Skrifter från SMDF, Nr. 17

Bringing Nordic mathematics education into the future

Proceedings of Norma 20
The ninth Nordic Conference on Mathematics Education
Oslo, 2021

Editors:
Nortvedt, G. A., Buchholtz, N. F., Fauskanger, J., Hähkiöniemi, M., Jessen, B. E., Naalsund, M., Nilsen, H. K., Pálsdóttir, G., Portaankorva-Koivisto, P., Radišić, J., Sigurjónsson, J. Ö., Viirman, O. and Wernberg, A.

SMDF Svensk förening för MatematikDidaktisk Forskning
Swedish Society for Research in Mathematics Education

Skrifter från Svensk förening för MatematikDidaktisk Forskning, Nr 17

ISBN 978-91-984024-6-9
ISSN 1651-3274

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The report is published at matematikdidaktik.org

Preface

The story about the NORMA20 conference is the story about a conference that had to be postponed, as a pandemic prevented us from coming together to present and discuss our research. When we made the decision to postpone just after Easter in 2020, we were all sad and disappointed that all the efforts made by authors, reviewers, LOC and IPC members might be for nothing. However, during 2020, digital solutions for cooperation emerged so we could host NORMA20 as an online conference in June 2021.

Delaying the conference had both positive and negative consequences. On the positive side, postponing allowed us to make a call for contributions in both 2019 and 2020, substantially increasing the number of submissions. Not only was the number of submissions large but the number of accepted submissions also increased. During the conference, 72 regular papers, 30 short communications, nine poster presentations, four keynotes, and one working group were presented. This made the NORMA20 conference the largest NORMA conference to date.

Moreover, the prolonged period gave authors, reviewers, and the IPC more time to work on regular papers, resulting in the conference report comprising two volumes: the conference preceedings published in 2021 and the conference proceedings in 2022. A large number of proposals originally submitted following the first call was revised and resubmitted during Fall 2020, allowing the IPC time to process these papers before the conference. At the same time, the conference received a high number of new submissions during November 2020 in response to the second call, thus adding to the number of papers accepted for presentation. After the digital conference, many of these papers were revised and resubmitted, using both the reviewer feedback and the feedback received during the conference. After a second review process, the accepted papers make up the proceedings presented in this volume.

While the conference preceedings included 36 regular papers, the volume you are reading now contains another 37 papers, 73[1] in total. As such, the preceedings and proceedings taken together offer valuable insights into current research on mathematics education in Nordic countries. The conference theme concerned what it takes to bring mathematics education in the Nordic area into the future; consequently, topics such as programming, classroom teaching and learning, teacher education, early-years education, and assessment were frequent. The peer-reviewed paper written by Fauskanger et al. (included in this volume) analyses the contributions made by the regular papers' authors in terms of not only the significance of their work but also the conference.

The NORMA20 conference also differed from physical conferences in other aspects. Compared to the high number of presentations, the number of participants was low (N = 184), meaning that most participants gave presentations and chaired sessions. The NORMA community is close knit, and most participants also served as reviewers of one or more proposals. In our opinion, this involvement by the participants contributed to the conference's success and the active discussions following most presentations.

Approximately one third of the participants (N = 60) completed the conference evaluation survey. Overall, the participants were pleased with the technical solutions used for the conference and the organisation of the conference programme. More importantly, the participants were satisfied with the quality of the scientific presentations. In total, 78% of respondents ranked the quality of keynotes and regular papers as high.

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[1] One paper accepted for presentation and publication in the preceedings could not be presented during the conference.

Seemingly, the digital format did not hinder participation. In total, 77% of respondents attended all or most keynote sessions, and 67% attended all or most regular paper sessions. Even though many stated that it felt exhausting sitting in front of a screen all day, participation was high. Fewer respondents attended short communications, the poster session, and the working group; however, among those who reported having attended, 7 in 10 participants ranked the quality of these sessions as high.

A digital conference comes at a cost. While an online forum can foster scientific discussions after presentations and opportunities for informal talks, networking and continuing scientific discussions outside the lecture room are scarce, which is reflected in the rating of networking opportunities. Merely 3 in 10 respondents ranked these opportunities as high.

The digital format is environmentally and financially friendly and saves time, since a large number of academics do not travel to the conference venue by air, rail, or road. Not preparing lunches and coffee breaks similarly saves money and prevents waste. The past year has likely opened our eyes to the possibilities for online events and the ease with which they might be organised and attended compared to physical events. Still many of us prefer physical conferences. This is reflected in the opinions of respondents, where the majority find it likely or highly likely that they will attend a physical (85%), hybrid (75%) or digital (63%) conference in the future.

Judging by our own experiences and the feedback from the questionnaire, we were surprised by how well discussions ran even though we were not physically in the same room; participants were warm, inviting, and welcoming toward each other while offering constructive feedback. Thus, organising and participating in the NORMA20 conference was rewarding.

The IPC warmly thanks all the participants, authors, and reviewers who contributed to the conference proceedings.

Oslo, June 2022

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The ”Project Forum” as a learning community for mathematics teacher educators in Denmark

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In Denmark, there is no formal education of educators (lecturers) of teachers for the compulsory school. The National Knowledge Center for Mathematics Education (NAVIMAT) developed, tried out and evaluated ”Project Forum” (PF) during 2008–2011 to support the professional development (PD) of lecturers involved in projects with research connection (RDP). We studied the potentials for PF to become a learning community (LC) and how it contributed to the lecturers' PD from a participationist perspective on learning. Although most lecturers saw a potential for PF to become a LC, other lecturers' attitudes were a main obstacle for building a LC.

Keywords: Teacher trainers, Professional development, Mathematics education, Learning communities, Participationist perspective.

Objectives

This paper focuses on mathematics teacher educators. We present a study of Project Forum (PF), which was a structure for organizing research and development projects (RDPs), developed and tried out in Denmark by The National Knowledge Center for Mathematics Education (NAVIMAT) during 2008–2011 as a model for professional development (PD) for mathematics teacher educators. The present study is an extension of an external evaluation of PF (Ejersbo, 2010). The reason for this re-examination was the desire to gain general insight into processes of PD from the perspective of the participants' reactions. Whereas Ejersbo's evaluation reported on *how* the participants responded, we focus on *why* they reacted to PF the way they did. Our focus is the question: *To what extent did PF possess the potential to become a learning community?* The object of study was PF as an interactive way of organizing sustainable and continuous professional development. As a means for structuring and analysing the data we present a new tool (Table 1), which was constructed in line with recent research on ”how theories affect the practical knowledge of teachers and how teachers perceive that their practice fits with theoretical issues” (Hospesová, Carillo & Santos, 2018, p. 190).

The Danish context

Teacher education for compulsory school is at bachelor level and takes place at university colleges (UC) in contrast with the research based upper-secondary teacher education at the universities (Dahl, 2010). There is no governmental programme for educating UC teacher educators (henceforth: lecturers); traditionally, they are recruited amongst experienced schoolteachers, often with supplementary education. There is a tradition for developmental projects (DPs) which often have a double aim: to improve school teaching, e.g., by classroom interventions and, indirectly, to improve UC teaching. Time to do DPs is included in the lecturers' workload, frequently co-funded by external projects. DPs might be research based, and they might involve lectures from more than one UC. There was no tradition for a formal dissemination of the results; the newly gained professional knowledge had an impact only on the individual participants (EVA, 2003).

Furthermore, in the early 2000s, there was a growing political awareness of the quality of (teacher) education and in-service training. The Danish Evaluation Institute (EVA) published critical reports (2003, 2006) about the relations between research and the professional context, and about the quality of continuing education. NAVIMAT was therefore established 2008–2011 by UVM with an obligation to create, organize, evaluate, and disseminate pedagogical content knowledge in mathematics education. NAVIMAT was a collaboration project between all Danish UC's, funded by UVM (total budget: 1.5 million EUR).

The PF model

It is important to notice that the PF model was established without reference to research literature or experience. NAVIMAT and the PF model must be understood in its historical context. Collaborative environments and models for sharing experiences were ideas reflected in mathematics education research (cf. Hošpesová et al., 2018). Although not research-based, the PF model was inspired by mainstream ideas at that time in the professional community of educators about teachers' collaboration, action learning and peer-supervision.

NAVIMAT's activities encompassed 17 RDPs (45 lecturers in total); almost all involving mathematics schoolteachers and classes. The RDPs were completed within six focus areas: 1. Teacher education with production of new teaching materials (2 projects); 2. Continuing education for teachers (1 project); 3. Professional development for lecturers (8 projects); 4. Elementary school teaching (1 project); 5. Collaboration and counselling school authorities (3 projects); and 6. International collaboration (2 projects). The lectures received reductions of 350-1600 working hours, on average 650, to compensate for the time spent in the RDPs. Although the 17 RDPs were organized in PF, each of them had lecturers as local project managers, individual budgets, time schedules and expected deliveries such as written reports, seminars, posters etc. PF was managed by the director of NAVIMAT, three regional NAVIMAT leaders who were responsible for evaluation and support of the projects, and a PF project leader, all experienced mathematics teacher educators. A team of four to six university researchers in mathematics education, the managers and all the project members (lectures) participated in PF's five 1-day seminars per year. The seminars intended to be a platform where the lectures could exchange experiences, present work in progress, receive critique and discuss in groups and in plenary sessions. A typical 1-day seminar would be from 10-16 o'clock and the programme would alternate between 1-hour talks from external plenary speakers, 20 minutes presentations from some of the RDPs, hours alone for the RDPs to work and get feedback from the researchers, and more general status discussions and evaluations of the PF. The 2-day meeting consisted of similar elements, besides a writing course. The researchers had two tasks at the seminars: give plenary talks and participate in the subsequent discussions, and advise the projects on e.g. the dissemination of results. The lecturers were invited to take part in planning the seminars including choosing topics for the external plenary speakers. Also, due to the lecturers' feedback, in one case, a meeting was changed from a 1-day meeting to a 2-day meeting (Ejersbo, 2010).

Theoretical framework

A "learning community" as a framework for PF

Although not established with reference to research, PF was in line with Jaworski's (2005) descriptions of a LC as far as the purpose of PF was to establish an inquiry-based community where all members were seen as learners in the professional discourse. Also corresponding to Jaworski (2005), the participating lecturers, university researchers and NAVIMAT leaders should learn from each other during inquiry, in peer relations without anyone taking the leadership based on formal authority (NAVIMAT) or scientific expertise (university researchers). The intention was to launch PF interactively as a joint effort, with the double aim to improve both the individual RDPs and the level of shared knowledge. This was reflected, e.g., in the above-mentioned fact that the lecturers were encouraged to come up with ideas about the topic of the plenaries.

A discursive approach to study the lecturers' reactions to PF

The study, of which only one part is presented in this paper, focused on the lecturers' reactions to PF as they appear in the old evaluation report, but taking a discursive approach (Forman & McCormick, 1995; Sierpinska, 2005). In this approach, "discourse" is understood as sets of linguistic material that is coherent in organization and content, and which enables people to construct meaning in social contexts (Cohen, Manion & Morrison, 2000). This implies that we perceive the lecturers' written reflections on PF as incidents of meaning construction in the lecturers' and researchers' professional discourse. From the participationist perspective applied in this paper, the lecturers' learning is associated with their progressive participation in the professional discourse (Erath, Prediger, Quasthoff & Heller, 2018; Sfard, 2008). Therefore, the object of study was neither the individual RDPs nor their outcome, but PF seen as a professional development (PD) activity for the lecturers. PD is here understood "in relation to the knowledge construction or the incremental refinement of practice or both" (Lin & Rowland, 2016, p. 499). According to Ejersbo (2010), PF intended to set the scene for meta learning from the RDP activities. In PF, the members should learn about working in, completing, evaluating, and reflecting on an RDP. These activities are all elements of the professional discourse. The meta-learning was supported by the sharing of, and collective reflection upon experiences from the RDPs they were involved with. In retrospect PF, hence, was established in accordance with in what Hošpesová et al. (2018) refer to as the co-learning inquiry paradigm in which "being involved in action and reflection collaboratively enables the participants, teachers and teacher educators, to achieve a deeper understanding of both their own world and the world of the other participants in the community" (p. 189). According to Lin and Rowland (2016, p. 506), "the notion of *inquiry community* brings together characteristics of being together and exploring for triggering professional development" with reference to ideas from activity theory as its basis. At the time, collaborative exchanges and reflections in PF were the means for the lecturers' progressive participation in the professional discourse.

PF's potentials for success

NAVIMAT can be perceived as an element of a top-down educational policy aimed at higher quality in education by means of PD activities including PF. Sowder's (2007) review of research results, essential for the successful implementation of professional development pinpoints the importance of determining the purpose of a PD programme, teachers in deciding on foci, and collaborative problem solving. These were all essential elements of PF. Fullan and Hargreaves (1992) pointed out four main elements of a framework crucial for the understanding of teacher development: 1) The teachers' purpose, 2) The teacher as a person, 3) The real-world context in which teachers work and 4) The culture of teaching; the working relationship teachers have with their colleagues inside and outside the school. The PD activity must resonate with all four elements. Applying the discursive and participationist perspective on the lecturers' development allowed us to adopt and adapt Cobb's framework (Yackel, Gravemeijer & Sfard, 2011). This framework was developed for analysis and interpretation of students' learning and interaction in mathematics classrooms from a sociological and a psychological perspective. We also wanted to analyse and interpret lecturers' learning from PD from those perspectives. We took Fullan and Hargreaves' four elements into account by incorporating real world context and culture of lecturing in the sociological perspective and lecturers' purpose and lecturer as a person in the psychological perspective. Our adapted version of Cobb's framework was operationalised into a tool (matrix in Table 1) for analysis of the lecturers' reactions to the goals and means of PF. Each of the matrix's cells would serve as a code. Thereby, the matrix induced a structure on data which allowed for a fine-grained analysis based on the adapted framework of the lecturers' reactions to key elements of PF:

- Column A entails PF's key ingredients sorted out in the matrix's rows with the following headings: main goals, activities, and interactions.
- Column B contains the lecturers' professional context. This captures reactions and reflections rooted in the lecturers' professional context like, e.g., shared norms amongst peers about lecturers' role in RDP, formal regulations for lecturers' position and (prior) experiences with RDP interaction. Column B, hence, can be associated with a sociological perspective on the lecturers' development.
- Column C contains the lecturers' professional profile. This captures reactions and reflections rooted in the lecturers' professional profile meaning their background, personal professional visions and situation including, e.g., their view on RDP, personal beliefs about teaching and learning, interaction with peers and self-confidence. Column C, hence, can be associated with a psychological perspective on the lecturers' development.

Methods

NAVIMAT had hired a professional evaluator to observe and collect qualitative and quantitative data about the participants' experiences of PF. Data for the present study was the resulting evaluation report including all responses (Ejersbo, 2010) plus background documents from NAVIMAT forming the content of column A in Table 1. Data was analysed by qualitative coding (Kvale, 2001) of the participants' responses to the evaluation's questionnaire. Out of 38 recipients (including four researchers and the NAVIMAT manager), 30 lecturers responded to the questionnaire and gave at most one answer per question. The matrix in Table 1 was used to organise the qualitative coding. In the following, coding like (1.A.c) refers to the according cell in the matrix whereas for example (2.B.x) refers to the cluster (2.B.a, 2.B.b, etc.) of cells.

Perspective *PF participant's learning*	***A. The project's (PF's) theoretical perspective:*** *Content and organisation of the project, underlying principles and decisions*	***B. Sociological perspective on RDP activities:*** *Professional context for the PF participant's work with his/her project*	***C. Psychological perspective on RDP activities:*** *PF participant's ideas and values, emotions, beliefs and visions*
1. What: The main goal to provide to each participant	1.A.1: Personal experiences and new insights	1.B.1	1.C.1
	1.A.2: Core mathematics education competencies with regard to: 1.A.2.a: Assessing and developing the level of research- and development projects' quality within the field	1.B.2.a	1.C.2.a
	1.A.2.b: Completing of the project including evaluation of it, and dissemination and appreciation of its results	1.B.2.b	1.C.2.b
	1.A.2.c: Competently taking the role as mentor or supervisor in peer to peer relations with colleagues within the field	1.B.2.c	1.C.2.c
2. How: Structure of PF. The PF activities included:	2.A.a: Presentations of individual projects in progress with response	2.B.a	2.C.a
	2.A.b: External talks in plenum	2.B.b	2.C.b
	2.A.c: Group mentoring by researchers and local leaders from NAVIMAT	2.B.c	2.C.c
	2.A.d: Plenum discussions about the individual projects	2.B.d	2.C.d
	2.A.e: Midway-evaluation sheets and reports	2.B.e	2.C.e
3. How: Interactions in PF	3.A: The PF participants were supposed to: 3.A.a: Share their experiences with the work on their project under and after the projects working phases	3.B.a	3.C.a
	3.A.b: Present their own work in plenum	3.B.b	3.C.b
	3.A.c: Listen to the other groups' presentations and give response to the groups	3.B.c	3.C.c

Table 1. Operationalization of theoretical framework, including the codes

The questionnaire's questions were: 1. What elements of PF have influenced your personal competence development? 2. What elements of PF have had the strongest influence on your project's development? 3. What are the phases where your project has developed mostly? 4. How far was the project disseminated? 5. Did PF meet your expectations? Split into: A: What did you expect from PF? and B: Did PF meet your expectations? 6. PF as a learning community. Split into: A: What are the strengths? B: Weaknesses? C: How could the model be improved? and D: Comments?

The data was analysed in two steps. First step was to get an overall impression of how PF suited the lecturers' ideas about their own PD by summarizing the answers to the questionnaire's question 1 about the lecturers' perceived support from PF. Thereafter, the matrix in Table 1 was applied to code all responses. The coding was non-disjunctive and based on meaning condensation in our interpretations. Each cell served as a code; hence all responses were assigned to one or more cells. In the second step, statements from each cell in columns B and C in Table 1 were compared to column A, seeking for signs of correspondence, synergy, and discrepancies between the lecturers' points of views and the purposes of PF. The purpose of the second step was to analyse the lecturers' progressive participation in the discourse with the aim to see the potentials of PF for becoming a LC.

Analysis

Elements of PF pinpointed by the lecturers

The personal experiences with collaboration (1.A.x) were valued. Quote: ”It was good to get the opportunity to collaborate with others – particularly with the researchers” (2.A.c). The PF structure (2.A.x) gave them the opportunity to participate in concrete developmental work. Some clearly found that they had received good responses from others when presenting (2.A.a). Quote: ”The working day with the researchers and the 2-days meeting was very efficient.” (1.A.1) The paper-writing course was highly valued as immediately applicable. Overall, the collaboration with the researchers was valued, (2.A.x) e.g., in one case the lectures needed, and got, help with the methodology. They also pinpointed insights into other's projects (1.A.2.c) and plenary talks (2.A.b). However, some found that they had received poor and/or insufficient responses (2.A.a), and scarce contact with the researchers (2.A.c). One mentioned a lack of continuity, i.e. no chance to follow the progress of others' projects (2.A.a). Start-up help for the projects was called for in one answer. In general, most of the lecturers valued elements of PF that are also characteristics of a LC.

Lecturers' participation in the PF discourse

The lecturers' progressive participation in the discourse was analysed with regard to some expected norms for interaction in a LC (3.A.x). During the analysis, all data coded with 3.B, i.e. the sociological perspective, were extracted, interpreted and their meaning condensed. Each of the 30 respondents gave none, or a single answer to each question. There is a pattern recurring in each question which, to some degree, must reflect the variations of views amongst the respondents: The responses divided into three groups: i) More or less specific negative responses, ii) More or less specific positive responses, and iii) Responses concerning missing opportunities (or disappointed expectations).

<u>i) The negative responses</u> focus mainly on too much time spent on PF, irrelevance of other groups' projects, poor performance in PF by other groups, poor contact with researchers, and the organisation of PF. Quotes: ”Discrepancy between time for project work and time for PF. … It is not effective enough - the main effect is meeting with the researchers, and this will be done better within the project with focus on one's own project's issues. … The attendance rate has been too volatile”; and ”It was difficult for the groups to provide sufficient grounds for the discussions in plenary to bring the projects forward. Interest in and benefit of knowing and discussing others' projects was limited”, and finally: ”A change needs to be made if the project participants really have to engage in other people's projects, and exercise in providing sparring”.

Compared to the ideas of the LC regarding expected activity, the negative group is, apparently, far from agreeing. The expressed irrelevance of other groups' projects is a clear obstacle. The poor contact with the researchers is intertwined with the very organisation of PF because the researchers only had obligations with relation to the projects via PF. The wording is remarkable; some of the respondents ask for researchers' guidance although the term is normally used in relations with students. This is in contrast with the LC's peer relations where no one should take leadership.

<u>ii) The positive responses</u> focus mainly on meeting with colleagues from other UC's, insight into others' projects, response and inspiration from others, contact with researchers, and input from external lecturers. Quote: ”Everyone gets the opportunity to be … in contact with the other projects and project participants, and thus gain insight into a number of issues and works. Everyone gets the

opportunity to formulate thoughts and ideas in connection with their own project, get exercise in giving sparring to others and get input from the research team and from external presenters”.

The positive group ii) is thus more in line with the ideas about a LC.

iii) Missed opportunities mentioned are mainly: time to spend on own project, motivation for engagement in others’ projects, closer contact with others’ projects, and guidance by researchers. The missed opportunities could be interpreted as a (little) more positive wording of the same concerns as the negative responses express.

Results and discussion

In the analysis, we saw that the lectures overall were positive towards the impact of PF on their learning, although some critique was also apparent. Interaction with the researchers and with the group of colleague lecturers were valued features of PF: (1) the researchers contributed twofold: a) direct involvement and interaction with the lecturers and the RDP, and b) more indirectly through plenary lectures; (2) the group of colleague lecturers gained knowledge in two ways: c) by direct feedback from colleagues and d) by listening to colleagues’ presentations of their own RDPs.

All these four elements, a)-d), are essential for a successful realization of the PF model. Our results point to the balance between them as an issue for discussion. These results are in accordance with the evaluation report (Ejersbo, 2010) and thus the re-analysis reveals an apparent discrepancy between some of the lecturers’ expectations and the goals of PF. The re-analysis was an attempt to ”zoom in” on the lecturers’ reactions to PF, to get a clearer and more detailed picture of both promotional elements and obstacles for the establishment of a LC in that context.

The analysis revealed a split in the group of respondents, one large group expressing interest for interaction with the other members and RDPs, and the other group expressing a lack of interest. The issues of time spent in PF and, more generally, the distribution of resources, overshadowed largely all discussion about the content and potentials of a LC community in the negative answers.

Conclusion

The results from the re-analysis of data from the evaluation report do not contradict the old report (Ejersbo, 2010). The large group of positive lectures pinpoint elements that would support the establishment of a LC. The re-analysis adds new results by pointing out the reactions by a group of lecturers as a main obstacle for building a LC in Jaworski’s sense (2005). Use of the matrix structure (Table 1) leads to a more detailed insight into the lecturers’ reactions compared to the old evaluation report. The re-analysis, thereby, provides a more nuanced answer to the question about the potential of PF. The establishment of a LC would not only take more time than spent in PF, it would also request stronger agreements between all the participants and the management about means and goals for mathematics teacher educators.

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Representational infrastructures' mediation on mathematical communication

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When implementing digital technologies into mathematics teaching and learning situations, new representational and communication infrastructures arise. The paper presents an example of functions as covariation, taken from a task in a Danish 8th-grade classroom. Results show that infrastructures mediate on students' mathematical communication. When students drag, continuous treatments between representations take place. The dragging not only influences on the transformation and manipulation of mathematical objects – it also mediates on the students' language. Looking at verbs and adjectives, the students' words become more dynamic and focus on the movements of mathematical objects within the digital tool. From a competency point of view, the students must be aware of different mathematical registers in which communication takes place. When working with digital technologies, students must thus be able to co-activate several mathematical competencies.

Keywords: Representational infrastructures, digital technologies, communication, competencies.

Introduction

When introducing digital technologies (DT) into the mathematics classroom, e.g. Dynamic Geometry Environments (DGE) or Computer Algebra Systems (CAS), new representational infrastructures follow (Hegedus & Armella-Moreno, 2009). From the literature, it is well-known that these new representational infrastructures both alter and provide new opportunities for mathematical communication (e.g. Arzarello & Robutti, 2010; Hegedus & Armella-Moreno, 2009). For example, in Danish upper secondary school, the CAS-word "solve" is now an integral part of students' mathematical vocabulary, and they can even conjugate the word in Danish, saying "jeg solver", meaning "I solve" (Jankvist et al., 2019). Other examples concern students who refer to "buttons" or functionalities within DGE, e.g. that of "dragging" which is not a usual paper-and-pencil process (Ng, 2016), or students who provide dynamic explanations due to the influence of the DGE in use (Antonini et al., 2020; Ng, 2016). In fact, several research studies indicate that the use of dynamic representations mediates on the students' language (Arzarello & Robutti, 2010). Dragging holds potentials, such as allowing students to "see the behaviour of (and even interact with) the mathematical objects over time" (Antonini et al., 2020, p. 10). Nevertheless, in mathematical communication, dragging reveals changes in verbs and adjectives: Verbs are expressed in continuous tense by saying it "is increasing" or "decreasing", or involve some kind of change, such as "change" or "move" (Ng, 2019, p. 1190). Students may also indicate movement and change using adjectives, such as "if", "then" and "as" (Ng, 2016).

Hence, several studies indicate that when using DT, and in particular dynamic DT, students' mathematical explanations come to involve some kind of motion, which the usual static paper-and-pencil approach does not. In fact, research on both oral and written communication finds that students' expressions are mediated by the dynamic aspects of the DT (e.g., Antonini et al., 2020). Furthermore, the new representational infrastructure, which contains dynamic representations of mathematical objects (Arzarello & Robutti, 2010), makes it possible to manipulate mathematical representations as real objects (Duval, 2017). From a development (and assessment) of mathematical competencies (Niss & Højgaard, 2019) point of view, it is still an open question to what extent we should accept such new ways of communicating mathematically, when we offer our students new digital means for representing and communicating mathematics. As phrased by Duval (2017),

> [...] the use of a computer for everything that concerns mathematical visualization, both in geometry and in analysis, and geometrical or graphical software opens considerable possibilities of creation and visual exploration. But does software suffice to develop in the students the ability to anticipate the different possible transformations of a given figure into others completely different? Does it make students aware of the one-to-one mapping between graphic visual values and the terms of the equations they represent? (p. xii)

The question we ask in this paper is: *How can students' participation in activities with digital tools' involving representational infrastructures' mediation on the students' mathematical communication be addressed from a mathematical competencies point of view?* As part of our answer, we shall also touch upon some of the issues raised by Duval in the quote above. We address the research question by means of 8th-grade students' answers to a task involving GeoGebra. Yet, before we present and analyse these data, we first provide the theoretical constructs that we rely on in the analysis.

Theoretical constructs in use

The Danish competency framework, KOM, defines *mathematical competency* as "someone's insightful readiness to act appropriately in response to a specific sort of mathematical challenge in given situations" (Niss & Højgaard, 2019, p. 14). Within KOM, eight mathematical competencies are defined, among them: aids and tool competency; communication competency; and representation competency. The aids and tool competency concerns all kinds of tools used in mathematics (e.g., rulers), including DT. This competency's key element is the student's ability to use a tool in solving tasks in a critical manner (both about one's own use and others'), and to know the limitations and potentials of the tools; including reflectively choosing a tool for a task (Niss & Højgaard, 2019).

On the one hand, *communication competency* involves the ability to interpret others' mathematical expressions. On the other hand, it involves the ability to express oneself mathematically. Mathematical communication exists in different media (e.g., written, oral, visual or gestural), in different genres, registers, and to different target groups of people (Niss & Højgaard, 2019). Building on linguists, Pimm (1987) defines communicative competence, which primarily focuses on the language itself and how it is applied appropriately in different social contexts. It "requires an awareness of the particular, conversational or written, context-dependent conventions operating, how they influence what is being communicated, and how to employ them appropriately according to context" (p. 4). The concept of register becomes important due to students' awareness of different registers with different meanings, grammar and utilizations. Pimm (1987) draws on Halliday (1974; 1978), who argues that registers are defined as sub-structures of natural language. A *mathematical register* is the meaning belonging to the language of mathematics, i.e., to express oneself with a mathematical purpose, where language is used in a particular way (Pimm, 2007).

The two competencies of communication and representation are closely related. *Representation competency* is students' ability to reflectively choose and translate between different mathematical representations. It ”involves relating to the scopes and limitations—including strengths and weaknesses—of the representations involved in given settings” (Niss & Højgaard, 2019, p. 17). For Duval (2017), the single representation in itself is not important; rather the relations between different representations are. Mathematical objects are abstract, making the representations of them our only way to access them. Duval proposes four *semiotic registers* of mathematical representations, defined as either discursive (oral and written languages) or non-discursive, mono-functional (algorithm controlled, e.g., symbolic language or coordinate systems) or multi-functional (for communication but never changed to be algorithmic). Note that Pimm’s (1987) use of ”register” is different from Duval’s (2017). For Duval, an element within a register can be transformed into another register. As an example of transformations between representations, the expression $y=2x$ is a mono-functional, discursive representation. A transformation of $y=2x$ within the same register (e.g., $x=y/2$) is a *treatment*, while a transformation to another register is called a *conversion* (e.g., to a graph).

Within DT, students can manipulate mathematical objects as if they were real objects. The dynamic features of DT create constant treatments of the representations in play. With the implementation of digital tools, new representational infrastructures offer the possibility ”to create a social network and enhanced communication” (Hegedus & Moreno-Armella, 2009, p. 400) —also referred to as *communicational infrastructures*. The use of digital tools *re-mediates* new methodologies and perspectives on mathematics (Arzarello & Robutti, 2010; Bolter & Grusin, 2000). For example, when constructing a circle in a DGE, it is a re-mediation of the construction using paper and pencil. It is within the intersection of representational and communicational infrastructure, new ways of expressing oneself arise via the transformations of the structures. This is referred to as *representational expressivity*. The representational expressivity makes it possible for the students to

> […] express themselves through the representational layers of software and where a participatory structure enables learners to express themselves in natural ways through speech acts (e.g. metaphors, informal registers and deixis) and physical actions (e.g. gesture or large body movements). (Hegedus & Moreno-Armella, 2009, p. 400)

Educational setting

As part of a teaching experiment, data were collected in a Danish 8th-grade classroom in October 2020. The teacher collaborated with the first author and had an interest in the use of GeoGebra. The teaching was part of a four weeks course (4-5 hours per week) about linear functions in which students worked with GeoGebra. The data presented in this paper stems from a teaching sequence from the 4th week of the course. The students worked in pairs at one computer solving a task adapted from Johnson and McClintock (2018). The task included concepts and formulas that the students were already familiar with, such as rectangle, height, area, length and coordinates. The students worked in a pre-set GeoGebra template, see figure 1.

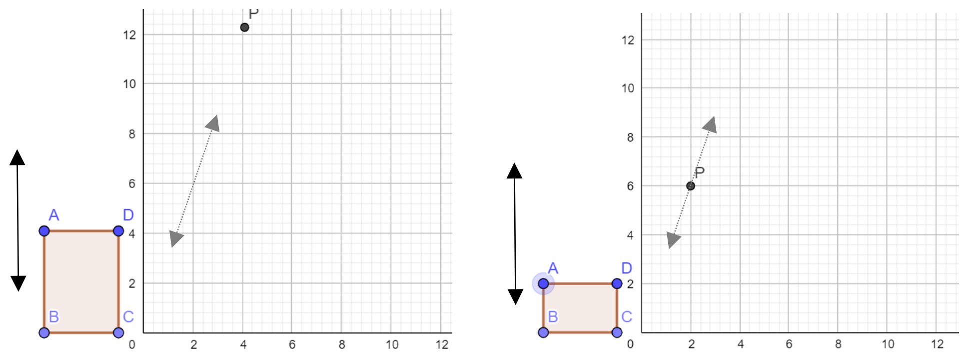

Figure 1. The pre-set GeoGebra template. You can drag Point *A* vertically. The length of *AD*/*BC* is fixed to be 3. Point *P* is defined as (height of *AB*, area of *ABCD*)

Visually, dragging point *A* increases or decreases the area of the rectangle *ABCD* depending on the students' movement. The rectangle is related to *P*. *P*'s coordinates are: (height of *AB*, area of *ABCD*). *P* ”moves” on a linear graph defined as $y=3x$. The students worked on the following task, and were asked to write down their answers by hand:

> Investigate the construction on your own computers by dragging in point ”*A*” in the figure. Describe the relationship between point *P* and the shown figure. For point *P*, you have to describe what characterizes the *x*-coordinate and *y*-coordinate. Explain why the relationship exists.

The task is at the beginning of the task sequence and students did not know the coordinates for *P* in advance. They had only identified the figure and its measures beforehand.

We present five students' written answers, which bring forth both new and different information from the literature. The students were selected to show a variety of different answers, and we chose students from different pairs. The students' answers have been translated from Danish. Within the analysis, the students' representations and expressions are analyzed by identifying three components. 1) The mathematical object in play: How the students write about the functional relation in general or state examples (e.g., one coordinate). 2) Representations in use: Signs of treatment or conversions done by the students. 3) The representational expressivity of GeoGebra: Students' physical acts and speech acts. Physical acts include gesturing (Hedegus & Moreno-Armella, 2009). Based on Ng (2019), we regard dragging as a gesture. For speech acts, the students' word use and symbolic representations are analyzed. In particular, we look for verbs and adjectives indicating movement (Ng, 2016; 2019). In the discussion, we provide a competency analysis. For communication, we consider how representational expressivity influences the students' abilities to communicate in different media and in different mathematical registers. For representation competency, we use Duval (2017) to discuss the related aspects of the competency when using GeoGebra. For the aids and tools competency, we investigate how the students' use of GeoGebra relates to the two other competencies in play.

Analysis of data

We present students' answers while analyzing them. Verbs indicating movement are underlined; and adjectives indicating movement are in italics (Ng, 2016; 2019).

> Student S When *a* <u>is</u> <u>moved</u> 2 up the *y*-axis, *p* <u>moves</u> 6 up the *y*-axis and 2 along the *x*-axis.

Student S's description is quite concrete and provides an example of the change, when the rectangle's height increases by 2. The description is in natural language, also when referring to (2,6). Translating the coordinate into natural language is a conversion, either from the graph or point in the coordinate system or from the point stated as (2,6) into a natural language in Student S's answer. Student S does not distinguish between letters in majuscule and minuscule—(s)he writes point *A* as *a* and *P* as *p*. The expression contains both adjectives and verbs referring to dynamic relations. The student writes "move" referring to a physical act (i.e., gesturing) and that *P* "moves" refers to the constant treatments done by GeoGebra because of dragging point *A*.

Student T — *When* you move the point "*A*" up the *y*-axis, "*P*" moves *along* the *x*-axis. However, if you take one of the other points and move them on the *x*-axis, *P* moves on the *y*-axis

Student T's expression generalizes the "movement" of *P*. However, the student's expression indicates that *P* only moves horizontally (on the *x*-axis), and not diagonally. The expression further indicates that the student not only dragged in the point *A*, but also other points in the rectangle, thus performing treatments that make unwanted transformations in *P*. All sentences have both adjectives and verbs with dynamic features, describing the objects' movements while dragging.

Student U — *When* you drag in Point *A*, the rectangle gets *bigger and smaller*. Point *P* follows.

Student U's expression is mainly focused on the rectangle. The statement "*P* follows" is shallow and indicates that the student focuses on dragging the rectangle, not on the functional relationship. The student's representations are primarily multifunctional, not deeply drawing on the coordinate system (i.e., mono-functional and non-discursive). The student uses "*when… drag*" to state that something is happening as you do it. For the rectangle, it is getting "bigger and smaller", also referring to the treatments handled by GeoGebra as it is a result of the dragging.

Student V — The relationship is that the point *P* always stands on the number that is the area in the figure *when* you drag it. The side length on the figure times three gives the area of the figure and *P*'s position. x=4 y=12 $y{\cdot}3=x$

Student V combines a static and a dynamic description. *P* is described statically, while the rectangle is the dynamic part able to be dragged. This mirrors the dragging done within the task and GeoGebra since they can only drag point *A*, not point *P*. Student V is able to formulate a general relation in natural language about *P*. However, the student cannot transform the natural language of the conversion into symbolic expressions (i.e., $y{\cdot}3=x$). Verbs expressing the static are "is", "stands", and "gives". The dynamic verb is "dragged" accompanied by the adjective "*when"* indicating temporality and that something is happening. The formulation "stands on the number" could be more precise in mentioning that it is the value on the *y*-axis and not just a "number". The symbolic representations are Student V's conversions from the points within the coordinate (i.e., transformations between two registers, see Duval, 2017).

Student W — *If* you change the rectangle's height to 0, the intersection (*P*) ends in 0,0. I think that the intersection is determined by the height and area of the rectangle. The area moves *p* up the *y*-axis and the height moves *p* by the *x*-axis. *When* you change the height, you also change the area, and therefore *p* moves obliquely away from (0,0).

Student W uses and refers to different representations; the rectangle and its measures, coordinates, point *P* within the coordinate system and the axes. The student argues "if.." and "the intersection is defined …”. The dynamic features are marked using different verbs (e.g., ”move” and ”change”). Furthermore, Student W states that the rectangle causes changes for *P*. The student inconsistently uses letters in majuscule and minuscule for *P*.

Within the pre-set GeoGebra template, the students dragging in point *A* causes the mathematical objects to be manipulated continuously. Dragging in point *A* leads to constant conversions between the height/area of *ABCD* and the coordinates of *P*. The students express themselves based on the new representational infrastructures provided by GeoGebra, which then come to serve as communicational infrastructures. Dragging thus provides the students with new representation expressivity as pointed out by Hegedus and Moreno-Armella (2009), which brings both new physical acts and speech acts. The students' dragging (i.e., gesture) is used to show the relationship. In our case, dragging is a particular dynamic feature of DGE, thus making opportunities for the students to explore relationships in another way than without GeoGebra, as also argued by Johnson and McClintock (2018). However, in written communication, students cannot drag and show the dynamic features, for which reason we then look at their speech acts. All six students refer to movement mediated by their dragging. A feature that is also part of the task description. GeoGebra makes it possible for students to investigate the relationship. More advanced mathematics is presented by Students V and W, who both combine static and dynamic descriptions. The use of ”it moves” or ”when you drag” refers to the treatments happening when dragging in point *A*.

Discussion

GeoGebra handles a myriad of operations and manipulations for the students. For instance, when dragging, GeoGebra performs several treatments as well as conversions (e.g., from rectangle to point *P*). This leads to the outsourcing of an important aspect of the representation competency: The translation *of* and *between* representations. Instead, another aspect of the competency comes into play: The ability to understand the involved representations (i.e., the rectangle and its relation to point *P*) and their reciprocal relations. In addition, being able to connect geometrical properties for *ABCD* with functional relationships and point *P* becomes essential. This is a problem for some of the students (e.g., Students T & U), and student S only expresses this for one example, not the whole situation. Even though student V makes an error in the symbolic expression, V's ability to state a general relation ($y=3x$) in natural language indicates an understanding of the relationship between the area and height of *ABCD* and the point *P*.

From a communication competency perspective, it is primarily the students' ways of expressing themselves mathematically, which is analyzable when they write up answers. For student T, however, data indicates that the student does not understand the task description, or rather the aim of the task, by dragging in another point than *A* and explicitly writing about it. Hence, the ability to interpret and understand others' communication. When Student T dragged *A* horizontally or dragged any other point, the length *AD* changed which also changed the relation to *P* and an unwanted answer appeared. This leads to a focus on the limitations and capabilities when using GeoGebra as well as the influence of how dragging transforms the mathematical objects in play solving this task. Thus, this case marks an intersection of all three competencies: communication, aids and tool, and representation.

As stated in previous research (e.g., Hegedus & Moreno-Armella, 2009; Arzarello & Robutti, 2010), the representational and communicational infrastructure is mediated on the students' representational expressivity. This means that the environment within GeoGebra both makes it possible for the students to drag objects (physical act) as well as to express themselves in dynamic terms (speech acts). The students' mathematical register (see Pimm, 1987) changes from being static to dynamic. A problem with the students' dynamic communication in relation to the representations and objects, is that they occasionally come to communicate about the construction and its visual apperance rather than actually describing the relationship between the rectangle and the point *P*.

The existing difference between mathematical terms, the use of static/dynamic words (verbs and adjectives) and the involvement of different representations (and how these are handled, e.g., treated or converted), are results of the representational expressivity provided by the representational and communicational infrastructures. Student V utilizes the term ”slope” to define how *P* moves, which is more mathematical than the use of obliquely as that of Student W. For the mathematical communication competency, expressions must be of mathematical nature. Hence, *slope* is more precise (Niss & Højgaard, 2019). There can be different reasons for communicating statically or dynamically. Some students tend to describe functions statically (e.g., Student V) and not covariational (Johnson & McClintock, 2018). From a competency perspective, each of the three mathematical competencies that we have addressed above gives rise to an observation. For the communication competency, students must learn about differences between mathematical registers, and which register that is the most appropriate in a given situation (see Pimm, 1987; 2007). For the representation competency, students must come to understand relations between representations in semiotic registers and be able to convert to and from natural language (Duval, 2007). Finally, aids and tool competency include knowledge about different tools and that tools bring representational expressivity (Niss & Højgaard, 2019).

Concluding remarks

These five examples of students' mathematical communication indicate that the use of DT mediates different mathematical registers than normally accepted and addressed when operating and talking in static environments (Niss & Højgaard, 2019; Pimm, 1987). The dynamic representational infrastructure of DGE is an important mediator of that representational expressivity and influences both students' actions (i.e., dragging and general gesturing) as well as their speech acts, which come to focus on movements and changes (Hegedus & Morona-Armella, 2009). For students, the communication competency thus becomes more complex, thus increasing the demands on both students and teachers to become aware of registers and discourses. However, this study also raises the question of whether we should accept an expansion of mathematical registers to involve dynamic features as the digital tools mediate these, and how to cope with multiple ways of communicating when assessing communication competency.

Acknowledgement

This paper is part of project 8018-00062B under *Independent Research Fund Denmark*.

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An inquiry of different interpretations of programming in conjunction with mathematics teaching

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This article discusses computational thinking and programming in mathematics teaching and aims to shed light on different interpretations of what programming entails. The literature review revealed that there are mainly two ways of approaching programming in an educational context either by a narrow interpretation or a broader interpretation of programming. These different interpretations of programming are manifested in different ways in mathematics teaching. The narrow interpretation is manifested in an activity that focuses on learning to write code on a computer, while the broader interpretation of programming is displayed in activities with a focus on learning how solve problems in such a way it can be executed by a computer or by a human. Also, this article explores an appropriate programming activity within the context of mathematics education in Sweden.

Keywords: Mathematics education, computational thinking, programming, coding, self-efficacy

Introduction

The background to this article is a survey conducted in two municipalities in Sweden in the spring of 2020. The purpose was to map the primary school teachers' perceived professional needs regarding mathematics education. The results showed that the teachers especially expressed a need to develop their knowledge in how to use problem-solving and programming in conjunction with mathematics teaching.

Computational Thinking (CT) involves the skills of reformulating a seemingly complex problem into smaller parts and finding patterns, making abstractions, and designing algorithms (Wing, 2006). However, the definition of CT, and the question of its usefulness outside the computer science context, are still under debate (e.g., Denning, 2017; Li et al., 2020). Programming activities are considered to be one way to develop CT-related skills in mathematics education. But, there are different interpretations of what a programming activity can entail in practice because, in some settings, programming is synonymous with ”coding” on a computer. And this might create ambiguities in how programming activities should be planned so that students also learn mathematical concepts and strategies. For example, Lu and Fletcher (2009) suggest that the first students encounter teaching CT in schools should be built upon concepts and symbols that are familiar to students rather than being introduced to a programming language.

Digital competence is defined as one of the eight key 21st-century skills for teachers and students. According to the Swedish National Agency for Education (Skolverket, 2017), digital competence involves four aspects, based on EU key competencies: Understanding of digitalization and how it affects individuals and society, understanding and knowing how to use digital devices and media, critical and responsible approach to digitalization, and finally, understanding of how to solve problems and implement ideas in practice. Consequently, digital competence has connections to mathematics education, perhaps primarily through the problem-solving aspects. Also, CT is often implicitly mentioned in the same context as digital competence (Bocconi et al., 2018).

The main purpose of this article is to discuss the various interpretations of what programming entails within the context of mathematics education in Sweden.

The research questions for this article are as follows:

1. What kind of interpretations of the concepts of programming and CT can be found in the literature?
2. In the Swedish context, what kind of interpretations of programming are especially relevant and justifiable based on the national goals for mathematics education in primary school?

The research questions are mainly answered using a literature review, which aims to identify the important discussions about CT and programming. And, to discuss an appropriate interpretation in a Swedish context and how this interpretation can be manifested and evaluated in mathematics teaching. First, however, the method of inquiry is explained, followed by a presentation of the main results.

Method

The first research question was answered using a literature review, which aimed to identify the key interpretations of CT and programming restricted to mathematics education. Google Scholar and Web of Science were used. From each search result, relevance was determined by the title, and further if the content seemed to discuss or relate to different interpretations of CT or programming, its full reference was obtained for further evaluation. First, the first twenty pages of search results from a search on Google Scholar were reviewed using the broad keyword ”computational thinking”. Then the search results were refined including a search on Web of Science using the keywords ”computational thinking”, ”education”, ”mathematics”, and ”programming”. After combining the search results from the two sources and removing duplicates, the abstracts were read to further decide their relevance. Mainly peer-reviewed articles were included in the review. In the end, a total of ten articles were deemed relevant since they discuss or contrast different interpretations to the understanding of the concept of CT and/or programming. Accordingly, many articles were excluded from the literature review. For example, articles that presented results from interventions or experimental studies were excluded because they are already framed in a certain type of interpretation of CT and programming. Further, the second research question is answered by using the results from the literature review and a written clarification by the Swedish National Agency for Education regarding programming in mathematics teaching.

Literature review

Research articles that include discussions of different interpretations of CT and programming in an educational context are rare. Among the relevant articles, most of them discuss the usefulness or transferable of CT outside of computer science, which essentially is about how the concept of CT should be interpreted (e.g., Denning, 2017; Nardelli, 2019). However, two articles were found that contrast different approaches to the understanding of CT, which also includes interpretations in an educational context. Bocconi et al. (2018) and Li et al. (2020), divided different interpretations of CT into categories that reflect different perspectives on CT and programming. Li et al. (2020) describe different perspectives on interpreting the concept of CT found in the literature (based on the historical development of the concept), and according to the authors, these perspectives also have a great impact on school practice. Bocconi et al. (2018), on the other hand, categorize the different interpretations based on an analysis of policy documents and interviews with experts in Nordic countries.

Based on this literature review, essentially, there are mainly two ways of interpreting programming either in the broad sense as something more than just ”coding” on a computer, or in a more narrow (or technically) sense as identical to ”coding” on a computer. Further, these interpretations may affect how programming activities are manifested in mathematics teaching. Most of the excluded articles from the literature review have framed their studies in a narrow interpretation of programming i.e. that programming is solely about writing code on a computer.

In the following sections, before a presentation of various approaches to the understanding of CT and programming, the historical conceptual development of the broad interpretation of CT and how it is connected to algorithmic thinking and programming is presented.

An interpretation of CT and programming

CT has many times been presented as a thinking model (e.g., Li et al., 2020; Wing, 2006). But, this is, however, a quite ambiguous interpretation of CT, and since programming is considered to foster CT, this ambiguity also influences how programming is interpreted. In the literature, the concepts of programming and CT are difficult to separate because the impression is that CT requires the use of programming (Voogt et al., 2015). Further, more research is needed to frame CT as an internal process (manifested in a certain behavior) instead of a predefined external process, which ultimately frames the learning activity (Lyon & Magana, 2020).

The concept of CT can be traced back to Seymour Papert’s idea of how children can develop procedural thinking through programming in LOGO (Papert, 1980). This concept got renewed attention when it was presented by Jeannette Wing in 2006, who explained that CT involves the skills of reformulating a seemingly complex problem into smaller parts and finding patterns, making abstractions, and designing algorithms (Wing, 2006). Further, abstraction is considered to be the central thought process in CT, which mainly refers to the special process that strives to reduce the information until the most relevant information for understanding remains (Wing, 2011).

Wing (2006) argued that CT is useful not only for computer scientists but for everyone; CT should be associated with how people think rather than computers think, and proposes that CT should be valued equally as much as reading, writing, and arithmetic in school. But also, to clarify the misleading apprehension that computer science would be equivalent to computer programming. However, although Jeannette Wing’s article from 2006 became influential in how we interpret CT, its usefulness outside computer science contexts is an ongoing debate (Denning, 2017; Li et al., 2020; Nardelli, 2019).

CT has similarities with other thinking skills such as algorithmic thinking and mathematical thinking. For example, the mathematician and computer scientist Donald Knuth, who was the creator of, among other things, the TeX computer typesetting system, explains algorithmic thinking by saying:

> It has often been said that a person does not really understand something until he teaches it to someone else. Actually a person does not really understand something until he can teach it to a computer, i.e., express it as an algorithm. (Knuth, 1974, p. 327)

Consequently, algorithmic thinking is more of a general mental tool to reach a deeper understanding rather than a skill in how to write code on a computer. Because, according to Knuth (1974) reformulating a problem and construction of algorithms force precision in thinking, which in turn leads to a deeper understanding.

In 2011, Jeannette Wing refined the definition of CT by citing Cuny, Snyder, and Wing:

> Computational thinking is the thought processes involved in formulating problems and their solutions so that the solutions are represented in a form that can be effectively carried out by an information-processing agent. (Wing, 2011, p. 20)

Aho (2012) also presented a description of CT, which is similar to the definition above, where CT is described as the ”thought processes involved in formulating problems so their solutions can be represented as computational steps and algorithms” (p. 832). Consequently, there are similarities between CT and algorithmic thinking, and perhaps that is also why algorithmic thinking is considered to be the core process of CT. But at the same time, there is also a clear difference between them because ”computational thinking includes the design of the model, not just the steps to control it” (Denning, 2017, p. 33). Mathematical thinking is also considered part of CT. That is because mathematical thinking involves the process that reformulates the problem so that it can be handled mathematically, while CT involves the process that reformulates the problem with clarity so that it can also be handled by a computer (Wolfram, 2016).

Various approaches to the understanding of CT and programming

Li et al. (2020) present three different approaches to the understanding of CT: Discipline-based, psychology-based, and educational-oriented. The discipline-based approach has its roots within computer science and mainly describes CT as a method of how computer scientists think and go about solving problems, and also involves the idea that the associated skills need to be developed through programming. This approach understands CT as a balance between computing and thinking skills. In the psychology-based approach, the focus is rather on thinking skills, since the understanding of CT has been influenced by the results stemming from research in human cognition. Human thinking in general and CT are considered separated because CT is more about creating effective solutions in computational steps.

The educational-oriented approach, on the other hand, seeks to define CT more practically so that it is applicable in educational contexts. Further, they also divided this approach into certain subcategories. In the first category, CT is considered to be able to be developed only through programming (cf. discipline-based approach). In the second category, CT is thought to be useful even outside computer science contexts. And finally, the last subcategory involves the idea that computational literacy is important for everyone, not just computer scientists, and that CT can be developed by other means than just through programming. For example, the International Society for Technology in Education (ISTE) and the Computer Science Association (CSTA) developed a practical definition of CT for K-12 education (ISTE, 2011), after collaborations with teachers and researchers. They defined CT as a problem-solving process that involves decomposition, logical organizing and analyzing data, abstractions, algorithm design, and generalization. Attempts have also been made to create a uniform definition of CT within computer science. For example, based on existing literature, Shute et al. (2017) defined CT using six categories: decomposition, abstraction, algorithm design, debugging, iteration, and generalization.

Bocconi et al.'s (2018) analysis of policy documents and interviews with experts in the Nordic countries revealed that there exist two approaches to understanding CT and programming (although the term CT was only implicitly used). The first approach was interpreted as a broad understanding of CT and programming, that is, CT is considered to be useful outside of computer science, and with the idea that CT is not the same as programming. The other approach was associated with a more technically oriented understanding of CT. This approach instead encourages the development of the necessary skills needed in our digital society.

Interpretation of programming in Sweden

In 2018, programming became part of the national goals for mathematics education in Sweden. The reason for this was that the knowledge of programming and the use of digital tools was considered to foster digital competence (Skolverket, 2017). Similar implementations of programming or algorithmic thinking in the national goals for mathematics education could be seen in all of the Nordic countries around that time (Bocconi et al., 2018).

Further, the Swedish National Agency for Education attempted to clarify the meaning of programming in the national goals for mathematics education in primary school, by emphasizing the broader perspective of programming:

> Programming includes writing code, which has great similarities with general problem-solving. However, programming should be seen from a broader perspective, which also includes creative makings, control and regulation, simulation, and democratic dimensions. This further perspective of programming is an important starting point in teaching, and programming thus includes all aspects of digital competence. (translated from Skolverket, 2017, p. 10)

According to this written clarification, programming in mathematics teaching should be interpreted as something more than just the ability of writing code on a computer. Using the results from the literature review, this can be seen as an example of a broad interpretation of programming. In addition, since digital competence is mentioned in the same context as programming, the interpretation of CT is implicitly technical (cf. Bocconi et al., 2018). Hence, a broad interpretation of programming should be applied in mathematics teaching. Thus, in the following section, an appropriate programming activity that matches this interpretation will be discussed.

An appropriate programming activity

One type of programming activity that shifts focus from the ”coding” on a computer, is the so-called ”unplugged” (without the use of computational devices) programming activities or ”paper- and pencil” activities. These types of activities are focused on practicing students in using their mental tools (logical representations) to effectively solve problems, rather than focusing on learning how to write code on a computer. In these kinds of activities, students learn how to decompose a seemingly complex problem into discrete steps, design an algorithm for solving the problem, evaluate solution efficiencies and optimize in a simulation (which includes ”coding”). In these activities, students have the opportunity to, for example, practice designing algorithms using different logical representations such as flow charts.

Perceived self-efficacy is an important factor for understanding students' performance and evaluating effective learning strategies (Dweck et al., 2014), and has also shown a positive correlation to mathematical achievement in previous research (e.g., Bonne & Johnston, 2016; Skaalvik & Skaalvik, 2004; Tossavainen et al., 2021). Previous research that examines the impact of ”unplugged” programming activities on students' mathematics self-beliefs in conjunction with mathematics teaching is rare. Some research studies, however, examine the impact on mathematics self-beliefs. For example, Psycharis and Kallia (2017) examined the relationship between programming, reasoning skills, problem-solving, and self-efficacy. The programming activity consisted of designing a solution to a math problem and then implementing it in MATLAB. The results showed that students' self-efficacy score was significantly improved after teaching mathematics in conjunction with programming.

Some studies investigated the impact of ”unplugged” programming activities on students' CT. For example, Kim et al. (2013) conducted an investigation aiming to improve students' CT skills and their interest in learning computer science. The student's logical thinking was used as a measure of the students' CT. By comparing a traditional programming course with LOGO, they wanted to investigate if students' logical thinking is affected by an intervention (”the paper-and-pen course”) where students practiced translating their mental models into logical representations such as flow charts. The results showed a statistically significant improvement in the students' overall logical thinking in both the paper-and-pen strategy and the traditional programming course. However, statistically, they could not claim that the improvements were greater in the paper-and-pen course compared to the traditional course. Although, the scores in the post-survey of students' understanding of CT were statistically significantly higher than in the traditional course.

Discussion

In this paper, various interpretations of CT and programming are presented. Li et al. (2020) made a thorough categorization of the different approaches for understanding CT, and since programming is mentioned as a tool to promote CT, this implicitly also applies to interpretations of programming. Although Bocconi et al.'s (2018) analysis of policy documents and interviews showed two different ways to approach CT, these fit very well within Li et al.'s (2020) suggested education-oriented perspective.

In summary, the various approaches to understanding of CT and programming indicate some confusing aspects. Because, in some contexts programming has a broad definition and thus contains several aspects of CT, and in others, programming is equal to ”coding” on a computer (i.e. the narrow interpretation), and by that only a phase of CT. These different interpretations of programming are likely manifested in different ways in mathematics teaching. The former would likely be manifested in an activity based on learning to write code on a computer. While the latter is more likely expressed in a learning activity where students focus on learning to solve problems in a way so it can also be executed by a computer i.e. in computational steps. In fact, the narrow interpretation of programming might be grounded in a misconception that algorithmic thinking only can be fostered through coding on a computer. According to Knuth (1974), algorithmic thinking should be considered as a mental tool rather than a specific skill in how to write computer programs. The narrow interpretation of programming was found to be the most frequently used among the excluded articles in the literature review.

Further, the literature review showed that the Swedish National Agency for Education interprets programming from a broader perspective similar to CT. Thus, appropriate programming activities enable students to practice a wide range of CT-related skills. For example, this broad interpretation of programming can be manifested in ”unplugged” programming activities that practice students in using logical representations to effectively solve problems. Additionally, the literature review showed that there was a need for more research that explores the effectiveness of ”unplugged” programming activities, in particular, concerning the impact on students' mathematics self-beliefs.

The classroom time for the teachers is a scarce resource and thus extremely valuable. Therefore, it is important to consider whether too much coding on a computer in connection with mathematics teaching risks leading to a shift of focus from the skills needed by a mathematics student to the skills that are especially needed by a computer scientist. According to Denning (2017), there are many similarities between the description of CT-related skills and George Pólya's suggested mental disciplines that make it possible to solve problems. For that very reason, the broader interpretation of programming might be more easily combined with regular mathematics teaching.

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Teachers’ choice of a challenging task through collaborative learning

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Challenging mathematical tasks are important for all students’ learning processes, and the demanding job of finding and developing such tasks is preferably done through teacher collaboration. Using cultural-historical activity theory, we analyze three upper secondary teachers’ collaborative learning process in choosing and rejecting tasks for a collection of challenging tasks they have agreed to develop. They collaboratively chose one task to fulfil the criteria of challenging tasks; one task was rejected as it did not fulfil the criteria, and another was temporarily rejected as not all the teachers knew how to solve it themselves. The analysis revealed a positive and open atmosphere among the teachers, with content-focused discussion highlighting mathematical content and teaching related to the tasks discussed. While their discussion showed several signs of collaborative learning, whether their work will result in changes to teaching practice remains to be explored.

Keywords: teacher collaboration, secondary school mathematics, mathematical enrichment

Introduction

Challenging mathematical tasks offer learning opportunities for all students (Nolte & Pamperien, 2017; Sheffield, 2003). However, textbooks usually offer a small proportion of challenging tasks (Jäder et al., 2020). Hence, teachers must find suitable tasks to offer their students. Searching for, analyzing, and developing challenging tasks is time-consuming work, which is preferably done together with other teacher colleagues (Mellroth, 2018). Swedish teachers’ conditions for collaborative learning are often poor (Nordgren et al., 2019). But when the right conditions are provided, for example regularly scheduled times, collaborative learning benefits student learning (e.g., Timperley, 2011) and reduces stress among teachers (Nordgren et al., 2019). We assume that, under such conditions, teachers can work in a more structured way and focus on developing teaching activities together.

In this paper we report from an ongoing project aiming to improve mathematics education in terms of meeting students’ different learning needs, including students with high abilities in mathematics. Eight mathematics teachers at a Swedish public upper secondary school participate in the project. They have decided to develop a collection of challenging mathematics tasks, with guidelines for classroom enactment and with clear connections to the Swedish national curriculum. We will present results from an activity in which three of these teachers worked together on choosing a task for further development. To frame, analyze, and understand the complexities of this collaborative learning process, we use Cultural-Historical Activity Theory (CHAT) (Engeström, 1999). Our guiding question is: What characterizes the collaborative learning process that leads to teachers’ choosing or rejecting tasks for further development?

Literature review

Learning by failing is an accepted view among educators. Each student needs to be exposed to mathematical tasks that are challenging. Naturally, what is challenging will be different for someone who struggles with mathematics compared to someone with high ability in mathematics. As textbooks include few challenging problem-solving tasks (Jäder et al., 2020) and reasoning tasks (e.g., Bergwall, 2021), students with high ability are at risk of never encountering any challenges unless the teacher deliberately offers them. In addition, students with learning difficulties are at risk of mostly solving procedural tasks, as textbooks' enrichment tasks are often too hard for them (Jäder et al., 2020).

Designing mathematical tasks that offer challenges for all students is a demanding job (Mellroth, 2018). Sheffield (2003) has developed criteria that such tasks should fulfil. Four of the criteria are: (1) everyone should be able to start working with the task, (2) it should be possible to solve the task in several ways, (3) the task should be engaging, and (4) the task should offer an open end. Nolte and Pamperien (2017) found that tasks developed to be challenging for highly able students create learning opportunities for all students. Some researchers (Hoth et al., 2017) claim that the teacher must have deep mathematical knowledge to orchestrate teaching with challenging tasks. Mellroth (2018) showed that teachers themselves propose collaboration with teacher colleagues when they feel that they lack the mathematical knowledge themselves. It is also shown that in countries where it is part of the teacher culture to collaboratively reflect, discuss, and develop teaching, students perform better than in countries where it is more common for teachers to work alone (Hargreaves & Fullan, 2012). A school culture in which collective responsibility is taken for students' learning is necessary to create, or withhold, sustainable strategies for teaching development (Hargreaves & Fullan, 2012).

This paper reports results from one of several closely related projects carried out in the same municipality. All the projects center around *collaborative learning*, defined as follows: a group of teachers' jointly and systematically organized work, consisting of content-focused, inquiry-based, creative, and tentative communicative processes, that continuously change the work itself, and/or the group, and/or the individual teachers' professional practice and classroom practice (Andersson et al., 2019).

Research has shown that, to create sustainable collaborative learning processes, it is important that the participants themselves identify a problem they deem important to solve (Harvey & Teledahl, 2019). The communicative process of collaboratively working to solve the problem is a human activity with great complexity.

Theory

We use CHAT as our analytical framework, as it offers the possibility to understand a complex human activity system in relation to its context (Engeström, 1999). The theory is often presented in the form of the CHAT triangle, with six nodes that are of importance in interpreting the outcome of the activity system, see Figure 1.

CHAT offers opportunities to study how *subjects* in a *community* influence each other towards joint development. In line with Engeström (1999), we consider the collective processes by focusing on the subjects' object, mediating artefacts, and rules. More precisely, the *object* is the subjects' agreed goal. The *community* is everyone who belongs to the same environment and shares the object. *Mediating artefacts* are the tools that are used or developed by the subjects in order to reach the object. *Rules* are explicit and implicit norms for actions and interactions between the subjects in the community that may influence the outcome. *Division of labor* includes hierarchical power structures in the community (Harvey & Teledahl, 2019). The rules, community, and division of labor nodes place individual and collective actions in their context. Thereby, the subjects' joint actions can be explained in the context to which they belong. In practice, most individuals are part of several different activity systems with their own objects, rules, artefacts, etc.

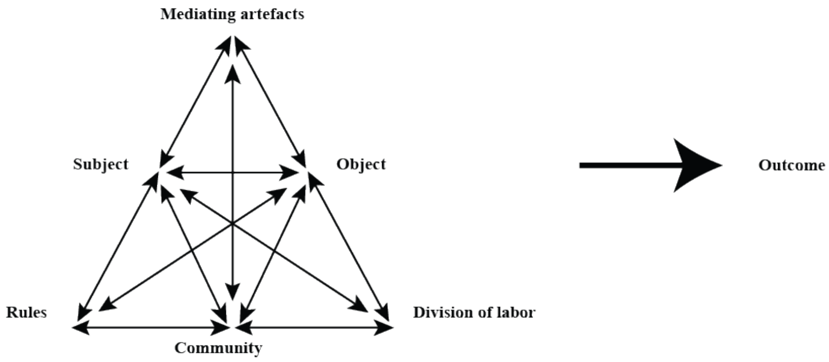

Figure 1. Activity system, illustrated by Engeström (1999)

In this study, we analyze the joint work of three of the participating teachers when they choose or reject tasks for further development. In the terminology of CHAT, these teachers are the *subjects* of this study. The other five teachers participating in the project are part of the subjects' *community*, along with the researchers, other colleagues at the school, the principal, politicians, and others who make decisions regarding the project's conditions and share its goals. Regarding the *division of labor*, the teachers participate on equal grounds in the following sense: They have all volunteered for the project, have no reduction in other tasks to make room for project work, and none of them have been appointed a particular role in the project. It is assumed that they will collaborate on project tasks. The principal supports the project, and has dedicated conference time for the teachers to work on it.

The other three CHAT nodes (object, mediating artefacts*,* and rules) are only partially determined by the project's design. Early in the process, the participants decided to create a collection of challenging problems, including introductory tasks as well as tasks that offer enrichment. This collection, which the participants refer to as the problem bank, is the subjects' general object. The collaborative learning process during the work with the problem bank is also a means to achieve the project's long-term object that teachers should become better prepared to meet the differing needs of all students, including those with high ability in mathematics. The meetings, seminars, and literature on collaborative learning and highly able students, and a task analysis protocol with criteria for challenging tasks are mediating artefacts provided by the researchers. Explicit rules include the assumption that the subjects will participate in meetings and activities as well as cooperate in the creation of the problem bank. The transcript analysis aims to reveal further details related to these three nodes.

Regarding rules for actions and interactions in the community, these include norms for the interplay between subjects as well as norms for what content they find worthwhile to discuss. We refer to the first kind of norms as *social rules* and the second as *mathematically oriented* or *didactically oriented*, depending on whether it is purely mathematical aspects that are discussed or if the discussion concerns teaching or learning.

Method

This study is part of an ongoing, 2.5-year long project aiming to improve mathematics education in terms of meeting students' different learning needs. It is explicit that this should include the needs of highly able students, that is those who quicker and easier reach the learning goals. The project is a collaboration between the local university and the municipality. Two researchers (the authors of this paper) serve as facilitators. Eight mathematics teachers who teach technology students at a public upper secondary school have volunteered to participate. They have been informed of the intention of the related research, and have all signed an informed consent form. The teachers expressed that they lacked a source of tasks that are ready to use when students (including the highly able ones) need extra challenges. Therefore, they decided to focus on developing a problem bank of challenging tasks and that the problem bank should involve tasks that introduce new concepts as well as tasks for enrichment.

According to national steering documents, Swedish upper secondary schools offer five mathematics courses called Mathematics 1, 2, 3, 4, and 5. This paper shares the preliminary results from when three of the teachers teaching Mathematics 3–5 met to choose their first task for further development for the problem bank. Earlier in the same semester, the researchers had arranged three meetings, 90 minutes each. These included a mix of literature studies, seminars, and discussions, with the aim of increasing the participants' knowledge about collaborative learning (Harvey & Teledahl, 2019) and challenging mathematical tasks (Mellroth, 2018; Sheffield, 2003). All meetings were audio-recorded. To analyze, choose, and develop tasks, the researchers and teachers together developed a task analysis protocol with criteria for challenging tasks (e.g., Sheffield, 2003). Immediately before the analyzed activity, the participants analyzed one task together with the researchers to create a shared understanding of the protocol's different criteria.

The following procedure was followed to select and analyze the data for this paper: 1) All audio-recordings from the three teachers' meetings were listened through, and all episodes in which they gave reasons for choosing, rejecting, or revising a task were marked; 2) The marked episodes were listened through once more and by both authors; 3) Transcripts were produced on the parts still judged to give reasons for the choice or revision of a task; 4) Verbatim transcripts were produced from the whole episode in which the teachers started their process of developing tasks for the problem bank; 5) Each author separately identified nodes from the CHAT triangle in the transcripts; 6) The authors compared their analyses and discussed the parts they had different opinions on until agreement was reached; 7) The complete analysis of the outcome using CHAT theory was performed jointly by the two authors.

Analysis

The analysis is exemplified in three paraphrased transcripts (translated from Swedish to English) that follow the chronology of the analyzed meeting. The agenda for the analyzed meeting was to initiate the work of choosing, analyzing, and developing tasks for the problem bank (i.e. the subjects' general object). The analysis focuses on how the subjects address the CHAT nodes called object, mediating artefacts, and rules.

The extracts are from the first 30 minutes of the meeting. During this part, all the teachers present possible tasks for the problem bank and ask one another about what they have brought to the meeting.

This indicates the existence of the social rule that all the teachers should (and do) contribute. Teacher T1 presents *Ferris Wheel*, a task in which the student is asked to investigate how a gondola's height above ground level varies as the Ferris wheel turns. The task is meant to serve an introduction to trigonometric functions, which relates to the object that the problem bank should include introductory tasks. In the episode below, the teachers discuss the difficulty of the task in relation to what is taught before it. That they deem it important to discuss such things is a didactically oriented rule.

69	T2:	It depends on whether you've introduced circular arcs and that kind of stuff.
70	T1:	Yeah, you usually do that first, don't you?
71	T3:	Yeah, that comes immediately before. Or at the beginning of the chapter.
73	T2:	I think it's after.
74	T3:	What comes first, then?
75	T2:	I think it starts with circular arcs, before radians.
77	T1:	The other way around, I think. You introduce radians, then circular arcs.
79	T1:	Yes, and then functions with radian.

Several social rules come into play here. It is accepted behavior to question one another (73, 77), ask for (74) and give (75) clarification, and express support (70, 71, 79). In addition, the discussion does not end until the ambiguities are sorted out and consensus is reached (after 79). The explicit textbook reference (71) shows that the textbook is a mediating artefact in this process.

In the next episode, the teachers discuss the possibility to develop (which in itself is an object) *Ferris Wheel* to be an enrichment task (which is another object). In the discussion, x and h refer to the horizontal and vertical positions, respectively, of a gondola on the Ferris wheel.

149	T2:	Here we only study h. What if we study x, so that you do the same thing as in part 1, but for x instead of h?
151	T2:	It's the same kind of task, but it becomes a two-dimensional motion. We call them $x(t)$ and $h(t)$, and get two different formulas that are rather similar. And then, maybe make a connection back to the unit circle. Provided that we place the origin at the center of the Ferris wheel.
153	T3:	Yeah, that can be part 3, and where you said $x(t)$, part 4 maybe. Changing a function's values when you move the center should be straightforward.
154	T2:	Finally, you could add these two: Can you go from your two functions to the equation of a circle? Both when the center isn't at the origin and … So, from the beginning we should've placed the center at the origin. That makes it a little easier.
155	T1:	Yeah, absolutely. It has potential, this task.
157	T3:	It gets extensive but that's good. Not everyone will have time to do it, but that's not the intention either.

Here we see how the social rules allow all teachers to contribute ideas, ask for opinions (149), and confirm each other's thoughts (153, 155). The mathematically oriented rules allow for a discussion of purely mathematical details (151), and didactically oriented rules are seen when the teachers express the belief that their students should find some steps ”straightforward” (153) or that certain changes should make the task ”a little easier” (154). The predetermined object that the tasks should also include challenges for students with high ability is seen in the last utterance (157). The reference to the unit circle (151) indicates that the ordering of content in the mathematics course serves as a mediating artefact when additions to the task are suggested.

Teacher T2 presents four different tasks. The teachers agree that the most promising one is *The Beam*, which asks for the maximum length of a beam that should be possible to carry through a right-angle turn in a corridor. Finally, Teacher T3 presents *Cut Cones*, in which the student is to cut a circular sector from a circular sheet of paper and form a cone with maximal volume. The teachers agree that *Cut Cones* offers opportunities to use different representations. They refer to a seminar held earlier the same day; i.e., this seminar is a mediating artefact. They also suspect that the task will be too easy and not meet the object of being a challenging task. In the episode below, the discussion is coming to an end and the teachers decide which task to develop first.

264	T3:	I thought *Cut Cones* was kind of fun. You can actually build these things.
276	T1:	It's a very good task.
277	T3:	But it doesn't meet the criteria.
296	T2:	And I think the first step isn't very difficult.
298	T3:	The first step isn't difficult in *Ferris Wheel* either. But it's hard to make *Cut Cones* more difficult. It's possible to develop *Ferris Wheel*.
303	T3:	I think both *The Beam* and *Ferris Wheel* have potential.
310	T2:	Should we start with *Ferris Wheel*? We can analyze it and then solve *The Beam*.
311	T3:	Yeah, because I need to solve it first.

In this episode, several motives (relating to different objects) for choosing or rejecting tasks are presented. T3 considers that the fact that *The Cones* offers an opportunity to work with physical representations speaks in favor of this task (264). This might be an individual object of T3, who also refers to applicability on several occasions. The overall object of developing tasks that can challenge students on different levels is used against *The Cones* (277, 298) as well as in favor of *Ferris Wheel* (298, 303). The criteria from the task analysis protocol serve as a mediating artefact in this discussion (277, 310).

Results

What characterizes the collaborative learning process that leads the teachers to choose *Ferris Wheel* and reject the other tasks? They chose *Ferris Wheel* because they believed (1) it has a first step that is not difficult, (2) it can challenge students on several levels, and (3) it can be developed to become even more challenging. Thus, their choice was driven by their agreed object to find a task that can challenge all students, and that can be used as an introductory task but can also be developed to offer enrichment. Their discussions were mediated by tools supplied within the project, such as the task analysis protocol and its criteria for challenging tasks, but also by other tools such as the textbook, local teaching plans, national steering documents, and the teachers' shared understanding of their students' prior knowledge.

During their discussions, rules of various types came into play. The teachers expected everyone to contribute, and asked for and questioned each other's opinions. They prioritized discussions about mathematical details related to the task and students' common understandings and difficulties in relation to the material. They rejected (for the time being) a task they all liked, *The Beam*, as they wanted to first solve it on their own; they had no problems admitting to each other that at first glance they did not know how to do this.

Discussion

This study focuses on three teachers' collaborative learning process when choosing a task for a problem bank of challenging mathematical tasks, which are rare in textbooks (Jäder et al., 2020). The teachers analyze and develop the task to ensure that it meets the needs of students who struggle with mathematics as well as those who are highly able and need extra challenges. They do this by providing a low entrance level and combining it with a sequence of subtasks of increasing complexity and an open end. Leaning on Nolte and Pamperien (2017) and Sheffield (2003), *Ferris Wheel* is expected to challenge all students in the classroom. The analysis shows how Sheffield's (2003) criteria, operationalized as the analysis protocol, provide guidance and help the teachers in choosing, rejecting, and developing tasks. This is an example of how a mediating artefact helps to keep the work organized and content-focused, which are two defining features of a collaborative learning process (Andersson et al., 2019).

The study sheds further light on the importance of creating opportunities for teachers to collaborate (Nordgren et al., 2019). The analysis of the CHAT node called *Rules* (Engeström, 1999) shows that the discussion when choosing a task involves the sharing of mathematical and didactical knowledge. As Hoth et al. (2017) mention, deep mathematical knowledge is important, and the analysis indicates that this is true, at least among upper secondary teachers when working to develop challenging tasks. Although we have not explored how an individual teacher would choose or reject a task using the same process, in line with Mellroth (2018) we assert that choosing and developing challenging tasks benefit from teacher collaboration.

Thus far, the teachers' work has complied with the first part of our definition of collaborative learning (Andersson et al., 2019). They have worked jointly and systematically, and their communicative processes have been content-focused, inquiry-based, creative, and tentative. The second part of the definition of collaborative learning, relating to changes in their professional practice, is an important next step in this ongoing research project. However, in line with Harvey & Teledahl (2019) and Nordgren et al. (2019), we see promising results regarding how a clearly articulated and common goal, regularly scheduled time for collaborative work, and the provision of operationalized frameworks (such as the task analysis tool) can stimulate collaborative learning processes. On a practical level, our study shows how such processes can guide and support teachers in the demanding work of supplementing textbook materials with suitable tasks that can challenge all students.

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Identifying the nonverbal mathematical exploration in preverbal children playing with a commercial toy

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Young children often engage with commercial toys, many of which are now connected to a television or online video program that the children (and their caregivers) can watch. Children who are preverbal are often engaged (and targeted) with these toys, benefitting with a range of learnings and demonstrations of understandings. This paper uses 360-degree videos to explore the learnings and understandings demonstrated when a preverbal child engages with a commercial toy with an educator. The 360-degree video shows the preverbal child's behaviours, both in response to the educator's speech and actions and self-initiated. Children were observed engaged in events, made up of behaviours that were delineated into actions to help the researchers identify what mathematical exploration and understandings each child may demonstrate. Of particular interest is the mathematical understandings and exploration that the child may demonstrated via their actions.

