regarding the use of mathematics in society. Skovsmose argued that mathematics colonizes and rearranges parts of reality, a phenomenon he termed "the formatting power of mathematics" (p. 6). Gutstein (2006), building on Freire, suggested an exploratory orientation towards building curriculum in which community knowledge and critical and classical mathematical knowledge are integrated in order for students to become empowered citizens. He underlined that in developing mathematics curricula, interconnections between these components have been undertheorized. Biesta (2009) emphasised that mathematics education should not only facilitate for students to become proficient in mathematics, but also offer students the possibility to use the power of mathematical reasoning to gain a more autonomous and reflective position when facing tradition and common sense. As teacher educators, we are interested in investigating to what extent the mathematics behind mathematical applications in the society are attainable and applicable for student teachers, and how student teachers display abilities to understand, evaluate, discuss, and relate critically to them. An application which is widely used in the society and therefore could be regarded as familiar to many student teachers, is the Body Mass Index (BMI). This is the background for investigating a situation with in-service primary teachers discussing this particular index.

The present BMI formula, w/h^2, is a result of several scientific and mathematical attempts in deriving a suitable measure of relative body weight. In the 19th century, the Belgian pioneer in anthropometry and statistics, Quetelet, discovered that for army conscripts of a given height, the corresponding weights distributed like a bell-formed curve. He explored both squaring and cubing the height in respect to weight, concluding that weight was approximately proportional to height squared for conscripts situated in the middle part of the population (Oliver, 2006). Since weight increases relatively disproportionally with height, several other indices were later suggested, for instance the Ponderal index, w/h^3, and the ratio w/h. For a more comprehensive introduction to these and other investigations, see Hall and Barwell (2015). Comparing Quetelet's formula to other proposed indices, Keys, Fidanza, Karvonen, Kimura, and Taylor (1972) confirmed that weight divided by height squared, still was the best formula. However, during the last decades, both popular, natural, and societal sciences have raised major objections to using BMI as a standard, and socio-political arguments regarding this can be found in e.g. Oliver (2006). A core argument in many texts is that for most bodies, weight will scale neither quadratically nor cubically with height. In relation to this, models building on body density and waist circumference are tried out (see e.g. MacKay, 2009).

In school mathematics, teachers work with concepts and processes that are important for understanding the mathematics involved in the BMI formula. However, the assumptions and mathematics behind a model, constitute a comprehensive amount of information which is often not possible to teach in a limited amount of time. Studying an index like BMI may contribute to research on how to include models in education. To analyse the teachers' discussions about the BMI, we have applied the cognitive process dimension from a revised version of Bloom's (1956) taxonomy suggested by Anderson and Krathwohl (2001), and parts of Niss' (2003) modelling mathematically competence. The latter framework relates to the mathematical literacy concept (see Skovsmose, 1992). The focus of this paper is to 1) identify mathematical knowledge in the teachers' discussions, and thereafter 2) analyse which cognitive process dimensions the teachers enter into in their discussions and to what extent they make use of Niss' modelling mathematically competence.

Theory

Bloom's (1956) taxonomy is a framework designed to categorize educational objectives. Originally, the main categories were general objectives involving knowledge, comprehension, application, analysis, synthesis, and evaluation. In the revision of the taxonomy, Anderson and Krathwohl (2001) distinguished between the knowledge dimension containing the categories factual, conceptual, procedural, and metacognitive knowledge, and the cognitive process dimension containing the categories remember, understand, apply, analyse, evaluate, and create. Each category refers to several cognitive processes, for example understanding is based on the cognitive processes interpreting, exemplifying, classifying, summarizing, inferring, comparing, and explaining. A reason for introducing this distinction was that learning objectives generally contain a verb (e.g. solve) and a noun (e.g. linear equation). The distinction provided the possibility to associate the cognitive process dimension with the verb and the knowledge dimension with the noun. The revision also implied a greater focus on content in addition to process.

Educational objectives are situated in between instructional and global objectives. Anderson and Krathwohl (2001) characterized instructional objectives as narrow, for instance exercises given by a teacher. Global objectives are broad, like the planning of a multiyear curriculum in elementary reading. In general, objectives distribute along a continuum, and the relative position of an objective is a matter of interpretation. In our analysis, we focus on the educational level.

A framework focusing particularly on mathematics learning is derived from the project *Competencies and the Learning of Mathematics* (KOM) presented in Niss (2003). This framework relates to the literacy concept by addressing the question "what is the counterpart in mathematics of mastering a language?" (p. 6). Mathematical competence is then explained as "the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role" (p. 7). The framework suggests eight competencies divided in two groups, and an activity often facilitates only some of the competencies. The modelling competency, which is of particular interest in an index model project, is part of the group concerning the ability to "ask and answer questions in and with mathematics" (p. 7). "All competencies have an analytical and a productive nature" (p. 9), where the former involves a focus on understanding, interpreting, examining, and assessing mathematical phenomena and processes. The latter is about actively carrying out mathematical processes.

The two frameworks are combined in the analysis. The analysis involves an identification phase (what mathematical knowledge) and a cognitive process phase (how do the teachers cognitively relate to the mathematical knowledge). As a project involving modelling, we also use parts of Niss' (2003) modelling mathematically competency, which includes the sub-competencies analyzing and decoding existing models and performing active modelling. Analysing refers to "foundations and properties of existing models, including assessing their range and validity", and decoding existing models refers to "translating and interpreting model elements in terms of the 'reality' modelled" (p. 7).

Method

The data were collected from a multidisciplinary course on numeracy across the curriculum, where twelve in-service teachers from different subjects in primary school discussed indices, and body mass index in particular. The teachers had previously completed from 0 up to 30 ECTs in mathematics or mathematics education. They were asked in advance if they were willing to participate in the project, and it was emphasized that it would not be part of their exam. The project is approved by the Norwegian centre for research data, and all the data is anonymized.

The teachers were randomly divided into two groups, given 45 minutes for discussion. After about five minutes, each group received an information sheet with the BMI formula including a diagram showing cut off points for six weight categories (https://no.wikipedia.org/wiki/BMI) and a question sheet encouraging them to discuss four questions: a) What is BMI? b) What does the index measure? c) What reflections do you do concerning BMI's formula and application? d) Could the index look differently – if so, how? The discussions of these questions were audio recorded and transcribed.

To answer the first research question, we started by counting all formal mathematics words used by the teachers. Formal mathematics words are words included in a formal mathematical vocabulary. Some words belong to a formal mathematical vocabulary and an everyday language (e.g. "curve" and "measure"). Such dual use is common in the subject of mathematics, and we classified such words as formal mathematical ones. We also identified many wordings representing informal mathematical thinking, like "measure number", "mean value thinking" or "bend on the curve". Wordings regarded as informal were not included in the counting. Neither were adjectives like "high", "low", and "less", which were frequently used by the teachers. This does not mean that we think of these wordings as unimportant for understanding mathematics. There are many examples of mathematics education research focusing on communication in mathematics (e.g. Usiskin (1996)). According to Rubenstein and Thompson (2002) teachers can give attention to mathematical language learning, for example by inviting students to invent their own terminology as a step towards formal mathematics.

We then classified the identified mathematical words into nouns and verbs. Mathematical nouns included words referring to mathematical facts, concepts, and procedures, like “ratio” and “long division”. Similarly, we classified all processes associated with mathematics, for example “measure” and “calculate”, as mathematical verbs. This classification was an effective way to gain an overview of the mathematical content of the discussion.

The next step was to answer the second research question. Data for answering this research question were excerpts selected on basis of the counting procedure. The criteria for selecting an excerpt was that it contained at least one of the two most frequently used mathematical words (we did not differentiate between nouns and verbs in this selection process). We emphasized choosing excerpts containing mathematical reasoning that could be investigated in order to gain insights about the cognitive processes the teachers applied.

The excerpts were analysed by use of the revised version of Bloom’s taxonomy and Niss’ (2003) framework. Associating the knowledge dimension in the taxonomy with the nouns and the cognitive process dimension with the verbs, made it possible to map to what extent the teachers’ discussion were situated in each dimension. Guided by the second research question, we only comment on elements belonging to the cognitive process dimension. The knowledge dimension is to some extent covered by the results from the counting process. Hence, we have used the revised taxonomy to differentiate and assess learning processes instead of educational learning objectives, which was the original intention of the framework. The teachers’ engagement in mathematical formulas other than the BMI introduced by Quetelet, like for instance the formula suggested by Ponderal, were considered to be a part of the create category in the taxonomy. We also identified what Niss classified as analysing and decoding skills within modelling competency. Since the teachers were not intended to perform any active modelling, we concentrated on their analysing and decoding skills. In the following, the selected excerpts are analysed chronologically.

Analysis and discussion

During the counting process the following mathematical nouns were identified (the number in parenthesis is the number of times the word is used): “mean” (9), “ratio” (9), “a measure” (5), “curve” (4), “representative sample” (4), “sample” (2), “ratio” (1), and “parameters” (1). Similarly, the identified mathematical verbs (in different tenses) were: “measure” (38), “square” (10), “weigh” (5), “assess” (4), “calculate” (3), “register” (1), and “compare” (1). The counting shows that the teachers move within both the knowledge and the cognitive process dimension in their discussions.

Based on this counting procedure, the two most frequent words were the two verbs “measure” and “square”. Below is an excerpt from the beginning of the teachers’ discussion where the word “measure” (38) is used:

T4: I have played with such a calculator on the internet, where you could just insert the height and weight. Then it [the calculator] will measure. …and I have played … how low could my weight be before I will be outside the normal?

According to the taxonomy, we interpret this teacher, T4, as being in the category of understanding. The teacher tries to understand the cut-off point leading to the underweight category on the information sheet. One of the cognitive processes involved here is "classifying", which is about "determining if something belongs to a category" (Anderson & Krathwohl, 2001, p. 67). T4 is also exemplifying using his own weight, which is about "using a specific example or illustration of a concept or principle" (p. 67). Leaving the calculations to the internet calculator, the cognitive process "executing" within the apply category in the taxonomy is paid less attention to. According to Niss (2003), the teacher is trying to analyse the model, by "assessing its range and validity" (p. 7).

Focusing on the second most used word, "square" (10), we present a transcript from later on in the discussion where this word appears several times. At this point, the teachers have not yet received the sheet with the BMI formula:

T2: Why is the weight squared?

T4: Yes, that is what I am wondering about as well, why is it squared? Why don't we just find the ratio between kilos and height? Why do we have to sq [does not complete the word] …?

T3: In a way, this makes the weight more influential than the height. And … that leads to …

The teachers make the mistake of thinking that the weight is squared. At this point, they are unaware that height is the variable to be squared in the BMI formula. If this mistake is temporarily disregarded, they have some interesting utterances concerning squaring. The utterances of T2 and T4 are expressed as questions, asking why one of the parameters in the formula is squared. According to the framework proposed by Niss (2003), the teachers try to decode the model by discussing the process of squaring in terms of relating this operation to the empirical context (weight and height). By discussing squaring as a property of the model, they also apply the process of what Niss refers to as analysing existing models.

Using the other framework, the revised taxonomy, the question "Why is the weight squared?" shows that the teachers are in a differentiating process, a process belonging to the category "analyse" in the cognitive process dimension. This occurs when "a student discriminates relevant from irrelevant information […] and then attends to the relevant or important information" (Anderson & Krathwohl, 2001, p. 80). According to the knowledge dimension, the differentiated knowledge can be interpreted as factual, since the teachers refer to the fact that the denominator is squared. In the following utterances, they elaborate on these reflections when T3 says: “this makes the weight more influential than the height.” By differentiating something (the weight) to have more influence than something else (the height), they extract an important property of the formula. By the comment “And … that leads to…” they are about to start a cognitive process of inferring, to draw “a logical conclusion from the present information” (p. 67). From their version of the formula, the teachers try to conclude what is implied from weight being more influential than height.

After having received the sheet with the correct BMI formula and the cut-off points, the teachers follow up the squaring issue:

T6: Could it [the BMI formula] look different?

T3: Perhaps something will happen with the curve. Will it be linear, then, if it isn’t …?

T4: If it isn’t squared?

T3: If it isn’t squared. You see it is, more like a … bend on it, that might make it more correct. I don’t know.

T6 is asking if the BMI formula can “look different” and is shown by that to still be in the process of trying to understand the formula. Discussing if the BMI formula can look different, like T4 and T3’s suggestion about not squaring the height, can be interpreted as the teachers being in the cognitive process of comparing (a process within the understand category). They are “detecting similarities and differences between two or more […] ideas” (Anderson & Krathwohl, 2001, p. 74). However, there are also elements from the evaluate process dimension as they are checking the formula for potential weaknesses and trying to test the idea of not squaring the height.

In their search for alternatives or possible adjustments to the BMI formula, the teachers are in processes from the create category. This category contains the cognitive processes generating, planning, and producing. Prior to the next excerpt, the teachers had come up with an alternative formula: w/h (as an answer to question d) on the question sheet). They decided to try out some different heights and weights to see what results they would get with the two different formulas:

T4: Yes, what did you find out?

T5: No, it is not very … very ground breaking, but I get 43.6 if I … the formula where I just divided the weight by the height, then I got 43.6, like it is now. And 47.9 if I increased the weight by 10 %. With this one [formula with the height squared] it increased from 22.3 to 24.6.

T4: Increased 2.3 and then it increased 4.3.

T2: I don’t see why, but …

T5: If you are short, if you are one meter … If you are one meter, then one squared is one.

T6: Yes.

T5: Then it plays a huge part.

The teachers compare the results by using the two different formulas, w/h and w/h^2. They relate to the create category in the taxonomy, since they have produced the alternative formula w/h. By investigating potential weaknesses of these two formulas, they are in the evaluate category. By comparing how the outcome, the BMI value, is influenced by using two different formulas, they are also in the category of understanding. T5 investigates what happens if the weight increases by 10 % for both versions of the formula, and T4 follows up on this by calculating both increments. Through these initiatives, they are in the category of understanding through the process of exemplifying. In this excerpt, the sub processes executing, and exemplifying are closely related, because by executing concrete calculations to see how the BMI values are affected, they are also in the apply category.

T5's reflection "If you are short, if you are one meter" is about investigating an extreme. In this last part of the excerpt, the discussion shifts from checking different weights to varying the height. This shift of focus indicates that the teachers are not able to generalize about the role of the weight for a given height, how the BMI increases proportionally with weight in both formulas. As a general comment, we could say that they are trying to analyse the BMI formula by breaking "the material into its constituent parts and determining how the parts are related to each other and the overall structure" (Anderson & Krathwohl, 2001, p. 79). Niss' (2003) decoding and analysing competencies continue to be in play, the teachers still struggle with the properties and interpretation of the model.

Concluding comments

We identified the squaring of one of the components in the BMI formula as one of the first mathematical issues that occupied the teachers. Squaring was the second most referred mathematical concept. To identify and reflect on squaring is highly relevant considering the origin and historical development of this formula. The most frequent used formal mathematical word was measuring. They used measuring, significantly more often than for instance calculating, a concept from mathematics terminology that is natural to compare with. A reason for the frequent use of measuring can be that this word is used in the question sheet and in everyday language. Themes like proportionality and normal distributions, which constitute an important mathematical foundation of the formula, were not mentioned throughout the discussion.

Considering the second research question, the teachers entered various cognitive processes to interpret the BMI formula and its applications. They sometimes were at an analytical level with respect to squaring, but their discussions of how squaring influenced general calculations by the formula, relied heavily on concrete examples. It was an unused potential for mathematical generalizations. According to the framework by Niss (2003), the emphasis on finding answers in and with mathematics was only partly fulfilled. In addition to entering the cognitive process of understanding the squaring, the teachers moved between several other categories in the cognitive process dimension. Their suggestion of the formula w/h showed the ability to get into create processes and reflect on other alternatives than the present formula. Engaging in comparisons of the two formulas, w/h and w/h^2, are in line with some of the formulas investigated in the literature (Keys et al., 1972). The teachers had few possibilities to find answers to the question of squaring, because this would require knowledge of the history behind the BMI formula. Their only source of information was the information sheet, and if by coincidence one of them knew about the controversies concerning the design process of the index (which the omitted parts of their discussion showed they did not). The cognitive processes we have detected and analysed in this study could be different from cognitive processes in play when students work with mathematics in traditional ways (e.g. working with text-book tasks). If so, specific forms for educational facilitation could be required to support these processes. Our investigation also indicates a need for further research on how to assist students in decoding and analysing different society-related mathematical models.

To study how a group of in-service teachers decode and mathematically interpret a specific mathematical model like BMI gives knowledge about how to work with models and societal use of mathematics in education on a more general basis. Inclusion of indices in teacher education therefore deserves further investigations.

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Teachers' attention to student thinking, mathematical content and teachers' role in a professional learning community

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The past decade has witnessed increased efforts in studying what mathematics teachers do to improve their teaching. This study builds on and contributes to the research on collective learning in professional learning communities. It aims to uncover the teachers' attention to different aspects related to their work and how these have been developed during one year of collaboration. The focus is on teachers' norms and their attention to student thinking, mathematical content and the teachers' role. By videotaping teacher's participation in the Boost for Mathematics project (Matematiklyftet) five cycles of collective planning and reflection are analysed. The findings indicate a shift in the teacher groups' attention from their role as an organiser to the mathematical content and student thinking.

Introduction

The past decade has witnessed increased efforts in studying what mathematics teachers do to improve their teaching. One of the specific interests has been the nature of various collaborations through which mathematics teachers are engaged in working and learning, particularly after the report about the Lesson Study in Japan from the TIMSS classroom video study (Robutti et al., 2016). Based on the idea that teachers' professional development can foster improvement in their teaching, many professional development (PD) programmes have been developed. Most of these programmes conform to the concept that a group of teachers should meet regularly, share their expertise and work collaboratively, which can be labelled as a "professional learning community" (PLC). According to Brodie (2014a), PLCs refer to "groups of teachers collaborating to inquire into their teaching practices and their students' learning with the aim of improving both" (p. 501). This goal is also compatible with the aim for the Boost for Mathematics (Skolverket, 2017). Several successful characteristics of a PLC have been identified, such as a productive relationship, a de-privatised practice, fostering collaboration and a collective responsibility for teachers' learning (Brodie, 2014a). Despite an agreement on the importance of PD programmes, there is little consensus about how it is expected to alter teaching practices and how it fosters teachers' learning (Kennedy, 2016), which Goldsmith, Doerr, and Lewis (2014) described as the black box of teachers' learning.

This study builds on and contributes to the research on collective learning in PLCs, specifically in the field of mathematics education. Although some studies (Goldsmith et al., 2014) have examined several successful characteristics of a PLC, there is hardly any consensus on the processes of how teachers develop their knowledge, as well as the interactions through which a PLC constitutes a resource for teachers' learning and innovations in teaching practice. As such, this study provides additional insights into how PLC groups collaborate to improve their teaching. It aims to uncover the teachers' attention to different aspects related to their work and how these have been developed during one year of collaboration. This study analyses the discussion of one group aiming to attain the Boost for Mathematics during one school year.

The research question guiding this paper is as follows: How has the teachers' attention to student thinking, mathematical content and the teachers' role been changed in a PLC during one year of collaboration?

PLC in mathematics education

Efforts to understand what teachers do to improve their teaching have led to an increased interest in studying different activities, processes and the nature of various collaborations through which mathematics teachers are engaged in working and learning. The organisation of work within schools has undergone changes over recent decades towards more collaborative forms. Teachers are expected to open up their practices to collective investigation. This modification has led to a stronger emphasis on school-based development and collective approaches to practice (Flitton & Warwick, 2013). The focus has shifted from individual autonomy to the development of teachers' practice as a collective enterprise. A central tenet of much of the PLC literature is that collaborative settings allow for individual and collective learning, critical examination of existing practices and joint development of pedagogical/mathematical ideas and artefacts. Collaboration implies that teachers work together and can also learn as a group. It involves teachers performing joint activities for a common purpose (for instance, a shift in practice and its implications for the mathematics learning of students), supporting one another in addressing issues that challenge their existing teaching practice and students' learning. Such collaboration will offer the teachers the possibility to reflect on their role in school (Brodie, 2014a).