Keywords: Mathematical exploration, observations, behaviours, actions.

Introduction

In this paper we study very young children (age 1-3 years) engaging in mathematical explorations during daily activities. For preverbal children, or children who do not speak during their activities, their actions can be used to help identify mathematical exploration (see Björklund 2008 for more references). Franzén (2015) investigated the physical actions a very young child engaged with whilst playing with a climb-in toy car. She identified actions that she believed indicated the child engaging in mathematical exploration. Clements and Sarama (2010, p. 2) stated that the "development of mathematics abilities begins when life begins" but delineated between the mathematical ideas of young children and those of adults. Sarama and Clements (2009, p. 313) argued that children's free play experiences involved mathematics, and this forms the foundation for the mathematical understandings that later mathematical ideas would build on.

The importance of early mathematical thinking and the impact on later achievement has been recognised. Clements and Sarama (2011, p. 968) signposted the impact of preschool children's mathematical thinking and knowledge of mathematics on their achievement as they progress through their school years. Likewise, Andrews and Sayers (2015, p. 257) identified children's number sense as a predictor of their later mathematics achievement. Therefore, ensuring young children have opportunities to engage with mathematics from a young age is beneficial. This places emphasis on identifying how very young children can engage with mathematical exploration.

Research investigating the play of young children has identified mathematics that may be evident. For example, Helenius et al. (2016) observed four young children (aged 6 years) engaging in free play with lego. In their analysis of the videos of the children, features indicated there were observed actions that could be identified as mathematical, such as rule negotiation, that could lead to children developing modelling. They cautioned that a narrow view of mathematics, which considered only content, could be problematical, particularly if it leads to a greater focus on formal mathematical activities in preschool.

Seeing mathematics as a cultural and social phenomenon that occurs in the everyday life is one way of expanding the view on mathematics. According to Björklund (2008) this implies that learning mathematics is then seen to be engaging in certain ”situations where some principles and tools are generally known” (p. 92). She studied toddlers' (children age 1-3) opportunities to engage in some basic mathematical concepts like part, whole relations. Bishop's (1988, 1991) seminal work describes six mathematical activities that may be evident in everyday or mathematics-specific activities. He stated that these six mathematical activities were universal, evident in all cultures, and essential requirements for development of mathematical knowledge (1988, p. 182). These mathematical activities are counting, locating, measuring, explaining, designing, and playing. Bishop (1991, pp. 18, 58) used language, as a form of technology, when identifying and describing these six activities. Bishop's mathematical activities have subsequently been used to identify the mathematics involved in young children's activities. For example, Johansson et al. (2016) used Bishop's mathematical activities when identifying the mathematics that occurred in young children's play at Swedish preschools. Johansson et al. (2016) found that the children's play could often be connected to more than one of Bishop's mathematical activities – reflecting what Bishop, himself said, that there will be overlapping between the mathematical activities (1991, p. 108).

As the learning that children express with his or her actions is influenced by what is in the room (Franzén, 2015), it is important to also take into consideration the toys the children engage with. According to Meaney (2016) the characteristics of artefacts, in this case the toys, will also influence learning. In the video at hand for this paper, the children are playing with a commercial toy called Babblarna. The ”Babblarna” toys is a Swedish concept and are a colourful group of six characters created with a purpose to be used by children with autism to develop language. Toddlers in Sweden usually recognise these six characters and hence it is interesting to use this specific video because the toy itself does not have a purpose of mathematical exploration.

Purpose and objectives of the research

Bishop (1991, p. 3) described the importance of viewing learning of mathematics in terms of mathematics education, rather than teaching mathematics, as the latter could focus more on doing instead of what the former would focus on – knowing. When linked with Clements and Sarama's (2010, p. 2) idea of early childhood educators interpreting mathematics from the child's perspective, it highlights the importance of early childhood educators being able to interpret what the children know – within the child's way of knowing. To be able to do this, the early childhood educator needs to have a way of interpreting their observations of children's actions to uncover the children's mathematical exploration. When young children are preverbal, this becomes more crucial.

Although Bishop's (1988, 1991) mathematical activities are effective for identifying mathematics, as it is based on and reflecting language (Bishop, 1991, p. 35), it can be more difficult to use with preverbal young children. Cooke and Jay (2021) have therefore reframed Bishop's (1988, 1991) mathematical activities to be used with observations of children's actions, rather than their language. Much as described by Franzén (2015), Helenius et al. (2016), and Johansson et al. (2016), Cooke and

Jay were interested in identifying the mathematical thinking that young children engage in terms of Bishop's (1988, 1991) mathematical activities. As with Franzén, these children were preverbal.

The purpose of this paper is two-fold, the first is to investigate and identify which of Bishop's (1988, 1991) reframed mathematical activities (Cooke & Jay, 2021) are evident in preverbal young children's explorations when they engage with their everyday environment in a Swedish preschool and the second is to connect the identified actions to the reframed mathematical activities in order to find a possible way of supporting teachers.

Perspective and theoretical framework

As suggested by Clements and Sarama (2010), the perspective of the child should drive research into the mathematical understandings of young children. Likewise, Franzén (2015) called on Barad (2007, as cited in Franzén, 2015, p. 46) to de-emphase language and the power given to it; rather, observing both the language **and actions** young children engage with when exploring their world. These, Franzén proposed, would enable the researcher to focus on the actions of the preverbal child – and to interpret these actions as a mean to identify the child's mathematical exploration. This would move the emphasis from language, an aspect that would be essential when working with preverbal children.

Methodology

In order to analyse the actions of preverbal children to identify whether Bishop's (1988, 1991) reframed mathematical activities were evident in their actions, video of young children engaged in everyday activities in a preschool was required. The video needed to be collected in an inconspicuous manner to ensure that the children's behaviours were not impacted or influenced by the process. The use of video enabled repeated viewing by the researchers, an element which Lynch and Stanley (2018, p. 58) considered a strength of the use of video in research involving observation of young children.

A three-layer hierarchy was used that deconstructed what was observed into events, behaviours, and actions. The first layer involved identifying the events – outcome-driven situations that a child engaged in. The next layer addressed the behaviours that were evident within that event. The behaviours are a collection of actions, both physical and verbal. These behaviours where the broken down to actions, which was the last layer that we considered.

Data collection

Data was collected via a 360-degree camera that was hung from the ceiling in the room where the children and educators were. The camera was unobtrusive but central in its placement and collected video (including audio) of everything that occurred within the room. Educators were not asked to engage in any specific actions or behaviours. The video was started prior to the children entering the room. The way the 360-camera was used, and the video was collected created an unstructured observation (Bryman, 2012, p. 273).

Participants

Staff at preschools in Sweden were approached to participate in the research. As Björklund (2008, p. 87) states, video of children must be collected carefully, with adults being provided with appropriate information regarding the project to provide informed consent – in this research, this involved the manager, staff, and parents of children at the preschool. The staff at the preschool where also given the possibility to view the videos before consenting to use them. The 360-degree video that is the focus of this paper was from one preschool. The preschool has 34 children aged 1-5 divided into two sections, with their focus on giving the children adventures every day. The staff consist of both preschool educators with an academic education and childcare staff. There are two staff and three children (all female) in the video. All of the children are one year old. Child 1 has some verbal language, being described by the staff as speaking a little bit. Child 2 and Child 3 are preverbal (described by the staff as not speaking at all). The analysis for this paper focuses on two children, Child 1 and Child 2, who are engaging with the Babblarna toys with one staff member (Child 3 is with another staff member at a wooden train set – this will not be considered in the analysis).

Process for analysis

A five-phase process was used to improve the veracity of the analysis. The process was developed as a way to address a potential concern raised by Franzén (2015, p. 47) – the power held by the researcher – by reducing the power of any individual researcher through emphasizing the focus on what is observed in the video in each of the phases of the multiple-phase process. The first phased involved two of the researchers viewing the video and documenting what was observed – one considering both language (Swedish and then translating to English) and behaviours and the other noting behaviours. In the second phase, one of the researchers deconstructed all noted behaviours into actions and linked the actions to Bishop's (1988, 1991) reframed mathematical activities, using the video as the first source referral point. The third researcher reviewed the actions (both physical and verbal, including confirmation of the translations), and the connections to Bishop's (1988, 1991) reframed mathematical activities in the third phase and compared these to the video. The actions were checked a third time by one of the researchers during the fourth phase. It was noted that there were natural groupings of the actions that had a clear beginning and end. To identify them as separate from actions, they were called ”events”. For the two children, the researcher created a chronological list of events that occurred in the section of the video and compared them to the video. The final phase involved the two researchers not involved in the fourth phase reviewing the content from the fourth phase, specifically, the events, actions, and connections to Bishop's (1988, 1991) reframed mathematical activities, comparing these to the video.

The reframed activities from Cooke and Jay (2021) were used, which revised *counting* as a child identifying how many objects can be touched, using one hand to pick up one item, or selecting a specific item; *locating* as the movement, positioning, or placement of their body or objects; *measuring* as identifying how much in terms of an attribute or measure; *designing* as recognising familiar objects and the functions of features; *playing* as engaging in imaginative or rule-bound activities or using hypotheses to work out or predict their environment; and *explaining* as posing and checking their beliefs about the environment by attending to the environment, repeating actions, or expecting specific actions to gain understanding about the environment.

Results

The 360-degree video enabled a very detailed analysis to be undertaken. The focus of the analysis for this paper was the first play experience in the video with the Babblarna toys. This created a section of video that was 1 minute and 30 seconds long. Not all of the observed activity involved the children physical holding the Babblarna toys. The following table describes a short section approximately 16 seconds of the 1 minute and 30 seconds long analysed part of the video. The focus is on combining actions made by the children to the reframed six mathematical activities (Cooke & Jay, 2021).

Table 1. Activities connected to actions

The actions	The mathematical activities
Educator walks towards dollhouse and crouches down to reach in. Child 1 walks towards dollhouse while Child 2 watches. Child 2 walk towards the Educator and doll house. Child 1 bends knees to look in the dollhouse. Child 1 reaches right hand in and takes a toy bed, made of wood, out from the dollhouse in her right hand; she then passes it to her left hand and then places it on the floor. Child 2 walks towards Educator and looks at the toys she has as the Educator is speaking. Educator takes purple <is this blue?> who is this (Dadda) and red who do we have here <is this yellow?> (Bobo) Babbler from the dollhouse. Child 1 reaches with her right hand towards the purple/blue toy the Educator is holding, takes it then places it in her left hand. Child 2 is watching the Educator. Child 1 takes the yellow/red toy from the Educator, places the yellow/red toy on the yellow bed and the blue toy on the floor.	Child 2 – *explaining*, watching the educator (event 1) to see what is happening as a way of creating understandings of the world; *playing*, focusing on the educator to see what will occur next. Child 1 – *locating*, positioning body at dollhouse; *measuring*, lowers body to look in the dollhouse (event 1); *designing*, using properties of form to know she can look in through the window; *playing*, planning the strategy to lower herself to look through the window. Child 1 – *locating*, reaching hand in (event 2); *measuring*, reaches into the dollhouse; *counting*, one-to-one correspondence using one hand to grasp one toy bed; *designing*, using properties of form grasp the toy bed; *playing*, planning the strategy to lower herself to look through the window Child 2 – *locating*, moving body towards Educator; looking at toy; *measuring*, moving towards Educator; *designing*, recognises the Educator is holding a toy (event 2); *playing*, following a rule-bound activity in attending to what the Educator is showing. Child 1 – *locating*, reaching with right hand (event 3); *measuring*, reaches for the toy; *counting*, one-to-one correspondence using one hand to grasp one toy; *explaining*, knowing how to grasp toy from Educator; *playing*, places toy on yellow bed; *designing*, properties of form (flat surface) to place the toy on the bed.

Table 1 shows that the actions and the six mathematical activities do not always reflect one another. We can see from this example that there were five types of actions that were observed. Table 2 lists the five actions together with the number of total occurrences and the number of occurrences for each child, as well as the six mathematical activities that can be associated with the actions.

Table 2. Actions connected to activities

Actions	Occurrences	Analysis
Moving around the room	five occurrences, three for Child 1 and two for Child 2	*locating*, moving body towards Educator; looking at toy; *measuring*, moving towards Educator; *designing*, recognises the Educator is holding a toy; *playing*, following a rule-bound activity in attending to what the Educator is showing.

Looking at or watching	two occurrences, one for each child	*locating*, positioning body at dollhouse; *measuring*, lowers body to look in the dollhouse; *designing*, using properties of form to know she can look in through the window; *playing*, planning the strategy to lower herself to look through the window.
Reaching	seven occurrences, five for Child 1 and two for Child 2	Analysis – *locating*, stretching up to move towards bathtub and toy Child 1 is holding; *measuring*, stretching for the bathtub; *playing*, devising a plan to get the bathtub and toy from Child 1.
Engaging with the toys	eleven occurrences, nine for Child 1 and two for Child 2	Analysis – *locating* explaining the positioning of the toy in relation to the bathtub (”take away”); *counting*, one-to-one correspondence with toy in one hand and bathtub in the other, saying ”take away”; *playing*, showing rule bound activity of take away; *explaining*, saying ”take away” to indicate the positioning of the toy and bathtub; *designing*, conceptualising the relationship between the two toys as one is taken away from the other.
Passively accepting the toys from the Educator	one occurrence, for Child 2	*locating*, grasps trolley, looks up towards Educator; looking at toy; *designing*, recognises Educator speaking of the toy she is holding; *playing*, rule-bound activity where the educator is offering the toy for the child to take; *explaining*, grasping trolley to show understanding of offer.

Table 2 shows there could be several ways that Bishop’s (1988, 1991) mathematical activities might be evident within the actions comprising the behaviours for one event. Also evident is that all of Bishop’s (1988, 1991) mathematical activities were identified during the short time period where the children first played with the Babblarna toys. Specifically, *counting* was identified 12 times, *locating* 27 times, *measuring* 15 times, *explaining* 18 times, *playing* 22 times, and *designing* 13 times.

Summary of results

In the video that is the focus of this analysis and paper, there was no traditional mathematical content. The educators did not engage in actions or speech that was mathematical nor designed to get the children to engage in mathematics. Using Bishop’s (1988, 1991) mathematical activities enabled the actions of the children to be viewed through a lens, much as Franzén (2015, p. 47) proposed, where focusing on the actions of the preverbal child potentially gave ”voice” to their mathematical exploration. However, there was much mathematics that was evident from the children’s actions. One of our results is that the reframed mathematical activities and the actions does not always reflect one another, hence it is not possible to say that a specific action is always a specifical mathematical activity, the actions need to be considered as part of an event in order to determine the mathematical exploration but also the other way around, there could be several ways that the mathematical activities can be evident.

Although these children are preverbal, there was a range of potential mathematical exploration that have occurred and each of the mathematical activities where evident. This may not be considered ”Mathematics” by the wider community, but it does show elements of ”what mathematicians do” (Helenius et al., 2016, p. 144). The most frequent of Bishop’s (1988, 1991) revised mathematical activities was *locating*, echoing the prominence found by both Meaney (2016) and Franzén (2015).

At the other end of the frequency scale, Bishop's (1988, 1991) revised mathematical activity of *counting* was identified least. The children did not count items but did demonstrate one-to-one correspondence and ”take away”. This lack of verbal counting may reflect the hierarchy Clements and Sarama (2010, p. 3) outlined in their learning trajectory for counting, where no verbal counting is evident for the pre-counter.

Discussion

Knowing what children know is a critical part of planning. Early childhood educators who work with preverbal children need to be able to identify the children's mathematical understandings. Using the revised version of Bishop's (1988, 1991) mathematical activities gives the children ”voice” (Franzén, 2015, p. 47) and enables researchers to link the actions of preverbal children to potential mathematical thinking. This mathematical thinking can be used by educators in their planning of activities and the environment for preverbal children. The environment created for young children, especially preverbal children, is what ”supports and stimulates children's learning and development” (Franzén, 2015, p. 43); in specifics, as Meaney (2016, p. 21) observed in her research, ”the objects being climbed on affected what could be learnt”. Using actions as a means to interpret preverbal young children's mathematical thinking is, perhaps, enabling the focus on doing as a means to knowing (Meaney, 2016, p. 24), rather than a push-down of formal mathematics as a way of schoolification (Helenius et al., 2016, p. 154). In working with preverbal children, it must be recognised that the observation of the children, the interpretation of what is observed, and the identification of mathematical exploration that may be evident are always framed within the perspective of the researcher (Franzén, 2015), without the possibility of member checks. The experience of the researchers in working with and observing young and preverbal children, and their experience with Bishop's (1988, 1991) work, can provide some justification, however, this remains a limitation of the research.

Acknowledgment

The research was funded through the Curtin University School of Education Research and Innovation Support Program (RISP) Round 3 2018. The researchers would like to thank the staff, children, and parents and guardians of the children who participated in the research.

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Compromises between required and preferred features of mathematical definitions in mathematics education

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The mathematics education literature distinguishes between the preferred and the required features of mathematical definitions. However, it is unclear whether and how this distinction differs between mathematical and didactical contexts. Here, we address the didactical context in which the purpose of the definition is to help students learn and understand the meaning of a new concept. We administered a comparative judgement study to assess how 12 mathematics teacher educators value the relative importance of required and preferred features of mathematical definitions. We found that the educators did not value the required features as more important than the preferred features of mathematical definitions. Furthermore, we found that the educators valued some required features more than others. These results suggest that, in a didactical context, the categorization into preferred and required features of mathematical definitions may not be as clear cut as indicated in the mathematics education literature and depends on the context or purpose of the definition.

Keywords: definitions, comparative judgement, teacher education, teacher educators, teacher lecturers

Introduction

To develop proficiency in mathematical reasoning, students must understand the essential ingredients of mathematics; one of these ingredients is *mathematical definitions*. Research shows, however, that students struggle with definitions (Edwards & Ward, 2004), partly because definitions are multi-faceted by nature. Morgan (2006), for instance, explained that definitions are used both for *deductive reasoning* and to *understand* a new concept. In previous work we found that scholars in mathematics education commonly label some features of mathematical definitions as *required* and others as *preferred* but not required (Forbregd et al., Submitted).

However, students and teachers of mathematics must balance these, sometimes, contrasting features of mathematical definitions. The literature does not address how the categorization into preferred and required features may depend on the purpose of the definition, such as whether it is intended for logical reasoning in a pure mathematics context or for learning and understanding a new concept in a didactical context. As a first step to clarify this issue, this paper addresses the didactical context by assessing how university lecturers who train mathematics teachers value the required and the preferred features of mathematical definitions.

A naïve interpretation of the categorization is that (1) if all preferred features can be omitted from a definition, every required feature is more important than every preferred feature, and (2) if no required feature can be omitted from a definition, all required features are equally important. Using comparative judgement, we answer these corresponding research questions:

1. *How do Norwegian mathematics teacher lecturers balance the required and the preferred aspects of mathematical definitions?*

2. *How do Norwegian mathematics teacher lecturers value the relative importance of the required aspects of mathematical definitions?*

We point out that this is a preliminary study which is part of a larger project.

Aspects of mathematical definitions

From a literature review (Forbregd et al., Submitted), we found five main themes on how scholars in mathematics education ($N = 74$) describe mathematical definitions. In this study, we focus on two of these themes, namely, the required and the preferred aspects of mathematical definitions.

In the review, we found three strands of the required features of mathematical definitions. First, several scholars emphasise *formal* requirements, for instance, that definitions must be consistent and non-contradicting (e.g., Johnson et al., 2014) and that they must be unambiguous (e.g., Foster & de Villiers, 2016). Second, multiple authors describe *existence* as a required aspect of definitions. That is, for every definition, at least one example or instance must exist (e.g., Avcu, 2019). Also, although definitions exist within representation systems, the definitions should be invariant under change of representations (e.g., Sánchez & García, 2014). Third, all mathematical definitions must exist within deductive systems (e.g., Cansiz Aktaş, 2016). Hence, all mathematical definitions employ only previously defined concepts (Van Dormolen & Zaslavsky, 2003).

The review identified three strands also for the preferred features. Definitions should be minimal, a feature that was highlighted by, for example, Vinner (1991). In our study, we coded *minimality* as a preferred feature, although we appreciate that scholars disagree on this issue (e.g., Van Dormolen & Zaslavsky, 2003). A second preferred strand comprises *aesthetic* features, such as elegance (e.g., Zazkis & Leikin, 2008), precision (Levenson, 2012), and clarity (Leikin & Winicki-Landman, 2000). Finally, definitions should be didactically suitable (Winicki-Landman & Leikin, 2000), intuitive (Ouvrier-Buffet, 2011), and match students' knowledge and needs (Leikin & Winicki-Landman, 2000).

Methods

From the aspects discussed above, we gathered 31 statements—each corresponding to either a required or a preferred feature of mathematical definitions. The statements were quotes from the literature about mathematical definitions; however, to reduce bias, we removed words such as ”must” and ”should” from the statements. The statements were also rephrased so that they had roughly the same wording. Examples of statements are: ”That it is consistent and non-contradicting” and ”That it matches the target group's knowledge and needs”. See Table 1 for a complete list of statements.

The statements were uploaded to the web application ”No More Marking” (abbr. NMM) (Wheadon, 2020). Respondents had to meet the criteria of being active teachers in mathematics teacher education at university level. Subsequently, 12 assistant professors in mathematics teacher education were enlisted as judges to conduct a comparative judgement study.

Each judge conducted 40 comparisons[1] of two statements—one pair at a time—with the following question: ”Which of the two statements about mathematical definitions are more important in mathematics education?”. For each judging session the pairings of statements to be compared were

[1] There were no possibility for ranking statements as equally important, that is a judge must rank one statement as more important than the other.

randomized by NMM[2]. The data were collected on Android tablets. The tablets were administered by the researchers, and hence, no person-identifying data or IP addresses were stored. The respondents compared 480 pairs of statements in total.

In the analysis, we used a Rasch model for paired comparison (e.g., Wright & Stone, 1979). This model expresses the likelihood that a statement n with measure β_n beats a statement m with measure β_m as

$$P(n \text{ beats } m) = \exp(\beta_n - \beta_m) / [1 + \exp(\beta_n - \beta_m)] \quad (1)$$

The numerator (Eq. 1) informs that, as β_n increases relative to β_m, the likelihood that n beats m increases. The denominator ensures that the likelihood is constrained between zero and one. The comparison data were used to fit the Rach model, yielding a (logit) score for each statement.

To assess the level of agreement between the judges, we conducted two analyses: First, we assessed the loss of invariance from differential item functioning (DIF). Specifically, we allocated each judge into one of two groups, and then we assessed whether the reported measures depended on which group we used in the analysis. Here, we used the Rasch-Welch t-test with a critical p-value of .05. Second, we examined the judge Infit Mnsq. Infit Mnsq is based on mean square standardised residuals of observations and the Rasch model (Bond et al., 2020). Roughly, persons with Infit Mnsq higher than one tend to respond unpredictably relative to the rest of the respondents. The analysis of Infit Mnsq was also conducted for each statement, and here, Infit Mnsq greater than one indicates that the sample of respondents disagreed on the relative importance of this statement.

Finally, after we had estimated and validated the measures of the statements, we conducted two tests, each responding to the research questions. To examine the null-hypothesis that required and preferred features were valued equally by the respondents (RQ1), we conducted a t-test on the mean values of the required and the preferred aspects. To test the null hypothesis that all required features of mathematical definitions were valued equally (RQ2), we conducted 1000 simulations in R (R Core Team, 2020) using the *Rwinsteps* package (Albano & Babcock, 2019). Here, we constrained the measures of every required statement as equal, and then, for each simulation, we returned the standard deviation (*SD)* of the required statements after 480 simulated comparisons. These simulated *SD*s were compared with the *SD* in the empirical data.

Results

Teacher lecturers' compromises between required and preferred features of definitions

Scholars in mathematics education distinguish between features of definitions that are required and features that are preferred but not required. As we have argued, if we applied only mathematical principles when we evaluated a definition, we should value all the required features more than the preferred features. However, the empirical results in this study show that lecturers in mathematics education balance these features. That is, it is not always the case that a required feature is regarded as more important than a preferred feature of a mathematical definition.

Examples of such compromises are shown in Table 1. As a first observation, we see that the two features that the lecturers valued highest were both labelled as required. Accordingly, the respondents

[2] The randomization algorithm aims to equalise the frequency of each statement in the total number of comparisons.

in this study were not willing to prioritise didactical principles (e.g., that the definition should be didactically suitable for the target group) at the expense of these two required features.

The following four features, however, were labelled as preferred. When the lecturers compared one of these preferred features (e.g., that the definition matches the target group's knowledge and needs) with required features of lower measures (e.g., that a definition fits into and is part of a deductive system), the teacher educators were more likely to select the former than the latter.

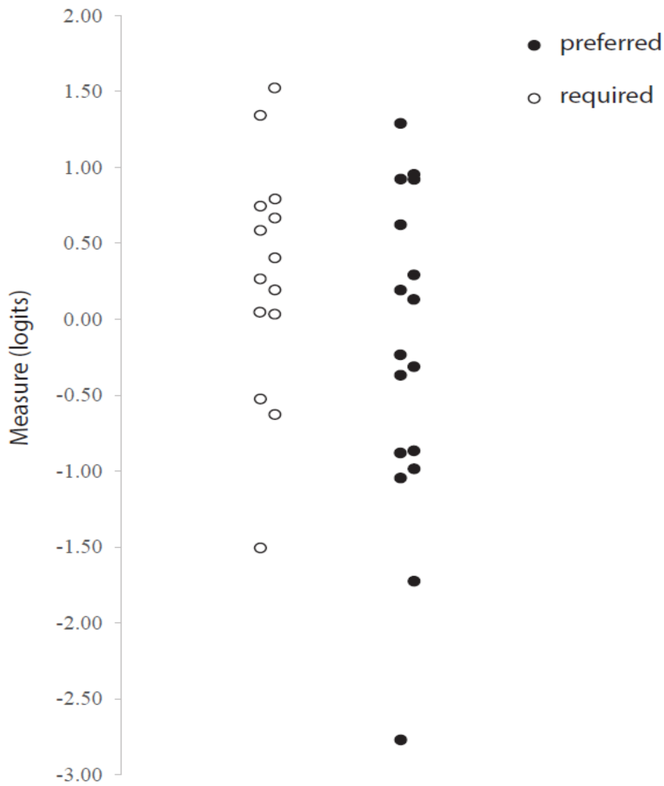

Figure 1. Values of required and preferred features of mathematical definitions

Moving beyond particular examples, the results, which are summarised in Figure 1, contradict the hypothesis that, in education, required features are always perceived as more important than preferred features of mathematical definitions. That is, we found no significant difference (p = .14) in how lecturers in mathematics education valued the required features (M = 0.28 , $S.E.$ = 0.21) and how they valued the preferred features (M = -0.23 , $S.E.$ = 0.26) of mathematical definitions.

Table 1. The relative importance of required and preferred features of mathematical definitions

Measure	s.e.	Infit Mnsq	Aspect
1.52	0.34	1.0	R That it is well-defined, that is to say, the meaning is unambiguous.
1.34	0.32	1.0	R That it is consistent and non-contradicting.
1.29	0.33	1.1	P That it matches the target group's knowledge and needs.
0.95	0.30	0.9	P That it captures and synthesises the mathematical essence of the concept.
0.92	0.30	0.9	P That it is didactically suitable to the target group.
0.92	0.30	0.9	P That it is understandable to the target group.
0.79	0.29	0.9	R That if there are multiple definitions for a given concept, they must be mathematically equivalent.
0.74	0.30	1.0	R That it allows to discriminate between instances and non-instances.
0.66	0.29	1.1	R That it is consistent with the mathematical theory formed thus far.
0.62	0.29	0.9	P That it only mentions necessary terms and properties so that it is possible to distinguish an instance from a non-instance.
0.58	0.29	1.1	R That it has a unique interpretation; in other words, it is well-defined and unambiguous.
0.40	0.29	0.9	R That it can be proven that at least one instance of the defined concept exists.
0.29	0.29	1.1	P That it has easily identifiable examples.
0.26	0.29	1.1	R That it is part of a deductive system, in a hierarchical and noncircular manner within itself and across existing axioms, definitions, and theorems.
0.19	0.29	1.0	P That it is hierarchical, in the sense that the terms used in the definition are known to the target group.
0.19	0.28	1.1	R That it allows instances and non-instances of the concept to be discriminated with certainty, consistency, and efficiency.
0.13	0.29	1.1	P That it only employs previously defined concepts known to the target group.
0.04	0.28	1.0	R That at least one example of the defined concept exists.
0.03	0.28	0.9	R That all the properties stated in the definition can coexist.
−0.24	0.29	1.0	P That it is precise.
−0.32	0.29	1.0	P That it is clear.
−0.37	0.29	1.0	P That it does not contain properties which can be mathematically inferred from other parts of the definition, i.e., it is minimal.
−0.53	0.30	0.8	R That equivalence can be proven if more than one definition is given for the same concept.
−0.63	0.30	1.1	R That it fits into and is part of a deductive system.
−0.87	0.31	1.1	P That it is useful, for example, for proving theorems.
−0.88	0.31	0.9	P That it is minimal. Minimality means that no conditions in a definition can be inferred from the other conditions; that is, there is no redundancy.
−0.99	0.32	0.9	P That it does not contain superfluous words or symbols, and that it ”looks nice”.
−1.05	0.32	1.2	P That it is intuitive.
−1.51	0.35	1.1	R That it describes any new concept as a special case of a more general concept.
−1.73	0.37	1.1	P That the name of the concept must be closely related to its natural-language usage.
−2.77	0.53	0.8	P That it is elegant.

Moreover, the results, presented in Table 1 and Figure 1, indicate that, in education, some required features are valued more strongly than other required features. To see if this difference was statistically significant, we compared the empirical standard deviation (*SD)* of the required features (*SD* = 0.90) to a simulated dataset in which all required features were modelled as equally important. Under simulated conditions, where all *true* measures are equal (i.e., when the *true SD* is zero) and all respondents make their judgements entirely in accordance with the Rasch model, we found that the empirical *SD* that can be expected due to measurement errors was 0.61. Since this difference was significantly different from the SD of the empirical measure (p =.04) we conclude that the reported *SD* in the required features is not due to measurement errors alone.

On the stability of the results

The judge Infit Mnsq values (not to be confused with *item* Infit Mnsq in Table 1) indicate the extent to which the lecturers agreed about the relative importance of the aspects of mathematical definitions. In our study, most Infit Mnsq values were around one, which is the expected value when all judges value the aspects equally. One judge stood out, however, with an Infit Mnsq value of 1.5. Although this value suggests that the respondent valued some of the aspects differently than most of the other respondents, the difference did not have practical consequences for the results on which we report in this paper. The correlation between the measures in the full sample and the measures when this respondent's judgements were excluded was r = .99.

To assess the stability of the results in more detail, we split the sample of respondents randomly into two groups. A DIF analysis on these groups showed that two features had measures that differed significantly (p = .03) between the groups. These features were ”That at least one example of the defined concept exists” and ”That it is useful, for example, it is useful for proving theorems”. Apart from these aspects, the respondents seemed to have an overall agreement on the relative importance of the features of mathematical definitions.

The reliability (analogous to Cronbach's alpha) of the measures was .77. Roughly, this value suggests that, while we obtain measures for individual features of mathematical definitions, we cannot make fine-grained inferences. For instance, within a cluster of features around zero logits, we cannot tell whether the respondents valued some of the features more highly than others.

Discussion

There are many features to consider when one formulates, or chooses, a definition formulation for a given mathematical concept in an educational setting. This study gives a glimpse into the ranking of features of mathematical definitions that teacher educators must take into account when teaching mathematics.

That some features of mathematical definitions are labelled as *required* in the mathematics education literature, can give the impression that there is no leeway for lecturers in mathematics education to emphasise preferred features of mathematical definitions at the expense of required features. However, we have shown that Norwegian lecturers in mathematics education valued many preferred features—especially those of a didactical flavour—as more important than mathematical requirements (e.g., that a mathematical definition must be part of a deductive system). Furthermore, we have shown that mathematical requirements are not necessarily valued equally by mathematics educators, suggesting that the required features are not all regarded as required when the purpose of the definition is to learn and understand new concepts.

The ranking of statements in Table 1 reveals a thematical distinction of the statements. Many of the aspects on the higher end in Table 1 relate to the clarity of the concepts with respect to the target group. By contrast, many of the statements on the lower end are non-functional features and applications of definitions (e.g., that they are ”useful”, ”elegant”, and ”special cases of more general concepts”). Another point that may be inferred from Table 1 is that statements with a clear didactical agenda are held in higher regard than those which do not (e.g., ”That it is hierarchical, in the sense that the terms used in the definition are known to the target group” was perceived as more important than ”That it describes any new concept as a special case of a more general concept”).

In a pure mathematical context, it is possible that there are no or few compromises to be made between requirements and preferred features. However, in a didactical context, we have shown that there is a measurable tension between mathematically important features and didactically important features of mathematical definitions. A ”good” definition in pure mathematics, might not be considered ”good” didactically, while a ”good” didactical definition might not be ”good”, perhaps not even valid, mathematically. Mathematics educators are willing to relax some mathematical requirements to allow for some didactical considerations. Insofar as some of these compromises are necessary, teachers and lecturers in education are faced with a challenging task: to teach mathematics by relaxing requirements of mathematical definitions.

Our work raises some new research questions. A larger study on the aspects of definitions from the literature seems worthwhile, both in an educational context as in our case and in a pure mathematical context. For example, it is not clear to us that all of the features labelled as *required* in the mathematics education literature are literally or objectively required in the purely mathematical context. Another relevant question is whether features can be usefully characterized as more ”didactical” or more ”mathematical”. Qualitative studies could further help to explain, exemplify, and explicitly describe our observations and the tension between mathematical features and didactical features of definitions. This would hopefully be valuable for teachers and educators in their work on mathematical definitions and introduction of new concepts for students to use in their deductive reasoning and proving.

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Changes in grades on the Norwegian lower secondary school mathematics exam

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Exams based on Norway's 2006 math curriculum were given every spring from 2009 to 2019, and the distribution of grades varied widely. Based on document analyses of the mathematics exams, we identify some traits of the exams' form that may contribute to these variations in grades. The answer formats and weighting of the two parts of the exam seem to be factors that should be taken into account in an in-depth analysis of the exams.

Keywords: Mathematics exam, assessment, language.

Background

The Norwegian Curriculum for Knowledge Promotion was implemented from 2006, and the first corresponding lower secondary school mathematics written exam was given in 2009.[3] Since then, there have been changes in the distribution of grades, with a remarkable decline in the lowest grades and an increase in the highest grades in the last four years (Figure 1).

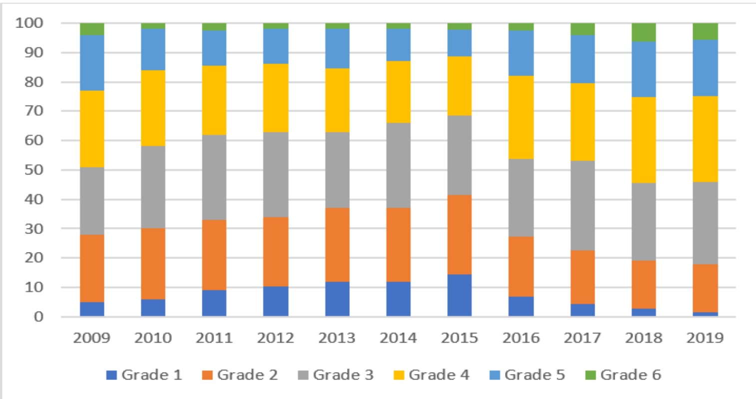

Figure 1. Distribution of grades for examinations in Year 10[4]

The proportion of examinees who were awarded the three lowest grades (1, 2, or 3) increased from about half to over 60 percent during the period from 2009 to 2015, while there has been an increase in the three highest grades (4, 5 or 6) since 2016. The proportion who achieves these grades is now approaching 60 percent, while the lowest grade has hardly been used in the last two years (Figure 1).

2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019
3.4	3.2	3.1	3.1	3.1	3.0	2.9	3.3	3.4	3.6	3.6

Table 1. Average grade for the mathematics exam in Year 10[5]

[3] Although the exam in 2008 was based on this curriculum, it had the same format as exams based on the previous curriculum. It is therefore not included in our analysis. Due to the Covid-19 pandemic, the last ordinary exam under this curriculum was administered in 2019. All exams are available at https://matematikk.net/side/Eksamensoppgaver. English version of the mathematics curriculum: https://www.udir.no/kl06/MAT1-04?lplang=http://data.udir.no/kl06/eng

[4] Results for the period 2015-2019 were retrieved from www.skoleporten.udir.no, while the results for the period 2009-2014 were taken from graphs presented in articles from the Directorate for Education and Training.

[5] Source: https://www.ssb.no/statbank/table/07498/

As Table 1 shows, the average grade has changed throughout the curriculum period. An increase of one tenth means that, on average, every tenth student has moved up one grade, so both the decrease from 2009 to 2015 and the later increase must be characterized as noteworthy changes.

Exams can have more roles than those formally enshrined in legislation. They certify competence and rank students, and they are used by different groups for a variety of purposes (Newton, 2007). Exam results are published at the school level and influence the standing of the school in the community. They are also used in aggregate form as the basis for political decisions about mathematics teaching (Nortvedt & Buchholtz, 2018). In addition, we know that most teachers use previous exams in their teaching (Andresen et al., 2017). The tests represent the ceiling, in terms of what is expected in most classrooms (Burkhardt & Schoenfeld, 2018), so exams in mathematics may be of great importance to everyday teaching.

Changes in results may be due to an increase in students' competence, in the exams' content, or in the formats of the exams. It is therefore of interest to investigate the changes in exams and grades more closely.

Theory and previous research

The curriculum used in Norway during the period addressed in this paper was competency-based. Among other things, competency frameworks should demonstrate that learning mathematics comprises more than just acquiring an array of facts, and that mathematics involves more than merely carrying out well-rehearsed procedures (Kilpatrick, 2014). To assess the total competency required by the exam sets, a comprehensive analysis based on a framework for mathematical competence is required. Turner et al. (2015) give examples of how this can be done based on the PISA Mathematics Frameworks, and they describe theoretical and practical issues relating to these analyses.

The formulation of tasks can also affect their difficulty. Andresen et al. (2017) include a summary of international research on language traits that make math tasks difficult to understand and show that these traits interact in complex ways. Such traits include many words, many low-frequency words, passive voice, and long noun clauses. Illustrations can sometimes be helpful but might also make the text more challenging. In their article investigating changes in high-stakes mathematics examinations, Morgan and Sfard (2016) developed an analytic framework with lexico-grammatical aspects and visual mediators as a central part. These are much in line with the literature review presented by Andresen et al. (2017).

Sangwin and Jones (2017) refer to research on answer formats that report impact both on students' results and on possible discrepancies between what the examiners intend to assess and what is actually assessed. They investigated the impact of answer formats for reversible mathematical tasks (e.g., verifying a solution, versus solving an equation) and found that students performed better on multiple-choice items than on constructed-response items, and that this effect was larger than the effect of guessing. How the mathematics exam is perceived by the students is also dependent on the knowledge they have acquired from their teaching and learning materials, such as textbooks. Lithner (2008) developed a framework to determine whether creative mathematically founded reasoning is necessary, or whether imitative reasoning is sufficient for solving a task. To be able to use imitative reasoning, the task must be familiar to the student to some extent. Non-identical tasks regarded by a mathematically well-versed person as having ”the same mathematical content” may be seen by the student as anything but equivalent (Morgan & Sfard, 2016, p. 89).

The distribution of tasks over the main mathematical domains in the curriculum may contribute to the total difficulty of the exams. In the PISA tests, there are four content categories with an even score distribution, and items in each category have a range of difficulty and mathematical demand (OECD, 2019). Exams in Norway shall give the student the opportunity to show competence in as much of the subject as possible (Forskrift til opplæringslova, 2006, §3–22), but there are no guidelines for the distribution of tasks over the mathematical domains like those found in PISA tests.

The full Norwegian examination sets are published on the day of the examination. While the PISA tests are composed of clusters of trend items in mathematics that have been kept confidential (OECD, 2019), a similar approach is not possible for the Norwegian national mathematics exams.

Despite this, many aspects of the exams can be compared from year to year, and the significance of the exam makes it interesting to investigate these changes more closely. To paint a full picture of these changes, a document analysis based on one or more of the frameworks mentioned above is needed (Lithner, 2008; Morgan & Sfard, 2016; Turner et al., 2015). Such analyses are extremely time-consuming, so, in a preliminary phase, we found it useful to carry out some simple document analyses of selected traits in *the form* of the exams that could contribute to explaining the changes in grades. We do not intend to draw conclusions about the extent to which each of the traits contributes to the changes in grades. Rather, we use the variations in grades to look at and discuss factors that should be taken into account in such analyses.

Our research question is: Which traits in the form of the exams are possible explanations for the changes in grades on the mathematics exam given in Norwegian lower secondary schools?

Method

Both mathematical and non-mathematical aspects may contribute to an exam's degree of difficulty. If these aspects vary, they may be possible explanations for variations in grades. Language traits and illustrations can shed light on how easy the task is to understand, while variations in the main areas covered may tell us something about how predictable the exam is. The answer formats provide information about the degree to which mainly procedures and specific knowledge are tested. The mathematics exam given in Norwegian compulsory schools consists of two parts. Part 1 must be completed in two hours using no tools other than pen and paper, and Part 2 must be handed in after another three hours. In Part 2, students are free to use whatever tools they wish, except communication. Adopting the framework developed by Lithner (2008), an analysis of the exams from 2017–2019 showed that creative mathematically founded reasoning was necessary to solve more tasks in Part 2 than in Part 1 of these exams (Bjørnset et al., 2020), so the relative weighting of Parts 1 and 2 could therefore be of interest. These aspects are investigated through a document analysis of all 11 ordinary Year 10 exam sets under the current curriculum.[6]

In the exams, tasks are numbered and sub-tasks are labeled a, b, c, and so on. Although multiple answers are required in some sub-tasks, we still count them as a single answer. For each main task, the maximum score that can be awarded is stated on the exam paper. The pre-grading report shows

[6] The exams given in fall 2013 and in spring 2020 and 2021 have not been included, since they were only distributed to external candidates.

how the points are distributed among the sub-tasks. We had access to these reports for the period 2014–2019; for the period 2009–2013, we distributed the points for the sub-tasks based on experience from recent years. Other researchers might distribute the points differently, but this would only affect the analyses to a small extent. Our analyses are based on the percentage of the total score achievable for the exam.

Since different language traits interact in complex ways, looking at only a few traits will not give a complete picture of how language affects the exam's degree of difficulty. For the purposes of this article, we have chosen some aspects as examples. The amount of text in the tasks contributes to their difficulty, especially for weak readers; therefore, the proportion of sub-tasks that has little text is of interest. We have identified sub-tasks with no more than 10 words, including the introduction to the task, if there is one. Of these sub-tasks, some come late in a main task and after sub-tasks or introductions containing more than 10 words. We call the sub-tasks that have no more than 10 words and which do not come after tasks/introductions with more than 10 words "real short tasks." Tables are included when analyzing illustrations. We distinguish between illustrations crucial to solving the task and those that are not.

Each sub-task is compared to competency objectives in the curriculum to determine to which main area it belongs. If a sub-task covers competency objectives from several main areas, the points are distributed among them. The tasks are also divided into different answer formats: multiple-choice, answer only, open, drawing, spreadsheet, and digital graphing tool. True/false tasks are categorized as "answer only," as is filling in certain cells in a pre-arranged table. We categorize lines of symmetry, finding the vanishing point, drawing a graph without digital tools, and coloring parts of a figure under "drawing."

Results

There have been major changes in the answer formats for the exams which have affected what is required regarding showing calculations, mathematical reasoning, and communication. Figure 2 shows that the proportion of multiple-choice tasks in the exams has increased markedly in the last three years, at the expense of tasks where the students have to justify their answers (open tasks). The changes in answer formats and grading can be exemplified by how linear equations are tested in Part 1. Typically, there have been two equations that must be solved by hand, with the most difficult one counting twice as much as the simplest one. This was changed in 2017, so the two equations were given equal scores, and then changed again in 2018, when the simplest one was presented as a multiple-choice task. The same pattern is seen for testing students' ability to manipulate algebraic expressions.

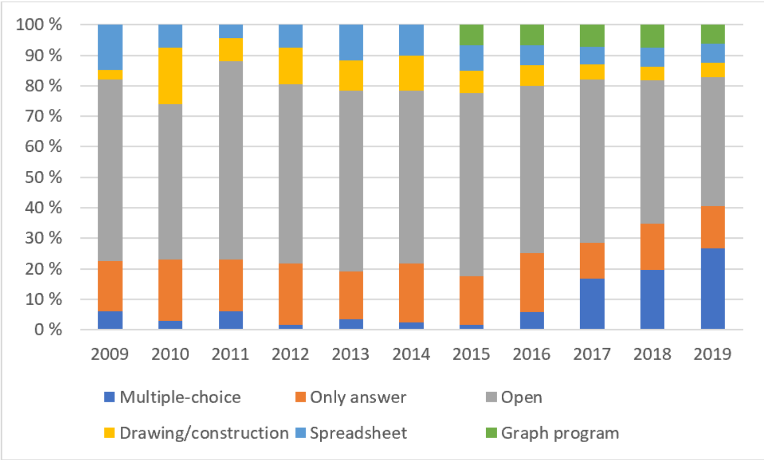

Figure 2. Proportion of different answer formats in exams, 2006–2019

Until 2016, the proportion of the total time allocated to Part 1 (two-fifths), was reflected in the scoring of the two parts, with 40 percent of the points being awarded for Part 1. From 2017, the weighting shifted, with more weight being given to Part 1; in the 2019 exam, 48 percent of the points were allocated to this part. The limits for awarding individual grades remained stable during the same period.

Figure 3 shows that the proportion of ”real short” sub-tasks was stable, at about 20 percent, from 2010 to 2017, but has decreased in recent years. In 2019, students were presented only six ”real short tasks.” That represents 9 percent of the points, which is a pronounced decrease from 2017.

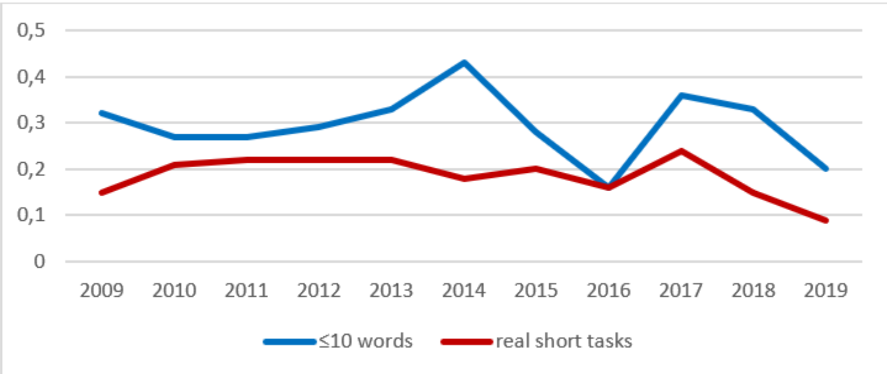

Figure 3. Proportion of sub-tasks with few words and ”real short tasks” in exams, 2009–2019

Figure 4 shows the development of the use of illustrations over the period. The numbers have varied, with a decrease in the number of illustrations from 2009 to 2015, and then a jump in 2016. The number of illustrations that are not crucial for solving the problem has also increased in recent years.

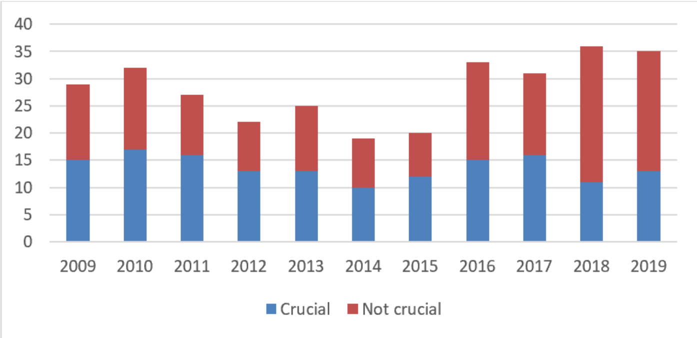

Figure 4. Distribution of illustrations deemed crucial and not crucial for solving the task

The curriculum is divided into five main areas, and Figure 5 shows that these are tested to varying degrees in the exam. There has been less variation in recent years.

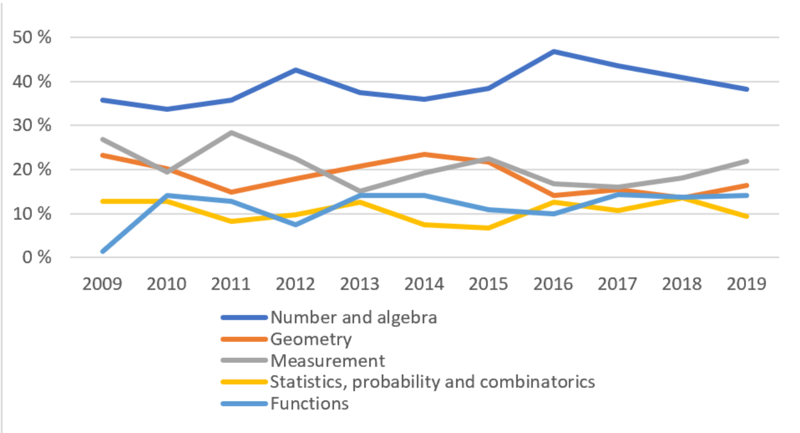

Figure 5. Proportion of tasks testing different areas, 2009–2019

Analysis and discussion

Below we discuss some of the changes presented in Figure 1 and Table 1. Many of the aspects addressed in our analyses of the mathematics exams vary considerably, which may therefore contribute to hypotheses about the reasons for the changes in grades during the period. In our materials, we find two distinct changes in the exam that could explain the decrease in the average grade from 2009 to 2010. Tasks that required drawing and construction were at their lowest in 2009 and highest in 2010 (see Figure 2). From Figure 5, we note that, while the main area ”Functions” was hardly tested in 2009, tasks in this area accounted for 14 percent of the exam in 2010. These changes might have resulted in more tasks being unfamiliar to the students, since teachers may have adjusted their teaching based on previous exams (Andresen et al., 2017; Burkhardt & Schoenfeldt, 2018).

The variations in the main areas tested from year to year was greater until 2016 than it has been in recent years. This might help to explain the further decrease in the average grade beginning in 2010 and the increase in grades since 2016. Less variation in the main areas tested may result in a more predictable mathematics exam and might lead to a narrowing of the curriculum (Gurskey, 1994), and to the exam being perceived as easier. Based on the average score on each item in the exams for the period 2017–2019, there seems to be a range of difficulty within each of the main areas in the curriculum (Andresen et al., 2017; Bjørnset et al., 2020). A more thorough analysis of the content and complexity of the tasks in each mathematical domain will be necessary to determine to what extent the main areas tested affect the overall difficulty of the exam.

The lowest average grades are associated with the 2014 and 2015 exams (see Table 1). In both of these years, we find a minimum number of illustrations, and in 2014 there was a larger proportion of tasks that used fewer words than in any other year. These are both features that could point in the direction of *less* demanding exams. Considering the other traits, we find that, in the 2015 exam, there are fewer multiple-choice tasks and tasks that only require an answer with no justification. This is also the first exam with the mandatory use of a digital graphing tool (see Figure 2). The years 2014 and 2015 were also special for having less focus on statistics, probability, and combinatorics, and placing more emphasis on geometry than the rest of the period.

In the last three years, the average grades have increased, and the lowest grade was hardly used in 2019. In the same period, we can observe a change of traditional tasks from open to multiple-choice ones, and a change in scores, with simple and more demanding tasks given the same score. Some of these items are of the reversible type, and the answer format might influence the results (Sangwin & Jones, 2017). Those who are awarded the lowest grades for the exams generally earn their points in Part 1, with a large proportion of their scores coming from multiple-choice tasks and those that require an answer with no justification (Bjørnset et al., 2020). As more tasks are converted into a multiple-choice format and assigned heavier weights, compared to those that require calculations or some other form of justification of the answers (as exemplified with equations and algebraic expressions), this will benefit these students at the exam. The decrease in the number of tasks that are short in words points in the opposite direction. The same applies to the increase in illustrations that are not crucial for solving the tasks, since this, arguably, increases the competency required of students, since they have to give more consideration to whether or not the provided illustrations are necessary.

Conclusion

Our analysis of different aspects of these exams paints a diverse picture. Some aspects are likely to impact some groups of students more than others. The analysis of language and illustrations points toward a more difficult exam for weak Norwegian readers in recent years, while the other traits analyzed here mostly points towards a decreased level of difficulty for all students. Thus, it is possible that the exam has become less equitable by being more difficult for the group of weak Norwegian readers, while other aspects camouflage this. It is not possible to draw explicit conclusions about the extent to which each of the traits we have included here contributes to the changes in grades in the Norwegian lower secondary school mathematics exam; however, we argue that these traits concerning the exam items' form are relevant.

Nortvedt and Buchholtz (2018) refer to research finding that policy-makers tend to focus on overall results, rather than on utilizing the detailed information that assessments offer. If the changes in grades in recent years are not the result of increased mathematical competence among students but are due to changes in the form of the exams, the results may mislead policy-makers. The form of the exam in Norway is in a revision process at the time of writing and our findings might be considered relevant in this process.

More research is needed to explain the changes in grades on the Norwegian lower secondary school mathematics exam. A comparative analysis of mathematical competency required by the exams, and of the mathematical content and distribution of difficulty between and within the mathematical content domains represented in the exams, is needed. These will provide valuable information concerning whether or not there have been changes in the degree of difficulty. In addition, there is a need for more research on the variations in the different aspects we have pointed out and how they work together. The effects of answer formats and the weighting of tasks and parts of the exams should also be included in an extensive analysis of the Norwegian lower secondary school mathematics exam.

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Sámi traditional measuring: Sámi preservice mathematics teachers' approach to the new curriculum

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Most Sámi traditional measures for length are body related. This paper focuses on Sámi preservice mathematics teachers' presentation of Sámi traditional measuring of length, a way of measuring that can contribute to primary school mathematics. Norway's new mathematics curriculum includes one competence aim for Grade 2 regarding measuring. Preservice mathematics teachers made videos about Sámi traditional measuring, and they supported their videos with a text. Analysis of a) one video and b) one video with its corresponding text shows a variety of body measurings; some are dynamic, and some are static. Traditional Sámi measures are still in use. They are individual and depend on individual body sizes.

Keywords: Curriculum, Sámi, measuring, preservice mathematics teachers.

Introduction

Sámi[7] traditional measuring is today an integrated part of Sámi handcrafters as well as reindeer herders' mathematical thinking and reasoning (Jannok Nutti, 2007), and it is functional for their professions. In addition, these people are familiar with moving between the metric system and traditional measuring. Most Sámi traditional ways of measuring length are body related. According to Kaijser (1994), all cultures have traditionally used the human body as a basis for measuring length. Bodily measuring units have been used worldwide, and standardized lengths of feet and other body parts vary between countries. Lakoff and Núñez (2000) point out that "[h]uman mathematics is embodied; it is grounded in bodily experience in the world" (p. 365). Dewey (1933/1998), as well as Freudenthal (1973), claims that children's use of their own body is important for their intellectual development. Sámi traditional measuring units are not standardized but vary from individual to individual. Individual measuring units are used by other Indigenous peoples as well; for instance, Lipka et al. (2013) describe how body-related, non-standardized measuring works for the Yup'ik people in Alaska. Sámi traditional measuring is a cultural activity that invites children to include experiences with their own body as basis for their mathematical reasoning.

Norway's core curriculum (Ministry of Education and Research [KD], 2017) is an overarching part of the curricula for all subjects, including mathematics. Regarding Sámi students all over Norway, it claims that "[t]he Sami School shall ensure that the pupils receive education and training based on Sami values and the Sami languages, culture and societal life" (p. 4). The core curriculum also

[7] The Sámi is an Indigenous people of the Arctic. They live in Norway, Sweden, Finland and the Kola Peninsula of Russia.

includes aims for students in Norway regardless of whether they are Sámi or not: ”[t]hrough the teaching and training the pupils shall gain insight into the indigenous Sami people's history, culture, societal life and rights" (p. 6). This means that there is a need for teaching material about Sámi mathematical issues for teachers all over Norway. Our paper shows how Sámi preservice mathematics teachers can contribute to the need for knowledge about this curriculum aim.

Norway's new mathematics curriculum (KD, 2019) was laid down in November 2019. KD (2018) stated beforehand that the new curricula for all subjects should contain Sámi issues, either implicitly or explicitly. The final mathematics curricula had no Sámi content, but the last hearing's version included one Sámi competence aim (Directorate of Education and Training, 2019, authors' translation): ”After Grade 2, the students shall be able to experiment with traditional Sámi measuring and with measuring length using different non-standardized units of measurement, describe their approach and talk about the results." Because of this, an editor/journalist from the TV 2 broadcasting company's school department e-mailed the first author regarding references to teaching units about Sámi traditional measuring (G. A. Ludvigsen, personal communication, October 4, 2019). That e-mail is our point of departure. Author one and author three are affiliated with the Sámi University of Applied Sciences. The institution is located in Northern Norway, and its official language is North Sámi (Sámi Allaskuvla, 2019). The teacher education program is based on Norway's curriculum, but the preservice teachers come from Norway, Finland, and Sweden. The other authors are Sámi preservice mathematics teachers. Preservice mathematics teachers worked out a reply to the journalist's wish for supportive material for teachers. Our research question is: How do Sámi preservice mathematics teachers present Sámi traditional body measuring? The preservice mathematics teachers created videos and texts about traditional Sámi measuring. The analysis presents how i) two different videos and ii) one video together with a written text, create an image of Sámi traditional measuring.

Measuring

According to Lindquist et al. (2019), *measurement* is the process of quantifying attributes of objects and phenomena like length and time. Norwegian mathematics curricula also use the term ”measurement". By contrast, Bishop (1988) uses the term ”measuring" instead of ”measurement". This is because he wants to move the focus in the curriculum toward actions and processes and away from nouns (quoted in Fyhn, et al., 2018). ”Measuring" is an activity that is found in all cultures studied so far (Bishop, 1988). According to Lipka et al. (2013) the mathematics of Yup'ik elders is situated in the context of solving everyday problems, so ”[t]hey established measuring not as measuring in an absolute sense, to find the length of an object, but measuring as a set of relationships and proportionality. Indeed, when elders are asked what Yup'ik word or concept best describes mathematics, 'cuqete' (to measure) is given" (p. 133). Like Sámi measures, Yup'ik measures of length are body related.

Freudenthal (1991) points out that palpable teaching materials are of great value, provided that the children can structure it themselves: ”[t]he best palpable material you can give the child is its own body" (p. 76). According to Lakoff and Núñez (2000), human ideas are to a large extent based on sensory-motor experiences. When the child uses parts of its own body in performing measuring, it is structuring the palpable material, as Freudenthal (1991) recommends. In addition, the child's sensory-motor experiences are the basis for what goes on. Clements and Battista (2001) call measurement a real-life application of mathematics. Van den Heuvel-Panhuizen (2005) points out that mathematics is not only connected to the world of numbers, but also to our physical world and those phenomena that occur within it. She claims that measurement is the connecting link between arithmetic and geometry, as shown in Figure 1.

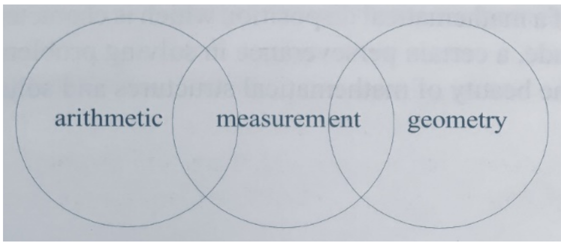

Figure 1. Measurement. Reprinted from *Measurement and geometry in line* (p. 9), by M. van den Heuvel-Panhuizen, 2005, Freudenthal institute. Reprinted with permission

For Sámi mathematics educators, it is important to approach ”measuring’ from a perspective of Sámi culture and social life. Fyhn and Hætta (2019) show how Sámi traditional measuring of length is an activity where the unit of measuring is dynamic; it is determined by the maximum distance between two moving parts of your body, like the forefinger tip and thumb’s tip in figure 2. This supports Bishop’s (1988) focus on the verb *measure* (measuring) instead of the noun *measurement.* Figure 2 presents the activity *goartilastit*, which means to measure by *goartil.* One *goartil* is the maximum distance between the thumb’s fingertip and the forefinger’s fingertip. One relation between measuring and arithmetic, is your counting while your fingers ”walk” along the item you measure. The segment shaped by the strengthened band you measure, is a relation to geometry. Other Sámi measuring activities are *sállut* (to measure by fathoms, *salla*) and *lávkut* (to measure by steps, *lávki*).

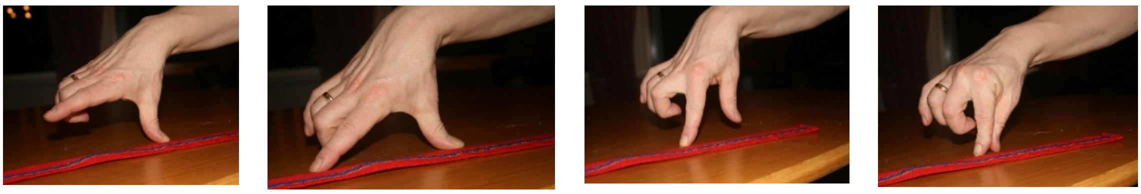

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Figure 2. *Goartilastit* – to measure by *goartil.* Reprinted from «Samisk språk og kultur som matematikkressurs» by A. B. Fyhn and O. E. Hætta, 2019, *Tangenten – tidsskrift for matematikkundervisning, 30*(3), 22. Reprinted with permission

In 2007, Norway ratified UNESCO’s (2003) Convention for the Safeguarding of the Intangible Cultural Heritage. The convention points out that the cultural heritage also includes traditions or living expressions inherited from ancestors and passed on to descendants. Traditional Sámi measuring is part of the Sámi intangible cultural heritage.

Methods and results

Six preservice mathematics teachers (three from Norway and three from Finland) got an open task: ”make a video (for example by your phone) that presents Sámi traditional measuring of length meant for people who do not know what this is.” Some weeks later they got a written task: Write 4–5 pages about Sámi traditional measuring to support your videos. The preservice teachers worked in pairs and created videos and texts about Sámi traditional measuring. Two of the pairs agreed to publish their videos online by the university's YouTube channel[8]. The six preservice teachers constituted the complete group of preservice mathematics teachers in the basic 30-ECT mathematics course at the Sámi University of Applied Sciences for the study year 2019/2020.

All preservice teachers were invited to participate in writing this paper, and two of them, the second and fourth authors, volunteered. The analysis is based on their video and text, and on one of the other videos. We attempt to show how i) two different videos and ii) one video accompanied by a text, reflect different aspects of Sámi traditional measuring of length. The analysis identifies a) the number of measures presented, b) what each measure is used for and c) whether the unit of measure is dynamic or static.

One important reason for inviting preservice teachers to become co-authors, is quality control. Since neither the first nor the third author has Sámi background, there will be a risk that they can misinterpret important details about Sámi traditional measuring and how it is practiced. The preservice teachers can participate in Sámi speaking conversations with practitioners; this is in line with the UNESCO's (2003) intangible cultural heritage convention, which underlines respect for the practitioners.

The two videos

The preservice teachers solved the video task in different ways. Taina and Iina-Marja's video lasts 56 seconds. It presents six different bodily units of measuring. Five of these are presented by drawings and one (*goartil*) is presented by a still photo. The six illustrations are categorized as analytical, according to Fyhn (2008) an analytical drawing extracts an issue from its context. Each drawing presents the front of an upstanding child, dressed in clothes for warm classroom temperatures (20°C or more), and a traditional measuring unit is marked by a black freehand line with arrows in the ends. A voice adds context by informing what each measure is used for. Two of the measures, *salla* and *goartil* are used for culture specific purposes (measuring lasso[9] length and measuring the length of bands), while the other measures are used for purposes that are not necessarily related to culture: measuring classroom length, snow depth and the distance from where you stand and up to your armpit. *Goartil, salla* and *lávki* are examples of dynamic body measuring units. By contrast, – *buolvvarádjái*, *bahtarádjái* and *gieđavuollái* – are examples of static body measures.

Sanna and Inga Berit Karen's video lasts more than twice as long, but it presents just three different traditional body measurements, which are dynamic. They are presented through a combination of video and still photos and linked to examples in a cultural context. The video starts by explaining that the authors chose to limit the number of examples. A voice explains that in earlier days, the Sámi themselves created their traditional measuring. They had to come up with something, because they did not have measuring instruments like those we have today. Here the video exemplifies what Bishop (1988) claims: "just as *each* cultural group generates its own language, religious belief, etc., so it seems that each cultural group is capable of generating its own mathematics" (p.180). The video explains that those who sew traditional Sámi clothes today still use Sámi traditional measuring: *goartil* is used for measuring when making Sámi traditional fur shoes, while *salla* is used for deciding

[8] Links to Sanna and Inga Berit Karen's video https://www.youtube.com/watch?v=0fH-iwfO6r4 and Taina and Iina-Maja's video https://www.youtube.com/watch?v=0IwHJE5i8nI

[9] A lasso (*suohpan* in Sámi) is a loop of rope that is thrown around a target and tightened when pulled.

the length of the clothing's lower edge when making *gákti*, the traditional Sámi dress. The video distinguishes between the two kinds of *goartil*, *čuvdegoartil mihttu* (measure from thumb tip to forefinger tip) and *gaskagoartil mihttu* (measure from thumb tip to middle fingertip). The video presents photos of two different hands that show the length of their *goartil* along a centimeter measuring tape; one *goartil* is 16 cm while the other *goartil* is 18.5 cm. These photos are supported by a voice: ”looking at the photos we can see that the length is changing from person to person”. Here the video underlines that the body measures are individual, similar to the Yup'ik people's body-proportional measuring described by Lipka et al. (2013).

How a text supported one of the videos

The preservice teachers' texts add information and references to their videos. This section presents some of this information, translated into English. While Sanna and Inga's video presents just three ways of measuring, their text adds five more measuring units. *Lávkki* is not much in use nowadays but was frequently used in earlier times for instance when measuring distance between the posts when building a storehouse for food. *Állan* or *gardnjil mihttu* (alen or underarm measure) is used to measure the length of a traditional shoe band and for measuring a rope or a lasso. An individual's *állan* goes from the tip of the long finger to the elbow tip. Other length measures are *buolvvarádjái*, *bahtarádjái* and *gieđavuollái*: up to your knees, butt, and armpit, respectively. These three are used for measuring snow depth and for measuring a child's height (Jannok Nutti, 2009).

In addition, the text elaborates on the video's content. *Goartil* is an old way of measuring that is established in sewing *gákti*, the traditional Sámi dress (Hætta, 1994), and for measuring snow depth (Jannok Nutti, 2007). This measure is in daily use nowadays in traditional work. The middle finger *goartil* is a bit longer than the forefinger *goartil* and the middle finger *goartil* is used when sewing clothes for men (Fyhn et al., 2013). One *salla* is the distance from fingertip to fingertip when your arms are stretched out, and your *salla* is approximately similar to your height. This means that if your height is 150 centimeters then your *salla* is 150 centimeters too. *Salla* is used by handcrafters and for measuring the length of tent poles and lassos. When measuring snow depth, you can put your ski pole through the snow to the ground and measure by *salla* or *goartil* how deep the snow is (Jannok Nutti, 2009).

Discussion and closing words

Our research question concerns Sámi preservice mathematics teachers' presentation of Sámi traditional *measuring*. Analysis reveals that the videos present several traditional Sámi ways of measuring length through examples that employ body parts. A written text might be more thorough than just a short video, a text might provide more examples and a text provides references. The videos were made because of a question from a journalist. The two presented videos together constitute a first reply to her e-mail.

Pais (2012) points out that mathematics empowers people not so much because it provides some kind of knowledge or competence to them, but because it gives people a value. Scandinavian mathematics teachers can present their students for the two videos made by Sámi preservice teachers. By doing so, the teachers may give a value to Sámi teachers and parents who share their cultural knowledge about Sámi traditional knowledge.

Sámi traditional measuring of length is still in daily use. According to Jannok Nutti (2007), all Sámi traditional measures for length referred to in this paper are still in use among Sámi handicrafters and Sámi reindeer herders. The static *meter* in the SI system was introduced in order to make all meters equal (Holtebrekk, 2020). This is in contrast with Sámi and other Indigenous people's traditional measures for length, which a) depends on body size and b) are not intended to make all measures equal. Yup'ik elders use body-proportional measuring as a generative solution strategy to solve everyday problems; the aim is a proportional measure and not a universal unit of measurement (Lipka et al., 2013). During the work with this paper, the second author posed some questions about lasso lengths to individuals with knowledge about lasso use. They claim 13 *salla* is the length of your main lasso, which in turn means that the lasso length is proportional to your body height. In addition, you have other lassos for special purposes, with different lengths.

Jannok Nutti (2007) refers to three body measures for length that were not referred to by the preservice teachers, *suorpma govddu* (finger width), *giehta govddu* (hand width) and *čuvdemihttu* (from a pointing finger's tip to its knuckle). Fyhn and Robertsen (2020) explain how the fisherman Håkon uses fathoms for measuring sea depth. Fathom is a dynamic measure, as opposed to most traditional Norwegian measures for length, which are static.

Sámi traditional measuring have one advantage that the SI system cannot achieve: children always bring their bodies with them. They do not need to worry about forgetting to bring their bodies when they go somewhere outside the classroom, and no child needs extra time to search for its body parts somewhere in a cupboard or a bag. The two videos can be used as teaching material in mathematics.as introduction to Sámi traditional measuring. Such a use of the videos as teaching material, might be in line with Dewey's (1933/1998) and Freudenthal's (1973) recommendations that children should use their own bodies as a tool for intellectual development. The videos and this paper may open for new research projects about children in lower grades' work with Sámi traditional measuring as well as about measuring as a bridge between geometry and arithmetic. Measuring lasso length, band length, snow depth and other distances by Sámi traditional measuring is part of the Sámi intangible cultural heritage. A research project about Sámi traditional measuring in primary school may contribute to implementation of UNESCO's (2003) intangible cultural heritage convention. The convention points at the need to build greater awareness, especially among the younger generations, of the importance of the intangible cultural heritage and of its safeguarding.

We expect the preservice teachers' videos to contribute to existing supporting material for primary school mathematics teachers, because Sámi traditional measuring of length represents an individual way of measuring that will be new to many teachers. The third author is from Ethiopia, from where he is familiar with traditional ways of measuring that are similar for instance to *goartilastit* and *gardnjil mihtu* (elbow measuring). This opens for future comparative research projects.

Acknowledgment

Thanks to the four other preservice teachers who contributed to the discussions during the lessons.

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Mathematics in physics classrooms: a case study on mathematical reasoning in relation to the notion of specific heat capacity

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This study is part of a larger study of mathematics in physics teaching at upper secondary school. In this paper we focus on the use of mathematics in relation to the teaching of specific heat capacity of water. We report from two physics lessons with first-year students (lecture, problem solving, and labwork). One observation is the frequent technical use of mathematics during the lessons, which is consistent with the teachers view on the role of mathematics in physics teaching. Another observation is how mathematical notions are misrepresented when theoretical frameworks become dominant in the interpretation of data, e.g. notions as "points on a line", "the equation of a line" or "function" with implied consequences for the students' understanding of mathematics and physics. One conclusion is that more emphasis should be made on a structural use of mathematics, i.e., the meaning of concepts and models, in physics teaching.

Keywords: Mathematical applications, Mathematical concepts, Upper secondary school, Physics education, Structural use of mathematics.

Introduction and framework

Because of the significance of mathematics both in the development and application of physical models and theories, mathematics has a vital role in physics teaching and learning. Krey (2014) reports on the importance of students' conceptions of the role of mathematics in physics studies. He also advocates for more in-depth studies and further research in the area. Several studies point to students' problems in transferring mathematical knowledge to new and applied situations (Michelsen, 2006; Kaiser & Sriraman, 2006; Uhden et al., 2012). Moreover, in relation to teaching of physics, students' skills in mathematics are widely discussed (cf. Uhden et al., 2012). It has also been reported that even though students themselves do not view mathematics as a big problem in physics class, the teachers view weak mathematics skills as a problem (Angell et al., 2004). Despite this, research focusing directly on the role of mathematics in physics classrooms is scarce. The purpose of this article is to further explore the role of mathematics for physics education in upper secondary school.

A semantic view on theoretical models (Adúriz-Bravo, 2012) was used in this study, where theoretical models are viewed to form families or classes linking theories with experiments and practices, and where the focus is on the explanatory powers of the theoretical models. From this perspective, the focus of physics teaching would be shifted towards creating meaning making and discussion about the relationship between the theoretical models and the real world, in which mathematics plays an important role.

This study adds to the line of research on the use of mathematics in physics classrooms. We have developed an analytical framework to analyse the relations made between the three entities Reality, Theoretical models and Mathematics, during classroom communication (for a more detailed account of the framework see Hansson, Hansson, Juter & Redfors, 2015). Reality refers to objects or phenomena (or observations of them) in the real world. Theoretical models refers to theoretical models in Physics and concepts related to them. The models could be mathematically or qualitatively formulated. Mathematics refers to mathematical concepts, theorems, representations, mathematical reasoning and methods. The aim is to apply the developed framework in the analysis of physics teaching to identify different foci in the classroom communication during different instances of a lesson or in different kinds of instructional situations. In addition, we aim to investigate the teacher's view on the role of mathematics during the lessons taught.

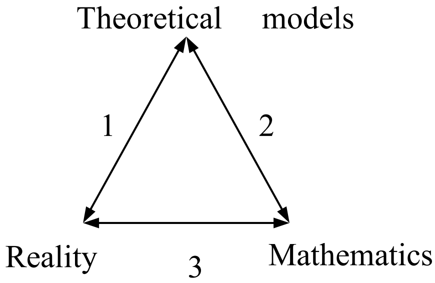

Figure 1. The links between Reality – Theoretical models – Mathematics

In Figure 1 the relations between the three entities Reality, Theoretical models and Mathematics are represented by the triangle's three sides in the form of bidirectional links 1, 2 and 3. The first type of link (1, in Figure 1) represents relations made between Reality and Theoretical models. We know from previous research that such relations are important in physics instruction (e.g. Lederman, 2007).

The second type of link (2, in Figure 1) represents relations made between Theoretical models and Mathematics. In the classroom communication we look for when a theoretical model is described in mathematical terms, or when a problem is transferred from a physics problem to a mathematical problem (e.g. manipulation of formulae, solving equations or constructing graphs). This link can be made in structural or technical ways (Karam, 2014; Uhden et al., 2012). Structural in relation to link 2 means that mathematics is used to support reasoning in relation to a theoretical model, while a technical use of mathematics is characterized by manipulations of formulae without discussing theoretical meaning, or when searching for the correct formula using a "plug and chug" approach to problem solving. Related dichotomies, or dualities, of technical and structural use of mathematics are instrumental and relational understanding (Skemp, 1976) or conceptual and procedural knowledge (Hiebert & Lefevre, 1986) in mathematics education. The third type of link (3, in Figure 1) depicts relations made between Reality and Mathematics. This could for instance happen when observations are discussed in mathematical terms (without contextualisation of physics concepts) when referring to experiences, e.g. it hurts more and more in the ears when diving deeper and deeper. Other examples can be various quantifications during labwork, e.g. measurements of angles, time or distances, or when a real-world phenomenon is related to a mathematical object, e.g. the slope of a hill is related to a right-angled triangle.