The analytical task entails showing how teachers' attention to student thinking, mathematical content and the teachers' role is developed in a PLC during their collaboration. Through interactions with one another, the teachers will present and discuss their experiences and knowledge as educators. A central task is to examine what aspects of teaching practice are taken up in conversations in a PLC. From a sociocultural perspective, the individual and the group contexts cannot be separated since knowledge does not occur in isolation. Knowledge is constructed through interaction and in a context (Vygotsky, 1978), not primarily through individual processes. How teachers engage in discussions about their practice can be considered examples of their knowledge as their various perspectives are presented and shaped through a year of collaboration. Teachers change by transforming their participation in sociocultural activities that are formed by individuals with other people in cultural communities.

Norms in PLC

Norms are regular patterns of behaviour that affect the nature of learning (Van Zoest, Stockero, & Taylor, 2012). Much of the research on norms in mathematics education draws on Yackel and Cobb's (1996) distinction between social and sociomathematical norms. Social norms are regular patterns of behaviour that are not unique to a mathematics classroom, while sociomathematical norms are specific to mathematical activities. Fostering what is often labelled as productive norms, particularly the sociomathematical type, can improve learning at any level in school, as well as in a PLC (Clark, Moore, & Carlson, 2008).

Elliott et al. (2009) drew on and developed a framework of norms when they designed seminar activities for PLC leaders. They built their framework on Yackel and Cobb's (1996) distinction between social and sociomathematical norms, suggesting that learning opportunities would be guided by patterns of interaction, both explicit and implicit, that would establish how a group could work together. During seminars (using video cases of teacher seminars, among others), the PLC leaders were prompted to notice the nature of questioning and the treatment of errors and confusions as a way of paying attention to sociomathematical norms. This focus on the nature of explanations led Elliott et al. (2009) to identify four productive social norms. The first is sharing, where the group's participants listen respectfully to one another and exchange ideas. The second involves justifying, where teachers describe and give reasons for their thinking. The third entails questioning, where teachers query one another. The last comprises responding to confusions and errors.

Despite the growing body of research on sociomathematical norms that might affect teachers' learning, research on more general norms that influence teachers' learning is less prevalent (Van Zoest et al., 2012). Therefore, this research focuses on social norms in a PLC, the specific ways that teachers engage with one another, and how these norms are related to the way that they interact and discuss student thinking, mathematical content and the teachers' role.

Categories for professional knowledge required for teaching

Ball, Thames, and Phelps (2008) emphasised mathematical knowledge for teaching, which they divided into two domains – subject matter knowledge and pedagogical content knowledge. To foster what Ball, Thames and Phelps (2008) label as effective teaching, the development of mathematical knowledge for teaching is an important factor. At least, familiarity with this model allows teachers in a PLC to reflect on the various domains of pedagogical content knowledge. According to Garet, Porter, Desimone, Birman, and Yoon (2001), PD programmes that focus on specific mathematics content and the ways that students learn are helpful, particularly regarding instruction designed to improve students' conceptual understanding. This is consistent with the study of Goldsmith et al. (2014), who reviewed articles related to professional learning and practising teachers of mathematics, searching for how and what teachers learn to provide high quality mathematics teaching for all students. They found six major categories related to teachers' learning, three of which are of particular interest for this present research. One category is teachers' attention to student thinking. Students often think about mathematics differently from teachers; therefore, it is important for teachers to understand and build on students' existing ways of knowing. The next category is teachers' instructional practice. Goldsmith et al. (2014) considered changes in teachers' instructional practice as evidence of professional learning, including lesson planning and post-lesson reflections, as well as classroom instruction as practice. Many of these studies included a PLC intervention, with different types of focus, such as mathematics, mathematics tasks, student thinking and pedagogy. The last category is mathematics content knowledge, and Goldsmith et al. (2014) identified particular ways that teachers' mathematical understanding affected practice and found a connection between mathematical knowledge and the ability to engage in productive professional conversations.

This section has presented some frameworks and constructs about norms in a PLC and several categories for the professional knowledge required for teaching. Based on this research, we have developed a framework to focus on teachers' norms when they contribute in the discussions and their attention to student thinking, mathematical content and the teachers' role. The next section, methodology, explains this framework.

Methodology

Between 2013 and 2016, the Swedish National Agency for Education launched a 649-million kr, curriculum-based PLC project. Called the Boost for Mathematics, this project aims to improve the mathematical classroom teaching. The most central components are 24 modules, eight per grade level 1-3, 4-6 and 7-9, developed to support teachers working in teams in planning, establishing and reflecting on mathematical classroom practices. The curriculum material is distributed digitally on a website (http://www.skolverket.se/kompetens-och-fortbildning/larare/matematiklyftet) and includes articles, instructions, images and video films. Each module is designed to support groups of teachers (during one semester) in engaging in eight iterations of (1) individual reading, (2) collective planning with colleagues, (3) individual classroom teaching and (4) collective reflections on classroom instruction. A coach guides each group of teachers. This paper focuses on one group of eight teachers, including one coach, teaching grades 4 to 6. The participants in this group came from three schools. The data were collected by videotaping four cycles – two in the autumn and two in the spring, a total of eight sessions. Each cycle included collective planning and reflections with colleagues. In the autumn, the group worked on the module "Understanding and use of numbers", and in the spring, they participated in the module "Relationships and change".

Framework

Based on videotaped records of the interactions among the teachers from one working group, we wanted to study a) how teachers' attention to student thinking, mathematical content and the teachers' role were developed and b) the norms of professional interactions and the ways that they were related to the elements described in a). The analytical task entailed showing how the interactions among the teachers revealed particular considerations of practice. We therefore developed a framework that could help us document the norms and the practice that would constitute the collective learning (Cobb, Zhao, & Dean, 2009) of a teachers' group.

Concerning norms of professional interactions, sharing refers to teachers exchange their ideas. One example could be that they discuss how the classroom are organized, the size of the students working groups and so on. Justifying involve the ways that teachers describe and explain their reasoning. Do they refer to the Boost for Mathematics project, their own experience, the textbook, the research literature or other factors involved? Questioning refers to how they query one another and what aspects of teacher attention they are asking about.

Concerning teachers' attention, one aspect involves students and their abilities and misconceptions (among others), what Ball et al. (2008) referred to as knowledge of content and students. Another aspect is the teachers' role, specifically, how they describe their own functions in the classroom and knowledge of teaching. The mathematical content category covers specialised and common content knowledge.

Analysis

The analytical task involved showing how teachers, through their interactions with one another, constructed their representation of practice. Representation of practice refers to the students, the teachers, mathematics or the organisation of the lessons that are taken up in the conversations among the group of teachers. The results were based on one incidental group of teachers teaching levels 4–6, who were chosen among six different groups. Therefore, the representativeness of this group could not be considered. Central to the analysis were videotaped records of eight teacher meetings in the course of a school year; these sessions' durations varied from 70 to 100 minutes each. We used the videotaped records to study the teachers' collegial interactions. The teachers' meetings were then transcribed, and the texts were coded using NVivo software. We identified relevant conversation episodes and categorised them according to the coding scheme presented in Table 1. A change in episode was registered when the teachers shifted their attention from one category to another or modified the aspect of an interaction (norm). These shifts were registered in each session and used to compare the sessions. One of my research members and I independently coded the first session, and we compared our results to adjust the rest of the coding of the materials. When disagreements occurred, we resolved them.

		Norms		
		Sharing	Justifying	Questioning
Teachers' attention to ...	Student thinking			
	Teachers' role			
	Mathematical content			

Table 1: Norms and teachers' attention to student thinking

Results

The first excerpt below is from the first videotaped collegial meeting in the autumn of the teachers' group. They are working with the module part "Number and number concepts, grade 4-6" and at this stage, the teachers are working on round 4 and session B, entitled "Reasoning". In the first 25 minutes, they have been discussing the questions about reasoning from the curriculum materials, for instance, "What do you mean by reasoning in mathematics?" In this excerpt, they are planning a lesson with the aim (given in the text from the Boost for Mathematics) that students should reason about fractions. Six teachers are working together, and Mary is the coach of this group.

Jenny: How many groups do you think we should form? How many groups could we gather?

Mary: Yes, that was what I was thinking; how many could be seated?

Frank: Four. With a large A3-sized paper.

Jenny: Then it would be five groups.

Frank: Yes.

Mary: I consider three … because I have attended some courses and heard ...

Nola: It should be three in every group?

Mary: I heard from some place that this will activate everybody.

Other: Mmm.

Mary: But I do not think that it is always correct. Sometimes, four could be right and sometimes, three.

Jenny: Sometimes, four could be too many.

Mary: Yes.

Jenny: Some will be passive, and some will push forward.

Mary: At the same time, everyone should also write.

Jenny: I would like to try with three.

Clara: We go for three.

The above transcript illustrates a typical focus on the interaction during the first collegial meeting in the autumn were the teachers mostly focuses on sharing how they should manage the lesson they are planning. They discuss whether the students should be given a fraction to consider or if the teachers themselves should choose for them. The excerpt also shows that the teachers are concerned about how to manage the lesson, including the size of the paper and the number of participants in each group. The questions they are posing are related to managing the lesson. They do not push for deepening understanding as a productive social norm (Elliott et al., 2009) since they are in agreement. There seems to be a lack of an opportunity to compare and re-conceptualise ideas and explore contradictions.

The next excerpt is from the module part "Relationships and change, grade 4-6," round 3 and session B in the spring, entitled "Evaluation of students who are showing their knowledge." Six teachers are participating in this meeting. Frank is not present. Beth is participating this time, and she was absent from the first videotaped collegial meeting. Mary is still the group leader. In the excerpt below, two of the Grade 5 teachers present the task that they will give their students. "*Thirty percent of the students in one school play handball. How many students are there in total in this school, and how many play handball?*" The concept is new for the students, but the teachers think that the students are familiar with it from everyday life.

Jenny: We were thinking that we should change 30% to 50% of the students in the school.

Mary: Why will you change to 50?

Beth: Since it is a new concept, and let them understand from the beginning, and then, we were thinking that 30% could be for them something that would need a further step.

Mary: Like an extra task.

Jenny: Yes, you start with 50 right, and then you could take 30%, and then you perhaps could choose on your own if you take

30%, and you can still choose your own percent number. And we thought that we should ask them if they had encountered the concept of percent and in which situations [...]. And we think they have done that in discount and when they load their mobile phones.

Beth: Yes, downloading a computer program.

Jenny: They see yes, 75% left in the battery and like that. When do you load your mobile phone? How much is left? You can start from that point.

Beth: Returning to when the mobile phone is fully loaded – how much percent is that? When is one full?

Mary: So for you, it will be some listening and some help?

Jenny: Yes, we base it on their own knowledge.

Beth: Connect it somehow to everyday [life].

Jenny: What do we want that they should have learned after this?

Beth: I think an understanding of percent as a hundredth at least. Eh, preconception, I don't know.

Jenny: No, that 50% is one-half.

Mary: That 50% is one-half and percent as a hundredth.

During the teachers' collegial work, there has been a shift from an emphasis on teacher role as managing lessons (the first module part) to a stronger focus on the mathematical content and student thinking (Ball et al., 2008). They are discussing what kind of mathematical content would be suitable for their students, along with the lesson's aim. The questions are also related to the mathematical "change to 50%" and to students in terms of "what [...] they should have learned". To a greater extent than the questions in the first excerpt, these are more productive norms (Elliott et al., 2009) since they push for a deeper understanding of student learning.

Conclusions and implications

The two preceding excerpts are presented to show how the teachers' attention and norms have changed in a PLC during their collaboration. When they started working on the curriculum materials for the Boost for Mathematics, the groups mostly focused on teachers' role as a lesson manager. In their subsequent meetings, their attention shifted to student thinking and mathematical content. Following the study of Gamoran et al. (2003), the teachers are now collectively concentrating more on student learning as opposed to their previously more common conversations about administrative details and lesson management. They are collaborating on ways to improve their students' understanding of mathematics, as well as engaging in dialogues about their role and the nature of teaching. The groups' norms have changed in the sense that their justifications are now more productive in pushing for a deeper understanding, particularly of student learning (Elliott et al., 2009). On the other hand, the quantity of situations where teachers questioned each other was very low and did not change during the year of collaboration. Questioning is important because if you always feel safe, you cannot learn (Brodie, 2014b). Teachers have to be challenged to move outside their comfort zones to create new ways of thinking about their own role as a teacher and their students.

In this paper, we have focused on how norms and teachers' attention to student thinking, mathematical content and the teachers' role have changed in the course of a year's collaboration. Further developments and studies would compare this evolution among different groups of teachers and emphasise how such a transformation could constitute the collective learning of a teachers' group.

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Teacher learning in Lesson Study: Identifying characteristics in teachers' discourse on teaching

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This study responds to a call for more theory-driven research that investigates how teachers learn from participating in Lesson Studies by using the commognitive theory. Learning is regarded as a change in discourse, and the study investigates teachers' discourse on teaching. From analysis of an empirical example, three characteristics of the teachers' discourse are identified. Firstly, students' learning is described as static conditions. Secondly, assumptions are made about prerequisites for developing understanding of students in these static conditions. Thirdly, dialogue between "weak" and "able" students are described as important for students' learning.

Introduction

Research on the development of teachers' professional teaching practice in mathematics has increased during the past decades, and recent studies have investigated the potential of practice-based approaches to contribute to this development (Thames & Van Zoest, 2013). Dudley (2015) and others suggest that Lesson Study (hereafter LS) should be implemented in schools as part of continued professional development. While the Japanese school system has applied LS as a sustainable form of teacher driven professional development for more than a century (Ronda, 2013; Saito & Atencio, 2013), researchers from other countries have become interested in LS the past two decades (Dudley, 2013, 2015; Lewis, 2002). Much of the interest in LS in the western world arose after Stigler and Hiebert (1999) wrote "The Teaching Gap".

Research question(s), *prediction* and *observation* are three important aspects of the LS process. Teachers investigate their own teaching practice (Olson, White & Sparrow, 2011), they plan, conduct and evaluate a research lesson in order to answer their own research question(s) (Chokshi & Fernandez, 2004). The entire process requires that the teachers are open minded and eager to better understand student learning or uncover new ideas of a particular aspect of the teaching of mathematics (Lewis & Hurd, 2011). When planning the research lesson, prediction and observation are crucial to help the teachers understand how students learn

(Bjuland & Mosvold, 2015), and to develop their own instructional interactions in their teaching practice (Lewis & Hurd, 2011).

Previous research on teacher learning in LS has focused on what teachers learn from planning meetings (e.g. Cajkler, Wood, Norton & Pedder, 2014), teachers' reflection as an important part of mathematics teachers' professional development (Ricks, 2011), and how observation of students influences teacher learning (e.g. Warwick et al., 2016). Xu and Pedder (2015) call for more research on how LS teachers learn and develop practice through participation in LS, within a clear theoretical framework. This study aims at contributing to this strand of research, by using the commognitive theory (Sfard, 2008) as a theoretical and analytical framework for investigating teacher learning in LS.

The data presented in this paper is taken from a larger ongoing study in a lower secondary school in Norway. The study regards knowledge as shared and collective rather than individual. Learning is considered to develop through social negotiation (Radford, 2008), and is visible as a change in discourse (Sfard, 2008). In terms of teacher learning in LS, a distinction can be made between discourse *on* teaching and discourse *of* teaching. The former refers to when teachers' talk about (their own) teaching practice and student learning. The latter, discourse *of* teaching, refers to the discourse and routines that the teachers use in the classroom. This study investigates change in teachers' discourse *on* teaching. A prerequisite for investigating change in discourse is to identify key characteristics of the discourse. The aim of this study is thus to identify key characteristics in the teachers' discourse on teaching from planning meetings in the first of three LS-cycles. The following research question is approached:

> What are some characteristics of teachers' discourse on teaching that might be relevant to investigate in terms of teacher learning in LS?

The characteristics of discourse identified in this particular LS group are intended to serve as exemplars of characteristics that might be relevant to focus on in studies of teacher learning in LS.

Theoretical and analytical framework

Sfard (2008) considers thinking as communication and she has developed the term commognition: a combination of *comm*unication and c*ognition*, which she claims are two processes of the same phenomenon. A discourse is defined as "different types of communication (and thus of commognition) that draw some individuals together while excluding some others" (Sfard, 2008, p. 91). A mathematical discourse is characterized by four critical properties: word use, visual mediators, routines, and endorsed narratives. Word use refers to how the user defines the meaning of words, and "is responsible for what the user is able to say about the world" (Sfard, 2008, p. 133). Sfard (2008) describes development of word use in four stages: passive use, routine-driven use, phrase-driven use and object-driven use. Passive use refers to hearing the word, without actively using it. Routine-driven use refers to using the word in a concrete situation. Phrase-driven use relates to being able to use the word in similar situations. Finally, object-driven use refers to “the users’ awareness of the availability and contextual appropriateness of different realizations of the word” (Sfard, 2008, p. 182). Visual mediators are visible objects, either iconic, concrete or symbolic. Narratives are defined as any sequence of utterances framed as a description of a mathematical object, and endorsed narratives are often by the discursants (participator in the mathematical discourse) labeled as true. Routines are repetitive patterns characteristic of the given discourse, and divided into three types: explorations, rituals and deeds. The first type of routine is a how routine, meaning you can recall, sustain and construct narratives. Rituals are when routines, referring to when it is appropriate to use the different narratives. Deeds are to consider as practical actions that result in a physical change. Sfard (2008) defines learning as a permanent change in discourse. The change can take place on two levels. She distinguishes between object-level learning and meta-level learning. On the object-level, the change in discourse (learning) expands by developing new routines, new objects or endorsed narratives. In contrast, meta-level learning involves a change in metarules, which can only occur if there has been a commognitive conflict (e.g. that two individuals use the same word, but with different meanings).

Objectification is important in discourse development (Sfard, 2008). It is a process where discourse on human behavior and actions develops into an impersonal discourse on objects. This process consists of two closely related – but not inseparable – sub-processes: reification and alienation. Reification is the first step in this process and refers to the process of turning a discourse into an object (Sfard, 2008). For instance, instead of saying, “A pupil has solved many of the tasks perfectly in the test”, one can state, “The pupil has developed a mathematical understanding of the subject”. To make this statement an alienation, the utterance must release the subject, then “mathematical understanding” is a way to simplify a long story about the students’ skills and activities. Subjectifying is an accompanying term which “refers to a special case of the activity of objectifying, the one that takes a discursive focus shift from actions and their objects to the performers of the action” (Sfard, 2008, p. 290). One trap of objectification of a person’s former actions and subjectification, is that it might affect as constrain to the persons’ abilities and motivation. As Sfard states, “Words that make references to action-outlasting factors have the power to make one’s future in the image of one’s past” (Sfard, 2008, p. 56).

Methods

The LS-group consists of four mathematics teachers, one participant from the school administration (the group leader), and one external expert (the author of this paper). The first LS-cycle took place in the spring of 2016. The main data sources are video-recorded observations from the group's meetings. Three meetings were conducted before the research lesson. The first meeting was an introduction to LS followed by two planning meetings. The presented examples are from the first planning meeting:

An overview of the data collection

Part of the first cycle	Video-recordings
Introduction to LS	73 min.
Planning meeting 1	**154 min.**
Planning meeting 2	162 min.

Table 1: Examples presented in the research lesson

In the first step of the analysis process, video-recordings were transcribed (by the author of this paper). In the second step, a data reduction was made. In this process, two particular aspects of discourse on teaching were isolated: 1) teachers' narratives on students and student learning, and 2) teachers' narratives on teaching practice. The third step was to identity characteristics in the teachers' discourse, within these two core aspects. The theoretical concepts that informed this third step of analysis were Sfard's (2008) four properties of mathematical discourse: word use, visual mediators, narratives, and routines, and the metaphors of objectification and subjectification.

Empirical example

The mathematical theme of the research lesson is the concept of volume. The teachers want the students to understand volume as the relation between the base area and height, not only to calculate the answer of some three-dimensional shapes (using formulas). The discussion arises in the first planning meeting, in which the transcript presented below is taken from. The discussion continues in the second planning meeting. The tasks for the lesson have not yet been selected, and the teachers have not yet decided how to organize the students. Early in the conversation, the teachers have two focus areas: *how to differentiate* and *how to pair the students in groups*. They stress that it is important to differentiate, because there is a significant gap in the students' mathematical understanding. The following dialogue takes place in this discussion[6]:

1	Teacher 4:	There are only students at the top and at the bottom in this class?
2	Teacher 1, 2, 3:	Yes (In unison).
3	Teacher 1:	But that is okay, it is like that in some classes.

[6] The transcripts have been translated from Norwegian by the author of this paper.

4	Teacher 4:	And then it is the bottom there, it is enough just to do the calculation.
5	Teacher 3:	It is like that in class C as well. It is the top and the bottom. In this class, students achieve all grades, except grade one.
6	Teacher 1:	But the differences, it is not in the same way.
7	Teacher 1:	I think the groups should be mixed. Slightly different levels, but not too big a gap. In addition, I think it would be better if we do not put all the weak students in the same group.

In the continuing discussion on how to pair the students, the teachers ponder whether the students should choose their own groups based on what task they want to elaborate upon, given tasks with different shapes, or if the teachers should set the groups beforehand. In the latter case, they have to consider whether they should group students homogeneously or mixed (7). One argument that the teachers present in support of "mixing students" is that, when a student explains something to a fellow student, both the explainer and the listener learn from the dialogue. They want the students to explain to each other how they got their answer – not only to exchange their answer, but to argue mathematically. The teachers assume that it is more difficult to find the volume of a shape with a complex base area than for instance a plain rectangular prism. They agree that when calculating the volume of a prism with different base areas, a rectangular base is easier than a triangular base; a cylinder is even more difficult. The teachers predict that the "weak students" need a shape with single base area, while the more able students can be given a shape with more complex shapes, for instance a shape with two or three different base areas, like a swimming pool with different depths. One of the teachers would like to hand out a concrete three-dimensional figure to each group, as a visual mediator. He proposes a task in which the students calculate the volume of the figure on the hand-out, first individually, then in groups, discussing their answers. To assess if the students have understood the relation between base area and height, the teachers want to study the students' discourse. The discussion proceeds as follows when the teachers plan on how to facilitate and observe student dialogue:

8	Teacher 3:	Do they understand how to calculate the volume?
9	Teacher 1:	Mm, and do they catch the connection between the base areas multiplied the height. We can check if they got it right, if we give the groups complex shapes.

10	Teacher 4:	I feel it is most appropriate to take “the house-task”[7].
11	Teacher 1:	Yes, but at the same time, they can be too caught up in that task.
12	Teacher 4:	Yes, they can.
13	Teacher 1:	So, I do not think we give them “the house-task”, we can rather find other geometric shapes.
14	Teacher 3:	So, is there a correlation between base area and volume. (Sitting and writing, reading what she has written)
15	Teacher 1:	Mm
16	Teacher 3:	We are wondering whether they can explain what they are doing in their calculation or not. Then they must be able to show their understanding, explaining to each other how they have done it.
17	Teacher 1:	Mm
18	Teacher 4:	I think it is a good idea that they can explain to each other.
19	Teacher 1:	Yes, I think so too.
20	Teacher 3:	I can write, “They must explain the procedures”.
21	Teacher 4:	Most likely, one of the group members is able to solve the task and explain how.
22	Teacher 1, 3:	Mm

Analysis and discussion

Analysis of the teachers’ discourse identify three potentially relevant categories of the teachers’ discourse on teaching: 1) narratives on students, 2) narratives on students’ learning and 3) narratives on teaching practice. The first relates to subjectification, whereas the two latter relate to teachers’ different expectations of the students’ routines and creating dialogues.

Narratives on students

As the teachers predict how students will respond to the given task, they are concerned about the significant gap in students’ mathematical skills and understanding. Teacher 4 (1, 4) and Teacher 3 (5) refer to the students as “students at the top” and “students at the bottom”. The dialogue (1–7) illustrates how the teachers categorize the students based on their grades. This kind of statement of

[7] “The house-task” is a practical task, which the students have elaborated on in an earlier project. They are supposed to build a model of a house, including mathematical calculations, in order to complete the task.

the students' understanding, describing and putting their skills as something (or someplace) the students are, on the behalf of their former actions, is by Sfard's (2008) term referred to as subjectifying. Another example of this kind of subjectifying is given by the teacher talking about "weak students" (7). Talking about "students at the bottom" or "weak students" is problematic as it might tend to function as a self-fulfilling prophecy (Sfard, 2008). If you are initially labelled as a student "at the bottom", it is hard to motivate the student for further development.

Narratives on student learning

As visual mediators, one teacher wants to hand out different three-dimensional figures to each group of students. Teachers' prediction indicates that "weak students" choose rectangular prism, whereas "able students" choose complex shapes. In addition, the teachers have different expectations of the students "at the top" and "at the bottom". Firstly, because the teachers are pleased if students "at the bottom" recall previously endorsed narratives (working on familiar shapes, e.g. a shape with rectangular base area). Secondly, following Sfard (2008), an interpretation can be made of the utterance by Teacher 4: "And then it is the bottom there, it is enough just to do the calculation" (4). "Just to do the calculation" (4) can be seen as a deed. If the students know the formula, they are able to calculate the volume without necessarily understanding the relation between base areas and height.

What does it mean "to understand the concept of volume", and how are the teachers going to find out whether the students have learned something or not? The first two lines in the second transcript (8, 9) indicate that in the teachers view, if the students calculate the volume correctly, they know the relation between volume, base area and height of the shape. These two first lines, viewed as separate utterances, one could recall as a deed (cf. Sfard, 2008). However, Teacher 3 (16, 20) and Teacher 4 (18) later stress that the students should explain their procedures to each other, and they expect students to use endorsed narratives. In this way, they want to observe students' utterances and evaluate their reasoning. The teachers thus, have different expectations to their students' routines. Routines for the "weak" students can be seen as a deed, and to recall narratives (4), in contrast to the "able" students that supposed to sustain and construct narratives as in an explorative routine (21).

Narratives on teaching practice

In the conversations from the planning meetings, narratives indicate that the teachers consider learning as participating in an activity. In the "house-task" (10, 13), students with practical skills were as much participants in solving the task (building the model) as the students who did the mathematical calculation. The teachers want students to explain to each other their mathematical thinking and understanding. Also, they claim that by listening to fellow students, it is easier to construct, substantiate or recall endorsed narratives. If some students do not understand the task, the teachers stress that in the learning process, the students' own mathematical language can be more helpful for fellow students than the explanations provided by themselves as teachers. A main focus in their discourse on teaching practice is to facilitate dialogue among students, where students help other students to develop new endorsed narratives (16, 18, and 20). Narratives from the reflection meeting, held after the lesson was conducted, is an account of the teachers' observations. The observations revealed that the students only focused on *what* the right answer was, not *why* it was correct, and mathematical conversations between students did not occur. This was one of the main goals of the teachers, they wanted to create dialogues that invite the students into explorative routines.

Concluding discussion

From analysis of these three categories of teachers' narratives on: students, student learning and teaching practice, three interrelated characteristics can be identified in the teachers' discourse on teaching. Firstly, there is the issue of how teachers talk about students' skills as something static – a condition – and categorize the students as being "at the top" or "at the bottom". There are different expectations about what the students are able to achieve. According to Sfard (2008), this kind of subjectification might have a negative impact on student learning, because it tends to function as a self-fulfilling prophecy and affect students' identity (see e.g., Mosvold, 2015; Mosvold & Ohnstad, 2016). Secondly, the subjectification of students influences the teachers' expectations of the students' performances (routines). The teachers predict that students "at the bottom" only understand plain shapes such as rectangular prisms and calculate the volume without understanding the relation between the base area and the height of the shape. Together these two aspects affect the teachers' teaching practice, for instance in the way the teachers organize the students to create student dialogues, and how they decide to differentiate. My interpretation of their way of differentiating, is that the teachers want to facilitate all students' opportunity to construct endorsed narratives (cf. Sfard, 2008), students need different three-dimensional shapes to work on, based upon their already known narratives. Thirdly, the teachers assume that learning develops through conversations between "able" and "weak" students, in which students use their mathematical language and explore their mathematical thinking and understanding. Warwick et al. (2016) support this kind of thinking on learning through dialogue. In their study, they accentuate how LS contributes to making a dialogical space amongst teachers in order to improve future teaching intentions. They advocate that inter-thinking – thinking out loud together – creates a good learning environment for the teachers. Analysis of the teachers' discourse in this paper indicates that the teachers desire this kind of learning environment for their students. This study claims that the findings reported on, might be of interests in further studies of teacher learning in LS. The three characteristics are examples of how teachers' discourse on teaching can be identified. If LS processes contribute to change these characteristics, thus change the teachers' discourse on teaching, interpretations of teacher learning in LS can be made (i.e. learning, cf. Sfard, 2008).

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Adopting the developmental research cycle in working with teachers

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This collaborative inquiry aims at learning to understand how teachers in schools and a mathematics teacher educator develop their work through participating in a developmental research project. Seven primary teachers worked at improving their mathematics teaching and researched their practice together with a teacher educator for three years. A model of a developmental research cycle, with two interconnected cycles of development and research, was used as the framework for the research. The main findings are that partnership between teachers in schools and teacher educators, where the knowledge both parties bring into the project is mutually respected, can add to our understanding of teacher development.

Introduction

Over the last two decades Icelandic teachers have been under growing pressure to adapt their work to changes in new curriculum guidelines and laws for schools. Teachers are now expected to meet the needs of diverse groups of children and improve their teaching competence. This paper reports on findings from a developmental study in which seven teachers in primary grades and a mathematics teacher educator collaborated. The aim was to investigate how the teachers and the teacher educator collaborated in researching their own practice, and the ways in which this collaboration impacted the work of both parties. The goal was to identify approaches to teacher education that could support teachers in meeting the needs of diverse learners in the mathematics classroom.

In former work with teachers I had found that many teachers lacked confidence in teaching mathematics in diverse classrooms. They lacked experience of focusing on mathematical processes and felt incompetent in using these approaches in inclusive schools (Guðjónsdóttir & Kristinsdóttir, 2011). In inclusive schools, emphasis is placed on the perspective that everyone is respected and noticed, their participation is valued, and an opportunity is created for them to achieve and show their strengths (Ainscow & Miles, 2008).

I decided to work with a group of teachers with the aim of assisting them in reflecting on the mathematics learning in their classrooms. The reflection should concern both their students' learning and their own learning, with regard to which I encouraged critical reflection. I contacted two schools with diverse groups of students. The study thus involved a) seven primary school teachers in grades 5-7 who examined their own practice as mathematics teachers, with my support and b) myself, where I focused on the collaborative process itself, as a whole, as well as the development I underwent throughout the research process, as a teacher and a researcher. Over a period of three years, we met at workshops on a monthly basis (17 in total) where we solved mathematical problems and discussed and reflected on our collaborative investigations. We also discussed the teachers' stories from their classrooms and reflected on their students' learning, as well as discussing how their experiences reflected findings from other research on mathematics teaching and learning and on teachers' professional development.

The focus here is on the processes that emerged throughout the project and the use of the developmental research cycle in answering the question: What learning processes emerge through long-term collaborative inquiry undertaken by classroom teachers and a mathematics teacher educator? The study involves a process through which teachers research their own practice with my support and myself researching this collaborative process and my development as a researcher.

Methodology

A model of a developmental research cycle as put forth by Goodchild (2008) was used as the framework for the research. In this model there are two interconnected cycles of development and research that model a linked dialectical growth of theory and practice. The model is based on Gravemeijer's (1994) description of developmental research, though with particular emphasis on the cyclical process between development and research. This diagram is presented in Figure 1.

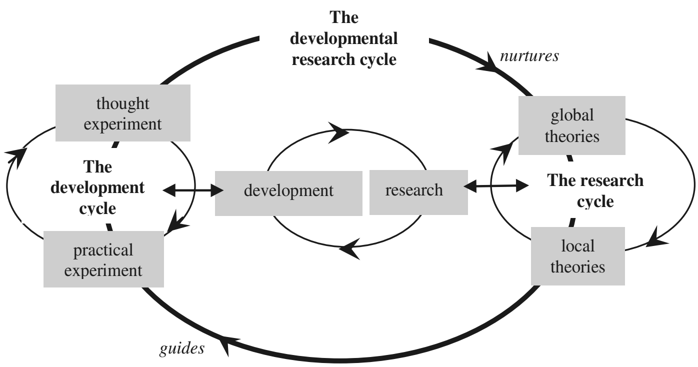

Figure 1: The developmental research cycle (Goodchild, 2008, p. 208)

The developmental research cycle represents two interconnected cycles that model the dialectical growth of both theory and practice. Components of the developmental cycle are presented as a thought experiment to accompany a practical experiment. The research cycle moves between global theories that are concretised in local theories. The cycles are interrelated; local theories are tried out in practice, when thinking through the consequences of some action and then implementing it in harmony with conclusions from the planning process. This leads to the adjustment and analysis of the local theory that informed the action, which then in turn, results in a reconstruction of the global theory. Consequently, the research cycle guides the development cycle, which in turn nurtures the research cycle (Goodchild, 2008; 2014).

To learn about the teachers visions for the project and the cultures in their mathematics classrooms I interviewed them and observed their classrooms at the outset of the project, after the first year, and one year after the last workshop. Data was collected of videotapes from 17 workshops, audiotapes from interviews and notes from classroom observations. The analysis of the results started at the outset of the study as a spiral of analysis developed over time (Creswell; 2007). The results from the analysis guided the process of the study as they were used to resolve what to focus on at each phase of the project.

Theoretical framework

The local theories that guide the study are based on former findings from research with colleagues where we found that teachers need support in reflecting on their students' learning of mathematics as well as on their own way of mathematics learning (Guðjónsdóttir & Kristinsdóttir, 2011).

The global theories that guide the study are sociocultural, in the Vygotskian sense, that individual cognition develops when people change their ways of understanding, perceiving, noticing and thinking through shared efforts with others (Vygotsky, 1978). During this development, they build on the cultural practices and traditions of communities such that participation is seen as both a social process and a personal experience (Lave & Wenger, 1991). When developing learning communities where the diverse background of the participants is respected, everyone's contributions must be valued. Jaworski (2006) argues that collective learning develops through a mutually reflexive process of knowledge growth between individuals and a community in which co-learning partnership is cultivated. Thus, through the process of sharing experiences and developing norms, the community provides supportive structures for individual inquiry and acts to mediate knowledge so that knowledge grows within the community, as well as for each individual.

Askew (2015) argues that in order to foster an inclusive approach in attending to diverse learners needs, it is important to begin with learning communities, rather than taking the individual as the starting point for planning learning experiences. In the learning communities, teachers work with the collective construction of mathematical knowledge while still ultimately addressing the needs of the individuals within that community. This is the position I took in working with teachers when attending to their different needs for improving their teaching.

Reflecting on and in one's own practice is an essential feature of teacher development and in inquiring into one's teaching. Inquiry refers to critical reflection and can be seen as a mode for critically reflecting on mathematics learning, mathematics teaching and research into the teaching of mathematics. Jaworski (2008) argues that in a community of inquiry the inquiry is seen both as a tool for developing practice and as a way of being in practice, and thus, inquiry becomes a norm of a community of practice. When individuals are encouraged to look critically at their own practices and to modify these through their own learning-in-practice, there will be a shift from "community of practice" to "community of inquiry". Through the shift a perspective emerges in which reflective development of practice by practitioners, individually or in groups, can be seen to result in the development of community.

The participants in the study belong to different communities, within a complex landscape of learning (Wenger-Trayner & Wenger-Trayner, 2015), that all affect how we interpret the learning that developed within our community and thus our own individual development as mathematics teachers and a teacher educator. The teachers' background and the experience they bring into our community, shape our collaborative work.

The quality of mathematics learning in classrooms depends on the teachers' capability of building communities that enable learners to develop their mathematical competences. To be able to enrich learning in mathematics classrooms, teachers need to be competent in approaching their teaching in such a way that all participants in their classrooms will gain from it. The competency model developed by Niss & Højgaard-Jensen (2002) explaining the ability to develop one's competency as a mathematics teacher was adopted for the project. The development of teaching in classrooms is seen as dependent both on the teachers' knowledge and their ability to learn together with others, both their students and colleagues. The teacher learns from participating with the learners about her own learning and of the collective learning in the classroom that shapes the classroom culture.

Findings

The developmental nature of the study entailed that the structure was flexible. The protocol for each of the workshops was based on the teachers' expectations for what to attend to and they were urged to come up with proposals for activities. I offered them tools to work with, entrusted them to decide what they found helpful and challenged them to rethink their teaching habits by participating in investigations into their practices. A sequence of six themes developed as our co-learning progressed. The following examples are representative for the learning processes that developed through our long-term collaborative inquiry.

Initial steps to an investigative approach: Based on the local theories and on the teachers' visions for the project our collaboration started with problem solving, discussions on our findings. The teachers' related to experiences from their classrooms. Vala was prompted to tell us about two boys in her group who always write checkmarks or count things at hand when they calculate. Dóra added that her pupils were not always willing to draw. They often said: "I think this in my head".

Edda: But can they explain it? That is often difficult.

Dóra: I know. It is often difficult to tease it out.

Jónína: Why is it difficult? Why is it difficult to explain one's thinking?

Vala: Is it not just a lack of practice?

Reflective practice, hindrances and opportunities: We continued exploring with problems I brought in and the teachers told about their work. I encouraged them to write about what they had noticed in their classrooms and to analyse their findings, based on a protocol for case and commentary writing (Kruger & Cherednichenko, 2006), that I introduced to them. A few days later I received an email from Gróa and two of her colleagues. They wrote that they were sitting together and reflecting on how they could explain their work. They felt, as teachers with wide-ranging experience, they were capable of assessing their work without writing in detail about it.

In my reply, I said that I was aware of the fact that in their work they always reflect on and attend to the needs of individual children. The goal of the task was to urge them to reflect on individual cases and support them in analysing what they have learned about their work. I urged them to write their own notes and keep for themselves. We would continue to discuss our teaching at the workshops.

A focus on interactions in mathematics classrooms: As the project developed we focused more on interactions in mathematics classes both by exploring together at the workshops and by discussing the work in the teachers' classrooms and analysing their cases together. The teachers planned visits to each other classrooms, observed and participated in lessons. They soon discovered that if they planned their visits together and met after them to discuss they gained insights into how to advance interactions in the classroom. Pála said:

> Yes, I felt we discussed this, how we grouped the pupils and how we are reflecting on each and everyone's learning. How we can activate them and how we have succeeded.

Many of the pupils in the teachers' classrooms are newcomers and have not mastered the Icelandic language yet. Vala mentioned that they have difficulties in discussing their work but the mathematical symbols help in communicating about their work. Dóra added:

> One girl did not understand anything and started crying. Then I talked to her with mathematical symbols. That is how we made contact and developed mutual trust.

The original plan for the project was approaching the end. The teachers felt that they were beginning to develop their practice and proposed to proceed for at least another year.

Focusing on pupils' learning in the classroom: Inga, a special education teacher, was concerned for her pupils' lack of self-esteem in mathematics: "These kids show such little initiative and they are so uncertain of their ability to learn". She was worried that the children who have learning problems are often told that they do things the wrong way and shared her concerns with us:

> Instead of getting the chance to explain their thinking, the teachers tend to explain to them again and again in a way that they do not understand. This makes them uncertain about themselves and they want the teacher to tell them what to do. But when the teacher listens to them they feel that they are capable of explaining their thinking like other children.

Teacher reflections lead our discussions: The teachers were taking more responsibility for what to attend to at the workshops. Gróa told about cultural days in her school and how she had decided to work with mathematics when an opportunity presented itself. We discussed and analysed her story.