Method and procedure

Observations of one mathematics and physics teacher were conducted during two physics lessons. The intention was to follow the normal practice of physics instruction, as defined by Duit et al. (2007), at upper secondary school. The students studied at the science program in an ordinary upper secondary school in Sweden and the teacher was teaching the students in both physics and mathematics. The study took place at the end of their first year of the three-year program. The first lesson (80 min, 13 students) consisted of a physics lecture about specific heat capacity with focus on water in different states, solid-liquid-gas, followed by a problem-solving session. The second lesson (80 min 14 students) consisted of labwork about the specific heat of liquid water. The labwork started with a brief introduction by the teacher, followed by work in groups of 2-3 students.

The teaching was video recorded. One camera focused on the teacher and the whiteboard, and other cameras focused on the students. During problem solving sessions and lab sessions we video recorded selected student groups. The teacher was interviewed before and after both lessons about the teacher's intent with the lesson and view of the outcome.

The communication during lectures and student-centred work was analysed using video recordings. The data was analysed from a perspective where we deductively identified relations between Reality, Theoretical models, and Mathematics communicated by teachers and students. A multi-step analysis process was used: watching a video sequence in its entirety, identifying major events within the sequence, transcribing the interactions (words and actions) and identifying the links made in the communication. For example, in Table 1 (Ex 1.3) links of all types (1, 2 and 3) are formed: e.g. link 1 related to water (R) and energy (TM), link 2 related to added energy (TM) and the interpretation of the diagram (M), and link 3 related to the experience of boiling water (R) with temperature of 100 degrees (M). Link 3 does not imply that observations are not theory-laden per se, only that the theoretical model is implicit. The first lesson was teacher driven and described with a detailed account of the distribution of links in Table 1. The second lesson consisted of labwork where the teacher continuously alternated between five lab groups with a repeated dialog when visiting the groups, in this case we have selected sequences that illustrate the teacher's reasoning with the students in a narrative form.

Results

Lecture

The lesson began with the teacher summarizing content from previous lectures related to thermal physics. The teacher then introduced the theme of the lecture as follows:

> Today we are going to introduce what is called specific heat capacity. We are going to look at how water behaves in different phases, in liquid state, gas state and solid state. And what happens between those [states].

The teacher then projected a diagram, see Figure 2, on the whiteboard and asked the students to discuss what information they could read from the diagram.

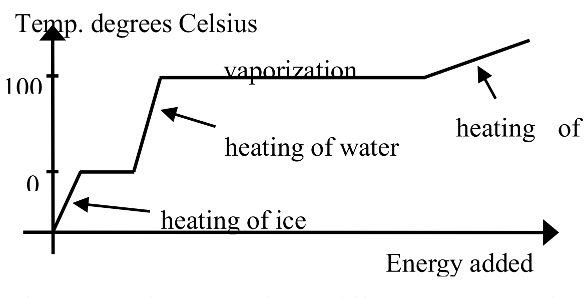

Figure 2. Diagram of the different phases of water illustrated by the teacher

After the teacher had listened to the students' discussions, the teacher held a joint discussion and asked the students to explain their conclusions. The teacher told the students that they, in future labwork, would study the slope of the line segment from 0 to 100 degrees in the diagram, which is called the specific heat capacity of water. Based on the diagram, the teacher introduced three equations. The first equation was $Q = m \cdot c \cdot \Delta T$ with thermal energy (Q), mass (m), temperature difference (ΔT), and specific heat capacity (c). The second equation was $Q = m \cdot I_s$, the energy required to change a substance state from a solid to a liquid. The third equation was $Q = m \cdot I_å$, energy required to change a substance state from a liquid into a gas. The last two equations contain the constants enthalpy of fusion (I_s) and enthalpy of vaporization ($I_å$).

During the rest of the lesson, the teacher gave the students two problems to work with as described below. Table 1 illustrates the implementation of the lesson (75 minutes long) with a detailed account of the distribution of links of type 1, 2 and 3.

Time	Activity	Link 1, R–TM	Link 2, TM–M	Link 3, M–R
0-5	Ex 1.1 The teacher summarizes previous contents.	The teacher comments on pressure and heat energy.		
5-7	Ex 1.2 The teacher projects a diagram (Fig. 2) on the whiteboard.	The teacher points out different phases of water with remarks on use in everyday life.	The teacher comments on the "special look" of the diagram. The grading of the axes is not important.	
7-22	Ex 1.3 The teacher asks the students questions about the diagram after the students have discussed the diagram with their peers in groups.	The teacher points out: During evaporation the molecules' bonds must be broken. Water has a tetrahedral shape at lower temperatures. The diagram's appearance is similar regardless amount of water, while energy will vary.	The teacher points out: The horizontal line segment evaporation stays at 100 degrees. It takes an equal amount of energy to melt ice as to heat water. The slope of the lines means different amounts of energy.	The teacher discusses the temperature of the water in a saucepan on a stove and the heat of the steam from boiling water.
22-26	Ex 1.4 The teacher sums up the content of the discussion	The teacher mentions that other substances, such as gold, have charts similar to water.	The teacher sums up the diagram, and inform on labwork regarding the slope of the line for liquid water, i.e. the specific heat capacity.	The teacher says different temperatures melt gold and water.
26-42	Ex 1.5 The teacher formulates the equation $Q = m \cdot c \cdot \Delta T$. Moreover, $Q = m \cdot I_s$ and $Q = m \cdot I_å$ for the melting and vaporization process, respectively.	The students look up heat capacity for various substances in their formula book.	The teacher reason on: Thermal energy (Q) depends on mass (m), specific heat capacity (c) and temperature diff. (ΔT). Energy (Q) for melting depends on, mass and (m) and enthalpy of fusion (I_s). Energy (Q) for vaporization depends on mass (m) and enthalpy of vaporization ($I_å$).	The teacher discusses with students what that can affect a pan of boiling water on a hotplate.
42-55	Ex 1.6 The teacher and students work with problem nr 1.		The problem is a direct application of formula $Q = m \cdot c \cdot \Delta T$.	
55-75	Ex 1.7 Work with problem nr 2, see section below.		The problem includes phase transition with application of $Q = m \cdot c \cdot \Delta T$ and $Q = m \cdot I_s$.	

Table 1. Relations made between Reality (R), Theoretical Models (TM) and Mathematics (M) during the lesson about specific heat capacity

To summarize, the lecture (see Table 1) is to a large part based on reasoning related to the graphical representation of the different phases of water (see Figure 2) with focus on theoretical models. Links of type 1 and 2 is frequent. Technical calculations, as described in link 2, dominated the solving of standard problems during the lesson.

Problem solving session

During the problem-solving session (30 minutes long) the students were assigned to work with two standard problems formulated by the teacher. The first problem was: *How much energy is required* *to heat 3.5 litres of water from 14 to 100 degree Celsius*. The students were asked to discuss the problem with each other and try to solve it. The teacher then discussed the solution to the problem with the students and they solved it jointly on the whiteboard, using the formula $Q = m \cdot c \cdot \Delta T = 3.5 \cdot 4.18 \cdot 86 = 1.3$ MJ. Where $c = 4.18$ kJ/kg K and the teacher pointed out that the temperature difference, ΔT, is equal for Kelvin and Celsius.

The second problem was: *How much energy is needed to heat 150 g of ice at -10 degree to 20 degree Celsius (water)*. The students were again asked to discuss the problem with each other and try to solve it. The teacher then summarized, and they solved the problem jointly on the whiteboard. This problem was related to phase transitions and the teacher first drew a diagram, similar to Figure 2, and calculated the total energy in three steps; heating of ice to 0 degrees (Q_1), melting the ice at 0 degrees to water (Q_2), and heating of water to 20 degrees (Q_3). Using the formula $Q = m \cdot c \cdot \Delta T$ for Q_1 and Q_3 and enthalpy of fusion $Q = m \cdot I_s$ for Q_2. Thus resulting in the total energy of $Q = Q_1 + Q_2 + Q_3 = 0.15 \cdot 2.2 \cdot 10 + 0.15 \cdot 3.34 + 0.15 \cdot 4.13 \cdot 13 = 3.3 + 50.1 + 13 \approx 66$ kJ.

The problem-solving session was focused on which formula to choose, and formula manipulation. The teacher's communication involved the theoretical model and the mathematics required to solve problems related to the theoretical model (link 2, technical). The communication rarely supported thinking or discussion about phenomena in reality.

Labwork

The labwork took place one week after the lecture. The students' task was to measure the specific heat capacity of water. The lesson started with the teacher giving an introduction to the labwork of a practical nature, that is, how the students were to handle the equipment and what to consider to obtain good results. The equipment consisted of a thermos bottle, a digital thermometer, an immersion heater, a timer, and a digital scale. After the introduction, the teacher handed out written lab instructions. The given task was as follows "*Your task is to determine the specific heat capacity of water, i.e. how much energy is needed to raise the temperature one degree in one kg of water.*"

The students (usually in triads) weighted the water (m) in their thermos bottles, heated the water with the immersion heaters (300W) and measured the temperature change (ΔT) in intervals of 30 seconds (link 3). They then calculated the added energy $Q = 300 \cdot t$, for $t = 0, 30, 60, 90, \ldots$ (to about 600 seconds) and got a sequence of pairs of values (ΔT, Q) to perform a linear regression on (link 2, technical). They knew from the lecture that $Q = m \cdot c \cdot \Delta T$, i.e. Q is proportional to ΔT, so from linear regression, say $Q \approx k \cdot \Delta T$, we get $m \cdot c \approx k$ and thus the requested value $c \approx k/m$. However, the students struggled to perform this procedure during labwork as shown below.

The students were instructed to use their graphical calculators (usually TI-82) to "draw your measured points" and "draw added energy [Q] as a function of the temperature difference ΔT". The formulation "draw … as a function of …" was frequently repeated by the teacher. The students discussed what variables should be represented by x and y on their calculators. The teacher's use of the concept 'function' in describing the task of plotting measured values (ΔT, Q) was in conflict with the students' conceptions of 'function' (as a formula). It caused students to try to find graphical representations of formulas on their calculators. Some students drew the graph of $Q = P \cdot t$, i.e. $y = 300 \cdot x$ before they asked the teacher for advice. This led the teacher to emphasize the function concept as a dependency relation between two variables. The teacher's intended procedure of adding measured values into lists on the calculator and plotting the values, and then perform linear regression seemed not to be a natural process for the students. During the lab, the teacher frequently told the students that their measured "points" (ΔT, Q) "lie on a straight line" and that they should calculate "the line's equation" with linear regression (link 2, technical). Apparently, the teacher had the theoretical model in mind, but the students did not question this description of their measured values or why they would use linear regression if the points were really on a straight line.

According to the teacher, the students had previously performed linear regression on their calculators, but it was a while ago and many students had forgotten how to do it. The teacher at one point gathered students from two groups and explained how to perform linear regression after entering values for Q and ΔT into lists on their calculators:

Teacher: … what we want now, okay, these points form a straight line. What we are interested in now, okay, what is the equation of this straight line? We do not need to calculate this by hand, but we have drawn it on the calculator for the calculator to calculate it. So, then we want to do what is called a regression, that is an adjustment to the curve, we want to do what is called a linear regression. Do you remember how to do this? It's been a while since we did this ...

In this case, the students focused on the teacher's handling of the calculator, how to do the calculations (link 2, technical). They did not ask any questions or comment on the teacher's description of linear regression.

To one group of students the teacher said that what they really are studying is a linear function, referring to the theoretical relation $Q = m \cdot c \cdot \Delta T$. Moreover, when the students made a plot of their sequences of measured values (ΔT, Q), the teacher often said that they drew "the function" – which is correct, if you look at the sequence of points as a function – but the teacher seemed to refer to the graph of the linear function $Q = m \cdot c \cdot \Delta T$. This is another example (link 2, technical) of the teacher's tendency to interpret data values based on a theoretical model and use mathematical concepts from the model in situations that do not correspond to the concepts.

Response from the teacher

Shortly after the two lessons the teacher filled out a questionnaire. One question was: *How would you describe the role of mathematics in the lessons taught?* The teacher answered:

> In the teaching you now have followed, mathematics plays a major role, both in terms of calculations, but mainly also the graphical mathematical connection to the understanding of physics. In many parts of physics, mathematics comes in as a natural tool, but mainly in the form of procedures/calculations. I like this section because you get the graphical connection more clearly than in many other sections and that the mathematical interpretation of graphs actually says a lot about the physical processes that take place.

Discussion and conclusions

The results show that the bulk of the discussion in the lecture is concerning the relation between theoretical models and mathematics. It is also shown that when such relations are made, the emphasis is often on technical use of mathematics. Links of type 2 that emphasise structural use are not frequent, in spite of the diagram (Figure 2) of phase transition that opens up for a structural use of mathematics. In the labwork situations the use of mathematics was predominantly technical and the main focus was on collecting data through measuring, followed by linear regression. The consequence of a focus on technical use of mathematics (Karam 2014) in the classroom communication, is that it can prevent opportunities to engage in deeper discussions about the meaning of concepts and theoretical models and their relations to real world situations and problems. Hence, not contribute to a semantic view on scientific models (Adúriz-Bravo, 2012).

The teacher's description of the role of mathematics after the lessons, i.e. "mathematics comes in as a natural tool, but mainly in form of procedures/calculations", is consistent with a frequent technical use of mathematics. Moreover, the view that "the mathematical interpretation of graphs actually says a lot about the physical processes that take place" is consistent with the introduction of the lecture with reasoning based on the diagram (Figure 2) of phase transitions. The graphical interpretation is strongly related to a theoretical view on the content (link 2). This view becomes even more prominent during the labwork where measured values were linked to the theoretical model (link 1), and the teacher's reasoning and dialog with the students were focused on the theoretical model of specific heat capacity of water. The model view that was applied to reality (link 1) affected the teacher's interpretation of data which also meant that mathematical concepts was not used properly, e.g. the notion of points on a line or the concept of function, with unsuccessful attempts to establish structural type 2 links to mathematics in the teacher's communication with the students. This does not support students' mathematical reasoning as described by Uhden et al. (2012) who advocates a shift from technical to structural mathematical skills while teaching physics.

One conclusion from the presented results is that more emphasis should be made on a structural use of mathematics, i.e., the meaning of concepts and relations to theoretical models. This would also mean an increased emphasis on the relation between theoretical models/concepts and reality, e.g. on how theoretical models could be used to describe and predict real world phenomena and events. This connects to the idea, from both science and mathematics education research, of increasing attention to mathematical modelling in physics teaching (cf. Kaiser & Sriraman, 2006; Michelsen, 2006; Uhden et al. 2012) – a way to provide meaningful ways for the students to make sense of complex situations.

Project funded by The Swedish Research Council 2015-01643

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Mathematics teaching in the Danish kindergarten class – an empirical study based on the tool CLASS

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In some Nordic countries, the first year of schooling has a unique feature that forms the transition from kindergarten to elementary school. The Danish kindergarten class has recently become mandatory, and for the 5 to 7 age group, a national curriculum for mathematics has been designed. This study is the first Danish study to investigate the taught curriculum in actual teaching performance. Therefore, the present study aims to provide a broad insight into the quality of classroom interactions between the teacher and the students during mathematics lessons. Methodically quantitative observations based on the tool Classroom Assessment Scoring System (CLASS) from two classrooms are used. The method is supported by educational theory, which compasses classroom interactions at this age level. The discussion focuses on aspects of reliability and validity connected with the chosen method.

Keywords: Kindergarten class, national curriculum, mathematics teaching, quantitative observations, classroom interactions.

Background information

In the Nordic counties such as Finland, Sweden, Faroe Islands (in a few villages), and Denmark, the transition from informal to formal learning is carried out by a year of attendance in a preschool institution. The target group is 5-7-year-old students. In 2014, the Danish Ministry of Education set standards divided into different areas of competence. The area of competence for mathematics covers four key skills: numbers, quantity, shapes and patterns, mathematical language, and thinking (Børne- og Undervisningsministeriet, 2014). A kindergarten class teacher is given 3½ years of training, which leads to the degree Bachelor of Social Education. Because the new curriculum for the kindergarten class was introduced recently, many professionals who teach in the kindergarten class have not learned about early mathematical education in their pre-service education. Up until now, there have been no Danish empirical findings on the taught curriculum in actual teaching performance. Therefore, the present study aims to give some broad insight into the different quality aspects of classroom interactions between the kindergarten class teacher and the students during mathematics lessons. Against this background, the research question is formulated thus: *Which challenges and opportunities are observed in classroom interactions when the kindergarten class teacher is teaching mathematics?*

Theoretical framework

Building on systems theory and empirical research in social development in classroom environments, Pianta (1999) investigates the role of student-adult relationships in developing social and academic competencies. Hamre and Pianta (2007) suggest that interactions between students and adults are the primary mechanism for students' development and learning. They consider students' interactions with their teacher and the classroom environment crucial for academic success. Based on theory and empirical research, Pianta and Hamre (2008) present a conceptual framework for classroom interactions suggesting three core domains establishing levels for the teaching quality in interactions between students and adults: emotional support, classroom organisation, and instructional support. Emotional support refers to ways in which teachers help students develop warm and supportive relationships, experience enjoyment and excitement about learning, feel comfortable in the classroom, and reach appropriate levels of autonomy. The other core domain*,* classroom organisation, refers to the organisation and management of students' behaviour, instruction time and routines, and learning formats. The third core domain*,* instructional support, is associated with students' cognitive development concerning higher-order thinking, process-oriented feedback, and language development (Pianta et al., 2008).

In Denmark, three essential competencies of teaching quality are used as a conceptual framework for teaching performance. These are competencies in relationships, classroom management, and didactics (Nordenbo et al., 2008) and, therefore, very similar to the three domains suggested by Pianta and colleagues. These domains/themes are exemplified in the following, drawing from the Encyclopedia of Mathematics Education and other studies in mathematics education research. The theme of emotional support deals with the subject of affect and emotions, such as studies examining the relationship between the teacher and student enjoyment and the relation between social norms and emotions (Hannula, 2020). The theme of classroom organisation is about didactic classroom management in mathematics and focuses on the connection between the organisation of the teaching and the learning objectives (Blomhøj & Højgaard Jensen, 2011). The theme instructional support covers three sub-themes. The sub-theme concept development constitutes a large research field in mathematics education. In the Encyclopedia of Mathematics Education, Vinner (2020, p. 123) deals with mathematical concept formation and development and suggests "not to isolate mathematical concept formation and development from concept formation and development in general." The sub-theme quality of feedback deals with the connection between students' experienced self-efficacy and achievements in mathematics (Andresen, 2017). The sub-theme language modelling deals with language and mathematics education from multiple perspectives, e.g., teaching and learning, resources and challenges (Moschkovich, 2010). In my view, the above examples justify that the work of Pianta and Hamre is in line with core works in mathematics education research. In addition, Hamre, Pianta, and colleagues provide a systematic tool, the Classroom Assessment Scoring System, for investigating classrooms based on extensive literature review and scales validated in large-scale classroom observation studies (Hamre et al., 2007). I have chosen to use their conceptual network and systematic tool in my research.

Introduction to the *Classroom assessment scoring system*

The Classroom Assessment Scoring System (CLASS) is a general quantitative observation tool developed to assess classroom quality, focussing on interactions between teachers and students. In this text, from now on, the abbreviation CLASS is used. Originally, CLASS was developed for research purposes but is now used in educational policy context as an observation instrument in the classrooms in over 40 American states to measure teacher effectiveness (Teachstone, 2013). Internationally, CLASS is increasingly used for research purposes, e.g., in Germany (Greve et al., 2020). The CLASS design offers a systematic observation and supports a common metric and vocabulary to describe quality as different aspects of the interaction.

The conceptual structure underpinning CLASS divides interactions between teachers and students into domains, and quality measurement takes place in each of these domains. The domains focus on their respective aspects of interactions between teachers and students (Pianta et al., 2008). The aspects are organised into three domains with a total of 10 dimensions. The three domains are described as emotional support, classroom organisation, and instructional support. Each domain is specified in 3-4 dimensions. There are four dimensions of emotional support: positive climate, negative climate, teacher sensitivity, and regard for student perspectives. There are three dimensions within the domain of classroom organisation: behaviour management, productivity, and instructional learning formats. The dimension of instructional support is also specified in three dimensions: concept development, quality of feedback, and language modelling. All CLASS dimensions are intended to be subject-independent. For each of the ten dimensions, indicators have been developed and through which the presence of observations must be identified (for further definition of indicators, see Pianta et al., 2008, p. 111). Most of the indicators focus on the teacher's actions. The manuals used in CLASS observations are age-related and cover the infant age group through secondary school.

Metrically CLASS is based on a scoring system using a 7-point scale, which considers both frequency and quality of teacher-student interactions: low range 1, 2; middle range 3, 4, 5; high range 6, 7. A set of scores results from the observed quality on each dimension during each observation sequence (20 min). These scores are averaged across cycles and consolidated to create domain and dimension scores. The following examples illustrate some characteristics of assigning low, middle, and high ratings in the dimension of concept development. A classroom scoring in the 1-2 range may provide students only with purely rote instruction, e.g., doing worksheets and focussing on "getting the right answer." Classrooms in the 3-5 range have occasional evidence of instructional interactions and activities that foster students' thinking and understanding, such as the teacher asking ”why” and ”how” questions and calling students' attention to broader concepts rather than only focussing on isolated facts. However, these interactions are only observed now and then. In classrooms, in the 6-7 range, teachers build on initial interactions to foster students' understanding, connections, and integration of learning (Pianta et al., 2008).

Method

The participants are two kindergarten class teachers (teacher A and teacher B). I consider two classrooms, each with its own teacher, to be sufficient to obtain an impression of teaching mathematics in the kindergarten class. The two teachers were selected because they are representative of the group of Danish kindergarten class teachers concerning: education (both teachers have the educational background, Bachelor in Social Education and have not learned about early mathematical education in their pre-service education); gender (both teachers are females as are 95% of the Danish kindergarten class teachers (Danmarks Lærerforening, 2021)); age group (both teachers belong to the age group 41-60 as are 74% of the Danish kindergarten class teachers (Danmarks Lærerforening, 2021)). In addition, both kindergarten class teachers are experienced and have worked in the same school for over ten years. About 20 students (5-7-years old) attended each kindergarten class. I did not choose the topics to be taught. According to the teacher's plan, the observations took place when both classes were taught numbers and quantity. The school is in a middle-class area.

In the USA, previous CLASS tests have used a large selection of teachers to obtain averages across the three domains to generalise findings to the entire population of teachers (Office of Head Start, 2018) and on that basis to be able to measure the progress of performance over time. In the present study, the analytic strategy is to obtain a snapshot of mathematics teaching and to analyse data that does not primarily focus on the exact numbers, but the trends in numerical values of the data material from both classrooms as the scores are divided into low range, middle range, and high range. In the procedure of observations, a CLASS manual version K-3 (kindergarten through third grade) was used (Pianta et al., 2008). Per cent-within-one (PWO) is used as an indicator of inter-rater reliability in the CLASS manual, and I have used the same reliability indices in this study. I am certified for the target group K-3 and the only certified Danish CLASS observer for this age group. The second rater is certified as a CLASS observer for the target group Upper Elementary (4-6 grades). The agreement indices of the domains for teacher A are 81,25 (emotional support); 66,67 (classroom organization); 75 (instructional support). The agreement indices of the domains for teacher B are 62,5 (emotional support); 75 (classroom organization); 66,67 (instructional support). The agreement indices of the dimensions for teacher A are 100 (negative climate, teacher sensitivity, regard for student perspectives, behaviour management, concept development); 75 (productivity); 25 (positive climate, instructional learning formats, quality of feedback). The agreement indices for teacher B are 100 (negative climate, behaviour management, productivity, language modelling); 75 (positive climate, teacher sensitivity, quality of feedback); 25 (instructional learning formats, concept development); 0 (regard for students' perspectives).

According to the recommendations in the manual, four sequences of 20 minutes were carried out in each class during mathematics instruction. The sequences were video-recorded by me, and the manual was used to code every video-recorded sequence from each teacher. At the end of each video-recorded sequence, I reviewed my notes and rated each CLASS dimension. The observations were carried out in spring 2019. First, an average score for each domain was calculated to assess differences in each class between the three different domains. Next, the ranking of the average score of the dimensions was examined and then presented to make the level of scores in the ten dimensions visually readable.

Results

The data on the average scores in domains and dimensions across each teacher's sequences are shown in tables 1 and 2.

	Teacher A	**Teacher B**
Domains	Average score	Average score
Emotional support	5.38	6.94
Classroom organisation	5.92	6.83
Instructional support	2.17	3.17

Table 1. Descriptive information across all sequences by the three CLASS domains

Table 1 illustrates that teacher A's average scores for all four sequences observed were: emotional support M=5.38; classroom organisation M=5.92; and instructional support M=2.17. Table 1 also illustrates that teacher B's average scores for all four sequences observed were: emotional support M=6.94; classroom organisation M=6.83; and instructional support M=3.17. These data indicate that both teachers scored well within the high range and the upper end of the middle range for emotional support and classroom organisation. However, both scored on the lower end of the middle range for instructional support. This indicates consistent, mean-level differences between scores on instructional support and scores in each of the other two domains. Following criteria for interpreting reliability data set up by Sandilos and DiPerna (2011), the three domain scores displayed marginally reliable to acceptable levels of per cent-within-one agreement. However, the average scores for the three domains are superior, and the dimensions can inform more detailed.

Teacher A			**Teacher B**		
Dimensions	Average score	Range	Dimensions	Average score	Range
Concept development	2	Low range	Concept development	1.75	Low range
Quality of feedback	2		Language modelling	3.5	Middle range
Regard for student perspectives	2.5	Low range/middle range	Quality of feedback	4.25	
Language modelling	2.5		Regard for student perspectives	6.75	High range
Instructional learning formats	5.25	Middle range/high range	Behaviour management	6.75	
Productivity	5.5		Productivity	6.75	
Positive climate	6	High range	Positive climate	7	
Teacher sensitivity	6		Teacher sensitivity	7	
Behaviour management	7		Instructional learning formats	7	
Negative climate	7		Negative climate	7	

Negative Climate scores are reverse and were recorded for the overall score of emotional support.

Table 2. Ranking of the average scores of the ten CLASS dimensions

Table 2 illustrates the three dimensions within the domain instructional support: concept development, quality of feedback, and language modelling appear for both teachers among the three lowest scores in a ranking of the average scores in all dimensions. Concept development is found among the two dimensions with the lowest average score of 2 for teacher A. Concept development is found to be the lowest average score of 1.75 for teacher B. Concerning these average scores: for teacher A the inter-rater agreement is acceptable; for teacher B the inter-rater agreement is unacceptable (Sandilos & DiPerna, 2011). In addition, table 2 illustrates 3 (out of 4) dimensions within the domain of emotional support: positive climate, negative climate (i.e., no negative climate is observed), and teacher sensitivity appear for both teachers among the three highest scores in a ranking of the average scores in all dimensions. The average scores (reversed) within the range of high, negative climate are found as the dimension with the highest average score for both teachers. Within the domain of classroom organisation, behaviour management is found for both teachers as a dimension within the high range. With regard to the average scores for the dimensions: negative climate and behaviour management, the inter-rater agreement is acceptable.

In summary, the results from the quantitative analyses based on CLASS from both teachers' classrooms show a pattern in which the average scores of the observations are in the upper-middle-range and high range for the domains of emotional support and classroom organisation. These scorings cover the high quality of students' social and emotional functioning in the classroom and the high quality of teachers' ability to provide clear behavioural expectations. The average scores of the observations are in the lower middle range for the domain of instructional support. The lowest average scores from observations in both teachers' classrooms are found in the dimension concept development, but the result is tentative. These scorings in the low range 1-2 cover observations of purely rote instructions.

Discussion

In my view, the observation instrument CLASS provides the opportunity to analyse and measure the quality of classroom interactions in detail and provides knowledge of domains and dimensions the teachers may develop or in which they show great strength in the quality of their teaching. Based on CLASS, the most significant challenges the two teachers in the present study face in classroom interactions when living up to new parts of curriculum intentions concerning mathematics are found within the dimension of concept development. At the same time, the theme of concept development constitutes a huge research field in mathematics education (Vinner, 2020). An average score in the low range of instructional support may be considered obvious since the teachers have not learned about early mathematics education in their pre-service education. In a Danish context, the results have, in any case, not previously been visible and are now documented from two classrooms. The most promising opportunities during mathematics teaching are found within the domain of emotional support. Different emotional aspects are acknowledged to substantially impact teaching and learning mathematics (Hannula, 2020).

When it comes to assessing classroom interactions in mathematics with a general tool, one must consider its validity. In contrast to the domains of emotional support or classroom organisation, cognitive activation/concept development within the domain of instructional support ”has to be defined for each academic subject based on specific findings from didactic research and cognitive psychology in that field” (Taut & Rakoczy, 2016, p. 47). An additional factor to instructional quality for mathematics instruction is suggested. The factor should describe ”the math-specific quality of the teaching-learning process in terms of relevant elements of understanding, forms of representation, and clear content structure” (Taut & Rakoczy, 2016, p. 48). My experience from the present study is in line with Taut & Rakoczy, e.g. the math-specific quality of teaching natural numbers and the related fundamental processes of counting have to be described.

The overall inter-rater agreement for the three domains in the present study can be described as relatively acceptable (Sandilos & DiPerna, 2011). I found that my results from CLASS observations are consistent with the aggregated scores in the three domains found in US studies, including 115 primary school teachers conducted with CLASS K-3 (Pianta et al., 2008). These US studies summarise the performance of the involved teacher in the domains of emotional support and classroom organisation in an upper mid-range and instructional support in a lower mid-range. The same pattern in results (covering different age groups) are also found internationally, for example, in Germany (Bihler et al., 2018) and the Netherlands (Slot et al., 2015).

Conclusion and implications

I have found that the observations seen through the lens of the tool CLASS have provided some broad insight into different aspects of classroom interactions and the challenges and opportunities involved when two kindergarten class teachers are teaching mathematics. This study calls for research on how instruction can be designed so that kindergarten class teachers can support students' concept development in mathematics teaching. In my PhD project, I will follow up on this study by a design-based study focussing on how kindergarten class teachers can scaffold dialogical teacher-students interactions concerning students' conceptual development in aspects of natural numbers.

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Teachers’ talk about attainment grouping in mathematics: the role of fixed ability beliefs

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This paper focuses on teachers’ beliefs about students’ learning in different attainment groups in the context of equal opportunities for learning. I report on interviews with four teachers, all teaching in different attainment groups in 9th grade at the same school. Applying dimensions 1 and 2 of Schoenfeld’s (2016) Teaching for Robust Understanding (TRU) framework (‘The mathematics’ and ‘Cognitive demand’), I focus on teachers’ beliefs about mathematics and mathematics learning, and how these relate to their experiences of attainment grouping and equal opportunities for learning. The analysis notes the role of fixed ability beliefs in teachers’ views of teaching and learning and its implications for their reported classroom practice.

Key words: Teacher beliefs, attainment grouping, fixed ability beliefs.

Introduction

The tradition of inclusive mainstream schooling, in which all children are to be given “equal opportunities for learning” (tilpasset opplæring, TPO), has deep roots in the Norwegian education system. There are different approaches to facilitating these equal opportunities for learning, and despite a long tradition of mainstream schooling and teaching in mixed classes, attainment grouping has become a common mode of achieving TPO, especially in mathematics (Vibe, 2012). One reason for this shift is an increased focus on student performance in mathematics based on OECD PISA results, leading to concern over Norway’s standing in world rankings (Kjærnsli & Olsen, 2013). Teaching in attainment groups has been suggested as a means of raising students’ marks. While the Education Act § 8-2 (Opplæringslova) states that “students shall not *normally* be organised according to level of ability, gender or ethnic affiliation” (my emphasis) (Opplæringslova, 1998), teaching students in attainment groups for limited periods is seen as acceptable practice. This paper focuses on teachers’ beliefs about students’ learning in attainment groups in the context of TPO. My research question is: *What mathematics do teachers aim to teach in attainment groups and how do they reflect on this?* I report on interviews with four teachers, all teaching in different attainment groups in 9^{th} grade at the same school; I argue that their views are heavily influenced by assumptions of fixed ability, particularly for lower attainers.

Background literature: attainment grouping

There is little research on the impact of attainment grouping in Norway, or its associated classroom level practices, but research from other countries shows that there are qualitative differences in teaching in different groups. Teaching in lower attainment groups tends towards more a traditional teacher-led approach where tasks are routine and repetitive, and questions are often closed and do not encourage students to reflect critically on their thinking. In high attaining groups, on the other hand, students may receive more “reform-orientated” teaching where opportunities for deep learning, open ended tasks and critical thinking are provided, although teaching can also be characterized by fast-paced work which focuses on algorithmic fluency (Beswick, 2017; Boaler, Wiliam & Brown, 2000; Francis et al., 2019; Solomon, 2007). Some research report that teaching organized in mixed groups is less restricted and more investigative than in attainment groups (Boaler et al., 2000; Francome & Hewitt, 2018).

In attainment grouping, students may therefore receive more restricted access to mathematics in terms of both pedagogy and content. This may relate to teachers' beliefs about students' abilities: in mixed attainment classes they are used to differentiating the work, while in grouped classes they may see students as a homogeneous group and so teach on one "level" (Taylor et al., 2017). Teachers' beliefs about students' learning in mathematics appear to be very important in determining how students experience learning. For example, Beswick (2017) asked teachers to describe "poor" and "good" students. Their descriptions of "good" students highlighted proficiencies in describing skills and knowledge, whereas "poor" students were seen as lacking proficiency, understanding and ability to explain. Teachers chose challenging and open-ended tasks for "good" students but more restrictive ones for "poor" students (Beswick, 2017). Relatedly, teachers may have lower expectations of students in lower attainment groups based on their beliefs that these students should not be over-challenged, leading to an "over supportive" pedagogy which limits development in the context of a nurturing approach (Francis et al., 2019; Mazenod et al., 2019). Teaching in attainment groups and the selection of different topics for different groups may communicate a labelling of students and a fixed ability view of both low and high attainers (Francis et al., 2017).

Theoretical framework: analysing mathematical content

I use Schoenfeld's (2016) TRU (Teaching for Robust Understanding) framework as my analytical framework in this paper. Although TRU is primarily designed for observing classroom teaching practice I use it here to focus on what teachers say about how they plan and organise their teaching and how they make curriculum and pedagogic choices in their teaching in attainment groups. TRU enables the observer to evaluate teaching along five different dimensions. In this paper, I concentrate on the two first dimensions, "the mathematics" and "cognitive demand", which concern the choice of curriculum content and the extent to which it provides students with opportunities for sense-making. In the TRU framework the "Mathematics" dimension is described as:

> The extent to which classroom activity structures provide opportunities for students to become knowledgeable, flexible, and resourceful disciplinary thinkers. Discussions are focused and coherent, providing opportunities to learn disciplinary ideas, techniques, and perspectives, make connections, and develop productive disciplinary habits of mind.

"Cognitive demand" is explained as:

> The extent to which students have opportunities to grapple with and make sense of important disciplinary ideas and their use. (…) The level of challenge should be conducive to what has been called "productive struggle. (Schoenfeld, 2016, p.4)

These two dimensions can be closely linked to teachers' beliefs about students' learning in mathematics and the related choices they make in attainment grouping. In this study I explore how teachers see the role of content with respect to equal opportunities for learning, and how they plan for learning in different attainment groups. Hence my Research Question is: What mathematics do teachers aim to teach in attainment groups and how do they reflect on this?

This study

This study involves observation of four different teachers, each teaching one of four different attainment groups in 9th grade at one school. Mathematics teaching is organized in whole class mixed groups and four different attainment level groups. Each student attends three mathematics lessons a week, two organized by attainment level, and one as a whole class mixed group. Each teacher is responsible for one attainment group and one whole class mixed group. This paper is based on semi-structured interviews with each teacher, focusing on their beliefs about, and experiences of, teaching and learning in whole class mixed and attainment groups. The interviews were audio recorded and transcribed in their entirety.

The teachers: background experience and roles

The school's practice of attainment grouping in mathematics was initiated by the teachers, as a means of addressing TPO. Over the last five years the school had organized teaching partially by attainment grouping, with some variations depending on economics and practical resources. In 9th grade, the students were placed in four attainment groups, where Group 1 was the lowest set and Group 4 the highest. I name the teachers according to which group they taught - Teacher 1 (T1) teaches Group 1 (G1) and so on. The teachers had agreed how they would distribute the teaching in the different groups. All four had their own ideas about which group they preferred to teach but all were also open to teaching in every group. T1 and T4 had permanently taught one particular attainment group for the last five years, coincident with their preferences.

Analytic approach

Preliminary analysis of the interview data involved coding according to the descriptions of Dimension 1 and 2 in the TRU framework. I looked for references to "mathematics" and "cognitive demand" in teachers' descriptions of their teaching practice and students' learning needs, and their views on appropriate curriculum content for different attainment groups. I found several references to what they considered appropriate content for different attainment groups and to appropriate cognitive demand and the students' needs in the context of the different groups. Having identified these references, I looked for emergent themes relating to teachers' beliefs about teaching and learning in attainment groups and their approach to mathematics teaching. In the course of the analysis, teachers' beliefs about student ability emerged as a repeated viewpoint, leading to the introduction of an additional coding category.

Analysis

In this analysis I first focus on teachers' accounts of teaching and learning in attainment groups, before moving to consider their views in terms of "mathematics" and "cognitive demand".

Teachers' accounts of attainment grouping

All four teachers argued for teaching organised by attainment grouping and agreed that it was a good way to organise for TPO, arguing that it was difficult to facilitate TPO in mixed groups due to attainment differences. They drew on both teachers' and students' perspectives. T3's and T4's support for attainment grouping was primarily based on TPO, arguing that students learn better and are more comfortable and able to participate in classroom activities:

a lot goes over their heads in mixed groups …in a mixed group it's a completely different kind of lesson…the gap is too big. … there is no gap at all in the attainment groups… students said they felt more comfortable in the group (T3)

Attainment groups across classes in 1-4 have been the best. … everyone gets something on their level. …in attainment groups students maybe feel a bit more comfortable. It's ok to be good…here all are good. … and then the low attainers also dare to raise their hands (T4)

T3 and T4 also emphasised the teacher's perspective. T3 felt that facilitating TPO in mixed groups was challenging because of the need for more individual work and differentiated tasks. Attainment groups meant that the teacher could 'get through' to students more readily:

. … instead of adjusting what is said in class into different types ... levels all the time, or tasks or pages … You get, ... you reach more of the students ... a little faster (T3)

For T4, attainment grouping made it easier to pitch lessons and manage workload:

[It's] easier for the teacher to adapt to their group. … Everyone is almost on the same level. In mixed groups we were frustrated by how much we had to differentiate [and we] … planned a lot for each of the classes … [We] wanted to try out attainment groups to make it easier for us teachers… have some more, control of the lesson. (T4)

Although T1 thought that exploration in mixed groups could be productive, he highlighted the benefits of attainment groups for working at a different pace and with different content:

there are some clear benefits for those students who want to… move forward faster. Even …those who are struggling ...finally they feel seen (T1).

T2 argued for a combination of both attainment grouping and whole class mixed groups, drawing on her experiences with lower attainers in G2. Teaching could be slow going in G2, whereas her mixed group included more motivated students who could drive the activities more; problem solving was more difficult in G2 compared to her mixed group. However, while there were no major gaps between students in attainment groups, these presented a challenge in mixed groups, and G2 students could fall back in the mixed group, so attainment groups were useful:

… it's ok with some variation, having both attainment groups and mixed groups …in attainment groups you can differentiate to the level more easily…it can go a little slow going on group 2...it's a lot of repetition. .... It's more exciting in the whole class mixed group with different levels…. But then maybe the group 2 students drop off. (T2)

Overall, the teachers' views were influenced by concerns about what was most efficient in the context of TPO; for them, attainment grouping addressed students' needs. In the following two sections I explore their view on mathematics and cognitive demand in these attainment groups.

Different approaches to "the mathematics"

The teachers' beliefs about "the mathematics" were strongly connected to their teaching approaches, and these differed considerably, ranging from procedural to conceptual approaches.

T1 taught the low attainers in class G1. He said that he focused on conceptual knowledge and understanding. He aimed to present ideas in at least three different ways in order to facilitate learning.

He emphasized the importance of students' participation in group discussions and explanations "to engage the students to learn from each other and to learn together". T1's support for attainment groups was closely linked to his experience of G1 students, who he described as less motivated and harder to engage. He felt that attitude and motivation were important, and that it was the teacher's responsibility to establish and building classroom practices; a major focus for him in G1 was developing motivation and a sense of mastery.

T1 took a conceptual approach to teaching and learning in his emphasis on teaching mathematics in a variety of ways and the importance of students' participation in discussions, described in TRU as "providing opportunities to learn disciplinary ideas, techniques, and perspectives, make connections" (Schoenfeld, 2016, p.2). In contrast to T1, T4 favoured a procedural approach to teaching, focusing on rules, algorithms and correct answers. She argued that, unlike G1 students, G4 students needed a more formal approach. Although T4 also talked about the importance of "in depth learning" and "getting the students to explain their strategies", she preferred procedures and was uncomfortable with teaching problem solving and exploratory activities. Applying the TRU framework to T4's procedural approach to "the mathematics", there was a lack of "opportunities for students to become…flexible, … [and] make connections" (Schoenfeld, 2016, p.2).

Like T1, T2's general approach to teaching and learning highlights the development of conceptual knowledge, use of classroom talk and investigation activities. In the TRU framework, this is described as "opportunities to build a coherent view of mathematics" (Schoenfeld, 2016, p.2). It is interesting that T2 referred to this approach to teaching in the context of teaching in mixed groups. When she described her teaching in attainment groups, T2 conveyed a procedural approach with a focus on explaining rules and student practice. Hence, T2's approaches to mathematics changed in relation to the organisation of teaching. T3 describes mathematics learning as a kind of language learning, developing in stages, and he highlights the importance of conceptual learning and mathematical discussion. He says he aims to teach in a way that motivates students to discuss and work together, in what he describes as "flow". He also emphasises the importance of connections between procedures, concepts and contexts. According to "the mathematics" in TRU, this can be seen in "Discussions (…) providing opportunities to learn disciplinary ideas, techniques, and perspectives, make connections, and develop productive disciplinary habits of mind" (Schoenfeld, 2016, p.2).

Cognitive demand and grouping

Analysis of the four teachers' accounts of the kind of teaching and content that they felt was appropriate for different groups showed some strong views on the nature of cognitive demand in the context of different attainment groups, and corresponding views on students' ability. T3 argued for teaching in attainment groups as the best way to facilitate students' learning in mathematics, because of the possibility for the teacher to teach, and the students to practice, at the same level. T3 is clear that students' learning is best facilitated in homogeneous groups based on their different levels:

> a lot goes over their heads in mixed groups …in a mixed group it's a completely different kind of lesson…the gap is too big. … there is no gap at all in the attainment groups… (T3)

Furthermore, T3's view of students' different needs based on their levels is also evident when he says that the G1 students are "in need of another type of mathematics" and how the students in G3 "are nearly on the G4 level, but they miss something". According to "cognitive demand" as described in

TRU, T3 appears to “provide room and support for growth, with task difficulty ranging from moderate to demanding”, but his view of “productive struggle” is restricted to its occurrence within different attainment groups (Schoenfeld, 2016, p.4).

Like T3, T4 also described the students in the context of the different groups. In attainment groups, “everyone is almost at the same level” and in one group “they all have the same perception of things, … and everyone actually thinks the same way”. Describing the G4 students she says: “they maybe need more theoretical stuff, because this is how their brains are created. They don’t need illustrations just as much … and they understand more quickly”. It is interesting how T4 includes herself in this kind of description, explaining why she is a good teacher for G4: “…I recognise me in those students … so it’s based on how I think”. Like T3, T4 matches cognitive demand to groups, but she also restricts cognitive demand according to what she considers appropriate pedagogic approaches. From the perspective of TRU, T4 limits students’ “room and support for growth” and “productive struggle” (Schoenfeld, 2016, p.4).

T1 also relates cognitive demand to attainment groups. This is evident in his view of G4 students but also in his description of appropriate content for the G1 students: “Group 4 is the most self-driven group…they are very self-propelled and … and there you go through the content at a much higher pace. In group 1 we can dwell more…we practice mostly with basic numeracy”. In these arguments T1 appears to be suggesting that the G1 students only need a low level of challenge and a limited of the content. T1 restricts what is the appropriate “level of challenge” and “productive struggle” for the G1 students (Schoenfeld, 2016, p.4). Interestingly, there seems to be a mismatch between what T1 says about the two TRU dimensions “mathematics” and “cognitive demand”. While he advocates limited content and arguably more restricted access to the mathematics for G1 students, this is different from what he says about students’ motivation and the importance of a feeling of mastery as crucial for learning.

As we have seen above, T2 advocates retaining mixed groups alongside attainment grouping. She argues for the role of whole class mixed group teaching where students with different abilities can learn from each other. She also highlights motivation, and the need to see mathematics relevant as an important basis for learning. T2 sees potential for G2 students to meet a greater cognitive demand in a mixed group where they are helped by high attainers. This view is less limiting than the others teachers’ views, fitting the TRU stance that: “Students learn best when they are challenged in ways that provide room and support for growth, with task difficulty ranging from moderate to demanding” (Schoenfeld, 2016, p.4).

As we can see, the teachers’ discussion is based on assumptions about what the students are able to do. T1 is concerned with making connections in mathematics, but he also sees the students as limited according to cognitive demand. Both T4 and T3 convey a strong fixed ability view of the students, although their approaches to the mathematics differ. T2 stands out from the three others in her view of teaching and learning in mixed groups. Overall, the teachers’ views of mathematics content and cognitive demand appear to be underpinned by some strong fixed ability beliefs for T1, T3 and T4, while T2’s views convey more nuanced beliefs.

Discussion

In this paper I ask: What mathematics do teachers aim to teach in attainment groups and how do they reflect on this? Focusing on teachers' beliefs about mathematics and mathematics learning, and how these relate to their experiences of attainment grouping and equal opportunities for learning (TPO), provides interesting insights into how all four teachers argue for attainment grouping to facilitate TPO, but how their approaches to the mathematics and mathematics learning differ. As the analysis has shown, the teachers' views are very much based on their beliefs about what the students are capable of, and assumptions they make about students' abilities.

Like Taylor et al.'s (2017) teachers, T4 and T3 both argue that attainment groups enhance the opportunity for TPO. T3 seems to have a more conceptual approach focusing on discussion in mathematics, but T4 favours a procedural approach, which is also rooted in her own approach to mathematics. This fits with research indicating that attainment grouping may lead to limited pedagogy and content because of how students are perceived (Boaler et al., 2000; Taylor et al., 2017). T1 argues for attainment grouping, but particularly for the low attainers, G1. It is interesting how T1 both clearly argues for limiting mathematics for G1 students, but also emphasizes a strong conceptual and investigative approach. T2 stands out compared to the other three teachers, in her argument for variation between attainment and whole class mixed groups. T2's different approach to teaching mathematics in mixed versus attainment groups is in line with literature which finds that the same teacher may take an investigative approach in mixed groups, but a more procedural approach in attainment groups (Francome & Hewitt, 2018). Clearly, teachers' beliefs about "the mathematics" will have implications for their assumptions about and use of "cognitive demand", the second dimension in TRU (Schoenfeld 2016).

The analysis notes the role of the teachers' beliefs about ability in their teaching. Teaching in attainment groups with limited content and pedagogy for different groups communicates a fixed ability view, both for low and high attainers (Francis et al., 2017). In the case of low attainers this can also be perceived as a nurturing approach by the teacher (Mazenod et al., 2019). It is therefore interesting how the four teachers' beliefs about attainment grouping can be linked to fixed ability beliefs. For some of the teachers there seem to be some contradictions in their beliefs. T4 clearly conveys a fixed ability view both in how she argues for different content in different groups, and how she labels the students in the different groups and describes "what they need". Like T4, T3 also conveys a fixed ability view as he clearly argues for attainment grouping as the best way to organize for TPO. At the same time, when T3 focuses on mathematics and mathematics learning inside G3, he may suggest a different approach to the students' learning. Looking at T2's beliefs and approach to mathematics and mixed groups, she seems less controlled by a fixed ability view, although there is some evidence of this in her views about attainment groups. Finally, while T1 seems to convey a conceptual approach according to his beliefs about mathematics and mathematics learning, when he advocates teaching only limited content in G1, he conveys a fixed ability view.

It appears that teachers' beliefs about what is appropriate teaching and learning for particular types of student seem to underpin a lot of what they say, and fixed ability beliefs seem prominent with respect to attainment grouping. It seems plausible that this will have considerable influence on their classroom practice and on how it would score on the TRU framework, particularly in terms of impact on the mathematics, cognitive demand, and teachers' responses to student contributions. One implication of this study is that, in terms of promoting equal opportunities for learning, working with teachers to explore the concept of fixed ability could have major implications for supporting them in developing inclusive mathematics teaching, in both mixed and attainment groups.

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Different ways of experiencing linear equations with a multiplicative structure

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Learning to solve linear equations of the form $a = b \cdot x$ with decimal fractions as coefficients is relevant to all secondary school students and especially for students in vocational education, as it can be significant for their future profession (e.g., Ohm's law). This study reports on students' different ways of experiencing equations depending on whether whole numbers or decimal fractions are used. In semi-structured interviews, students displayed an uncertainty towards equations containing decimal fractions and did not address them with the same powerful method as they used for equations with whole numbers, even though they had access to a calculator. The findings emphasize the importance of using different kinds of numbers when teaching equation solving.

Keywords: Linear equations, multiplicative structure, decimal fraction, vocational education, phenomenography.

Introduction

In the gathered knowledge of mathematics education, students' understandings of equations have been explored from many different angles, e.g., focusing on concepts such as equality, variables or what methods to use for solving equations (e.g., Kieran, 1992; Vlassis, 2002). Existing literature also recognizes the difficulties that students experience when handling decimal fractions, though often in an arithmetic context (Christou, 2015; Greer, 1987). However, students' learning to solve equations involving decimal fractions (e.g., $0.12 = 0.4 \cdot x$) is not a well-researched area. The topic of this study is students' ways of experiencing equations with different types of numbers as coefficients.

The study reported in this paper is part of a PhD project aiming at exploring critical aspects of learning algebra in vocational education. The students are attending vocational education in a class of 16-year-olds in a Swedish upper secondary school. Since the students train to become network technicians, the use of formulas (often with a multiplicative structure, e.g., handling Ohm's law) and decimal fractions will be crucial in their competence. The aim of this study is to reveal aspects, with a phenomenographic approach, that are important in students' understanding of equations by addressing the following research questions: (1) What different ways of experiencing linear equations on the form $a = b \cdot x$ (multiplicative equations) are expressed among the 16-year-old vocational students? (2) Does the use of different types of numbers (whole numbers and decimal fractions) as coefficients influence the students' awareness of the multiplicative structure and if so, are there indications why?

Background

The ability to identify structures in algebra involves recognizing properties of symbols and operations, identifying strategic groups and choosing the type of solution (Hoch & Dreyfus, 2006; Pierce & Stacey, 2004). If students can solve an equation with whole numbers but not solve a similar equation with decimal fractions, then their awareness of one of these three components must be different. Filloy and Rojano (1989) and Vlassis (2002) both report on situations where pupils fail to recognize the structure of an equation they are otherwise familiar with. Filloy and Rojano call it ”a temporary loss of ability” and explain it as a case of distraction. They suggest that when pupils derive equations from geometrical models and then are asked to solve an equation regarding these models, they ignore the linear dimension of the equations and thereby limit their use of established knowledge. Vlassis (2002) reports on a similar predicament in transferring knowledge of equation solving, although for multiplicative equations with negative numbers. Vlassis disagrees with Filloy and Rojano (1989) and argues that the reason for pupils' inabilities to apply earlier knowledge, rather than being a question of distraction, lies in the equations' degree of abstraction. These arguments suggest that using decimal fractions could either be interpreted as distracting students from perceiving the linear dimension or making the equation less concrete to the students.

There is an extensive body of research that confirms pupils' potential difficulties in using decimal fractions. In operations, decimal fractions less than one are considered difficult, as they do not conserve the common rule that ”multiplication makes bigger” and ”division makes smaller” (Christou, 2015; Greer, 1987; Vergnaud, 1988). This has led to what Greer (1987) calls *nonconservation of operations*, which refers to situations where students confronted with similar word problems change solution method if whole numbers are replaced by decimal fractions. Vergnaud (1988) concludes that ”bad numbers”, i.e., positive numbers less than one, make multiplicative problems more difficult for students even though calculators are used. In research on algebra learning, difficulties concerning decimal fractions primarily concern the issue that students expect the literal symbol to be a whole number (Christou & Vosniadou, 2012). That students apply characteristics from whole numbers to rational numbers is often referred to as the ”whole number bias” (Ni & Zhou, 2005). Duval (2006) uses the term *register* for semiotic systems that include specific rules for formation and enable transformation of representations. Duval (2006) argues that mathematical procedures ”depend just as much on the system of representation used for the numbers as on the mathematical properties of the operations” (p. 111). From a mathematical point of view, whole numbers and decimal fractions are both within the decimal system with base ten and are often considered as one kind of register. However, when students use whole number notation and decimal fraction notation, they learn that there are differences in the rules of notation (e.g., the impact of adding a zero on the right). Therefore, students may not consider the use of whole numbers and decimal fractions as the same register.

Learning to solve multiplicative equations systematically usually takes place during a period when the students are in the process of extending their mathematical knowledge from arithmetic to algebra. In order to put forward this delicate transition, Filloy and Rojano (1989) draw a distinction between arithmetical equations that can be solved by simply undoing operations based on arithmetical knowledge and algebraic equations, where the symbol for the unknown needs to be handled. In a similar way, but with a focus on the mathematical activity, Kieran (1992) makes a distinction between a procedural approach to algebra, where arithmetical calculations are used and numerical results are produced, and a structural approach that concerns operating on algebraic expressions. Even though the goal in algebra teaching is a structural approach, teaching equations to pupils is traditionally introduced by avoiding the algebraic symbolism (Kieran, 1992) and with equations that the students can solve solely using their arithmetical skills (Röj-Lindberg, Partanen & Hemmi, 2017). Kieran (1992) discerns two strategies for equation solving that are often perceived as structural from a teacher's point of view: performing the same procedure on both sides and transposing terms by inverse operations. Transposing terms can either be a shortened version of performing the same procedure on both sides or it can be a memorized method used with limited understanding. In this article, where the influence of using decimal fractions is analyzed, the focus will not primarily be on what method the students use, but rather on whether their approach displays an intention to balance the equation (structurally or procedurally) or whether they try to find the answer of the equations by trial-and-error.

Method

Solving linear equations is a part of the curriculum in mathematics both in lower and upper secondary school in Sweden. A test performed earlier in the project showed that several students found it difficult to handle $0.12 = 0.4 \cdot x$, despite being able to solve $42 = 3 \cdot x$ and having access to a calculator. To further examine why some students did not apply the division algorithm to the equation with decimal fractions, six students were selected to participate in semi-structured interviews, which is the most common method in phenomenographic studies (Marton & Booth, 1997). The interviews, made in early 2020, set out to investigate the students' conceptions of multiplicative equations in the beginning of algebra teaching in their first year in vocational upper secondary education. The test in the larger project served as a selection tool for the interviews. Four of the students were chosen as they only solved the whole number equation in the test, whereas the other two students solved both equations in the test. In the interviews, all students were asked to reflect on three equations that were presented on paper, one at a time. The three equations had identical multiplicative structure, but with different numbers used as coefficients. The equations were presented in the following order: $0.12 = 0.4 \cdot x$, $42 = 3 \cdot x$, and $0.25 = 5 \cdot x$. They were asked to solve the equations under similar conditions and the aim was to display differences in the way they perceive the same multiplicative structure with different types of numbers. During the interviews they all had access to calculators. Five of the six respondents displayed different approaches to the equations depending on the type of number in the equation and were therefore chosen for analysis. As the first student interviewed was not able to solve either equation in the interviews, he was excluded from the analysis.

The analysis was conducted with a phenomenographic approach (Marton, 1981). The research approach aims at revealing qualitative differences in how people perceive a phenomenon by taking a second order perspective. The phenomenon in this study is multiplicative equations. As some ways of experiencing a phenomenon are more powerful than others, it is necessary to identify what different features of the phenomenon that needs to be discerned for students to gain a more advanced way of seeing it. When students treat equations with whole numbers and decimal fractions in separate ways, they also perceive them differently, as they base their actions on how they see the equations. *Ways of seeing* the phenomenon will also be called *ways of experiencing* and *conceptions* synonymously. The variation in students' ways of experiencing multiplicative equations can be divided into *referential aspects*, i.e., how the meaning of the phenomenon as a whole is perceived, and *structural aspects,* i.e., how the different parts of the phenomenon are discerned and how the parts relate to each other and to the context. These different conceptions will be described in an outcome space (Marton & Booth, 1997).

In the interviews several equations were used to probe students' understanding of multiplicative equations when using different types of numbers. By transcribing the interviews and reading them repeatedly, focusing both on the interviews as a whole and certain tasks independently, several different ways to see certain features in the phenomenon could be identified and ways of experiencing were formulated. Phenomenographic studies explore understanding at a collective level. As there are a limited number of qualitatively different ways to experience a phenomenon in a group, these can be found and described by performing and analyzing interviews (Marton & Booth, 1997). However, six interviews are not enough to reach saturation in all possible ways to experience multiplicative equations in a larger group, even though an individual can exhibit different ways of seeing a phenomenon. Nevertheless, the conceptions found will contribute with knowledge on the role of using different types of numbers as coefficients when students learn to solve equations. The translation of the quotes aimed at being true to the intention of students' utterances.

Findings

To show that working with the equation $a = b \cdot x$ when a, b $\in \mathbb{Z}_+$ not necessarily implies a fully developed procedural understanding of multiplicative equations, this section will provide an overview of the outcome space. There were two conceptions expressed in the interviews. However, a third conception (A), which is the goal in learning multiplicative equations, is displayed in the outcome space even though this way of experiencing was not empirically found in the interviews. The two structural aspects presented in the outcome space (Table 1) were the two content-related aspects that varied between the different ways of seeing the multiplicative equations; the connection between multiplication in the equations and division as a solution method, and also the role of the type of number used in the equations.

Ways of experiencing	*Referential aspect* Multiplicative equations are discerned as…	*Structural aspects* The relation between multiplication in the equations and division as a solution method.	The role of the type of number used in multiplicative equations.
A. The same kind of equation	… a multiplicative structure independent of type of number.	Discerned as inverse operations.	Discerned as irrelevant for the structure.
B. Different kinds of equations	…equations to balance in different ways depending on type of number.	Discerned as inverse operations, but not next to decimal fractions.	Discerned as a part of the structure.
C. Different tasks	… tasks to find the value of the unknown.	Discerned as operations connected in some way.	Discerned as affecting the solution methods.

Table 1. The outcome space displaying three ways of experiencing multiplicative equations

The outcome space is hierarchical in the sense that conception A is more advanced than B and C. However, the two empirically found conceptions (B and C) do not include each other but have different qualities and will be described further.

Conception B: different kinds of equations

Multiplicative equations are seen as different kinds of equations that need to be understood and balanced in different ways depending on the type of number used as coefficients. The common attribute in this way of experiencing is students' intention to balance the equations independently of the method used: performing the same operation on both sides, transposing terms, or using number facts and counting techniques to figure out the unknown. Division is discerned as an inverse operation to multiplication when there are only whole numbers in the multiplicative equation (e.g., $42 = 3 \cdot x$):

Student five: Three times something is 42 […] divide it by three to reduce, it's the same thing here, and then the three goes away, so it is x on this side […]

This student performs the same procedure on both sides to maintain equality. The following example could be a shortened version of performing the same procedure on both sides, or a rehearsed procedure, still the student identifies division as an inverse operation for multiplication:

Student two: When I've taken times something, I can divide it with the number, to get what it was before.

Equations with the same multiplicative structure, but with one or two decimal fractions ($0.12 = 0.4 \cdot x$ and $0.25 = 5 \cdot x$) are also seen as equations to balance, but division is not instantly discerned as a possible solution method. The purpose displayed is still to preserve the equality of the equation, but now with the less powerful solution method of reasoning on how to balance the equation based on arithmetical knowledge.

Student five: Something times it will be 0.12…

Student two: x should be three because 3 times 0.4 is 0.12 […] But I can't describe how I get it.

The students' initial focus when addressing these equations is on the decimal fractions. For example, some students mention that something is made smaller when multiplied with a decimal fraction and one student even reflects on that 0.4 "is the same as 0.40".

It is significant that all students displaying this conception address the whole number equation by using inverse operation on one or both sides, but not the equation with two decimal fractions. Some of the students that experience in this way do not see the similarity of the equations with and without decimal fractions when the equations are juxtaposed. On the direct question if there are similarities between $0.12 = 0.4 \cdot x$ and $42 = 3 \cdot x$, one student thinks hard and then says "I'm not sure". In this way conception B is separated from conception A as it does not acknowledge the same structure in the multiplicative equations independent of the type of number used for the coefficients. Indications to the reasons for this are given by comments on the difficulty that decimal fractions bring to mathematical operations:

Student two: […] decimal fractions and times and divided are a bit tricky, how they affect them.

Student three: […] it's probably that it's not integers. It becomes trickier calculations, I think.

This suggests an anticipation that the solution procedure should change as the coefficients are decimal fractions, even though this is not the case when they have calculators. However, the students' skills in using calculators varied which could possibly influence their experience of the equations.

Conception C: different tasks

Multiplicative equations are experienced as tasks to be solved. To find the value of x that satisfies the equation is more important than understanding why it balances the equation. If there are only whole numbers in the multiplicative equation (e.g., $42 = 3 \cdot x$), this way of seeing is expressed rather similarly to the other conceptions as multiplication instantly is connected to division. An example of this is given by the sixth respondent, as he explains when applying division to the whole number equation that he ”saw what to do the moment I [he] saw it.” When addressing multiplicative equations with decimal fractions, a less developed understanding of inverse operations is displayed:

Student six: For I know that it had to do with multiplying or dividing with the correct number […] So I tested with multiplication, but it was wrong. But when I divide it’s correct.

Another example is given by student four, as he knows he should use division to solve the equation with two decimal fractions but is insecure of what numbers to use as numerator and denominator respectively. Multiplication and division are seen as connected in some way, but the nature of their relation is not clear.

The influence of the decimal fractions on the equations and what solution method to apply are also significant for conception C:

Student four: It’s also a decimal fraction since if you multiply two decimal fractions it becomes less.

Student six: Eh, it’s probably that it is with decimal fractions I think, probably. My brain becomes a little like, what do you say… eh, it gets harder when there are decimal fractions, to me at least […] it is probably that I’m not a hundred percent sure on how to calculate.

After having solved the equation with two decimal fractions by trying different operations, the sixth student recognize a certain similarity when addressing the equation with only one decimal fraction:

Student six: I think I will do the same way as I did with this one, but it is still decimal fractions so I… wait… I’m still not sure…

He goes on trying different operations on the calculator, but without success this time. This displays the volatility in the knowledge on inverse operations, as he found what to do with the equation containing two decimal fractions, but then suddenly is insecure again with the next equation. The insecurity due to the presence of decimal fractions is not caused by an uncertainty in using calculators for the students seeing the equations in this way. On the contrary, they found the solutions to some of the equations with decimal fractions as they flexibly tried different operations on the calculator.

Discussion

The results display that students' difficulties with solving equations are not necessarily connected to their understanding of variables – or the equality sign – but may instead be connected to the problem of recognizing the structure of multiplicative equations, due to the presence of decimal fractions. Several students in this study had trouble transferring the powerful method of equation solving to equations with decimal fractions, despite having calculators as an aid. The phenomenographic approach (Marton & Booth, 1997) made it possible to reveal two underlying qualitatively different ways of experiencing multiplicative equations (B and C, see table 1) as the causes of why some students handle equations with decimal fractions differently. Neither conception B nor C include a recognition of multiplicative equations with whole numbers and decimal fractions as the same structure that call for the same solution method. The change of coefficients from whole numbers to decimal fractions affects the two ways of experiencing the equations in two different ways. For students experiencing conception B the difficulty is to *recognize* the multiplicative structure of an equation they otherwise are familiar with, but now with decimal fractions as coefficients. In contrast, students' way of experiencing in conception C recognize multiplication and division as related even when there are decimal fractions as coefficients, but with an insecurity regarding how the decimal fractions *affect* the solution method. This suggests that the students have rote learnt the steps of the operation and might not have been challenged in their understanding by more difficult equations that require a more advanced way of seeing (Röj-Lindberg, Partanen & Hemmi, 2017).

An indication why the students do not recognize the structure in the context of decimal fractions is that several of them experience the use of decimal fractions in equations as using another register, a new semiotic system with a new set of rules. Students mentioning decimal fractions as making calculations more difficult aligns with earlier findings that pupils are used to whole numbers always producing a greater result with multiplication and lesser with division (Greer, 1987). Since the algorithms for performing a numerical operation normally depend on the notation system used (Duval, 2006), it supports the argument that the students in this case expect a difference in the solution procedure, even if there is none as calculators perform the calculations. However, insecurity in using the calculator might make some reject the option to divide with a decimal fraction without telling.

In short, this study indicates that decimal fractions in multiplicative equations can affect students' focus and make them unable to discern a structure that they are otherwise familiar with. However, the different conceptions have been interpreted from five interviews, and additional interviews could bring more qualitative descriptions. Therefore, this project aims at performing further interviews to explore the significance of the type of number of the coefficients for students' understanding of linear equations. The findings of this study support the argument that students' bias for whole numbers is not limited to learning arithmetic, but also concern algebra in upper secondary school. Consequently, teachers should be careful in presuming that students can generalize an algorithm for solving equations to equations with other types of numbers by themselves. To prevent whole number bias, an increased use of decimal fractions as coefficients in the teaching of equations should be encouraged. The different ways of experiencing equations due to decimal fractions can be critical to students at the same stage of learning as the ones presented in this study. However, the issues raised here are especially relevant to vocational students as it might become a predicament in their future profession.

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Guidelines for the teacher – Are they possible?

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We report on the design and implementation of teacher guidelines in a didactic sequence that was developed in the frame of a design-based research study. The guidelines are anchored in the frame of the Theory of Semiotic Mediation and the hypothesis was that the guidelines could support the teachers' management in the classroom discussions. We elaborate on the teacher guidelines design and present results of analysis from a classroom discussion as well as an interview with the teacher. Our results show an implementation that was not particularly effective, and that the teacher followed the guidelines only to some extent. We discuss the dilemma of developing condensed guidelines while trying to convey underpinnings of an elaborate theoretical frame.

Keywords: Teacher guidelines, dynamic geometry environments, the theory of semiotic mediation.

Introduction and conceptual frameworks

Dynamic geometry environments (DGE) are widely implemented in mathematics education in several Nordic countries, not least in Denmark, where the software GeoGebra is particularly popular (Højsted, 2021). Since the introduction of DGE, ample research has been conducted to outline the potentialities of the software to foster students learning of mathematics (for an overview see Højsted, 2020). While there are many studies that have shed light on the students' work in DGE from a cognitive point of view, not least in relation to mathematical reasoning, generalizations and conjecturing (e.g. Arzarello et al, 2002), there has been less research conducted in relation to the design of adequate tasks to utilize the potentials of DGE for specific mathematical learning aims, as well as on the role of the teacher in the mathematics classroom in which the DGE potentials are to be utilized (Højsted, 2020; Komatsu & Jones, 2018). A promising framework that acknowledges the essential role of the teacher in artefact-based activities is the Theory of Semiotic Mediation (TSM) (Bartolini-Bussi & Mariotti, 2008). Taking a semiotic perspective, the TSM provides a model of the teaching and learning process centered on the role that a specific artefact can play because of its semiotic potential with respect to a specific mathematical meaning. The semiotic mediation process develops from the use of the artefact and students' personal meanings emerging from that, towards the evolution of shared mathematical meanings, which is accomplished in a collective discussion managed by the teacher, who is expected to play a crucial role. The TSM frame describes organization of the teaching/learning sequence according to iterate didactic cycles, which comprise (1) activity with the artefact (2) production of signs (writing etc.), followed by (3) classroom discussion.

The paper reports on part of a project, set in a Danish context, whose overarching aim is to develop guidelines for the design of didactic sequences that utilize the potentials of a specific artefact, a DGE in relation to fostering students' development of mathematical reasoning competency (Højsted, 2020; 2021; Højsted & Mariotti, 2020). When we use the notion of mathematical reasoning competency, we refer to the Danish KOM framework description of reasoning competency (Niss & Højgaard, 2019).

In this paper, we report on our attempts to convey the rationale of the TSM by developing **guidelines for the teacher**. In particular, considering the delicate proactive role that the teacher is asked to play when managing a collective discussion, the guidelines intended to outline a general frame for the teacher's purposeful interventions. Such a frame, empirically developed (Mariotti, 2013), consists of four categories of actions that the teacher can perform with the aim of fostering the process of semiotic mediation centered on the use of a particular artifact.

1. The **back to the task** action asks the students to report on what they did. "The objective is that of reconstructing the context of the artefact" in order to make meanings "emerge in relation to that experience*."* (Bartolini-Bussi & Mariotti, 2008, p. 775). 2. **Focalize** is complementary to the previous action, and aims at focusing on relevant aspects of task experience with respect to the intended mathematical meaning. 3. **Ask for a synthesis** action aims at fostering the move towards mathematical meanings by asking students to generalize and decontextualize the experience from the specific artefact task. 4. **Give a synthesis** complements the previous action and is used to support students' synthetizing.

The interventions of the first pair of categories refer to the unfolding of the expected semiotic potential of the given artifact, and aims at fostering students move from their experience with the artifact towards pertinent aspects that can be related to the mathematical meanings that constitute the educational goal. The second pair of interventions refer to the move towards the introduction of the expected mathematical meanings.

In a previous experience (Højsted & Mariotti, 2020), students showed difficulties in grasping the hierarchical dependencies between objects in GeoGebra, while the teacher seemed not aware of this fact, and therefore could not help the students to interpret the onscreen phenomena sensibly.

Considering that the teacher could not manage to help the students overcome this difficulty, we designed guidelines for the teacher, by implementing the didactic cycle and the teacher actions from the TSM frame. Our goal was that of providing the teacher support to interpret students' difficulties and to intervene to make them overcome them. We propose the guidelines as the product of a design process, making explicit the principles of design and reporting on the a posteriori analysis of the implementation in the classroom. The analysis of the collected data is aimed at studying the effect of these guidelines. In this paper, we are therefore focused on answering the following research question:

In what ways do the teaching guidelines support the teacher in managing the classroom discussion and to what extent is the support consistent with our expectation?

We set out by briefly reporting on the methodological approach; the designed task and teacher guidelines; as well as the data collection. In the following results and analysis section, we look at the first classroom discussion after the students have worked on tasks 1 and 2 and at data from the subsequent interview with the teacher, in order to identify to what extent the teacher guidelines were used, and find out to what extent they were useful for the teacher. On this basis we suggest revisions and consider emerging issues in the concluding discussion.

Method, design and data collection

This study is part of a design-based research project, which is methodologically characterized by its dual purpose of developing both educational practice and theory about practice, by means of iterations of design and testing of educational activities (Bakker & van Eerde, 2015).

Based on an a priori analysis of the semiotic potential of DGE in relation to reasoning competency (Højsted, 2020), a didactic sequence was developed and tested in three iterations in Danish 8th grade (age 13-14) classes (Højsted & Mariotti, 2020). For the last iteration, we designed guidelines for the teacher. The criteria of design was to include the explication of key elements of TSM rationale: specifically, the notion ofdidactic cycle, of semiotic potential of a DGE in relation to reasoning competency, and specific instructions about the management of the collective discussion, e.g. the four actions described above. We opted for a concise text written in a simple language, refraining from using too many technical terms.

We met with the teacher prior to the experiment to discuss the guidelines, attempting to share hypotheses and design principles of the teaching material, which included tasks for the students, a task answer book for the teacher, and the guidelines for the teacher.

The task and the teacher guidelines

In this paper, we discuss data coming from the classroom discussion following the very first task, a ”dependency task” (Højsted & Mariotti, 2020): 1.a. Construct two points A and B in GeoGebra and the midpoint C between them. Use the midpoint command […] 1.d. What do you think happens with the other points when you drag point C? Guess and justify first. Investigate afterwards, what happens?

Task 2 was also a dependency task, which involved the construction and guided exploration of lines and parallel lines, which we will not report on in detail due to space limitations.

As mentioned, *the teacher guidelines* consisted of a two-page introduction describing the general structure of a didactic cycle and in particular, elaborated on the four categories of actions the teacher was expected to perform. The specific instructions that followed were organized into three main constituents, presented in a table: 1. a description of the educational aim of each task, 2. the personal meanings expected to emerge from the activity (hypothetical and experienced in previous design iterations), 3. and the corresponding possible teacher actions that the teacher is advised to perform, followed with specific examples and comments.

The first classroom discussion was planned to take place after the students had worked on task 1 and 2, marking the end of the first cycle. The guidelines advised the teacher to perform the first pair of actions (back to the task and focalize) in relation to each subtask, and then after discussing the whole of tasks 1 and 2, to ask for a synthesis and give a synthesis. We provide a sample from the table that was given to the teacher.

Task no.	**Purpose of the task (the educational aim)**	**Expected student meanings**	**Possible teacher actions in the discussion**
…	…	…	…
1d	Specific sub-goal: to become aware that in GeoGebra the derived points cannot affect the free objects that have defined them. I.e. that C cannot be moved because if you	Some students guess/expect that point C can be moved by direct dragging and that the other points will follow it as if the figure had a rigid/solid	**Back to the task** They probably won't say why C can't move. **Focalize**

	drag C, then A and B have to follow in order to maintain C as the midpoint, but this is not allowed in GeoGebra. A derived object (a child) cannot affect the objects from which it is defined (its parents). C is therefore a locked point. In contrast, derived objects can be moved if they do not affect the objects that have defined them, as we see in task 2.	structure. Possible student guesses "They move together", ”*A and B follow in parallel* ". They will probably be surprised that one could not move C. And they will probably have a hard time explaining why, but possibly someone mentions that C cannot be moved because it is made based on the others. This can be highlighted in the discussion.	Therefore, ask the students about the construction process. How did point C come into the world? In this way, the focus can be oriented towards C being a derived object from A and B. The terminology of children and parents can be used. C is a child of A and B (its parents). In GeoGebra you cannot move objects (children) that are derived from other objects (parents), if that requires that the parents follow.
…	…	…	…
After 1-2	The specific objectives in the sub-tasks above contribute to the overall purpose of tasks 1 and 2: Understanding that it is dependency relations between geometrical properties of objects in GeoGebra that determine the outcome of dragging, and that these dependencies stem from the construction method, and its logical consequences from the geometric rules that govern the program. And that these relationships remain when you drag points. The goal is for students to describe the objects and dependencies in geometric terms. That they can justify what they see based on the objects, their dependency relationships and the construction method.	After task 1 and 2, the students are expected to be thinking about the relationships in the constructions and to connect those with construction method. I.e. that they can describe how the figure behaves in GeoGebra referring to geometric properties.	**Ask for a synthesis** For example, ask students what determines how a figure moves in GeoGebra, try to get them to say something general about the construction process and geometric properties. **Give a synthesis** Offer that there are dependency relationships between the objects in GeoGebra that determine the specific points’ behavior, and that these relationships stem from the construction method that induces logical consequences based on the geometric properties defined by construction.

Table 1. The table from the guidelines for the teacher to hold classroom discussions

Our hypothesis was that the guidelines would support the teacher to hold effective classroom discussions and we expected the teacher to use the advice put forward in the table.

Data collection and analysis

The classroom data was collected in the form of video of the whole class and selected groups; in screencast recordings of all groups; audio recordings; and written products from the students. Data was also collected by interviewing the teacher before and after the teaching experiment and after each teaching session. Through a semi-structured interview, we asked the teacher about the guidelines, and the perceived usefulness of the guidelines from the point of view of the teacher.

Results and ensuing analysis

The teacher opened the discussion asking students to report on their responses to subtasks 1a-1c, which seemed straightforward for the students. GeoGebra is running on the whiteboard with the construction from the task performed. The discussion goes on until they reach subtask 1d. The teacher, apparently expecting the task to be more difficult, seems excited to discuss this task.

172	Teacher:	So I'm a little excited about this with point C, what do you think happens to points A and B when you drag in point C. What did you guess would happen here? Freja.
173	Freja:	We guessed that nothing happens.
174	Teacher:	That nothing happened at all?
175	Freja:	No, we just thought that if it was that you dragged C that, either it would move, but we did not think it would get bigger, or that the C would not move.
176	Teacher:	Well so, if I drag this point, it would all move as if it were one line? *[She simulates grabbing point C with her hand at the whiteboard, and simulates dragging it up and down]*
177	Freja:	Yes, or it would remain as it is.

The teacher starts with prompting students' guesses (172), but though apparently using a **back to the task** action, the intended aim of this action – fostering students to express personal meanings – is not fulfilled. It is herself that explains the task formulation, and the following interactions show her gently but firmly guiding the discourse; she reiterates what the students say, (174, 186, 198, 219, 223, 230), but changing her tone to a question tone provides an implicit evaluation of the students' suggestion. Freja seems to put forward two scenarios, it will not move, or it will move as a solid. The teacher simulates what Freja says, using the construction on the whiteboard (176). The teacher wants more signs to emerge, and asks others.

185	Maja:	Yes, so if you moved it up [point C], it would just straighten up like a triangle but still in the middle.
186	Teacher:	And get a triangle out of it, yeah okay interesting.

…

197	Freja:	And the C it will get out of control.
198	Teacher:	It will get completely out of control. Yes. Okay! Interesting, so there were actually more suggestions here, what happened when you dragged it. Dima?

Maja's guess indicates some misunderstanding of midpoints, while Freja's guess indicates that they did not have any explanation. To both answers, the teacher merely replies that it is interesting, and moves on, now to ask another student what actually happened. Several students describe that point C cannot be dragged. The teacher stops on this and asks students for a justification.

217	Teacher:	Why do you think that is? Why can't I do something about it Julie.
218	Julie:	Yeah because that's no point.
219	Teacher:	That is because it's no point. Okay, can you try to say a little more about that it's not a point. What is the difference between points A, B and then C?
220	Julie:	Because that's the midpoint.
221	Teacher:	Yes, but there's one such dot right there, so there's a point?
222	Julie:	Yes, but it's not blue.
223	Teacher:	It's not blue, okay.

The student has noticed that point C is different than A and B, it even has a different color. When Julia says it is not a point, the teacher asks her to elaborate, helping her by asking what is the difference between the three points (line 219). This can be considered the start of a **focalization** action. However, she does not ask about the construction process, trying to foster awareness of derived objects, which was the advice given in the guidelines. Instead, she just asks why they cannot drag point C. The teacher asks others for more suggestions.

229	Tobias:	That is, A and B are points that you have made.
230	Teacher:	It's something I have made.
231	Tobias:	Yeah, you didn't make point C.
232	Teacher:	Yes, okay. But I actually have, because I used that tool after all. Larso!?
233	Larso:	I think point C it just simulates the center of both points.
234	Teacher:	Yeah, okay. That's the same thing you would say Maja?
235	Maja:	Yes.

Tobias suggests A and B were generated by the user, in contrast with the fact that the user did not generate C directly. The utterance ”have made/didn’t make” could develop into the categories ”Free points” / ”derived points” and then into the relationship that derived points depend on free points. However, the teacher is not able to manage this evolution; she stops the semiotic process by talking about the use of the tool (232) in a manner that invalidates Tobias’ suggestion. Several students suggest that C is different to A and B (218, 220, 222, 231, 233, 235). The teacher did not perform further **focalizing** actions; thus, the students’ suggestions remained suggestions. Without reaching a consensus in class, or an accurate explanation, the teacher just moves on to the next task. It seems that the teacher has no clear view of the objective of the discussion.

After discussing tasks one and two, the teacher did **not** ask for a synthesis or give a synthesis herself, even if the teacher guidelines advised her to do so. It seems she does not feel the need of that.

Teacher interview

Actually, in the interview after the teaching session, when asked about the teacher actions, the teacher seems to realize the synthesis was missing.

114	Researcher:	Were there any of the four teacher actions that you used, or perhaps used the most? And were there some that you didn’t use so much?
115	Teacher:	Ehm, I don’t know if…
116	Researcher:	You mentioned that one ”back to the task” before.
117	Teacher:	Yeah, that one I think I managed to use it quite well. To ask ”what was the purpose”, ”what have you investigated”, ”ok, what was actually the purpose”, ”what is our conclusion then”, or like that. Ehm… Yeah I don’t know… I think maybe I, except that one about give a synthesis, I think I used the others pretty much.

During the classroom discussion, the teacher held the task answer book, and not the guidelines. She had read the guidelines beforehand, and decided that it was more important to have the answer book.

38	Teacher:	But that is mostly because when I am standing there, then I have to juggle, suddenly they have said something, and then I am about to confirm something wrong, then it is very nice to stand with the answer book. And if I should stand with both, then I would drown in papers, right.

When asked about the usefulness of the guidelines, the teacher described that she had found them useful. The teacher apparently felt she had followed them, especially the general description of the four teacher actions.

62 Teacher: It was perhaps some of the general, so the general setup [pointing to the first part of the teacher guidelines that contain the four teacher actions] […]

She indicated that the guidelines table contained too much information, and suggested that there should only be a few examples of questions to pose, in addition to the general teacher actions.

70 Teacher: For sure in the following guidelines, I think, the first two [tasks] can be very detailed, but in the following, just a few proposals for questions […]

Since the teacher did not use the questions or the terminology from the guidelines table during the classroom discussion, the question arises whether or not she had fully read them. Alternatively, perhaps she had forgotten some parts or did not understand everything, and since she did not look at the guidelines during the classroom discussion, she was unable to use them effectively.

Concluding discussion

From the data, we can conclude that only parts of the guidelines were acknowledged and only parts of the suggested interventions were performed. This is the case for the first pair of actions categories, **back to the task** and to some extent **focalize**, however, not for the second pair of actions, which were completely neglected. When guideline suggestions were followed what seems missing are some aspects of the general aim of the actions. For instance in the case of back to the task, although the teacher encourages students to express themselves, it seems difficult for her to let students elaborate on their own formulation without intervening and she does not leave the students to talk to each other. There is always an intervention of the teacher in response to the intervention of a student in a ”ping-pong” effect. The teacher recognizes that reacting ”on the spot” is difficult for her because she does not feel certain about the answer (line 38), which unveils that she feels that she has to confirm, or reject, what is said. Actually, this can explain the ”ping pong” pattern of the discourse; the reaction corresponds to the feeling of obligation for confirming. We wonder if the teacher would behave differently if explicitly advised to keep silent and wait for the students to fuel the conversation. Another example of missing aspects of the general aim is in the case of Tobias (229-232).

The teacher indicates that the guidelines need to contain less information to be effective (line 70). A possible revision of the guidelines could consider condensing and incorporating the guidelines table into the answer book, giving more examples of questions and statements related to the four teacher actions for each task (or set of tasks). However, it is questionable if further condensing the guidelines would be helpful: in fact, the actions of asking for a synthesis and giving a synthesis seem not being appropriated by the teacher. From the point of view of the TSM frame, reaching a synthesis shared and elaborated by the whole class is a crucial part of the teaching and learning process.

This leads us to a conclusion that emerges as a critical issue - to what extent should the theoretical frame be made explicit in the guidelines in order to give the teacher the conceptual tools to interpret the guidelines? If the teacher does not share the rationale of the TSM framework, it may happen that she disregards some key aspects, because she does not grasp their importance. We did not incorporate much of the underlying theoretical assumptions of the TSM frame in the guidelines, so we can make the hypothesis that it is one of the reasons the teacher did not use much of the guidelines. We suspect that the teacher was too far from this pedagogical perspective to really appropriate it.

The dilemma is that the teacher already suggests that the guidelines must be shorter, while the data suggests that more of the theoretical frame must be shared with the teacher, and the importance of key aspects must be elaborated.

Seeing as the teacher only followed the guidelines to some extent, neglecting two of the teacher actions, and did not use the advice from the table, we can conclude that it is difficult to communicate theoretical aspects in the form of guidelines, at least in the chosen design form of a condensed text and the table. Perhaps the guidelines somehow need more flexibility to be adapted to different teachers' pedagogical paradigms. Conversely, we can notice the important need of teachers' flexibility and pedagogical awareness in presenting new activities with technological tools.

More insights are needed on designing effective guidelines, which may be considered the core of the articulation between theory and practice. It requires reflecting on how to interface with teachers taking into account the diversity of their possible pedagogical paradigms, most of the time implicit. We intend to report more on this matter in forthcoming research.

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Relations between grade 1 students’ number sense and nonverbal and verbal reasoning

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This study investigated relations between number sense and nonverbal and verbal reasoning in 75 Norwegian grade 1 students. Number sense was measured by a digital assessment utilising eight components of the foundational number sense (FoNS) framework in combination with reasoning measures. Analysis revealed that number sense related moderately to verbal reasoning and moderately to strongly to nonverbal reasoning. However, number sense components correlated differently to the two reasoning types. Our results support, extend, and contradict previous research, perhaps due to how number sense was operationalised and measured. We found arithmetic to be the component with the strongest correlation to both nonverbal and verbal reasoning, while estimation did not correlate significantly to any reasoning types. We discuss individual variations and the use of visual and verbal task demands, particularly number lines, to support or estimate.

Keywords: number sense, reasoning, number lines, visual and verbal information, grade 1 students

Introduction

Number sense is ambiguously defined beyond being the ability to flexibly work with numbers and quantity and a predictor of mathematical competence (Andrews & Sayers, 2015). Several research fields focus on number sense, but there remains a need to resolve problems regarding polysemous and synonymous constructs thereof (Whitacre et al., 2020). This has led to varying foci on the perspectives of preverbal, applied, and foundational number sense. Preverbal number sense concerns the innate ability to estimate and compare small quantities without counting (Dehaene, 2001). Foundational number sense (FoNS) builds on preverbal number sense, is typically acquired during the first years of school, and requires instruction. Applied number sense builds on FoNS and consists of the basic mathematics needed in everyday life (Sayers & Andrews, 2015).

Developing number sense is cognitively complex and requires reasoning processes to make mental representations of numbers: to reason is to manipulate and work with information from sounds, signs, and symbols (Geary et al., 2018). Visual information is reasoned nonverbally, while spoken information is reasoned verbally. When combinations of visual and spoken information are given simultaneously, both nonverbal and verbal reasoning are required. Consequently, measures of nonverbal and verbal reasoning as cognitive prerequisites and focus on nonverbal visual and verbal task demands are being included more often in studies of children’s abilities with counting and number representations (Whitacre et al., 2020). However, it is unknown to what extent and in what ways nonverbal and verbal reasoning are interrelated and how they are involved with the different task designs in components of number sense. Therefore, this paper’s research question is:

How is the number sense of grade 1 students’ related to their nonverbal and verbal reasoning?

Previous research

Nonverbal reasoning is suggested to underpin the understanding of numbers and arithmetic (Dehaene, 2001). As such, subitising is assumed to be important for verbal counting skills and arithmetic (Sayers et al., 2016). Number line estimation and mental number line representations have been found to correlate with number sense (e.g., arithmetic), counting, grouping, discrimination, and comparison of quantities, as well as with school achievement tests (Dehaene, 2001; Schneider et al., 2018). Despite the fact that geometrical, nonverbal reasoning is presumed important for number line estimation (Olkun et al., 2019), research findings differ on whether nonverbal or verbal reasoning is predominant for and involved in number sense. Jordan et al. (2013) highlighted both reasoning types as important.

Like Geary et al. (2018), who found support for verbal reasoning in cardinal understanding of numbers, Cross et al. (2019) found that verbal reasoning is involved in number identification, counting, and arithmetic, but not in number line and quantity comparison tasks. This illustrates that both nonverbal and verbal reasoning are important for some number sense components, but only one type is implicit in others. Extended knowledge and clarification about what number sense is and its relation to cognitive reasoning abilities is needed to better identify children who struggle to learn mathematics and to improve teaching methods and content (Geary et al., 2018).

Methods

The relationships between grade 1 students' number sense and their nonverbal and verbal reasoning were measured using a digital assessment operationalising number sense, while two standardised assessments measured nonverbal and verbal reasoning.

Participants

Following informed parental consent, 75 grade 1 students were recruited from two typical neighbourhood schools in Mid-Norway. The 10% who scored the highest and the lowest on the digital number sense assessment made up the subgroups *proficient students* and *developing students*, respectively.

Assessments

Nonverbal reasoning was measured using Raven's Progressive Matrices 2 (RPM). The students manipulated and compared visually presented figures to fill in the missing part (Raven, 2000).

The New Reynell Developmental Language Scales (NRDLS) were used to measure verbal reasoning, which resulted in two separate standard scores: language perception and language production (Letts et al., 2014). This assessment was conducted as a play-based dialogue about animals (Letts et al., 2014). Both the RPM and NRDLS results were standardised for Norwegian samples.

Andrews and Sayers (2015) defined number sense as consisting of eight components; of these, the digital number sense assessment (Saksvik-Raanes & Solstad, personal communication, September 2020), which consists of 69 tasks, assessed seven of the components. See Table 1 for the operationalisation of Andrews and Sayers' (2015) FoNS framework. Subitising was also included because of its importance to FoNS (Sayers et al., 2016). The digital number sense assessment distributes number sense tasks over three sets, two of which were solved by all participating students and used in the study. Tasks within the representing number component were placed in the third set, which not all students did. Because of this, the representing number component was not included.

Table 1 shows the included components and the number of tasks and exemplifies content within each component. Each task included verbal (sound file), visual, or both verbal and visual instruction. To respond, students typically dragged and dropped, organised objects on the screen, or tapped the appropriate multiple-choice item response.

Table 1. Type of component, content, and number of items in the number sense assessment

Component	Content	N
Number identification	Recognise number symbols, vocabulary, and meaning	8
Systematic counting	Ordinality. Counting to twenty and back (arbitrary starting point)	8
Number and quantity	Cardinality. 1-1 correspondence between symbol and quantity	10
Quantity discrimination	Compare quantities. Vocabulary: larger, smaller, more/less than	8
Estimation	Estimate the size of a set and the position on a number line	6
Arithmetic competence	Transforming small sets by using addition or subtraction	15
Number patterns	Continue or complete a number sequence	3
Subitising	Perceive quantity without counting. Perceptual/conceptual. Timed	11

Table 1 shows included components from Saksvik-Raanes & Solstad's (personal communication, September 2020) number sense assessment.

Procedures

Each student was tested in two sessions that occurred within two weeks of one another in autumn 2020. Data collection was completed within seven weeks. The first session, where RPM and NRDLS were conducted, was a 45-minute, one-on-one session with the first author and the student. Number sense was assessed via digital assessment in a 20–30-minute second session, wherein students were divided into groups of eight.

Analytical procedures

Descriptive statistics and pair-wise correlation analysis of standard scores from the RPM assessment, as well as raw scores from the digital number sense assessment, were conducted in SPSS. Pearson's (2019) efficacy reporting framework clarified the strength of the relations, classifying r-values < 0.20 as weak, $r < 0.50$ as moderate, and $r < 0.80$ as large.

Strengths and limitations

Measures of cognitive reasoning in combination with the FoNS framework provided opportunities to investigate unknown number sense relationships. Including seven of Andrews and Sayers' (2015) FoNS framework components provided a broad operationalisation of number sense. Adding subitising further strengthened the study. Task design within each component varied, elucidating visual and verbal contributions to the students' understanding when content was the same. However, not including the FoNS representing number was a limitation.

Investigating language perception and production made it possible to discuss understanding and production as different and overlapping aspects of language related to number sense. While the sample size of 75 participants limited the statistical inference possibilities, especially regarding the small number of developing and proficient students, including grade 1 students at all ability levels was a strength.

Results and analysis

Of 69 possible points, the 75 students averaged 43.5 (*SD* = 10.5) on number sense. Developing students averaged 23.5 (SD = 4.4) points, and proficient students averaged 59.1 (SD = 1.7) points.

The average nonverbal reasoning score of all 75 students was 93.0 (*SD* = 9.5); developing students' average nonverbal reasoning score was 85.9 (SD = 9.3), and proficient students' average nonverbal reasoning score was 103.8 (SD = 6.4). Regarding verbal reasoning, the 75 students' verbal reasoning average was 99.4 (*SD* = 14.1). A tendency towards lower verbal reasoning in the developing students (*m* = 93.2, SD = 15.3) and higher verbal reasoning in proficient students (*m* = 110.8, SD = 10.9) was observed. Standard deviation showed that within group variation was larger for developing students than for proficient students in all three measures of number sense, nonverbal and verbal reasoning.

Regarding verbal reasoning, the 75 students' average language perception was 100.4 (SD = 17.9); language production was 98.7 (SD = 13.1). We observed lower language perception in the developing students (*m* = 86.8, SD = 23.9) and higher language perception in the proficient students (*m* = 116.6, SD = 15.1), which was also the pattern observed for language production in the proficient students (*m* = 104.9, SD = 12.4) and the developing students (*m* = 86.0, SD = 17.6).

Number sense's relations to nonverbal and verbal reasoning

Figure 1 displays the relationships between number sense variation and nonverbal and verbal reasoning for all 75 students.

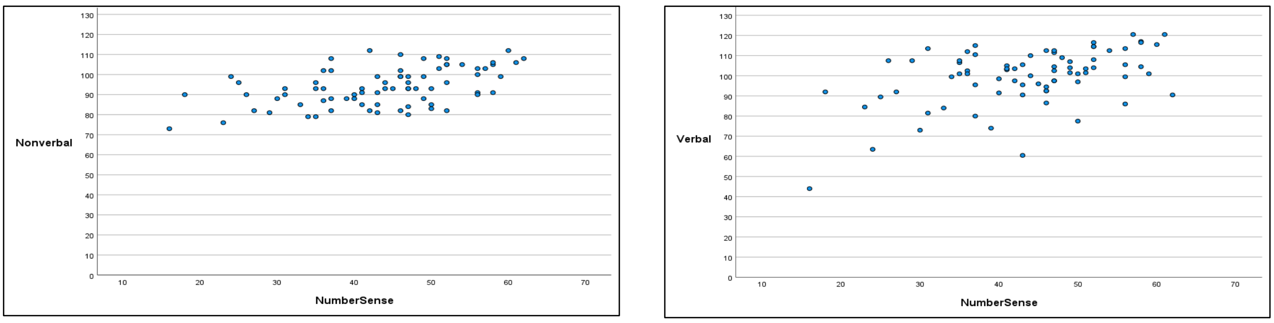

Figure 1. Number sense variation's relations to nonverbal and verbal reasoning

There was a moderate to strong correlation between number sense and nonverbal reasoning (r = 0.503, $p < 0.01$), and a moderate correlation between number sense and verbal reasoning (r = 0.463, $p < 0.01$). Nonverbal and verbal reasoning correlated significantly, and a thorough look into their aspects was sought. Investigating verbal reasoning showed that number sense correlated moderately to language perception (r = 0.397, $p < 0.01$) and moderately to strongly with language production (r = 0.447, $p < 0.01$). Due to the large variations in number sense and nonverbal and verbal reasoning within and between groups of students, we explored the different number sense components' correlations to nonverbal and verbal reasoning.

Number sense components' relations to nonverbal and verbal reasoning

The results support previous research indicating relations between number sense and both nonverbal and verbal reasoning (see also Cross et al., 2019; Olkun et al., 2019). Table 2 nuances the relations between the components making up number sense and their relations to reasoning.

Table 2. Correlations between number sense components and nonverbal and verbal reasoning

Reasoning	N	NI	SC	NQ	QD	ES	AC	NP	SU
Nonverbal	75	0.229 *	0.230 *	0.328**	0.238 *	0.121	0.504**	0.373**	0.433**
Verbal	75	0.238 *	0.435**	0.344**	0.314**	0.089	0.449**	0.228 *	0.273 *

Statistical significance of the correlation coefficients (presented in r-values) for number identification (NI), systematic counting (SC), number and quantity (NQ), quantity discrimination (QD), estimation (ES), arithmetic competence (AC), number pattern (NP), and subitising (SU) in all 75 students are marked * at $p < 0.05$, and ** at $p < 0.01$.

Focusing on each number sense component, we found that arithmetic correlated the strongest to nonverbal reasoning; number identification correlated slightly stronger to verbal reasoning; and systematic counting correlated the strongest to verbal reasoning. As such, we partly support the findings of Cross et al. (2019), who found that number identification, counting, and arithmetic mainly involved verbal reasoning. Our findings might be discussed in light of Jordan et al.'s (2013) findings that nonverbally mastering number line estimation strongly contributed to competence in arithmetic procedures.

Arithmetic competence, number patterns, and subitising correlated stronger to nonverbal reasoning than to verbal reasoning, supporting previous studies highlighting nonverbal reasoning in arithmetic (Dehaene, 2001; Sayers et al., 2016). Comparing the number sense components, arithmetic was the component with the strongest correlation to nonverbal reasoning, followed by subitising and number patterns. Estimation showed no significant correlation to either nonverbal or verbal reasoning. Our findings extend and contradict those of Cross et al. (2019), who did not find the importance of verbal reasoning in number line and quantity comparison tasks. Subitising is considered a type of nonverbal reasoning (Dehaene, 2001; Sayers et al., 2016). Still, arithmetic competence correlated stronger to nonverbal reasoning than subitising, while subitising and arithmetic competence correlated strongly, supporting previous findings of nonverbal reasoning being closely related to arithmetic competence (Olkun et al., 2019; Sayers et al., 2016).

Task design may give further information to the relations observed. Jordan et al. (2013) assumed that both nonverbal and verbal reasoning are important to master number line estimation tasks, but Cross et al. (2019) did not find verbal reasoning important for number line tasks. Number lines were included in arithmetic competence, number and quantity, subitising, and estimation but with different purposes: to estimate in estimation, and to support in the other components. Arithmetic consisted of some number line tasks and correlated strongly to both nonverbal and verbal reasoning, but more strongly to nonverbal reasoning. Subitising also correlated stronger to nonverbal reasoning than to verbal reasoning, but the opposite pattern was observed for number and quantity. Estimation exclusively consisted of number line tasks and had no significant correlations to either nonverbal or verbal reasoning. Estimation, arithmetic, and number and quantity tasks gave verbal instructions, while subitising did not.

Figure 2 shows three examples of nonverbal and verbal information provided to the students in the number sense assessment and exemplifies that task demand made students engage in both nonverbal and verbal reasoning. Tasks A, B, and C are examples from the arithmetic component.

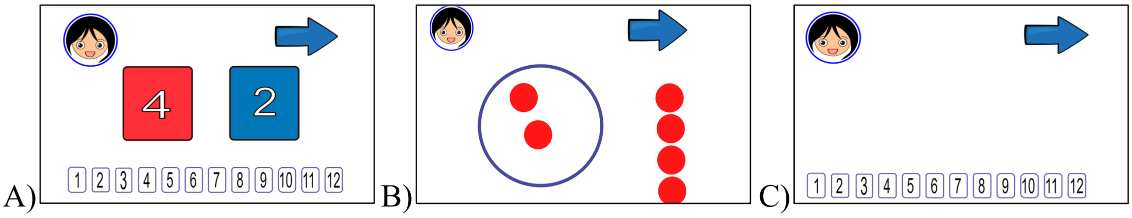

Figure 2. Examples of task instructions given in the arithmetic competence component

Task A provided visual and verbal information. The verbal instruction ”four plus two” and number symbols as nonverbal visual support were given to the students. They were then told to tap their answer on the number line. The number line might help execute the task mentally. Task B provided the verbal instruction ”Move enough balls so that there are four balls in the circle” without number line support. Tasks A and B differed in abstraction level of representation of numbers and the adding operation. Task C provided a picture of a number line with the verbal cue ”What does two plus three make?” and the verbal instruction to tap their answer on the number line. No visual representations of the numbers or the operation were provided.

Due to task demand, estimation is perhaps more appropriate to compare with Cross et al.’s (2019) study, as tasks in both studies gave quantities represented as numbers to estimate on number lines. Still, our findings contradict, as Cross et al. (2019) found number line tasks to strongly correlate to nonverbal reasoning, while estimation had little correlation to nonverbal reasoning in this study. The contradictory findings of number line tasks may be due to Cross et al. (2019) providing some numbers written and others verbally, while this study provided only written numbers.

Discussion

Our findings support, extend, and contradict previous findings of relations between number sense and nonverbal and verbal reasoning. Contradictory results may reflect different number sense and number line (estimation) constructs, as well as different focus on nonverbal visual and verbal demand in task design or as cognitive prerequisites for number sense. We operationalised verbal reasoning as both language perception and language production, while Cross et al. (2019) operationalised verbal reasoning only as language perception. Inclusion criteria for participation also vary across studies: in Cross et al.’s (2019) study, the participants had language difficulties. As we observed large variations within and between subgroups, we need to investigate whether their finding of no correlation between nonverbal reasoning and quantity comparison tasks can be observed in this study’s developing students.

Insight into interrelations between number sense and nonverbal and verbal reasoning might improve teaching methods and content in grade 1 mathematics instruction. The correlations we found suggest that working with subitising small quantities, considering spatial aspects of number sense, and developing mental number lines will probably ensure FoNS (Andrews & Sayers, 2015; Olkun et al., 2019; Sayers et al., 2016). They implicate classroom activities that strengthen connections between nonverbal and verbal representations of numbers and quantities and support interaction of students’ number and space systems to improve the development of mental representations and a mental number line to achieve a robust number sense (Geary et al., 2018; Olkun et al., 2019). Further consideration of correlations between number sense components in developing and proficient students in a longitudinal predictive perspective is needed. Consideration of estimation and subitising as nonverbal prerequisites or preverbal number sense instead of components of foundational number sense is needed to develop the research field (Whitacre et al., 2020), the teaching content, and methods.

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”I do not know much about programming, but I think that it is good for mathematics”: views of student teachers in Norway on integrating programming into mathematics education

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Since 2020, programming has been integrated into mathematics in the new Norwegian curriculum. However, research on the views of student teachers on integrating programming into mathematics education is scarce. The present research aimed to address this gap in the literature by analyzing a survey of the views of 408 pre-service teachers at three institutions in Norway, particularly the views of a subset of these respondents (N = 217) who were familiar with programming. The results indicated that the respondents were less convinced of the usefulness of programming in the case of geometry relative to other mathematical and pedagogical subjects. These findings are important for educators as they underline the need to emphasize the application of programming to different mathematical subjects in their preparation to work with the new curriculum.

Keywords: programming, online survey, teacher education, views of programming, usefulness of programming

Introduction

The importance of programming as a core competence for the future work force has become increasingly evident, and schools are bearing most of the responsibility for helping pupils acquire this competence (Balanskat & Engelhardt, 2015). Programming requires the ability to analyze, understand, and solve problem by verifying algorithmic requirements, processes that are commonly linked to mathematical and computational thinking (Grover & Pea, 2013). In the Nordic countries, programming is integrated into the curriculum (Bocconi et al., 2018); however, since 2020, programming has been explicitly included in the Norwegian mathematics curriculum. Nonetheless, research on effective pedagogical solutions for incorporating programming into mathematics education is limited. Although the role of mathematics teachers is clearly essential, effective pedagogical approaches may be challenging to determine whether teachers lack either knowledge of programming or experience of integrating programming into mathematics education (Kaufmann & Stenseth, 2021). The Norgesuniversitetet (2018) report for higher education concluded that a shift in perspectives on such technology is necessary: rather than treating technology as a tool, it must be regarded as central to new practices in developing new projects. Thus, the current study aimed to gain insight into the opinions of pre-service teachers about programming, more specifically their views on the value of integrating programming into mathematics education.

Literature review

Discussions on the role of programming in the curriculum have been ongoing for the last decade. The addition of programming to modern curricula indicates that most—if not all—mathematics teachers must teach this new content (Nouri et al., 2020). Consequently, many questions arise about what to teach and how to teach programming in the context of mathematics education. Although some limited research has examined K-12 teachers and programming (Moreno-León et al., 2016), relevant studies have been conducted on the views of teachers with regard to the use of technology. The latter have corroborated that the beliefs of pre-service teachers play a key role in their pedagogical decisions, thereby affecting whether they adopt technology and how they integrate it into their classroom practices (Tondeur et al., 2016). For instance, teachers who had successfully integrated technology into their teaching were more likely to describe themselves as having a passion for technology and a problem-solving mentality and to believe that these factors shaped their practices in using technology (Ertmer et al., 2012). The study of Dinçer (2018) on the pre-service teachers' knowledge of, skills in, and attitudes toward technology use is also relevant. In addition, this study verified that even if pre-service teachers had a low level of technology literacy (i.e., knowledge and skills), they still highly valued technology.

Research focusing on learning mathematical ideas through programming is scarce (Bråting & Kilhamn, 2021). In the ScratchMath project in the United Kingdom, Benton et al. (2017) asserted that programming activities could broaden the learning of students with regard to mathematical ideas, such as geometry and numbers. Other studies have found strong connections between computational thinking in programming and problem-solving processes, e.g., using debugging and tinkering as practices to explore the structure of an algorithm (Bråting & Kilhamn, 2021; Kaufmann & Stenseth, 2021). Similarly, there are few studies on the views of students regarding programming and the teaching of programming within the mathematics curriculum (Pörn et al., 2021). In their survey of 133 Swedish teachers, Misfeldt et al. (2019) argued that although teachers were favorable toward the use of programming in mathematics, not all could see the relationship between the two. Kilhamn et al. (2021) reached a similar conclusion and revealed that Swedish teachers who were early adopters of programming were favorable toward programming activities but did not see an apparent connection to mathematics, whereas some teachers found the inclusion of programming in other subjects easier. In addition, when Pörn et al. (2021) analyzed written answers on the definition of programming, they affirmed that teachers primarily emphasized writing, giving, and following instructions and that programming contributed to the development of logical thinking and served as a valuable tool in problem-solving. Nevertheless, explicit connections to specific mathematical content were scarce as no answers were related to the application of programming to arithmetic or algebraic expressions.

In Norway, programming is included in the mathematics curriculum with competence aims at each level from 2 to 10. Programming is not related to a particular topic in mathematics as, for example, in Sweden where it is related to algebra (Bråting & Kilhamn, 2021). The core element ”exploration and problem-solving” includes computational thinking, which is described as follows: ”Computational thinking is important in the process of developing strategies and procedures to solve problems and means breaking a problem down into sub-problems that can be systematically solved” (Ministry of Education and Research, 2019). The present study focuses on how programming can be used in distinct branches of mathematics and pedagogical topics. Given the emerging integration of programming into mathematics education and its status as a necessary skill for success in society (Moreno-León et al., 2016), addressing our limited knowledge of the views of teachers about using programming in mathematics education is increasingly crucial. Therefore, essential questions for this study relate to whether pre-service teachers view the integration of programming into their practice as necessary.

Methodological approach

The present research was part of a larger study investigating the views of pre-service teachers on digital technologies in mathematics education. The survey comprised 88 questions and required approximately 20 min to complete. The participants were enrolled in the following institutions: 408 respondents from the Western Norway University of Applied Science (57% of the respondents), The Arctic University of Norway (16%), and Østfold University College (27%). The participants were either enrolled in a master's level program for teachers of Grades 1–7 (208 students) or Grades 5–10 (200 students). Moreover, those three institutions were convenience sampled, and all of the pre-service students in their first, second, or third year of teacher education could take part in the survey. In terms of gender, 70% of the participants were women, and 30% were men. This gender distribution approximates the study population of pre-service teachers in these three institutions specifically and in Norway generally (roughly 70/30). On this basis, we may argue for the representativeness (a prerequisite for external validity) of the study sample.

The current research focused on two vital questions about programming. The first is about the views of pre-service teachers on the integration of programming into the mathematics curriculum (positive or negative). The second is about the views of pre-service teachers on the usefulness of programming in different topics (useful or not useful). The questions on the view of the usefulness of programming were in the form of multiple choices, and the participants were not asked about the reasons for their answers, nor were they asked about their interpretation of ”programming.” Furthermore, they were asked to rank the value of programming in teaching specific mathematical and pedagogical topics on a five-point scale (1 ”completely useless” to 5 ”very useful”) in the following five areas: (1) numbers, algebra, and functions; (2) geometry; (3) problem-solving; (4) multicultural classrooms; and (5) differentiated teaching. These different areas are related to the TPACK framework from Mishra and Koehler (2006). Topics 1–3 relate to content in the mathematics curriculum and thus form a part of the ”technological content knowledge” category. Topics 4 and 5 form a part of ”technological pedagogical knowledge”: multicultural classrooms and differentiated teaching are central components in all subjects in the Norwegian school system.

Focusing on these questions facilitated an exploration of whether the views of student teachers on integrating programming into the curriculum corresponded with their views on the usefulness of programming in mathematical subjects and corresponding pedagogical considerations. The survey results were analyzed in SPSS, first to identify any differences in the views on usefulness between student teachers with positive or negative views on integrating programming into mathematics (independent variables). Moreover, further analysis was conducted to compare the differences between pre-service teachers with a positive view on programming and pre-service teachers with a negative view on programming and the views of pre-service students on the usefulness of programming in mathematics education (dependent variables as assessed by a five-point scale: 1 ”completely useless” to 5 ”very useful”). As a nonparametric test of two independent samples, the Mann–Whitney U test was employed to identify any differences between how the respondents viewed the usefulness of programming. Further, ANOVA was used to determine if there was a significant difference between the mean scores of usefulness for the different topics.

Although there were 408 initial respondents to the survey, all the participants who answered ”I do not know enough about programming to answer this” to the question about their views on integrating programming into the curriculum (N = 135) were excluded from the analysis, as well as all the respondents who responded ”I do not know the tool” (N = 186) (many respondents answered that they did not know enough about both these two categories of questions). Following these exclusions, 217 responses remained for analysis, and there were both advantages and disadvantages in removing the answers of these participants. An advantage is the data sample included only those who knew about programming, which was integral to the present study. Conversely, only 217 of 408 (53%) participant responses could be included, suggesting a potential problem with the questions asked in the survey. The main hypothesis of the study was that pre-service student teachers evaluate programming as more useful in geometry and problem-solving in comparison with the other areas; numbers, algebra and functions, multicultural classrooms, and differentiated teaching.

Results

Table 1 presents summary of the views of the participants on integrating programming into the mathematics curriculum (positive and negative) and how they consider the usefulness of programming in different topics (left column). According to their responses to the question ”In the new curriculum, programming becomes part of the competence goals in mathematics. Do you agree that it should be?,” the participants were categorized either as positive (”Yes, this will be easy” or ”Yes, but this is going to be challenging”; N = 169) or negative (”No, but I am loyal to the curriculum” or ”No, I will try to avoid this in my teaching” or ”No, but I will consider it when the final curriculum is available”; N = 48). These 217 participants ranged the usefulness of programming in various topics and were grouped into those who considered programming useful (responses 4 and 5), those who regarded programming as not useful (responses 1 and 2), and those who were neutral (response 3).

Topics	**Positive view on programming**			**Negative view on programming**		
	Considers programming useful (response 4 or 5)	**Considers programming not useful (response 1 or 2)**	**Neutral (response 3)**	**Considers programming useful (response 4 or 5)**	**Considers programming not useful (response 1 or 2)**	**Neutral (response 3)**
Numbers, algebra, and functions	52% (N = 88)	16% (N = 27)	32% (N=54)	23% (N = 11)	48% (N = 23)	29% (N=14)
Geometry	31% (N = 53)	36% (N = 60)	33% (N=56)	17% (N = 8)	58% (N = 28)	25% (N=12)
Problem-solving	47% (N = 80)	22% (N = 37)	31% (N=52)	25% (N = 12)	42% (N = 20)	33% (N=16)
Multicultural classrooms	42% (N = 71)	25% (N = 43)	33% (N=55)	29% (N = 14)	52% (N = 25)	19% (N=9)
Differentiated teaching	50% (N = 85)	24% (N = 41)	26% (N=43)	33% (N = 16)	48% (N = 23)	19% (N=9)

Table 1. Pre-service teachers’ views on programming and on the usefulness of programming

Two interesting results can be observed in this table. The first is that the respondents with positive views on programming in the mathematics curriculum seemed more likely to positively respond (option ”4” or ”5”) to the usefulness of programming in various mathematical topics than those who held negative views, which is not very surprising. This is further elaborated by the Mann–Whitney U test presented in Table 2. Another surprising result is that the respondents evaluated programming as less useful in geometry compared with other topics, which is further elaborated in Table 3. Based on

Table 1, we may hypothesize that those pre-service student teachers categorized as ”positive” ranked the usefulness of programming higher. We also explored whether the respondents in the positive and negative groups had statistically significantly distinct responses to the five claims. In the Mann–Whitney U test, responses 1–5 were re-scaled to ranks that were then used for comparison. Table 2 presents the differences in the mean ranks between the participants categorized as positive or negative with respect to programming in the new curriculum. The results indicated a significant difference between the two groups in all the five questions. Hence, Table 2 supports the argument that the participants who were favorable toward programming ranked the usefulness of programming higher than those who were not favorable.

Usefulness of programming in:	N	Mean rank	Asymptotic significant (two-tailed)
Numbers, algebra, and functions	Positive (N = 169)	119,35	0,000
	Negative (N = 48)	72,55	
Geometry	Positive (N = 169)	115,64	0,003
	Negative (N = 48)	85,61	
Problem-solving	Positive (N = 169)	117,50	0,000
	Negative (N = 48)	79,06	
Multicultural classroom	Positive (N = 169)	114,45	0,014
	Negative (N = 48)	89,80	
Differentiated teaching	Positive (N = 169)	114,69	0,011
	Negative (N = 48)	88,98	

Table 2. Mann–Whitney U test to investigate if there are differences in view of usefulness of programming in different subjects between pre-service teachers with a positive view on programming and pre-service teachers with a negative view on programming

A logical next step in this analysis is to test whether any specific area of mathematics teaching stands out from the rest when it comes to the evaluation of the usefulness of programming.

Topic	Topic	Mean difference	Std. error	Sig.
Geometry	**Numbers, algebra, and functions**	-.58065*	.12356	<.001
	Problem-solving	-.45161*	.12356	0.02
	Multicultural classroom	-.58986*	.12356	<.001
	Differentiated teaching	-.36406*	.12356	0.27

Table 3. Post-hoc test between geometry and the other topics (* The mean difference is significant at the 0.05 level)

An ANOVA test was conducted with a result that there was a significant difference in the means for the different topics (F (4, 1080) = 7.6, $p < .001$). A post-hoc comparison using the Tukey HSD test shows that there are statistically significant differences between geometry and all other topics (table 3), and no significant differences between the remaining topics. The mean scores of pre-service teachers on the rank of the usefulness of programming in geometry were compared with the mean scores for each of the four other topics. The post-hoc comparison revealed that the mean score (of usefulness) for geometry were significantly lower than those for the remaining topics. This finding contrasted with the finding of Pörn et al. (2021) that teachers emphasize that programming is a valuable tool in geometry.

Discussion and conclusion

The present study aimed to investigate the views of pre-service math teachers on the usefulness of integrating programming into mathematics education. The results contradicted our main hypothesis that programming was viewed as more relevant to geometry and problem-solving than areas such as numbers, algebra and functions, multicultural classrooms, and differentiated teaching. Thus, the following two questions arise: (1) Why did so few respondents view programming as useful for teaching geometry? (2) Why did so many respondents perceive programming to be valuable for teaching mathematics in multicultural classrooms and differentiated teaching? The answers to these two questions could be related to the experiences of the respondents with programming. Only four participants answered that they have had some experience with programming in school, and nine participants (no overlap between these two groups) answered that they are programming during their leisure time; 6% of the 217 participants had some kind of experience with programming.

With regard to the first question, the literature and the curriculum emphasize the relationship between programming, problem-solving, and geometry. Programming is the ability to analyze, understand, and solve problems through an iterative process (Bråting & Kilhamn, 2021; Kaufmann & Stenseth, 2021), and similar processes, such as decomposing problems and applying algorithms, are involved in problem-solving. In addition, robotics and block programming use concepts such as geometrical figures and angles (Benton et al., 2017), and one of the main reasons for the popularity of some programming languages over others may be their visual nature, which may make programming more accessible to students compared with text-based languages (Grover & Pea, 2013). The development of the Scratch and Logo languages has allowed programming to be introduced as early as primary school and kindergarten. The answers of the participants might be affected by their lack of experience with programming. Students may also have practical and outdoor experiences with geometry in teaching and learning and, hence, find programming to be less useful.

A possible response to the second question is that the participants may view both programming and differentiated teaching as individual work. Thus, programming might support the right of students to differentiated teaching. This, however, conflicts with the core elements of the new curriculum and the importance placed on mathematical discussion (The Ministry of Education and Research, 2019). In addition to the study of Dinçer (2018), further information is necessary about why pre-service teachers had favorable views on programming in these aspects.

The limitation of this study is that we did not define programming in the survey nor ask the informants about their understanding of the concept of programming. Do the student teachers understand programming as related to everyday life? Or do they interpret it as computational thinking as described in the curriculum? This difference of perceptions may have influenced the results when asking them about their experience with programming in school and leisure.

Integrating programming into mathematics education is a political decision. Nonetheless, a comprehensive discussion on the required competencies for mathematics teachers is still essential to decide how best to integrate programming into the pre-service teacher education curricula. Programming is generally linked to mathematical thinking, and there has been a tendency in several European countries to include programming in their respective curricula to develop computational thinking (Grover & Pea, 2013). There is little doubt that programming skills are important and that their importance will increase in the future. Further discussion is required on the concepts of mathematical thinking, algorithmic thinking, and computational thinking in relation to programming. Nevertheless, our hypothesis from the present study is that in- and pre-service teachers may need additional competence and experience in programming itself, in programming in the context of mathematics, and in teaching programming. Therefore, learning programming for its own sake is insufficient. Teachers must learn how programming is connected to mathematics and teaching and must gain experience in how programming can be applied to different mathematical topics as well as in the context of various student populations. Further research is, thus, required to explore the connection between mathematics and programming (Kilhamn et al., 2021; Pörn et al., 2021) as teachers need more information on how programming can advance students' understanding of mathematics.

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”Where is my angle?” – students cooperating to make a square

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This paper investigates how students collaborate to make a square using micro:bit as a tool. By analysing data from 17 groups of four students in each group, from fifth and sixth grade, this study aimed to understand and identify students' collaborative computational thinking. A teaching experiment methodology was used to explore and explain students' collaborative actions. In our analysis, we identified themes related to collaborative computational thinking. The results show that the level of success of making a square depended on two central components. The first is the use of verbal interactions and how they manage to identify their part of the square (”where is my angle”), which is central for understanding the modularity of the task. The second is whether or not the group systematically checked their work to identify efficient improvements (debugging).

Keywords: computational thinking, collaborative problem solving, geometry, micro:bit

Introduction

The use of digital technology can help increase collaboration and bring more of an emphasis on practical applications for mathematics, through modelling, visualisation, manipulation and more complex scenarios (Bray & Tangney, 2017). The availability of technology will not in its own ensure the development of a collaborative and explorative classroom (Geiger et al., 2010). The challenge teachers face is not only how to practically use digital devices but also how to implement and use digital technology in a teaching and learning context (Engen, 2019). According to Hoyles (2018), the integration of digital technology has the potential to transform teaching and learning practices in mathematics classrooms in the way that students get more ownership of their learning and engage more in the classroom. However, she raises two problems regarding the use of digital technology in the classroom. Firstly, teachers and students have to learn how to use the new digital technology. Secondly, students might use digital technology to avoid ”mathematical thinking”; instead of learning mathematics, students use the digital technology only to get the job done more easily (Hoyles, 2018). To handle these problems, the use of technology in mathematics should be integrated in situations that could not have been completed without it (Hoyles, 2018), which was the idea behind our experiment where the students were dependent on micro:bit to collaborate and solve the problem (see also the chapter about micro:bit). In their systematic literature review on the usage of technology in mathematics education research, Bray and Tangney (2017) call for more research on the kind of mathematical knowledge and practices that might emerge through the use of digital technology in an inquiry-based approach. The aim of our research is to examine how micro:bit could be integrated in middle-school mathematics to solve a mathematical problem. We wanted to gain insight into if and how the use of micro:bits can support collaborative learning. In this context, we pose the following research question: What kind of collaborative computational thinking strategies can be identified when students use separate micro:bits in mathematical problem solving?

Relevant research

The research background will be presented in two parts. The first part focuses on computational thinking (CT). In the second part, we present the role of collaborative problem solving in relation to CT.

Computational thinking

Recently, there has been growing recognition of the importance of CT not only in the field of computer science but in all disciplines (Atmatzidou & Demetriadis, 2016). CT can be understood as the thought processes involved in formulating a problem and expressing its solution(s) in such a manner that a computer—human or machine—can effectively carry it out (Wing, 2006). Berland and Wilensky (2015) defined CT as ”the ability to think with the computer-as-tool” (p. 630). This definition suits well according to our experiment where students should try to make a square using individual micro:bits (see the descriptions of the experiment in the methodology chapter). Further, many agree that a CT framework should include (Lavigne et al., 2020): Algorithmic thinking, which is creating a set of ordered steps and then performing them in a particular order to accomplish a task in a way that could be repeated by others (i.e. an algorithm); Modularity, which is breaking down problems into smaller modules and then identifying opportunities to adapt these parts to attend to the larger problem; Debugging, which is noticing when a solution is not working the way it was intended, reflecting pattern recognition, pattern generalisation and abstraction. Hoyles and Noss (2015) sought to operationalise CT and its implications for mathematics learning. They defined CT as entailing abstraction (examining a problem at different levels of detail), algorithmic thinking (considering tasks in smaller, connected, discrete steps), decomposition (solving a problem involves solving a set of smaller problems) and pattern recognition (considering a new problem as related to problems previously encountered). This implies that CT involves concepts and practices primarily from computer science, which are shared across other disciplines such as mathematics (Kafai & Burke, 2013).

The role of collaborative problem solving in computational thinking

A lot of research has focused on the individual student in his/her interaction with a computer. There has been less focus on how collaborating students develop their CT working together with a computer or digital tool (Forsström & Kaufmann, 2018). When students engage in collaborative problem solving, they create and maintain knowledge to help them make sense of CT. How the problem is solved is important in the process of problem solving, with active learners engaging and generating thoughts and ideas in the process. Collaborative problem solving can be seen as a method in CT by which students work together in small groups scaffolding each other, while working towards achieving a common goal (Albert & Kim, 2013). In our project the students collaborated in groups of four, each with their micro:bit, to make a square. All participants depended on each other to solve the problem. If they were not coordinated, the group were not able to come to a solution. Through both verbal and non-verbal interactions that occurred in their collaborative problem solving (such as speaking, listening, pointing and pushing the buttons on the micro:bit), students were able to solve the given mathematical problem.

Methodology

We will begin with giving a description of the experiment. Then we will present our rationale for using micro:bit, before we describe the context and analytical focus.

The experiment

The purpose of this article is to investigate cooperation and strategies when a group of students is trying to solve a common problem. Each member of the group is given a tool for solving a part of the problem. This is relevant in an approach to cooperation in problem solving in general, and it is relevant when we look into CT and programming.

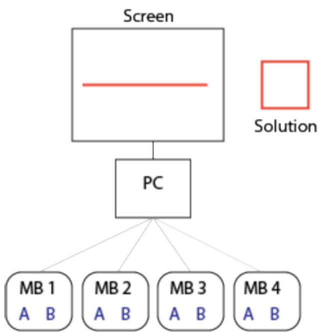

Figure 1. The connection between the components using micro:bit

Each participant has a micro:bit (abbreviated as MB in Figure 1), and they are all located in front of the same screen. The micro:bits are connected to the PC with radio. The screen displays what seems to be a continuous line, and the task is to transform the line into a square using the micro:bit's two buttons, A and B. The basic functionality is that the line is made up by the four sides of a potential square. Each click will change one of the angles that connect two sides, or the direction of the first side. The four angles are changed +/- 5 degrees on each click, (A or B).

How can a group develop an efficient strategy? How do they describe it, how do they communicate it and how do they implement it? The setup with four clickers has, before this controlled experiment, been tested in various settings, for participants in all ages and with different background. The common reactions are interesting and reveal attempts and patterns of different strategies, different roles and usually a fascinating discussion. In this experiment, we have documented the cooperation in all groups on video and analysed the videos. All clicks are logged. We have been focusing on the communication within the group, and we have documented and analysed the numbers of clicks and the time used. Examining the problem from an algorithmic point of view, we can see that we have decomposed the algorithm, defined four components and implemented those as four functions. Each function is owned by one group member, and we challenge them as a group to implement a complete algorithm that use these four basic components. If we look at the problem from outside it is fairly easy to find good and efficient strategies. The theoretically optimal strategy is to leave the first angle untouched and click the three other angles consistently in the same direction until they all are 90 degrees. The sequence is not important. When each click changes an angle 5 degrees, this strategy will involve 3*90/5=54 clicks. This is, of course, a non-reachable practical result for any of the groups. They have to start with a ”what happens when I click” approach, and the form of the geometrical shape is normally changed in a non-systematic way for a number of clicks. The problem cannot be solved without the four students cooperating, because each student has full control of his/her micro:bit, i.e. algorithmic component. The assignment can also be categorised as a problem-solving assignment. The students do not have a clear strategy on how they can solve the assignment when they start.

Micro:bit

Micro:bit is a programming platform developed in England (BBC) and has become very popular to introduce primary school students to programming in many countries. In addition, in Norway, micro:bit is pointed out as an important tool for introducing programming, and the distribution of the equipment has been done in connection with the regional science centres, carrying out training of students in primary schools. Additionally, the training program offers teacher training courses and support materials freely available online so that everything is in place for good and interdisciplinary use for many years. Further, we will emphasise that programming the micro:bit was not a part of this experiment. It is, however, extremely simple, and a setup of this type could easily be programmed by the students in an introduction to programming. This experiment is an example where the use of micro:bit as a tool plays an important role in decomposing the problem and connecting the components. Due to the dynamic properties of the line segments that are connected it will not be possible to solve this problem by pen and paper. In addition, this is significant because the students must identify their angle to solve the problem, which would have been impossible by pen and paper.

Context

Data were video recordings of 17 groups of students (total 68 students) from fifth and sixth grade two different schools. The teacher set up the groups. The students were not familiar with micro:bit in a school context. They worked with the problem in groups of four. All groups were given sufficient time to complete the assignment. The 17 groups were divided in two based on the conditions they were given to solve the problem. Thirteen of the groups received a very brief verbal introduction given from the authors of this paper: ”You all have one micro:bit with two buttons. Use these buttons to make a square”. Four groups were randomly chosen to observe the group that was working with this problem. Immediately after, they had to solve the task themselves. For practical reasons, there were not as many in both of these abovementioned categories. The reason why this division was chosen was because we were investigating whether they used other strategies and whether the collaboration was different for those who had observed a group solve the problem first. The focus in this paper is on collaborative problem solving and not so much on differences between these groups, although we have observed that in average the four groups that were allowed to observe first used shorter time and fewer clicks.

Analysis

A teaching experiment methodology was used to explore and explain students’ actions through collaboration using micro:bit about making a square. According to Steffe and Thompson (2000), a primary purpose for using teaching experiment methodology is for researchers to experience students’ learning and reasoning. Our data were videotaped, showing groups working with the given problem: making a square with micro:bit. We used thematic analysis procedures (Braun & Clarke, 2006) to analyse our data. Thematic analysis is proposed as a flexible method for identifying, analysing and reporting patterns like, for instance, themes within data (Braun & Clarke, 2006). Our analysis led us to identify a number of themes related to collaborative CT. Based on the foregoing review, we have operationalised CT in our project as follows. To effectively use micro:bit to solve the problem, students must understand the nature of order and sequences (modularity). To generate solutions for the problem using modularity, students must understand that the problem can be decomposed into smaller, independent parts (problem decomposition). To fix or improve a final solution (for instance students were making a rhombus instead of a square), students must be able to move backwards to find an error or systematically check their work to identify the room for improvement (debugging). To solve the problem the students must cooperate (collaborative problem solving) using verbal (i.e. the use of mathematical concepts) and non-verbal (using the micro:bit buttons or pointing at the figure) interactions.

Results

The 17 groups were divided into two categories:

Category I, 13 groups	These groups were given a very short verbal introduction.	
Time used	average: 342 s	min: 63 s max: 670 s
Clicks used	average: 509	min: 87 max: 1067

Category II, four groups	These groups had observed another group before they started.	
Time used	average: 156 s	min: 55 s max: 250 s
Clicks used	average: 235	min: 146 max: 347

Table 1. Average time used and clicks used

The groups that had observed another group seems to have an advantage in their work. It is noteworthy that one of the groups in category I solved the problem with 87 clicks, which are only 33 clicks above the theoretical minimum of 3*18=54.

Sequences

We can identify four different steps (problem decomposition) in the work towards a solution:

AB: Identify the ownership of the Microbit and the direction of the rotation

BD: Close the figure to a square.

BC: Close the figure to a rhombus

CD: Correct the rhombus to a square

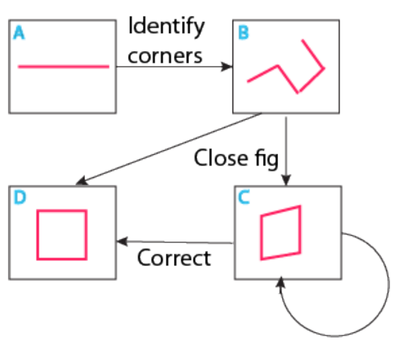

Figure 2. Phases in the solution

Several groups did create a rhombus during their solution procedures. Seven of the groups solved the task without making a rhombus, whereas 10 did. Five of 13 groups in category I did not make a rhombus, whereas the number in category II is two out of four. Many groups were in situation C (rhombus) a number of times during their process of creating a square.

It is interesting to ask why the step CD caused so much trouble for the groups that needed this final correction. If we apply a mind-set of correction or debugging, we see that step BD has a relatively clear dominant goal: close the construction. That means the clicks that bring the construction closer to being closed are welcome and considered a step in the right direction. However, the concept of a right angle is less controlled. When we look at step CD, this is more complicated. Any click will destroy the closed construction, and a successful strategy must be part of a more sophisticated plan, involving more than one click. The observations of the videos and the analysis of click sequences reveal that this is not expressed verbally by the students. It seems that the students do not have a terminology that makes it possible to express a strategy involving more than the next click. The expression ”right angle” is not used, even if the students identify C, the rhombus, as a wrong or uncompleted solution.

Typical paths from the groups’ working processes

We will take a closer look at three of the groups that typically represent the different steps in Figure 2. We will give a description of one group that went through the steps ABD; then, another group that went through the steps ABCD; finally, a third group that went from ABC (and then back to C several times) before ending up with D.

The fastest group, which belongs to category II, is an example of a group going straight to a square (ABD). This group did relatively quickly identify their ownership of the micro:bit. After 25 seconds, one student says ”it's me” while another immediately responds ”no it's me”, identifying their own component of the micro:bit. Further, the different parts of the line were also rotating consistently in the same direction, and the group closed the figure to a square (BD) directly. It should also be noted that this group’s ”baseline” (they left the first angle almost untouched) was most of the time in its horizontal position and nearly did not move. This strategy is close to the optimal strategy described earlier in the experiment chapter. The group did cooperate well, and the students seemed to identify the task and their individual role after a short time. They completed the task according to a strategy that is close to optimal.

Another path is ending up with a rhombus before correcting the figure to a square. A representative group with these steps is a group from category II. They start out by identifying their ownership of the micro:bit, in that one student say before they start clicking that they have to find out ”who is who”. They quickly manage to make a rhombus (after 59 seconds), and they recognise this figure, i.e. one student says ”it may be a bit skewed”. Then the group makes relatively small movements to close the gap in one corner and to make right angles, rotating the figure; at the same time, the corners are closing. They spend more time correcting the skewed figure, and finish their square after 2 minutes and 20 seconds. This group seems to identify what role each member plays after a short time and makes a rhombus fairly quickly. The debugging process to make a square takes a little more time and seems more troublesome for the students.

Interestingly, eight of the ten groups who first made a rhombus also made this figure two several times before they managed to create a square. The next example illustrates such a path from a group that ended up with a rhombus several times before they managed to make a square. This group used almost ten minutes to finish the task; however, already after 1 minute and 37 seconds, they had made a rhombus. When they tried to correct the solution, they used relatively small movements, in that one of the corners opens and closes while another rotates the figure. During their solution process, they end up with a rhombus eight times. It does not seem that they have any strategies for correcting the skewed solution. This may be because they have ”lost” the ownership of the micro:bit and the direction of the rotation, i.e. after about four minutes one student says, ”maybe it's me”, and later the same student says, ”maybe it's you”. In summation, this group seems to get lost in the debugging process; some of the students in the group lost the track after almost finishing the task several times.

Discussion

In this experiment, we investigated how a group of students with micro:bit collaborated to create a square. There were two issues in particular in this solution process that were clear. The first issue concerns problems in modularity (Hoyles & Noss, 2015) based on the lack of terminology in using geometric concepts. The first step, closing the figure, was relatively easy for all the groups. The groups that had made a rhombus recognised the shape and realised that it was not a square. In the subsequent step when they had to modify the figure to a square, they had problems because they could not communicate a strategy that involved a series of clicks. They did not use the concept of ”angle”, and none of the groups communicated that the angle must be 90 degrees. Observing a property like ”right angle” is easy, but in this experiment we found that the use of the concept in construction was more difficult.

The second issue is debugging. With debugging, the students are trying new solutions, and they often are unable to gather relevant data or use systematic processes that address a specific problem (Lavigne et al., 2020). Rather than being systematic, many of the groups tend to use trial-and-error approach to correct problems as they notice them; further, they tend to judge the success of a solution through observation rather than through reasoning. The effect of this is negative when they are in the process of correcting a rhombus because each click opens the construction and seems to bring the group further from a solution.

Our conclusion is that most of the groups did not succeed with their collaborative problem solving because of the lack of communication in formulating hypothesis in the debugging process. Instead, individual trial-and-error attempts dominated the activity.

Based on earlier studies, debugging strategies can be more efficient when the teacher interacts with the students in the problem-solving process (Forsström, 2019). If the purpose of the experiment had been to teach geometric concepts, we would have involved the teacher, especially to help the group to focus on the geometrical terminology in their communication. Further research should focus on how collegial discussion must be supported to achieve a better strategy and algorithmic reasoning.

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Professional development program in Iceland based on the Swedish program *Matematiklyftet*

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Teachers need to have access to diverse professional development. This action-research study reports on a progress of professional developmental program with the aim of responding to teachers' need for strengthening their mathematics teaching competences. The program is based on the authors' experience as mathematics educators and "Matematiklyftet". A pilot study was conducted in 2017–2018, and the three subsequent years we proceeded with two yearly modules with leaders from 20–45 schools. Data consists of the authors' notes during the process, retrospective data, and recordings from workshops and interviews with leaders and teachers in their schools. The results indicate that cyclic reflection on former experience and the developmental process has improved the structure and the quality of the course. The findings add to the growing field of knowledge about mathematics teaching and professional development within the Nordic countries.

Keywords: Professional development, learning communities, teacher educators learning.

Purpose of the research

Over the last two decades, Icelandic teachers have been under growing pressure to adapt their work to changes in new curriculum guidelines and laws for schools. Teachers are expected to meet the needs of diverse groups of children, improve their teaching competence, and their use of information and communication technology (ICT). This paper reports on findings from an ongoing study of a professional developmental program for mathematics teachers in compulsory schools, offered by the University of Iceland and the University of Akureyri. The goal with the program is to support learning communities within schools in response to the need for improved mathematics teaching.

The program is based on theories of teachers' professional development and mathematics teaching and learning, the authors' experience as mathematics educators, and *Matematiklyftet* from Skolverket Sweden. Through our experience as teachers, teacher educators and curriculum writers we have gained knowledge of mathematics teaching and learning in schools. It has supported us in structuring a developmental project with teachers as leaders of mathematical learning communities within their schools. Through collaboration with Nordic mathematics educators, we became acquainted with *Matematiklyftet.* We found that this professional developmental program for all mathematics teachers in Sweden resonated with our goals. Skolverket (The Swedish National Agency for Education) in close collaboration with NCM (Nationelt Centrum för Matematikutbildning) was responsible for the implementation. The universities and teachers' colleges participated in the building of the program, and the content of a web-based platform. The aim was to boost the quality of mathematics teaching and learning throughout the country by making a professional development program accessible for all mathematics teachers. It was based on the idea that if teachers strengthened their competences, they could improve the learning culture within their mathematics classrooms. The fundament was the building of collegial learning communities with support from experts and school leaders.

The goal with studying our program is to gain knowledge that guides us in developing opportunities for mathematics teachers to participate in learning communities with collegial support. The aim is that they will improve their practice as mathematics teachers. Our research question is:

- How has the professional developmental program for leaders of learning communities of mathematics teaching and learning, progressed?

To answer the question, we focus on in what way the authors former experience as mathematics educators, the material from *Matematiklyftet,* and the participants reaction and feedback guides the project. Data were collected from teacher educators' notes, program descriptions, recordings from workshops and interviews with leaders and teachers in their schools.

Theoretical background

In our former studies with teachers, we learned that partnership between teachers and teacher educators, where the knowledge both parties bring into the project is mutually respected, can add to our understanding of teacher development. Teachers need opportunities to develop and enhance their knowledge about teaching and learning in an environment that reflects the very same aspects they are expected to foster in their schools. If teachers are given opportunities to collaboratively investigate mathematics and solve mathematical problems, they discover how the different experiences can contribute to their own understanding of the mathematics involved, as well as how individuals learn mathematics (Gunnarsdóttir et al., 2013; Kristinsdóttir, 2017).

We base our work on theories of (i) professional development and (ii) learning communities in mathematics education. In professional development projects, teachers are encouraged to analyse and deal with factors in their own working environment both the learning environment in the classroom and the role of the teacher in a wider sense (Darling-Hammond et al., 2009; Loucks-Horsley et al., 2010). Research indicates that learning communities can play an important role in supporting teachers on their continuous path to improve their teaching and sustain their professional knowledge (Darling-Hammond et al., 2009; Loucks-Horsley et al., 2010).

- Learning communities, where teachers share their understanding of the nature of good teaching and work together at improving their practice, seem to create fruitful conditions for teachers' professional development (Borko et al., 2011; Desimone, 2009).
- In learning communities, competences of its members can develop in several regards such as professional communication and collaboration (Loucks-Horsley et al., 2010).

In a Lesson-Study project in Iceland (Gunnarsdóttir & Pálsdóttir, 2019) learning communities were created that built on trust and the teachers were not afraid to voice their ideas and concerns. The teachers were not used to discuss mathematics and mathematics teaching and did not seem to have the vocabulary to talk about mathematics teaching and pupils learning in detail. In a study of the characteristics of teacher learning communities in Iceland, similar results were found. Teachers were enthusiastic to talk about their work, but their discussions were more descriptive than analytic or reflective (Gísladóttir et al., 2019). Findings from the evaluation of *Matematiklyftet* revealed that participation in the project helped individual teachers to develop their teaching but a collegial support and development of learning communities within schools was less visible. The concern was raised that the effect of the participation in the project would not be sustainable (Österholm et al., 2016).

Teachers have opportunities for professional learning both within formal professional development settings such as courses and in-service days, and informal settings as common planning and discussions of lessons, self-reflection and reading of professional journals (Borko et al. 2011; Desimone, 2009). Darling-Hammond et al. (2009) define effective professional development as development that leads to improved knowledge and instruction by the teachers and improved student learning. Darling-Hammond and colleagues presented four main principles for designing professional learning based on Darling-Hammond's work. They proposed that professional development needs to:

- be intensive, ongoing, and connected to practice,
- focus on student learning and address the teaching of specific curriculum content,
- align with school improvement priorities and goals,
- build strong working relationship among teachers (Darling-Hammond et al., 2009).

Darling-Hammond and colleagues emphasize that intensive professional development rooted in practice is likely to lead to changed teaching practices and to increased student learning. They focus on building a shared vision in the school culture. Loucks-Horsley et al. (2010) have analysed research findings about high quality professional development. Their results are in congruence with the findings of Darling-Hammond et al. (2009) about the importance of learning communities and close connection to practice that include possibilities for teachers to deepen their professional expertise.

In 2014, Gunnarsdóttir, studied the professional development opportunities for in-service mathematics teachers in Iceland. Her analysis was based on five main features put forth by Desimone (2009) about effective professional development; focus on content, active learning, coherence, duration, and collective participation. She found that most of the courses had a strong content focus, both on mathematical knowledge and pedagogical content knowledge. Active learning and collective participation were emphasized and teachers from the same school took part in collective learning. The courses were planned for only few hours and diverse content and teaching approach covered in a short time. There was a lack of coherence and time for teachers to gain deep understanding in mathematics teaching and learning, and progress in their work in the mathematics classroom. She suggested:

> It is therefore also important that the universities design courses that meet the needs of teachers to collaborate with other practicing teachers and create learning communities. (Gunnarsdóttir, 2014, p. 169)

We found that the features building learning communities, coherence, duration, and connection to practice are inherent in *Matematiklyftet*. Our decision to build the professional developmental program on *Matematiklyftet* and adapt it to our conditions is based on our analysis of these core features.

Methods

Data are collected from teacher educators' notes, program descriptions, recordings from workshops, interviews with leaders and teachers in their schools and notes from meetings in schools.

Throughout the project we, the teacher educators, write memos to help us focus on important features in the development of the program. We refer to our previous developmental projects with teachers and what we have learned from them. We also discuss our experience from previous sessions and how our observations help to progress our work with the leaders. Through this process we are developing our own way of supporting the leaders and adapting *Matematiklyftet* to our conditions. By collaboratively recalling experience from previous sessions, we try to prevent missing out relevant information that can support us in developing the program (Saldana, 2009). This cyclic way of working with data from a longitudinal project helps in improving the project.

Findings

At the initial phase of the project, we decided to start with a pilot group, based at the university sites in Reykjavík and Akureyri collaborating and teaching via the internet. The following years we drew conclusions from the pilot study based on our experience for developing our further work.

Year 1: The pilot-project, *Mathematics teachers learning communities with support from mathematics teacher leaders*, was carried out in 2017–2018 with eight schools, four at each university site. We contacted heads of schools, who we knew had interest in professional development for their mathematics teachers, and offered them to participate. In each school one or two teachers were chosen to participate in a course for leaders, and they then would work with their co-teachers. We started with the module *Relation and change* (Samband och förändring, åk 4–6). Our choice was based on the didactical approach of the module. We studied each part within the module and concluded to focus on open tasks and communication, both in group work and whole class discussions. Working with open tasks leads the teachers' attention to communication in the classroom. From former work with teachers, we had learned that teachers need support in creating powerful communication culture in their classrooms. We therefore found it important for the leaders to strengthen their competence as mathematics teachers by taking part in workshops, read texts and discuss their experience of working with open tasks in a group. Equally important it was to have a space to learn about leadership and discuss how to take on the role of a leader in their schools.

We translated texts from Matematiklyftet into Icelandic. We also used texts in English that we had used within the teacher education program that resemble the content of the module. We met the leaders seven times for three hours from August to May. At the first two meetings we focused on theories and research on mathematics teaching and learning and the role of a leader in a learning community. At the next four meetings emphasis was on one part of the module, and discussions and reflections on the experiences made. At the last meeting, the focus was on the teachers' learning, the learning communities within each school, and future development.

Based on our experience from the pilot year we planned courses for the following year. The leaders in the pilot group showed interest to proceed within the project and establish this way of professional development within their schools. They found that progress was made within their groups in teaching mathematics and discussing their own teaching. They felt that a learning community was beginning to be established and it was essential to advance with another module. To be in the role of a leader within this project gave the participants access to the teachers' time and possibilities to participate in the teachers' preparation for mathematics classes as expressed by Anna, a pilot leader:

> It opened ways for me to guide others in an encouraging way that led to reflection and discussions among colleagues.

Year 2: In collaboration with the leaders in the pilot group we decided to work on the module *Mathematics and ICT* (Matematik och digitala verktyg I). The main reason for choosing this module was that ICT is an effective tool to use in all mathematical topics and the schools in Iceland are gradually using ICT more in their teaching. The leaders from the pilot group believed that their colleagues would appreciate support in that field. We decided to focus on the teachers' use of ICT as a tool to present mathematical ideas, analysing computer programs and apps, and the use of ICT as a learning tool for pupils. ICT tools improve rapidly, and teachers need to be able to analyse programs and apps and try new tools. The pilot leaders felt that participation in a learning community on this topic could be the support teachers needed to be able to expand their use of ICT in mathematics teaching. Jóna, a pilot leader, said:

> We need more consensus in the mathematics teaching at the school and fruitful discussions around mathematical concepts and ICT. The teachers appreciate when I bring new ideas into the learning community within my school.

The leaders saw the participation in the course as a possibility to lead the development of the mathematics teaching in their school.

We decided to run the course *Relation and change* (grades 4–6) for another year and offer it to all schools in Iceland. Leaders from 22 schools throughout the country took part in the course, some through real-time participation via internet. We ran the courses simultaneously at each university using the same program, preparing together, and supporting each other teaching. During the pilot year we discovered that the teachers preferred to read texts in Icelandic, and it facilitated their discussions around the topics. The leaders confirmed our analysis of the importance of having the material in Icelandic. We used the Swedish videos and added some videos from Icelandic classrooms taken by the pilot leaders.

Experience from this year revealed that support from principals was essential for making space in the schools for teachers to work with the modules and for the leaders to become leaders. Some of the leaders experienced problems with getting all their co-teachers to participate. This resulted in those leaders mainly focusing on the teaching in their own classes. In other schools, monthly meetings were scheduled from the beginning and there, learning communities were established. There seemed to be a common understanding that focus on preparation and deliberation of approaches to mathematics teaching and learning is needed. The leaders concluded that the didactical approach aligned with their visions for mathematics teaching and the competence criteria for mathematics in the national curriculum. In some municipalities, emphasis has been on ICT and teachers are encouraged to strengthen their ICT competences and use as a tool in their classrooms.

Year 3: We ran two courses *Relation and change* (grades 1–4) and *Mathematics and ICT (*grades 4–6). At the University of Akureyri, the pilot teachers met with a teacher educator every month to sustain their learning community and support each other. At the University of Iceland, three teachers from the pilot group became a part of the teaching team. Their experience of participating in two programs was valuable in the development to this approach to professional development.

We focused on the leaders' competence in exploring with mathematics. In an interview by the end of the third year, Sunna expressed:

> It was helpful to work with problems at the course, try ourselves and then we discuss them with our colleagues at the school. I think I gained most from that.

We experienced difficulties with using the ICT module. Parts of the texts were outdated, and many computer programs and apps recommended were not functioning and much effort was needed to find quality replacement. Consequently, focus partly moved from the mathematics to the use of ICT in general. The use of GeoGebra resources were appreciated as the leader Kristín said:

> The teachers appreciate the support. They are pleased, like with the GeoGebra web, it is a welcomed resource.

Year 4: The fourth year is now running and based on our analysis of the three first years we chose to work with language in mathematics (Matematik och språk) for teachers in grades 1–7. The experience from the first three years confirmed our gained insight into teachers' lack of experience in discussing mathematics, both amongst themselves and with their pupils. This year, financial support has enabled us to work with 45 schools all over the country. We work closely with leaders in four schools, participate in meetings with their co-teachers and learn about the learning communities developing within their schools. When they share their work with their pupils with us, we gain insights into how their didactical approach is progressing. Two quotes from the leaders Lóa and Björk raise their views:

> You don't do any major changes unless you have something concrete to work with, some support.

> The course for leaders gives me a solid ground for becoming a leader with peers.

We have managed to foster discussions about mathematics teaching and learning. Whether the learning communities the leaders have built with their co-teachers are sustainable, the future will tell.

Discussion

This program for leaders in mathematics teaching is a fruitful experience both for the participants and us the teacher educators. Conditions for teachers' professional development have been created and sustained by emphasising learning communities, coherence, duration, and connection to practice (Borko et al., 2011; Desimone, 2009). According to Darling-Hammond et al. (2009), professional development should be intense, ongoing, and connected to practice. In all the courses the leaders were expected to develop what they learned and experiment with their co-teachers. They welcomed the new way of working with teaching mathematics and collaborating with colleagues. The repeated cyclic experience was vital for the development of their role as leaders. The leaders valued the support for becoming a leader in a learning community.

The structure of *Matematiklyftet* is based on a strong connection between theory and practice and we have emphasised this focus in our courses. The course material supports the participants in discussing mathematics teaching. It opens for enhancing their vocabulary and understanding of mathematical concepts and didactical approach. By focusing on the preparation phase with individual reading, discussions about the texts and collegial collaboration in making a lesson plan, the teachers are supported in connecting theory and praxis. Research in Iceland on learning communities has shown that Icelandic teachers need support in deepening their discussions and strengthen their analytical and reasoning competences (Gísladóttir et al., 2019; Gunnarsdóttir & Pálsdóttir, 2019; Kristinsdóttir, 2017). The same seems to be the situation in Sweden with building sustainable and rich learning communities in the schools (Österholm et al., 2016).

The adaptation of *Matematiklyftet* was made in close collaboration between the authors. All of them have worked with mathematics education and professional development in Iceland for many years. It was valuable to have access to *Matematiklyftet* and choose what we found suitable in Icelandic context. The collaboration with the pilot leaders was valuable in designing our program. They gave insight into what could work and what was needed to boost the mathematics teaching in schools. This knowledge is essential in creating realistic opportunities for teachers to help them deal with factors in their own environment (Loucks-Horsley et al., 2010; Darling-Hammond et al., 2009).

Emphasis was put on building learning communities within the program (Desimone, 2009). The leaders reported that they built relationships with each other and found ways to communicate on collegial basis as found by Loucks-Horsley et al. (2010). The opportunity to work closely with four schools gave us valuable insight into the leaders work with their co-teachers, and how learning communities developed within their schools. The leaders have strengthened their competence as facilitators of professional development and their co-teachers have improved their instructional practices by focusing on classroom discussions about mathematical content.

Conclusions

The experience gained from carrying out this adapted version of Matematiklyftet was rich and rewarding. We gained knowledge about the leaders' schools and their work with their co-teachers, their opportunities to progress and the limitations in developing effective ways of teaching mathematics. In addition to basing the courses on *Matematiklyftet* and our former developmental work with teachers, the program was built on results from research about professional development and learning communities.

We found that we succeeded in building a learning community with the leaders and give them substance to develop their role as facilitators of development in mathematics teaching and learning within their own schools. The motivation for starting this developmental program was our vision for rich learning culture in all mathematics classrooms. We had little financial support the first three years. Being true to our belief that partnership between teachers in schools and teacher educators is essential in developmental programs has raised the interest of educational policy makers in our work. We have got financial support that enables us to expand the program to develop courses for pre-, primary- and secondary schools. It also enables us to publish courses on a web-based platform for teachers to use within their mathematics learning communities. Our close collaboration with teachers in four school this year gives us insight into their work that enables us to structure the program further.

For our future planning of professional development programs, we will highlight the importance of collaboration between program facilitators and teachers as well as principals who play an important role in making learning communities a resource for professional development. Reporting on our adaptation of the Swedish program is a contribution to the Nordic mathematics education.

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This study is supported by *The Ministry of Education and Culture* and *The City of Reykjavík.*

Student teachers' beliefs shaped by history of mathematics: the case of the construction problems of antiquity and Euclidean geometry

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The purpose of this study is to uncover two categories of Norwegian student teachers' beliefs that are cultivated during a Historical and Philosophical Aspects of Mathematics course, particularly by working with the construction problems of antiquity and Euclidean geometry. The first category concerns beliefs about the nature of mathematics and the second beliefs about the use of history in mathematics teaching. Awareness of these beliefs can be useful in the design and implementation of a course that aims to utilize the full potential of history of mathematics in mathematics teacher education programs.

Keywords: beliefs, nature of mathematics, history of mathematics, teacher education.