Jónína:	When we as teachers think as you do, reflect on our conditions and then respond to the situations, what are we then doing?
Gróa:	What am I doing? … You do not think when you are in the action, you just, you see that something needs to happen.
Jónína:	Yes, and why do you do that?
Dóra:	So, the pupils will understand.
Gróa:	To try to make the pupil understand. Particularly when you see that one pupil understands, and the other does not. What can I do?
Jónína:	This is what we are looking into, how professionals work, your response did not come out of the blue. … And this is what teachers do. What I am asking you to do is to look into how you do this. What you reflect on and how it is represented in what you do. How you come to these conclusions, because when you are in the classroom you are not thinking about how you reached the conclusion.

Towards an investigative approach and inclusion: At our final workshop Pála told about a mathematics lesson in which her pupils worked with word-problems. They were required to write their solutions to the problems with algebraic expressions. She gave examples of the pupils' discussions about the problems and how they wrote the expressions. She had recorded these examples, showed us how the pupils calculated and how she interpreted their thinking about the problems.

Pála was particularly keen to hear my opinion with regard to the way she had accepted her pupils' way of solving a problem instead of telling them to think about it in the same terms she did. We discussed how the value of the unknown variable in Pála's equation was different from the value in her pupils' equation and she was confident in accepting their way of writing it.

Pála:	These were just my thoughts. I found it interesting to see how they understood and thought about this.
Jónína:	Yes, and their discussions about what they did.
Pála:	Yes, they discussed a lot. They all enjoyed this and found it easy.
Jónína:	And still this is algebra. … This problem was in the form of a story that the children could visualise. When we teach this in abstract form without context many pupils have difficulties with this abstract form.
Pála:	It is important that the problems are about something, something they know.

Pála's story mirrored a discussion where she inquired into mathematical problems with her pupils and nurtured reflective discussions. In the final visits to the teachers' classrooms and interviews with them I found that they had been strengthened to review their work and taking on an investigative approach in their classrooms and into their own practice. They also had questioned norms at their schools, like testing children on memorising facts and grouping them into ability groups in mathematics classes and taken actions to influence the culture in their schools about these norms.

Discussion

The sequence of six themes developed as we moved between the developmental and research cycles and reflected on the global and local theories we based our research on as we developed our learning community. The mutual trust we built supported the learning process and nurtured our collaborative progress as tensions arouse. The teachers were concerned about their pupils' way of learning mathematics and I challenged them by asking them probing questions to support them in inquiring into their practices (Jaworski, 2008). The first confrontation in our work presented itself when I required the teachers to write about and analyse cases from their classrooms. I realised that I had been too quick to step into the role of a teacher and needed to respect that these teachers are professionals who belong to other communities within their schools (Wenger-Trayner & Wenger-Trayner, 2015). I decided to give space for discussing the cases at the workshops and collectively analyse them. I needed to align myself to the community we were shaping together and respect the teachers' values (Lave & Wenger, 1991).

When we critically reflected on our work the teachers learned how inquiring into their teaching can support them in developing their practice (Jaworski, 2008).

Their reflections prompted them to find ways to include all learners in the mathematics discussions in their classrooms (Askew, 2015). By sharing their work, they developed their competency as mathematics teachers when reflecting on their experience of communicating with their pupils and sharing it with us (Niss & Højgaard-Jensen, 2002). Additionally, they nurtured our learning and added to building our community of practice (Vygotsky, 1978; Lave & Wenger, 1991).

By bringing in a story from cultural days Gróa cultivated our co-learning partnership (Jaworski, 2006). I responded to her story and urged her to critically reflect on her contribution (Jaworski, 2008) and analyse her own learning with our support, thus responding to her reluctance to write about it. Pála's story from her classroom gave an insight into how she had changed her way of teaching as she had gained experience in solving problems together with us and discussing her own thinking. Instead of describing step by step to her pupils how to write algebraic equations, as she had done before, she felt confident in accepting their way of doing it and discuss with them how their thinking about the problem was different from her own. Her story was an indication of that we had succeeded in building a community of inquiry where we reflected on our practice and this experience affected the culture the teachers built in their classrooms (Jaworski, 2008).

The teachers aligned critically to established norms at their schools as they adopted an inquiry stance to their teaching (Jaworski, 2006). The tensions that arouse within our community became a source of creative innovations for the teachers and myself as we learned to question established norms within our professional culture and initiate creative innovations (Goodchild, 2014).

Conclusions

Adopting the developmental research cycle (Goodchild, 2008) was vital in analysing the gradual progress of our community building. The spiral of reflecting on the developmental cycle in reference to the research cycle supported actions taken in our collaborative work. As we learned to accept the knowledge that each of us brought into the community and think of ways to cultivate it, the developmental cycle affected the research cycle. The global socio-cultural theories about co-learning and community building affected the local theories about teachers' need to rethink their own way of exploring with mathematics and work with their pupils. The local theories in turn affected the developmental cycle when we decided what to attend to at the workshops and how to communicate about our work (Goodchild, 2008).

The results of this collaborative inquiry into mathematics teaching and learning showed that partnership between teachers in schools and teacher educators, where the knowledge both parties bring into the project is mutually respected, can add to our understanding of teacher development. In particular when the aim is to support classroom inquiry where pupils in schools learn mathematics through exploration, as conceptualised in Jaworski (2006). The learning gained from the study aligns with the findings of Askew (2015) about teacher development that aims at inclusive practices and mutual understanding. Teachers need opportunities to develop and enhance their knowledge about teaching and learning in an environment that reflects the very same aspects they are expected to foster in their own classrooms.

The overall results indicate that teachers are professionals who can work at developing their mathematics teaching in order to cultivate inquiry in mathematics within their classrooms when provided with support on discussing and interpreting their work in classrooms. The findings support the view that teachers' opportunities for further empowerment to participate in educational research needs to be facilitated.

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In-service teachers' positioning when discussing the body mass index

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In this paper, we present insights about the complexity of in-service teachers' shifting of perspectives when indices and the society's use of indices are discussed. The data are collected from an in-service course on numeracy for teachers in primary school. The concepts polyphony and centripetal and centrifugal forces from Bakhtin's dialogism are used as theoretical framework to investigate how the teachers position their utterances and how they move between different perspectives, aims and ways of expressing knowledge about indices. The in-service teachers position their utterances from the perspectives as teachers, as students who focus on learning, and as critical citizens who can be a mother, friend etc. The findings indicate potentials for learning mathematics and critical awareness in the tension between ways of understandings and mathematical representations expressed from different positions.

Background

The focus in this paper is to investigate how in-service teachers communicate and express understanding about indices when they work with a task that encourages them to discuss different aspects of the body mass index (BMI). The purpose is to gain insights into teachers' different mathematical voices and the polyphony that develops when participants discuss indices and position their utterances from different perspectives. Polyphony is a key term in Bakhtin's (1984; 1986) dialogism and concerns how dialogues can be described and analyzed by investigating the multiple voices taking part. It is a space where different opinions, understandings and linguistic settings are expressed. Through choice of words, expressions, voices and gestures, participants can position their utterances. Rangnes (2012) identified how aims for doing mathematics in different settings influence how the participants speak in and about mathematics, and which tools they see as appropriate to use. In the school setting the aim was to learn mathematics, while in a construction enterprise the aim was to use mathematics as a tool to do their work. In our study, the in-service teachers participated in a numeracy course in which they were expected to learn about indices, their use in the society, and how this topic could be implemented in primary school. The insights into their conversations can contribute to teacher educators' knowledge of how positioning and different perspectives influence discussions about complex issues such as indices.

Skovsmose (1994) described how mathematics influences and structures how we think and behave, influences that can be both explicit and implicit. Behind new technologies, there is often complicated and invisible mathematics that only a handful of experts have access to (Skovsmose, 1994). It takes critical citizenship to unpack how mathematics forms our lives. The Norwegian curriculum in mathematics states: “Active democracy requires citizens who are able to study, understand and critically assess quantitative information, statistical analyses and economic prognoses” (Ministry of Education and Research, 2010, p. 1). Educating children to become critical citizens requires teachers who can develop methods for introducing and stimulating critical citizenship in mathematics classrooms. As teacher educators, we have a responsibility to support such development. However, we need more understanding about how teachers (re)act when discussing issues where mathematics is involved. In this study, indices are brought into focus because they often involve mathematical models that can have an impact on people’s lives.

An index can be interpreted as an average value based on measures of two or more quantities from a relatively large sample taken from a population or a set. The present BMI-formula is a result of scientific attempts to derive a suitable measure for relative body weight. The formula BMI=w/h^2 is a person’s weight in kilograms divided by the square of the person´s height measured in meters. In the 19th century, Quetelet investigated the weight and height of conscripts. He found that weight was approximately proportional to height squared (Oliver, 2006). This was criticized and other formulas like the Ponderal index w/h^3 and the ratio w/h were suggested, but in the 1970s, the BMI-formula as we know it today was accepted as the best formula. During the last decades, the BMI-formula has again been criticized because weight does not scale quadratically with height.

Teaching about indices can be viewed as an example of working with mathematical models that are build to structure our society. Indexes are used to measure different phenomena and, in that way, to make important decisions that have consequences for people’s lives. The BMI is a particularly interesting example, because the number you get when you use the formula has the power to define people as underweight, normal, overweight, or obese. Furthermore, the BMI is an index people face in the news, in the health care system (also in schools), and as a political factor that influences the decision-making in health care politics. The components in the formula are relevant for the teachers to investigate. The index’s validity in different contexts, whether it is a good measurement of a phenomenon or not, is also something that can be discussed. We consider the use of tasks involving indices as a fruitful entrance for students’ critical discussions, to both evaluate the validity, discuss mathematics in the index, and discuss their use in the society.

Theoretical framework

From a Bakhtinian perspective, an utterance will always be an answer, but it will also demand an answer. It involves a statement and an evaluation. When a teacher explains a concept in a classroom, he will take into account his students' possible questions and responses. In that way, the students' voices can be identified in his explanation by his choice of words, mathematical representation or gestures. According to Bakhtin (1986), the voices of others in an utterance can be described as a polyphony of voices. Bakhtin did not explicitly define polyphony (Morson & Emerson, 1990), but he described it as a space in which different opinions, understandings, and linguistic settings are expressed. The voices can be identified through content, expressivity and purpose. If the teachers talk about BMI in terms of learning, in terms of the mathematics involved, or in terms of teaching children about it, we can expect them to emphasize different topics and express mathematics in different ways.

Bakhtin used the concepts *centripetal* and *centrifugal forces* to describe the tension and the dynamics between voices (see Bakhtin & Holquist, 1981). Some forces point to the center, they are the centripetal forces. Voices that point towards the right answer, towards consensus and a mutual aim, can be regarded as centripetal forces. Centrifugal forces point outwards and contribute with diversity and even more voices. A lengthy reflection that to a little degree is restricted to a given task can be regarded as a centrifugal force. The two forces can interplay between utterances but also within a single utterance. A dialogue needs both forces.

Utterances must be considered in light of their social, cultural and historical context (Bakhtin, 1986). Utterances in different mathematical practices comprise language shaped by underlying history and culture, a language developed according to the aims and motives of the practices. Within a practice, participants develop social norms for which language to use, which tools are allowed, and which approaches are regarded as best (Rangnes, 2012). In mathematics education practice, oral explanations, written symbols, manipulatives, real-world events or pictures are used. The ability to choose meaningful representations, to move between different representations, and to combine them in order to obtain a better understanding of a mathematical concept or situation, is an important component of mathematical competence (Niss & Jensen, 2002). From a Bakhtinian perspective, differences between voices are considered as opportunities to open up the dialogue for critical discussions and learning. The aim is not to overcome differences by synthesizing for joint agreement and, through that, encourage uniformity (Barwell, 2016). From a dialogical perspective, the ability to be critical can be identified through the tension between voices. Alrø and Skovsmose (2006) argued that criticism, if directed at your own or others' work, reflects an engagement and a willingness to participate. Thus, students' criticism can be considered as fundamental for taking ownership and making choices.

Method

The data for this study is collected as part of a project aiming to develop the teaching of critical mathematics with indices in teacher education. The project group had two meetings where ideas about teaching indices were discussed. Two teacher educators from the group, one from mathematics education and one from social science, collected data as a part of their own teaching about indices in an interdisciplinary course on numeracy across the curriculum. Twelve primary school in-service teachers from different parts of Norway, with varying educational backgrounds in mathematics, were enrolled in the course and agreed to be a part of the project.

The teacher educators organized one day on campus with the in-service teachers. One of the teacher educators lectured about indices in general, exemplifying with the human development index. Afterwards, the in-service teachers were divided randomly in two groups and given approximately one hour for discussion. They were given a sheet with questions about indices in general and BMI in particular, the mathematical construction and appropriateness of the BMI, the uses and the meanings of the BMI and other indices in society, and possible uses of indices in their own practice.

The BMI was chosen as focus of attention for these discussions because of its relatively simple mathematical formula and its extended use and presence in the media. The two teacher educators observed one group each and interrupted only to make sure that all the points were taken into consideration in the discussions. A picture of a rugby player was provided together with the questions, and with a BMI of 35.98, he would be placed in the overweight group. This was done to direct the attention to the fact that BMI does not take into account that muscles weigh more than fat.

The in-service teachers' discussions were audiotaped and then transcribed. The focus in this paper is on one of the groups in order to make the analysis easier to follow, and because it makes it possible to draw a comparison between the utterances. The analysis started with noticing that the teachers seemed to position their utterances. In order to identify in more detail what kind of positions or positioning took place, the teachers' arguments and choice of words were analyzed. In this paper we include only three excerpts due to space limitations, and these three excerpts are chosen because they show how the teachers enter and move between particular positions. The positioning of an utterance can be reflected upon according to aims, motives, and norms of the practice the teachers are part of, and to the content and context they include in their utterances. Questions guiding the analysis refer to students, classroom, school topics, societal issues, the use of BMI outside the classroom, language and words from a mathematics learning perspective. The teachers' use of mathematical concepts, their choice of mathematical representations, and the contexts they refer to, are used in the process of identifying and characterizing different positions and how mathematical understanding is expressed.

Analysis and discussion

Three excerpts are analyzed and discussed in this section. In the first excerpt, the teachers discuss what BMI is. Next, they discuss how a public health nurse uses the BMI. The last excerpt concerns the in-service teachers focus on how h^2 in the BMI formula can be understood and explained.

What is BMI?

This first excerpt is right from the beginning of their discussion:

T1:	Question 1a. What is BMI? [Reads the task]
T2:	BMI stands for body mass index or kroppsmasseindeks [body mass index in Norwegian].
Several:	Mm.
T3:	Balance between height and weight …
Several:	Mm.
T3:	for people.
T2:	I don't quite remember the formula, but it had something to do with weight and height squared or something like that.
T1:	Is it height times height divided by weight or something like that? Or the other way around?
T2:	It's for sure a formula to find the ratio.

They start by reading the first question from the task sheet and try to clarify what BMI is. Then T2 explains what the letters in the acronym stand for and directs by that the discussion towards a joint aim of finding an answer to the question – the utterance has a centripetal force. In the following utterances, the teachers focus on the mathematics in the formula by expressions like "balance between height and weight", "weight and height squared", and "height times height divided by weight". This indicates a search for the formula of the BMI. The expressions "Or the other way around" and "I don't quite remember the formula" show that the teachers emphasize the formula, to remember it. A key purpose of these utterances concerns learning. Two of the teachers end their utterances with "or something like that", and such choice of words indicates they are thinking aloud. Expressions like "I don't quite remember" and the suggestion about the ratio weight and height squared followed by a suggestion about the inverse ratio, show the teachers' effort to make sense of the mathematics behind the BMI. They build upon each other's ideas by partly repeating or reformulating what the preceding participant says. In the end, T2 states, "it's for sure a formula to find the ratio", and emphasizes by that what they are sure of when it comes to the BMI.

The teachers' search for an answer or a formula they know exists, their thinking aloud and emphasis on remembering, indicate that they position their utterances as learners and students. The acronym BMI and the ratio "balance between weight and height" are two different representations that can refer to different aspects of the BMI concept: BMI as a value representing body mass and the other as a balance between two measurements. The teachers use words such as formula and ratio, which are concepts emphasized in the mathematical curriculum. The aim, to find the right formula, works as a centripetal force. The teachers' openness about their insecurity can be regarded as a centrifugal force that open up for critical voices and more than one understanding of the concept BMI. This openness facilitates the continuous aspect of their dialogue.

The public health nurse's use of the BMI

The following excerpt takes place after four minutes of the discussion.

T1:	I know that when my girls were in grade three, we received the printout in an envelope from the public health nurse and then the curve was from since they were born, from level one, quite simply. It is about looking into the curve to see if they follow their curve, they are interested in. Then we could see and discuss, but I know that in another school a friend of mine was called in … They had to do some small changes … she was too much above, she was outside her range. And it was a good thing, even though it was a bit tough message to get. They did some changes, and then …
T2:	Was the curve based on BMI … the one the public health nurse used, or …?
T3:	Probably weight, I guess.
T2:	They have always measured that.
T1:	It was weight according to height.
T2:	That is BMI.

T1 uses an example from her own family when talking about an envelope from the public health nurse, and continues by talking about a friend who was called in (to the health nurse). This narrative involves polyphony. In the beginning, there is the voice from the public health nurse identified through the "printout" with "the curve". Later, the friend's voice is identified through the changes the friend's family had to do. The voices of the health nurse and the friend are intertwined with T1's voice as a parent and the reflection about the friend having to do "small changes" as a good thing. This is a polyphonic example where the in-service teacher positions the utterance from a parent, a friend and a citizen perspective who critically discuss different sides of the use of BMI.

If the aim is to answer the question "What is BMI", T1's first utterance can be regarded as a centrifugal force that pushes the dialogue in a new direction. However, T1 refers to the "curve", and if they "follow their curve". This is yet another representation for the BMI, in addition to the formula and the number resulting from it. The curve can be considered as a graph to be followed over time, or as something you can be "too much above" at a given time. In addition, the friend's family "had to do some small changes". This shows an understanding that the curve can be influenced, you can do something to get back on track. The curve is something the teachers bring into the discussion – it was not mentioned in the task.

T2 asks "Was the curve based on BMI … the one the public health nurse used, or …?" By questioning if the curve was based on BMI, she seems to search for the link between the curve that the public health nurse uses, and the task about BMI they are working on. It is a turning point in the discussion, the participants' utterances change from a citizen to a student perspective. The utterance also acts as a centripetal force that pushes the discussion back to the initial aim of understanding what BMI is. The answer from T1, "it was weight according to height", is understood by T2 as a confirmation of the relationship, because T2 then concludes: "That is BMI". The polyphony in this excerpt, the voices positioned as citizens and students, brings forward different representations and nuances about what BMI can be for the teachers in their attempt to grasp the BMI concept.

Area measurement?

This last excerpt takes place after 22 minutes. Prior to this excerpt, the teachers positioned themselves as citizens by discussing the use of BMI in society and its limits.

T4:	I think it's a little bit difficult to think that you
Several:	Yes …
T4:	measure us in area.
Several:	[Laughing] Surface area [in Norwegian: flatemål] yes. [laughing]
T4:	Talking with students about this and then you take kilos and then you divide by the area of the body …

To make sense of the BMI concept and its formula is a recurring topic in the discussion. In this excerpt, T4 seems to try to find out what the index really measures. He refers to the use of the area measurement, and says he finds it difficult to think that way. The laughter indicates that several of them recognize and question the same issue. They translate area into "surface area" (flateinnhold) which is a Norwegian word for area often used when introducing the area concept in school. T4 associates area measurement with school and his work when he takes a teacher perspective by saying: "Talking with students about this and then you take kilos and then you divide by the area of the body". The words "talking with" indicate a conversation where students can ask questions and give responses. He is picturing how he would explain something he himself struggles to make sense of. In this utterance, he is answering a possible question from his students with an instruction about what the students have to do. However, by saying "divide by the area of the body", he also adds his interpretation of what h^2 can be. Thinking about teaching this to students, and having in mind students' possible responses, stimulate teachers to reflect about their own understanding.

The polyphony is present in T4's utterance where BMI is identified as something the teacher is struggling with while trying to find an explanation that both he and his students can accept. However, if a centripetal force is seen as finding a correct explanation in a mathematical sense, then his explanation "divide the area of the body" can be seen as a centrifugal force that disturbs an understanding of index as an average value based on measures of quantities from a sample taken from a population.

The teachers' positioning

The teachers bring forward different understandings of the use of BMI when positioning their utterances from different perspectives. As learners, the aim was to find out what BMI can be. As citizens, they use narratives about how BMI used by a health nurse can influence a family's lifestyle. When positioning their utterances as teachers, the explanation of the formula was the focus of attention. They took ownership by including their own experiences and by critically discussing how BMI could be understood and how the use of BMI in society could affect people's lives.

When discussing from their positions as students, teachers, and citizens, the teachers use the following representations for BMI: 1) the acronym and English and Norwegian words for BMI; 2) "weight according to height"; 3) curve, "to follow their curve" (from a current situation perspective and as development over time); 4) as a value, a number; 5) as a formula, w/h^2; and 6) as area, surface area, divide weight by the area of the body. These diverse representations are linked to different positions in the discussion, and there can be tensions between the representations. To follow a curve can support an understanding of the BMI as a function and a development over time. This can be problematic when the aim is to develop an understanding of the BMI as an average value based on measures of quantities from a sample taken from a population. Seeing the formula as weight divided by the area of the body can interfere with an understanding of BMI as a number expressed by a ratio. However, the different voices, the identified polyphony, can also be regarded as a potential for learning. Through different positions and understandings, the teachers open up for a deeper understanding rather than simply accepting the mathematical formula.

Concluding comments

Polyphony takes place through the teachers' change of positions, but also within utterances. It is not something static, you move in and out of positions and the utterances are positioned through different perspectives. Centripetal and centrifugal forces are important components of this polyphony. Centripetal forces come to the fore when the participators focus on finding correct answers, while centrifugal forces are characterized by utterances bringing in to play diversity and new directions. These two forces generate a dynamic dialogue that gives space to critical voices. To be critical about an index means to reflect about it from different perspectives and to use different voices. Using open tasks about BMI, opens up for different voices where critical reflections are present. It also opens up for different understandings and mathematical representations, which provides a potential for deeper understanding of the mathematics behind the index.

There is no doubt that the questions the teachers were given influence how they position their utterances. For instance, the formulation of the first question about what BMI is, invites the teachers to position themselves as students who will answer it. However, the teachers could very well use arguments from a teacher perspective regarding not only potential use in school, but also when they discuss what BMI is and its role in society.

The participants' shifting of positions brings about a polyphony that, based on a Bakhtinian perspective, can offer a valuable contribution to their learning processes. To understand these perspectives, the shifts and interplay between different perspectives, is an important part of teacher educators' knowledge about facilitating discussions about topics like indices that promote students' understanding of mathematics and its role in society. Different ways of orchestrating such discussions in teacher education and schools are something that deserve attention in future research.

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Characterizing Swedish school algebra – initial findings from analyses of steering documents, textbooks and teachers' discourses

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The paper reports the first results of an ongoing research project aiming at characterizing Swedish school algebra (grades 1-9). Both diachronic and synchronic studies are conducted to identify the specific teaching tradition developed in Sweden and different theoretical approaches are applied in the overall project in order to obtain a rich picture of the Swedish case. The results reported here are based on the analyses of mathematics curriculum, textbooks and focus group interviews with teachers in seven schools. The initial results indicate that, since 1980s, algebra is vaguely addressed in the steering documents and the progression of algebraic thinking is elusive in teachers' discourses. We discuss the implications of the initial findings for our project.

Background

Profound knowledge in algebra is important for the understanding of several mathematical areas as well as topics in other disciplines. A multitude of studies among adolescents document students' difficulties with algebra and the serious consequences of these difficulties. Due to its role as a critical gatekeeper and to recent research results that question the earlier school mathematics traditions, many countries including Sweden have revised their curriculum[8] attempting to integrate algebra in school mathematics from the very beginning (Cai, Lew, Morris, Moyer, Fong Ng & Schmittau, 2005; Prytz, 2015). Yet, the international evaluations like The Programme of International Student Assessment (PISA) and The Trends in International Mathematics and Science Study (TIMSS) show that Swedish students' results in algebra have not improved. The overall purpose of the ongoing project is to contribute to the international research field concerning the complex issue of implementing algebra in school mathematics by investigating the

[8] With curriculum we refer to the national steering document concerning the contents and goals in mathematics.

Swedish case. More specifically, we attempt to find possible reasons for the failure of raising the quality of algebra teaching by examining how algebra is traditionally treated in the Swedish school curricula and textbooks in Grades 1 through 9 (the diachronic perspective). We also explore the current situation (the synchronic perspective) by analyzing the treatment of different algebra-related items in the current mathematics curriculum, textbooks and teacher guides (cf. Cai et al., 2005; Hemmi, Lepik & Viholainen, 2013), and investigate how teachers at different school levels relate to these issues and the materials. The focus of the research project is to identify the expected student progression in algebra as interpreted in different *arenas* of the Swedish school system. The school system is regarded as stratified into levels and a basic distinction is made between arenas of formulation and realization (cf. Lindensjö & Lundgren, 2000). Examples of the former are the group of people who decide content in policy documents, for example curricula, but also people producing textbooks. Examples of the latter are the teachers who interpret texts and design and carry out lessons. Our particular interest is also to reveal how more or less tacit traditions in textbook production and teaching practices are related to the intentions of the curriculum in order to find possible mismatches and contribute to the future development at different levels of the educational system. In order to achieve this, the project is built upon three sub projects probing 1) the diachronic perspective on the formulation arenas, 2) the synchronic perspective on the formulation arenas, and 3) the synchronic perspective in the realization arena.

In this paper, we report the first steps of the project taken during the first year and discuss the initial results and their consequences in terms of how to proceed during the following years.

Relevant literature

In many countries, algebra has traditionally been postponed until adolescence partly because of former assumptions concerning child cognitive development, and partly because of the parallels made between the learning trajectories of students and the history of mathematics (cf. Carraher, Schliemann & Brizuela, 2006). The dominant view of child cognitive development connected to constructivist learning theories was already challenged by the classroom studies of the Vygotskian based Davydov team showing that Russian children who received instruction in algebraic representation of verbal problems from Grades 1 through 4 performed better than their control peers throughout later school years (Carraher et al., 2006). Also, the suggested similarity between a child development and the history of mathematics has been questioned (see for instance Bråting & Pejlare, 2015) and recent studies show that it is possible and even beneficial to start working with algebraic ideas and generalizations in parallel with arithmetic already in early grades (e.g. Cai et al., 2005; Carraher et al., 2006; Blanton, Stephens, Knuth, Murphy Gardiner, Isler

& Kim, 2015). Blanton et al. (2015) found that children are capable of engaging successfully with a broad and diverse set of algebraic ideas. The idea of early algebra is to facilitate students' progression towards understanding more formal algebra. Scholars agree that algebraic thinking in early grades should reach beyond arithmetic and computational fluency "to attend the deeper underlying structure of mathematics" (Cai et al., 2005). Kieran (2004), for example, addresses the following adjustments that students need to make in developing an algebraic way of thinking: 1) A focus on relations and not merely on the calculations of numerical answers; 2) A focus on operations as well as their inverses, and on the related idea of doing/undoing; 3) A focus on both representing and solving a problem rather than on merely solving it; 4) A focus on both numbers and letters, rather than on numbers alone; and 5) A refocusing of the meaning of the equal sign. Although these adjustments are in the domain of arithmetic they represent a shift toward developing fundamental ideas of algebra (cf. Cai et al., 2005).

In Sweden, algebra became a part of all students' schooling after the introduction of the nine-year compulsory school in the 1960s. Moreover, the 1969 policy documents prescribe that algebra should be a part of school mathematics from grade 2 (Prytz, 2015). The algebraic content in upper secondary textbooks has changed from being dominated by algebraic manipulations and expressions to becoming more integrated with other school subjects and thus being more anchored with reality as well as everyday activities (Jakobsson-Åhl, 2006). Besides the directives in the curriculum documents and changes in the textbooks, there have been various attempts to improve algebra teaching in Sweden through in-service training projects for teachers and in teacher education for some decades. However, it is not possible to discern a general positive effect of these efforts on Swedish students' learning in algebra, at least not if we consider the results in the TIMSS evaluations and in FIMS and SIMS that preceded TIMSS. Since 1964 (FIMS) Swedish students have always performed below the international average in algebra.

Häggström (2008) compares algebra tasks used in Chinese and Swedish mathematics textbooks (grade 8) and finds an extensive variation in many relevant aspects in the Chinese textbooks while tasks in the Swedish do not open many dimensions of variation. Concerning the realisation arena, there is some research about how teachers interpret and relate to national mathematics curriculum documents in Sweden in general (cf. Boesen et al., 2014) but studies focusing on certain mathematical areas are largely lacking. An exception is a small case study of Kilhamn (2013) who identifies different approaches to the introduction of variables in grade 6 of two teachers referring to the same piece of national curriculum text and using the same textbooks.

Methods

As mentioned in the introduction we conduct three studies that both separately and related to each other help us to discern important aspects of the issue of implementing algebra in school mathematics. Next, we briefly describe them and indicate what is done so far (the focus of this paper).

Study 1: The diachronic perspective on the formulation arenas (1960–2015)

The motivation for having a diachronic perspective is based on the observation that we cannot assume that all actions are based on people's awareness of explicit goals. People might also act according to traditions in a more or less conscious manner. Thus, there is a tacit dimension for us to handle. Our point is that if we want to understand people's action today, we also have to consider the possibility that they act according to a tacit tradition. The purpose of the study is to deepen our understanding of the tradition in Swedish school mathematics and the position of algebra within this tradition. The data material consists of steering documents issued by the central school authorities, textbooks, teacher journals and official reports. The category steering documents includes the syllabi and commentary materials.

Thus far, material issued by the central school authorities, i.e. syllabus and commentary material, have been studied. The analysis is focused on how knowledge in mathematics is described by different types of terminology, mainly expressions for mathematical concepts and expressions for competencies. The results reported in this paper are based on these analyses.

Study 2: The synchronic perspective on the formulation arenas

The second study is synchronic and focuses on the formulation arenas. The aim is to find out and characterize the hypothetical learning trajectory/trajectories (cf. Hemmi et al., 2013) and the typical ways of integrating algebra in the current Swedish compulsory school instruction for grades 1–9 (age 7-12). The main data for this study comprises the current steering document in mathematics and mathematics textbooks with teacher guides.

In an initial study we have identified and classified the algebraic content in the current Swedish curriculum in mathematics and in the two textbook series Matte Direkt and Matte Eldorado for grades 1-6. The two textbook series were chosen on the basis of high popularity (Neuman et. al., 2015), and because they represent different approaches to organization of teaching (Neuman et. al., 2015). Moreover, Eldorado is relatively new at the Swedish textbook market compared to the more established Matte Direkt.

As a starting point for our analysis we used a classification that is based on the analytical framework of Blanton et al. (2015) regarding how algebraic content can be characterized at compulsory school level. They identified the following four main categories (that they call ”big ideas”) in school algebra: 1) Equivalence, expressions, equations & inequalities (EEEI); 2) Generalized arithmetic (GA); 3) Functional thinking (FT); 4) Variable (Var).

Study 3 The synchronic perspective on the realisation arena

The third study is also synchronic, and it focuses on the realisation arena. The aim is to find out how teachers talk about algebra progression and texts and tasks produced in the formulation arenas. In the first part of Study 3 we have conducted focus group interviews in seven schools with, in all, 33 certified teachers from grade 1-9 (mean 15.9 years of teaching experience, SD=9.4). The schools were situated in different socio-economic contexts. An interview guide containing 14 open questions steering the conversation into two themes; 1) what is algebra (pre-algebra), and 2) what mathematical tasks are suitable for teaching algebra (at some specific school level). In the second theme we used tasks from Blanton et al. (2015) in order to cover the big ideas to be developed throughout the school years. Moreover, we selected tasks from mathematics textbooks to investigate how the teachers relate to specific aspects, such as informal/formal methods, everyday mathematics/pure mathematics. One project assistant conducted the interviews, and one took notes and collected background data (i.e., a questionnaire). The interviews were audio recorded and transcribed, and the initial thematic analyses were conducted with NVivo software using both a priori categories concerning Blanton's big ideas and the specific aspects Blanton et al. (2015), and an open approach to capture items that may be invisible in the documents or in previous research.

In the following sections, we will first display the first-year results from these three studies, and thereafter draw some conclusions for further studies. The results can be understood both separately and related to each other and can help us discern important aspects of algebra teaching and learning in Swedish schools.

Results

The result from the diachronic study concerns how knowledge has been expressed in syllabus. As regards algebra, our preliminary observations indicate that progression has been expressed differently in different topics; especially in the syllabus of 1962 and 1980. Since 1980 progression in arithmetic was expressed more clearly than in algebra. However, from the syllabus of 1994 and onwards, progression in all topics was expressed more vaguely.

Regarding how algebra is addressed in the current Swedish mathematics curriculum for Grades 1-6 the result of our initial study reveals that three of Blanton et. al.'s (2015) categories, namely EEEI, FT and VAR are well-represented in the content of the curriculum. Meanwhile, statements connected to category GA is not represented at all in the Swedish mathematics curriculum for Grades 1-6. Functional thinking (FT) is the most represented category where the dominating items are "proportional reasoning" and "construction of patterns".

The result of our initial study of textbooks for grades 1-6 shows that EEEI is the most represented category in both textbook series, especially in grades 1-3 and in Matte Direkt. However, in both textbook series the EEEI content decreases from grades 1-3 to grades 4-6. The categories FT and VAR are also well-represented in both series, especially in grades 4-6. Apparently, the tendency in both textbook series is that FT and VAR increases from grades 1-3 to grades 4-6 while the amount of EEEI decreases from grades 1-3 to grades 4-6. The category GA is the least represented in both textbook series, especially in Matte Direkt. As mentioned above, the category GA is not represented at all in the current mathematics curriculum for grades 1-6 which probably is one reason behind the low representation of GA in the textbooks.

Concerning the teachers' ways of talking about algebra at different school levels, the initial analyses indicate that teachers' considerations about the expected progress of students' algebraic thinking at different grade levels are vague. The next extract illuminates this.

A teacher: Often, I think, that we lack this, what one expects of the students when they leave the 3th grade… and also the other way around, what I can expect when the students come to me in the 4th grade. What have they done? What skills do they have? I can't start somewhere the students have not yet arrived at. And yet, we [the teachers] are often in the same building… what about when the students leave for the 7th grade, and they change the school completely. Then it's even more difficult to know what we teachers can, kind of expect from each other.

The excerpt above is representative of the way in which the teachers talk about the lack of consensus concerning what to expect of students in different school grades. Moreover, when they talk about goals they talk in general terms in a tentative manner without specifying what students should actually learn at different levels. We find this interesting as the diachronic studies show that algebra is traditionally vaguely addressed in the Swedish curriculum. Moreover, in line with the results from the analyses of the current steering document and the two textbooks, items connected to EEEI, such as the meaning of the equal sign and informal and formal methods of equation solving dominate the teachers' discourses. The following extract represents a common way of discussing the topic.

A teacher: …one sometimes thinks that a child has no clue of what the equal sign actually means, they think that it results to something, instead of balancing on both sides. If one has understood that it weighs evenly, then one can use that knowledge in almost all mathematics later on.

Understanding of the equal sign and simple equations are stressed as especially important in grades 1-3 but also raised as an important and difficult topic to continue working with in the following grades. Working with patterns sometimes followed by a formulation of a rule (FT) is another item that the teachers in the focus groups raised when discussing algebra at different school levels.

Conclusion

The absence/low occurrence of generalized arithmetic in the curriculum, textbooks and teachers' discourses can be something important to investigate further in search of reasons for the low results at TIMSS and PISA in Sweden, especially considering that the results in algebra is, and has been, the weakest of all mathematical content areas. Generalized arithmetic is stressed as an important part of algebra by several researchers and it can be seen as a bridge between arithmetic and algebraic thinking. The term "generalized arithmetic" has emerged from the part of algebraic thinking that considers the study of structures and relations arising in arithmetic (Kaput, 2008). Previously, generalized arithmetic has been associated with a letter-symbolic algebra, with its equations and unknowns (Kieran et. al., 2016). However, during the years and within the research field of early algebra the term has acquired a much broader sense in that the relations and properties inherent to arithmetical operations are explored and seen by students as being generalizable, without necessarily involving alphanumeric symbols (Kieran et. al., 2016). We believe that a progression in ”algebra as generalized arithmetic" throughout compulsory school is necessary in order to improve the algebraic skill of Swedish school students. The high representation of FT may be due to an international trend where "study of change" has been identified as a key area of mathematics in for instance PISA:s framework for school mathematics. This is reflected in the current Swedish curriculum where "Relationship and change" constitutes a separate category within the mathematical content for both compulsory and upper secondary school. Earlier this type of content has been spread out over different content categories.

Methodologically our ongoing project is unique, as we approach the issue from both diachronic and synchronic perspectives, and also investigate how algebra is addressed in the different arenas. Relating the results from the different studies to each other helps us to find explanations for the separate findings in different arenas and we believe that it will help us to increase our understanding both of the specific Swedish tradition and of reasons for why a diversity of mathematics initiatives concerning algebra have not been successful. All the three studies are not yet synchronized, for example only textbooks from grades 1-6 have been analysed so far and the analysis of algebra in the diachronic study has just been started. The initial results are promising and the next step regarding the diachronic study is to consider the status ascribed to algebra by the central school authorities. Another type of future studies concerns textbooks and not at least how and to what extent textbooks have realized the syllabus. A selection of the textbook series will then be made according to their popularity. By using a database regarding textbooks in mathematics published in the period of 1930–2015, constructed for the project, we can do this selection in a reliable way. The results from the diachronic studies can also be related to the variation of Swedish students' results in national and international evaluations and offer us interesting information about the effects of different kinds of steering documents and textbooks on students' learning of algebra.

As to the synchronic textbook studies, we will identify and classify the algebraic content also in the curriculum and textbooks for grades 7-9. We will deepen our knowledge concerning the specific character of the activities/tasks identified in the Swedish materials within the big ideas, as these categories are quite general. We aim to do this by conducting comparative studies with our colleagues in two countries that seem to have different approach to algebra, in order to find more nuanced picture of the Swedish situation. Thereafter the expected student progression within the categories across the grades 1-9 will be investigated in order to understand the hypothetical learning trajectory in current school algebra in Sweden. Finally, concerning the realisation arena, we will gather more data and also deepen the analyses of the categories. We will continuously relate the results from our three studies to each other.

We aim to apply Bernstein's (2000) theories on classification and framing in the entire study, to understand our results from a broader perspective. Drawing on Bernstein, the analysis focuses on how boundaries related to classification and frames are, and have been, created and maintained at different levels, in our case different arenas. By identifying differences in the creation and maintenance of these boundaries, both within arenas and between arenas, we achieve a better understanding of the implementation problem regarding algebra in the Swedish educational system. We can also provide an explanation, at least to some extent of the Swedish results in international comparisons.

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A cross-cultural study of teachers' relation to curriculum materials

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This study investigates teachers' relation to mathematics curriculum materials in three different cultural-educational contexts; in Sweden and in Finnish- and Swedish-speaking parts of Finland. The results are based on a survey among teachers (N = 603) who work in compulsory schools. The results support the previous findings which show that curriculum materials are experienced by teachers as a guarantee of good quality in mathematics education, but, at the same time, as a burden. Some notable differences were found between teachers with various experiences in different contexts. The findings are discussed in terms of pedagogical design capacity and the specific character of the three contexts.

Introduction

Recent studies have raised the role of curriculum materials as an important factor, not only for improving the quality of teaching and students' results (e.g. Stein & Kim, 2009), but also for influencing teachers' conceptions and teacher change (e.g. Remillard & Bryans, 2004; Pehkonen, 2004). The focus has also been on the interaction between a teacher and curriculum materials in relation to emerging mathematics classroom practices (e.g. Roth McDuffie & Mather, 2006).