Introduction

There are several arguments for employing the history of mathematics in teachers' training programs. Among those, we distinguish two that are in focus in our study. The first one is that the history of mathematics can contribute to developing views about the nature of mathematics and mathematical activity (Tzanakis & Arcavi, 2000). History may help students understand how mathematics has evolved in time through inquiry and mistakes and reveal its tight connection to society and culture. The second one is that it can enrich their didactical repertoire by offering a number of different explanations, examples, problems, and approaches (Tzanakis & Arcavi, 2000). Thus, the history of mathematics can be used as a tool towards the achievement of two of the goals of mathematics teachers education in Norway: a) to develop the students' knowledge of mathematics and their attitudes towards it; and b) to develop their ability to teach mathematics (Christensen & Nordberg, 2007, as cited in Smestad, 2011). Beliefs about the nature of mathematics and mathematical activity are related to the first goal as being part of the knowledge of mathematics[10] and the use of history in teaching mathematics can be linked to the second.

Besides the increased interest regarding the integration of the history of mathematics in mathematics education, most of the contributions remain still on a theoretical level. In addition, few of them have teacher education in focus (Clark, 2014). Our study can contribute to increasing the volume of empirical studies related to the use of the history of mathematics in teacher education. We provide part of the preliminary results after the first year of a pilot study which had as a goal to understand the learning of prospective mathematics teachers (PMTs) of grades 5 to 10 in the Norwegian school system gained by a Historical and Philosophical Aspects of Mathematics (HPAM) course. Furinghetti (2007) points out that the method history of mathematics is integrated into teachers' training programs accounts for the different outcomes documented in research. It is reasonable to believe that different topics from the history of mathematics can also produce different outcomes. Our focus is on Greek mathematics, particularly on the three famous construction problems of antiquity, the quadrature of

[10] Like Phillip (2007), we consider knowledge as beliefs held with certainty or justified true belief.

the circle, the duplication of the cube and the trisection of an angle, and Euclidean geometry. Thus, our research questions are: 1) How does working with the construction problems of antiquity and Euclidean geometry contribute to PMTs' beliefs about the nature of mathematics and mathematical activity? 2) What are the PMTs' beliefs about the use of the history of construction problems and Euclidean geometry in teaching?

Theoretical background

Teachers' beliefs about mathematics and the impact those might have on their teaching practice have been the focus of several studies since the '80s (Thompson, 1992). In this study, we think of beliefs as "psychologically held understandings, premises or propositions about the world that are thought to be true" (Philipp, 2007, p. 259). A teacher's beliefs about the nature of mathematics form the basis of the teacher's view of mathematics as a whole. Based both on empirical studies and advances in the philosophy of mathematics as a field, Ernest (1989) distinguished three categories of views mathematics teachers might hold. Two of them consider mathematics as a static body of knowledge to be learned: the Platonist view where mathematical knowledge is structured and unified and the instrumentalist view where mathematics is a collection of unrelated tools. The third one is the dynamic problem-driven view of mathematics as a continuously expanding field of human inquiry (Ernest, 1989). The underlying philosophy of mathematics in the problem-driven view is the one proposed by Hersh (1994). Central ideas in this philosophical trend are that mathematical objects are cultural-historical-social objects, and that mathematical knowledge is fallible. Mathematics is historical because history has played an important role in the evolution of the field, social as mathematicians are working organized in institutions and their products are shared ideas, and cultural in the sense that it originates and develops according to the pressures of society. Mathematical knowledge is fallible in that mathematics can advance by making mistakes and correcting them. The fallibilist view comes in contrast to the absolutist view of mathematics which remained the dominant epistemological perspective of mathematics until the beginning of the twentieth century and according to which "mathematical knowledge consists of certain and unchallengeable truths" (Ernest, 1991, p. 7). The main representative of this view is formalism which considers mathematics as a "meaningless formal game played with marks on paper following rules" (Ernest, 1991, p. 10).

There is a growing body of literature surrounding the role of history in teaching mathematics (Clark, 2014). A systematic review of the arguments for using history and ways that this can be accomplished is provided by Tzanakis and Arcavi (2000). In their classification they divide the arguments for using history into five categories: arguments related a) to the learning of mathematics, b) the development of views on mathematics and mathematical activity, c) the didactical background of the teachers and their pedagogical repertoire, d) the affective predisposition towards mathematics, and e) the appreciation of mathematics as a cultural-human endeavor. A different approach is offered by Jankvist (2009). Jankvist observes that the whys and hows of using history often get mixed in the literature, making it difficult to clarify where the focus is. In order to resolve this, he proposes a categorization of the whys and the hows that can be used both for "intellectual clarity, understanding the interconnections between the whys and the hows and making it easier to analyze teaching material applying history" (Jankvist, 2009, p. 236). The whys are divided into two categories: arguments referring to history a) as a tool, and b) as a goal. Typical history-as-a-tool arguments include the use of history to provoke affective effects, that is effects related to the student feelings like motivate, engage or comfort them in their struggles, but also the use of history as a cognitive tool to support the student's actual learning. History-as-a-goal arguments are related to the learning about developmental and evolutionary aspects of mathematics. The distinction between history as a tool and history as a goal reflects the distinction between in-issues and meta-issues of mathematics. Jankvist (2009) also discusses how the arguments collected on the classification of Tzanakis and Arcavi (2000) can fall in one or sometimes both of his two categories.

Participants, the content of the course and the data collection

This research is part of the pilot study entitled "Understanding student teachers' learning and development in the Master course of HPAM". The intention of the course is to enhance PMTs' knowledge of mathematics from both a subject matter and a pedagogical matter point of view. The participants of the study were in their first year of a two years master program for Grades 5-10 at the department of teacher education. The group, who took the course and voluntarily agreed to participate in the above-mentioned pilot study in the academic year 2018/19, consisted of 18 PMTs. The teaching was organized in five seminar weeks. Approximately every third week PMTs came to campus and had two days devoted to in-class teaching. Between the seminar weeks, literature studies, assignments, data collection in schools, and contact through the online learning platform were conducted. The content of the third seminar week, taught by the first author of this paper, was the three construction problems of antiquity and Euclid and the Elements. In the first session the PMTs were presented the construction problems, worked hands-on with the efforts made by prominent mathematicians of the time, and discussed how mathematics had to develop so that the problems could be solved. In the second session the focus was on what constitutes an axiomatic-deductive system, what a valid deductive proof is, and how this relates to school mathematics. After each seminar week, the PMTs were given a compulsory assignment composed of several tasks. The data in this paper comes from the assignment related to the third seminar week, but here we only focus on PMTs' answers related to one of the tasks, in which they had to write a short reflection (250 – 500 words) based on three questions: 1. What does the story of the three famous construction problems of antiquity teach us about mathematics and its development? 2. How can you use knowledge [from the construction problems of antiquity] in your teaching in school? 3. To what extent can one use the ideas of Euclidean geometry in school? Not all PMTs have answered all these questions, so the number of answers analyzed in this paper varies between 15 to 17. We underline that these questions were not addressed in the classroom due to time limitations, thus we consider PMTs' answers as their own views that were shaped without the influence of the course teachers or their peers.

A hybrid approach, modeled after the inductive-deductive thematic analysis method described by Fereday and Muir-Cochrane (2006), was used to code PMTs' responses. The thematic analysis began with the development of an a priori (theory-driven) coding template based on categorizations by Ernest (1989), Hersh (1994), Tzanakis and Arcavi (2000), and Jankvist (2004). The coding template was later updated, as we encountered new content that was not covered by the theory-driven codes.

Analysis

The analysis of our data is divided in two parts, one for each of the types of beliefs we are studying. In the first part we analyze the PMTs' beliefs related to the nature of mathematics. Using Hersh's and others' philosophy of mathematics (Hersh, 1994) we identify elements that that can contribute to

Ernest's problem-driven view (Ernest, 1989). The beliefs concerning the use of history in mathematics education are analyzed in the second part using Jankvist's (2009) categorization of the why's and how's of using history and Tzanakis' and Arcavi's (2000) classification of arguments for using history of mathematics.

Beliefs about mathematics

The analysis of the data coming from PMTs' answers to question 1 showed that the PMTs' beliefs can be placed into two categories. The first one is Ernest's (1989) *problem-driven* view. 12 out of 15 PMTs gave answers that fitted in this category in various degrees. For the remaining three students, the work with the three construction problems seemed to strengthen *formalist* beliefs of mathematics. We identified four key elements that contribute to a problem-driven view of mathematics. Answers that contained one or more of these elements were placed in the problem-driven category.

The first three elements concern the nature of mathematical objects and reflect the three dimensions given to mathematics by Hersh (1994), the social, the cultural, and the historical dimension. The *historical* dimension was shown to be the most straightforward to grasp, but apparently not that straightforward to be included in the answers of all PMTs. There were 13 out of 15 PMTs that included in a way this element in their answers. Due to space limitation, we only show some examples of parts of different answers that reflected a historical dimension. Answers included phrases like "the knowledge about geometric understanding has been built stone after stone within thousands of years", "it is not made over night but [mathematicians] have worked on mathematical problems through a period of time" and "a project that stretches over thousands of years in the human history". The *social* dimension was found in four of 15 answers. One for example states that "mathematics is a cooperation project", and another that "others in a later time have taken up again the same problem". The *cultural* dimension went unnoticed by most PMTs. There was only one PMT that gave the construction problems a cultural characteristic by stating that "it tells us that the ancient Greeks were fascinated by abstract mathematics and not just applying it to concrete situations." While this does not completely capture the way that mathematics is cultural, we feel that as an observation it does point in the right direction. The fourth key element concerns the mathematical knowledge and the process of acquiring it. In a problem-driven view mathematical knowledge is *fallible* and corrigible (Ernest, 1991). Answers that captured the fact that mathematics progresses through mistakes fall in this category. There were five PMTs that included in their answer this key element. Examples are phrases like "even if one has not necessarily solved the problem, one learns from the process" and "it is not always the right answer that should be in focus" and "through our failures we discover some important concepts that can be useful to … the field of mathematics". In Figure 1 we summarize the key elements that were used to categorize an answer as problem-driven view together with the frequency.

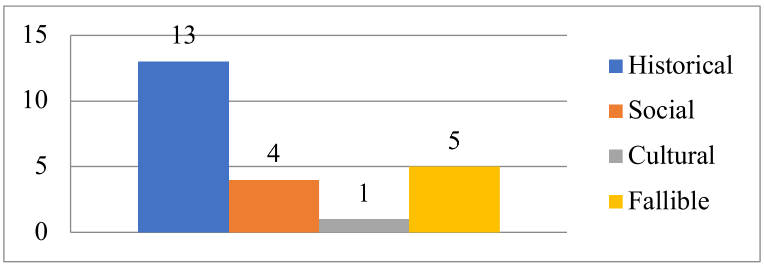

Figure 1. Key elements related to a problem-driven view of mathematics

The three PMTs whose answers were categorized as formalist all seemed to stumble upon the strict conditions under which the three construction problems were to be solved and the realization that if it were not for those strict premises the problems could otherwise have been solved. This could have been interpreted as an acknowledgment of the tight bond between culture and mathematics and therefore recognized as an element that contributes to a problem-driven view. But in all these cases it was commented in a negative way: ”their rigid rules for construction in some ways actually held them back” or ”mathematics is not natural; it is something that we have decided that it should be the way it is”. Instead of acknowledging that mathematics can progress with questions coming both from inside and outside mathematics, the students seemed to distinguish these two: ”[the three unsolved problems] will illustrate a conflict between the world of mathematics and the real world”. There was no trace of the elements the problem-driven view comprises, and the focus was turned towards the artificial rules of the construction problems and were therefore related to formalism.

Beliefs about the use of history of mathematics in mathematics education

Although the questions 2 and 3 from the assignment are stated as *how* to use the two parts of the history of mathematics in class, a closer look in the PMTs' answers revealed that they were mostly arguing about *why* to use them. Almost all PMTs, 16 out of 17, saw the potential of using history as a tool. Among those, there were two PMTs that believed that history can be used in addition as a goal. There was one PMT that didn't see any potential in using history of construction problems/Euclidean geometry in teaching school mathematics. The *history-as-a-tool* arguments that are present in PMTs' answers can be divided into two categories. The first category concerns the use of history as an *affective* tool with a purpose to motivate, engage, and encourage students and is related to the topic of the three construction problems. The second category concerns the use of history as a *cognitive* tool to support the actual learning of students and is related to the topic of Euclidean geometry.

Looking at the categories of Tzanakis and Arcavi (2000) our PMTs' arguments for using history as an affective tool fall mostly in the category related to the learning of mathematics and in particular to history as a resource. There were six PMTs that acknowledged that problems coming from history have ”the potential to motivate, interest and engage the learner” (Tzanakis & Arcavi, 2000). Some representative PMTs' answers here are that ”[learning the history] can contribute to motivation”, ”you can use the trisection of the angle as a fun fact” and ”[the construction problems] can contribute to curiosity”. Three PMTs' arguments fall in the category of affective predisposition towards mathematics. According to them by working with the construction problems students can learn that ”mathematics is not always easy and people before them have struggled for years” and ”that we learn from mistakes”. Last, there was one that falls into the category related to the nature of mathematics and mathematical activity. This PMT commented that the students can ”get an insight in that [inquiry] can contribute to the discovery of new relations”. History as a cognitive tool dominated the PMTs' answers about the use of Euclidean geometry, and they all fall into Tzanakis and Arcavi's category (2000) related to the learning of mathematics and in particular the content. All PMTs that answered this part related the use of ideas from Euclidean geometry with the development of mathematical reasoning in students.

Both *history-as-a-goal* arguments that appeared in the answers of the PMTs refer to the use of the construction problems. According to one of the PMTs, the problems are possible to use to ”show the students how mathematics is made. They will see that it's not made over night”. The second PMT argued that the story of the construction problems can be used to show to the students how ”there was a need for developing further our mathematical understanding to solve those problems”, which can give them an ”insight in how the evolution of mathematics has happened in history”. Both those arguments deal with meta-issues of mathematics and are therefore categorized as history-as-a-goal arguments.

When it comes to the *how* of using history the PMTs’ answers were not explicit enough to be categorized in a systematic way. The illumination approach (Jankvist, 2009) seemed to be the preferred one for working with construction problems. The Euclidean geometry part was somehow thought of being used in the way it was presented in the class in various degrees. Among 12 PMTs who stated that they could use the ideas of Euclidean geometry in their teaching in school, there were 5 that thought that teaching Euclidean geometry in school the way it was presented in ”The Elements” would be beneficial for students: ”we could in practice actually teach geometry from a complete deductive reasoning perspective”. The remaining seven stated that they could use some ideas without being explicit, for example: ”we can use Euclidean geometry in different situations to promote students’ understanding of geometry”, ”I’m thinking to use some of the ideas from Euclid’s construction of an equilateral triangle and congruence, in order to work on students’ argumentation and their understanding of constructions”.

Discussion

In this paper, we focused on PMTs’ beliefs about mathematics and about using the history of mathematics in mathematics teaching. When looking at beliefs about mathematics emerging from the work with the construction problems, our findings reveal that the socio-cultural side of mathematics stayed in the background. We believe two reasons might have contributed to this outcome. First, the way this topic was implemented. During the in-class time, there was more focus on the mathematics involved whilst the cultural context within which the questions were asked and attempted to be answered was left for the students to be explored as homework. The second reason could be that the difficulties the students met with the mathematics moved them away from the historical context around them, something that Smestad (2012) has also observed. In any case, the tight connection between culture and mathematics should not be ignored. As Radford (2018) states ”mathematics always refracts ideas, values, interests and needs of the society from which it emerges”. The lack of addressing the role of the culture in the development of mathematics might also lie behind the formalist views three of the PMTs expressed. Understanding the socio-cultural context within which Plato’s ideas about mathematics arose could have helped them give meaning to the strict premises of the problems and acknowledge that asking questions from inside mathematics can also be a way mathematics advances as a discipline. Whole class discussions and/or assignments with focus on the way mathematics responds to the pressures of society, can help to bring to the surface this important dimension of mathematics.

The PMTs showed in general positive but limiting beliefs about the use of the history of mathematics in school. The construction problems were almost exclusively thought of as stories that can be told in order to motivate, engage or comfort the pupils in their struggles, a finding that is in line with results from other studies (Alpaslan & Haser, 2008; Gonulates, 2008). Euclidean geometry was thought to be used as a cognitive tool to promote reasoning in school but the PMTs’ beliefs of how to use this tool were unrealistic. Rigorous deductive proofs are not in focus in either schools or teacher training programs in Norway and for most PMTs this course was their first opportunity to become acquainted with this type of mathematical reasoning. Thus, the PMTs' difficulties in relating this topic to school mathematics shouldn't be surprising. History-as-a-goal arguments were underrepresented. We see this as being related to missing important elements of the view of mathematics as a continuously expanding field of human enquiry (Ernest, 1989) as discussed previously. The limiting beliefs concerning how to use history in teaching mathematics is probably linked to the way the course was implemented. Besides Jankvist's article (2009) that was discussed at the beginning of the course and one assignment related to PMTs' school practice where they had to design and implement a lesson grounded in the history of mathematics, the PMTs were left alone in reflecting over the use of the material in school, which proved to be insufficient in enabling the students to propose interesting ways of integrating the history of mathematics in teaching.

The data we use in this study is directly related to the topics of construction problems of antiquity and Euclidean geometry and were obtained soon after the relevant seminar. On one hand, this is a limitation of our study, as one cannot draw general conclusions about the PMTs' beliefs formed from the whole course. The same questions concerning other topics of the course or even the same questions given at the end of the course could have led to different outcomes. On the other hand, we found that even those two closely related topics can produce different beliefs as in the case of how to use the history of mathematics in teaching mathematics. Being aware of the types of beliefs that are formed by the different topics within the history and philosophy of mathematics can inform the design and implementation of such a course in teacher training. In the future we plan to carefully redesign the HPAM course, so that we address the two observed issues related to the lack of cultural dimension in PMTs' beliefs and their difficulties in integrating the history of mathematics in school mathematics.

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From the perspective of a student-driven teacher: compromises when teaching a new class in mathematics

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Mathematics teachers have many opportunities to design, use and redidactisise their own teaching resources according to the way they are driven. However, especially when teaching a new class, mathematics teachers must make compromises. In the research presented in this paper, one competent student-driven teacher (Siedel & Stylianides, 2018) was video-observed and interviewed during a semester in which she began teaching a new class in mathematics. Her teaching is first analysed according to the written curriculum, intended curriculum and enacted curriculum. Then, different compromises are discussed with respect to social and sociomathematical norms (Cobb & Yackel, 1996). The findings indicate that the student-driven teacher was very aware of the need to make compromises in order to initiate and guide the renegotiating process establishing social and sociomathematical norms.

Keywords: Mathematics teachers' resources, sociomathematical norms, redidactisation, intended/enacted curriculum.

Introduction

Approximately 75% of fourth- and eighth-grade students experience mathematics lessons based on textbooks, according to international studies (Stylianides, 2016, p. 1). Studies in Denmark have reached similar conclusions (Mogensen, 2012). In mathematics education research, attention now focuses on how mathematics teachers design and use teaching resources in their classrooms (Trouche, Guedet & Pepin, 2018). Teaching resources can be textbooks as well as print-outs, digital teaching materials, videos, elaborative representations, diagrams and a linguistic approach (Remillard, 2018). As such, teachers have many opportunities to find, select and adapt teaching aids to use in the mathematics classroom. As designers, teachers also have opportunities to develop teaching aids or to create a ”redidactisation” of existing teaching resources (Hansen & Gissel, 2017).

In this article, we examine the compromises one mathematics teacher made when orchestrating mathematics textbooks and other teaching resources (Brown, 2009) whilst teaching a new class. More specifically, we focus on her redidactisations of tasks by concentrating on her intentions for the teaching resources selected, on the task as presented by the teacher and on the task realised in the interaction in the classroom (Stylianides, 2016).

Theoretical background

A central role for teachers as designers and users of teaching resources is to make transformations between the written, intended and enacted curriculum (Stein, Remillard & Smith, 2007). The written curriculum is the official curriculum; the intended curriculum is the written curriculum as interpreted by textbooks and other teaching resources as well as in the way the teacher selects transformations and considers orchestrations with the teaching resources. The enacted curriculum is the actual orchestration in the classroom with respect to student learning. When focusing on tasks in document analyses and classroom observations, it is relevant to distinguish between tasks as they appear in the textbook, as they are arranged by the teacher and as they are enacted in the classroom (Stylianides, 2016). In all three phases, one must consider the product, resources, operations and accountability (Stylianides, 2016). The product is the result achieved, and the resources are the conditions and teaching resources available with which to manage the task, which include conversations with other students and modelling by the teacher. The operations are the actions students take or are supposed to take to generate the product, and accountability considers the overall importance of the task.

Siedel and Stylianides (2018) identified the following six types of mathematics teachers based on their driving forces: student-driven, teacher-driven, mathematics-driven, constraints-driven, resource-driven or culture-driven. The types of mathematics teachers can be analysed and categorised based on classroom interactions or by interviewing the teacher. Each approach leads teachers to consider the teaching resources and the didactic intention that already exists in the teaching resources in a specific way and to make a redidactisation of those materials (Hansen & Gissel, 2017) according to their own driving forces. Using Cobb and Yackel's (1996) framework for analysing collective activity from a social perspective at the classroom level (social norms, sociomathematical norms and classroom mathematical practices), it is possible to investigate the entire didactical situation by considering the use and design of teaching resources.

This leads us to the following research questions:

What kind of compromises does a student-driven teacher make when teaching a new class in mathematics? Specifically:

- How does the teacher redidactisise a task from the textbook?
- How do the intentions of the redidactisised task align with how it is enacted in the classroom?

Methods and data

The case in this paper concerns a teacher named Mia. She was selected to participate in this project by the researchers (the authors of this paper) based on their intention to select a competent teacher. The authors considered Mia competent because she had participated in earlier in-service training development and research projects during which she appeared reform oriented (Skott, Jess & Hansen, 2008) and skilled. Mia also had a diploma in mathematics supervision (60 ECTS), and she served as a supervisor of mathematics at her school. She had taught mathematics in elementary school for 16 years when we first visited her class.

The research group observed Mia for two years in two classes (a class called 5X and a class called 5Y) in connection with a research project called ”Practical Interpretations of Teaching Resources” (PAL). In this paper, we focus on 5Y, which we only observed within one semester. The 5Y class was new to Mia when we began our observations. We observed eight lessons in the class. These lessons were video-recorded with two or three handheld cameras focusing on different groups of students. The student groups were selected based on who had given permission to be video-recorded. The observations were conducted with a non-participating approach (Fangen, 2010). An observation manual was created that explicitly emphasised that the camera should focus on the students' activities with the resources. Immediately after each observation, a log was created that contained a rough transcript in which the lesson content and the different students' activities were described in chronological order. Data were also obtained through two semi-structured interviews with Mia—one before the observations in 5Y and one stimulated recall interview after the observations (Mia, Personal communication, Interview 1 and 2). The interviews were audio-recorded and transcribed in full.

The research design strategy was based on a qualitative case study design: ”The case of Mia teaching 5Y”. The aim of this case study was to obtain a detailed and specific description of how the teacher interacted with the students and resources. The underlying rationale for this approach was that the case study is viewed as a typical strategy for empirical exploration of a selected phenomenon in the context in which the phenomenon takes place (Johnson & Christensen, 2014). We used a case study method for this investigation because it allowed us to go into much more detail and depth, and we were able to use a multi-sided approach that may not have been possible if we had been dealing with a large number of research participants.

In this paper, we first address Mia’s predispositions (Siedel & Stylianides, 2018). Next, we highlight two situations: the first observation we made in class, which we call ”Prior Knowledge”, and the final observation we made three months later, which we call ”Round Things”. We analyse the specific tasks in both situations as the tasks appeared in the textbook, how they were set up by the teacher and how they were enacted in the classroom by considering the product, resources, operations and accountability (Stylianides, 2016). Mia told us that her teaching was based on a specific textbook (Lathi, Mogensen, Sperling & Westfall, 2013) that the school provided. Finally, we discuss the compromises she made with respect to social and sociomathematical norms (Cobb & Yackel, 1996) when teaching a new class in mathematics.

Mia’s predispositions

We define Mia as a student-driven teacher (Siedel & Stylianides, 2018) due to various features we observed in her teaching as well as information she shared during the interviews: ”The textbook should not dictate; rather, the students should. I need to plan lessons according to the students I teach. It will never be a good teaching experience if you insist on following the textbook . . .” (Mia, Personal communication, Interview 1). ”If you have some students that cannot handle a particular teaching resource, then there is no point in being insistent . . . then you have to find another teaching resource or redidactisise it” (Mia, Personal communication, Interview 1). These quotes indicate that the most important part of Mia’s planning and orchestration is to focus on the group of students in the class, their specific needs and how these particular students learn mathematics. Mia planned many different activities for each lesson to make sure that the resources fit the students’ needs and level of understanding. This all points to Mia being a student-driven teacher who emphasises resource variety (Siedel & Stylianides, 2018; for more specific documentation, see Hjelmborg et al., 2020).

Two situations

The first observation we made in class, called ”Prior Knowledge”, will be analysed according to the redidactisations of the task as it appeared in the textbook, how it was set up by the teacher and how it was enacted in the classroom by considering the product, resources, operations and accountability (Stylianides, 2016). The final observation, ”Round Things”, which we made three months later, is analysed using the same framework.

Observation 1. Prior Knowledge as it appears in the textbook

The task called ”Prior Knowledge”, as it appears in the textbook (Lathi et al., 2013 p. 38), is a starting task consisting of a picture with young people asking mathematical questions. The questions concern real-life situations. In the first, a young girl is holding the reins of a horse.

Young girl: How much fence do I have to buy for my 5,000m^2 horse field?

The next picture presents a triangular wood base with details noting that it is 8m high by 4m wide, along with a 1L can of paint and a notation that 1L covers 6m^2.

Young boy: How much paint is needed if I have to paint my gables twice?

According to the teacher's guide (Mogensen, Sperling & Westfall, 2014, p. 38), the students' *products* were their solutions to the problems as well as their discussions about solutions to the problems or, more generally, a discussion of how to solve real-life, open-ended problems. The pictures and the information in the speech bubbles comprised the only *resources* given to the students, who worked in pairs. The *operations* that the students were expected to take to generate the product were to solve the problems by extracting the relevant information from the pictures and to activate their own prior knowledge about characteristics of polygons, including the area and circumference of a polygon. With respect to *accountability*, the teacher's guide was vague about the intended outcome as well as the time to use for this task and whether the students were to make notes and/or to present their work (Mogensen, Sperling & Westfall, 2014, p. 38).

Observation 1. Prior Knowledge as it was set up by the teacher

Mia redidactisised the *resources* by electronically creating a table with the textbook's images and the information in the speech bubbles. Next to each picture was an open space where the students were told to write their results. Mia began the task with a class discussion about the new mathematical concepts concerning characteristics of polygons as well as area and circumference. Mia introduced the specific task and the associated *resources* like this: ”Please find your Chromebooks. You are going to work in pairs. The pictures are from the book. The children in the pictures ask questions”. To address the *operations,* she said, ”You have to help them and write your best suggestion next to the picture. Please share your suggestions with me”.

Mia also transformed the *product*. ”Today we focus on what we can remember as well as the new mathematical concepts”, she said. She clarified the purpose of the task as well as the *product* when the students asked for help: ”We are about to find out what we know”. She also said, ”I know that you love to find results, but here we have to suggest ways to help; the calculations are not the important part”. In general, when she helped students, she tried to concentrate on the meaning of the mathematical concept and its connection to a real-life situation. From her actions when helping students we interpret that, to address the *operations* and *accountability*, she wanted the students to discuss in pairs, write their suggestions and be as precise as possible when using words for mathematical concepts.

Observation 1. Prior Knowledge as it was enacted in the classroom

Whilst observing the task as it was enacted in the classroom, we noticed that the students did not know how to interpret the *resources*. They were aware that they had to share the document with their teacher, so they seemed to try as hard as they could, although they seemed a bit frustrated, making comments such as ”I don’t get it—did she cut off the text?” and ”How long are those fences? There’s not a lot of information here!” The students appeared puzzled about *product* and *operations*. They seemed to prefer to solve the problems and, therefore, they asked for help. In most cases, they wrote exactly what Mia told them to say when they asked for help, such as ”Measure it” or ”Find out how big YOUR field is”. Some students also engaged in practical solutions, according to *accountability*: ”Use a measuring wheel”, ”Just buy one tin of paint at a time” or ”Try to paint the first half and see how much paint you used”. The teacher’s intention for the *product*, focusing on the meaning of the relevant mathematical concepts, was not realised in the students’ work.

Observation 2. Round Things as it appeared in the textbook

The task ”Round Things”, as it appears in the textbook (Lathi et al., 2013 p. 67), is an activity for the whole class. Each student cut a circle from a copied template (12 circles with radii varying from 2cm to 7cm); this was the *resource*. The *product* was to line up the round things in order according to the largest area (or the largest circumference). The teacher’s guide (Mogensen et al., 2014, p. 67) contained comments about *accountability*, such as possible rules for communication, including designating two students as problem solvers. The students’ *operations* were making visual estimates of the circumference and area of the circle or directly measuring by placing one circle on top of another.

Observation 2. Round Things as it was set up by the teacher

In the classroom, Mia began the lesson by referring to the students’ preconceptions about circles. She drew a circle on the board, noted its radius, diameter and perimeter, and made a list: ”diameter, radius, circle perimeter, 360 degrees, no edges”. This was a part of the student´s *resources*. Next, she said, according to the *product*, ”We are going to learn how to find the area and circumference of a circle”. Then, she presented the task: ”Today we will have a competition. There are seven different objects in the competition. You must start by guessing which of them has the smallest circumference. Make a table—one for circumference and one for area”. She pinpointed the *operations* and drew a table, another *resource*, with seven columns and two rows. The students had to put the seven figures in order; the *product* was filling out the two tables. She placed *resources* in the form of seven round objects on the floor, two of which seemed to have the exact same radius. Many *resources* that could be used to put the objects in order were also available: pen and paper, a string, the Internet, and other helpful items. A student suggested using a tape measure, so Mia found some. They were also allowed to copy the circles to a piece of paper to compare them, and they had to discuss the order in pairs. ”Maybe you can use your former knowledge about circles”, Mia told the students. In the interview, Mia told us that, according to *accountability*, she chose the competition to motivate the students.

Observation 2. Round Things as it was enacted in the classroom

The task as enacted in the classroom was characterised by students working in pairs, trying to guess and check which objects were the smallest and largest and filling out their tables. They used a variety of *resources*: they drew, named, estimated and measured the circles, actively discussed and listened to other groups and did research on the Internet. Some of the pairs dissolved, and two girls formed a new pair. The *product* in the form of the table gave the students an opportunity to extend their *operations* by reasoning: ”The circumference is always larger than the area, but the area becomes larger when the circumference becomes larger”. Another student observed: ”The flat plate has a larger circumference, but less area [than the deep plate]”. His friend answered, ”I think you mean volume”. Yet another student asked if he should use the perimeter to calculate the area, but Mia did not answer the question. Instead, she referred all students to the fact that they must investigate for themselves whether that was possible, for example, by searching on the Internet. Most of the students worked hard, engaged themselves in the *operations* and seemed to focus on investigations and reasoning according to *accountability*, or the general importance of the task.

Discussion about compromises and conclusions concerning the compromises in the two situations

During our interviews, Mia was very specific about her view on social norms and the characteristics of mathematical activity (Cobb & Yackel, 1996). She described her role as a facilitator of students’ learning processes. Mia was also explicit about what it means to be a student: ”Students must help to formulate their own mathematical questions so that they can make them [the problems] their own”. Mia’s conception of being a student in mathematics is not just about solving already posed mathematics questions. ”Mathematics is primarily something you need to understand, which is why the focus on the mathematical register is important” (Mia, Personal communication, Interview 1). Hence, it is important for her to focus on the meaning of concepts as well as let students investigate in groups.

In the ”Prior Knowledge” situation, it was clear that even though Mia used teaching resources with only slight redidactisation, creating a sort of subset of the original task, the students were not able to decode what the teacher would accept as a reasonable answer. They demonstrated this by asking each other ”What is it that she wants?” The students were met with an alternative approach to teaching compared to what they had encountered the year before, with a different teacher. They were met with demands of working together in new ways and seeing fellow students as *resources*. They also experienced different *products* and *operations* from those to which they had become accustomed. The compromises for Mia occurred, especially in this situation, by her gently initiating a renegotiation process concerning the reflexively related social norms and beliefs about mathematical activity in class. She made compromises according to the *product*, the mathematics the students were expected to learn, as well as the *operations*. However, her intention to focus on the meaning of the relevant mathematical concept was actually not realised in the students’ work. Mia gently guided them toward discussion of mathematical concepts instead of solving problems, but she allowed them to answer in imprecise terms, such as measure ”them”, and let them focus on problem solving.

She also allowed them to ask for help individually, even though they were instructed to work in pairs. Mia was aware of this conflict, as she explained in interview 2: ”Of course, you have to be acquainted with the students and try to orchestrate with respect to the group of students. But there may well be a gap between the group of students, the requirements as seen in the curriculum or task and your own ideas about didactics. Sometimes you redidactisise tasks according to your own didactics, even if it may not suit the group of students, and it leads to possible conflicts”. She added, ”In this class [compared to a class with which she was much better acquainted] I often have to decide: where should the tension [students-curriculum/task, didactics] be today?” (Mia, Personal communication, Interview 2)

By redidactisising and orchestrating the task ”Round Things”, she also systematically made some compromises with respect to social and sociomathematical norms. Her way of listing many possible resources created an opportunity for joint orchestration, with the students negotiating the *product* and the *operations* needed. She also focused on students' motivation and their opportunity to engage in mathematics whilst solving the task. She knew that most of the students still focused on the correct result, and her announcement of the competition acknowledged that. However, here it comes with positive side effects, because students interpreted that they could win the competition in many ways, such as by being precise and being able to explain the order rather than finding the correct result. The way the students worked and discussed their findings in pairs demonstrated that they solved the task on a higher cognitive level, as Mia had intended.

Mia is very aware of her role in initiating and guiding students toward this renegotiation process: ”I think you have to spend most of a year before they know what you want . . . and it will not be as instructive as it was in the beginning” (Mia, Personal communication, Interview 1).

Although we did not observe particularly instructive teaching, Mia was generally true to her student-driven approach when she transformed tasks, i.e. by letting the students discuss the meaning of concepts as well as letting students investigate in pairs. In the interview, she shared ”It is not by getting the right answer that we learn something” (Mia, Personal communication, Interview 1). Her immediate compromises mainly occurred when tasks were enacted in class. She made the necessary immediate compromises regarding what constituted *products* in a mathematics lesson, the kind of *resources* needed and available for the tasks and the kind of *operations* the students expected in the classroom, especially when she noticed that the students struggled with her intentions for the tasks. Mia must consciously balance these compromises because creating too much distance in the renegotiating process toward the social and sociomathematical norms may cause frustration amongst students. At the same time, however, the students must be slowly nudged in the direction of the teacher's ideas about social norms. Absent that, change will not occur.

We are aware that we must refrain from generalising based on this single case, but it will be interesting to observe whether we see similar tendencies for compromises when using teaching resources for other student-driven teachers in further research projects. It will also be relevant to address the possible importance of driving forces for a teacher's way of compromising when using teaching resources.

In our case, the teacher, not the quality of the teaching resources, seems to be the key to improving the quality of teaching. We therefore find it important to focus on the significance of the various driving forces at the level of teacher education as well as in in-service programs by addressing the complex interplay between teaching resources, the teacher's redidactisation of the resources, and the immediate compromises made by the teacher during planning and teaching in terms of initiating a renegotiation process of social and sociomathematical norms.

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“Should the turn be counted, or not …?”

Analysing learning loop contexts of preservice teachers professionalising in the teaching of mathematical modelling

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This paper presents an analysis of preservice teachers (PTs) teaching mathematical modelling to seventh-grade pupils (Ps) in a school-based practice setting. An analysis of shared contexts in the Learning Loop of PTs and Ps shows that PTs and Ps can situate each other’s learning. The contexts can strengthen PT learning and professional development at a later stage. One shared context episode provides valuable PT insight into risk-taking reflection. Risk-taking can stimulate Ps to think critically about mathematics and demonstrate autonomous teaching. An important result of this study should be to inspire PTs to take risks in teaching mathematical modelling.

Keywords: Mathematics teacher education, mathematical modelling, risk-taking, preservice teachers, primary school.

Introduction and aspects of theoretical approach

Several studies point towards the significant impact that teachers have on student application and modelling processes and that, to perform quality teaching, the teachers require numerous competencies for modelling-based activities in the classroom (Barquero et al., 2018; Blomhøj & Kjeldsen, 2006; Blum, 2011; Doerr & Lesh, 2011; Stillman et al., 2013). This study examines the professionalisation process of a group of four second-year preservice teachers (PTs) while teaching mathematical modelling in seventh grade and trying out different ways to promote pupils’ (Ps) engagement in the mathematical modelling process. As I intend to contribute greater understanding about ways to support PT learning and development in the quality teaching of mathematical modelling, this investigation aims to identify contexts that indicate where PT learning is situated.

During the last few decades, Barquero et al. (2018) have identified two main perspectives on approaches to contributions in the field of mathematical modelling and applications for teacher education: (1) professional development programmes that improve competencies and professional knowledge and (2) constraints that impinge mathematics as a modelling activity in the current educational systems at all school levels. Barquero et al. (2018) studied an enquiry-based online course for in-service teachers within the framework of *study and research paths for teacher education* (SRP-TE) methodology. Greefrath and Vorhölter (2016) stated that apart from being convinced of and trained in the use of mathematics, teachers must overcome numerous obstacles, which, according to Schmidt (2011), include a lack of time, complexity of performance and a lack of material. Blomhøj and Kjeldsen (2006) identified three dilemmas in the teaching of mathematical modelling: (1) the degree to which the teacher understands mathematical modelling competency and the educational goals of modelling the practices, (2) how the teacher copes with student autonomy and (3) the responsibilities during all phases of the modelling process. As beginners in the field of the teaching of mathematical modelling, the PTs’ situation involves dilemmas corresponding to those identified by Blomhøj and Kjeldsen (2006). Jaworski (2007)-viewed (1) Ps’ learning of mathematics, (2) teacher learning of ways to promote Ps’ learning of mathematics and (3) didactic learning of ways to promote teacher learning and teaching as three complex interrelated layers. I view my research development according to this complexity and within a frame of enquiry approaches according to Alrø and Skovsmose (2004). Hence, PTs undergo a process of determining how to teach mathematical modelling. This paper, therefore, uses the term teaching to address ways to promote P learning.

The concept of practice connotes doing or "taking part" in a historical and social context, providing structure and meaning to our actions. We use the language, words or gestures of the practice or activity to engage and communicate with those around us. Learning can be seen as participation in such contexts, as we engage mentally and physically, and as deeply situated within these contexts (Jaworski, 2007). Johnsen-Høines (2010) developed a methodological tool called a Learning Loop, which makes it possible to describe different learning activities, as well as the learning community in which students participate. The loop is characterised by ongoing activities and indicates where student learning is situated. The idea is that each activity influences the other activities. Student learning is situated both within contexts of activity and between the different contexts. Johnsen-Høines (2010) demonstrated a loop with two contexts associated with a school development project, i.e. the learning activities were organised and related to the (i) classroom and the (ii) workplace. This paper investigates ways in which PTs promote Ps' learning of mathematical modelling (teaching), devoting attention to certain school activity contexts in which both the Ps and PTs participate. The activity contexts that indicate where P learning is situated therefore characterise the Ps' Learning Loop. According to Jaworski (2007), a complex inter-relation exists between Ps' learning and PTs' learning; hence, the contexts simultaneously characterise the PT Learning Loop. Teaching and learning problems that occur in specific educational settings in which students engage in modelling activities can be studied on micro levels; however, much more research is needed on that level (Kaiser, Blomhøj & Sriraman 2006). According to the initial description of recent decades' contributions to the research, this paper contributes a greater understanding of micro-level teaching. In line with the aim of this study, this paper examines ways in which preservice teachers promote Ps' learning of mathematical modelling in school setting contexts in the seventh grade and analyses the contexts of the PT Learning Loop to identify what characterises their teaching that could promote Ps' learning of mathematical modelling.

Quality in learning to teach and the learning of mathematical modelling

A mathematical modelling process is characterised by several sub-processes: problem formulation, system delimitation, mathematising, mathematical analysis, interpretation and assessment of results, and validation of the model. Some of these sub-processes may occur several times in the overall process (Blomhøj, 2003). Blum (2011) claimed that teaching quality in a modelling process should offer ample opportunities for Ps to acquire mathematical competencies and make connections within and outside of mathematics. Three metaphors characterising P development of competence in mathematical modelling can be distinguished (Hansen & Hana, 2012): "modelling as content" (Julie, 2002), which is related to the learning modelling; "modelling as vehicle" (Julie, 2002) which relates to the learning of mathematical concepts and procedures through the modelling process; and "modelling as critique" (Barbosa, 2006), which reflects the role of mathematics in society. Blum (2011) used the term *teaching effects* about P development of mathematical competencies and claimed that these effects, at most, can be expected on the basis of quality mathematics teaching. He characterises crucial quality in teaching as dependent on maintaining a permanent balance between (minimal) teacher guidance and (maximum) student independence. In modelling contexts, strategic interventions are often the most adequate. This entails giving hints to students on the meta-level: “Imagine the real situation clearly”, “Make a sketch”, “What is your goal?”, “How far have you come?”, “What is still missing?”, “Does this result fit in with the real situation?”, etc. (Blum, 2011). These meta-level strategic intervention questions are viewed by Blum as quality criteria, especially in everyday mathematics teaching. Meta-level questions like these are independence-preserving and often lacking, as there is virtually no stimulation of student solution strategies (Blum, 2011). Consequently, the PT process of balancing guidance and P independence can also promote a change to student engagement and critical involvement (Elbers, 2003; Hansen & Hana, 2012). I also believe that, if addressed properly, Blomhøj and Kjeldsen's (2006) dilemmas can characterise quality in teaching mathematical modelling. For instance, how do PTs cope with Ps’ autonomy and their responsibilities in the modelling process? In teaching directed autonomy, the Ps need to be responsible for most of the decisions, but the decisions also need to be the “right ones” for the Ps to be able to carry through the modelling process (Blomhøj & Kjeldsen, 2006). Within the framework of the enquiry-based paradigm approaches (Alrø & Skovsmose, 2004)–, the idea of transferring knowledge in teaching is disassociated from characterising quality in education. PT and P learners should therefore be involved in processes of “finding out” something in teaching and the learning of mathematical modelling. They come to know through their engagement in practice (Jaworski, 2007), which includes engaging in reflection on meta-cognitive concepts and outcome constructions (Elbers, 2003; Hana, 2014). Meta-reflections relevant to this study pertain to the teaching and learning of mathematical modelling. Johnsen-Høines and Lode (2007)-identified two modes of focusing in meta-level mathematics reflection aimed at improving quality in teaching practice: (1) *Subject-based reflection* focuses on pursuing a subject-based interest generated in the practice teaching situation, (2) *an evaluative approach* is a retrospective reflection perspective directed towards the past, focusing on an evaluation of what has been done good and what is not so good. These modes of focus in meta-reflections can improve quality in the PT Learning Loop. The evaluative mode is characterised by a tendency to undermine the subject-based mode, but both modes are necessary to facilitate and focus on the subject-based mode in reflections.

Context and methodology

The PTs’ course participation includes three weeks of school-based practicum supervised by an in-service teacher (IT). The practicum period is preceded by an introduction to learning mathematical modelling (Blomhøj, 2003) and teaching mathematical modelling (Barbosa, 2006; Julie, 2002; Hansen, 2012; Hansen & Hana, 2012). A task for planning, teaching and reporting on mathematical modelling based on school practice was given to the PTs before their practice period to support their professional development. In this study, the goal is to explore ways in which learning through practice can be conceptualised. The methodology is based on the notion of exploring the Ps’ Learning Loop. Searching for contexts, the PTs and Ps could situate each other’s learning in shared contexts. Contexts in the PT Learning Loop that indicate that P learning is situated are suggested for future PT reflection. This paper presents data from one of the PT groups. I (ME) observed the group teaching mathematical modelling to a seventh-grade class. To analyse contexts in the PT Learning Loop, I (ME) used episodes from school activity contexts in which the PTs were attempting to promote Ps’ learning of mathematical modelling. The school activity contexts suggested were characterised by P engagement and involvement and provided a base and potential for future contexts of reflections in which the PTs and I (ME) participated. The transfer of knowledge across contexts of practices can be problematic (Jaworski, 2007). Meta-level reflections on practice can be helpful for some transformation to meta-level cognition or learning. The mode of *subject-based reflection* and the mode of an *evaluative approach* are tools for focusing in meta-reflections (Johnsen-Høines & Lode, 2007) and can be helpful for analysing the contexts of the PT Learning Loop, as well as PT learning. The main interest in attempting to position within the subject-based reflections should focus on meta-cognitive concepts and outcome constructions in accordance with the teaching and learning of mathematical modelling. From the field notes, discussion notes and photographs, including public drone photographs, the four episodes discussed in the following section were compiled. The first two episodes served as the basis for an ensuing practice conversation and a workshop on case reflections from which episodes of metalevel reflection were taken. The fourth episode then provided informative documentation of situated learning contexts involving both the PTs and Ps and reflection contexts involving PTs and ME.

Analysis of contexts of preservice student Learning Loop

Episode 1: The episode starts in the classroom and is initiated by a PT introducing a mathematical modelling project that the Ps are going to work on. Apart from some repetition of useful concepts, the PT draws a sketch of the school building on the smartboard and instructs the Ps to measure their school building to make a smaller scale model of it. The Ps are placed in four predetermined indoor working groups.

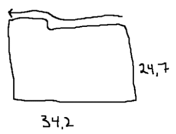

Figure 1. Sketch and measurements of school building

One member of each group carries out the required outdoor measurements and brings the results back to the group. The Ps measuring outside are required to cooperate with each other. Two PTs supervise the Ps measuring outdoors and give such advice as “Try looking at the school from above to see where to measure” and “Everyone should first draw a sketch of the school. This shows you which side of the building is located where”. In addition to the PTs’ advice, observations offer the Ps the opportunity for problem-solving and mathematical discussions: There is a small annex along one of the long sides of the building. One P rolls a measuring wheel along the building, curving around the annex without lifting the measuring wheel from the ground. Another P asks, “Should the turn be counted, or not …?” A discussion follows about what is correct. After several measurements and discussions about how to measure, the Ps end up creating what is shown in Figure 1.

Analysis of Episode 1: When the PTs add such suggestions as “Try looking at the school from above to see where to measure” and “Everyone should first draw a sketch of the school” to the Ps’ measurement problem, they are incorporating meta-level strategic intervention questions that Blum (2011) considers adequate for independence-preserving in everyday mathematics teaching. Consequently, PT teaching is characterised according to Blum’s (2011) quality criteria. The activity of applying the measuring wheel falls into the category of “applying mathematics in practice” according to the core element of Modelling and Application in the National Curriculum (2020). This activity can be viewed as a mathematical skill that can help the Ps assign meaning to the concept of measuring. In this meaning construction, the Ps gain insight into and competencies in the learning metaphor of "modelling as a vehicle" (Hansen & Hana, 2012; Julie, 2002). The episode fits in with the sub-process of mathematising, according to Blomhøj (2003). The discussion about whether the curvature should count as part of the building's length coincides with the metaphor of "modelling as a critique" (Barbosa, 2006; Hansen & Hana, 2012), i.e. it characterises the development of critical competencies for everyday life. The discussion has intrinsic value in challenging the concept of measuring. This phenomenon is characterised by involvement and motivation and has a positive influence on learning processes (Elbers, 2003). Episode 1 and the analysis generate a basis for future contexts of subject-based reflections related to the PT Learning Loop.

Episode 2: After measuring outside, the groups are supposed to continue working on the modelling project. In the classroom, the PTs attempt to increase P engagement to proceed in the project. As the Ps do not have the required measurement of the building height, one PT makes a meta-level strategic intervention comment about the quantification of the height and hints at a strategy in which the result can correspond to the actual height: "The height has to be realistic!" he Ps indicate that they want to find an answer that is close to the actual height. Suddenly, all four groups make a decision and rush out of the classroom, discussing how they can find a number that is close to the actual height of the building. The PTs let them go out of the classroom. While discussing, an idea is put forward to measure the height of a single step and then count the number of steps from one floor to the next and, finally, sum up all three levels. The Ps measured one step to be 17 cm in height. They then count 18 steps from one level to the next and calculate the height from one level to the next as 17 cm x 18 = 306 cm. Because the school building has three levels in addition to the ground level, the Ps determine the height to be 9.18 m, and they conclude that the height of the school building is approximately 10 m. The Ps justify this approximation using the explained procedure with loud and convincing voices.

Analysis of Episode 2: The PT's comment that the height must be realistic stimulates independence-preserving, according to Blum (2011) and can have various other dimensions. The comment can indicate knowledge by (i) referring to the core element of Modelling and Application in the National Curriculum (2020) in which modelling must be close to reality, (ii) reflecting on system delimitation (Blomhøj, 2003), (iii) encouraging interpretation of the task and (iv) aspiring to increase Ps' motivation and engagement. When the PT says that the preferred height should be "realistic", there is movement towards "modelling as a critique" (Barbosa, 2006; Hansen & Hana, 2012), in which modelling can enable Ps to critically engage with mathematisation and the application of mathematics. Allowing the Ps to rush out of the classroom indicates that the PTs perceive their enthusiasm and respect their decision. There is an almost imperceptible moment when the PTs decide whether to let the Ps take control. The PTs are willing to take a risk, as the Ps' engagement and decision to rush out of the classroom while discussing how to solve the height problem indicates taking over responsibility for and increased involvement in the modelling process, and it coincides with genuine enquiry (Wells, 2000). During this phase of the modelling process, the PTs introduce the Ps' responsibilities, which are linked to dealing with Ps' autonomy, as requested by Blomhøj and Kjeldsen (2006). The episode relates to the sub-process of mathematising (Blomhøj, 2003). The discussion aligns particularly with the metaphor of "modelling as a critique" (Barbosa, 2006; Hansen & Hana, 2012) and shows the potential for developing critical competencies. The strategy of measuring and the subsequent calculations fall into the category of 'applying mathematics in practice' as the core element of Modelling and Application in the National Curriculum (2020). The activity enables meaning construction to the concept of measurement, from which the Ps gain insight and competence in accordance with the learning metaphor of "modelling as a vehicle" (Hansen & Hana, 2012; Julie, 2002). It can also be assumed that there is potential for the Ps to learn within the metaphor of "modelling as content" (Hansen & Hana, 2012; Julie, 2002). Episode 2 and its analysis generate a basis for future contexts of subject-based reflections in relation to the PT Learning Loop.

Episodes of meta-level reflections (A) and (B): After Episodes 1 and 2, an ensuing conversation (A) takes place among the PTs and ME. The PTs pursue the subject of Ps' motivation and characterise the Ps as little motivated in general. One PT emphasises the beginning of Episode 2 to illustrate how difficult it can be to motivate the Ps and how hard she works to achieve this. The ME offers details about the Ps' modelling process from Episodes 1 and 2 as the subject base for the conversation, highlighting their "taking over control" and involvement, indicating the emergence of a high level of motivation and learning. Other concepts that are put forward as subject-based interest are how teaching can encapsulate different modelling sub-processes and metaphors of learning through mathematical modelling. Consequently, the PTs evaluated their teaching according to the lesson plan and considered the start of both Episodes 1 and 2, which took place inside the classroom, as better than the outdoor and staircase activities. A few weeks later, back at the university, the ME prepared a written descriptive case based on Episodes 1 and 2 to facilitate subject-based reflection as part of a meta-level reflection workshop (B) (Johnsen-Høines & Lode, 2007). The PTs participated in smaller groups. The meta-level reflection outcomes during this part of the PT Learning Loop culminated in the need for an evaluative approach (Johnsen-Høines & Lode, 2007) through the PTs' articulations:

> In practice, we did not perceive all the positive dimensions of the modelling learning activities that were going on. We were a bit hesitant to deviate from the lesson plan. We were concerned about spending too much time on measuring and did not understand the purpose of the minor critical mathematical discussions. We also undervalued what Ps were capable of performing in terms of mathematical modelling. And... we did not realise the value of the aspects that the Ps were reasoning and arguing about.

During the conversation context (A), the PT evaluation failed to take into account the perspective of learning mathematical modelling. The PTs' conversational approach to teaching is in the mode of an evaluative approach (Johnsen-Høines & Lode, 2007), emphasising "whether they followed the lesson plan". In the context of the workshop (B), the PTs' roughly commented "we undervalued the Ps' and 'we did not realise the value of reasoning and arguing". The PTs' comments on Episodes 1 and 2 were in an evaluative mode but were influenced by the subject-based reflections during the workshop and by subject-based reflectiveness. The PTs' meta-level reflections referred not only to their Learning Loop but also to the Ps' Learning Loop, based on the same episodes of teaching and the learning of mathematical modelling. The Ps and PTs have therefore situated each other's learning within the shared contexts of their Learning Loops, and the PTs have developed their professionalisation in teaching mathematical modelling through "finding out something".

Conclusions

An analysis of the PTs' Learning Loop in relation to the Ps' Learning Loop identified several shared contexts which the PTs and the Ps situate within each other's learning. When the Ps are undergoing the learning processes of mathematical modelling, the PTs can learn about how to teach them quality.

During one unexpected classroom episode, the PTs' risk-taking in managing the situation resulted in the Ps taking over ownership of the modelling project and engaging in critical mathematical thinking. Risk-taking can prompt autonomous teaching and the Ps to critically think about mathematics. Consequently, I find that risk-taking should be a significant component in mathematical modelling quality teaching. The PTs comment on their own learning based on the PT and P shared Learning Loop contexts, showing the potential for learning that is situated in these contexts. Considering the contexts as important contributions to the PT Learning Loop, in which they should be involved later, can significantly support the PTs' professional development. The present case could inspire PTs to take risks in teaching mathematical modelling and to consider unexpected classroom situations as having significant potential to enhance the quality of the learning of mathematical modelling.

Acknowledgment

This research is part of the LATACME project which is funded by the Research Council of Norway. Empirical data was collected autumn 2018.

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Communicating mathematics in a real-life context

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This article presents a case study of communication in mathematics in a seventh-grade classroom of a Norwegian primary school. The main aim of this study is to investigate characteristics of student communication in mathematics in the context of their realities. Video recordings of conversations were analyzed using a framework of various speech acts and interaction patterns that may indicate students' approaches to mathematics. The students explored body movements to describe different rotations in a real-life context. The analyses show that the students' interaction displayed an investigative approach to mathematics that amplifies their voices by facing others' opinions and thoughts. This is important for the development of students' mathematical knowledge and engagement in the learning process. The results also indicate that real-life contexts may influence the flexibility of communication.

Keywords: Communication, speech acts, investigative approach, real-life context.

Background and purpose

Traditional approaches to mathematics teaching are characterized by authoritative communication forms (Mortimer & Scott, 2003), where little space exists for different thoughts and opinions. In contrast, Skovsmose (2001) presented an inquiry approach called ”the landscape of investigation” as a qualitatively different approach to learning mathematics. Contexts in a landscape of investigation invite students to an inquiry process that is often characterized by different speech acts. We can consider conversation and dialogue as communicative actions that students use for different purposes. There is broad consensus in mathematics didactics about the importance of communication in learning mathematics (Nilsson & Ryve, 2010). Sfard (2001) and Lee and Johnston-Wilder (2013) concluded that there is a connection between communication and thinking in mathematics. Students' learning processes often occur in a social context where interactions can make their voices heard. One's voice expresses their thoughts, ideas, perceptions, attitudes, and other personal positions (Dysthe, 1999).

This study is part of a pilot project related to the event, Vitenuka (2014), organized by a research group at UiT, which aims to contribute to science and mathematics recruitment among primary school students in Finnmark, Norway. ”Communication” is a comprehensive term that includes drawing, the use of objects, mimicry, body movements, and short speeches or texts. The analyses in this study cover students' dialogues and texts. By studying communication in each context, we can better understand students' learning processes. The research question is as follows: What characterizes the communication of seventh-grade students in a real-life context?

By ”real-life context,” we refer to a task or problem pertaining to reality. Student engagement and social action are foundational for mathematics learning and form the basis for experience acquisition (Cobb & Bowers, 1999). Therefore, it is crucial that students experience mathematics as relevant and meaningful. Thus, a context with reference to their everyday lives may support their engagement in communicating mathematics. The learner develops mathematical knowledge by constructing concepts through a reference context (Steinbring, 2005). Here, we investigate student communication that expresses thoughts and ideas about mathematical concepts, using motion as a reference context.

Theoretical framework

Pupils encounter the concept of angles in the fourth grade, where they measure angles and examine the properties of geometric figures in two and three dimensions (Directorate of Education, 2020). The concept of angles can mainly be perceived in three ways: static, dynamic, and angular sector (Devichi & Munier, 2013). The static definition of ”angle” expresses two rays with a common point forming the angle. The dynamic aspect involves rotation as a movement. The third aspect is the area bound by two rays with a common vertex (ibid.). Students should gain experience with various aspects of the concept of angles to develop a rich understanding. Argumentation for choosing conversations as a unit of analysis relates to meaning-making as a social process (Bakhtin, 1984; Ernest, 1994).

Mathematics as a field of knowledge involves abstract concepts that can be mediated through semiotic symbols and signs (Steinbring, 2005). The meanings students construct for mathematical concepts (mental objects) often occur in a social setting, such as the classroom or home. Thus, communication in mathematics involves individual’s thoughts and perceptions about these mental objects (Sfard, 2008), where conversation and text are central communicating units. According to Mortimer and Scott (2003), a conversation has a dialogic character if it includes different perspectives and thoughts. They (ibid.) defined a conversation as authoritative if only one perspective was focused and only one voice was heard. The analysis also involves the concept of the interaction pattern, presenting the order of different utterances concerning initiative (I), response (R), and feedback (F) in a conversation. The interaction pattern of a conversation may indicate different approaches to learning in mathematics classrooms. The teacher taking the initiative and the student responding, followed by the teacher’s evaluation (E), may characterize an example of interaction within the task paradigm. This interaction pattern is of the type IRE and reflect the interaction among students (Manshadi & Lysne, 2013).

Alrø and Skovsmose (2002) considered dialogue as a communicative action involving different speech acts that may indicate the investigative characteristics of an interaction. These are getting in touch, locating, arguing, identifying, thinking aloud, reformulating, challenging, and evaluating. For example, through ”getting in touch,” the student tries to convey their ideas, thoughts, and opinions to get a response. The student can make a statement that expresses their opinion on something and is an invitation for dialogue. Therefore, identifying speech acts provides a better picture of students’ approach to learning mathematics. The speech acts as well as interaction patterns form the framework for analyzing conversations.

Research methods

Participants

All schools in the area could register for Vitenuka (2014) via open enrollment. Vitenuka’s management chose three participating schools: Schools A (with a class of seven students), B (with sixteen students), and C (with only two students).

The teachers were mostly observers but could assist students with practical issues in the classroom. This is because some students might be shy when meeting other students, teachers, and adults with whom they were not acquainted. Two external participants, a researcher and a representative from the Mathematics Center in Finnmark, initiated the activity. The conversation regarding rotational movement based on the pupils' own anatomy occurred on day 1, where Schools A and C participated. The total data material (days 1 and 2 combined) consisted of five hours of video recording. Two desktop video cameras were installed in each corner of the classroom, and a third camera was used to record the students' work. This guaranteed that we could capture moments that the other two cameras could not. Each camera had a wireless transmitter, ensuring that the pupils' conversations could be recorded with good quality. Data analysis consisted of the following three steps. An overall interpretation of the video recordings resulted in preliminary interpretations in the first step. Next, data reduction was performed by choosing interactions with explicit mathematical contents. Finally, the interactions were analyzed in depth using speech acts and interaction patterns. Observer triangulation (Robson, 2002) was conducted by the research group by discussing interpretations.

The activity

The activity in this study is based on a sociocultural perspective of learning and knowledge. Cobb and Bowers (1999) argued that students' learning of mathematics is a process that requires active participation in the classroom's learning environment. The activity mostly involves geometry and was designed to focus on the concepts of rotation and angles. It also invites students to explore movements, participate in discussions, and become involved in a problem-solving task. Regarding mathematical content, this will mostly involve proportional reasoning. Initially, the students were asked to identify rotations based on their own movements. They should also determine the center of rotation of the movement. The activity focused on the dynamic and static aspects of the concept of angles. Physical activities involve rotation in most of the movements we perform, making the body relevant as a representation of the dynamic aspect of the concept of angles.

Results and analysis

Discussing body movement

Students were asked to suggest movements that could express rotation. The teacher who had participated in the conversation encouraged her students to begin the activity:

14	Martin:	What is—what is the point? (thinks aloud)
15	Eva:	That's the whole thing, after all.
16	Ole:	To know the body that way.
17	Teacher:	You might bend your stomach.
18	Ole:	You bend your neck, stomach, and arms.
19	Martin:	Butt, tailbone... The tailbone that is not. It's like—it's the tailbone that is in the middle. (Rotated with one hand to visualize body rotation around the hip.)
20	Ole:	It will not be backward.
21	Teacher:	Can you come up with more suggestions—
22	Martin:	The eyes. You cannot spin your hands like that. (Martin moves his arm and indicates the arm's limit of rotation. The rest of the group moves their hands to test the hand's rotation limit, that is, to know the maximum value of the rotation angle.)
23	Martin:	Can you rotate your arm 360 degrees?
24	Simon:	Yes. (rotates the wrist)

25 Simon: The ankle.
26 Martin: Kneel, knee.
27 Ole: The back.
28 Martin: The back? Can you?
29 Ole: (moves upper body from side to side)
30 Eva: When you bounce, you bend down.
31 Martin: Bounce? Then you do not rotate. You [are] bouncing.
32 Eva: Now, you're bending down. (smiles)
33 Martin: Yes.
34 Eva: Yes, you also bounce.
35 Martin: Yes, it is bouncing, not rotated — not rotation.
36 Simon: Yes, but what about the jaw? (moves his jaw)

Martin thought aloud and wondered what the aim of the conversation (activity) was: "What's the point of this?" (14). Eva may have believed that the whole conversation involved movement, rotation, etc.: "That's all it is" (15). Ole argued that this is a way of knowing the body. The teacher intervened in the conversation to motivate the students to respond to her statement: "You might bend your stomach" (17). Ole located rotational movements when the upper body rotates around the hips (18): "One bends the neck, abdomen, and arms." Martin identified the axis of rotation, and he was eager to express his thoughts (19). He attempted to reformulate his utterance to convey his argument more clearly. In primary school mathematics, the term "rotation" is often illustrated with two-dimensional examples. Examples of geometric tasks in a two-dimensional space may be the rotation of a geometric figure on a sheet. Martin may have used his experience of the rotation concept from primary school mathematics to identify the axis of rotation in three-dimensional motion. He says, "Butt, and tailbone. It is the tailbone that does not move. It's like—it's the tailbone that's in the middle." Martin expresses his idea of a rotation (salto) and argues that the imaginary line passing through the tailbone is an axis of rotation and that it is invariant. The teacher encourages the students to discover several suggestions, and she is interrupted by Martin, who suggests that eye movement is a rotational motion. Martin challenges others in the group to rotate the arm 360 degrees (23) about the elbow. He is aware that this rotation has a maximum limit of approximately 180 degrees, but Simon moves his wrist and illustrates 360 degrees of rotation. Lines 22–36 are the students' suggestions for rotational movements based on their anatomy. Eva mentions jumping as a rotational motion, with which Martin disagrees (31).

37 Ole: It's so easy to roll your tongue when we eat.
38 Martin: One uses the eyes for . . . (Martin moves his eyes. The teacher asks Simon what he thinks. He listens to the discussions but does not participate.)
39 Martin: The fingers. (Moves his fingers)
40 Eva: (Bends her index finger)
41 Martin: There is still rotation. (All three students bend their fingers.)
42 Eva: Yes, isn't it? You bend down and jump up.
43 Martin: It's the bouncing. These are fingers.

During the conversation, the students constantly produced several movements that they identified as rotations. Martin believed that bending fingers is also a rotational movement. When he bends his finger, the center of rotation of the movement becomes visible. Eva perceived Martin's statement as support that bouncing is a rotation (42). Martin's response (39) serves as feedback for Eva. To determine a movement as rotation, students must be able to identify the center of rotation. Eva viewed several rotational motions that enable a jumping movement (bouncing), but she likely could not identify a jump as a rotational motion. Martin could not identify any rotation center for the bouncing motion. This may be why he did not accept bouncing as a rotation and resisted Eva's stance (43). The conversation likely had an IRF pattern with a dynamic that differs from the IRE pattern, where an initiative receives a response and is evaluated. The interaction appeared to involve speech acts such as locating, arguing, identifying, thinking aloud, reformulating, and challenging, all of which may indicate that the interaction had investigative characteristics.

Students' different mathematical voices

Students were given a mathematics assignment in PISA 2012 (Kjærnsli & Olsen, 2013, p. 56; OECD, 2012, p. 74) after working on movement with rotational properties. The task is called the "revolving door" and involves the static and dynamic aspects of the concept of angles. It also requires students to imagine rotational movement (revolving door rotation) at a certain speed. The task had a reference to the students' reality as all the students had previously seen and experienced a revolving door. The task involves a rotating door or port with three sections, where each section has a maximum capacity for two people. In Question 1, the students were asked to calculate the angle between the door leaves. Question 2 states that the revolving door rotates four times per minute, and they must determine how many people can pass through the door in 30 min. The task originally consisted of three questions, where one of the questions in geometry concerned the arc length of the revolving door. This part was omitted to adapt the task to the students' knowledge level, because the PISA tasks are aimed at the tenth grade. The students were informed of this edit and were advised to ignore the information about the diameter of the door in the text. Three of the pupils' solutions to Question 2, which is more demanding than Question 1, are mentioned here. Few students explained their solutions because they provided multiple-choice answers. Martin sat quietly in thought. He chose option D (720 people) but did not write anything down. The researcher (R) asked about his argument:

45	R:	And what is the argument?
46	Martin:	The argument . . . If it goes four times a minute and can hold two in each of these (two people in each sector), then there can be six in one of them there, so within a minute, [it can] have twenty-four people. In addition, 24 times 30 is 720.
47	Martin:	Seven hundred twenty people can walk within half an hour here.
48	R:	Okay.
49	Martin:	I might think a little too big.

The pupils' interpretation of the task implies that they could imagine the projection of an object (the door) from a three-dimensional space to a two-dimensional space. Ole has control over the number of turns and the maximum number of people who can pass through the door in 30 min. Simon's solutions differ slightly from those of Ole and Martin.

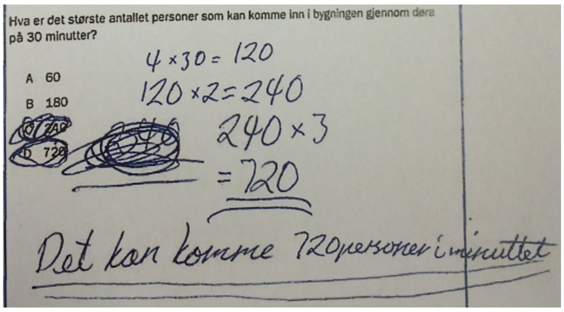

Figure 1. Ole's calculations

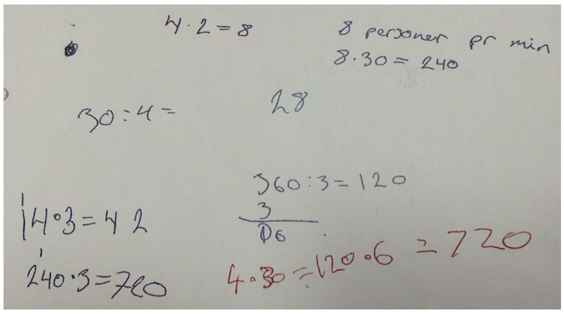

Figure 2. Simon's calculations

Considering the students' answers as their own voices, their expressions of their understanding of the task is reasonable. We may track the indications of proportional reasoning through the students' calculations. This type of reasoning involves multiplicative relationships between different quantities, such as the number of people per turn (rotation), rotation, and time. The students used multiplicative structures and expressed their own strategies in each calculation with different nuances.

Discussion

The students' involvement in the activity is apparent by speech acts that move the conversation forward. These speech acts constructively influence the dynamics of the interaction. This indicates that the conversations had a dialogic (Mortimer & Scott, 2003) character, in which the students' thoughts and ideas were expressed, met, and confronted by one another. This occurred without anyone having an exclusive right to decide what the absolute truth should be. The interaction patterns can be interpreted as IRFRF…, which characterizes the conversation as a dialogue. The absence of evaluations of the form ”good, wrong, right . . .” in the conversation can be observed. A combination of various speech acts and interaction patterns may indicate a conversation with investigative characteristics. Students attempted to connect mathematical objects (concepts) and real-life contexts by exploring their anatomical movements. The closeness of the context to the students' real-life experiences may have reinforced this connection. Their statements were open to criticism, and their responses were not negatively loaded. The students supplied several suggestions because they could sense why these movements constituted rotation. The proximity of context may have been important in supporting the students' awareness of a center of rotation that was, in some cases, not physically visible (22, 27, 29, 36). The risk of whether students accept the context's invitation to the learning process is often evident. Lee and Johnston-Wilder (2013) referred to their research involving students as co-researchers to gain knowledge of how schools can improve learning in mathematics. Students had the opportunity to choose their own contexts, which generated a degree of freedom, and their voices became noticeably heard. This also greatly influenced the pupils' involvement and communication in the activities. In the current study, the students used their voices during the problem-solving task (the revolving door task) and expressed their understanding of the problem. Sfard (2008, p. 81) defines thinking as a form of communication: ”Thinking is an individualized version of (interpersonal) communicating.” Although thinking is an invisible human activity (ibid) for others, one's voice expresses their thoughts and ideas (Dysthe, 1999). The pupils' calculations as an expression for their interpersonal communication can indicate two interesting and important areas in mathematics: proportional reasoning and mathematization of a situation. The development of proportional reasoning is central to students' cognitive development in mathematics (Kastberg, D'Ambrosio & Lynch-Davis, 2012), as proportional reasoning can be traced to several contexts, such as economics, physics (such as the concept of speed), and other sciences. Another important element is the process by which students translate possible connections and structures from a context to mathematical expressions. Using relevant information from a context and then translating it into mathematical expressions can be a demanding process for students. The utterances may indicate that the context influenced the flexibility of the interaction. Sfard (2001, p. 36) considered communication as effective when various interlocutors' statements elicited responses that were consistent with the speaker's expectation. The results suggest that students understood one another during the conversation. They used movements (22, 24, 29, 36, 38, 40) as a statement that supported their verbal expressions. The analysis shows that reformulation as a speech act rarely occurs in conversations. Students' use of body movements may have dampened the need to reformulate their statements. The context's direct reference to the students' reality likely makes their communicative space more flexible and their interaction more efficient.

Conclusion

The students' involvement in the activity reinforced the encounter between their perceptions and thoughts about mathematical concepts. In this case, expressing students' opinions and arguments can be important for constructing a richer understanding of mathematical concepts. Therefore, pupils' voices are an important resource for teachers in the learning process. The choice of a real-life context with reference to the students' reality can have a significant impact on their active participation. The study shows that the interaction displayed investigative characteristics that are reflected by speech acts and the interaction pattern of the conversation. The context likely contributed to the effectiveness of student communication, which, in turn, was beneficial for the visibility of their voices, reflecting their conceptual perceptions. This is the motivation to further study the impact of a variety of contexts on student communication and investigate deep learning processes in elementary school mathematics.

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Examining TPACK among 8^{th}–10^{th} grade teachers after introducing a Use-Modify-Create programming approach

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This paper reports on teachers' technological, pedagogical and content knowledge (TPACK) related to programming in mathematics after participating in an intervention study, implemented as five workshop sessions. In this setting, we introduced a Use-Modify-Create (UMC) approach to programming and provided examples of how programming could be utilised in the classroom. We used TPACK as the theoretical framework to analyse the participating teachers' interview data. Our findings reveal that teachers reflect on UMC as an approach to afford low threshold to programming, support mathematical discussion, facilitates problem-solving and strengthen the relationship between programming and mathematics.

Keywords: Mathematics education, programming, teaching knowledge, TPACK.

Introduction

In recent years, programming has been introduced in primary education across Europe. In Norway, programming has become a core part of mathematics education from autumn 2020 (Bocconi et al., 2018). However, previous research emphasises that mathematics teachers have little or no knowledge of programming or how to introduce programming in mathematics (Forsström & Kaufmann, 2018; Stigberg & Stigberg, 2019). There is a substantial need to support teachers in acquiring basic programming knowledge and pedagogical knowledge on how programming can be utilised in the classroom to teach mathematics. It is important that programming knowledge is not viewed as separate from content knowledge or pedagogical knowledge, since "good teaching is not simply adding technology to the existing teaching and content domain" (Koehler & Mishra, 2005, p.134). Higher educational institutions play an important role in carrying out this challenging knowledge promotion both in practice and through research.

To participate in this undertaking, we arranged a series of five workshops introducing programming to 8^{th}-10^{th} grade mathematics teachers from our practice schools. We applied a Use-Modify-Create approach (Lee et al., 2011) during the workshops to provide examples for all programming-related competence aims defined in the mathematics curriculum. We conducted a series of focus group interviews with the participants to investigate what technological, pedagogical and content knowledge they expressed after participating in these workshops providing critical implications for teacher educators as requested by Cox & Graham (2009). We analysed these interviews using a framework based on the TPACK model. Finally, our research question can be framed as: *How do mathematics teachers reflect on Use-Modify-Create to support their technological, pedagogical and content knowledge related to utilising programming in mathematics for 8^{th}-10^{th} grade?* The paper continues with an introduction to the Use-Modify-Create programming approach, followed by a description of the TPACK model used as the theoretical framework in the paper. Then, we present our research design including context, data collection, analyses and critical reflection. Finally, we present our findings and conclude with a discussion of our results and provide ventures for future work.

Use-Modify-Create programming approach

The Use-Modify-Create (UMC) approach has been widely promoted to scaffold student engagement in computational thinking (CT) (Lee et al., 2011; Martin et al., 2020). Students are first introduced to programming through example code (Use), exploring and altering that code (Modify), and then designing and building their own code project (Create). Previous research indicates that the UMC approach provides a lower threshold to programming compared to instructional approaches (Lytle et al., 2019). Furthermore, Franklin et al. (2020) found that a UMC approach provides both structure (during use and modify activities), which encourages content learning while maintaining flexibility and open-endedness (during create activities), which encourages constructionist ideals of student-driven learning and engagement. In addition, previous research indicates that the approach enables teachers to adopt integrated CT curricula more readily, letting them learn CT and programming along with their students (Lytle et al., 2019). However, there is scattered research investigating how UMC can be utilised in teaching math and science in schools, e.g., biology (Lytle et al., 2019). There is some evidence that the approach ”helps teachers gradually learn how programs represent their disciplinary knowledge, enabling them to make those connections just in time with students” (Lytle et al., 2019). In our project, we investigate how UMC can be utilised by teachers in mathematics education to add more substantiation to the body of knowledge.