Teaching is widely considered to be a cultural activity (Pepin, Gueudet & Trouche, 2013). This study adds to our knowledge of the complex relation between a teacher and curriculum materials (Remillard, 2005; Brown, 2009) in different cultural-educational contexts (Hemmi & Krzywacki, 2014). The term 'curriculum materials' in our study refers to commercially produced materials used in school education, such as student textbooks and teacher guides. The focus of the paper is to investigate how compulsory school teachers in Finland and Sweden relate to mathematics curriculum materials. We consider the teacher as part of the social practices embedded in certain cultural norms (cf. Hill & Charalambous, 2012). Hence, the study joins a fairly large body of work that aims to compare systematic mathematics teaching and learning practices across different cultures (e.g. Andrews, 2007).

Swedish and Finnish cultural-educational contexts resemble each other in many ways; for example, the national steering documents set only a non-specific outline for the school system, and teachers have free choice and use of curriculum materials and how to implement the curriculum. In both countries, commercially produced materials are in accordance with the core curriculum but neither regulation of curriculum materials nor inspection take place. (Hemmi & Ryve, 2015; Kaasila, Hannula, Laine & Pehkonen, 2008). Yet, there are considerable differences in how teachers organize mathematics teaching, in the character of curriculum materials, and how they are utilized by teachers in these two countries (Hemmi & Krzywacki, 2014). In Finland, mathematics teaching at the lower secondary level appears rather teacher-centred (Andrews, Ryve, Hemmi & Sayers, 2014); while at the primary level, a certain cultural script (see e.g. Andrews, 2007) with various reoccurring lesson events have been identified both in the Finnish context (Hemmi & Ryve, 2015) and the curriculum materials (Hemmi, Krzywacki & Koljonen, in press). This is not necessarily the case with the Finnish Swedish mathematics materials and classrooms. In Sweden, students usually work with their textbooks at their own pace without any teaching (Boesen, Helenius, Bergqvist, Bergqvist, Lithner, Palm & Palmberg, 2014) and the Swedish curriculum materials vary greatly, at least at the elementary school level (Neuman, Hemmi, Ryve & Wiberg, 2013).

Approximately 90% of Finnish teachers are qualified (Opettajat Suomessa, 2013). However, in the Swedish-speaking part of Finland, there are slightly less qualified teachers than the Finnish-speaking part. Over 30% of Swedish teachers teaching mathematics in compulsory school are not qualified for teaching mathematics (Skolverket, 2015). In Sweden, neither curriculum materials nor teaching methods have been the focus of the teacher education. In Finland, for decades, the aim of teacher education has been to educate autonomous independent teachers who research and reflect on their own work (Krzywacki, Pehkonen & Laine, 2016).

This paper draws on a quantitative survey of compulsory school teachers (grades 1-9) in Finland and Sweden and focuses on how teachers in different cultural-educational contexts relate to mathematics curriculum materials. In our study, we look at the Swedish-speaking and Finnish-speaking teachers in Finland separately due to the existence of possible differences in the teaching cultures between the language groups. Research questions are:

1. To what extent do teachers think of the curriculum materials as means to guarantee the even quality of mathematics teaching? Are there differences between cultural-educational contexts?
2. To what extent do teachers perceive the curriculum materials as burdens in mathematics teaching? Are there differences between cultural-educational contexts?

The relationship between teacher and curriculum material

The complex relationship between teachers and curriculum materials has been examined through the use of several theoretical frameworks (Brown, 2009; Remillard & Bryans, 2004). Remillard (2005) distinguishes theoretical perspectives characterising teachers' relation to curriculum materials in terms of fidelity to, interpretation of, or participation with curriculum materials. This study engages with the third approach, the participatory relationship view, which highlights the dynamic interrelationship between teachers and materials. The activity of using or participating with the curriculum resource is influenced by various individual factors such as teacher knowledge, beliefs and goals, perception of curriculum and students, tolerance for discomfort and professional identity (Remillard, 2005). Furthermore, general pedagogical trends and cultural traditions may affect teachers' views on teacher professionalism and thus their relationship with curriculum materials (e.g. Hemmi & Krzywacki, 2014). Therefore, research results should be considered in the light of different education cultures

Brown (2009) proposes the construct of Pedagogical Design Capacity (PDC) to describe a teacher's capacity to perceive and customize curriculum resources in order to design and enact instructional episodes, meet perceived student needs and achieve instructional objectives. Teaching experience is influential in enhancing teachers' readiness. According to Brown (2009), pedagogical design capacity may emerge over time, as familiarity with the pedagogical affordances of available resources and ability to use them increases. In addition to factors related to teachers as users, the character of the materials — for example, their flexibility and structure (Brown, 2009) — naturally has an impact on the participatory relationship.

The materials can both afford and constrain teachers' actions in mathematics classrooms (e.g. Brown, 2009). Roth McDuffie and Mather (2006) stress that teachers should use the instructional materials to support instruction, rather than allow them to prescribe instruction. According to Pehkonen (2007), teachers may feel guilty leaning solely on textbooks rather than their own planning when teaching. Although the Finnish teachers found the materials to be of high quality, they thought they had 'given up a part of their professional competence to the textbook authors' (Pehkonen, 2007). Remillard and Bryans (2004) show that teachers have different orientations toward using new curriculum resources, which influence the way they utilize them in practice. The orientations depend on the extent to which teachers familiarize themselves with the teaching material. Inexperienced teachers are most likely to engage fully with available resources (Remillard and Bryans, 2004), whereas teachers with more self-confidence are less dependent on curriculum materials (Stipek, Givvin, Salmon & MacGyvers 2001).

Methodology

The respondents in this study were comprehensive school teachers in Finland and Sweden (N=603) who voluntarily agreed to answer. The sample consisted of Finnish-speaking (N_{FIN}=209) and Swedish-speaking teachers (N_{FINSWE}=200) in Finnish schools, and Swedish teachers (N_{SWE}=194) working in Swedish schools. Female teachers were overrepresented in the sample (N_f=529, N_m = 71). About 74% of comprehensive school teachers (at the population level) are women in both Finland and Sweden, while 85% of the Finnish and 96% of the Swedish respondents of the study were women. In addition, the respondents were more qualified than teachers at the population level. The most prominent difference was in the Swedish data with unqualified teachers comprising only less than 2% compared to the over 30% reported by Skolverket (2015). In the Finland-Finnish data set, a little less than 4% comprised unqualified teachers, and that figure was about 8% in the Finland-Swedish data. Based on a survey carried out by Statistics Finland in spring 2013, the number of unqualified teachers in Finland is around 10% among Finnish speaking and around 20% among Swedish speaking teachers (Opettajat Suomessa, 2013).

The data was collected via e-questionnaires by announced on various teachers' professional network forums with a request to participate. In addition, the Swedish data was partly collected during in-service teacher education. The data collection instrument for the study was created from previous qualitative studies of interviews with Finnish teachers (Pehkonen, 2004; 2007). In those studies, three qualitatively different ways to speak about the use mathematics curriculum materials had been identified: 1) justification (assuring the even quality of teaching, supporting changes); 2) criticism of textbooks and the use of them; and 3) expressions of guilt. The questionnaire was constructed based on those dimensions and the items were formulated convergent with the teachers' statements. The instrument was modified through testing pilot versions in various data sets based on different teacher populations.

The questionnaire comprises 39 items (statements) that were shown in blocks of five statements in a random order. Thus, the respondents could focus on five statements at a time. No headings was shown labelling the blocks. The respondents were asked to take a stand on each statement on a five-point Likert-scale (1 = totally disagree, 5 = totally agree).

The three dimensions (factors) with the resemblance to the original dimension were extracted in explorative factor-analysis (GSL and Varimax-rotation) and found in all used data sets. We omitted the items with loadings over .40 on two factors, and the items with loadings under .40 on each factor. The first factor was named 'quality guarantee', and the constructed subscale was consisted of nine items. In the entire data set, the Cronbach's alpha was .87, and it varied from .85 to .89 in the three separate data sets. The second subscale 'burden' comprised eight items (of the second factor) with the alpha coefficient of .83 in the entire data, and in separate data sets $.80_{FINSWE}$, $.84_{SWE}$ and $.85_{FIN}$, respectively. The third constructed subscale (based on the third factor) measured teachers' self-confidence in mathematics teaching. It consisted of six items, and the Cronbach's alpha in the entire data set was .728 with variation from .720 to .751 in separate data sets. In this paper, we concentrate on reporting the findings regarding the first and second subscales.

SUBSCALE	Quality guarantee	Burden	Confidence
ITEMS	N = 9	N = 8	N = 6
Alpha TOTAL	.874	.831	.728
Alpha FIN	.892	.852	.720
Alpha FINSWE	.854	.804	.751
Alpha SWE	.858	.840	.739

Table 1. Subscales and Cronbach's alphas in various data sets

Results

Our first research question concerns the extent to which teachers conceived the curriculum materials as a means to guarantee the high and even quality in mathematics teaching. The scale contained nine items, like 'Textbooks help me to assure the quality of instruction'. In total, the teachers found that curriculum materials are somewhat helpful in assuring the quality of mathematics teaching. The arithmetic mean on this subscale was 3.38 (SD = .77). However, differences were found between teachers working in different cultural-educational contexts. The Finland-Swedish teachers had the highest mean (M = 3.63) and smallest standard deviation (SD = .70), whereas the Finland-Finnish teachers had the lowest mean (M = 3.18) and greatest standard deviation (SD = .82) (see Table 2).

The differences between the groups were statistically significant ($F(2, 597) = 18.296$; $p < .001$). The effect size was mediocre (eta squared = .06). The variances between groups were not homogenous, so the mean differences were localised by Tamhame's T2-test. It indicated that the differences between means were due to the Finland-Swedish teachers, who differed both from their Finnish and their Swedish colleagues. The Finland-Swedish teachers in our data had the highest confidence in using the mathematics curriculum materials as quality guarantees in mathematics teaching. Teachers' gender, age and teaching experience were not related in this respect.

	Mean (QG)	Std. Dev	Mean (B)	Std. Dev.
Finland-Finnish teachers	3.18	.82	2.55	.84
Finland-Swedish teachers	3.63	.70	2.71	.71
Swedish teachers	3.35	.72	2.55	.76
TOTAL	3.38	.77	2.60	.78

Table 2. Curriculum materials as means to guarantee high and even quality in mathematics teaching and as burden

Secondly, we answer the question 'To what extent do the teachers conceive the curriculum materials as burdens in their work?' The subscale measuring this dimension included eight items like 'Since the mathematics textbook keeps us so busy, we do almost nothing else in mathematics classes'. On the five-point scale (from 1 to 5, where 5 refers to a very high burden), the mean of the burden scale in the entire data set was somewhat below the middle point ($M = 2.60$; $SD = .78$). The means and standard deviations are presented in Table 2 above.

On average, the Finland-Swedish teachers found the curriculum materials the most burdensome with the highest scale mean of 2.71 and lowest standard deviation (= .71). The Finnish and Swedish teachers scored somewhat lower ($M_{FIN} = 2.55$ and $M_{SWE} = 2.55$; SDs .84 and .76, respectively). However, the differences between cultural-educational contexts were not statistically significant, though the Finland-Swedes were borderline outliers. To obtain a somewhat sharper picture of the situation, we selected teachers with a scale mean slightly above the middle point, i.e. $M>3.5$. Of all the teachers, 11.4 % ($N = 69$ out of the total $N = 603$) who scored above this limit found that the curriculum materials put a strain on them. Most of these were Finland-Swedish teachers. On the whole, around 13% of Finland-Swedish teachers in our data shared these experiences. We continued by selecting those teachers with a relatively high mean scale ($M > 4$), which indicated that they found the materials even more burdensome. In the whole data set, approximately 4% of teachers reported that curriculum materials created a considerable burden for their work.

Overall, the length of teaching experience was found to be related to experiencing curriculum materials as a burden. Teachers with little (under two years) or a significant amount of (more than ten years) teaching experience found the curriculum materials to be much less of a burden ($M_{te<2} = 2.64$; $M_{te>10} = 2.49$) than the teachers with teaching experience between two to ten years ($M_{te2\text{-}10} = 2.88$); $F(2, 599) = 16.033$; $p < .001$, eta squared = .05). Female teachers found the curriculum materials more burdensome ($M_f = 2.65$, $SD = .786$) than their male colleagues ($M_m = 2.33$, $SD = .67$). The difference between the means was statistically significant ($t = 3.04$, $p = .002$), but the effect size was small (eta squared = .02).

Discussion and conclusions

Curriculum materials are important tools for teachers when designing and enacting teaching (Brown, 2009). The way teachers relate to curriculum materials plays an important role for how productively they utilize these resources. All the teachers of our study found curriculum materials somewhat helpful in assuring the quality of mathematics teaching. However, the Finland-Swedish teachers differed significantly from both their Finnish and Swedish colleagues in that they had the highest confidence in the curriculum materials as a quality guarantee in mathematics teaching regardless of gender, age or teaching experience. In the Swedish part of Finland, it has been common to use restricted number of curriculum materials that are typically developed by the teacher educators who also educate future teachers in the only Swedish elementary teacher education in Finland. This might explain why the Finland-Swedish teachers put more trust in the quality of available curriculum resources.

Curriculum materials are not considered a heavy burden by any group of teachers. Although it is not a statistically significant difference, it is worth noting that the Finland-Swedish teachers also stood out from the other teacher groups by finding curriculum materials more burdensome than the others. It is possible that teachers who consider the curriculum material a guarantee of quality feel guilty if they cannot follow the material in the way that they conceive the underlying idea. On the level of the entire data, teaching experience seemed to have the most powerful impact on experiencing burden (Brown, 2009). Teachers with either a little or a lot of experience in teaching mathematics found the curriculum materials significantly less burdensome than the teachers with two to ten years of experience. On the one hand, newly graduated teachers possibly appreciate curriculum materials especially because the materials help them in teaching by familiarizing them with the contents and goals of particular grade levels. On the other hand, teachers with a long teaching experience hardly feel stress for the way they utilise the available materials. As stated by Brown (2009), pedagogical design capacity emerges over experience and practice, and the more experienced teachers have developed their capacity to customize the materials for their purposes. Therefore, the material is not found as a burden but rather a support for teaching (Remillard and Bryans 2004; Hemmi & Krzywacki, 2014).

The constraints and affordances experienced by teachers utilizing curriculum materials should also be discussed in terms of different teaching traditions. We expected to find differences between Finland and Sweden particularly due to the differences in classroom cultures, teacher education (Hemmi & Ryve, 2015) and curriculum materials (Hemmi et al., in press; Neuman et al., 2013). Contrary to our expectations, we found no particular differences between the Finnish and Swedish teachers' relation to curriculum materials. The difference could be found, however, within Finland between two language groups. A possible explanation could be that the curriculum materials are developed within a certain cultural-educational context and, therefore, could be in line with the prevailing teaching tradition and social practices within the cultural norms internalised by teachers (cf. Hill & Charalambous, 2012).

There are some limitations resulting from self-selection that generates a special sample of three cultural settings. The respondents were those who voluntarily decided to answer to the questionnaire, which may have resulted in some biases in the data. First, the female teachers are over-represented in our data. Second, the respondents in our study were somewhat more qualified than teachers on average.

Curriculum materials can be experienced as a burden rather than an affordance if pedagogical design capacity is undeveloped and a teacher cannot utilize resources flexibly and struggles with achieving fidelity between the written and enacted curriculum (cf. Brown, 2009; Pehkonen, 2007). Our findings indicate that there could be some general cross-cutting patterns connected to teachers' experience of curriculum materials as a burden. Those might possibly be connected to pedagogical design capacity (Brown, 2009) but also to the general view of teacher professionalism and the material to which they are accustomed. Further investigation could enlighten both the similarities and differences in the teachers' relation to curriculum materials in the three different educational contexts. For example, it would be interesting to study deeper how teachers perceive and customize curriculum materials in practice and what the role of the curriculum materials is as a part of everyday work in the classroom.

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Estonian and Finnish teachers' views about the textbooks in mathematics teaching

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This paper explores Estonian and Finnish compulsory schools' teachers views about mathematics textbooks. The data consist of compulsory schools' teachers' responses on a 36-item questionnaire that was analysed using quantitative methods. The main findings show that both the Finnish and the Estonian teachers found the textbooks somewhat important in assuring the quality of teaching mathematics. The textbooks did not particularly strain the teachers. However, the findings reveal some interesting differences in this respect between contexts and between teachers with different teaching experience. Findings from this study contribute to the research-field by adding research-based knowledge about the relationships between teachers and curriculum materials.

Contradictory views about textbook use in mathematics teaching

Remillard and Taton (2016) state in their recent research that one of the common myths about curriculum programs and teachers is that good teachers reject textbooks and develop their own curriculum materials. There are somewhat contradictory views about the use of textbooks and other curriculum materials among teachers and mathematics educators. Many mathematics educators emphasize the textbooks' role as the teachers' aid (e.g. Lepmann, 2005, pp.25–32) and connect teacher professionalism with teachers' independence from the guidance of textbooks or teacher guides (Oates, 2014; Hemmi & Krzywacki, 2014). O'Keeffe and O'Donoghue (2011) reported that mathematics textbooks had an overdominant influence in Irish classrooms and that teachers relied on the textbooks without knowing their effectiveness on teaching or learning. However, many empirical studies have raised the role of textbooks and other curriculum materials as important factors, not only for improving the quality of teaching and students' results (e.g. Stein & Kim, 2009), but also for influencing teachers' conceptions and teacher change (Remillard & Bryans, 2004; Pehkonen, 2004). The term 'curriculum materials' refers not just to student textbooks and teacher guides but to a wider package including other supplemental resources the teacher might use. A number of studies show that, depending on the character of the materials and how teachers relate to and interact with them, the materials can both afford and constrain teachers' actions in mathematics classrooms (e.g. Brown, 2009; Nicol & Crespo, 2006). Despite of the central role textbooks are claimed to have in mathematics classrooms (e.g. Lepik & Kaljas 2010), we have surprisingly little research-based knowledge on how and what teachers think about textbooks.

This paper draws on previous qualitative studies about Finnish teachers (Pehkonen 2004: 2007) and is part of the Nordic project on curriculum materials in mathematics education (Pehkonen, Hemmi, Krzywacki & Laine, 2017). The project focuses on how compulsory school teachers relate to mathematics curriculum materials (text-books, teachers' guides etc.) in different cultural contexts. Finland and Estonia are neighboring countries with close cultural and educational ties. Even the first Estonian post-Soviet national core curricula (1996) was developed in cooperation with the Finnish National Board of Education. Contemporary curriculum discourse in both countries emphasize teacher autonomy. However, there are more tensions among Estonian teachers in respect to experienced curricular autonomy than among Finnish teachers (Erss, Kalmus & Autio 2016.) The aim of this paper is to report Estonian and Finnish compulsory schools' teachers views about mathematics textbooks and how they perceive the textbooks in mathematics teaching.

Teachers have different orientations towards teaching mathematics

According to Remillard and Bryans (2004), teachers have different orientations toward using curriculum resources and this influences the way they utilize them in practice. The orientations depend on the extent to which teachers familiarize themselves with the teaching material. Remillard and Bryan define the orientation as a set of perspectives and dispositions about mathematics, teaching, learning, and curriculum that together influence how a teacher engages and interacts with a particular set of curriculum materials. Teaching experience seems to be crucial in this matter. Inexperienced teachers are most likely to engage fully with available resources (Remillard and Bryans, 2004). Confident teachers use a maths textbook when it supports their teaching principles, while insecure teachers mainly rely on the textbook and often also use the key given in the book to check the students' answers (Stipek, Givvin, Salmon, & MacGyvers, 2001). Different orientations are probably the reason the same materials can be experienced as constraints by some teachers, while others see them as affordances (Pehkonen, 2007). This reflects the contradictory image of the use of teaching materials.