Technological, pedagogical and content knowledge framework

Mishra and Koehler (2006) present a theoretical model called TPACK to describe teachers' technological, pedagogical and content knowledge, based on previous work by Shulman (1986). This model has been extensively used in research on technology education and teachers' professional development in a technological context (Chai et al., 2013; Rosenberg & Koehler, 2015; Voogt et al., 2013). The model presents the relationship between technological knowledge (TK), pedagogical knowledge (PK), and content knowledge (CK) needed for teaching. TK is defined as knowledge about new technology in education. We also include knowledge about a programming language as TK. PK is ”teachers' deep knowledge about the processes and practices or methods of teaching and learning” (Koehler & Mishra, 2009, p. 64). CK comprehends ”knowledge of concepts, theories, ideas, organizational frameworks, knowledge of evidence and proof, as well as established practices and approaches toward developing such knowledge. Knowledge and the nature of inquiry differ greatly between fields, and teachers should understand the deeper knowledge fundamentals of the disciplines in which they teach” (Koehler & Mishra, 2009, p. 63). CK is independent of pedagogical activities (Cox & Graham, 2009). In the intersection between technological knowledge (TK) and content knowledge (CK) lies technological content knowledge (TCK), e.g., knowledge of how to utilise new technologies to represent content in multiple ways (Mishra and Koehler, 2006). Furthermore, we interpret teachers' reflection on how programming is related to mathematical content as TCK. In the intersection between technological knowledge (TK) and pedagogical knowledge (PK) lies technological pedagogical knowledge (TPK). Mishra and Koehler (2006) describe TPK as knowledge of how new technologies can be used to support and change teaching. We understand teachers' reflection on TPK as how they use their pedagogical knowledge to teach programming. Finally, in the intersection between pedagogical knowledge (PK) and content knowledge (CK) lies pedagogical content knowledge (PCK). PCK identifies the characteristic parts of knowledge for teaching covering the content didactic knowledge for teaching (Shulman, 1986). Combining all three dimensions, technological, pedagogical and content knowledge, encompasses good teaching (Koehler & Mishra, 2009). The model has been criticised for being too general and not explicitly demonstrating the connection between technology, pedagogy and content knowledge (Angeli & Valanides, 2009). Moreover, Archambault and Barnett (2010) argue that the boundaries between the components are unclear. To our knowledge, the model is used frequently for understanding teachers' professional development in a technological context (Chai et al., 2013; Rosenberg & Koehler, 2015; Voogt et al., 2013) and is therefore best suited for analysing our interview data. We strive to precisely define each category in our method section.

Research design

In the following, we will describe the context, data collection, data analyses for the research project, as well as provide a critical reflection of our chosen methodology.

The participants in this study are eight 8^{th}-10^{th} grade mathematics teachers from three different practice schools, all with solid long-term mathematics teaching experiences. In agreement with their principals, they participated in this development and research project. They had varied knowledge of programming: two of them had previously programmed, the others had no prior experience. The teachers met for five one-day workshops over the course of one school year at the university college. Between the workshops, they tested what they had prepared during the workshops in their teaching. The workshops were planned and conducted by two researchers from the Faculty of Education and Languages, and one from the Faculty of Computer Sciences, Engineering and Economics. During the workshops, we introduced both block-based (Scratch) and text-based (Python) programming languages as required by the national 8^{th}-10^{th} grade mathematics curriculum (Ministry of Education and Research, 2019). We introduced UMC as a pedagogical approach during the workshops, focusing on the first two phases (Use & Modify), since we assumed that it would be too time-consuming for teachers to reach the level of programming competence required to create their own programs. Previous research also indicates that it is not necessary for students to become programmers, and that programming should be introduced in schools in order to develop knowledge about coding and how it can support mathematics learning (Kaufmann et al., 2018; Kaufmann & Stenseth, 2020). During each workshop, teachers were given a short introductory lecture and completed a programming exercise, before they discussed and incorporated the materials presented in the preparation of their own teaching. The examples were designed to cover the competence aims defined in the national curriculum (Ministry of Education and Research, 2019). We present two examples below:

- We presented two programs simulating throwing a dice in Scratch[11] and Python[12]. The participants explored these programs and modified them, e.g., throwing dices with nine sides or throwing three dices. Competence aim for 9^{th} grade: simulate results in random tests and calculate the probability of something happening by using programming.

[11] https://scratch.mit.edu/studios/25998548/

[12] https://repl.it/@SusanneStigberg/maprogterning

- We introduced programs related to drawing geometric figures[13] and participants modified code, e.g., drawing circumscribed and inscribed circles[14]. Competence aim for 10th grade: investigate mathematical characteristics and connections by using programming (Ministry of Education and Research, 2019).

Data collection

We collected empirical data in semi-structured focus group interviews conducted at the beginning of workshops 2-5. We chose to divide the teachers into two groups of four teachers according to Wilkinson's (2009) recommendation to engage "a small number of people in an informal group discussion, focused on a particular topic or set of issues" (p. 177), resulting in a total of eight recorded focus group interviews. We chose this method as an expedient means of exchanging experiences, providing grounds for opinions, presenting new opinions and challenging each other (Wibeck, 2015; Wilkinson, 2009). The participants were informed of this purpose in advance. They were asked to discuss and reflect on their own experience of teaching programming in mathematics in their classroom based on what they had planned and learned during the workshops. We used the same set of questions[15] for each interview focusing on the challenges teachers and students experienced related to programming and experiences of using the UMC approach. The interviews lasted between 30 and 45 minutes and were recorded and transcribed in full. In the following, we refer to the interviews as I1 – I8.

Data analysis

The empirical data was coded deductively according to the framework developed by Mishra and Koehler (2006). We used a unique colour for each component of the TPACK framework and created a table with statements and related components. First, we coded the data separately and then discussed differences and agreed on code statements.

We applied the following coding rules:

- To code a statement as TPACK, the teacher explicitly mentioned the technology or relates the content to technology when talking about how to teach programming in mathematics class e.g., a teacher was talking about how she is using the example code and how she is challenging the students to extend the program to handle two dices for exploring probability in mathematics.
- When general pedagogical concepts were related to mathematics, the statements were coded as PCK, e.g.: "[…] but I challenged them to try and find how many throws were needed to make it even [...]" (I5)
- When teachers discussed the relationship between technology and how it affects or is related to pedagogy, the statements were coded as TPK, e.g. "Now we have gone straight to the try-and error method, we have got something done [a ready-made program]". (I8)
- When teachers discussed how technology or programming is related to mathematical content, but not relating it to teaching. Those statements were coded as TCK, e.g., "...it was the

[13] https://repl.it/@SusanneStigberg/GeometryTurtlePatterns

[14] https://repl.it/@SusanneStigberg/GeometryInnerOuterCircleSquare

[15] https://docs.google.com/document/d/1PJWv2CZ4zCrwyAZBtUKdX5AMFWUBs6aGPNPM5MGbLhI/edit?usp=sharing

> conceptual thinking, you must build it step by step and attack the problem, and especially debugging to try to examine if it ends up right, so that you can relate the process to solving a mathematical problem" (I6)

Critical reflection on methodology

Our methodology follows a qualitative research paradigm documenting idiographic statements and not generalizable knowledge as demonstrated by the positivist paradigm (Kvale & Brinkmann, 2017; Wibeck, 2015). We are interested in meaning: gathering and analysing teachers' reflections to build understanding to support the development of teachers' technological, pedagogical professional knowledge when integrating programming into mathematics teaching. As qualitative researchers, we are the primary instrument for our data collection and analysis. To minimise individual researcher bias, we followed a predefined structure for data collection and performed the interpretative analysis as a group. The design of both workshops and data collection may have influenced our findings. However, using this approach enables us to develop a relationship with the participating teachers as trusted parties, and we can gather in-depth reflections that would not otherwise be possible. The use of informal, semi-structured interviews during the workshops supported our aim to create a safe and trustworthy environment for the participants. Finally, the interdisciplinarity of the research project with experts from mathematical didactics and computer science provided a strong background for investigating the interconnected research problem.

Findings

The results highlight the technological, pedagogical and content knowledge teachers demonstrate after participating in the intervention study, categorised as TPACK, TCK and TPK. PCK is omitted, since we found little evidence in the interviews (see 1). The findings presented are based on what teachers report they are doing in the classroom and their reflections on how the UMC approach affords programming in the mathematics classroom.

Component in the TPACK framework	Total number of statements
PCK: pedagogical content knowledge	5
TCK: technological content knowledge	25
TPK: technological pedagogical knowledge	27
TPACK: technological, pedagogical & content knowledge	26

Table 1. Summary of number of statements related to PCK, TCP, TPK and TPACK

Technological, pedagogical and content knowledge

Teachers relate UMC to problem-solving, clarifying the connection between programming and mathematics, and adapted teaching as types of technological, pedagogical and content knowledge. Several teachers stated that UMC enables students to practise problem-solving when they change the code (Modify): e.g. "The ability to then retrieve, sort and find the way forward, that is, to check, troubleshoot then, …" (I8), or "… realised that they had to learn by trial and error and replace...and tell them that it doesn't matter if you make changes, because we always go back to it" (I8). These statements are linked to both problem-solving strategies and explorative ways of working, which are listed as core elements in the national curriculum (Ministry of Education and Research, 2019).

Teachers explicitly mention trial and error (Pólya, 1971) as one problem-solving strategy. Another strategy is to divide up a problem (Lester, 2011), which can be recognised in the teacher's statement ”to sort and find the way forward” (I8). Training students in problem-solving has previously been challenging for teachers, and they state that using programming and UMC in particular may be one approach to reaching this objective. Teachers also highlight that applying UMC enables students to immediately start modifying programs presenting mathematical concepts. This appears to strengthen the relationship between programming and mathematics. Moreover, teachers reported that their students liked the UMC approach because it provided a starting point (I5). Changing ready-made codes is easier, ”you get everyone involved somehow” says one of the teachers (I6). This may indicate that UMC can contribute to adapted education. Teachers also describe UMC as fostering mathematical discussions among students. Teachers reported that using the programming examples for throwing a dice, both in Scratch and Python, led to the students discussing what different codes could mean and what changes they had to make to construct programs for multiple dices or dices with alternative faces, as one teacher summarised: ”this led to good discussions in class and raised the quality of the conversations” (I8). Communication and argumentation are core elements in the national curriculum (Ministry of Education and Reserach, 2019) and have been challenging to achieve in conventional teaching (Stein et al., 2008). Finally, teachers reported that ”students improve their ability to endure difficulties in mathematics when programming” (I6). This is also in line with the new curriculum that states that the opportunity to solve problems and master challenges on their own, contributes to developing the students’ endurance and independence (Ministry of Education and Research, 2019).

Technological content knowledge

Teachers refer to how they started to learn programming themselves in a mathematical context. They reflect that students can focus more on the mathematical subject by using UMC (I7 and I8). One of the teachers stated that if they had worked exploratively trying to create their own program (e.g., throwing the dice), ”I wouldn’t have stood a chance, and it wouldn’t have been fun either” (I8). Throughout the workshops, the teachers demonstrated that they could use and modify the example programs provided and relate them to mathematical concepts. However, they did not feel capable of creating text-based programs to present mathematical concepts to their students. As one teacher critically reflected: ”with the experience I have so far, it’s difficult for me to use programming to create something useful in mathematics” (I7). Yet, they are more confident using block-based programming (I7).

Technological pedagogical knowledge

Teachers’ reflection on how UMC lowers the programming threshold in the mathematics classroom demonstrates their technological and pedagogical knowledge. ”It kind of lowers the entry threshold, … if you’re given something prepared that you have to change, it's easier to see what you have to do, than if you have to start from scratch with text programming” (I7). Teachers focus on the modify phase, where students change program code, and how it supports the students’ understanding of programming. Teachers express that UMC is needed more when working on text-based programming, as it is less intuitive and visual compared to block-based programming.

Conclusion

In this paper, we have presented how mathematics teachers reflect on Use-Modify-Create to support their technological, pedagogical and content knowledge when utilising programming in mathematics for 8th-10th grade after participating in a development and research project. To analyse our empirical data, we chose the TPACK model developed by Koehler and Mishra (2009). Our interpretation of the model is explicitly based on coding rules in the method section. We found no prior research examining UMC for programming in mathematics education. Our study indicates that teachers reflect on UMC to support adapted education, mathematical discussions, problem-solving and exploration, which are core elements in the mathematics curriculum (Ministry of Education and Research, 2019), as well as strengthening the relationship between mathematical content and programming. For future work, we propose to conduct research in the classroom to validate our findings, as well as to develop programming resources covering the competence aims that can be used by teachers. In our study, we only explored the use and modify aspects of mathematical programs. An additional point of departure for future work could therefore be to investigate means of utilising the whole UMC approach.

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Identifying mathematical problem solving in a preschool activity

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In this paper, we report on a study aimed at identifying mathematical problem solving in a preschool activity. Observations of mathematical activity with 9 preschool children have been analyzed in accordance with Variation Theory principles for how new aspects of phenomena can be discerned to identify necessary conditions for an activity to be considered problem solving. We have identified two such conditions. The first is attentiveness to different approaches in completing a task. The second is the availability of different types of activities open for children to transcend the given features of a task. The results show how the planned activity in empirical focus gives children limited opportunity to distinguish and work with problem solving. We argue that the results of this study can be used as a starting point in the planning of mathematical problem solving and to help preschool teachers in developing their work, but also to facilitate future research on the topic.

Keywords: Mathematics, Preschool, Problem solving, Variation Theory

Introduction

In this paper, we direct attention to early childhood education and aim to investigate problem solving as part of basic mathematics education. Preschool is an important part of the educational system since children's understanding and ways of thinking have a starting point in the early years when they encounter both formal and informal learning situations. Problem solving is often considered to be both a means and a goal for learning, particularly in mathematical learning contexts. In the Swedish preschool curriculum, problem solving is also highlighted as something children need to become familiar with and develop knowledge about (National Agency for Education, 2018). However, even though research on problem solving in mathematics education is extensive, how to conduct and implement problem solving as a means and goal for learning in preschool mathematics education is not evident. This implies a motive is needed to specifically direct attention to problem solving in preschool mathematics education by focusing on the conditions for learning brought about through different material conditions. To plan and conduct such a study, it was necessary to identify and conceptualize the phenomenon of ”problem solving in preschool education”. Thus, in this paper, we aim to deepen knowledge of what constitutes problem solving in a designed mathematical activity based on empirical observations by identifying necessary conditions for an activity to be considered as one that is problem solving. This paper thereby has both a theoretically driven aim – to conceptualize problem solving in relation to preschool mathematics – and a methodologically driven aim – how to generate data that allows the identification of the characteristics of problem solving in preschool mathematical activity. To identify what criteria emerge as constituting an activity as problem solving, we analyze observations from preschool through the lens of Variation Theory of Learning (Marton, 2015). Variation theory contributes to our analysis with theoretical tools that allow the recognition of contrasting ways of experiencing how to solve a task as an indicator of potential problem solving.

Research on problem solving in mathematics education

There is a large body of research on problem solving. Pólya (1957) stated early on that problem solving is the heart of mathematics and described it as an art form. Since then, several researchers have contributed to our knowledge of problem solving in relation to learning. For instance, Stanic and Kilpatrick (1989) identified three ways of using problem solving in teaching: as an independent skill, as means to develop other skills, or as a motor in the teaching process. The latter method closely resembles Pólya's view on problem solving as an art form and a practical activity. Based on these descriptions, problem solving can be understood as forming part of any teaching practice for children at any age.

Research has shown that mathematical problem-solving processes are similar regardless of age, suggesting that young children use the same kinds of strategies as older ones, but that limitations in terms of experience result in the use of different strategies (Askew & Wiliam 1995). Clements and Sarama (2004) conclude that children participating in problem solving activities may facilitate mathematics learning, the development of cognitive processes, and increased motivation, but may also facilitate social and cultural knowledge. For this to be true, however, children must not only be able to solve tasks in familiar situations, but must also be able to transfer their experiences and skills to those contexts that are more novel. Therefore, a central question for the field of knowledge about problem solving in education is how to conceptualize problem solving that supports learning in the early years.

According to Hattie (2017), one way of describing problem solving is to compare routine and problem situations. If a child makes use of strategies that they know well to solve a task, it can be considered a routine situation or task. In such situations, mathematical concepts and skills can be taught and further developed. If, on the other hand, the child approaches a task and does not have a strategy ready to use, it may be considered a problem and the child must find new strategies or new ways of applying familiar strategies to new situations. This means that what is a routine task for one child may be a problem for another child. In summary, problem solving is described in many ways in the literature, but for the purposes of this study, we consider an appropriate problem in mathematics teaching to be open to an encounter with different approaches and multifaceted, but still realistic and meaningful for the child so it is engaging and arouses curiosity (Smith & Stein, 2014). In sum, to facilitate learning, an appropriate problem should balance elements of familiarity and novelty.

Methods

With the aim of conceptualizing and identifying the characteristics of problem solving in preschool, an activity was designed and conducted to allow the first author of this paper to observe the extent to which problem solving took place and how it was achieved in interactions among children and teachers. In the process of designing the activity, specific attention was paid to maintaining an openness to different problem-solving approaches, the realism of the task in relation to children's experiences, and to the task being meaningful for the child to solve (see Smith & Stein, 2014). Furthermore, the activity was designed to include mathematical areas that are common learning objectives in preschool. The starting point for the activity design was Stanic and Kilpatrik's (1988) way of using problem solving in teaching as an independent skill and as a motor in the teaching process. The designed activity tasked children with helping an imaginary figure ”Kim” to find his way on a map while encountering different obstacles (see Table 1).

	A hand puppet mouse, Kim, has moved into the preschool. Kim has a map of the city where the children live. On the map are various buildings such as a shop (Ica), the preschool, the bus station, the football field, etc. Kim has three assignment cards for the children. Each card contains a task that he needs help with. First task Kim has been to preschool and is on his way home. Today, Kim will exercise and needs to get home quickly. Can we help Kim find a way to get him home quickly? Second task Kim has to drive to the store first and after that, he has to go to preschool. Kim does not like to go past the bus station in the area. There is always a lot of traffic there and Kim has to sit in the car for a long time. Are there other paths he can take? Third task Kim wants to go visit his grandmother, but there are road works in the area. Can we help Kim find different ways to Grandma's house and not get stuck in traffic?

Table 1. The map and examples from the assignment cards (translated from Swedish)

The research was conducted at a municipal preschool that follows the national curriculum and was not engaged in any extra initiative for mathematics at the time of data collection. This means that the children in the study had not received any mathematics-specific instruction. Observations were conducted at the preschool in a room familiar to the teacher and the children. The teacher presented the task to the children and supported their completion of the task. The activity began with an introduction to the activity to ensure that all the children had sufficient prior knowledge to understand what the activity was about. The teacher presented the mouse, Kim, and his problems and asked open-ended questions to the children, helping them to express their ways of understanding the task and thereby seeing alternative ways to complete the task. Examples of questions that the teacher used are: What do you think? What do you mean? Describe your thoughts etc. This facilitation was suggested by the researcher and intended to help the children develop their analytical skills, and their ability to reason in extended chains of argument.

The data generated for analysis consists of three separate video observations of 14, 17 and 19 minutes in length. Each video features a teacher and three children between the ages of four and five. The parents of all the participating children were informed of the study and the kind of data that would be collected. They gave their written consent before the start of the observations. To protect the integrity of participants, the names of the children have been replaced with pseudonyms and the names of teachers and the preschool where the research took place have been omitted from publications.

Analysis

Our starting point for understanding learning and teaching is that there is only one world and that different people experience phenomena in this world differently (Marton & Booth, 1997). The way in which a person experiences certain phenomenon depends on which aspects of a phenomenon the person in question can discern. Each phenomenon (e.g., a task) can thus be experienced differently by the children in a preschool group as each of them acts to try to complete a task in a way that reflects their understanding (or experiencing) it. In the activity in focus in this study, there are different ways to understand the task, to find paths to its completion, and to justify the choices made. This gives us a theoretical framework for revealing the criteria for problem solving in a preschool activity through the contrasts that occur among children's attempts to complete the task as they express and explore their own and others' ways of experiencing it. When contrasts appear in ways of understanding and handling a task, there is an opportunity to observe and identify the problem solving in relation to the designed activity. This means that cases where children express different ways of solving a given problem, both in words and in deeds, become relevant units for analysis. From this perspective, to learn something is to change one's perception of a given phenomenon. This is done by bringing new aspects of the phenomenon into focus and providing new possibilities for interpretation. A child who has not discerned the necessary aspects of a phenomenon does not necessarily have a lack of ability, but may instead have missed relevant critical features in the learning object, perhaps because the teacher has not made it possible for the children to perceive the critical features (Marton, 2015). Through analysis of observations of different ways in which children solve a task, we can find the specific dimensions that are relevant and discern which aspects are critical for children's reasoning within the planned activity. In this way, we can identify the extent to which the planned activity can be interpreted as problem solving, and thus recognize the criteria by which that activity can be understood as problem solving.

Results

While investigating what constitutes problem solving in the designed activity, we can first conclude, based on observations, that the children participating in the study had different ideas of how to complete the task. This is an important insight since the task was designed to offer the children opportunities to make use of experiences from similar activities, but also provides for different ways of completing the task to be displayed in their joint actions. To be able to present children's different ways of understanding and completing the task, we have chosen excerpts with an orientation that indicates the use of spatial thinking when children find different paths on the map.

Teacher: Now we need to help Kim go to the supermarket without passing the bus station. Kim does not like the bus station, as he has to sit in the car for a long time.

Eliot:	I know, I know. He can go this way (draws a road that goes via the preschool and then to the supermarket)
Anna:	No, I know another way. He can go to the forest where there are no buses and then to Ica.
Teacher:	Can you think of anything else? (asks Maja the question)
Maja:	Yes, he can take the bus (she looks around and then moves the map in front of her. She points to the photo of the preschool and then to the photo of the bus station).

The sequence above shows that all the participating children have distinguished the objects on the map, as well as the starting point where the mouse, Kim, is located and the end point. They have also distinguished that different roads reach the same end point and that the path is not predetermined. The children have identified all the necessary aspects to complete the task, which can be interpreted as the children's previous experiences of similar phenomena appearing in new contexts. However, following Smith and Stein (2014), the excerpt above cannot be considered to be problem solving because it does not show that the children experience any contrast in their ways of perceiving the task. For an activity to be interpreted as problem solving, we assume it must pose a challenge to children and contain something unknown. Hattie (2017) emphasizes that if a child uses a new strategy, it can be considered as problem solving, and that a child must find new ways or apply familiar strategies to new conditions to solve the task. In this sense, the knowledge that children have before they work with the task determines whether the task becomes one involving problem solving or not.

In the sequence below, the children are challenged by the teacher in their way of thinking to compare and motivate their answers. For example, when the children found the shortest way, they were asked to justify their choice. By comparing different answers, the children became aware of alternative approaches and differences became apparent. In this way, contrasts emerge that arise between the children's solutions. To be able to call an activity problem solving, children need to be able to distinguish different aspects of the same task, and contrast is used to present differences between solutions, with a focus on what distinguishes theses from each other.

Teacher:	Kim has trained today and needs to get home quickly. Can you find the fastest way home from preschool?
Albin:	I can, just do this (he draws a straight line from the preschool to Kim's house.)
Teacher:	Is this the fastest way home?
Albin:	Yes, the house is so close to the preschool that he can only run straight.
Teacher:	What are you thinking about? (Asks the question to Tim.)
Tim:	I think he should go to Ica first if he is a little hungry and then home.
Mimi:	No, that will take a long time. I can show you! (Draws a path from preschool to home that follows the first child's solution.)
Teacher:	Aha, you think the same as Albin.
Mimi:	No, you don’t see mine going in the door and not to the tree. You have to enter through the door.

Tim:	He’ll be hungry after training and he has to go to Ica first.

In the present example, the children distinguish the core of the task, the connections between the whole of the problem and different parts such as the objects on the map, the start and end points, and the fastest path. The contrast between the children's different perceptions means that the task can be interpreted as problem solving. For an activity to be called problem solving, children thus need to pay attention to different approaches when completing the task. The answers from Albin and Mimi highlight the contrast in the mathematical content, i.e., the fastest way. Tim is more focused on the contextual content and makes a remark about needing to buy food when you are hungry. The contextual content of the task is something that children may notice, but in this activity, it can be largely ignored because it is not clearly linked to the mathematical content and does not readily contribute to a solution. Therefore, our conclusion is that being able to identify the problem and find an adequate solution requires an ability to sort and prioritize important content in the task. The children in the excerpt above do not display agreement that all alternatives to hand are suitable ways to complete the task.

In the following example, the teacher repeats the children’s proposed alternatives for completing the task and asks questions about the strategies they were using.

Teacher:	Which path did you draw? (asks Linda)
Linda:	This one!
Ebba:	No, that was mine!
Teacher:	How do you know it’s your path?
Ebba:	I drew it like this (shows in the air) and so it turned out like this (the road became crooked in one place).
Teacher:	But, do all your solutions work? Has Kim reached Grandma’s?
Noa:	My path is the fastest!
Teacher:	What do you think now? How do we know that road is the fastest?
Noa:	Because it is!

In the excerpt above, the teacher acts to make it possible for the children to distinguish differences between different solution strategies and to express agreement when they lead to the same result. This makes it possible for the children to see the task from different perspectives and distinguish alternative ways of solving it. In another example of accomplishing an agreement on a solution, the children have once again distinguished the necessary start and end points and found ways to grandma's house on the map. They seem to have difficulty interpreting the two-dimensional map as a plan projection of a three-dimensional space because they argue that the house is high up and they need a ladder. Grandma's house is located at the top of the map and behind the house is the blue sky.

Teacher:	How can Kim go to grandma and avoid the road works?
Albin:	He can ride like this… He needs to use the ladder and climb first.
Teacher:	Now, he's going to drive and find his way to his grandmother.

Tim:	He should go like this (draws the road behind the house) and then use the ladder to go up.
Teacher:	It's difficult to drive on a ladder.
Mimi:	He can have a car that can fly.

To handle a map, a child must be able to depict objects from different positions in order to imagine something from different perspectives. In the example above, it seems that the children's experiences are in contrast to the task in hand and so they try to find solutions involving a flying car. A deeper understanding of the properties of the map is required to solve the task, necessitating several aspects of the phenomenon "map" to be differentiated. This became an issue because children had to come up with a solution that was not previously presented, and it became a problem solving task because children went beyond what was given and used a new approach. We can interpret that the nature of the task largely limited the opening up of the necessary dimensions of variation to be discerned, but interestingly, the children themselves opened up other dimensions of variation based on their previous experiences, which gave them solutions that were unforeseen. This underlines how important it is to consider the children's perspective to gain an understanding of how children experience a task. In the situation above, Albin solves the task presented on a two-dimensional map as if it were three-dimensional. It becomes a problem because the children do not have a ready-made solution, but instead, draw on their experiences to find an alternative.

Discussion

The purpose of this study was to identify problem solving in a preschool activity and see to what extent the planned activity can be interpreted as problem solving. The purpose of the study was also to understand the criteria for an activity to be called problem solving. Following Hattie (2017), we have assumed that to be able to identify problem solving in mathematics activities in preschool, we must find instances where children use new strategies so that the activity can be considered to be a problem, i.e., that the child must find new ways or apply familiar strategies to new conditions to complete the task. Taking these criteria as necessary prerequisites for an activity to become an instance of mathematical problem solving, we have identified two necessary conditions:

- The first is attention to different approaches to solving a task.
- The second is the presence of different types of activities that are open to children to exceed the given aspects of the task.

Our analysis of the designed activity confirms what has already been shown in earlier research on the importance of previous experiences and interest in children's approaches when solving problems (Öhberg, 2004; van Bommel & Palmér, 2015). This was clearly identified in the first example presented where Eliot solved the task without major problems while Maja seemed to take the available solution image on the map without being able to explain it in more detail. This indicates that children's previous experiences in part determine whether a task becomes a problem or not.

A contribution from this study is the identification of situations that indicate problem solving and situations where problem solving does not occur. All the examples presented have the common element that the children discovered the connections in different parts of the task. In the planning of a task intended to produce problem solving, it is necessary to understand how the parts of that task will be comprehensible to the children who will work with it. However, this is also a challenge for the children as the ones is on them to ultimately figure out which parts of an activity answer the question in hand and which parts are irrelevant and do not need to be considered. The children in this study show this by looking for solutions that go further than expected. Our results show that an activity must contain necessary variation to make it possible for children to distinguish problem solving within it. To be able to plan and carry out activities that can be considered as problem solving, we need to evaluate children's previous knowledge in mathematics and plan challenging activities that make it possible for them to see a problem and complete the task. This and the other insights from this study can both be used by preschool teachers in the planning and implementation of mathematical problem solving and to better orient research into the conditions for mathematical problem solving in early years education.

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On the value of interthinking for mathematical learning

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Collaboration, reasoning and argumentation are viewed as important processes for in-depth mathematical learning, and there is a need for more knowledge on how these aspects are interconnected and can be studied in conjunction. The aim of the current work is to use the concept of interthinking as a frame to discuss collaboration, reasoning and argumentation in mathematics. We argue that linking Toulmin's model of argumentation with a proposed framework on collaborative processes can provide detailed insights into students' interthinking. These ideas are explored though an example from a continuing education class in calculus, and we also discuss the need for future research on what a productive interthinking process may entail.

Keywords: Interthinking, collaboration, argumentation, reasoning, mathematical learning.

Background and aim

In-depth learning in mathematics comprises understanding concepts, mathematical ideas and relations, fluency of procedures and algorithms, ability to use this knowledge in various situations, reason and argue, evaluate and critically reflect, among others (Hiebert, 2013; Lithner, 2008). Research has shown that social interactions have great potential for in-depth learning in general (e.g. Mercer, 2004), and in mathematics in particular (e.g. Weber, Maher, Powell & Lee, 2008). The ways learners interact and participate in group work are therefore central elements for understanding how learning can occur and evolve. According to Mercer (2004, p. 139), however, ”When working together, we do not only interact, we ”interthink””, and Littleton & Mercer (2013) describe interthinking as *the process of using spoken language to think creatively and productively together*. We believe that this definition captures important social elements involved in deep learning of mathematics, such as collaboration, reasoning and argumentation, in addition to suggesting that these elements are closely related. There is a need for more detailed knowledge on *how* these elements are related, as this will link the learning potential of students' argumentation on specific mathematical content to the different stages of their collaboration. There are not many studies on how collaboration, reasoning and argumentation are interconnected, but we build on the work of Hansen (2021). Hansen links reasoning, collaboration and agency together in her study, which focuses on the interplay between these interactional aspects as students collaborate on a functional problem. She found two distinct interaction patterns (bi- and one-directional interactions) emerging from the roles in how students engaged or refrained from engaging in collaborative work. In our study we delve deeper into the details and role of students' argumentation. More precisely we propose the frame of interthinking to investigate the interplay between the argumentative function of utterances and collaborative processes. The role of *arguments* is therefore important when studying reasoning processes, emphasized also by Lithner (2008). He describes reasoning as ”… the line of thought adopted to produce assertions and reach conclusions in task solving. It is not necessarily based on formal logic, thus not restricted to proof, and may even be incorrect as long as there are some kinds of sensible (to the reasoner) reasons backing it" (Lithner, 2008, p. 257). A distinction is also made between *imitative reasoning* and *creative mathematically founded reasoning* (CMR). The former category involves recalling memorized or previously learned rules and algorithms. In the latter category, students are to a much larger degree prone to construct new knowledge, as the reasoning should be *novel* to them (a new – to the reasoner – reasoning sequence is created), and the arguments they use should be *plausible* (supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible), and have a *mathematical foundation* (anchored in intrinsic mathematical properties of the components involved in the reasoning). Thus, the role of arguments becomes particularly prominent when performing, evaluating, and studying creative mathematical reasoning. The creative thinking part of the definition of interthinking is in our study captured by Lithner's (2008) CMR-framework with the inherent focus on the plausibility and anchoring of arguments.

The definition of interthinking emphasizes the use of *spoken language*, and further the use of spoken language to think creatively *together*, in our study captured by how students collaborate and express their reasoning via language. Wilkinson et al. (2018) claim that in the case of students' mathematical learning, a careful analysis of interactions among students "…supports a close examination of how students build mathematical ideas and communicate them via language" (Wilkinson et al., 2018, p. 2). Roschelle & Teasley (1995) define collaboration as "… a coordinated, synchronous activity that is the result of a continued attempt to construct and maintain a shared conception of a problem" (Roschelle & Teasley, 1995, p. 70), and highlight collaboration as a mutual engagement in a coordinated effort to solve the problem together. Explaining "shared conception" of a problem, they talk about "… an emergent, socially-negotiated set of knowledge elements" (Roschelle and Teasley, 1995, p. 70). Thus, Roschelle & Teasley (1995) consider learning to take place in a negotiated and shared conceptual space. Other researchers have studied the relation between argumentative and interactional aspects (e.g. Krummheuer, 2015), however not with detailed analysis of collaborative processes as discussed in this paper. Krummheuer (2015) addresses how an argument is structured in the course of interaction by studying the group members' participation. Krummheuer's focus is more on the individual, and related to who takes responsibility for an utterance, whereas this paper is more concerned with the group as a whole and how groups interthink, through studying how they build shared conceptions.

Thus, with an overarching aim to promote in-depth learning of mathematics and increase our understanding of how students come to learn mathematics, the paper's research question is: How are collaboration and arguments in reasoning interconnected, and how can interthinking be studied through the interplay between these elements?

The process of mathematical interthinking

Toulmin's (2003) model of argumentation can be useful for studying students' arguments in collaborative interactions (e.g. Krummheuer, 2015; Weber et al., 2008). In this model, an argument consists of three main parts: The claim, the data, and the warrant. A *claim* is when a person presents an argument and tries to convince others of its value. *Data* refers to the evidence being used to support the claim. Finally, a *warrant* refers to an explanation of why one can deduce the claim being made from the data presented. In addition, if the validity of the warrant (the plausibility) is being questioned, a *backing* (further justification of the warrant) may be needed. Toulmin's model presents two components in these cases: A *modal qualifier*, which qualifies the conclusion by expressing degrees of confidence, and a *rebuttal*, which states conditions under which the conclusion does not hold (Weber et al., 2008).

That arguments should be plausible and have a mathematical foundation are important aspects of what Lithner (2008) calls creative mathematically founded reasoning. Linking these ideas to Toulmin's (2003) model of argumentation, there is particularly in the warrants, the backing of the warrants, the modal qualifiers and rebuttals, that the deep mathematical foundation lies, as well as the mathematical validity of the plausibility of the students' arguments. Weber et al. (2008) discuss how group discussion can contribute to learning by focusing primarily on warrants. They focus on how presenting and defending mathematical claims, carefully attending to each other's arguments, and challenging these arguments if they did not find them convincing, can lead to learning opportunities. The role of warrants became highly important in their study through the parts of the communication where the students challenge each other's thoughts and ideas. We go further into the idea of interthinking (Mercer, 2004), focusing on shared reasoning in the collaboration between students. Building on other well-known frameworks for studying collaboration and the co-construction of ideas (Alrø & Skovsmose, 2004; Mueller et al., 2012; Roschelle & Teasley, 1995), we distinguish between five collaborative processes in our project. To study the interplay between argumentation and collaboration, it is necessary to discuss how the frameworks focusing on collaboration can be studied in conjunction with Toulmin's (2003) framework on argumentation. We view the starting point of a collaborative reasoning process to be when someone introduces an (1) *initial idea:* An idea which has to be relevant for the problem context and a starting point of a process towards a solution. Similar to Lithner's (2008) CMR-framework, an initial idea should be plausible to at least one group member, but it need not be a well-developed idea. It could be what Toulmin (2003) refers to as a claim, but it may also be a vaguer strategic idea of how to proceed with the problem. (2) *Expanding:* In order to maintain and develop the collaborative reasoning process, the participants need to further build their shared conceptions from the initial ideas. Here, the students build further on other's thoughts and ideas, and contribute to develop and expand the group's shared understanding. Typically, in this collaborative process, we see warrants for the claims and backing of the warrants. As the expanding occurs by communicating with the group, we can by observation extrapolate insight into how the students understand the problem, as well as their own and other group members' arguments. (3) *Confirming*: Confirming and supporting other's thoughts contributes to connecting each participants' conceptions and reasoning with the rest of the group. It helps the group gather around joint ideas, thus creating a feeling of shared understanding of where they are in the process, which is important to develop interthinking. (4) *Correcting:* Correcting and challenging input can lead to divergences and conflicting ideas, where reformulation, repairing, and further explanation and deeper anchoring (further warrants and backing) of arguments is important. Finally, instances of (5) *insecurity*: A gap in the students' understanding and a need for clarification may occur, oftentimes through questioning each other's and one's own ideas. Such questions and input spur further argumentation among the other participants and are thus important for building and developing interthinking.

The above preliminary framework describes how the collaboration in a group can be coded as a starting point for further qualitative analysis. We have also argued how the collaborative processes relate to the argumentative function of utterances as described in Toulmin's (2003) framework. The application of the framework is now illustrated through an example.

Exploring the process of interthinking through an example

In this example we follow four students A, B, C and D (all women) in a continuing education class in calculus at a Norwegian university, in a passage from their collaboration on the problem in Figure 1. Both the problem and the discussion were in Norwegian and have been translated to English for the current work. Student D is silent in this particular excerpt, and we therefore discuss the interthinking between A, B and C. Important concepts from the theory are italicized. Prior to the passage below, the students used one and a half minute to introduce some first interpretations of the graph and context, and to negotiate a *shared conception* of the situation that the graph represents. The shared conception is a verbalization of what grew from this process. This shared conception is not formally mathematically correct, as it states that the area of the graph with a positive slope indicates water being filled into the bottle, the areas of the graph with a zero slope means that water is neither being filled nor poured out of the bottle, and that the areas of the graph with a negative slope indicates water being poured out of the bottle. Nevertheless, the students *confirmed* and supported this interpretation, indicating that it was *plausible* inasmuch as it was accepted by the group. Their interpretation can be seen as an *initial idea,* a starting point for their process of interthinking, and the following passage shows the students' interthinking growing from this initial idea.

Anne wants to fill an empty bottle with exactly 3 dl of water. The figure below shows how much water is entering or leaving the bottle at time t. Discuss if Anne is succesful.

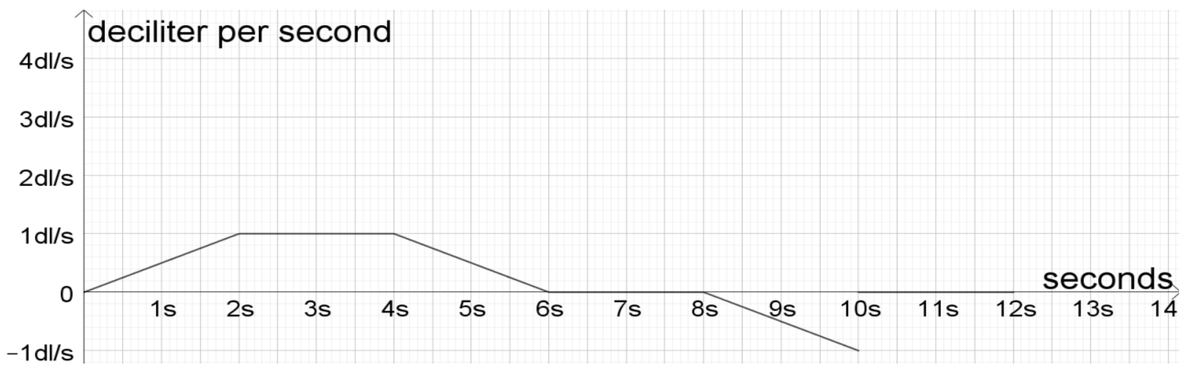

Figure 1. The problem given to the students

A: So, here [*pointing where the graph is increasing],* we fill it with one deciliter, watch it for a while, think to ourselves that "shit, this is wrong", pour one deciliter from the bottle again – just the same as I filled it with, watch it for a while, and then pour another deciliter from it. So, if you have a bottle that takes half a liter, then you have … You've poured out two deciliters from it.

B: Yes … It doesn't say how big the bottle is, though. It just says that you want to fill it with exactly three deciliters. But do you also interpret that as when the graph increases, we add water, and when it decreases, we pour water out? [*C confirms*].

Here, A verbalizes a *claim* that, in total, two deciliters of water will be poured from the bottle, through reformulating the interpretation from their initial idea. This way, she ensures that her claim is in line with their *shared conception.* The justification of her claim (the *warrant*) is *anchored* in their (faulty) interpretation of the graphical representation and is *plausible* by supporting their strategy and being founded in their initial idea. B brings a new piece of information about the bottle to the *shared conception* (thus *expanding* their initial idea), which she connects to the problem formulation. She then offers some *confirmation* to A's warrant, through reformulating it as a question, which invites the group members to connect their conceptions to each other's through *expanding* or *challenging* the claim. As the following extract shows, B's combination of information on the bottle size and a question initiates collaboration on a shared reasoning sequence that serves useful in the development of the group's interthinking:

A: It doesn't say how much water there is in the bottle, just how much you add, so it's kind of difficu- or, what I'm thinking is that it's a little difficult to know if she succeeds. I mean; if it is four deciliters, no, yes, if there's four deciliters in the bottle to begin with – then she succeeds. If the bottle is empty- it cannot be empty, 'cause it isn't…

C: It says here that the bottle is empty when you start. *[A confirms]*

Here, A *expands* their shared conception with attention to B's comment on the bottle size, highlighting the importance of B's input and how the students think *together* by expanding on a joint idea. In this expanding-phase, she (again) *anchors* her reasoning to the negotiated interpretation of the graph, but adds a new element (*expanding*) through outlining a scenario where Anne will succeed. This scenario is quickly followed by a *rebuttal* – the outlining of another scenario where their idea does not hold. This can be seen as the early beginning of *challenging* their shared conception. C then *corrects* A's utterance, revealing that the rebuttal is the actual case (and this discovery is *confirmed* by A). Now, their shared conception is challenged and needs to be repaired through the introduction of new initial ideas that can serve as a new starting point in their process of interthinking:

B: But how can … Let's see, how much water is added…

C: I'm thinking; is this [*the first area with zero slope*] really a pause? Isn't it that it's filled with one deciliter all the way here? *[A and B both confirm]*

C: So, here, it is filled with…

A: Ok, so, if we look at it like that. Then it is filled with 0.5 deciliter per seconds for – can I draw it? – for two seconds. So that's 0.5 deciliter times two. Right? So you've filled one deciliter there. And then there is one deciliter for two seconds. Times two… That's two deciliters. But then she pours out … No, no, no, wait, it isn't the same all the way here.

A critical moment for the interthinking occurs when their *shared conception* of what the graph represents is *challenged* and a new interpretation – a new *initial idea* – is introduced. C offers a *claim* (which becomes a new initial idea) that the zero-slope area represents a period where the bottle is filled with a constant rate of one deciliter per second (instead of no water being neither filled nor poured from the bottle). Her *data* for this claim is the sequence of the graph with zero slope represented by $y = 1$. This claim receives instant *confirmation* from A and B, who then seem to accept the claim and data as *plausible*, without anyone offering any justifications to it (*warrants*). It shifts the ”interpretational paradigm” in which they are working, from the slope of the graph showing the amount of water entering/leaving the bottle, to the y-value of the graph showing the amount of water entering/leaving the bottle. As a result, a *refined shared conceptio*n is established that is *creative,* as it is *anchored* in the mathematical properties of the graphical representation and *plausible* by supporting their interpretation. This process does not involve much negotiation, as they all seem to agree on this shift (meaning that no one expressed any questions regarding the plausibility of this claim). Further, A *expands* on C's claim with a *backing of the warrant,* with more formal mathematical *anchoring* and with *plausible* justification (this time being both plausible for the group accepting it and from a strictly mathematical view). In the last part of her *backing* of the *warrant*, A (again) interprets the negative slope as water being poured from the bottle, but quickly corrects herself.

The process described in this example captures the students' *interthinking* through highlighting how collaboration and arguments helps them *think creatively together* through sharing, connecting and expanding their conceptions. Following the excerpt, their interthinking continues with *increasing creativity* through more formal mathematical *anchoring* and *formal plausibility*. They *collaborate* in constructing a *shared conception* that a negative slope means that <u>less</u> water is being filled to the bottle, and that the graph has to take negative y-values when water is being poured from the bottle. Further, they collaborate on strategies for calculating the total amount of water in the bottle at given times. In this, they introduce mathematical terms like *slope*, *constants* and *delta x and delta y* to their *shared reasoning.*

Concluding discussion

This paper has discussed how collaboration and argumentation can be studied in conjunction within the frame of interthinking, where creative reasoning and shared constructions are central elements, and further how this interplay can facilitate in-depth mathematical learning. We view the ideas presented in this paper as an important step towards developing a framework for analyzing students' mathematical reasoning in collaborative situations. In the example we relied on Toulmin's (2003) model of argumentation linked with a proposed framework on collaborative processes. To capture the process aspect of interthinking through a problem-solving session, we promote the importance of studying sequences of utterances, and herein link the argumentative function of utterances to the evolvement of collaborative processes. The turn-taking elements between the utterances are therefore important. In the example we observed that the elements and development of the group's interthinking enabled them to reason from a co-constructed and socially negotiated (though mathematically incorrect) initial shared conception, through phases of challenging (here: external challenging by the rebuttal). This resulted in the group reestablishing their shared conception through co-construction of new arguments in line with more formal mathematical anchoring and plausibility. Thus, it seems like collaborative aspects grew ”deeper” – the group moved gradually towards co-construction of arguments (a negotiating discourse). This leads us over to another important word in the definition of interthinking (Littleton & Mercer, 2013), that of *productivity.* We propose that by combining Toulmin's (2003) model of argumentation with a broad and detailed categorization of collaborative actions and processes, one can study not only the interconnection between collaborative aspects and the mathematics foundation at play, but also the development of this interplay throughout a problem-solving session. What are fruitful, thus ”productive” in this sense, steps in order to develop interthinking that lead to in-depth mathematical learning? Mercer (2004) described children’s talk through three types: *Disputational talk* (disagreement and individualized decision making, with few attempts to joint decision making and constructive criticism to each other); *Cumulative talk* (building on each other’s suggestions, however uncritically, through repetitions, confirmations and elaborations); *Exploratory talk* (engage critically and constructively with each other, statements and suggestions are often challenged, counter-challenged and justified, and decisions are jointly made). Cumulative, and in particular, explorative talk, have in previous research (e.g., Mercer, 2004) been called productive, whereas disputational talk is considered unproductive. Viewing collaboration and reasoning as evolving processes, we argue that all forms of collaborative actions described here are productive on a path to gradually more sophistication in both collaboration and argumentation. Non-acceptances and disagreements are often present in productive collaboration and are then often followed by periods of intense interaction important to incorporate individual insights into shared conceptions (Roschelle & Teasley, 1995). According to Alrø & Skovsmose (2004), such ”unproductive” collaborative actions can contribute to clarity of what is already known and create a safe classroom environment, and therefore have a learning purpose. We see it as highly important for future research to discuss what a productive interthinking process may entail, and link this to how interthinking may develop in a process from initial idea to the end product of a problem-solving session. We claim that all forms of interthinking processes play an important role in developing new knowledge and shared conceptions, and that to understand more of the evolvement of interthinking, it is important to study the whole process and what characterizes the interthinking in all stages of this process. We propose that productivity is not determined by how formally valid the reasoning is but lies in the interplay and evolvement of argumentation and collaborative processes, and shared understanding – the interthinking.

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Lávvu as a teaching arena: identification of mathematical activities

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Teachers at Guovdageaidnu lower secondary school cooperated with researchers in translating Bishop's mathematical activities from English into North Sámi. They found that at least three Sámi words are needed to cover the meaning of the activity "locating". In this paper, we conduct a further investigation of how the Sámi translations of mathematical activities work, and we explore whether the suggested verbs are sufficient. The context in our study is a teaching unit in lávvu—a traditional Sámi dwelling—where the teaching was carried out by lower secondary school students. The analysis reveals that one more Sámi word is needed to fully cover the meaning of the activity "locating". In addition, two Sámi words and not one, are needed to cover the meaning of the activity "designing".

Keywords: Bishop's activities, measuring, designing, locating, Sámi education.

Introduction

Teachers at Guovdageaidnu lower secondary school in Norway have developed an interdisciplinary teaching unit related to building a *lávvu*, a traditional Sámi[16] dwelling, with a focus on mathematics (Fyhn, Meaney, Nystad & Jannok Nutti, 2017). As part of this project, their Grade 10 (final year of compulsory school) students teach younger students about *lávvu* and mathematics. We present a teaching unit in which the students teach at another school. Both schools are Sámi schools located in rural areas in the northern part of Sápmi. A large amount of the people who live in these areas are experienced with raising and living in a *lávvu*. As such, this approach to teaching provides opportunities for the school to connect knowledge and experiences from local everyday practices to concepts in school mathematics. The teaching unit focuses on a) the organisation of the poles that shape the framework of the *lávvu* and b) the size (area) of the floor inside the *lávvu*. The teaching unit is based on the mathematics curriculum for Grade 7, "[t]he aims of the studies are to enable pupils to select suitable measuring tools and carry out practical measurements in connection with day-to-day life" (Ministry of Education and Research [KD], 2013, p. 8).

Guovdageaidnu lower secondary school is located in the northern part of Sápmi, and the language spoken in this area is a dialect of North Sámi[17]. The school's teachers believe in what local, culture-based mathematics education can provide. One reason for this is the "glow" they observe in students' eyes when lessons in mathematics are based on local culture (Fyhn, Jannok Nutti, Nystad, Eira & Hætta, 2016). Their experiences are in line with Matthews, Cooper and Baturo's (2007) observations that many Indigenous students perceive mathematics to be a subject in which they must become "white" to succeed. This paper is about young students' mathematics teaching in an Indigenous

[16] Sámi is an Indigenous people who live in the northern part of Fennoscandia. Sápmi is the North Sámi name for the Sámi people's historical settlement area. Retrieved from https://snl.no/samer

[17] When the term Sámi is used in this paper it means north Sámi unless otherwise stated.

language and in a cultural context, and is in accordance with Trinick, Meaney and Fairhall's (2016) point that Indigenous mathematics education should relate to both culture and language.

Bishop (1988) claims that just as every cultural group generates its own language and religious beliefs, each cultural group is also capable of generating its own mathematics. He argues that mathematics is the outcome of six activities—*counting, locating, measuring, designing, playing* and *explaining*—that have appeared in every investigated culture to date. Bishop's activities constitute a framework for projects where Guovdageaidnu lower secondary school's mathematics teachers cooperate with researchers. They developed a mathematics teaching unit based on Sámi braiding (Fyhn, Jannok Nutti, et al., 2017), where the activities *counting* and *playing* were found relevant for describing the Sámi cultural practice of making *ruvden* (round-shaped) cords. This led to the need to translate Bishop's (1988) activities into North Sámi. Fyhn et al. (2018) encountered several challenges during the translation process; for instance, Sámi languages do not use overarching terms in the same way as Germanic languages do. The north Sámi language has several meanings of the mathematical activity *locating* (Fyhn et al., 2018). One meaning is finding a suitable area to place a *lávvu*, while another meaning of the word is to search for and find something. A third meaning is to find something that is lost. One aim of our study is to elaborate further on possible North Sámi translations of Bishop's (1988) activities. Another aim is to quality check Fyhn et al.'s (2018) findings that three different North Sámi words—*bidjat*, *gávdnat* and *ohcat*—are necessary to cover the meaning of the English word ”locating” (Sámi translations are listed in infinitive form). It is important to test the translations in different contexts to increase their accuracy.

To avoid misunderstandings, we refer to the teaching students as 'teenage teachers (TTs)', and the school where the teaching took place as the 'learning school'. The learning school is a compulsory school comprised of 1st–10th grades. Grade 5–10 students at the learning school are referred to as 'learning students (LSs)'. The TTs included eight 10th grade students: four girls and four boys. Our study focuses on the TTs' teachings of mathematics and seeks to answer the following research question: How do the TTs' teachings about *lávvu* relate to Bishop's (1988) activities of ”measuring”, ”locating” and ”designing”? For pragmatic reasons, we chose to narrow our focus to only three of the six activities, due to the restricted size of this paper. The research question is illuminated by a presentation of how the North Sámi version of these three activities works in the analysis of the TTs' teaching unit.

Intangible cultural heritage of the Sámi

An intangible cultural heritage includes traditions or living expressions inherited from our ancestors and passed on to our descendants (United Nations Educational, Scientific and Cultural Organization [UNESCO], 2018). According to the core curriculum in Norway (KD, 2017), intangible cultural heritage is an important topic in Sámi schools. The traditional upbringing of children is one part of intangible cultural heritage. Learning through observation, participation and trial-and-error is central to the upbringing of Sámi children (Balto, 2005). Children may try out a task, and success or failure to accomplish the task is valued in the community as part of the learning process.

The intangible cultural heritage of the Sámi includes how to raise a traditional *lávvu*. When raising a *lávvu*, one starts with three poles, the *válddahagat*, which are Y-shaped in one end, as shown in Figure 1. The *válddahagat* are assembled together in a triangular construction that is stable and reliable

(Keskitalo, Fyhn & Nystad, 2017). When the *válddahagat* and the remaining poles with straight ends are assembled, they constitute the framework of the *lávvu* (Buljo, 1994). How to arrange the poles around the fireplace in the *lávvu* centre is also part of Sámi intangible cultural heritage.

Figure 2. *Válddahagat*. Right: drawing by K. M. E. Buljo. Middle and left: photo by K. Nystad. Reprinted from "Sámi Cultural Properties of the Numbers Three and Four" by J. H. Keskitalo, A. B. Fyhn and K. Nystad (2017), Journal of Mathematics and Culture, 11(3), pp. 81–111. Reprinted with permission

Culture and mathematics

Mathematics is an outcome of human activities (Bishop, 1988). Work related to the practical experiences of Sámi cultural traditions allows for inductive approaches to mathematics, but in school, mathematics is often taught through a deductive approach. *Counting* is defined as organizing and comparing discrete entities. *Playing* is defined as participating in games and play while following more or less fixed rules. *Locating* is defined as placing and finding things, people and events in space. *Measuring* is comparing, arranging and quantifying values that cannot be counted. *Designing* is creating or processing a material that can be used for something, either aesthetically or practically. *Explaining* is about moving away from the activity itself and putting it in context with other phenomena. Bishop (1988) also noted that one should never exclude one activity when performing another activity. One activity may share features with another activity, and one mathematical activity can lead to another (Shockey & Silverman, 2016) (see Figure 2).

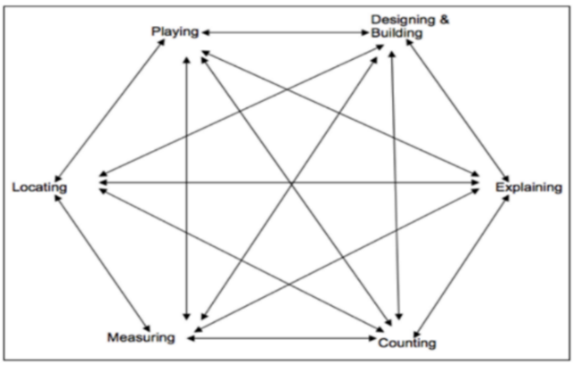

Figure 3. Bishop's (1988) six activities. Reprinted from "Ten Years Later: A Look Back on the History of the Journal of Mathematics and Culture" by T. Shockey and F. Silverman (2016), Journal of Mathematics and Culture, 10(2), p. 10–35. Reprinted with permission

Lipka, Adams, Wong, Koester & Francois (2019) point out that the Yup'ik tend to use converting asymmetric forms into symmetric ones in their measurement. This is similar to how you adjust the poles around the fireplace when raising a traditional *lávvu*.

Method

Our study is a collaboration between two researchers and one teacher. While the researchers are native speakers of Norwegian, the third author is a native speaker of Sámi. Due to language differences, discussions in the analysis process were carried out in Norwegian. The third author was one of the professional teachers who worked with the TTs. The first author is a Norwegianized Sámi who was educated in the Norwegian school system and speaks Norwegian as their first language. The Norwegian school have no tradition of including Sámi culture in their teaching. Thus, for generations, many Sámi children have neither been educated about the Sámi language nor the Sámi culture (Høgmo, 1986).

The analysis presents a selection of Sámi words, which best categorizes the data in relation to Bishop's (1988) three activities: *locating* (*bidjat*, *gávdnat, ohcat*), *measuring* and *designing* (*hábmet*). Traditionally, searching and finding (*gávdnat*) where to place (*bidjat*) the *lávvu* is a process that depends on the terrain. Wind direction must also be taken into consideration. The Sámi word *ohcat* means to search for something for which one does not know the whereabouts. There was no need for this locating activity, *ohcat*, during the teaching unit; no search was conducted for anything that was lost. The data consist of several sources that provide descriptions of the teaching unit. An audio recorder was placed near the fireplace, close to the centre of the *lávvu*. In order to avoid misunderstandings, the audio recordings were translated from Sámi to Norwegian by an interpreter who spoke the same Sámi dialect as the TTs. In addition, some key portions of the recordings were retranslated by the third author during a meeting between the three authors. These sentences were immediately transcribed in Norwegian by the first author. The third author's choice of verbs was preferred in order to avoid a loss of information related to cultural knowledge. To supplement the recordings of the TTs' teaching activities, the analysis was also based on photos from a small part of the teaching unit and the first author's field notes. The field notes were handwritten in the form of keywords and figures, and later the same day, the notes were written up as continuous text on a computer.

Findings

The mathematical focus in the analysis in this paper is on measuring, but the analysis also revealed that activities related to Bishop's (1988) activity *locating* involved *measuring* (*mihtidit*) and *designing* (*hábmet*). The TTs brought along the equipment necessary to build a traditional *lávvu*, and sorted it on the ground before the teaching started (Nordkild, field notes, 2017, author's translation). Equipment for building the *lávvu* included three *válddahagat*, poles for the rest of the framework, canvas, a chain, stones for the fireplace and a stepladder.

> Then, the teenage teachers put the straight end against the ground, causing the Y-shaped poles to be standing on their own (comment: may look like a pyramid). When this is done, they move the Y-shaped poles around. (Nordkild, field notes, 2017, author's translation)

The TTs placed (*bidjat*) the straight end of each *válddahagat* on the ground, as shown in Figure 3. They adjusted the poles by moving them in order to shape a regular figure, just as Lipka et al. (2019) described that Yup'ik elders turn asymmetry into symmetry. The first author's comment in the field notes, "may look like a pyramid", refers to the outcome of the Sámi verbs *hábmet* and *heivehit*, which mean to create and to adjust [what you have created]. They then moved (*heivehit*) the poles, which involved the activity *designing* (*hábmet*). A pyramid is neither a Sámi construction nor concept; however, the first author's reference to a pyramid reflects how she has been socialized into mathematics as it is taught in Norwegian schools.

Figure 3. Raising the framework of the *lávvu* (author's own photo)

> … one of the other teenage teachers grabs the three *válddahagat* on the underside and budges the construction of the Y-shaped poles that are set up. (Nordkild, field notes, 2017, author's translation)

By budging the *válddahagat*, the TTs were *measuring* the stability of the *válddahagat* without saying anything. This is an example of how the activities of *playing* and *measuring* are intertwined. As Keskitalo et al. (2017) surmised, there are no formal proofs in traditional Sámi knowledge. It is a well-known and accepted cultural truth that the three-pole structure shaped by the *válddahagat* is stiff and stable.

TT: This is hanging in the middle of the opening in top of the lávvu.

This quote refers to a chain that hangs just above the fireplace from the top of the inside of the *lávvu*. The chain is used to cook food over the fireplace. The TTs were adjusting (*heivehit*) the poles and the chain from the top of the *lávvu*.

One issue that arose in the teaching unit was the area, or the amount of space available on the *lávvu* floor:

TT: Yes, then we test how many people there is room for in here.

The LSs looked around and started counting and suggesting different numbers between four and ten (Nordkild, Field notes, 2017, author's translation). After some time in silence, the TTs asked the LSs to test (*playing*) how many (*measuring* and *counting*) individuals there was room for within the *lávvu*. During the testing process, they also had to investigate whether or not they could make enough room for all of them inside the *lávvu* by adjusting (*heivehit*) the poles. A teacher who instructed the LSs asked the TTs if they could have everyone lie down. While everyone was lying on the ground, one TT asked:

TT: Is it comfortable to lie like this? I'm not comfortable.

The TT asked if the LSs felt comfortable and commented that she herself was not. In other words, there was a need for more space in the *lávvu* when they were lying down. The TT's question encouraged the LSs to reflect on how many people there was room for (*measuring*) when living in a *lávvu* (including sleeping space). The TT commented that, when there are five individuals lying on each side of the fireplace, there was room for ten people in one *lávvu*. The TT then asked how many *lávvus* would be needed for the entire class of 23 LSs. The LSs' first suggestion was two, which was later revised to three. The TTs closed the session by claiming that they could adjust the number of individuals in each *lávvu* so that there would not be ten individuals in two of the *lávvus* and only three in the last one.

Discussion

The study revealed that, in order to cover the activity of *locating,* there is a need for a fourth Sámi verb, *heivehit,* in addition to the three verbs *gávdnat*, *ohcat* and *bidjat* that are suggested by Fyhn et al. (2018). Although adjusting (*heivehit*) the *válddahagat* and the other poles was part of the *locating* activity, it could also be interpreted to be part of the *designing* activity. When the TTs raised the *lávvu*, the activities *heivehit* and *hábmet* were performed simultaneously. Because the Sámi word *heivehit* overlaps with Bishop's (1988) *locating* and *designing*, this indicates a contribution to the development of Bishop's (1988) mathematical activities in Sámi. Our analysis supports the findings of Bishop (1988) and Shockey and Silverman (2016) that several mathematical activities can be performed at the same time, and that there are no clear borders between the activities.

One of the mathematical tasks provided by the TTs involved *measuring* how many people could lie down in the *lávvu.* This is an example of an inductive approach to *measuring* the size of an area by asking students to test through trial and error. This process corresponds with Balto's (2005) description of traditional Sámi upbringing. *Measuring* how many people there is room for in a *lávvu* depends on context: the number of people who can sit comfortably in a *lávvu* will not be the same as the number of people who can sleep comfortably in one. This is also an example of how there are no clear borders between the activities of *counting* and *measuring*. The TT's work is in line with the student-centred approach to mathematics in real-life contexts described by Skovsmose (2001). It is also connected to Fyhn et al.'s (2016) description of teaching mathematics within a cultural context, because the tasks given in the teaching are about the *lávvu*.

Closing words

In the context of raising a *lávvu*, the words *heivehit*, *gávdnat* and *bidjat* are needed to determine the meaning of *locating*. In addition, the word *ochat* covers one more meaning of *locating*. So, at least four Sámi verbs are needed in order to translate Bishop's (1988) activity *locating* into the North Sámi language found in Guovdageaidnu. Because *heivehit* can also be interpreted as *designing*, the present study indicates that Bishop's (1988) activity *designing* probably needs *heivehit*, in addition to *hábmet,* to fully capture every aspect of the activity. Our findings suggest that a Sámi categorization of Bishop's (1988) activities may result in different categories and varying borders for those categories. Subsequent studies may contribute additional insight into the Sámi mapping of such activities. Translating mathematical activities to North Sámi and testing these translations may also contribute to the development of a Sámi mathematics curriculum.

Acknowledgments

The present study was supported by the Local Culture for Understanding Mathematics and Science (LOCUMS) research project, UiT The Arctic University of Norway, the Research Council of Norway and the Research and Innovation in the Educational Sector Program. We thank Per Øystein Haavold for comments on the manuscript and we thank the teachers at Guovdageaidnu Lower Secondary School.

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Mathematical thinking competency when using GeoGebra in investigations of differentiability

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This paper is a part of the initial work of a project investigating the interplay between the mathematical thinking competency of the Danish KOM framework and the use of digital technologies in the teaching and learning of differential calculus. I present an empirical case with two students working in a premade dynamic worksheet in GeoGebra, representing the process of the difference quotient approaching the differential quotient. The results illuminate that it is through the students' work with different examples of differentiability and their reflections upon the reciprocal relations between the given examples that they show aspects of the mathematical thinking competency.

Keywords: Mathematical thinking, mathematical competencies, instrumental genesis, digital technology, differentiability.

Introduction

In the past decades, a shift in focus for teaching and learning mathematics has taken place, from focusing on skills and procedures to focusing on mathematical competence (Niss et al., 2016). In Denmark, the school curriculum (e.g., UVM, 2017) is heavily influenced by the Danish competency framework (the KOM framework) (Niss & Højgaard, 2019). The KOM framework is a description of what it means to master mathematics across level and subject matter and focusses on the processes of actively doing mathematics.

Simultaneously, digital technologies have entered the teaching and learning of mathematics and are now an integrated part of the mathematics classroom (Trouche et al., 2013). In learning situations, digital technologies are implemented to assist students in their learning processes of mathematical thinking and understanding (Weigand, 2014). The KOM framework does not include the perspective of using digital technologies for the students to exercise and develop mathematical competencies, and research on the potential interplay between students' use of digital technologies and mathematical competencies is rather limited (e.g., Geraniou & Jankvist, 2019).

An attempt to address the interplay between mathematical competencies and the use of digital technologies is the notion of mathematical digital competency (Geraniou & Jankvist, 2019). The notion of mathematical digital competency is based on analyses using the KOM framework, and the perspectives of the instrumental approach (Trouche, 2005) and conceptual fields (Vergnaud, 2009).

The mathematical digital competency builds on the entire KOM framework and is rather broad, concerning general mathematical activities using digital technologies. To focus more explicit on students' learning processes of mathematical thinking competency, my PhD project in progress focusses on the mathematical thinking competency of the KOM framework in interplay with the use of digital technologies in relation to the concept of differentiability.

This paper reports on the pilot study of my PhD project, where students work with a premade dynamic GeoGebra worksheet intended for examination of differentiability. The aim of the pilot is three fold. First, to examine the method and the data in relation to identify which aspects of the mathematical thinking competency, the students engage in; second, to examine in which ways the students use the dynamic GeoGebra worksheet and third to inform the further task design. Therefore, this paper addresses the questions: *Which aspects of the mathematical thinking competency can be identified in situations of examining differentiability by using a dynamic GeoGebra worksheet? And, what is the relation between these identified aspects and the students' use of the dynamic GeoGebra worksheet?*

The mathematical thinking competency and its overlap with other competencies

The KOM framework outlines aspects of the processes of doing and dealing with mathematics, and is divided into eight distinct but intertwined competencies: mathematical thinking; problem handling; modelling; reasoning; representation; symbols and formalism; communication; and aids and tools (Niss & Højgaard, 2019). Niss and Højgaard (2019) define a mathematical competency as ”someone's insightful readiness to act appropriately in response to *a specific sort* of mathematical *challenge* in given situations” (p. 14, italics in original). Each competency comprises aspects that cover different mathematical situations, in which some of the competencies overlap. In fact, more than one competency are at play in every mathematical situation.

Considering the mathematical thinking competency, its aspects involve activities of mathematical inquiry, such as:

> being able to relate to and pose the *kinds* of generic questions that are characteristic of mathematics and relate to the nature of answers that may be expected to such questions. It further involves relating to the varying scope, within different contexts, of a mathematical concept or term, as well as distinguishing between different types and roles of mathematical statements (including definitions, if-then claims, […] statements concerning singular cases, and conjectures) […]. Finally, it involves relating to and proposing ”abstractions” of concepts and theories and ”generalisation” of claims (including theorems and formulae) as processes in mathematical activity. (Niss & Højgaard, 2019, p. 15, italics and quotation marks in original)

In relation to the teaching and learning situation of examining differentiability, the mathematical thinking competency means to engage in the inquiry of determining when a function is differentiable. In other words, it means to understand which answers are expected, when asked to examine if a function is differentiable at a single point and for an interval, as well as considering the conditions for differentiability and non-differentiability of a function. Part of these aspects are related to the reasoning competency, posing arguments and justifying the answers to the tasks.

As the students are to work with a premade GeoGebra worksheet, the students have to be able to put the worksheet into constructive use for the inquiries, which are part of the aids and tools competency. Furthermore, working with a multi-representational tool such as GeoGebra, the students have to be able to interpret, translate and move between different representations.

Considering the aids and tools competency, it simply focusses on mastering aids and tools for mathematical activity, rather than on how tools are used when doing mathematics. Jankvist and colleagues (2018) argue that the aids and tool competency is rather limited as an analytical framework on its own, even when adding aspects of the representation competency. To gain deeper insights into the students' work with a digital tool, the authors suggest to apply the perspective of instrumental genesis, which focusses on students' interaction with a digital tool.

The perspective of instrumental genesis

Instrumental genesis is the on-going process of a non-trivial artefact becoming a meaningful instrument, or part of an instrument, for a user. Drijvers et al. (2013) describe the process of instrumental genesis through three dualities, the artefact-instrument duality being the first. An artefact is the tool as object, non-trivial in the way that, for the student, the tool does not have a clear purpose and use at first. Then through the instrumental genesis and the student's cognitive development using the tool, it becomes an instrument for a certain purposes and situations.

The second duality is instrumentation-instrumentalisation, which concerns how the user shapes the artefact, and how the artefact shapes the user. Instrumentation is the process in instrumental genesis where the artefact's affordances and limitations influence how the user carries out given tasks, and in that way influence the user's conceptions. Instrumentalisation works in the opposite direction, i.e. the activity of the student shapes the artefact based on the student's mathematical and digital knowledge. (Drijvers et al., 2013)

The third and last is the scheme-technique duality. Here, Drijvers et al. (2013) draw on Vergnaud's (2009) notion of scheme. Scheme refers to thinking and technique to observable actions carried out with the tool. Schemes are the cognitive foundations behind the techniques. A students' scheme both guides the activity for a given task and at the same time develops through this same activity (Drijvers et al., 2013). Hence, the notion of scheme in the process of instrumental genesis connects the use of the tool with the user's thinking. Thus, analyzing students' techniques for using the artefact can potentially provide access to their way of thinking with and of the mathematical concepts in play.

The perspective of instrumental genesis with the three dualities focusses on a student's interaction with a tool. The tool influences the student's way of thinking, and the student's way of thinking shows in the use of the tool. Therefore, this perspective does not only give deeper insights into the student's aids and tool competency, as argued above, but also into their mathematical thinking competency.

Method and data generation

In the pilot study, 27 students of second year in upper secondary school (11th grade) participated. They were working with tasks in a premade dynamic GeoGebra worksheet (see figure 1), which design is inspired by an example given by Hohenwarter and colleagues (2008). The orange graph on figure 1 represents the function f, which is given by

$$f(x) = \begin{cases} -\frac{11}{72}x^2 + 5.5x, & 0 \leq x \leq 18 \\ 49.5, & 18 < x \leq 20 \end{cases}$$

The purple graph represents the secant going through the points $(x_0, f(x_0))$ and $(x_0 + \Delta x, f(x_0 + \Delta x))$. The blue point named x_0 can be moved along the x-axis and the red slider underneath the x-axis represents the value of Δx. By moving the slider towards 0 for both positive and negative Δx, such that the point $(x_0 + \Delta x, f(x_0 + \Delta x))$ moves towards the $(x_0, f(x_0))$ along the graph, the students can access the concept of differentiability as a numerical approximation to the slope of the tangent line (Hohenwarter et al., 2008).

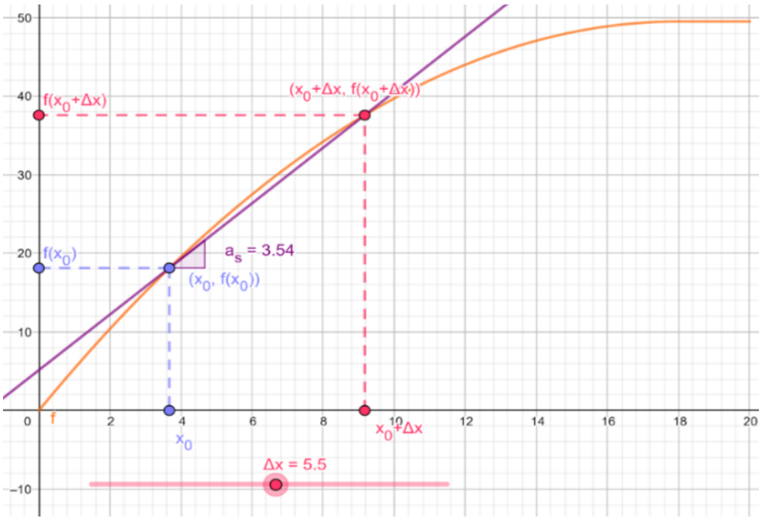

Figure 1. Snapshot of the worksheet with a changeable secant line to the given function

To access the students' mathematical thinking, the students worked in groups of two or three to be able to articulate their thoughts through mutual discussions with each other during their work. Screencasts of the students' activities working with the dynamic GeoGebra worksheet were recorded, including their voices and webcam recordings of the front of the students to observe who is talking and potential hand gesticulations. The observed lesson was the third on the subject of differential calculus, and in the two previous ones, the class had reviewed concepts and representations of the linear function and worked with some introductory tasks on the idea behind differential calculus, including the concepts of secant and tangent lines.

The screencasts were coded using the aspects of the mathematical thinking competency to identify relevant video sequences related specifically to this competency. Due to the competencies being intertwined, other competencies may show relevant as well, but not as a first lens, navigating the first selection of data material. Afterwards, these sequences were analyzed using the three dualities of instrumental genesis to examine how the students use the dynamic GeoGebra worksheet in their work.

From the first coding, a case of two students, Alex and Ben, stands out from the rest of the groups with very explicit reflections upon the answer to the task of examining differentiability and non-differentiability. At the same time, their obstacles during the work and their sequence of actions working with the task illustrate well the discussions and behavior of the other groups.

Findings: the importance of examples when investigating for differentiability

In the following section, the case of Alex and Ben is analyzed and illustrates the importance of working with different examples, when introducing the definition of differentiability. In the task, the students work with two sub-tasks (examine differentiability for $x_0 = 5$ and $x_0 = 18$), through which they work with two particular examples of differentiability at a point.

The importance of different examples

To investigate differentiability for $x_0 = 5$, the students move x_0 to 5 on the x-axis. Despite the teacher's demonstration of using the slider, Ben, finds it easier to write changing values of Δx in the algebra window instead. By first changing Δx from 0.05 to 0.01 by subtracting 0.01 at the time, they observe the secant slope changes from 3.96 to 3.97. Afterwards, starting with Δx to be -0.01 changing it slowly to -0.07, they observe the secant slope changes from 3.97 to 3.98.

01 Ben: Yes, then it [the secant slope] moves away from 0.97.[…] So, 3.97 is the point it moves closer and closer to, and then it moves farther away when you move away from 0. That, I don't know what we should use for.

02 Alex: Is that the answer, or…? We are to answer if it is differentiable, right.

03 Ben: That it is [differentiable], because there is such point. Hm… so, A is that real number, which it can move closer and closer towards, and farther and farther from, and that 3.97. [Talking to another group] So, 3.97 is that right for task 3? I just wanted to know if it was correct.

After concluding that the function is differentiable at $x_0 = 5$, they move on to investigate differentiability for $x_0 = 18$. They move x_0 to 18 on the x-axis and set $\Delta x = 0.5$. They observe the secant slope to be 0, and changing Δx to 0.4 and 0.3, the secant slope is still 0.

04 Ben: Here A is 0, that's clear, it shouldn't be that.

05 Alex: Right here, it looks like it is not differential. It doesn't move towards a point.

Trying for negative Δx, they get the secant slope is 0.03 for $\Delta x = -0.2$, and 0.02 for $\Delta x = -0.1$, but when they try $\Delta x = -0.01$ the secant slope is again 0. Ben consider it as an error that the secant slope is 0 and asks for help. After clarifying what it means for the slope to be 0, they conclude the secant slope approaches 0 for negative and positive Δx, which means the function is also differentiable for $x_0 = 18$. Their investigation ends with the conclusion of the function being differentiable for $x_0 = 5$ with slope 3.97 and for $x_0 = 18$ with slope 0.

From the perspective of the mathematical thinking competency, the above excerpts indicate that the students relate to the nature of the expected answer, when asked to examine for differentiability, and reflect upon the role of the performed inquiry and its result (line 01-03). Second, as part of these reflections, they distinguish between the roles of the definition and the two particular examples ($x_0 = 5$ and $x_0 = 18$) given in the task. By referring to the denomination A in the formal definition (line 03 and 04), Ben shows that he is able to relate the two examples to the definition. Alex, on the other hand, shows he is able to relate the two examples of differentiability from the two sub-tasks to each other, by observing that the example given in the second sub-task ($x_0 = 18$) does not behave in the same way as the first example did ($x_0 = 5$), when they use the slider (line 05).

The activities and reflections shown in these excerpts indicate that the students also possess aspects of other competencies. Alex and Ben's careful work changing Δx using the slider shows aids and tools competency. Working with the slider and interpreting how the changes adapt to the algebraic expression of the secant slope, show representation competency. Alex' explanation of why the function is differentiable at $x_0 = 5$ also contain aspects of the representation competency, as well as of the reasoning competency. The aspects of the other competencies are part of the students' interaction with the tool, and help them reflect upon and engage in the inquiries, showing aspects of the mathematical thinking competency.

From the perspective of instrumental genesis, with the dynamic GeoGebra worksheet being the artefact for investigating differentiability, the data illustrate instrumentation of how the configuration of the dynamic GeoGebra worksheet shapes the students' conception of differentiability. In the first sub-task ($x_0 = 5$), the students observe the changes of the secant slope while changing the values of Δx, which leads them to a successful conclusion. Transferring the technique used for the first sub-task to the second sub-task challenges each student. In the second sub-task, the secant slope is constantly 0 for positive Δx. Hence, it does not change from other values towards one value, which Alex interprets as non-differentiability for $x_0 = 18$. This indicates that based on the technique and conclusion of the first sub-task, Alex' scheme contains "Letting Δx approach 0, the secant slope has to change from different values to one value." At the same time, Ben interprets the slope being 0 as an error. Here Ben draws on a scheme of slope, which plays a crucial role in the field of differentiability, but might stem from outside this topic.

Watson and Mason (2007) argue that variation of examples can generate conscious or subconscious expectations, and introduce the term of sequences of examples. For students to make sense of such examples-sequences the students need to see patterns across the examples, from these form expectations and conjectures and then generalize. The analyses of the case illustrates that the students do develop an initial technique-scheme duality, where they implicitly set up conditions for the secant slope—to change as Δx approaches 0, and to be different from 0, for a function to be differentiable at a point. For the students to conclude differentiability for $x_0 = 18$, they need to adjust their conditions for differentiability in their initial scheme. Hence, by working with more tasks introducing the students to different examples, the students work with processes towards generalization, which are aspects of the mathematical thinking competency. In this case, it is not obvious whether the students manage to change these conditions for the concepts of slope and differentiability. Yet, concluding differentiability for $x_0 = 18$, the students do show progression in their concept formation.

The students need to simultaneously investigate different examples of the concept as well as reflect upon the relations between the single examples and the generalized definition of the concept. These relations might be investigated in more depth by introducing an example of non-differentiability. Such an example can be termed "non-example" (e.g., Bardelle & Ferrari, 2011), despite it might be misleading as it is in fact an example, but one that does not satisfy the definition of a given concept (Bardelle & Ferrari, 2011).

The importance of non-examples

Having seen only examples of differentiability, Ben raises the question of non-differentiability:

06 Ben:	So, for it not to be differentiable should there then just be… nothing. Or, how it is with that?
07 Alex:	Hmm… That, I actually don't know.

With this question, Ben shows the mathematical thinking competency aspect concerning the ability to pose generic mathematical questions characteristic to mathematics. This question indicates that Ben lacks a non-example of differentiability to emphasize the characteristics of the concept.

Working with both examples and non-examples plays an important role in concept formation, as they link between the particular and the general (Bardelle & Ferrari, 2011). The alternation between the particular (the examples met through the tasks) and the general (the formal definition) happens simultaneously with the process of instrumental genesis, with the artefact becoming an instrument and the development of techniques and schemes.

The analyses of the case with Alex and Ben, using the aspects of the mathematical thinking competency and the perspective of instrumental genesis, illustrate the importance of letting students work with different examples and possibly non-examples, for them to engage in the activities comprised in the mathematical thinking competency.

Conclusion

This paper addresses the interplay between the mathematical thinking competency of the KOM framework and the use of digital technologies through a case of two students using a dynamic GeoGebra worksheet to examine differentiability of a given function. The first analysis, using the perspective of the mathematical thinking competency, indicates that it is in the reflections upon the actions with the digital tool that mathematical thinking comes into play. The analysis, using the perspective of instrumental genesis and the duality of technique schemes then illustrates a string of activities, where the students develop techniques and schemes for differentiability. In this way, the students develop concept formation of differentiability in forms of developed schemes. For the students to do so, they need to engage in the processes of

- Understanding the role of the dynamic GeoGebra worksheet in the work with the tasks as single examples compared to the generalized mathematical concept in question, since the digital tool can only provide examples and non-examples of the concept and cannot illustrate the concept as a generalized whole.
- Reflecting upon the collection of examples and non-examples provided by the tasks using the GeoGebra applet and relating these to each other and to the generalized concept, e.g., by posing questions that emerge from the dynamic processes represented in the GeoGebra applet, and consider and conjecture on the conditions for the different tasks.

The first of these two processes is related to the aids and tools, the representation and the reasoning competency, being able to connect the actions carried out with the tool to the formal definition to answer the tasks and give reasons herefore. It is in the second of these two processes, the students use aspects of the mathematical thinking competency such as relating to the nature of the expected answer and posing mathematical questions; distinguish between the roles of mathematical statements, using the definition in their reasoning, and relating to generalization as processes in mathematical activity. The students develop techniques using the tool, and schemes reflecting upon the results of their techniques. This initial process of instrumental geneses happens simultaneously with the students using aspects of the mathematical competencies to engage in their work with the tasks, and it is from such engagement that new questions, like the one of non-differentiability, emerge.

Acknowledgment

Sincerely thanks to the reviewers for their time and constructive and detailed feedback, and to the chair and the participants at the conference presentation for useful comments as well.

Supported by Independent Research Fund Denmark [Grant no. 8018-00062B].

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Developing a formative, digital tool to assess children's number sense when starting school

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We investigate how digital technologies can enrich teachers' formative assessment of number sense by describing the development process of a digital assessment tool for children starting school (five- and six-year-olds). Studying different aspects of validity, we focus on scoring and digital affordances. The quantitative analysis of the preliminary data from 101 children evidences the technical validity of the tool. We find that interactive assessment items add to the content validity of the tool. The interactive items provide qualitative data about students' number sense which cannot be captured by quantitative measures. At the same time, children may have greater difficulty interpreting more complex items. Our results support the view that a digital tool can be a useful supplement to the assessment of number sense. Further developments and approaches to investigating additional aspects of validity are discussed.

Keywords: Number sense, formative assessment, educational technology, elementary school mathematics.

Introduction

Children start school with considerable knowledge related to learning mathematics (Clarke et al., 2006). Assessing each child's number competence enables the teacher to plan effective and engaging teaching, but assessment can also be a challenging and time-consuming task. The rapid advance of digital technologies brings new opportunities for assessment in mathematics education. Developing digital formative assessment instruments has been highlighted as an area to prioritise in early mathematics education research (Ginsburg & Pappas, 2016). Specifically, to inform teachers about children's Foundational Number Sense (FoNS), Sayers et al. (2016) point out the need to "develop a diagnostic tool for teachers to assess individual grade one children's FoNS-related understanding" (p. 389).

We address this need by developing and analysing a digital assessment tool. The purpose of the tool is to inform teachers about the mathematical knowledge that children have already acquired when they start school. In a classroom setting, a digital tool can be an efficient supplement in the initial assessment process, especially since time typically does not permit one-to-one assessment interviews. In this paper, we focus on the opportunities and challenges associated with technology supplementing the teacher's formative assessment of students' number sense. We discuss how interactive tasks and records of qualitative data about the children's solutions and solution processes can strengthen the evidence for the construct validity of a digital assessment tool. Based on preliminary data from an on-going study, we address the following research question:

How do digital tasks support aspects of validity in the assessment of first-graders' number sense?

Frameworks

The FoNS model for operationalising number sense

To operationalise number sense we build on the FoNS model which describes the number-related skills that require instruction (Andrews & Sayers, 2015). The FoNS model provides a multi-layered, flexible, and relational definition of the number sense concept with eight interrelated categories: Number identification (NI), systematic counting (SC), number-quantity relationships (NQ), quantity discrimination (QD), representing numbers (RN), estimation (ES), simple arithmetic competence (AC) and awareness of number patterns (NP).

Assessment validity

In our context, assessment validity concerns the extent to which theory supports inferences made from test scores and how the evidence supports interpretations (Wolfe & Smith, 2007a). Evidence for validity can be found by looking at eight different aspects of validity: 1) content, 2) substantive, 3) structural, 4) generalisability, 5) external, 6) consequential, 7) interpretability, and 8) responsiveness (Wolfe & Smith, 2007b). In this paper, we focus on the content, substantive, and interpretability aspects of validity. The content aspect of validity addresses the relevance, representativeness, and the technical quality of the items. Documenting the purpose of the tool and the development process with expert reviews are part of the evidence that relates to the content aspect of validity. The substantive aspect of validity relies on the theoretical model we based the item development on. Here, we look at how the different items reflect children's overall number sense as described in the FoNS model. We focus on how the digital format can enhance content and substantive validity. Finally, we address how the number sense score can be interpreted by considering features that affect item difficulty.

Formative assessment of young children's knowledge

Formative assessment can be considered both an instrument and a process (Bennett, 2011). In this context, formative assessment refers to how the results of the assessment process are used to promote further learning (Black & Wiliam, 2018). Therefore, in addition to informing teachers about the children's present competence, a formative assessment tool should also support teachers in adjusting their teaching to meet the children's needs. Previous research has examined how task-based, one-to-one assessment interviews can provide crucial information to help teachers facilitate students' learning (Clarke et al., 2011). The digital medium can provide teachers with some of the features of assessment interviews, such as screen recordings of the students' solution processes on interactive items, as well as more traditional skill-based assessments characteristic of pencil-and-paper tests. Certain researchers highlight the transformative improvements that software can have on early mathematics education, both for helping children learn mathematics, and for providing guidance to teachers (Ginsburg, 2016). Hence, a digital assessment tool can guide instruction and improve students' opportunities to learn mathematics.

Methods

Procedure

About 50 schools in and around Trondheim municipality in Norway were invited to participate in the project, out of which eight of the interested schools were chosen. In this paper, we present preliminary data from 101 of the first-grade children (five to six years old) who participated in the study. A researcher visited the schools over a period of two months at the beginning of the school year. Groups of six to eight children carried out the assessment on separate tablet computers. The participants were seated so as not to get disturbed by each other's screens or sounds. All children were given the same instructions before they started the assessment and were free to finish at any time. Pre-recorded voice instructions were given for each item. For technical reasons, the assessment was presented in three separate units with different FoNS categories and increasing difficulty. Each child could decide whether to continue to the next unit or not. Most children completed the first two units, some completed all three units, and a few children completed only the first unit. There was no time limit for the items, but time on task was recorded for each item. The children typically spent between 15 and 25 minutes on the assessment in total.

Analytical procedures

Rasch measurement was used for quantitative analysis of the children's responses, using the Winsteps software. The Rasch model is a probabilistic measurement model that provides interval-scale measures of item difficulty and person skill on the same measurement scale in the unit of logits (Wright, 1977). On a basic level, Rasch analysis involves calculating the probability that a person with competence B answers correctly on an item with difficulty D. A person with higher competence always has a higher probability of successfully answering any item than a person with lower competence. An item that is more difficult, always has a lower probability of being successfully answered than an item that is less difficult, regardless of person ability.

All items were scored dichotomously, meaning that the children received one point for a correct answer and zero points for a wrong answer. In addition, to investigate and illustrate the potential of interactive items for enriching digital assessment, observations and screen recordings of the children's solutions underwent qualitative analysis.

Results and discussion

Development of the assessment tool: from framework to data collection

We developed items for each of the eight FoNS categories. The items were adapted from different cognitive and educational studies on number sense, standardised number sense tests, and formative assessment instruments (Ginsburg & Pappas, 2016; Davison et al., 2012).

A selection of items was presented to expert groups for them to adjust and determine the items that were the most suitable for measuring the number-sense construct. The expert reviews were performed by researchers in mathematics education familiar with the number-sense concept, target population and instrument development. Based on the first reviews, items from five number sense categories were selected for the first pilot study: Number identification, systematic counting, number and quantity, quantity discrimination, and arithmetic competence. Several pilot studies were conducted to refine the content and selection of items. The qualitative observations obtained from the first pilot study led to further changes, predominantly associated with technical issues and voice instructions. Certain items that were often misunderstood by the children were removed. The first pilot study was carried out at the end of the academic year, whereas the assessment was designed for children who were almost a year younger. To provide better estimates of the difficulty parameters, we conducted a second pilot study in a preschool right before the end of the academic year. This second pilot study led to further adjustments. The level of difficulty had to be adjusted to enable the overview of the number sense competence of all first graders. Subitising had also been included as a separate category at this point since conceptual subitising has been highlighted as a central aspect of number sense (Sayers et al., 2016). After analysis in expert groups, the decision was made to include the last three FoNS categories in the assessment tool: Estimation, number patterns, and representing number. The items from these categories were piloted in a fourth pilot study, along with the rest of the assessment, before commencing the main data collection in the fall of 2020. An overview of the final 78 items included in the tool is presented in Table 1. Two items did not contribute to the number sense measure as intended and were removed from the present analysis. We based the subsequent analysis on the remaining 76 items. The items varied from typical skill-based items, asking the child to identify a certain number symbol or quantity, to items capturing the child's solution process (later referred to as process items). The process items were designed to exploit the digital medium's potential for capturing some of the characteristic aspects of one-to-one assessment interviews. Of the 78 items, 10 were process items, and two of them will be discussed in more detail.

Scale	**Content**	**N(SB)**	**N(P)**
Number identification (NI)	Recognise numeral and meaning.	8	0
Systematic counting (SC)	Ordinality. Count to twenty and back from an arbitrary digit.	6	2
Number and quantity (NQ)	Cardinality. One-to-one correspondence between symbol and quantity.	6	4
Quantity discrimination (QD)	Compare quantities. Vocabulary: larger, smaller, more than, less than.	6	2
Representing number (RN)	Different representations of numbers and part - whole aspects.	4	1
Estimation (ES)	Estimate the size of a set and position on a number line.	6	0
Arithmetic competence (AC)	Operate on small sets by using addition or subtraction	14	1
Number patterns (NP)	Continue or complete a number sequence.	7	0
Subitising (SU)	Perceive quantity without counting. Perceptual and conceptual. Timed.	11	0

Table 1. Overview of the number sense items within each FoNS category. N(SB): Number of skill-based items. N(P): number of process items

Development of a specific item involving children's solution strategies

NQ10 was developed within the category of number and quantity (NQ) to assess the ability to count a number of objects and identify the number symbol that represents that amount. The item was adapted from similar items used in interview settings (Malofeeva et al., 2004). We present the first version of the item in Figure 1A. Expert groups discussed the specific number of objects that would enable insights into the child's structuring of quantities, how to give the most succinct and accurate voice instructions, and how the objects could be placed to encourage the child to manipulate them.

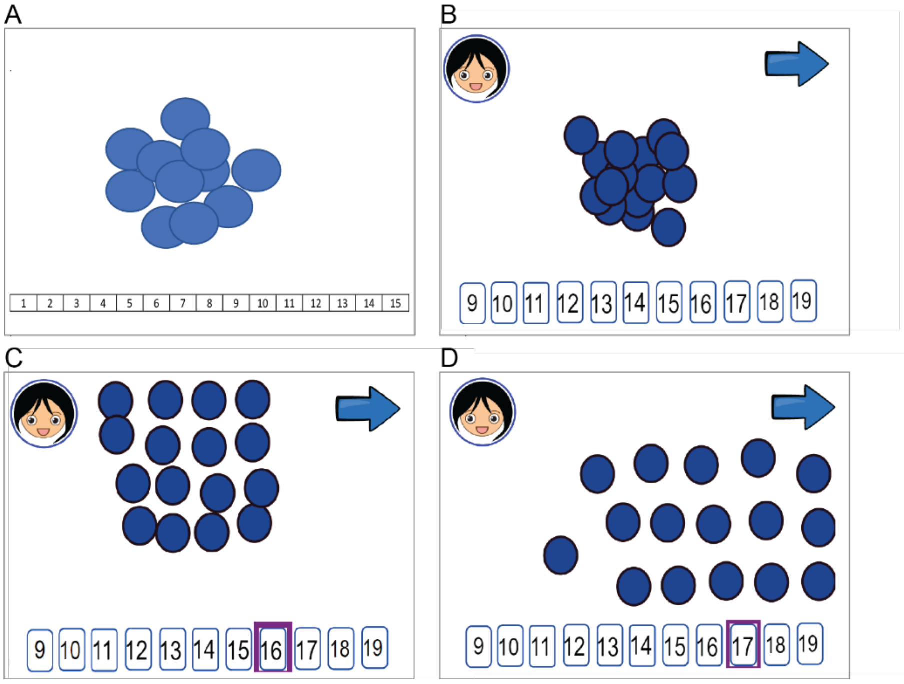

Figure 1. Development and responses to Item NQ10
A. First version; B. Final version; C. Final response of Child 1; D. Final response of Child 2

In the final version of item NQ10 (Figure 1B) the children are asked to arrange the objects in a manner that is easy to count. The software provides both a recording of the motion and the final positions, which allowed us to study how the child rearranges the objects to simplify the counting process.

In the given examples, we see how two first-graders structured the objects. Child 1 structured the 16 objects into four groups of four and selected the correct answer. Child 2 structured the objects into three groups of five, placing one object separately, and then selected 17 as the answer. If not just a random mistake, the error might be related to the number symbols (rather than the quantity involved), the aspects of cardinality, or the structure of odd and even numbers. Such access to the qualitative aspects of the responses can give the teacher valuable insight into each child's solution process and more information about the child's understanding than simply knowing whether the response was right or wrong. Therefore, these process items can provide qualitative data to support the assessment process and contribute to the evidence for both the content and substantive aspects of validity. Process items cannot replace the insights gained through human interaction in one-to-one assessment interviews. However, the qualitative recordings can support the validation of the assessment and be valuable for screening, individual follow-up, and teaching purposes.

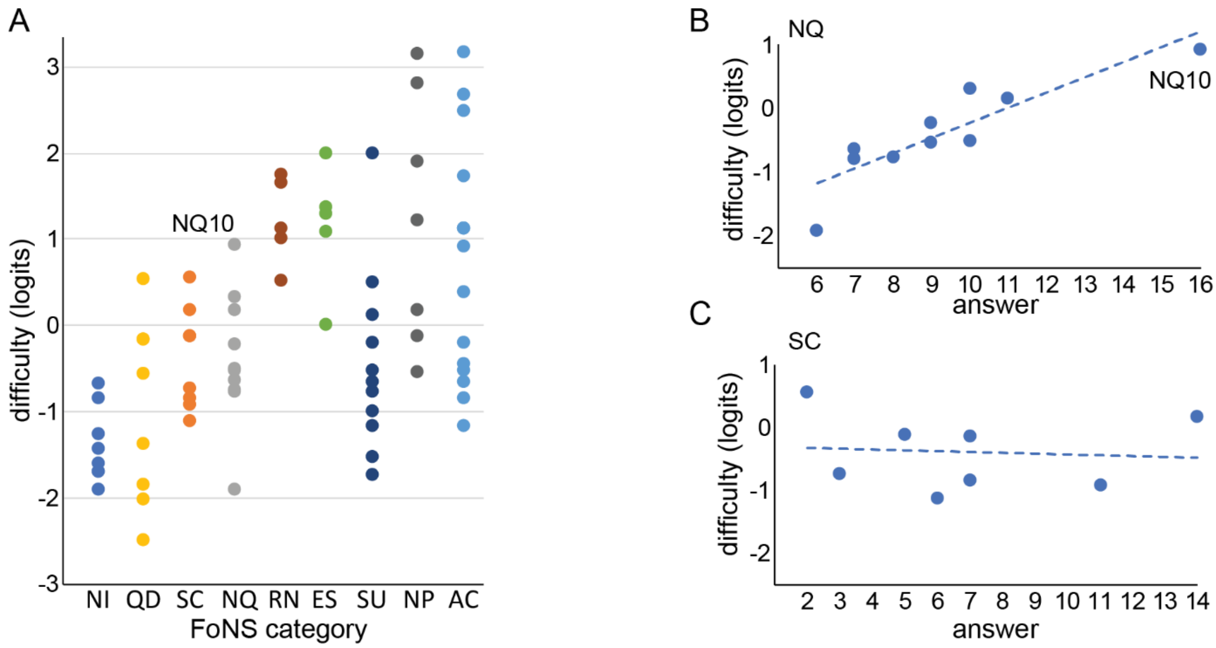

Figure 2. Item difficulty. A. Plotted as a function of the FoNS categories arranged by maximum item difficulty. B, C. Plotted as a function of the correct answer to the item in the categories number and quantity (NQ) and systematic counting (SC). The dashed line is the linear regression line. Item NQ10 is indicated in the plots (difficulty = 0.93; answer = 16). The abbreviations are defined in Table 1

Test quality: technical aspects of construct validity

The person reliability value of the 76 items analysed in this paper was 0.88 (corresponding to a Cronbach's alpha of 0.91), which is typically considered productive for measurement.

As a whole, the items were well targeted to the group of school-starters and were slightly easy, having the value of 0 ± 1.31 logits (mean ± sd), while the children scored 0.51 ± 0.93 logits (mean ± sd). The larger standard deviation of the items indicates that only a few children scored at the top and bottom of the measurement scale. This gives the assessment tool the necessary range to track the children's number sense as it develops during their first year of school, for example, by comparing the assessments from the first and second half of the school year.

If a single aggregate score is used to measure number sense, individual items need to measure the same number sense construct. Evidence for this can be found in the infit and outfit measures of the items. An item with fit values close to 1 is considered to measure the same construct as the rest of the items. For this assessment tool, most items fit well with the Rasch model, with a mean item infit mnsq of 0.98 ± 0.15 (mean ± sd) and mean item outfit mnsq of 1.02 ± 0.42.

The group of process items also measured the same construct as the other items. One potential exception was item RN4, with infit mnsq 1.21 and outfit mnsq of 2.2, which is high. Excluding RN4, the process items had a mean infit mnsq of 1.02 ± 0.11 and mean outfit mnsq of 0.99 ± 0.17.

Item RN4, in which the child uses dice to compose the quantity ”four” in three different ways, was adapted from a composing number task used in interview settings (Clements et al., 2008). Formulating clear instructions for such complex tasks was a general challenge of the item development and might be related to the misfit of this item. Representing numbers seems to be different from other number sense categories, as three out of five items from this category had infit values greater than 1.2 (one process and two skill-based items). One reason for items from this category to stand out as more misfitting than items form other categories

might be the complexity of this domain. Representing numbers in different ways is a rich domain which includes several other categories of number sense, such as arithmetic competence and connections between number and quantity (Andrews and Sayers, 2015).

How can the number sense score be interpreted? Here, we give two examples supporting the interpretability aspect of validity of the assessment. First, different FoNS categories had different ranges of difficulty (Figure 2A). This means that a child's aggregated measure is indicative of whether that child can solve the tasks from each category. Second, for some FoNS categories, such as number and quantity, the difficulty correlated with the numerical value of the answer (Figure 2B). The relation between numerical value and item difficulty suggests that the aggregate measure is predictive of the range of numbers the child can process confidently. For other categories, such as systematic counting, the difficulty was not clearly related to the numerical value of the answer (Figure 2C), indicating that task difficulty was largely determined by the content of the tasks in some categories.

Conclusions

Presenting parts of an on-going study, we have described the development of a tool to assess first grader's number sense. The digital assessment tool is based on the FoNS model (Andrews & Sayers, 2015) and has been subjected to expert reviews. The use of interactivity provides opportunities for simulating aspects of interview tasks, which adds to the content aspect of the validity of the assessment. The Rasch analysis of the first graders' responses indicates that the technical quality of the assessment tool is high. Taken together, these results indicate that a digital assessment tool has the potential to provide teachers with information about their students' number sense.

We argue that the digital format can supplement the teachers' informal assessments and offer a valid, reliable, and more complete alternative to paper-and-pencil assessment by incorporating aspects of one-to-one assessment interviews. The use of interactive process items gives us the opportunity to adapt tasks that were previously reserved for one-to-one assessments to a setting in which teachers can gain information about their students' number sense in a less time-consuming manner. It remains to investigate how the digital assessment may affect test conditions and whether the level of engagement in the digital assessment is different from that of comparable written assessments.

An important validity aspect of a formative assessment tool is how the tool is used to improve children's learning. Describing formative assessment as a process and validating a tool to be used in this process, entails presenting the scores in a manner that is useful for the teacher. In a classroom setting, the teacher's informal assessments are often based on several different interpretations, which leads to a measurement issue in the formative assessment (Bennett, 2011). The integration of fundamental measurement principles with digital technology may help ameliorate this issue. To be able to "develop curriculum support tools for teachers to plan an explicit incorporation of FoNS categories in their teaching" (Sayers et al., 2016, p. 389), the other aspects of the validity of the tool must be considered. As a next step, we need to assess the consequential validity of the claim that the tool can usefully supplement the teacher's formative assessment of number sense.

Acknowledgements

We would like to thank everyone involved in this project for their contribution to aspects such as technical implementation, design, expert reviews, data collection and analysis.

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Developing genres of educational programming videos to support mathematics teachers utilize programming in the classroom

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In this paper, we report on the state of the art of educational programming videos and suggest three genres of educational programming videos to support mathematics teachers and teacher educators utilizing programming in the classroom. In a literature review, we report on a scattered but trending research field, synthesizing three distinct genres of education programming videos: video lectures of programming concepts, video tutorials of step-by-step instructions of educational technology, and video podcasts discussing teaching examples of how to use programming in the classroom. Finally, we present example videos for the suggested genres and conclude our work with a call for more research into educational videos for programming, suggesting two specific paths for future work.

Keywords: educational technology, video, programming, mathematics education

Introduction

Programming became a compulsory part of mathematics curriculum in Norway from fall 2020[18]. The arguments for the inclusion of programming in the curriculum (e.g., problem solving, logical thinking and reasoning) remind us of when Logo was introduced in the 80s. Logo was both a programming language and a teaching philosophy that wanted to develop active, explorative, and creative children who should become good at problem solving. However, Logo did not become a lasting change, mostly because it did not focus on giving teachers training in programming (*Fagartikkel*, 2019). To avoid repeating the past, we need to support teachers to acquire programming competences as well as provide knowledge on how programming can be utilized to teach mathematics in schools. Online material, accessible for large audiences always, can be an essential resource to establish an endurable shift. In this project, our aim is two-fold: We explore *what educational videos are used to introduce programming to a novice audience*; and *what genres of educational programming videos can be synthesized from our findings*. To answer the first objective, we conducted a scoping review. To operationalize the second objective, we created example educational programming videos illustrating different declared learning outcomes and video formats. In the following section, we will present a brief overview of research related to the introduction of programming as part of the mathematics curriculum, before reporting on the scoping review of educational videos for programming. We continue by presenting examples for the suggested genres of educational programming videos. We conclude this paper with suggestions for future work.

Teaching programming and mathematics in practice

In a recent article, Berggren and Jom (*Fagartikkel*, 2019) describe teachers' positive attitude for programming being introduced as part of the mathematics subject, and that they believe it can help students become more motivated in mathematics. However, there is greater uncertainty about how it will affect teaching, based on teachers' lack of programming knowledge. The authors point out that

[18] https://www.udir.no/laring-og-trivsel/lareplanverket/fagspesifikk-stotte/nytt-i-fagene/hva-er-nytt-i-matematikk/

their findings indicate that teachers would rather learn programming and how to teach it, than get ready-made assignments to use in the classroom. Stigberg and Stigberg (2019) propose two key actions to support the introduction of programming in mathematics:

- Teachers need more resources to teach programming in mathematics.
- The relation between programming and mathematics must be sufficiently motivated.

We experienced similar findings in our ongoing research project raising programming competences for 8^{th} to 10^{th} grade mathematics teachers[19], where the participating teachers suggest providing more materials online as an extra resource for themselves and their colleagues. These studies highlight teachers' requests for online resources to support them to implement programming in mathematics. However, there is little information about the type and content of such resources.

Educational videos for programming

Educational videos are often understood as a genre in which content is explained. Descriptions of video types tend to focus on content delivery, e.g., Greenwood (2003) defines educational video as a demonstration or a report on a topic. Winslett (2014) presents an investigation into what counts as an educational video and finds that there is an extensive range of what can be considered as educational videos. He proposes a two-dimensional framework to analyze educational videos based on declared video outcomes and video production type. Declared video outcomes can be grouped by different learning objects (e.g., show factual and procedural content, directly instruct, provide exemplars, show real life practice, or show complexity) or educational topics (e.g., accessible delivery, flexible delivery, to engage learners, or to move from shallow to deep learning). Found video production types included are: fly on the wall, interviews, video games, recording a teaching event, capturing hard to see processes, video diaries, dramatic works, mashing up or presenting to the camera. However, Winslett finds no distinct patterns mapping video production approaches and declared outcomes. Hansch et al. (2015) interviewed practitioners in the field of educational video production and found that most videos used in MOOC courses are recorded lectures. Especially, two video production styles, including (1) *talking head style* in which an instructor giving a lecture is recorded, and (2) *tablet style*, which uses a voiceover mode, have been significantly featured in recent years. They introduced a new typology of video production styles as following: presentation slides with voiceover, picture in picture, text overlay, table capture, whiteboard, screencast, animation, classroom lecture, recorded seminar, interview, interview, conversation, live video, webcam capture, demonstration, on location, green screen. To comprehend educational videos for programming in more depth, we performed a scoping review investigating what types of educational videos are used to support introductory programming education as declared learning outcome. We followed a three-step iterative critical analysis. During the first two steps we focused on finding appropriate papers. In the third step we analyzed the selected papers based on video characteristics.

Step 1. Retrieving publications

We selected papers from the ACM (Association for Computing Machinery) digital library, archiving work from influential conferences and journals in the field of computer science education as well as publications from the Special Interest Group on Computer Science Education (SIGCSE). To widen

[19] https://www.hiof.no/it/forskning/grupper/interaction-design/prosjekter/digital-fabrication-and-programming-in-schools/programmering-i-matematikkfaget/index.html

our search, we included papers from Google Scholar as well as related UDIT (The Norwegian Conference on Didactics in IT education) publications. Digital technology is constantly changing how we interact with the world. Research in educational videos must adapt to reflect these changes. Liu et al. (2014) show that popular research topics in e.g., HCI (Human Computer Interaction) are replaced by new topics after only a decade. To explore state of the art in educational videos for programming, we included studies from the last 20 years. We searched for publications using the term *"educational video"* in title, abstract or keywords. Furthermore, publications should contain the term *introductory programming* and not *games* in title, abstract or keywords. The final search query used in the ACM digital library and Google scholar:

[All: "educational video"] AND [All: introductory programming] AND NOT [All: games] AND [Publication Date: (01/01/2000 TO 12/31/2020)]

Step 2. Appropriate papers

The literature search retrieved 86 papers from ACM digital library, 19 papers from Google scholar and 2 papers from UDIT. 7 papers were duplicates found in two different searches. In the second step, papers that were not available (1) and not appropriate for the review based on the following exclusion criteria: papers mentioning educational video without any further details (75 publications), were eliminated. The elimination process resulted in a final list of 24 papers to be included in the results presented here. The selected publications can be downloaded here: https://shorturl.at/beovK

Step 3. Analyzing the papers

Quantitative and qualitative data was extracted from the papers and gathered in a database. The analysis of each paper included video type, video content and video context.

- Video type: We reviewed what video types, that were described, both predefined video types as well as explicitly declared formats.
- Video content: We were interested in what programming content was conveyed in the videos.
- Video context: We assessed how the video was included in a teaching context such as target audience, usage and intended learning objective.

Review findings

From the review several observations are made possible. Firstly, the majority of retrieved scientific papers on educational videos (75) do not provide information on what is meant by the term educational video, making it impossible to analyze the videos using the proposed dimensions by Winslett (2014). Only 13 out of the remaining 24 papers explore explicitly video genres for teaching programming. Secondly, the literature review retrieved papers published between 2011 and 2019, with 2016 as both median and average, proving our research in educational videos to support mathematics teachers utilize programming in the classroom is timely, reflecting a change in available video production and distribution platforms as well as an increasing need for programming skills for a larger audience.

Video type. The selected papers describe a variety of videos using diverse video types. However, two prevalent styles can be distinguished:

- A black board /tablet style lecture, a video of a slideshow or a whiteboard with the voice of a presenter explaining the content of the slide (Atapattu & Falkner, 2017; Dodson et al., 2018;

Fitzgerald & Tisdell, 2019; Fong et al., 2019; Hegarty-Kelly et al., 2015; Kim et al., 2014, 2015; Monserrat et al., 2013; Ohashi et al., 2019; Pal & Iyer, 2015; Tøssebro, 2017)
- A recorded classroom lecture (Anderson et al., 2012; Bednarik & Kauppinen, 2013; Consiglio & van der Veer, 2011; Kharitonova et al., 2012; Minnes et al., 2019; Pal, 2016; Thornton et al., 2017; Tøssebro, 2017).

Other video types found in the papers were: scripted video presentations recorded in a studio with a presenter and digital background (Carro et al., 2014); animated videos (Thornton et al., 2017) and storytelling (Guidazzoli et al., 2016; Peters, 2018); how-to videos and video tutorials (Dodson et al., 2018; Gordon & Guo, 2015); dialogue-based videos (Ohashi et al., 2019); and demonstration, interviews, and scenarios (Molnar, 2018). Several papers focus on providing *brief* educational videos: podcast highlights (Minnes et al., 2019) as distilled versions of classroom lectures, knowledge pills (Carro et al., 2014) as pre-recorded video clips, mini lectures (Molnar, 2018), micro lectures (Tøssebro, 2017), or microcontent (Fitzgerald & Tisdell, 2019) as short black-board style lectures. Another research focus is student participation through interactive elements in educational videos (Kim et al., 2015; Peters, 2018) or using students to generate educational video content (Carro et al., 2014; Gordon & Guo, 2015).

Video content. 13 papers concerned with learning programming, contain theoretical concepts, students' misconceptions, programming examples and how-to tutorials for computer science students or people aiming to learn programming online. We did not find research papers that discuss educational videos on programming didactics, something that is requested by mathematics teachers or teacher educators.

Video context. The educational videos were integrated in different learning environments:

- videos for online learning platforms (Atapattu & Falkner, 2017; Bednarik & Kauppinen, 2013; Gordon & Guo, 2015; Guidazzoli et al., 2016; Kim et al., 2014; Monserrat et al., 2013; Thornton et al., 2017; Weir et al., 2015)
- videos as additional resource to support traditional classes (Consiglio & van der Veer, 2011; Hegarty-Kelly et al., 2015; Kharitonova et al., 2012; Minnes et al., 2019; Ohashi et al., 2019; Pal, 2016; Pal & Iyer, 2015; Peters, 2018; Tøssebro, 2017)
- videos used to prepare for labs or as flipped classrooms (Carro et al., 2014; Dodson et al., 2018; Fitzgerald & Tisdell, 2019; Kim et al., 2015)
- videos used in the classroom (Anderson et al., 2012; Fong et al., 2019)

Towards genres of educational programming videos

Based on the findings from the review and our previous engagement in a research project raising programming competences for 8th to 10th grade mathematics teachers[Fel! Bokmärket är inte definierat.], we formulated three distinct genres of educational programming videos with different video types and content, that could be relevant to support mathematics teachers utilize programming in the classroom:

- **Pre-recorded lecture video** used to teach programming concepts: Lecture type video is the most frequent video type, often providing brief introductions to concepts and theories.
- **How-to tutorial** with voiceover to give step-by-step instructions on how to use educational technology: Tutorials that prepare students for lab exercises or can be used for troubleshooting in the classroom were another common video genre in the review
- **Dialogue style video**, that illustrates examples of teaching programming in mathematics: The third proposed genre is an example of an alternative video type and highlights the need for educational videos on programming didactics.

Figure 4. Lecture describing basic programming concepts

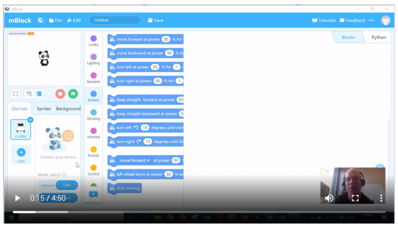

Figure 5. Get started with mBot - Tutorial

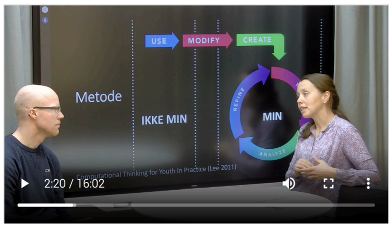

Figure 6. Use-modify-create conversation

In the following we will present the example videos we created for each genre[20].

Video lecture. Basic programming concepts

The video (**Fel! Hittar inte referenskälla.**) describes basic programming concepts such as variables, sequences, and algorithms. It could be used by students as supporting material, or by teachers and teachers' educators in the classroom. The video is structured in three parts: an introduction at the studio, followed by a trip to the grocery store and closing with a screencast with voice overlay. During the introductory part, the lecturer is placed at a desk in a studio, presenting the concepts of variables and arrays in combination with names, terms and examples written as text overlays in the background. The concepts are described using everyday metaphors. During the trip to the grocery store, the lecturer uses shopping activities as metaphor for further programming concepts. Again, text is added, to combine written and audio-visual elements. The concluding section is a screen-based animation representing the relationship between the concepts, overlaid with a narrator voice. The video's total length is 10:08 min and requires both time and resources to create. It is produced using a recording studio, on-scene filming, audio recording, text overlays as well as screen capture of slideshow animations.

Video tutorial. Getting started with mBot

The idea for this video tutorial (**Fel! Hittar inte referenskälla.**) arose from our teaching experience. We use the mBot robot to introduce programming to mathematics pre-service teachers. In the beginning of a programming lecture, we demonstrate how to get started with the robot before continuing with the main objective of the lecture. We often experience, that students just sit and wait, because "the mBot didn't work." Frequently the problem occurs when connecting the mBot to the computer using Bluetooth.We spend lots of time helping individual students to connect the mBot to their computer. The video is an alternative to prepare for the lecture or to be watched during the lecture as a step-by-step tutorial on how to get started with the mBot including how to connect the robot to the computer using Bluetooth, how to move forward and stop and to demonstrate common problems. We chose a screencast with a talking head as video type and kept it concise (4:50 min). The video was easy the create at the office, using the built-in recording functionality on the computer of both screencast and web camera. We had prepared a script and practiced the procedure a couple of times, to be able to record the tutorial in one scene and minimize post-editing.

[20] http://www.it.hiof.no/~toremake/programmering-for-alle/vip/survey1/

Video dialogue about Use-Modify-Create teaching methodology

In previous projects raising programming competences for 8th to 10th grade mathematics teachers**Fel! Bokmärket är inte definierat.**, we were often asked for teaching examples, lecture plans or best practices to utilize programming in mathematics classroom. There is a need for mathematics teachers, not only to learn programming, but also to develop a didactic mindset around programming in mathematics teaching. This video (**Fel! Hittar inte referenskälla.**) is intended to provide such a resource. It presents a conversation between two lecturers about their experiences from teaching programming. A teaching methodology called *Use-Modify-Create* (Lee et al., 2011) is presented and explained using examples from their teaching. The lecturers reflect on their experiences and discuss advantages, and limitations regarding the methodology. The video is long (16 min) and was produced in a studio with two lecturers placed around a coffee table to assure suitable audio quality during their conversation. Additionally, a slideshow was displayed on a screen behind the lecturers that could be maneuvered through touch. The video resembles a podcast with added visual elements in the background.

Future work

There is limited research exploring educational video genres for programming. We are interested to build on our review, including discussions with practitioners, to evaluate our set of educational programming video genres, and provide a lingua franca supporting practitioners to choose suitable video types for intended learning outcome, content, and context. Finally, we will integrate these videos into our mathematics teacher education and evaluate how these videos are perceived by pre-service teachers.

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Problem posing in mathematics education: a comparative study of textbooks in Ethiopia, South Sudan and Norway

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This paper presents a comparison of textbooks on Problem Posing (PP) activities. The importance of PP activities is motivated in relation to problem solving (PS), a principal component of mathematics education. Using a framework that has nine categories, problem-posing activities are analyzed in six different mathematics textbooks from Ethiopia, South Sudan, and Norway on the topic of algebra. The finding shows that there is sparse or even insignificant number of PP activities in the textbooks, and these PP activities are not comprehensive enough in their types and forms. A deliberate action is needed to embed such activities in mathematics textbooks.

Keywords: Problem Posing, Problem Solving, Textbooks, Curriculum, Comparative Study.

Introduction

Comparative studies in mathematics education document, analyze, contrast or juxtapose similarities and differences across all aspects and levels in the field (Tesfamicael & Lundeby, 2019). Usually, one starts from *a construct* and tries to investigate similarities and differences of that construct in two or more contexts (Artigue & Winsløw, 2010). In this work, problem posing (PP) activities is the construct under comparison, while the mathematics textbooks from Ethiopia, South Sudan, and Norway define the context for comparison. The aim is to improve the quality of education by facilitating the dissemination of different effective frameworks, principles, and models across nations (Tesfamicael & Lundeby, 2019). Especially, the recent reform-oriented mathematics education influence in Norway may serve as a context for reference, not just on the quality of the physical features of the textbook but the contents of the textbook, in this case, PP.

The contexts in comparison

Textbooks have a crucial role in the teaching and learning of mathematics. The textbook curriculum plays the role of mediation between the intended curriculum and the implemented curriculum (Valverde, 2002). Especially the teaching and learning of mathematics has long been heavily associated with textbooks and curriculum materials (Remillard, 2005). In fact, textbooks are the main resources used in mathematics classrooms in many countries (Valverde, 2002; Pepin & Jablonka, 2007). Hence it is meaningful to engage in the improvement of textbook research for improving student learning (Cai & Jiang, 2017).

The contexts that we are engaged in for comparison are textbooks from three different countries, two from Sub-Saharan Africa, Ethiopia and South Sudan, and two from developed countries, Norway. Most schools in Ethiopia and South Sudan use textbooks produced by the respective ministries of education in the countries. This means for a given grade level one textbook fits all in these countries. While in Norway, textbooks are prepared by authors and publishing companies. A variety of textbooks for a given grade level are available for teachers, parents (in the Norwegian case), and schools to choose from.

The construct under comparison

What is PP? (Silver, 1994, s. 19) defined it as follows: “It refers to both the generation of new problems and the re-formulation of given problems. Thus, posing can occur before, during, or after the solution of a problem.” (Kilpatrick, 1987) indicated that problem formation is an important companion to PS. Further, it has been advocated by many researchers that PP and PS are central to mathematical thinking, creativity, discourse in mathematics (Cai et al., 2015; Kılıç, 2017), in such a way that both are used as a means of instruction as well as an object of instruction to improve student learning (Kilpatrick, 1987; Silver, 1994; NCTM, 2000; McDonald & Smith, 2020). Furthermore, in recent reform movements PP is a focus on its own (NCTM, 2000), that is, the purpose of the instruction is to formulate or reformulate a problem, not just to solve it. Hence PP has become part of the mathematics curriculum and pedagogy (Cai & Jiang, 2017; English, 2020).

Research question

Cai et al. (2015) synthesized answered and unanswered questions concerning research on PP in mathematics education. Among the 14 unanswered questions, two of them related to PP activities and mathematics curricula are considered in this article. These are “How do the actual textbooks include PP?” and “If curriculum designers intend to integrate PP into textbooks and teaching materials, what are the best ways to do so?” Based on these questions, we aspire to start addressing the following research question. How do the textbooks in Ethiopia, South Sudan, and Norway include PP activities related to Algebra? In addition, we aim to investigate if PP activities are embedded in textbooks from these countries, focused on Algebra, just to limit the scope of our investigation. But first, we present the conceptual framework we have used in the study.

Conceptual framework

Recently, researchers have focused on teaching through PS than on learning about PS (Schoenfeld, 1992; Lester, 1994). Hiebert and colleagues (1996) emphasized that reform in curriculum and instruction should be based on PS. Hence, there is a huge interest on doing research on PS as compared to PP (Cai & Jiang, 2017; Deringöl, 2020). Cai and colleagues (2015) argue that PP is different from PS. In PS the students are asked to solve tasks, whereas in PP students are asked to generate new problems or re-formulate the existing ones. This does not mean that the students are required to solve the problem. Cai & Jiang (2017) reflected that the correlation between PS and PP is one reason why PP activities should be included in textbooks. Even for teachers to encourage students to pose problems, it is unlikely that they will have time and skills to do that unless teachers are offered such an opportunity, at least supplemental materials are needed (Cai et al., 2015).

How and when PP activity happens in relation to PS can be explained using the diagrammatic presentation as given in **Fel! Hittar inte referenskälla.**, as an extension of the work of Stoyanova and Ellerton (1996). Box 1 stands for a PP activities induced by teachers, as usually is done in traditional mathematics instruction, or given in a textbook, or it can even be generated by students. Hence it is prior to the PS process. The next phase is the PS process, and one option is doing it using George Polya’s (1945) four steps: understanding the problem, making a plan, executing the plan, and looking back. These are reinterpreted as three Boxes (2, 3, and 4) in the diagram.

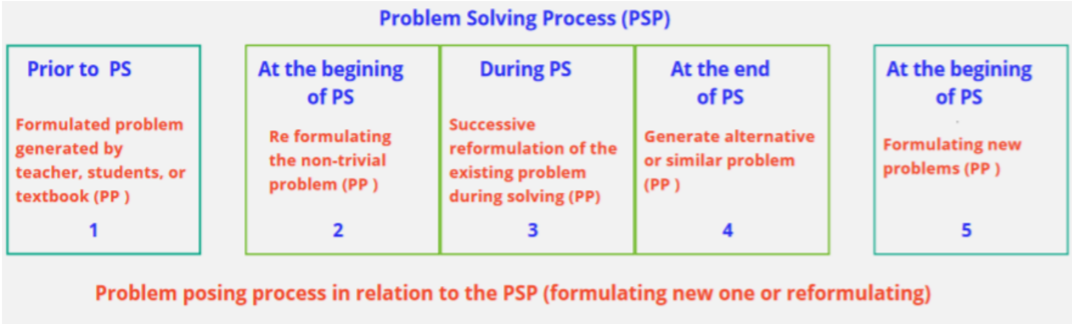

Figure 7. Diagram showing the occurrence of PP relationship to PS

At the beginning of solving the problem, a student can reformulate a given complex/non-trivial problem to a simplified version so that it can easily be solved with known strategies. During PS the expert solver tries to reformulate the existing problem in a successive reformulation and transformation of the problem. Somehow the solver is engaged in a series of posing problems (Box 3). Further, at the end of PS steps, as Polya called it, looking back, the solver can generate alternative or similar problems (Box 4). Formulating a new problem can also happen after a problem is solved. That new problem can demand a new imagination and creativity. It can be based on the previous solved problem, that is why we have post PS Box number 5. As in Box number 1, this can also be independent of the PS process. Since the study is on textbooks, it might not be possible to observe the PS process in boxes 2,3&4, unless it is made explicit. Mathematics textbooks can therefore be designed in such a way that teachers and students have access to PP activities (as described at Box 1 and 5 phases above).

General framework for problem posing types (Stoyanova & Ellerton, 1996)	Categorgy of PP Activitys	Specific Criteria (Adopted from, Cai & Jiang, 2017; Deringöl, 2020)
Free Problem Posing Activity (FPP)	#1	The student is asked to pose a problem without providing any data and without limitation.
Structured Problem Posing Activity (SPP)	#2	A Similar Problem Posing: The student is given a problem. The student is asked to pose another problem similar to this problem.
Semi-Structured Problem Posing Activity (SSPP)	#3	Problem posing in accord with the given **operation** (**symbolic**)
	#4	Problem posing in accord with the picture or using the information in the **picture** (**diagram**).
	#5	Problem posing in accord with the information given in the **table**
	#6	Problem posing in accord with the **graphic**
	#7	Problem posing in accord with **manipulatives** (**physical tools**)
	#8	Problem posing by completing **problem sentences** (or **verbal text**) through writing the missing information or through writing the appropriate question sentences.
	#9	Problem posing through given **context**

Table 2. Framework for PP analysis used in this work

Stoyanova and Ellerton (1996) proposed a framework that can link PP, PS, and curricula. The framework has three components. (i) *Free problem posing (FPP):* Students are asked to pose a problem based on a natural situation. (ii) *Semi-Structured problem-posing (SSPP):* “Students are given an open situation and are invited to explore the structure and to complete it by applying knowledge, skills, concepts and relationships from their previous mathematical experiences (p. 520)". (iii) *Structured problem-posing (SPP):* It occurs when a well-structured problem or problem situation is given, and the task is to construct similar problems (see also **Fel! Hittar inte referenskälla.**). Cai & Jiang (2017) classified PP tasks into five categories according to what it required students to do, in connection to the information provided in the task. Further, Deringöl (2020) has provided a framework for analysis joining these two frameworks. We have proposed a new way of sub categorizing *SSPP.* These seven subcategories under SSPP are based on the concept of multiple representation as given in Van de Walle (2018, s. 45). As SSPP activities include given conditions, and these conditions could be given in context, via diagram or graph, and so on, providing opportunity for students to make connections among representations as a way of showing mathematical understanding (see **Fel! Hittar inte referenskälla.**).

Method

In order to set up a reasonable comparison we took topic-based selection of textbooks. To start with, the topic Algebra is considered. It is a tool for generalizing arithmetic and representing patterns in our world, almost similar in all the textbooks from the three countries. This concept is presented in grade 7 and 8 mathematics textbooks in both Ethiopia and South Sudan. That means unit 2 in grade 7 (U2-G7-E), unit 2 & 3 in grade 8 mathematics textbooks (U2-G8-E & U3-G8-E) in Ethiopia; unit 4 in both grades 7 and 8 of South Sudan textbook are included (U4-G7-SS & U4-G8-SS). In Norway, there are a variety of textbooks produced by publishers. It means the sample space for choice of the textbook is broader in contrast to the two countries, where only one textbook for each grade level produced by the respective governments is used. Further, we base the choice of textbook for our research in Norway solely on one criteria. That is, the most recent textbook published in the country. The rationale behind the selection of recent textbooks is that we assume the textbook is influenced by the recent developments in mathematics education. That means the reformed mathematics education movement influence could be visible in the contents of the textbook. Textbooks considered, therefore, are Matemagisk and Maximum, and they are among the actively used and recent textbooks (Tesfamicael & Lundeby, 2019). Chapter 4 in both grade 7 Matemagisk and grade 8 Maximum (C4-G7MM-N & C4-G8MX-N) presents the topic Algebra, and these are included in the data analysis. Many textbook producers and publishers are revising the older versions of their textbooks after the implementation of the new curriculum in Norway in 2020. But these textbooks were not available to be considered in our study.

A qualitative research method, content analysis, is used in this study. Cohen and colleagues (2018) explained that qualitative content analysis helps to set up a set of procedures for the rigorous analysis, examination, inference, and verification of contents of original texts. Each of us has analyzed the textbooks from our respective countries using the 9 categories provided in Table 1 above. We agreed to use a subtask as a counting unit of analysis, just to simplify our comparison. We have also shared a google drive file to cross-check with each other's analysis. In addition, we selected some tasks and compared the result at the beginning for the sake of boosting the reliability of the study.

Findings

From the investigation of 7 chapters/units in algebra, in the 6 mathematics textbooks, around 1567 tasks were analyzed. And only 62 tasks were identified or approximated as PP tasks or activities, which is only 3.93% of the total task (See **Fel! Hittar inte referenskälla.**). Among these PP tasks most of them are categorized under SSPP (16 under category #3, 4 under category # 8, 34 under category # 9 and the rest 0). That is 87% of these PP activities are SSPP. There are 8 tasks that are SPP activities, but 0 tasks under category FPP.

Textbooks	Number of Tasks	Number of PP activities				Percentage of PP activities
		FPP	SPP	SSPP	**Total**	
U2-G7-E	148	0	0	4	**4**	2.7%
U2-G8-E	253	0	4	7	**11**	4.35%
U3-G8-E	219	0	0	0	**0**	0%
U4-G7-SS	72	0	1	4	**5**	6.94%
U4-G8-SS	134	0	0	0	**0**	0%
C4-G7MM-N	219	0	0	5	**5**	2.28%
C4-G8MX-N	531	0	3	34	**37**	6.97%
Total	1576	0	8	54	**62**	3.93%

Table 3. Number of tasks and PP activities in the 6 textbooks

Most of the PP activities are from the Maximum textbook (C4-G8MX-N). This coincides with our expectation that it will be influenced by the recent development in the teaching and learning of mathematics, as this textbook series is published in recent years.

The percentage of PP activities in U4-G7-SS is closer to that of C4-G8MX-N. However, looking closer to the tasks, we have a doubt if PP activities are embedded in U4-G7-SS intentionally. We can provide two reasons. First, four of the five tasks are actually subtasks under a task which are counted as one PP activity in our investigation. Second, there is no PP activity in U4-G8-SS, a textbook from the same country. Somehow this can indicate the absence of intentionality.

Fel! Hittar inte referenskälla. shows example of tasks found in Matemagisk (C4-G7MM-N) and Maximum (C4-G8MX-N) in connection to patterns and generalization in algebra. Task 89 is coded as SSPP activity, since it asks the pupils to create a pattern that contains 12 circles in figure 3 and 27 circles in figure 8. This is considered as SSPP activity, as it requires some kind of reformulation of the previous task based on the given conditions, as explained in PSP in **Fel! Hittar inte referenskälla.**. Task 5.29 is categorized as SPP activity since it asks a to formulate similar task. These are not FPP activity, since the students are asked to make a similar pattern, that means some kind of limitation is embedded in the formulation of the task.

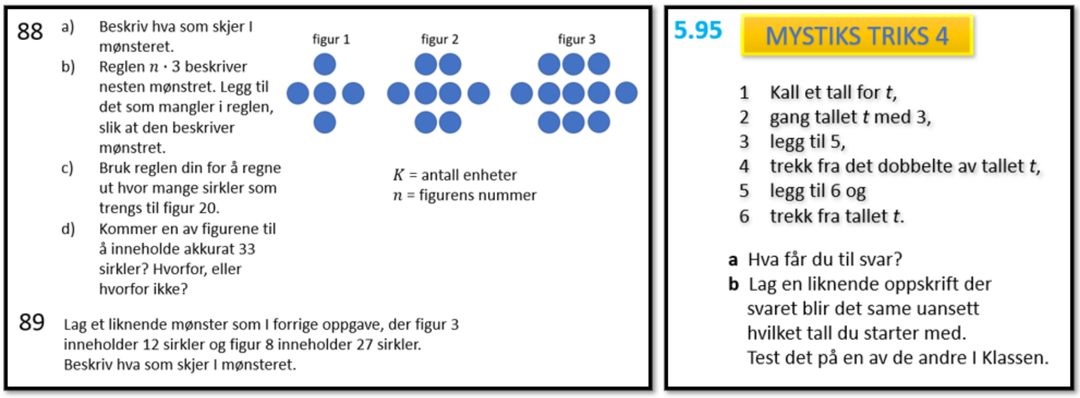

Figure 8. Task 89 in Matemagisk categorized as SSPP, Category #3 (Left). Task 5.95 b in Maximum 8, Grunnbok categorized as SPP (Right) (All figures recreated by the authors)

Discussion

The findings show that there is little PP activity in relation to the number of tasks provided by the textbooks in the area of Algebra in these textbooks from the three countries. This was a surprising result for us. Except Maximum, the textbook from Norway (C-G8MX-N), there are few PP activities or even not present in the case of U3-G8-E in Ethiopia and U4-G8-SS South Sudan textbooks (see **Fel! Hittar inte referenskälla.**). What could be the cause of such minimum or absent PP activity in these textbooks? Could it be the intended curriculum of the countries have no such mention or emphasis? Or is it due to the topic we selected, Algebra? Or is it just assumed that teachers would pose the tasks for the students and maybe teachers will ask the students to pose tasks to each other? All these need deeper investigation. Actually, in our preliminary investigation after our data collection, we found that the new Norwegian curriculum includes pupils to pose problems and uses the phrase "stiller matematiske spørsmål," translated as asking mathematical questions (LK20, 2020). But does this refer to PP activities? This needs further investigation, since asking questions might not be the same as problem posing.

The textbooks that include PP activities related to Algebra are mostly under the SSPP category. They do not provide the opportunity to students to make FPP activities, while there are 8 tasks (0.5% of the total tasks) of the category SPP, which is also a negligible amount. Hence these textbooks tend to embed the categories of the type SSPP activities mostly. Even the two textbooks in Norway differ in how they integrate the PP activities: Matemagisk contained only SSPP, while Maximum provided both SPP and SSPP activities. Deringöl (2020) reported only one task that is of the category FPP. This might limit the creativity and intuition of the learners. In addition, similar to the findings of Cai and Jiang (2017) on Chinese and US elementary mathematics textbooks, the PP activities across the grade levels of Ethiopia and South Sudan varies.

In the process of analyzing the tasks, identifying and categorizing a task as a PP activity was somehow problematic for us. This might be due to many reasons. For example, if the curriculum of a particular nation does not mention PP (implicitly or explicitly), then textbook producers might not consider including such tasks. In that case, looking for something that was not intended to be there in the first place, is meaningless. Such study is more appropriate when PP activities are considered at all levels of the curriculum (intended, textbook, implemented, and attained).

Conclusion

In this preliminary investigation of these 7 textbooks from 3 different countries, we found sparse PP activities among the tasks related to Algebra. These *PP activities are restricted in form.* That is, most PP activities are of type SSPP, and FPP activities are nowhere in all the 7 books. This needs a shift in including PP tasks in curriculum at all levels (Valverde et al. 2002). As discussed in the beginning, to improve student learning opportunity, and students critical thinking and reasoning, PP has been supported by several research (Cai & Jiang, 2017; Cai et al., 2015). If students and teachers are going to engage in such activity, the textbooks in the respective countries should provide opportunities for it. A deliberate action is needed. Especially, in the case of Ethiopia and South Sudan textbooks, where teachers adhere using only one textbook for each grade level, including PP actives is critical if students are expected to engage posing problems. One can argue that in Norway, teachers have more flexibility in choosing textbooks, and hence tasks and activities, and therefore the problem may not be as severe as compared to Ethiopia and South Sudan. Nonetheless, researchers have also indicated that most mathematics teachers, including those in Norway are heavily depending on textbooks (Pepin & Jablonka, 2007).

In our next step, data from the curriculums of other Nordic countries, teacher guides, and several mathematical topics will be included for deeper understanding.

Acknowledgment

The study was partially funded by the Norwegian Program for Capacity Development in Higher Education and Research for Development (NORHED).

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Norwegian, Swedish, and Finnish first-year engineering students' motivational values and beliefs about the nature of mathematics

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We report from a Nordic research project that has investigated first-year engineering students in Norway, Sweden, and Finland, and the relationships between their task performance, motivational values, and beliefs about the nature of mathematics. The present paper focuses on the covariance between their motivational values and beliefs from the perspectives of gender and nationality. The results show that female students' motivational values are more strongly related to their beliefs than is the case for male students. Further, the Finnish students' motivational values are only weakly related to their beliefs, whereas the Norwegian and the Swedish students' motivation to mathematics is quite strongly related to how much they appreciate the applications of mathematics.

Keywords: beliefs, engineering students, gender, motivation, nature of mathematics.

Introduction

This study is an outcome of a larger project investigating first-year engineering students in Norway, Sweden, and Finland, and how students' task performance depends on their motivation, self-efficacy, and beliefs about the nature of mathematics. The project was motivated by the well-known 'Transition problem' which refers to the challenges related to transition from studying school mathematics to studying university mathematics. Often this includes also changes in the social setting when a student moves to a new location but, especially, concrete and practical changes in what and how to learn in mathematical courses (Gueudet, Bosch, diSessa, Kwon & Verschaffel, 2016) and their connection to a student's motivation and performance (e.g., Anthony, 2000).

The present paper focuses on the relationship between the participated students' motivational values and their epistemological beliefs about the nature of mathematics from the perspectives of gender and nationality. Our research questions are as follows.

1. How do the Nordic students' motivational values correlate with their beliefs about the nature of mathematics across gender?

2 How do the correlational relations vary across the national cohorts?

Theoretical framework and literature review

We consider students' motivation values with aid of the Expectancy–value theory which explains how expectations and values affect an individual's learning behaviour. In this theory, the motivational values are categorised into four classes: 1) *intrinsic value* refers to the enjoyment of and interest in studying the subject, 2) *attainment value* represents the perceived importance of being good at mathematics, 3) *utility value* is related to the perceived usefulness of knowing mathematics for short- and long-term goals, and 4) *cost* portrays how much an individual is ready to invest his or her resources in studying mathematics (e.g., Wigfield & Eccles, 2000).

The role of motivation for success in studying mathematics is a topic that has been extensively studied and the positive effect of high motivation is well known. Due to the limitation for the length of the NORMA articles, we review only one Nordic study which is most relevant to us; for a more thorough review, see Tossavainen, Rensaa, Haukkanen, Mattila, and Johansson (2021). Bengmark, Thunberg, and Winberg (2017) examined Swedish engineering students' motivation, beliefs about and expectations on studying mathematics, and study habits as well as their relative importance for the students' performance in the transition to tertiary mathematics. They noticed that students' characteristics, e.g., motivation and beliefs, play a significantly more important role than, for instance, their study habits when predicting students' achievement during the first year in university.

The epistemological beliefs about the nature of mathematics are also called orientations. Felbrich, Müller, and Blömeke (2008) have formulated and applied them as follows.

- A *formalism-related* orientation is mathematics viewed as an exact science with an axiomatic fondation and being developed by deduction (e.g., 'Mathematical thinking is determined by abstraction and logic.')
- A *scheme-related* orientation is mathematics given as a set of terms, rules and formulae (e.g., 'Mathematics is a collection of procedures and rules which precisely determine how a task is solved.')
- A *process-related* orientation is mathematics regarded as a science which in most parts consists of problem solving processes and where finding structure and regularities is important (e.g., 'If one comes to grip with mathematical problems, one can often discover something new (connections, rules and terms).')
- An *application-related* orientation is mathematics regarded as a science which is relevant for society and life (e.g., 'Mathematics helps to solve practical tasks and daily problems.').

The formalism and scheme-related orientations are of a static nature, taking mathematics merely as a ready-made construction, whereas process- and application-related orientations are more dynamic embracing developing and discovering activities. Students usually acknowledge several aspects of mathematics simultaneously, yet most of them can name which orientation best corresponds to their view of mathematics (e.g., Tossavainen, Rensaa, et al., 2021).

For example, Harris et al. (2015) have noticed that freshmen engineering students recurrently highlight the application-related orientation to mathematics. In their study, the students accentuated that the ”use-value” of mathematics is lost if mathematics is not taught with an emphasis on the applicable examples in engineering. Similarly, the dynamic orientations were three times more often mentioned as a primary orientation than the static orientations in the study of Tossavainen, Rensaa, et al. (2021). However, previous studies that we are aware of have not focused on the relationship between the orientations and the motivational values. In this sense, the present study is the first of its kind.

Sax and colleagues (Sax et al., 2016) have surveyed 40 years of the changing gender-related dynamics among engineering students. They found that female students' self-efficacy in mathematics has long been lower than that of male students. Concerning self-concept, performance expectations, and motivation, similar findings from the adult learners have also been reported in the Nordic context by Skaalvik and Skaalvik (2004). However, Tossavainen, Rensaa, et al. (2021) showed that this is not necessarily true anymore; female students outperformed male students in a set of seven mathematical tasks and set more demanding learning goals for themselves than male students. Further, they expressed higher intrinsic motivational values, whereas male students emphasised utility values.

Method

The present investigation is based on 431 first-year engineering students' responses to a questionnaire; out of these 71 were from Norway, 88 from Sweden, and 272 from Finland. The students were just starting their engineering studies; the data were collected during the first weeks of their first basic calculus course. The same questionnaire was used in all countries, yet translated to the native language. The universities taking part in Norway and Sweden are medium-sized and situated in northern, sparsely populated areas, while the university in Finland is one of the largest and most popular ones in Finland situated in a densely populated area in Southern Finland. This explains the difference in the sizes of the national cohorts. It is important to notice that this kind of design does not automatically lead to a bias. Differences, for instance, in the entrance requirements could easily affect, e.g., first-year students' task performance across the cohorts, but there is no a priori reason to believe that the differences between the participating universities have an effect on the relationship between the students' motivational values and orientations. A main motivation for studying different kinds of Nordic cohorts was to increase a variety of educational settings in our data.

The cohorts represent 17% (NOR), 18% (SWE), and 32% (FIN) of all first-year engineering students at the universities. The proportions of female students in the cohorts are 13% (NOR), 19% (SWE), and 34% (FIN). In the whole data, 27% are female students, 71% male students, and 2% did not express their gender. For more details, see Tossavainen, Rensaa, et al. (2021).

The questionnaire used in the study contained a set of propositions equipped with a five-step Likert interval scale measuring a student's self-efficacy, motivation, and view of the nature of mathematics. The items used to measure the distribution of orientations were based on the instrument developed by Tossavainen, Viholainen, Asikainen, and Hirvonen (2017). The items are shown in the first column of Table 1. Similarly, the eight items measuring the motivational values are shown in Table 1. The statements were deliberately formulated in positive terms to encourage engineering students to reflect on their motivation. The letter in the parenthesis at the end of each item indicates which value or orientation the item is intended to measure. For each of value and orientation, there are two items. The Likert scales were directed so that 1 corresponded to ”strongly disagree” and 5 to ”strongly agree”.

For the statistical analyses, the variables standing for the orientations and motivational values have been computed as mean values of the corresponding items, see Table 1. The Pearson correlation analyses between these notions have been conducted using SPSS Version 25. For interpreting the effect sizes, we have used the critical values given by Cohen (1988).

Orientations	**Motivational values**
O1. Mathematics is about describing the real world (A)	M1. I really like mathematics (I)
O2. It is not mathematics if it cannot be proved theoretically (F)	M2. I am motivated to study maths mainly because it is useful to other studies (U)
O3. Mathematics is a collection of formulas and concepts (S)	M3. I want to succeed as well as possible (A)
O4. Mathematics is problem solving (P)	M4. I would suspend a hobby in order to succeed in a maths exam (C)
O5. The purpose of mathematics is to maintain functionality in society (A)	M5. I would do extra exercises to guarantee that I succeeded well (C)
O6. Mathematics is about discovering structures and regularities (P)	M6. I would study maths voluntarily because every engineer must know it (U)
O7. The main task of mathematics is to give correct rules for calculations (S)	M7. If I get a low grade in mathematics, I want to take the exam again (A)
O8. In mathematics, all concepts must be defined in a precise and clear way (F)	M8. Mathematics is full of interesting problems and results (I)

Table 1. Items in the questionnaire

Results

In the following tables, the number of participants vary slightly between columns and rows because a few students' entries in certain items of the questionnaire were empty or ambiguous. These entries have been excluded from the analyses.

Table 2 answering the first research question shows some significant differences between female and male students. For female students, the correlations between the motivational values and orientations are higher and there are more significant relations between the motivational values and orientations than for male students. For both groups, all significant correlations are positive, which indicates that a higher motivation (in general) in mathematics is related to a higher appreciation of various natures of mathematics. It is not surprising that the utility values are related to appreciating the usability of mathematics in applications, but a less obvious relation is the relatively high correlation between Cost and Applications for female students. In other words, those female students who appreciate mathematics for its applications are prepared to invest extra time for studying mathematics. This relation is more than double higher for female students than for male students. Female and male students differ from one another even more in the relation between Attainment and Application; the female students who appreciate mathematics for its applications also want to perform well in mathematics. The effect size in this correlation is already medium. A finding worth mentioning is also the fact that the orientation that measures appreciation of exact reasoning and defining mathematical notions precisely has no significant correlation with motivation in mathematics.

	Application	**Process**	**Scheme**	**Formalism**
Male (N=298–305)				
Intrinsic	0.17**	0.08	–0.10	0.01
Attainment	0.11*	0.06	0.06	0.00
Utility	0.21**	0.05	0.15*	0.03
Cost	0.12*	-0.03	0.04	0.01
Female (N=116–118)				
Intrinsic	0.26**	0.16	–0.04	0.16
Attainment	0.30**	0.05	–0.03	–0.02
Utility	0.20*	0.19*	0.21*	0.06
Cost	0.27**	0.09	0.07	0.14

* = p<0.05; ** = p<0.01

Table 2. Correlations for the male and female participants

Table 3 summarizes differences between the Finnish, Norwegian, and Swedish students. The main finding is that the Finnish students differ significantly from the Norwegian and the Swedish students in one issue, but the Norwegian and the Swedish students differ from one another only slightly in a few issues. For the Finnish students, none of the orientations is highly correlating with the motivational values although some of the correlations are statistically significant even at the level $p < 0.01$. But the effect size is small in each case. This means that the Finnish students' motivation values do not depend on their orientations; recall that zero correlation between the normally distributed variables is an evidence for the independence of the variables. An interesting detail is that the correlation between the formalism orientation and interest in mathematics is statistically significant and positive for the Finnish students as opposed to the other students.

A common feature for all nationalities is that the relationship between the motivational values and orientations is first and foremost established via the application-related orientation and two or more values. When it comes to the Norwegian and the Swedish students, there are several quite high correlations with medium or almost large effect size between the motivational values and the application-related orientation. Other orientations are not related to their motivational values. The relationship between the motivational values and orientations is broadest for the Swedish students, there are altogether four significant correlations in the Swedish part of Table 3, of which two are significant even at the level $p < 0.001$.

The most remarkable difference between the Norwegian and the Swedish students is that those Swedish students who are interested in and appreciate the applications of mathematics often indicate also that they want to perform well and they are prepared to invest extra time in studying mathematics, whereas the Norwegian students do not have this kind of relation between their motivational values and orientations. Their appreciate mathematics merely for its usability.

	Application	**Process**	**Scheme**	**Formalism**
Finnish (N=270–272)				
Intrinsic	0.04	0.08	–0.09	0.13*
Attainment	0.14*	0.02	–0.03	–0.01
Utility	0.12*	0.01	0.08	0.09
Cost	0.17**	–0.08	–0.09	0.11
Norwegian (N=65–68)				
Intrinsic	0.46***	0.12	–0.04	–0.14
Attainment	0.10	0.08	–0.23	–0.02
Utility	0.35**	0.17	0.03	0.01
Cost	0.01	–0.02	–0.13	–0.04
Swedish (N=85–86)				
Interest	0.42***	0.16	–0.13	–0.08
Attainment	0.31**	0.04	0.03	0.05
Utility	0.42***	0.12	0.05	–0.01
Cost	0.26*	–0.02	–0.03	0.02

* = p<0.05; ** = p<0.01; ***= p<0.001

Table 3. Correlations for the Finnish, Norwegian and Swedish participants

Discussion

Our findings related to the first research question are somewhat surprising. If the application orientation becomes more central in the Nordic engineering students' view of mathematics, it increases female students' motivation in mathematics clearly more than male students' motivation. This outcome cannot be explained by a larger variation in female students' motivational values or orientation variables because there are no such differences in the data. However, the difference between male and female students lies in the magnitude of the relationship, not in the quality of the relationship. Further, the findings show that the Nordic first-year engineering students get motivation for mathematics from studying meaningful and relevant applications. This finding seems to be universal, cf. Harris et al. (2015).

Concerning the second research question, we found some correlations with larger effect sizes when compared with the results to the first question. The main result however is that, for the Finnish students, the motivational values and orientations are only weakly related, whereas for the Norwegian and Swedish students, there are quite strong relationships between the values and orientations.

Again, it is difficult to find a self-evident reason for this outcome. Why the first-year students from a university, which is situated to a more populated area and, therefore, has got somewhat higher entrance requirements, should have a weaker relationship between their motivational values and orientations? A possible reason might be that the study programmes provided by the participating universities are not completely similar. As the Finnish university offers many theoretical alternatives, it could explain the positive correlation between the motivational values and the formalism-related orientation. On the other hand, also the other two universities provide theoretical programmes. Moreover, the descriptive statistics for the formalism orientation are similar for all national cohorts (Tossavainen, Haukkanen & Rensaa, 2020; Tossavainen, Rensaa & Johansson, 2021). Hence this explanation is not completely convincing.

A more plausible explanation for the above-mentioned unexpected findings is as follows. We computed, both for the whole group and across the national cohorts and the gender groups, the Cronbach alphas for the items used in the sum variables. Surprisingly, we found some remarkable differences which above all are related to the items used for measuring the utility value. For the Scandinavian students, $\alpha \approx 0.5$, whereas $\alpha \approx 0$ for the Finnish students. In other words, the internal consistency of these items for the Scandinavian students differs remarkably from that for the Finnish students. There is a similar yet somewhat milder difference in the alphas between the groups organised due to gender. This phenomenon has an effect on the correlations of the utility sum variable with other variables in Tables 2 and 3. The observed correlations are likely weaker than they would be if the utility value had been measured using a more consistent instrument. This is seen as we compute the Pearson correlations between the single utility items and the orientation sum variables. For example, for male students, the Pearson correlation between the item M2 in Table 1 and the scheme-related orientation is 0.25 ($p<0.001$), whereas the correlation coefficient between the other utility item M6 and the scheme-related orientation is –0.05 ($p>0.05$). These variations damp one another which explains why the corresponding correlation for the utility sum variable is only 0.15 in Table 2.

Now afterwards, we can only say that we should have used more than two items for each value and orientation so that the sum variables would have been more coherent, and then we might have found more significant relationships. For the reason discussed in the above paragraph, it is very unlikely that our instrument would come up with correlations that were not real. In this sense, the low internal consistency is not a too serious problem now. On the contrary, it reveals that the motivational values in the real world make a more complex structure than the four dimensions described by the Expectancy–value theory. Moreover, this incident revealed a very interesting phenomenon: the same instrument may work differently in different contexts.

To sum up, our study succeeded to reveal gender- and culture-related variation in the relationship between the motivational values and the orientations. These relationships are far from being examined completely. Consequently, this topic deserves further investigations.

Acknowledgment

We thank Eivind Kaspersen for his valuable comments and for helping us to discuss our findings in a more correct way.

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Sámi concepts of *pattern* in the mathematics curriculum

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Norway's new mathematics curriculum took effect in 2020. The word mønster *(English: "pattern") appears 13 times in the curriculum text. This paper aims at providing insight into how the curriculum's term "pattern" can be translated into North Sámi in a way that supports a mathematics teaching rooted in local Sámi culture and language. Five Sámi words for patterns are analysed to highlight different meanings. The analysis reveals that* minsttar *is the word most frequently used in direct translations. A possible explanation for this is that the meaning of the word* minsttar *has evolved to cover more aspects of the meaning of "pattern". Context plays an important role in determining which Sámi word to choose; some words are for instance related to visual patterns and one is related to Sámi handicraft. The choices depend on what kind of pattern is examined. Because of language differences, it is important to translate "pattern" with great care.*

Keywords: Pattern, translation, Indigenous, curriculum, language.

Introduction

In this paper, we identify challenges related to translating one central word in the mathematics curriculum from Norwegian to Sámi. According to Norway's Sámi[21] act (Ministry of Local Government and Modernisation, 1989), Sámi and Norwegian languages are equal. This means that Norway's mathematics curriculum (Ministry of Education and Research [KD], 2019) will be translated into Sámi languages[22]. The translation must be made so that it supports Sámi teachers' choice of relevant cultural contexts for their teaching. Language and culture differences make translations of terminology complicated, not just in the field of mathematics education. Cole (2009) discusses challenges related to translations of Vygotsky's word *obuchenie* (teaching, learning, education, et cetera) from Russian to English; the English translations do not fully capture the meaning of the original word. According to Barton et al. (1998), Māori has more words than English for *large*; all of these can be used for numbers, but only some of them refer to size. Omitting nuances like this in translations from English to Māori might contribute to language deprivation. Fyhn et al. (2011) explain how the meaning of the Sámi term *bealli* is richer term than just *half*; it describes both the ratio 2:1 and the ratio 1:2. A direct translation of *bealli* to Norwegian (or English) might cause unclarity or be misleading. It is even more problematic to translate "half" into Sámi languages, because there is a risk that "half as much" is translated into "twice as much". Sámi students had to face that obstacle at the 2018 national test in mathematics (Fyhn & Hætta, 2019). Fyhn and Hansen

[21] The Sámi are an Indigenous people of the Arctic. They inhabit Northern Scandinavia and the Kola Peninsula of Russia.

[22] November 2021: The curriculum is not translated.

(2019) revealed that when translating the mathematics curriculum's word ”pattern” from Norwegian to Sámi, one must consider at least the three words *minsttar*, *hearva* and *girji*.