Curricula in different countries describe important principles for teaching mathematics. Both Estonia and Finland have national core curriculum frameworks, but teachers and schools have to interpret and adapt them for the specific school and learning contexts. In addition, both the Finnish (Perusopetuksen opetussuunnitelman perusteet 2014) and the Estonian (Põhikooli riiklik õppekava,

2014) curricula include the requirement of the differentiation of instruction, which teachers have to consider while planning and carrying out the teaching. Taking into account the specific nature of each learner means the preparation of a lesson for a class with students of different levels and from different backgrounds. It means that the teacher has to plan different ways of introducing a new topic to make it understandable to a possibly large proportion of students. According to many mathematics educators, the textbooks should contain tasks of different levels of complexity that provide more able students with an opportunity to solve more difficult tasks while offering simpler tasks to the students less able in mathematics (Lepmann, 2005, pp.25–32; Lepik & Kaljas, 2010). If, however, the textbook does not meet the various expectations (concerning e.g. ease of use, quality and/or quantity of tasks, quality of performance) teachers set on them, they may avoid it. In addition, textbooks can burden teachers if the books are overloaded with various task materials or if teachers had to do much extra work to find tasks and compile task instructions.

According to the recent OECD report (Echazarra et al. 2016, 44-47), the teaching strategies (i.e. teacher-directed instruction, student-oriented interaction and cognitive-activation instruction) among Estonian and Finnish teachers in mathematics seem to be very similar compared to the other participating countries in PISA (Programme for International Student Assessment). In both countries, teachers are experienced and have full autonomy to choose the textbooks (Mathematics teaching in Europe, 2011). The teachers in the Finnish compulsory schools have, on average, around 15 years teaching experience and in Estonia around 22 years. The teacher profession is also highly appreciated in Finland: in the recent international comparative study, around 60 % of all Finnish compulsory teachers reported that they believed their work is valued in the society. In Estonia the percentage (14 %) was considerably lower (Taajamo et al. 2014; 2015).

Research methodology

In both countries, the sample consisted of teachers of general education schools who were teaching mathematics at the time the survey was conducted and who voluntarily agreed to participate. In Finland, the data was collected via an e-questionnaire and an announcement requesting to participate was provided on various teachers' professional network forums. The Estonian sample was based on accessing the teachers whose contact data was available on schools' websites and on the portal of Estonian Education Information System (EHIS). In addition, the Estonian researchers sent invitations to the teachers with whom they had had an earlier contact, asked them to participate and to forward the survey link to other teachers teaching mathematics at their school. In total, 420 teachers participated in the study. Of them, 198 were Estonian and 222 Finnish teachers. Most of the participants (83% of the total 420) were female teachers. In Finland (at the population level), about 74% of teachers in comprehensive schools are women, but in Estonia the percentage of female teachers is higher (86%). The Finnish sample roughly follows the teacher gender division in the compulsory schools, since 73% of the respondents were female teachers. In the Estonian sample 94% of the teachers were female. Hence, it seems that male teachers are somewhat under-represented in the Estonian sample.

The data collection instrument we used in this study has been created on base of previous qualitative interviews with Finnish teachers (Pehkonen 2004; 2007). In those studies, three qualitatively different ways to speak about the use mathematics textbooks and curriculum materials had been identified: 1) justification (assuring the even quality of teaching, supporting changes), 2) critics towards textbooks and use of them and 3) feelings of guilt (or insecurity) concerning teaching of mathematics. The questionnaire has been developed based on those dimensions and formulated the items directly from teachers' statements. The instrument has been modified through several pilot versions with different amounts of items and through testing with different teacher populations. The version used in this study consisted of 36 items on a five-point Likert scale, where only the end points of the scale where given, 1 – completely disagree and 5 – completely agree (Pehkonen, Krzywacki & Laine 2014). The statements were divided into blocks with five statements in each. Such division allowed the respondents to focus on only the five statements at a time they could see on the computer screen. The question blocks did not have headings dividing them into topics, and the statements were presented in a random sequence in the questionnaire.

In explorative factor analysis (GSL and Varimax rotation) three dimensions (factors) were extracted with the resemblance to the original dimensions. The three factor solution explained 39,75% of the total variance. The first factor was labelled as Quality guarantee and it explained 18,6 %, the second factor (Burden) 13,88%, and the third factor (Self-confidence) explained 7,27% of the total variance. We constructed tree scales based on the factors and items with factor loadings more than .40. We omitted the three items with the loadings (over .40) on two factors, and the two items with the loadings under .40 on each factor. The three items with negative loadings were recoded before the scale construction.

Scale	N of items	Cronbach's alpha
Quality Guarantee	14	.89
Burden	13	.86
Self-confidence	4	.66

Table 1. Constructed scales with the alpha coefficients

The constructed scales with the alpha coefficients are presented in Table 1. The first scale based on the first factor (Quality guarantee) comprised 14 items concerning the role of textbooks in the quality assurance of mathematics teaching, such as ‘Textbooks help me to assure the quality of instruction’ with Cronbach’s alpha of .89. The second scale was constructed of the second factor (Burden) and consisted of 13 items (dealing with various matters related to how the textbooks made mathematics teaching more difficult and strained teachers), such as ‘Since the mathematics textbook keeps us so busy, we do almost nothing else in mathematics classes’. Its alpha efficient was .86. The scale (Self-confidence) comprised four items of the third factor and had the alpha coefficient of .66. The third scale included items concerning teachers’ self-confidence in mathematics teaching, such as ‘I consider myself an expert in teaching mathematics’.

Findings

The means and standard deviations on each scale are presented below in Table 2. We first consider the importance teachers attached to textbooks in ensuring the quality of teaching. The first scale (Quality Guarantee) consisted of items which were concerned with the extent which teachers conceived the curriculum materials as a means to guarantee the high and even quality in mathematics teaching. On average, both the Finnish and the Estonian teachers in our study found the textbooks somewhat important in assuring the quality of mathematics teachings. On the 5-point scale (from 1 to 5 were 5 refers to very great importance) the mean was 3.25 among the Finnish teachers ($SD_F = .75$) and about the same ($M_E = 3.22$; $SD_E = .64$) among the Estonian teachers, indicating no differences between the two countries. Teachers’ gender, age and teaching experience were not related either, in this respect.

	Quality Guarantee		Burden		Self-confidence	
	Mean	SD	Mean	SD	Mean	SD
Estonian teachers (N=198)	3.22	.64	2.61	.62	3.69	.69
Finnish teachers (N=222)	3.25	.75	2.87	.80	3.93	.73

Table 2. Means and standard deviations

Secondly, we will consider to what extent teachers perceive the mathematics textbooks as burdens that restrict their working in classrooms. The second scale consisted of items concerning the straining effects of textbook in mathematics teaching. In total, the teachers in our study did not conceive the textbooks as particularly burdensome. On the five-point scale (from 1 to 5, where 5 refers to a very high burden), the mean of the burden scale in the entire data set was somewhat below the middle point ($M = 2.74$; $SD = .73$). However, there were differences between the teachers. Estonian teachers found the mathematics textbooks less burdensome ($M_E = 2.61$; $SD_E = .62$) than their Finnish colleagues ($M_F = 2.87$; $SD_F = .80$). The difference between the two countries is statistically significant, but the effect size is small ($t = 3,66$, $p < .000$, eta squared = .03). There also seems to be a small difference between male ($M_m = 2,54$, $SD_m = .61$) and female ($M_f = 2,76$ $SD_f = .74$) teachers ($t = 2.05$, $p < .05$), but the effect size (eta squared = .014) is very small. In the full sample, the teaching experience was statistically significantly related to the experienced textbooks' burden ($F(2,397) = 13.48$, $p < .000$). The effect size is medium (eta squared = .064). Teachers with more than 10 years' teaching experience found the textbooks less as burdens ($M=2.6$) than their colleagues with shorter teaching experience ($M_{<2years} = 3.02$, $M_{2\text{-}10years}=3.02$). The differences are similar in both countries, with the exception that Finnish teachers with medium teaching experience from two to ten years found the textbooks significantly most burdensome ($M_{F2\text{-}10years} = 3.2$) than the other teachers.

Finally, we take under consideration the participating teachers' self-confidence. In total, the self-confidence in teaching mathematics was above the middle point among the participants. The self-confidence was higher among Finnish teachers ($M_F=3.93$, $SD= .73$) than among their Estonian colleagues ($M_E = 3.69$, $SD = .69$). Although the difference between the means is statistically significant ($t = 3.38$, $p < .001$), the effect size is small (eta squared = .027). Female teachers indicated somewhat lower self-confidence ($M = 3.79$, $SD = .71$) than male teachers ($M = 4.01$, $SD = .73$). The difference is significant ($t = -2.02$, $p < .05$), but the effect size is very small (eta squared = .012). Hence, we must be careful not to make any conclusions considering the gender effect in this respect. The teaching experience was related to the self-confidence in mathematics teaching. ($F(2,406) = 10.12$, $p < .000$). The most experienced teachers with more than 10 years experience had higher self-confidence ($M_{>10years}= 3.92$, $SD = .69$) than their colleagues with less experience ($M_{<2years} = 3.43$, $SD = .79$ and $M_{2\text{-}10years} =3.64$, $SD = .71$). The effect on self-confidence is near to medium (eta squared = .05).

Discussion

We found small or medium differences between the two countries concerning comprehensive schools' teachers' views on mathematics textbook use. Teaching experience had the most powerful effect in making the differences visible. The findings indicate that teachers rely on mathematics textbooks and find that textbooks do have significance in assuring the quality of mathematics teachings. Somewhat unexpectedly, in both samples there are vague signals that teachers with minor teaching experience rely less on mathematics textbooks than teachers with more experience. This may reflect the intentions of teacher education to encourage student teachers to become critical users of texts. The connection between teaching experience and confidence on textbooks and other curriculum materials should be elaborated in future studies.

Textbooks may have different roles in different pedagogical and cultural contexts. During the Soviet time, there was a shortage of textbooks in Estonia and teachers had to prepare the teaching materials themselves. The new time brought along a variety of textbooks to support teachers and help them with the workload. This may at least partly explain the difference in experienced burden of textbooks between the two countries.

According to Remillard and Bryans (2004), inexperienced teachers are most likely to use all the resources of the reform-oriented curriculum material. Although it helps the new teachers to learn about mathematics education, it requires hard work. We have evidence from the previous studies (Brown 2009) that teachers' skill to perceive the affordances of the materials develops over time and teaching experience. Remillard and Bryans (2004, 384) concluded that, "experienced teachers seem to develop pedagogical repertoires that include the ways they read and use curriculum resources". Curriculum materials can be experienced as a burden rather than an affordance if pedagogical design capacity is undeveloped and a teacher cannot utilize resources flexibly and struggles with achieving fidelity between the written and enacted curriculum (Brown, 2009; Pehkonen, 2007).

Our finding about the slightly lower self-confidence among the Estonian teachers is in accordance with the TALIS 2008 (OECD 2009) results, were Estonian lower secondary school teachers' self-efficacy scores were under the mean of all participating countries (see also Erss et al. 2016). We found that teaching experience was related to self-confidence in teaching mathematics, and the teachers with higher confidence in mathematics teaching found the curriculum materials less burdensome. On average, the teachers in our study felt confident teaching mathematics. It is a good signal, since there is evidence that teacher's self-confidence in mathematics education facilitates high-quality learning in pupils (Jamieson-Proctor & Byrne 2008).

The findings of this study help us to understand that textbooks can serve both as affordances and contraints in teaching mathematics. They give us reasons to conclude that Estonian and Finnish teachers rely on mathematics textbooks to help them in maintaining a high quality in of teaching, but textbooks also stress teachers. In this study, as well as in the previous study of the same project (Pehkonen et al. 2017) the teachers with medium teaching experience found the textbooks most burdensome. Hence, more research is needed about the teachers' experiences on curriculum materials as a burden to elaborate this connection.

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Inquiry-based Learning in Mathematics Education: Important Themes in the Literature

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From a grading list of 28 of the highest ranked mathematics education journals, the six highest ranked journals were chosen. A systematic search for inquiry-based mathematics education and related keywords was conducted. This led to five important themes for inquiry-based learning in mathematics: communication in the mathematics classroom, mathematical competence; moving in and out of the mathematical domain; tools and resources for planning and implementing inquiry-based learning; professional development and collaboration. From these five themes, three principles were developed to determine which implications were important for the didactical intervention of the design in the Quality in the subjects Danish and Mathematics (KiDM) project.

Introduction

The movement towards open problem solving and the investigation of mathematical situations has been prominent in mathematics education for several years. This has been expressed in the key frameworks of mathematics education (Freudenthal, 1986; Brousseau, 2006), in empirical literature about reforming mathematics education (Boaler, 2012), and in descriptions of the key competences in mathematics (Niss & Højgaard, 2011). Over the last decade, this emphasis has focused on bringing inquiry into the teaching and learning of mathematics as well as on creating continuity between mathematics and science education. Inquiry-based science and mathematics education (IBSME) is the term that is used to signify this movement.

A number of projects and initiatives have been launched in order to implement IBSME as a method to improve the teaching and learning of mathematics. This is also the case in Denmark where the Ministry of Education has begun the Quality in the subjects Danish (first language) and Mathematics (KiDM) project, a three-year program involving 150 schools that will develop and test a didactical design for inquiry-based teaching. The design considers distinct ways to scaffold teachers when planning inquiry-based learning (IBL) and to scaffold students as they become actively involved in inquiry-based activities. It also addresses professional development for teachers.

We have chosen to review the most recent and prominent literature as a preliminary investigation of KiDM to gain insight into the main concerns and issues associated with inquiry-based science and mathematics education. To investigate the most important issues for IBL in mathematics education. This paper reports on this literature review attempts to answer these research questions: Which themes, issues, and concerns are prevalent in the mainstream mathematics education literature? What are the implications of the didactical intervention design for KiDM?

This paper describes the method we used and the specific choices we made to obtain an overview of the most important knowledge about IBSME. It then describes the most prominent findings of the literature review, and it concludes by discussing the possible implications of the didactical intervention design for KiDM.

Methods

The literature review was conducted to map the issues concerning the possibilities, conditions, and processes that are emphasized in the mathematics education literature. The review is oriented towards excellence rather than emphasizing specific methods. Thus, this review focuses on assembling an international consensus of the knowledge about the importance of IBSME. We chose articles in the highest ranked journals in mathematics education (Toerner & Arzarello, 2012) that were published from 2010 to 2016 (see Table 1 for information on the authors, the articles and the specific journals).

We conducted a systematic search for inquiry-based mathematics education and related keywords. The keywords were chosen by screening abstracts in special issues of some of these journals regarding IBL, problem posing, creativity and mathematics education, and application and modeling. Finally, we also randomly chose an issue from each journal and screened for relevant keywords. The search resulted in 170 studies, which were screened for relevance, first by the title, which had us discard 51 studies, and then by reading the abstract, which had us discard another 57 studies. The issues raised and discussed in the remaining 62 studies were then condensed to five main themes: 1) communication in the mathematical classroom, 2) mathematical competence, 3) moving in and out of the mathematical domain, 4) tools and resources for planning and implementing IBL, and 5) professional development and collaboration. These themes are described below. If a study discusses more than one theme, it is included in each relevant theme. The reviewed articles were primarily from Western countries. The age focus of the studies ranged from daycare to university level as well as professional development of teachers. In most of the studies, the focus was on the age from 6 to 18.

Thirty-seven of the 62 reviewed studies used a case study research design and nine of the studies were systematic reviews (n=62). A complete list of the studies is included in Table 1 in the appendix.

Communication in the Mathematics Classroom

Communication between the teacher and students plays a major role in inquiry-based teaching. The culture and norms in the classroom and the situations situated for the students have a significant impact on the student's learning and the student's mathematical creativity. A process-oriented culture in the classroom is desirable; this enables the students' self-generated representations to be included, and it makes it possible to uncover their difficulties and eliminate them (44, 47, 48; note: throughout the remainder of this paper, the numbers in parentheses refer to the journals listed in Table 1).

The interaction between the teacher, the class, and the content must be facilitated as open, inquiring, and related to the students' activities. The teacher must create an environment where students feel comfortable about expressing their mathematical understanding; this also enables them to feel ownership and responsibility of the content and to trust that the teacher takes their findings seriously (28, 9; 6).

For inquiry-based mathematics education, it is vital that the interaction between the teacher and the students has its starting point in the students' prior knowledge; this allows the students to develop their own strategies. When the teacher plays a guiding role, it has a positive effect on the students' deeper learning (3, 4, 23, 26, 50, 53). There is a need for the teacher to scaffold the students' learning, particularly in relation to analyzing the students' solution process and their deductive reasoning, and to ensure that the teacher promotes cognitive conflicts for the students, as this enhances their learning, so they can investigate the mathematical content (5, 51).

Mathematical Competence

When using an inquiry-based teaching approach in mathematics, some mathematical content is more obvious than others, and students benefit from engaging in different mathematical tasks. Problem solving, problem posing, and modelling activities are core areas in inquiry-based teaching. The student's mathematical creativity is improved when inquiry-based teaching methods are used.

Problem solving is categorized as: 1) traditional, 2) traditional with an open approach to real-world phenomenons, and 3) modelling (58). The study found that the modelling view of problem activities hold the greatest learning potential for students in inquiry-based teaching. The development of a student's problem solving competence is significantly improved by using "standard-based" curricula in comparison to using traditional curricula, which do not articulate aims for a student's mathematical skills and competencies. Therefore, a teaching approach that is oriented towards learning objectives offers greater opportunities to develop problem solving competence in comparison to curriculum-oriented teaching (2; 39). The literature reports that inquiry-based teaching with a modelling focus in problem solving is a better learning framework for students than other types of problem solving activities. In inquiry-based teaching that includes problem solving, a goal-oriented teaching approach is preferred over curriculum-based teaching, as it develops the students' problem solving competence.

Problem solving and problem posing are connected activities. A focus on one of these activities will also develop the other; however, it is important that both are specific goals in inquiry-based teaching. Problem posing is a core element in inquiry-based mathematics teaching either as a specific goal or as a means to achieving other goals. When students are encouraged to ask questions and pursue them in an inquiry investigation, they will develop the connection between mathematical competence and mathematical generalizations. This makes the problem-posing process an authentic mathematical inquiry, which increases the student's flexible thinking (35, 37, 38, 39, 41, 54).

Modelling activities are highly promoted for inquiry-based teaching in mathematics. In an inquiry investigation, mathematical objects can be a starting point for modelling activities (25). Model-eliciting is a special kind of modelling activity. In working with this activity, the student's role is important, and it provides the possibilities for an interdisciplinary and realistic approach, which supports the student's development of modelling competence (30, 29).

This realistic and interdisciplinary approach also relates to our theme of communication in inquiry-based teaching, where the students are handed control of the learning process; thus, they are the centre of the activities. Working with modelling activities has a positive impact on a student's emotions (52).

The benefits that students receive from inquiry-based mathematics teaching will, among other things, have a positive impact on their mathematical creativity (26). If the focus is only on developing the students' mathematical skills, it is most likely that their creativity will not be developed (45). However, there are multiple indicators that mathematical creativity cannot be developed without the general mathematical competencies (46).

Moving In and Out of the Mathematical Domain

The students' mathematical knowledge and their knowledge of the world that surrounds us are key aspects of IBL. Scaffolding the students' work, keeping their pre-understanding in mind, has a significant influence on their learning outcomes.

If the students are supposed to engage in inquiry-based work, they require some knowledge about the subject, especially if the task is problem posing (4, 5, 7, 40).

There is no difference in how the students value, enjoy, show interest in, or feel about math whether they are working on intra-mathematical problems, traditional tasks, or modelling problems. Helping students feel positive towards mathematics or be more motivated to engage in mathematical tasks, is not as simple as using problems from outside the mathematical domain (52). However, it is important to use a wide spectrum of activities in teaching, where less structured and more open-ended tasks require students to be flexible thinkers and prepare them to cope with situations and problems they encounter outside of school (26, 34, 38). IBL is significantly related to the world outside mathematics, as it includes emancipation and democracy as essential elements (25).

Tools and Resources for Planning and Implementing Inquiry-based Learning

The articles in this literature review noted that several didactical tools, technology-based tools, and resources are important for supporting teachers as they plan and implement IBL. It is important for teachers to initiate, orchestrate, and sustain collective learning by enhancing the students' communication and reasoning skills, by sharing control and allowing mathematical events to unfold, and by addressing the students' results and procedures (28, 59).

Visual representations help students be aware of their own mental processes when they are engaged in problem solving (48). Technology, especially dynamic geometry, helps teachers comfortably rely on their skills in order to initiate situations where students foster mathematical inquiry on their own (36). When planning lessons, teachers should try to predict the students' answers and prepare general and specific questions to scaffold the students' norms of argumentation-based inquiry and their ability to generalize mathematical ideas (8, 59). Developing teaching materials is time-consuming; therefore, several studies recommended that it is important to create a collection of teaching units with practical advice for developing and implementing activities to teach students how to solve mathematical problems (10, 13, 43).

Student-centred inquiry learning makes it possible for students to design and participate in mathematical experiments with dialogic approaches, so they can explain, discuss, and reflect upon their own ideas. However, it is crucial that teachers have the tools to extend and promote student thinking (7, 20). Students are urged to use *guess and check* methods when they lack ways to solve problems (56). Several analytical tools are relevant for teachers. One study mentioned that if the teacher scaffolds common solution patterns for students, they promote the students' modelling competency, in general (4). Other analytical tools help organize students' thinking while they are solving or posing problems concerning complex and interrelated phenomena (42, 54).

Inoue (59) addresses five important key points for effectively incorporating consensus building discussions in classrooms: 1) Know what you are asking; 2) Anticipate students' responses during lesson planning; 3) Release control to students; 4) Don't hesitate to provide traffic control; and 5) Always follow-up.

Professional Development and Collaboration

Professional development is an extensive theme that includes the development of teacher competencies, as well as how to plan and evaluate IBL and how to create and maintain teacher collaboration.

Teachers must develop their IBL teaching repertoire. Issues arise when teachers have not personally experienced an inquiry-based approach in their own education, when school hours are not sufficient for accommodating an IBL approach, when teaching is influenced by "teaching to the test", and when teachers are not supported by their colleagues (19).

Teachers need to broaden the repertoire they use to evaluate students. Evaluation is most likely constructed with no reference to theory. There is no easy way to evaluate students' mathematical competencies 14). A great concern for teachers is whether their students cover the curriculum when learning in the IBL paradigm. Another concern is that IBL does not offer the students the possibility to see and experience mathematics as a unique structure of interrelated concepts (17, 23; 1; 11).

Planning is time-consuming and difficult when using an inquiry-based approach, as almost every lesson generates a new interesting mathematical problem. The teacher needs to address problems with the potential of engaging all students in constructing and testing new mathematical hypotheses (22, 35, 47; 18). The teacher's own beliefs influence his/her choice of problems in an inquiry-based setting (15).

The teacher's knowledge of content and students (KCS) is central to his/her ability to listen and generate new questions. The teacher's specialized content knowledge (SCK) is also important, as he/she must engage in the students' thinking. It is not sufficient to simply have a high level of curricular content knowledge (CCK) (27, 55).

Professional development is a lengthy process when changing the teacher's approach to teaching (60; 24; 62). Longitudinal national projects have a better chance of following such an implementation strategy than time-limited international projects where the educational approaches often are different (21). Promoting IBL must be supplemented by a systematic and sustainable strategy to support the teacher's professional development (16). Utilizing small, already planned teaching units in an IBL context can contribute to a teacher's ability to reflect on how IBL is different from his/her own approach to teaching (13, 12). It is important that teachers feel that implementing IBL is meaningful (57, 61; 32). The support from school management is essential for creating positive and sustained teacher-beliefs about IBL (31, 61). The teacher's ability to involve parents has an impact on increasing the level of participation in model-eliciting activities with an interdisciplinary and realistic approach (30).

Summary and Implications for the Design of an Intervention

In this review, the relationship between the central issues in inquiry-based mathematics, as we have discussed here by describing five themes, covers a very broad area in mathematics education. The literature review includes journal articles about how to plan inquiry-based teaching, such as why it is necessary to choose different activities and why it is important to plan and predict communication. The review includes journal articles about how to teach using an IBL approach, such as how and why to give students ownership, how to make the teaching be activity-driven, how the teacher can take on a new role, and which tools teachers can use to make their teaching more inquiry-based. There are journal articles about the output students realize from working with inquiry-based mathematics, such as strengthening their creative skills and improving their flexible thinking skills, as well as improving other mathematical competences. Finally, some of the articles also focus on general professional development and, more specifically, on how to develop teachers, and why certain teacher knowledge is important for inquiry-based teaching. In general, we found that a number of issues have an impact on the possibilities for conducting inquiry-based mathematics teaching and learning. As described in the introduction, this literature review was conducted with the purpose of informing a large-scale intervention to improve mathematics and L1 education in Denmark. Therefore, it makes sense to consider the implications that we can draw from the overview generated by the review. The implications that the research results have on IBL practice can be viewed in many different ways. In general, we do not believe in a direct inference from research results to classroom practices, due to the complexity and situated nature of such practices. In this case, where the review was quite broad and was more concerned with the discussions that are prevalent in high ranked journals and less with identifying specific effective practices, we need to be particularly careful about drawing strong conclusions about translating knowledge into classroom sessions. To meet that concern and still relate our intervention to the knowledge presented in the review in an overt manner, we developed three design principles for our intervention. Those design principles are presented below, and they are discussed in light of our findings.

> Principle 1: An exploratory, dialogical, and application-oriented teaching method with room for student participation increases the effect of the student's understanding of mathematical concepts and develops appropriate ways of working.

By focusing on the exploration, application, the dialogic climate, the student's possibilities for participation, and the mathematical concepts, we integrate several of the key findings from the research survey into one principle. The dialogical climate and teacher-student communication have a major impact on whether investigative activities can be carried out. Student participation and a focus on the students' work process is one of the key didactical choices that the review showed will support inquiry. Finally, this principle stresses that we should not forget to focus on mathematical concepts; otherwise, there is a risk that the lack of mathematical knowledge will deprive students of the opportunity to participate.

> Principle 2: In order to enhance motivation and learning, we prioritize that the students' experience of the teaching and the content should be meaningful both from an internal mathematical perspective and from the perspective of the situation of application/inquiry.

We stress that our conception of meaningfulness should be related to several domains because the literature review shows that it is crucial that students be able to move in and out of the mathematical domain. By looking at *meaningfulness* across these domains, we support this movement and we also support that the communication between teachers and students should be based on the students' prior knowledge.

> Principle 3: An exploratory, dialogical, and application-oriented teaching approach with room for student participation increases the possibility of implementing mathematical competencies.

This principle is similar to principle 1, with one important difference. Principle 3 introduces the term mathematical competencies (Niss & Højgaard, 2011), which is a key concept in the Danish curricular standards. Principle 3 stresses the possibility of teaching mathematics in action through inquiry-based activities; it also stresses the necessity of students' mathematical competencies in order for inquiry-oriented teaching to succeed.

The three principles described above, which reflect the five important issues associated with IBL in mathematics, were the fundamental basis of the development of the design of a four-month inquiry-based mathematics teaching approach that was part of the KiDM didactical intervention design project.

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Appendix

Table 1: The 62 studies included in the review

Number	Author & year	Title	Journal	Pages
1	Jung, Hyunyi; Brady, Corey (2016)	Roles of a teacher and researcher during in situ professional development around the implementation of mathematical modeling tasks.	J. Math. Teach. Educ. 19, No. 2-3	277-295
2	Cai, Jinfa; Hwang, Stephen; Moyer, John C. (2016)	Mathematical problem posing as a measure of curricular effect on students' learning: a response.	Educ. Stud. Math. 91, No. 1	9-10
3	Kobiela, Marta; Lehrer, Richard (2015)	The codevelopment of mathematical concepts and the practice of defining.	J. Res. Math. Educ. 46, No. 4 (2015)	423-454
4	Schukajlow, Stanislaw; Kolter, Jana; Blum, Werner (2015)	Scaffolding mathematical modelling with a solution plan.	ZDM, Math. Educ. 47, No. 7	1241-1254
5	Tropper, Natalie; Leiss, Dominik; Hänze, Martin (2015)	Teachers' temporary support and worked-out examples as elements of scaffolding in mathematical modeling.	ZDM, Math. Educ. 47, No. 7	1225-1240
6	Stender, Peter; Kaiser, Gabriele (2015)	Scaffolding in complex modelling situations.	ZDM, Math. Educ. 47, No. 7	1255-1267
7	Calder, Nigel (2015)	Student wonderings: scaffolding student understanding within student-centred inquiry learning.	ZDM, Math. Educ. 47, No. 7	1121-1131
8	Makar, Katie; Bakker, Arthur; Ben-Zvi, Dani (2015)	Scaffolding norms of argumentation-based inquiry in a primary mathematics classroom.	ZDM, Math. Educ. 47, No. 7	1107-1120
9	Makar, Katie (2014)	Young children's explorations of average through informal inferential reasoning.	Educ. Stud. Math. 86, No. 1	61-78
10	Albarracín, Lluís; Gorgorió, Núria (2014)	Devising a plan to solve Fermi problems involving large numbers.	Educ. Stud. Math. 86, No. 1	79-96
11	Goodchild, Simon (2014)	Mathematics teaching development: learning from developmental research in Norway.	ZDM, Int. J. Math. Educ. 46, No. 2	305-316
12	Anthony, Glenda; Hunter, Roberta; Thompson, Zain (2014)	Expansive learning: lessons from one teacher's learning journey.	ZDM, Int. J. Math. Educ. 46, No. 2	279-291
13	Hähkiöniemi, Markus (2013)	Teacher's reflections on experimenting with technology-enriched inquiry-based mathematics teaching with a preplanned teaching unit.	J. Math. Behav. 32, No. 3	295-308
14	Frejd, Peter (2013)	Modes of modelling assessment – a literature review.	Educ. Stud. Math. 84, No. 3	413-438
15	Levenson, Esther (2013)	Tasks that may occasion mathematical creativity: teachers' choices.	J. Math. Teach. Educ. 16, No. 4	269-291
16	Dorier, Jean-Luc; García, Francisco Javier (2013)	Challenges and opportunities for the implementation of inquiry-based learning in day-to-day teaching.	ZDM, Int. J. Math. Educ. 45, No. 6	837-849
17	Schoenfeld, Alan H.; Kilpatrick, Jeremy (2013)	A US perspective on the implementation of inquiry-based learning in mathematics.	ZDM, Int. J. Math. Educ. 45, No. 6	901-909

18	Maaß, Katja; Doorman, Michiel (2013)	A model for a widespread implementation of inquiry-based learning.	ZDM, Int. J. Math. Educ. 45, No. 6	887-899
19	Chin, Erh-Tsung; Lin, Fou-Lai (2013)	A survey of the practice of a large-scale implementation of inquiry-based mathematics teaching: from Taiwan's perspective.	ZDM, Int. J. Math. Educ. 45, No. 6	919-923
20	Engeln, Katrin; Euler, Manfred; Maass, Katja (2013)	Inquiry-based learning in mathematics and science: a comparative baseline study of teachers' beliefs and practices across 12 European countries.	ZDM, Int. J. Math. Educ. 45, No. 6	823-836
21	Maaß, Katja; Artigue, Michéle (2013)	Implementation of inquiry-based learning in day-to-day teaching: a synthesis.	ZDM, Int. J. Math. Educ. 45, No. 6	779-795
22	Goodchild, Simon; Fuglestad, Anne Berit; Jaworski, Barbara (2013)	Critical alignment in inquiry-based practice in developing mathematics teaching.	Educ. Stud. Math. 84, No. 3	393-412
23	Swan, Malcolm; Pead, Daniel; Doorman, Michiel; Mooldijk, Ad (2013)	Designing and using professional development resources for inquiry-based learning.	ZDM, Int. J. Math. Educ. 45, No. 7	945-957
24	Stillman, Gloria Ann (2013)	Implementation of IBL in Europe from an Australasian perspective.	ZDM, Int. J. Math. Educ. 45, No. 6	911-918
25	Artigue, Michële; Blomhøj, Morten (2013)	Conceptualizing inquiry-based education in mathematics.	ZDM, Int. J. Math. Educ. 45, No. 6	797-810
26	Bruder, Regina; Prescott, Anne (2013)	Research evidence on the benefits of IBL.	ZDM, Int. J. Math. Educ. 45, No. 6	811-822
27	Johnson, Estrella (2013)	Teachers' mathematical activity in inquiry-oriented instruction.	J. Math. Behav. 32, No. 4	761-775
28	Towers, Jo; Martin, Lyndon C.; Heater, Brenda (2013)	Teaching and learning mathematics in the collective.	J. Math. Behav. 32, No. 3	424-433
29	Kim, Young Rae; Park, Mi Sun; Moore, Tamara J.; Varma, Sashank (2013)	Multiple levels of metacognition and their elicitation through complex problem-solving tasks.	J. Math. Behav. 32, No. 3	377-396
30	Mousoulides, Nicholas G. (2013)	Facilitating parental engagement in school mathematics and science through inquiry-based learning: an examination of teachers' and parents' beliefs.	ZDM, Int. J. Math. Educ. 45, No. 6	863-874
31	Krainer, Konrad; Zehetmeier, Stefan (2013)	Inquiry-based learning for students, teachers, researchers, and representatives of educational administration and policy: reflections on a nation-wide initiative fostering educational innovations.	ZDM, Int. J. Math. Educ. 45, No. 6	875-886
32	Wake, Geoffrey David; Burkhardt, Hugh (2013)	Understanding the European policy landscape and its impact on change in mathematics and science pedagogies.	ZDM, Int. J. Math. Educ. 45, No. 6	851-861
33	Pepin, B.; Gueudet, G.; Trouche, L. (2013)	Investigating textbooks as crucial interfaces between culture, policy and teacher curricular practice: two	ZDM, Int. J. Math. Educ. 45, No. 5	685-698

		contrasted case studies in France and Norway.		
34	Krummheuer, Götz; Leuzinger-Bohleber, Marianne; Müller-Kirchof, Marion; Münz, Melanie; Vogel, Rose (2013)	Explaining the mathematical creativity of a young boy: an interdisciplinary venture between mathematics education and psychoanalysis.	Educ. Stud. Math. 84, No. 2	183-199
35	Lehrer, Richard; Kobiela, Marta; Weinberg, Paul J. (2013)	Cultivating inquiry about space in a middle school mathematics classroom.	ZDM, Int. J. Math. Educ. 45, No. 3	365-376
36	Leikin, Roza; Grossman, Dorith (2013)	Teachers modify geometry problems: from proof to investigation.	Educ. Stud. Math. 82, No. 3	515-531
37	Van Harpen, Xianwei Y.; Sriraman, Bharath (2013)	Creativity and mathematical problem posing: an analysis of high school students' mathematical problem posing in China and the USA.	Educ. Stud. Math. 82, No. 2	201-221
38	Bonotto, Cinzia (2013)	Artifacts as sources for problem-posing activities.	Educ. Stud. Math. 83, No. 1	37-55
39	Cai, Jinfa; Moyer, John C.; Wang, Ning; Hwang, Stephen; Nie, Bikai; Garber, Tammy (2013)	Mathematical problem posing as a measure of curricular effect on students' learning.	Educ. Stud. Math. 83, No. 1	57-69
40	Van Harpen, Xianwei Y.; Presmeg, Norma C. (2013)	An investigation of relationships between students' mathematical problem-posing abilities and their mathematical content knowledge.	Educ. Stud. Math. 83, No. 1	117-132
41	Silver, Edward A. (2013)	Problem-posing research in mathematics education: looking back, looking around, and looking ahead.	Educ. Stud. Math. 83, No. 1	157-162
42	Singer, Florence Mihaela; Voica, Cristian (2013)	A problem-solving conceptual framework and its implications in designing problem-posing tasks.	Educ. Stud. Math. 83, No. 1	9-26
43	Leung, Shuk-kwan S. (2013)	Teachers implementing mathematical problem posing in the classroom: challenges and strategies.	Educ. Stud. Math. 83, No. 1	103-116
44	Sarrazy, Bernard; Novotná, Jarmila (2013)	Didactical contract and responsiveness to didactical contract: a theoretical framework for enquiry into students' creativity in mathematics.	ZDM, Int. J. Math. Educ. 45, No. 2	281-293
45	Sriraman, Bharath; Haavold, Per; Lee, Kyeonghwa (2013)	Mathematical creativity and giftedness: a commentary on and review of theory, new operational views, and ways forward.	ZDM, Int. J. Math. Educ. 45, No. 2	215-225
46	Kattou, Maria; Kontoyianni, Katerina; Pitta-Pantazi, Demetra; Christou, Constantinos (2013)	Connecting mathematical creativity to mathematical ability.	ZDM, Int. J. Math. Educ. 45, No. 2	167-181
47	Kim, Dongwon; Ju, Mi-Kyung (2012)	A changing trajectory of proof learning in the geometry inquiry classroom.	ZDM, Int. J. Math. Educ. 44, No. 2	149-160

48	Csíkos, Csaba; Szitányi, Judit; Kelemen, Rita (2012)	The effects of using drawings in developing young children's mathematical word problem solving: a design experiment with third-grade hungarian students.	Educ. Stud. Math. 81, No. 1	47--65
49	Erfjord, Ingvald; Hundeland, Per Sigurd; Carlsen, Martin (2012)	Kindergarten teachers' accounts of their developing mathematical practice.	ZDM, Int. J. Math. Educ. 44, No. 5	653-664
50	Jaworski, Barbara (2012)	Mathematics teaching development as a human practice: identifying and drawing the threads.	ZDM, Int. J. Math. Educ. 44, No. 5	613-625
51	Prusak, Naomi; Hershkowitz, Rina; Schwarz, Baruch B. (2012)	From visual reasoning to logical necessity through argumentative design.	Educ. Stud. Math. 79, No. 1	19-40
52	Schukajlow, Stanislaw; Leiss, Dominik; Pekrun, Reinhard; Blum, Werner; Müller, Marcel (2012)	Teaching methods for modelling problems and students' task-specific enjoyment, value, interest and self-efficacy expectations.	Educ. Stud. Math. 79, No. 2	215-237
53	Murata, Aki; Kattubadi, Sailaja (2012)	Grade 3 students' mathematization through modeling: situation models and solution models with multi-digit subtraction problem solving.	J. Math. Behav. 31, No. 1	15-28
54	Kontorovich, Igor; Koichu, Boris; Leikin, Roza; Berman, Avi (2012)	An exploratory framework for handling the complexity of mathematical problem posing in small groups.	J. Math. Behav. 31, No. 1	149-161
55	Johnson, Estrella M.S.; Larsen, Sean P. (2012)	Teacher listening: the role of knowledge of content and students.	J. Math. Behav. 31, No. 1	117-129
56	Capraro, Mary Margaret; An, Song A.; Ma, Tingting; Rangel-Chavez, A.Fabiola; Harbaugh, Adam (2012)	An investigation of preservice teachers' use of guess and check in solving a semi open-ended mathematics problem.	J. Math. Behav. 31, No. 1	105-116
57	Chapman, Olive (2011)	Elementary school teachers' growth in inquiry-based teaching of mathematics.	ZDM, Int. J. Math. Educ. 43, No. 6-7	951-963
58	Peled, Irit; Balacheff, Nicolas (2011)	Beyond realistic considerations: modeling conceptions and controls in task examples with simple word problems.	ZDM, Int. J. Math. Educ. 43, No. 2	307-315
59	Inoue, Noriyuki (2011)	Zen and the art of neriage: facilitating consensus building in mathematics inquiry lessons through lesson study.	J. Math. Teach. Educ. 14, No. 1	5-23
60	Chapman, Olive; Heater, Brenda (2010)	Understanding change through a high school mathematics teacher's journey to inquiry-based teaching.	J. Math. Teach. Educ. 13, No. 6	445-458
61	Towers, Jo (2010)	Learning to teach mathematics through inquiry: a focus on the relationship between describing and enacting inquiry-oriented teaching.	J. Math. Teach. Educ. 13, No. 3	243-263
62	Hunter, Roberta (2010)	Changing roles and identities in the construction of a community of mathematical inquiry.	J. Math. Teach. Educ. 13, No. 5,	397-409

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