Bishop (1990) points out that the language and logic of the Indo-European language group have developed layers of abstract terms within the hierarchical classification matrix, but this has not happened in all language groups. As a result, there are different logics and different ways of relating phenomena around the world. Sámi languages belong to the Finno-Ugric language group (Duolljá & Gaski, 2020), while Norwegian (like English) is a Germanic (and Indo-European) language. Sámi languages do not generalise by using overarching terms the same way Germanic languages do (Fyhn et al., 2018). Our paper provides examples of how one central term in mathematics is embedded differently in one Indigenous language than it is in Germanic languages like Norwegian and English.

Why we focus on *pattern*

Fyhn and Hansen (2019) studied mathematics teaching in a mixed Grades 1–2 class in a Sámi school. They focused on students' work with repeating linear patterns, which according to Threlfall (1999/2005) is the first step towards number patterns and algebra. The teacher and the students consistently use the Sámi word *hearva* for patterns (*mønster* in Norwegian[23]), while the mathematics curriculum (KD, 2013) consequently translates ”pattern” into *minsttar.* A linguistics professor supports the teacher's choice of word. Our study is rooted in Fyhn and Hansen's (2019) findings by highlighting challenges that may arise when translating the mathematics curriculum's term ”pattern” into Sámi[24]. According to Zazkis and Liljedahl (2002), patterns are the heart and soul of mathematics. Devlin (1998) claims that mathematics is the science of patterns; it is a way of looking at the world. The Organisation for Economic Co-operation and Development (OECD, 1999) was in agreement with Devlin when they claimed that ”[m]athematics is the language that describes patterns, both patterns in nature and patterns invented by the human mind. To be mathematically literate, students must recognise these patterns and see their variety, regularity and interconnections” (p. 48).

The word ”pattern” appears on 13 different occasions in the new mathematics curriculum (KD, 2019). In this paper we investigate possible translations of these 13 appearances of ”pattern” into Sámi. Our aim is to contribute to the wider discussions of how to translate mathematics curriculum texts to Indigenous languages. To avoid misunderstandings, it is of importance to Sámi mathematics education that the mathematics curriculum is translated with great care. The research question is: How can ”pattern” (*mønster*) in the mathematics curriculum be translated from Norwegian to Sámi?

Roberts (1998) claims that when translating from English to Indigenous languages, mathematical language should follow the structure of the Indigenous language, to avoid language deprivation. To consider this claim, we firstly provide an overview of Sámi words that can mean *pattern* when translated from Sámi into Norwegian. McMurchy-Pilkington et al. (2013) point to the importance of debating standardisation of terms and the place of dialectical differences in the development of Indigenous mathematics curricula. In many cases, there are significant differences between words from one dialect and words from another within the Sámi languages. Therefore, a first step of the analysis is to examine and compare translations of the curriculum sentences from Norwegian to the

[23] This paper uses the English word ”pattern” for the Norwegian *mønster* throughout the text.

[24] When we refer to Sámi language in this paper, we mean North Sámi unless something else is explicitly stated.

Kárášjohka/Karasjok (eastern) and Guovdageaidnu/Kautokeino (western) dialects of North Sámi. The second step is to discuss each appearance of Sámi words for *pattern* to a) validate if the translation is in accordance with the meaning of the original sentence in Norwegian, and b) investigate whether more than one word is needed to cover the meaning of ”pattern”. Standardisation of terms is included in the discussion section.

Sámi culture and language

Fishman et al. (1985) claim that language itself is a part of culture and that every language becomes symbolic to the culture with which it is intimately associated. Furthermore, Indigenous cultures include traditional knowledge about how to use nature and its resources, and how to adapt and transform purchased materials for use in the local community (Sara, 2004). Traditional Sámi livelihoods like hunting, fishing, trapping, and reindeer husbandry are important for sustaining Sámi culture and language. Sámi languages (there are ten in total) reflect the use of nature by reindeer herders, those who pick berries, hunters, and others. Local languages are rich in details about the terrain, nature, and the use of nature. Traditional knowledge is an important part of language preservation. *Duodji* (Sámi handicraft) constitutes a major part of Sámi traditional knowledge; it is handicraft intertwined with Sámi culture, traditions, and language. People who perform *duodji* work with cultural expressions that arise out of the culture's traditional knowledge (Guttorm, 2007). Traditionally, a purpose of *duodji* was to cover a need as well as having a decorative aspect. There are several *duodji* words that can be translated into ”pattern” in English. All these words describe specific aspects and properties of the handicraft process and product. The term *minsttar* describes a template, made from for example paper or wood and used when cutting materials for the handicraft. Craftsmen use *minsttar* when cutting materials for different kinds of garments. One traditional Sámi garment is *gákti*. The style or appearance of a *gákti* varies from place to place in Sápmi. The term *málle* or *gáktemálle* is typically used to describe the distinctive style of a *gákti* (Hermansen, 1993), which depends on its origin. This means that you can have a specific *minsttar* for cutting materials for a *gákti*, and the *gákti* can have a *málle* belonging to a particular area or family in Sápmi. In addition, the word *hápmi* refers to how the *gákti* appears on you as an individual.

A similar distinction as between *minsttar* and *málle* is present in Sámi weaving, knitting and braiding terminology. This distinction is between the two words *minsttar* and *hearva; minsttar* means a template, while *hearva* means decoration or ornamentation. Hætta (2016) consequently uses *minsttar* for weaving patterns. The woven products then have varying *hearva* and colours depending on their place of origin; they are symbolic and express a meaning. Additionally, Sámi weaving terminology includes the term *girji*. Figure 1 shows an example of a woven band where *girji* is highlighted. *Girji* describes the repeating unit of a woven band. Norwegian weavers use the similar term *rapport.* There are two kinds of visual patterns on gloves; gloves that are patterned all over have *girji*, while gloves with a repeating pattern around the wrist have *hearva* (Nielsen, 1962/1979; Fyhn & Hansen, 2019).

Figure 1. *Girji* on woven band. Photo: Márjá-Liissá Partapuoli

An overview of Sámi words for *pattern*

Dictionaries provide translations of ”pattern” from Norwegian to Sámi: i) The Sámi mathematics dictionary (Nystad et al., 2002) translates ”pattern” into *minsttar.* ii) The online Sámi dictionary (Giellatekno, 2021) translates ”pattern” into *málle* or *minsttar*. These translations do not cover all meanings of ”pattern”. The other way around: Translations from Sámi to English reveal more words than translations from Norwegian/English into Sámi, as Table 1 shows.

	Giellatekno (2021)	Konrad Nielsen (1962/1979)	Kåven et al. (1995)	Nystad et al. (2002)
minsttar	pattern	pattern, model	pattern, formula	pattern
hearva	decoration (*stas, pynt*)	finery, adornment, ornament(ation), amusing person	decoration, trimming, embroidery, amusement (*dekorasjon, pynt, broderi, fornøyelse*)	
girji	book, letter, fur dot (*bok, brev, flekk i pelsen*)	a spot of another colour (on an animal), coloured ornamental patterns, book, letter	book, letter, spot (*bok, brev, flekk*)	
hápmi	shape, appearance, figure, façade (*form, skikkelse, utseende, fasade*)	exterior, appearance, carriage, semblance, outward appearance, shadow, phantom	figure, character, attitude, appearance, shape (*skikkelse, holdning, utseende, form*)	shape *(form)*
málle	pattern, template, shape, design (*mønster, mal, fasong, design*)	pattern, design, style	design, pattern (*design, mønster*)	type, model (*type, modell*)

Table 1. Sámi ”pattern' words translated into English, Norwegian words in parenthesis

Four dictionaries provide a variety of Sámi words for ”pattern”: Giellatekno (2021), Konrad Nielsen’s (1962/1979) Sámi–English dictionary, Kåven et al.’s (1995) Sámi–Norwegian dictionary and the two-way Sámi–Norwegian mathematics dictionary (Nystad et al., 2002). Several Sámi words for *pattern* depart from *duodji*, so we include this context in our analysis. *Hápmi* means *shape* in a broader sense than just the shape of a geometrical figure; it includes for instance the patterned surface of a pinecone or a cactus. It is important to note that different Sámi words for *pattern* cannot be treated as discrete categories; there are often sliding transitions between them.

”Pattern” in the new curriculum

The first author is a native speaker of the Kárášjohka dialect, while the third author is a native speaker of the Guovdageaidnu dialect. The third author is also *duodji* master student at the Sámi University of Applied Sciences. Despite great differences between these two dialects, we found no significant dialect differences in the first translation of the 13 occurrences of ”pattern” in the new curriculum, here we aimed at one overarching word for each occurrence. The next step aimed at finding the most appropriate Sámi word(s) for *pattern* in each case. It turned out that *minsttar* was the most common choice of term: it occurred in ten out of thirteen cases. *Hearva, girji* and *málle* were, however, not chosen for any occurrence. *Hápmi* was chosen for the two contexts ”properties and structures in number- and figure patterns” for Grade 2 and ”structures and patterns in play and games” for Grade 4. The context ”investigate and describe symmetry in patterns” for Grade 6 was the only instance where the translation needed two Sámi words to cover the meaning. Both *minsttar* and *hápmi* were chosen for this meaning of ”pattern”. It is worth noticing that the Sámi word for symmetry, *symmetria*, is a new Sámi word that has recently been imported from other languages. The term appears in the Sámi mathematics dictionary (Nystad et al., 2002), but it is neither found in Nielsen’s (1962/1979) five-volume dictionary nor in Kåven et al.’s (1995) dictionary.

Because the Sámi words for *pattern* have different contextual meanings, the analysis focuses on which Sámi word to use when. To highlight these different contextual meanings, an alternative, developed translation is made for certain parts of the curriculum. The purpose of the developed translation is to illustrate how it may be more appropriate to use *hearva*, *hápmi* or *girji* instead of *minsttar*. Table 2 presents a competence aim for Grade 2 as an example. The leftmost column shows the original Norwegian curriculum text; then follows direct translations into English and Sámi. The rightmost column shows a translation that was developed by searching the literature and through discussion among the authors. This translation includes the two words *girji* and *hearva*; *girji* refers to the unit of repeat, while *hearva* is more appropriate for visual, cultural patterns.

Original text	English translation	Sámi direct translation	Developed translation
Eleven skal kunne kjenne att og beskrive repeterande einingar i mønster og lage eigne mønster.	The student should recognise and describe repeating units in patterns and create their own patterns.	Oahppi galgá dovdat ja čilget geardduheaddji osiid minstariin ja ráhkadit iežaset minstariid.	Oahppi galgá dovdat ja čilget girjjiid (geardduheaddji osiid) hearvvain ja ráhkadit iežaset minstariid.

Table 2. Translations of ”pattern”. Sámi pattern words are highlighted

Discussion

Fyhn and Hansen’s (2019) study focused on visual repeating patterns. The teacher in their study chose to depart from patterns that were closely related to the children’s experiences with local Sámi culture and language. This is probably an important part of the reason why the teacher consistently chose to use *hearva* instead of the curriculum’s *minsttar.* Their example highlights how the choice of which

Sámi word to use in translations may differ depending on what kind of patterns the students are investigating. If the children were to investigate other kinds of patterns, e.g. number patterns or repeating letter sequences, the choice of Sámi word might have been different.

A possible explanation for the many occurrences of *minsttar* in the translations is that the term *minsttar* has evolved and progressed past the traditional meaning (template) to become a term that covers more aspects of the meaning of ”pattern”. *Minsttar* would thus also cover more abstract patterns, such as patterns in algebra and algorithms. The contextual meaning of *hápmi, hearva* and *málle* describe visual aspects. Because of this, these terms may seem narrower in meaning than *minsttar*. Sámi and Finnish are Finno–Ugric languages, as opposed to Norwegian, Swedish, and English. The Finnish mathematics curriculum is published in Swedish as well as in Finnish, because Swedish is an official language in Finland. It turns out that the word ”pattern” (*mönster*) occurs only twice in Finland's mathematics curriculum (Utbildingsstyrelsen, 2015). Mathematical reasoning for Grades 1–2 and 3–6 is focused on finding similarities, differences, and patterns. Regarding algebra for Grades 3–6, the students investigate patterns in number sequences. Here the Finnish word ”regularities” (*säännönmukaisuuksia*) is translated into ”pattern” (*mönster*) in Swedish. However, according Giellatekno (2021), *säännönmukaisuuksia* means *njuolggadus* in Sámi, which in turn means ”rule” or ”guideline” in English.

Closing remarks

The mathematics curriculum in Norway will be translated into three Sámi languages: North -, Lule - and South Sámi. According to the United Nations Educational, Scientific and Cultural Organization (UNESCO, 2010), North Sámi is definitely endangered while South Sámi and Lule Sámi are severely endangered. This means that there are no strong populations who speak the Sámi languages and who can comment on potentially misleading or slightly wrong translations of terms in the mathematics curriculum. This demonstrates the importance of choosing words and terminology with great care when translating a new mathematics curriculum text; the curriculum's choice of words must make sure that Sámi children's teaching is rooted in their local culture. The analysis in this paper may contribute to a more appropriate translation of the mathematics curriculum. Omitting the huge variety of Sámi words for *pattern* in translations from Norwegian to Sámi might lead to the exclusion of words with more narrow and precise meanings. This must be avoided since it may in turn contribute to language deprivation. More research on Sámi mathematics teaching practice is needed to determine which Sámi words are most appropriate for *pattern* in different parts of the curriculum.

Acknowledgement

The work in this paper is part of the SUM project (Norwegian acronym for Coherence through Inquiry-based Mathematics Teaching), which is supported by the Norwegian Research Council's program FINNUT – Research and Innovation in the Educational Sector, Grant no 270764.

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Kindergarten preservice teachers evaluating mathematical apps

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In Norway, curriculum documents require early childhood teachers to integrate digital resources into their work. Consequently, teacher educators need to know how to support preservice teachers to evaluate apps that encourage young children's mathematical engagement. This requires investigating preservice teachers' initial evaluations. In this paper, thirteen screen recordings of kindergarten preservice teachers evaluating apps were analysed, using Artifact-Centric Activity Theory. The results show that preservice teachers could identify several aspects of the relationships between apps, children and mathematics, which could become the basis for future mathematics teacher education work in this area.

Keywords: Digital resources, preservice teachers, early childhood education, app evaluations, mathematical play

Introduction

In Norway, early childhood institutions are expected to support children to engage with mathematical ideas (Ministry of Education and Research, 2017). As well, the curriculum, known as the Framework Plan (Ministry of Education and Research, 2017), states that kindergarten staff should use different tools, with learning expected to occur through play, experimentation and everyday activities. Digital tools are specifically described as contributing to this play and learning – ”digital practices in kindergarten shall encourage the children to play, be creative and learn” (Ministry of Education and Research, 2017, p. 44). Yet to enact these requirements, kindergarten teachers need to evaluate digital tools in order to decide which ones best support children to play, be creative and to learn about mathematical ideas.

In regard to digital applications (apps), Cherner, Dix and Lee (2014) indicated that such requirements placed new demands on teacher educators to improve preservice teachers' digital competencies, because ”as these new technologies become more ubiquitous and available, teacher candidates need tools to guide their selection, integration, and effective use of apps as they begin their teaching careers” (p. 159). In Norway, preservice teachers (PTs) considered that they have not been made aware of how to use digital resources in kindergartens (Alvestad & Jernes, 2014) and there is limited research on PTs' evaluations of digital apps for young children, either before or after their participation in teacher education courses. This may partly be because of the newness of touch-screen tablets as well as the ever-expanding range of digital apps (Blackwell, Lauricella & Wartella, 2016). For PTs' digital competence to be developed in the limited time available in teacher education, there is a need to understand what they know at the start so their courses can be designed appropriately. Our research question is: What factors do preservice teachers raise when evaluating mathematical digital games for young children?

Artifact-centric activity theory

To identify the aspects of digital competency that preservice teachers have at the beginning of their teacher education, we chose to use Artifact-Centric Activity Theory (ACAT). The ACAT framework was developed by Ladel and Kortenkamp (2011) for designing digital resources. It shows how the interaction between the subject, the child, and the object, the mathematics, is mediated by the artefact, in our case the digital app (see Figure 1).

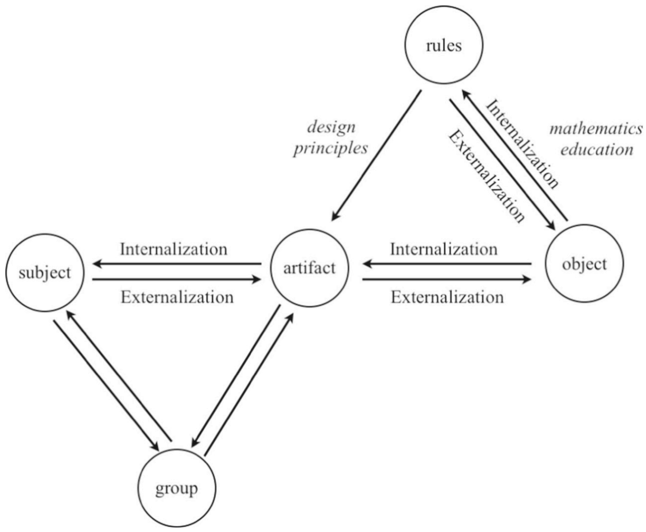

Figure 1. The ACAT framework from (Ladel & Kortenkamp, 2011, p. 66)

In the top right triangle, the focus is on the relationship between: the artefact (the app); the rules which govern the kinds of interactions that are possible; and the object, the mathematical content (Ladel & Kortenkamp, 2011). The mathematics learning made available to children is dependent on the design of the app. In the triangle on the left-hand side, the focus is on how the child (subject) would make sense of (internalize) the mathematics from what is made available for them to play with (externalize). The group, in the lower left triangle, is about how an individual interacts with an artefact as part of a society. This leads to expectations about how the artefact could or should be used in a social group. For example, the relationship between a teacher and child could affect the child's interactions with the artefact (Ladel & Kortenkamp, 2011).

Data collection

The data were collected from a workshop on mathematics as play, in a course on "Language, texts and mathematics", taken in the PTs' first year. The PTs, who had not previously had experiences with evaluating apps, worked together in groups (two to five people in each group). Five apps had been uploaded on to tablets by the PTs' normal mathematics teacher educator, who was a colleague. The teacher educator asked the PTs to evaluate the apps according to whether their use in a kindergarten would be in alignment with the aims of the Framework Plan, particular in regard to play (Ministry of Education and Research, 2017). Although the PTs were provided with the ACAT evaluation tool, only one group made any reference to it. As a result, we consider that the analysis of data allows us to identify the factors that the preservice teachers noticed with only limited input about how to do this.

In this paper, we focus on PTs' evaluations of two apps, Bimmy Boo, and Tella as most groups evaluated these apps to some degree. Both apps presented a range of mathematical ideas. Bimmy Boo is a commercially designed app for children, aged 2-6, for learning numbers from 1 to 20. (https://bimiboo.net/app/numbers/). Tella was designed by a set of government agencies, with the lead role being taken by programmers at our own institution. The Tella website (http://tella123.org/#/om) describes the app as developing a sense of quantity and then numbers, before simple addition and subtraction. The tasks must be completed in the same order. Originally aimed at children aged 5-8, it is also promoted as being for younger children.

With the consent of the PTs, 13 groups were recorded using a screen recorder app, which simultaneously recorded the discussion along with the screens shown on the tablet. The PTs determined how long their interactions were recorded. The recordings were then transcribed. Bimi Boo was discussed by eight groups. The time that each group recorded on this app, varied from 2 minutes 4 seconds to 8 minutes 40 seconds. Tella was discussed by ten groups, with recordings lasting between 2 minutes 42 second to 11 minutes 15 second.

The first author did the initial analysis and identified interactions about how aspects of the ACAT framework (Ladel & Kortenkamp, 2011) were discussed in the transcripts (see Table 1). The second author did a sample (about 10% of transcripts) to ensure reliability.

The ACAT nodes	Subject: Child(ren)	Object: Mathematical content	Rules: Pedagogical aspects of the app	Group: Individual/ group interaction
Key aspects of each node	Children's skills, abilities or needs	Numeral writing, one-to-one matching, size comparisons (bigger/smaller)	Feedback, (un)reasonable requirements of children	Need for children to play by themselves or have adult input

Table 1. The identification of ACAT nodes in the data

In the results section, we discuss the results in relationship to ACAT's two triangles (subject-artefact-group and object-artefact-rules) (Ladel & Kortenkamp, 2011). The examples in the results section were chosen as representative of the data as a whole.

Results and discussion

The results for each app are described individually, before we considered what the evaluations of the two apps told us about the input teacher educators may need to provide to PTs.

Bimi Boo: artefact – rules – object triangle

In Bimi Boo, the PTs identified the mathematical content (object) as quantity and number (cardinal number, number representations and symbols, including how to draw them). What they identified depended not just on what the digital game made available, but also on what they considered to be mathematics. For example, the PTs, in group 10, described how the app ”works very well with number”, exemplifying this by talking about writing numerals, seeing the symbolic numerals and hearing them being said in Norwegian. These aspects were valued by them as important mathematics for young children.

The PTs also identified when the mathematical content was not presented in ways that they considered it should be shown to or learnt by children. We considered these comments as being about the ”rules” node in the ACAT framework. For example, a PT in group 3 talked about the potential difficulties for children to draw the numerals because they did not have the fine motor skills, so it was, ”intuitively difficult to draw the number inside the lines”. This discussion happened after a PT, tried unsuccessfully to trace a numeral, as shown in Figure 2. Their understandings about what a child could do meant that the PTs queried the design principles in the app. They also discussed that the app gave no feedback to the children if they got something wrong and felt that this should have been incorporated into the app.

Figure 2. Screenshot from the preservice teachers trying to trace ”2” inside the lines

The PTs in Group 7 also stated that the app was good, but noted that it could have been more flexible. Figure 3 shows how the task about the number 2 required a child to place two ribbons on the cat’s head, placing them on other parts of the cat was deemed incorrect.

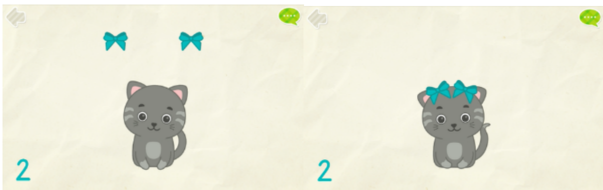

Figure 3. Screenshot from the preservice teachers placing the two ribbons on the head

When the mathematics in the app was what the PTs thought mathematics was for young children, such as number understanding, then they recognized it. They were also able to identify factors to do with how the mathematical content was presented in the app. This allowed them to identify some design principles, to do with feedback and how the children would interact with the mathematics, which they considered could be improved.

Bimi Boo: subject – artefact – group triangle

The PTs’ discussion about children was about their interests and how they learnt. For example, the PTs focused on the children’s ages as a way of determining if the app was appropriate. Group 19 described the app as being okay for two-year olds, but boring for three-year olds. Group 3 made judgements about ages based on the amount of the same or very similar games that the children should engage with, ”Boring after all because it is nothing new maybe that is why it is easier for the little ones”. The PT in group 7 felt that all children could learn from the repetitive tasks and so did not link the app to a specific age group.

The PTs appreciated the way the app provided opportunities to learn about writing numerals. In group 10, the PTs saw learning as being a result of doing, ”It is not memorizing - they do it and learn (how to write)”. The PTs seemed to value that the children did things themselves.

On the other hand, comments about the need for explicit feedback suggest that some PTs believed that children learnt by being told what they had to do and could not find it out themselves. The PTs in group 9 stated that ”it was very good that there were explanations in the game”. For learning to occur, they felt that children needed to know what they were learning. For example, group 13 discussed how tasks were displayed so that the children could not gain the ”right” answer without understanding why it was correct. Figure 4 shows a part of the game where children were to show their understanding of the number ”1”, by dragging one item into a basket, ignoring pairs or groups of items. All the single objects were in the middle, requiring one consistent movement to get the right answer. The PTs suggested the placing of items should be varied, so the right answer was not always in the centre.

Figure 4. Screenshot about choosing a single object

The PTs seemed to value that the children could play the game by themselves and that it gave them information about what they had to do. There were very few comments about how the social environment could support the children’s learning, except when the PTs commented that they saw the parent control for the game was useful.

It seemed that the PT groups were able to judge the value of the digital games based on their understandings about how children learnt. However, they did not comment on the role of the teacher when the children played the game, or whether the children could be creative when they were learning, only about the game being potentially boring for the children.

Tella: artefact – rules – object

Many of the PT groups were able to identify mathematical concepts in the Tella app, the object in the ACAT framework. Most groups identified the mathematical content as being about size, order, comparisons, classifications, counting and number. For example, a preservice teacher in group 17 states, ”is about size and numbers, measurement, volume and comparison”.

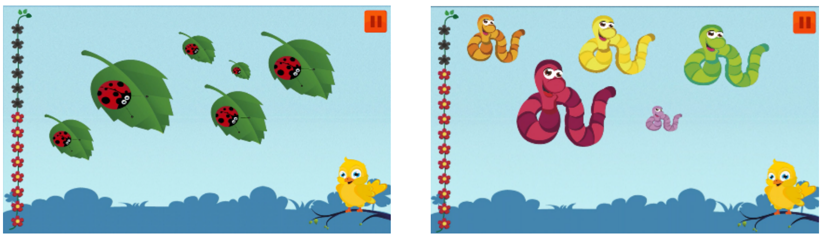

Figure 5. Screenshot about choosing the smallest and biggest object

They described the app as progressing gradually, starting with what was biggest, what was smallest and then on to numbers. Group 9 stated, ”(it) becomes more difficult when it goes further” when looking at the examples in Figure 5 where the children had to identify the smallest and the biggest figure.

However, some groups felt that the progression was too restricted in that there were too many tasks at the same level. The PTs in Group 10 did not consider it was good that all the first level tasks had to be solved, before moving to the next level. Some also felt the separation between concepts was not helpful. For example, the PTs in group 11 suggested that it would be better if the concepts were combined, ”It should be that you press the largest, press the smallest, press the pad colour, uses slightly different terms, which is largest, which is smallest, in between”. The preservice teachers considered that to only focus on the ”biggest” or the ”smallest” in a set of tasks limited the possibilities for understanding comparison.

Two of the groups, 7 and 17, discussed the way the app only presented the ”correct order” as being from smallest to largest. They considered this to be misleading, ”It should be ’start with the smallest’ and not ’the right order’, it is the wrong way for children to think and it is not right”. Figure 6 presents two examples where the task was to put the items in ”the correct order”.

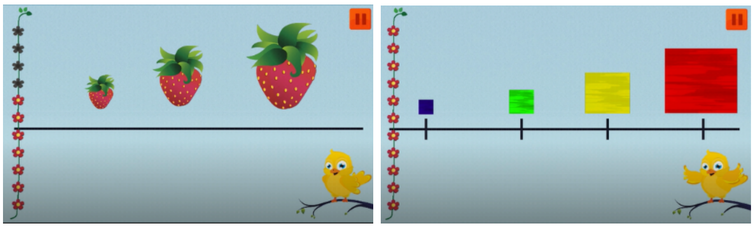

Figure 6. Screenshot showing ”the right order”

Therefore, the PTs were able to identify comparisons as well as ordering as part of mathematics. Some also discussed how the design principles connected to the tasks in the apps could affect how the mathematical ideas were externalised.

As had been the case with Bimi Boo, one group commented on the need for the app to include feedback if the task was done incorrectly, ”We do not get an explanation as to why we made a mistake” (Group 17). However, their point seemed to be more about their own wish for feedback than what they thought children may have needed.

Tella: subject – artefact – group

With Tella, the PTs in group 15 discussed how comparisons could lead to counting for some children, while others would just look at the height and not see the need to count, ”Some kids start to count to find what is biggest, some keep trying until they find (the solution)”. Figure 7 shows two examples which the PTs did not consider channelled children into using counting as a solution strategy, but would focus them instead on internalising comparison ideas about height.

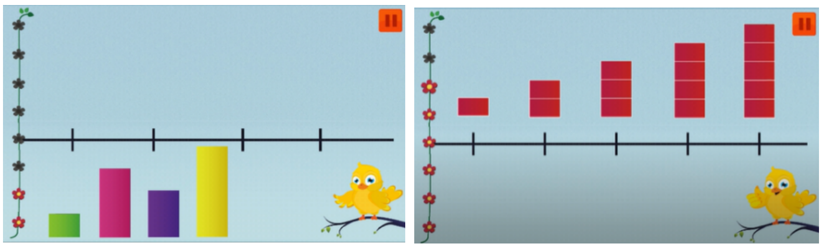

Figure 7. Two screenshot showing ordering by height

Unlike in the discussions about Bimi Boo, repetition was not highlighted as a way to learn. Instead, one group of PTs in Group 17 considered that the movements and the differences between objects required the children to explore. The PTs in Group 19 also thought that the children would enjoy it because it was easy, ”I think the young ones would think this is very fun, because it is quite easy”.

Some of the groups discussed that the children needed clearer instructions from the app. The PTs in Group 7 stated, ”They say move it here, but what do they mean by here?”. As had been the case with Bimi Boo, one of the groups highlighted that it was important for children to know what to do. They suggested that this was the role for the adult, ”the adults can explain to the smaller children about what is smallest and what is largest” and ”can adapt, then it is okay to sit with an adult”. This seemed particularly important, when the game moved into more complex ideas.

Although Tella required the children to engage in different tasks than those in Bimi Boo, the PTs identified similar factors about how the tasks could construct children’s mathematical understandings. There was some discussion about how children learnt and whether the app met the PTs’ expectations about how this should happen.

Discussion

The research question was ”What factors do preservice teachers raise when evaluating mathematical digital games for young children?”. From the analysis, it seemed that the PTs could identify different aspects of number understandings that children were expected to engage with in the two apps. One group in discussing how Tella separated ”bigger” from ”smaller” suggested that this was not an appropriate way to present these concepts as it limited children’s possibilities for seeing the connection between the two ideas. They were also able to discuss how the mathematics was presented in the apps and critique some of the design principles, such as the lack of feedback. Nevertheless, this criticism seemed to arise when the PTs themselves had difficulties with the tasks, such as tracing the numerals or placing objects in the right order. This is in contrast to Larkin et al.’s (2019) advice that apps should be evaluated in relationship to the mathematical goals for the children. Although the PTs were able to identify parts of the apps where they felt the mathematics was presented inappropriately, they did not suggest alternatives for how the mathematics could be better presented to children, such as using paper and pencil to practise writing numerals, when they identified issues with the app.

The PTs’ evaluations of the apps reflected how they considered children learnt mathematics, with comments about the value of repetition and to a lesser extent the need for more flexibility or variety in what was presented. There was no discussion about how children needed to use digital tools creatively or engage in play as a learning context as suggested in the curriculum for kindergartens (Ministry of Education and Research, 2017).

In discussions about what children bring to their interactions with the apps, some PTs talked about the ages of the children as a contributing factor to what they could be expected of their engagement with apps. These discussions were mostly about Bimmy Boo. The PTs also raised the need for an adult to be with the child when the game did not explain exactly what the child should do or what they should learn, such as to do with the biggest and smallest in Tella. There was also no discussion about how two or more children could interact together, when playing an app. This suggests that the PTs considered that the children could not explore the apps by themselves, because they would not understand why they made mistakes. These responses are similar to Handal et al.’s (2016) conclusion that the variation in responses that PTs gave when evaluating apps was likely due to their lack of experience with apps and that the PTs ”need to understand clearly the difference in instructional roles among explorative, productive and instructive apps” (p. 217). Apps where the children were expected to move through a series of restricted tasks, such as made available in Bimi Boo and Tella, can be considered instructive apps.

Conclusion

It is clear that when kindergarten PTs begin their teacher education, they have an awareness of some factors that can be used to evaluate apps. Nevertheless, their evaluations highlighted specific areas of both content and pedagogical knowledge that the PTs needed more knowledge about if they were to undertake appropriate evaluations in order to contribute to fulfilling the requirements of the Framework Plan (Ministry of Education and Research, 2017), especially in regard to play and being creative. The results suggest that PTs need to discuss different ways that mathematical content can be presented to children, of which apps could be one possibility. They also need explicit discussions about how to interact with children while engaging with apps, more than just telling them what the app expects. Similarly, PTs need to see value in children interacting with each other while playing an app. Such knowledge allows for the development of more targeted mathematics teacher education for kindergarten teachers, something noted as missing in earlier teacher education (Alvestad & Jernes, 2014). The next step is to determine how to do this given the limited time available in teacher education courses for developing this pedagogical content knowledge.

Acknowledgment

This project is part of Research Council of Norway funded KINDknow - Kindergarten Knowledge Centre for Systemic Research on Diversity and Sustainable Futures.

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Trends in everyday mathematics: the case of newspaper weather forecasts

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Making sense of weather forecasts in newspapers is a form of everyday mathematics with which many people engage. In this paper, I describe how newspaper weather forecasts have changed between 1945 and 2015, thereby indicating trends in everyday mathematics. Aided by social semiotic theory, a corpus of weather forecasts from two major Norwegian newspapers are analyzed. The findings indicate that newspaper weather forecasts have shifted towards more non-verbal forms of communication (maps, graphs, tables). This shift also changed the readers' role, from an interpreter of text to an organizer of information. I argue that students need to be better prepared for participating in an increasingly quantified public discourse. I suggest that more interdisciplinary schoolwork between mathematics and other school subjects where students use mathematical literacy skills to explore socially relevant issues is needed.

Keywords: Everyday mathematics, weather forecasts, social semiotics

Introduction and literature review

Data are more available than ever before and are also more often communicated in the shape of – sometimes innovative – data visualizations in news media (Engebretsen, 2017). The use of visual representations of statistical data in the media is a domain of everyday mathematics in rapid development. Further, the growing presence of quantitative data in the public means that mathematical literacy is getting more important for citizens' participation in public discourses. Everyday mathematics, that is, the forms of mathematics that are developed and wanted in everyday life (Wedege, 2010), has been a topic of interest in mathematics education research for a long time. So, what does everyday mathematics look like in practice, and how does it change over time? According to Zevenbergen (2011), mathematical practices and dispositions vary across generations. For example, younger generations are more comfortable using technology and more inclined to estimation and problem-solving, whereas older generations value mental, accurate calculations more highly (Zevenbergen, 2011). Hence, mathematical practices should be regarded as evolving entities, and the nature of these practices says something about the world for which school mathematics is preparing them. In Goos' model of mathematical literacy for living in the 21st century, elements of mathematical literacy include the capacity to use mathematics in diverse contexts and using representative tools such as symbol systems, graphs, and maps (Goos et al., 2014). In this paper, such tools are called *semiotic resources*, a term from the discipline called social semiotics that I will elaborate on later. The objective of this paper is to explore trends in everyday mathematics through the context of newspaper weather forecasts (NWFs), focusing on the use of semiotic resources and demands on the reader. The research questions are:

How has the use of semiotic resources in newspaper weather forecasts changed from 1945 to 2015?

How do the changes in newspaper weather forecasts change the readers' role, and what do the changes indicate for everyday mathematics?

Regarding people's abilities to read weather forecasts, Sivle and Aamodt (2019) found that many non-experts struggle to understand verbal meteorological jargon in online weather forecasts. Instead, readers prefer visual weather icons because they are more readily understood (Sivle & Aamodt, 2019). The respondents in Masson and van Es' (2020) study used internet weather forecasts to organize their plans, but they were nevertheless skeptical of the information provided by the service. The skepticism was mostly due to lay knowledge concerning the difficulties involved with weather prediction. However, it was also related to the nature of data visualizations: the informants regarded the information displayed as incomplete and selective, and the methods used to compile the data as obscured (Masson & van Es, 2020). Thus, at least some readers take a critical reader role.

To shed light on the research questions, I use data from the two leading newspapers in Norway, Aftenposten and Verden Gang. The data covers the period from 1945 to 2015. I understand newspaper weather forecasts as representations of meteorological data adapted to an audience of general news readers and presented in newspapers printed on paper. I analyzed the NWFs using an analysis tool inspired by social semiotic theories of communication (van Leeuwen, 2005), where the semiotic resources used were categorized according to type (verbal-numeric text, maps, graphs, tables) and degrees of visual salience (high, middle, low).

Semiotic resources, genre, and affordances

In my research, I use social semiotics to study communication. Semiotics is the study of meaning-making, and *social* semiotics emphasizes how humans use various means of expression to act in the social and cultural world (van Leeuwen, 2005). The things used to communicate (e.g., pictures, spoken and written words) are called *semiotic resources*. According to van Leeuwen (2005), semiotic resources are material, such as sound waves. The semiotic resources in NWFs are made by ink on paper. I take NWFs as a *genre*, that is, a class of expressions that is recognizable based on shared characteristics of content, form, and function (Miller, 1984). Engebretsen (2006) provides a model of genre evolution based on the concept of *affordances*. As Gibson (1979) defines, "[t]he *affordances* of the environment are what it *offers* the animal, what it *provides* or *furnishes*, whether for good or ill" (p. 127, emphasis in original). Hence, the affordances of the environment constitute a space of possibilities and constraints for the evolution of the genre. Engebretsen differentiates between social and instrumental affordances. Examples of social affordances are the needs, expectations, and literacies of the readers. Textual innovations will not affect the genre over time unless they match these social affordances. Printing technology is an example of instrumental affordances. Current printing technology affords a much greater range of semiotic resources than precursors. According to Engebretsen (2006), genre evolution reflects an interplay of social and instrumental affordances.

Methods

The object of study for this paper is the NWF genre. The research design is inspired by van Leeuwen (2005). I chose to focus on the period starting in 1945 because it captures the period when computerized modeling and simulations gradually became the dominant weather prediction method (Bauer et al., 2015). I chose to focus on the two leading Norwegian newspapers in this period, *Verdens Gang* (VG) and *Aftenposten* (Ap). Until 2012, Ap came in two daily issues, *Ap morgen* and *Ap aften*. Note that Ap aften was a regional paper that was distributed in the Norwegian capital region. Because Ap aften show some interesting deviations from the overall pattern, I decided to include it in this study. I opted to sample for 1 March every five years or the closest available date (1945, 1950, …, 2015) from all three issues. This sampling strategy yielded a corpus of n=44.

Next, the corpus was analyzed by (a) categorizing the types of semiotic resources and (b) assessing the degree of salience for each semiotic resource. The categories are verbal-numeric text, maps, graphs, and tables. I draw on Few (2012) to define the categories:

- *Verbal-numeric text* is text based on written words and numbers that is a separate compositional element in the NWF and not embedded in a table or graph.
- *Maps* are geospatial representations of (a part of) the world.
- *Graphs* are defined according to three characteristics, ”[v]alues are displayed within an area delineated by one or more axes”, ”[v]alues are encoded as visual objects positioned in relation to the axes” and ”[a]xes provide scales (quantitative and categorial) that are used to label and assign values to the visual objects” (p. 45). This includes line graphs and histograms.
- *Tables* are defined by two characteristics, ”[i]nformation is arranged in columns and rows” and ”[i]nformation is encoded as text (including words and numbers)” (p. 43), colors, or icons.

Two issues arose in the categorization that I will briefly elaborate. First, there were a few cases were verbal-numeric text was arranged in a table-like manner (e.g., Figure 2, left, ”været i morgen”). These cases were classed as verbal-numeric text. Second, some semiotic resources use a table-like structure but encode information with colors or icons. These cases were categorized as tables. Hence, I extend Few’s definition of tables by including colors and icons as possible means of encoding information.

Another aspect of the semiotic resources that I analyzed is visual salience. Salience refers to ”the degree to which [a semiotic resource] attract the viewer’s attention” (van Leeuwen, 2005, p. 284), that is, how a semiotic resource stands out or is obscured. Salience is an outcome of an interaction of several factors, including size, color, contrast, placement, and more (van Leeuwen, 2005). Because the degree of salience is the product of a complex semiotic interplay, it does not lend itself to stringent analysis. For this reason, I have opted to use three heuristic degrees of salience. *High* salience means an item clearly stands out, for example, by being placed partly over other components, conspicuous colors, large size, and so forth. Semiotic resources that are neither standing out nor being significantly downplayed are categorized as *middle* salience. *Low* salience is used for components that are much downplayed, for example, by being small, blending into the background color, and so forth. I use the following symbols to encode high, middle, and low salience: ●◐○.

The final step in the analysis concerns situating the semiotic resources in their social, cultural, and technological context (van Leeuwen, 2005). For this, I focus on the affordances of the four semiotic resources outlined above (Few, 2012; Kress, 2003) in accordance with Engebretsen (2006). Verbal-numeric text primarily affords narration (Kress, 2003). For displaying larger amounts of data, tables, graphs, and maps are better suited (Few, 2012). The main affordance of graphs and maps is that they can be used to display relationships among and between data, and maps have the additional affordance of showing the geospatial location of the data (Few, 2012). The main affordance of tables is that they make it easy for the reader to look up individual values (Few, 2012). The different semiotic resources assign different roles to the reader (Kress, 2003), which I will elaborate in the conclusion.

Results and analysis

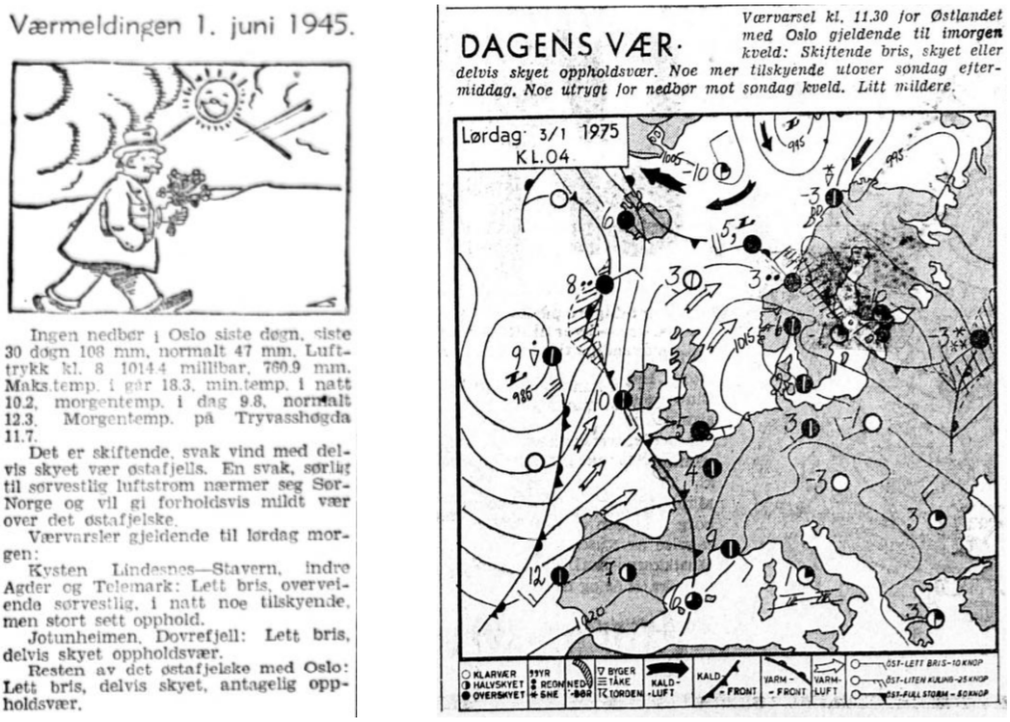

Figure 1. Examples of NWFs. Left: Ap morgen 2 June 1945. Right: Ap aften 1 March 1975. Reprinted with permission

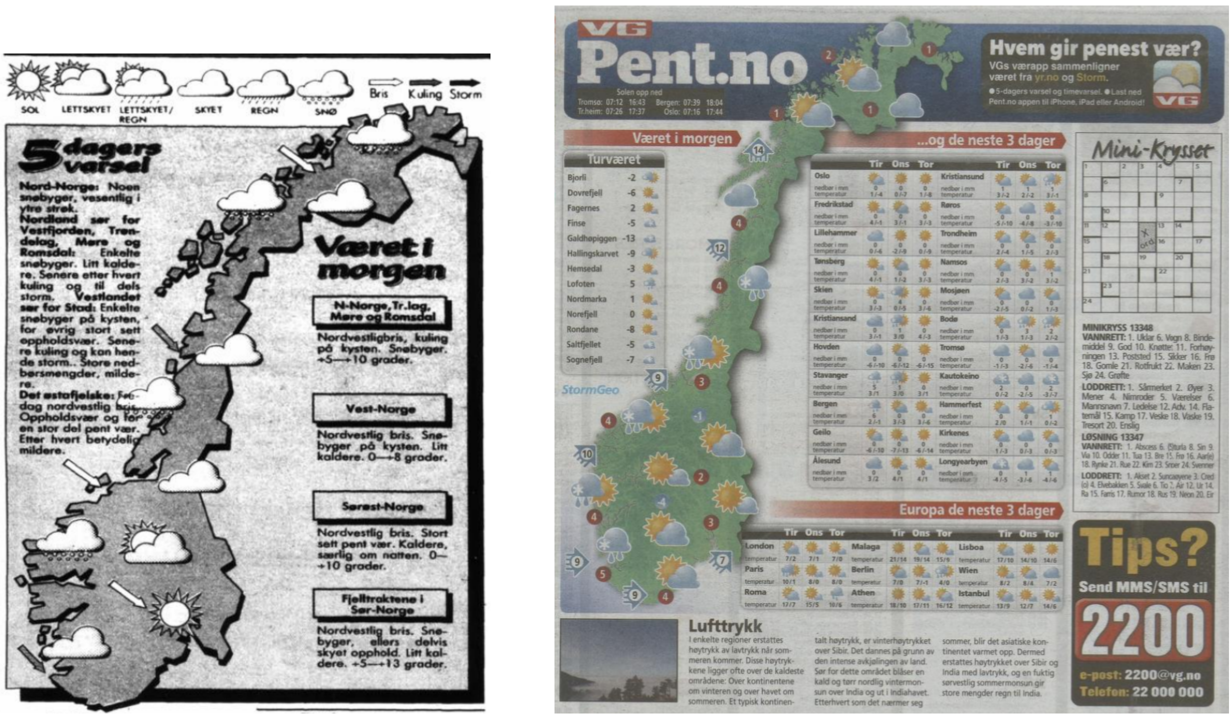

Figure 2. Examples of NWFs. Left: VG 1 March 1990. Right: VG 1 March 2015. Reprinted with permission

	Verdens Gang			Aftenposten morgen				Aftenposten aften			
Year	Text	Maps	Tables	Text	Maps	Graphs	Tables	Text	Maps	Graphs	Tables
45	●			●				●			
50	●			●				○	●		
55	●			●				○	●		
60	◐			●				○	●		
65	●			◐◐		●	◐	○	●		◐
70	●			●			◐	○	●		
75	●			●				○	●		
80	●			◐	●			◐	◐		◐
85	◐	●		●	◐◐◐		◐	◐	◐		◐
90	◐	●		◐◐○○			◐	◐			◐◐
95	◐	●		◐	◐◐		◐◐○○○	◐	○		
00	◐◐◐	●◐	◐○	◐◐	●◐◐		◐◐ ○○○○	◐ ○		◐	◐◐
05	◐○	●◐	◐◐○○	◐◐○○	◐◐◐	◐	◐◐◐◐◐ ○○○	○○	●	◐	◐◐
10		●◐	◐◐◐ ○○○	◐○	◐◐◐		●◐◐ ○○○○	◐○	◐◐◐		●◐◐ ○○○○
15	◐	●	◐◐◐○	◐	◐◐◐		●◐◐ ○○○	Discontinued			

Table 1. Overview of the data. Legend: ●: High salience, ◐: Middle salience, ○: Low salience

The results (Table 1) show that the four different semiotic resources have different distributions across the period. While verbal-numeric text was the dominant resource in VG and Ap morgen for most of the period until 1980, it was gradually phased out after 1985. Maps dominate in Ap aften from 1950 until 1975, and from 1985 they dominate in VG. Starting ca. 2000, tables are by far the largest category and continue to increase in frequency and salience towards 2015. NWFs from 2000 onward includes many tables, a few maps, and little supporting text. Graphs appear occasionally in Aftenposten, but never in VG. Over the period, the amount of data and the level of detail has significantly increased, and the information was gradually distributed across more semiotic resources.

Some possible explanations: social and instrumental affordances

The early NWFs, with verbal-numeric text only, may have been influenced by radio weather forecasts. In December 1960, weather forecasts became a regular feature in Norwegian television newscasts (Nilsen & Vollset, 2016), introducing maps, weather icons, and voiced narration to the presentation of weather forecasts. This development afforded TV watchers opportunities to learn to read weather maps. Around 1980, maps started to become a regular feature in the NWFs in VG and Ap morgen. Assuming that the readers of NWFs were also TV watchers, it is likely that the demands of reading weather maps were, at least in part, learned through watching TV.

Throughout the period this study covers, meteorological science underwent a mathematization and computerization that led to increasingly accurate meteorological data that covered longer time spans (Bauer et al., 2015). Hence, the meteorologists gradually produced more meteorological data, and the data they produced became more accurate. Since maps, graphs, and tables better afford to visualize larger amounts of quantitative data (Few, 2012), meteorological developments likely contributed to the shift away from verbal-numeric text and towards increased use of maps and tables. Further, printing technology has undergone major changes, making images, graphs, maps, and tables cheaper to print (Kress, 2003). Thus, printing technology also afforded the use of a broader repertoire of semiotic resources.

Conclusions

This paper aims to shed light on two research questions. The first research question asked how the semiotic resources used in NWFs changed between 1945 and 2015. The early NWFs were heavily reliant on verbal-numeric text. The use gradually transitioned towards using multiple forms of data representations where maps and tables typically were salient elements, and text, if any, only plays a supporting role. Graphs were used occasionally. Maps, graphs, and tables are semiotic resources that can efficiently express meaning by degree (Lemke, 2003). According to Lemke, meanings by degree are the starting point for the language of mathematics: mathematical language enables one to move smoothly between meaning by degree and meaning by category. The shift towards maps, tables, and graphs foregrounds quantitative data (e.g., millimeters of precipitation, meters per second of wind) and backgrounds the practical interpretation and evaluation of the data. This means that contemporary NWFs present more quantitative data and use more semiotic resources that are characteristic of mathematical language. Thus, NWFs have become more mathematical. This may be indicative of broader developments in everyday mathematics, but further research is needed. This brings me closer to the second research question, which asked for how the semiotic changes changed the role of the reader and what these developments suggest for everyday mathematics. To do this, I will borrow ideas from Kress (2003). According to him, verbal-numeric text on the one hand and maps, tables, and graphs on the other affords somewhat different ways of communicating. He expresses this difference with the metaphors *telling* and *showing*. Verbal-numeric text, which dominates the early NWFs, mainly affords *telling* how the world is, where the message is more or less explicitly formulated by the sender. The other semiotic resources used (maps, tables, graphs) affords *showing* the world to a greater extent, leaving the interpretation of the information more open. Kress argues that, around the turn of the millennium, the landscape of communication was characterized by a shift from *telling* to *showing* the world. Thus, NWFs fall into this broader trend. This shift implies a change in the role of the reader. Kress expresses this change by the metaphors ”reading as *interpreting*” (what has been told) to ”reading as *ordering*” (what has been shown). Before this shift, readers were told how to understand the data. However, after this shift, readers were shown the meteorological data to extract, order, and interpret it themselves. Thus, the role of the reader changed from an *interpreter* to an *organizer*. By taking the reading of NWFs as an example of an everyday mathematical practice, they provide an example where everyday mathematics has become more complex, active, and individualized.

Discussion: implications and significance for mathematics education

Being mathematically literate is not only about knowing procedures, conventions, facts, and concepts. It is also about having the capacity to draw out useful information from semiotic artifacts like NWFs and use this information to reflectively inform action in the world (Tønnessen, 2020). This study's significance is that it provides an example where everyday mathematics has become more complex, requiring a complex mathematical literacy from the reader. However, in Kennedy and Hill's (2018) work on people's interactions with data visualizations, they found that many of their participants experienced lacking confidence in their mathematical literacy, which hindered their reading of data visualizations. Kennedy and Hill attribute their participants' poor confidence to the way they learned to relate to data, which was through formal mathematics education. Due to the growing importance of quantitative data and visual representations of data in public discourses, here exemplified by NWFs, teaching for mathematical literacy ought to prepare students to engage effectively with such resources. NWFs, and data journalism more generally, are rapidly evolving genres. Hence, education should equip students with strategies for organizing, sense-making, and critical questioning of quantitative information and using this information to inform reflection and action in an evolving semiotic landscape.

The accountability of preparing students to make sense of mathematical artifacts in everyday settings, such as reading NWFs, is not clear cut. People learn and use mathematics in formal, institutionalized settings like schools and universities and in informal settings like everyday interactions and workplace activities. Kennedy and Hill's (2018) findings suggest that the mathematical literacy that students need outside of school should not be learned in the mathematics classroom alone. Kennedy and Hill even claim that formal mathematics education can be counter-productive in preparing students for engaging with data by making them anxious and unconfident when dealing with data. I suggest that students can benefit from more interdisciplinary work between mathematics and other school subjects, where students can use and develop their mathematical literacy to explore socially relevant issues. Vos and Frejd (2020) provide an example of such an interdisciplinary intervention, where students used Sankey diagrams, a mathematical concept, as a tool to explore issues related to waste management. The grade 8 students in Vos and Frejd's study readily appropriated Sankey diagrams as tools despite little prior knowledge.

The research presented in this paper has some limitations. While the use of four categories of semiotic resources does not provide a full description of NWFs, it is well suited to provide an overview of historical changes. Also, the analysis of the reader's role guided by affordances is limited because I do not rely on user data. It remains an open question if other forms of everyday mathematics have seen a similar development. More research is needed on the nature of everyday mathematics, the mathematical obstacles that people experience in everyday contexts, and how students and adults can be better prepared for the mathematical demands of living in the 21st century.

Acknowledgments

I am equally grateful to Martin Engebretsen and Pauline Vos for their contributions in the preparation of this paper. I also wish to thank the anonymous reviewers for their insightful comments.

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Characterisation of fraction representation transformations of Norwegian preservice teachers

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Learning to use a diverse set of representations to support teaching and understanding is an important and integrated part of Norwegian mathematics teacher education. This study uses a thematic analysis and Duval's theory of semiotic representations to characterise representation transformations of 53 preservice teachers' answers to three additive fraction problems. From the analysis, three themes characterising the transformations between representation registers emerged: connected, assimilated and inconsistent. The three themes are exemplified with student work, and their didactical implications are discussed.

Keywords: representations, fractions, transformations, preservice teachers, teacher education

Introduction and theoretical background

Unique to mathematics as a field of science is the ontological position that mathematical objects in and of themselves do not exist in the real world; the only way to access a mathematical object is through representations of that object. For example, a fraction can be represented symbolically as $\frac{a}{b}$ or as a point on a number line. The different representations of an object are not a form of decorative illustration but offer their own unique perspectives of what the object *is* and how it can be manipulated. Together, the representations form an amalgamation of the mathematical object. Hence, the role of representations in both doing and learning mathematics is significant (NCTM, 2000; Kilpatrick et al., 2001; Duval, 2006). Representations, in particular the idea of linking different representations and recognising what is involved in using a particular representation, are also emphasised as an important part of teachers' professional knowledge, for example, within mathematical knowledge for teaching (Ball et al., 2008). As teacher education is a primary arena for acquiring such specialised professional knowledge, it is important to investigate preservice teachers' (PSTs') knowledge about connections within and between mathematical representations.

Representations, their use and how they are connected are especially important for the learning of fractions, a subject that can be challenging for both students and preservice teachers (Ni & Zhou, 2005; Newton, 2008). There is empirical evidence that fraction knowledge predicts later mathematical academic success (Siegler et al., 2012), making fractions an essential object of study in mathematics education and teacher education. Therefore, it has been suggested that teachers should have an understanding of the range of representations, how they are used and how they relate to the concept of fractions so that they can better teach the subject (Siegler et al., 2010). These aspects of representations are encompassed in the theory of semiotic representations (Duval, 2006). Within this theory, mathematical activities consist of the transformations between representation registers (e.g., symbolic, diagrammatic, natural language and mathematical language). These transformations are divided into *treatments*, which are transformations within the same representation register, such as computing a fraction addition problem using symbolic notation, and *conversions*, a transformation from a source register to a (different) target register without changing the denoted object, such as drawing a number-line representation of a fraction addition problem that has been given in symbolic notation (Duval, 2006). In this theory, comprehension is the ability to coordinate or mobilise (at least) two representation registers simultaneously. Here, coordination is understood as the ability to use both transformations effortlessly and transform them into a suitable register for the task at hand.

Regarding research on preservice teachers' comprehension of fractions, it has focused predominantly on multiplication and division (Olanoff et al., 2014). This is also true for studies looking at transformations between representations (Son & Lee, 2016; Jansen & Hohensee, 2016), even though there are exceptions that focus on all arithmetical operations at the same time (Rosli et al., 2013). Generally, this body of research shows that PSTs often have difficulties with representing fractions (Olanoff et al., 2014). However, because the multiplication and division of fractions is more difficult than addition and subtraction (Newton, 2008), it is not fully known if their difficulties are related to performing a more complex operation or to the coordination of representation registers. Addressing this question, we investigate the conversions that a group of PSTs make between different representation registers for addition and subtraction of fractions. We state the following research question: What characterises the conversions between representation registers in Norwegian preservice teachers' written answers to additive fraction problems at the end of their first course in mathematics education?

Methods and analysis

A set of problems regarding the different representations of fractions was devised for 151 PSTs as part of the final mandatory assignment of their introductory course in mathematics education at the end of their first year of study. This topic had previously been covered in course lectures. The course description states that "…we will thoroughly analyse the foundational understanding of concepts in fractions," and one of the described learning outcomes of the course is that the student "..has knowledge of different representations, and the effects the use of representations can have on pupils' learning."

In the present paper, we analyse solutions to three additive fraction problems (Figure 1). The problems

Problem 1

Add or subtract the following fractions using an area model, a number line model and a set model.

a) $\frac{1}{4}+\frac{1}{2}$ b) $1\frac{1}{2}-\frac{3}{4}$ c) $\frac{2}{3}+\frac{3}{5}$

Figure 1. The three tasks (translated from Norwegian)

were presented symbolically using three fraction models (i.e., representation registers): the area model (A), the number line model (NL) and the set model (S). These registers are denominated as *diagrammatic registers* as opposed to the *discursive registers* of mathematical symbols and text in natural language.

A subset (N = 53) of the PST answers were selected purposively (the 53 first when sorting alphabetically) and analysed in the current study. In order to find new categories or themes within the PSTs' conversions, the analysis of the data was guided by a thematic analysis (Braun & Clarke, 2006). During the first phase, the PSTs' answers were imported to the software package NVivo for processing. Because Duval's theory of semiotic representations functions as an analytic framework in the current study, some preliminary ideas for codes were already noted. While familiarising ourselves further with the data corpus, more codes were generated. These codes were mostly semantic. The second phase of the coding process was inductive, deductive and nonsequential. The coding was done by one researcher. Eventually, saturation of the codes was achieved. Three themes of conversions between representations were identified. These themes were checked against the coding data, and subthemes were identified (often corresponding to some of the codes used).

Because the 53 PSTs were given three tasks, there was a total of 159 tasks to analyse. Each of the tasks asked for three representations; however, some PSTs only provided a single representation to each task, resulting in a total of 431 coded representations. These representations were distributed in the following registers: 145 area models, 144 number line models and 142 set models. The discursive registers were not counted because they often permeated throughout the answers.

There is not a one-to-one correspondence between the written product of the PSTs and the cognitive processes leading to that product. Therefore, a central premise for the analysis is that to some extent, the PSTs' work reflects their thinking or the communication of their mathematical knowledge.

Results

Three main themes characterised the PSTs' conversions between representation registers: assimilation of the different representation registers, inconsistency between the different representation registers and connected use of representation registers (see Table 1).

These themes encapsulated the answers from most of the PSTs (51 out of 53) and their representations (344 of 431), providing a functional way to analyse the relationships between representation registers in these types of fraction problems. Answers from the two PSTs not encapsulated by these themes included only a single representation for each task so that the relationship between models could not be evaluated and showed no sign of assimilation or inconsistency.

Table 1. Characteristics of the PSTs' conversions from symbolic to diagrammatic representations

Themes and *subthemes*	Representations	PSTs
Assimilation of different registers	**219**	**44 (83%)**
Symbolic treatment of diagrammatic register	190	42 (79%)
Number line model treated as area model	17	11 (21%)
Set model treated as area model	6	3 (6%)
Isomorphic representations	6	2 (4%)
Inconsistency between registers		
Discursive vs. diagrammatic	19	11 (21%)
Diagrammatic vs. diagrammatic	4	3 (6%)
Connected use of different diagrammatic registers	**22**	**12 (23%)**
All three registers	9	5 (9%)
Two registers	13	9 (17%)

Assimilation of different registers

In a large proportion of the representations, we found evidence of assimilation of two or more representation registers: either (i) discursive and diagrammatic or (ii) diagrammatic and diagrammatic. By assimilation, we mean that some elements of one representation register are brought into the other register during the conversion between them. Assimilation occurred most frequently in the conversion from the symbolic source register and then into either of the diagrammatic target registers (see Table 1).

Assimilation from symbolic registers to diagrammatic registers. Figure 2 shows two examples of how elements from the symbolic register, such as the equal or plus sign, have been brought into the diagrammatic register, where they have no denoted meaning by themselves and where there is no supporting text explaining what the symbols mean in the current context.

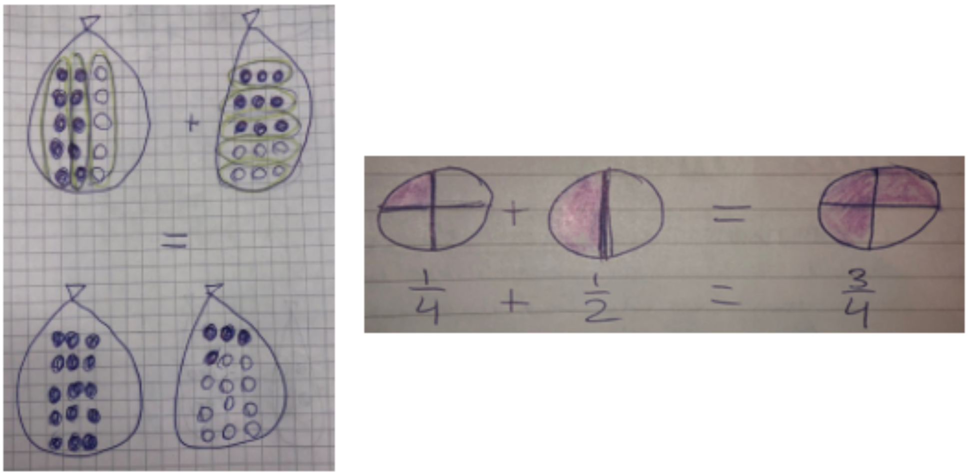


Figure 2. Typical examples of symbolic elements in the set model and area model

This type of assimilation may seem innocuous because the correct numerical answer is obtained and the PST may have taken the symbols as implicitly defined or understood. However, unless the unit and meaning of the signs are explicitly defined, such drawings can be confusing to pupils and contribute to misconceptions such as adding denominators. Indeed, assimilation also led to idiosyncratic representations in some of the PSTs' answers as exemplified in Figure 3.

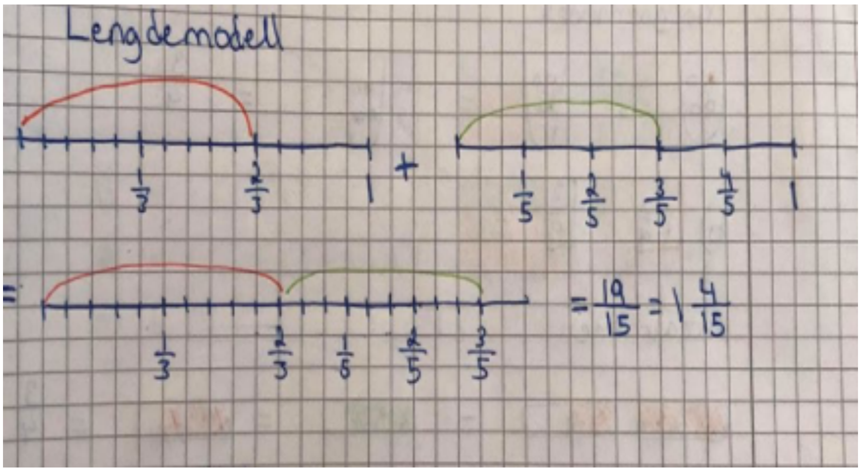
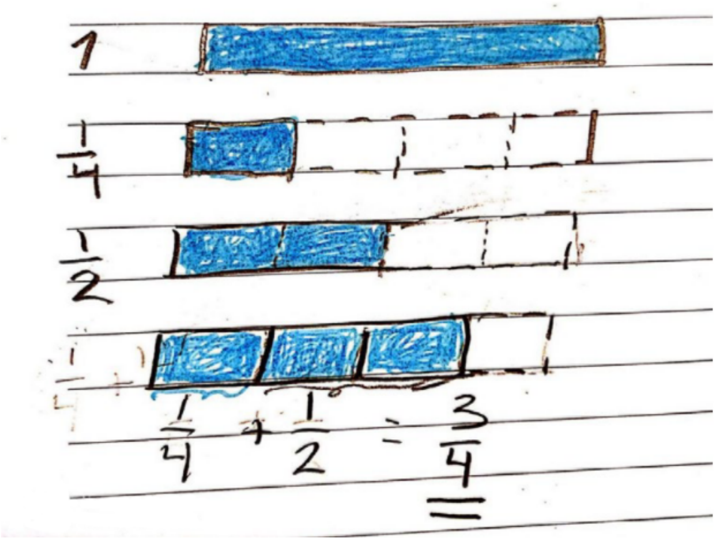


Figure 3. Addition is implicitly defined as "gluing together" two number lines, leading to a representation where one-fifth is greater than two-thirds. The numerical answer is disconnected from the diagrammatic representation

Assimilation between diagrammatic representations. In addition to the assimilation of a discursive and diagrammatic register, assimilation of two diagrammatic registers was also observed. In Figure

4 (left), we see an example where the elements from the area model were brought into the number line model. The unit has been implicitly defined as a filled area rather than, for example, a point on the number line.



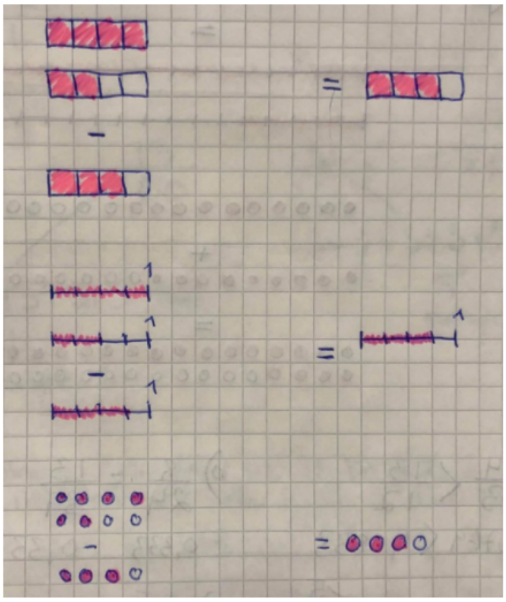


Figure 4. Left: A representation presented as a number line but treated as an area model. Right: Three representations that are isomorphic

Isomorphic representations. In the most extreme cases of assimilation, all diagrammatic registers were more or less collapsed into one register. An example is shown in Figure 4 (right), where none of the representations offers distinct or additional insights into the concept, and the conversion between representations can be described as a one-to-one mapping between different shapes. A hybrid of the set model and area model seems to dominate the expression of what a fraction is. Notably, the representations also have elements of the symbolic source register.

All of the subthemes contain naïve transformations between representation registers. By naïve, we mean that the relationship between the source and target representation is treated as congruent; the transformation between two registers is essentially reduced to an encoding process of symbols (Duval, 2006). Assimilation rarely led to an incorrect answer, potentially because the answer was found symbolically and directly translated to the diagrammatic representation. However, assimilation misses an opportunity to communicate the unique properties and uses of the different representations that can aid pupils' mathematical reasoning and problem solving in other contexts.

Inconsistency between registers

In some answers, we observed a lack of consistency between treatments in the different representations. An inconsistency between registers emerged as two subthemes: (i) an inconsistency between two diagrammatic registers, which was relatively uncommon, and (ii) an inconsistency between a diagrammatic register and a discursive register, which was relatively common.

In the inconsistency between different diagrammatic registers, the different diagrammatic representations used to represent the solution to the problem represented different answers. The most dramatic interpretation of this is that there is no connection (and therefore no conversion) between the different registers; the registers are independent of each other regarding the concept they are representing. Of course, it could also be the result of a simple mistake, but one would expect that the PSTs would notice or comment on the contradicting results.

The most common type of inconsistency of the latter subtheme was observed in the subtraction problem, in which the minuend is greater than one. The inconsistency observed in this problem has diagrammatic representations showing the correct subtrahend (Figure 5), whereas in the (self-provided) text, the subtrahend incorrectly described the problem as removing three-fourths of one and a half: ”We have a rope that is one and a half metres long. We are going to cut off three-fourths of the rope, how much rope will there be left?” In some sense, this appears to be a variant of the referent unit error (Lee, 2017) but only in an additive context. Although seemingly a subtle point, it is a fundamental one because in any practical context, the teacher will have to supplement the diagrammatic register with a natural language register. An inconsistency in what is verbally described and diagrammatically depicted is likely to confuse the learner in a classroom setting.

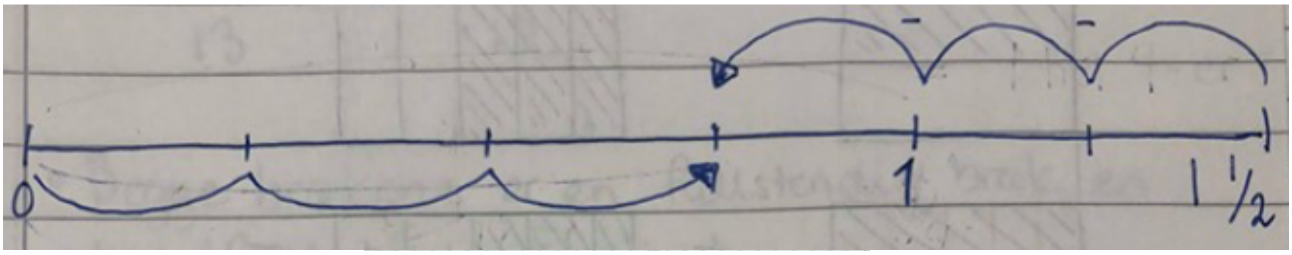


Figure 5. Correct representation of subtracting three-fourths from one and a half on the number line

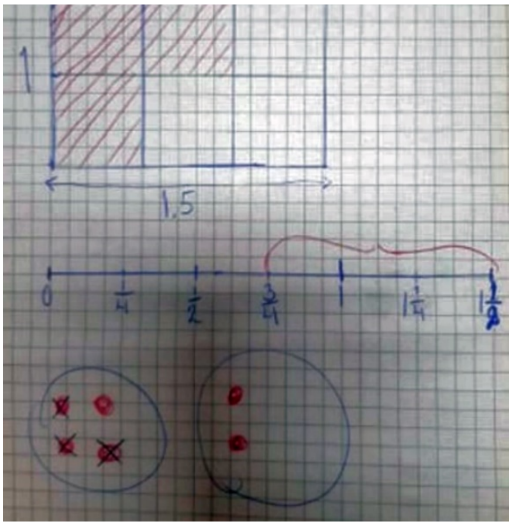


Figure 6. Connected use of the different registers. Note that these representations were also supplied with a text giving more context (e.g., viewing the area model as a lawn)

Connected use of different diagrammatic registers

The final theme can be understood as the ideal scenario. Here, the registers are treated as different entities but conceptually bound through the conversion between the representation registers. In this theme, there is evidence of transformational fluency, and each of the representations offers a distinct perspective of the fraction task. An example is shown in Figure 6, where the unit is explicitly stated or explained using supplementing text. Often, the representations are also framed in a real-life context (e.g., thermometer or cases of strawberries). Furthermore, there is a distinct difference between each of the diagrammatic representations, yielding a more diverse view of the concept of fractions. Only nine answers across five PSTs were coded as fully connected uses of different registers. In addition to these, 13 answers across nine PSTs were coded as successfully connecting two of the three diagrammatic registers. Note that some of these PSTs were also included in the other themes because for instance, there was assimilation in the area model but not in the other models.

Concluding remarks

Through a thematic analysis of 53 PSTs' answers to three additive fraction problems, three themes regarding the transformations between representation registers were identified. Going deeper into the transformations described in Duval's theory of semiotic representations, these themes were found to correspond to three different types of conversions between representation registers: naïve conversion (assimilation), inconsistent or lacking conversion between some registers (inconsistency) and fluent conversion (connected).

Each representation register has its own unique qualities. The PSTs that show a connected use of the representation registers are able to leverage the strengths and limitations of each register to communicate their thinking about a fraction problem. That teachers clearly distinguish between different representation registers themselves is essential if they are to guide their pupils towards making meaningful connections between mathematical subdomains and between mathematics and the real world. Therefore, these results are consistent with the idea that developing PSTs' appreciation of the unique strengths and limitations of different representations and representation registers should be a prioritized aspect of teacher education (e.g., Lamon, 2012). To better understand how to mitigate the assimilation of and inconsistencies between representation registers, investigating how PSTs think when they assimilate registers will be an important future endeavour.

Finally, because these are first-year PSTs at the end of their first course in mathematics education, their answers might reflect their previous education to a larger extent than their teacher training. Therefore, an analysis of later year PSTs would be of high interest to assess whether a larger proportion of PSTs acquire fluency with conversions in additive fraction problems during the course of teacher training.

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When beliefs about oneself hinder the development of beliefs about mathematics as a discipline

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As students' beliefs about mathematics seem to become more negative with age, an increased focus on students' beliefs about the role of mathematics in the world is suggested to counter this tendency. The development of one student's beliefs about mathematics as a discipline and about himself as a learner of mathematics during the first year of a longitudinal intervention study is analyzed using data from a questionnaire, classroom observations and interviews. With only one exception, the results show little change in the student's beliefs about mathematics as a discipline, and it is suggested that the centrality and stability of his beliefs about himself as a learner of mathematics might be an obstacle for such change.

Keywords: beliefs, middle school students, mathematics as a discipline, overview and judgment

Introduction

It is generally agreed that students' beliefs about mathematics have a significant influence on their learning, motivation, and approach to the subject (e.g. Furinghetti & Pehkonen, 2002; McDonough & Sullivan, 2014). In his study among secondary school students, Grigutsch (1998) distinguishes between a *schema-oriented view* on mathematics, where it is perceived as a tool based on rules and formulas, and a *process/application-oriented view*, which is characterized by a perception of mathematics as a method for understanding and solving problems, both mathematical and in daily life. To a higher extent, the process/application-oriented view is related to both enthusiasm for the subject and higher performance (Grigutsch, 1998). Unfortunately, students' attitudes towards mathematics seem to become more negative with age (Blomqvist, Elamari & Sumpter, 2012). Thus, it is essential for students' learning to focus on the development of their beliefs about mathematics and its relevance. One way to develop and expand students' beliefs in harmony with a process/application-oriented view of mathematics could be to connect the subject to the world outside school, illustrating its application and role in world. Such a connection might be established by giving the students a multifaceted image of mathematics as a scientific discipline, and not just as a school subject.

This paper presents preliminary results from a case study, which is part of an ongoing PhD project concerning the development of middle school students' beliefs of mathematics as a discipline. Drawing on successful results in an intervention study among upper secondary students (Jankvist, 2015b), the PhD project is based on the hypothesis that an implementation of perspectives on mathematics as a discipline will potentially provide middle school students opportunities to develop evidentially held beliefs about mathematics. The hypothesis is tested in two Danish middle school classes in an ongoing longitudinal intervention study, where the teachers implement teaching principles focusing on mathematical overview and judgment (to be explained later). The presented case concerns the development of one student's beliefs during the first year of the intervention. After an introduction to the applied theoretical framework, the student's beliefs are analyzed using different data sources, followed by a discussion of the results.

Theoretical framework

Students' mathematics-related belief system

In his review chapter of mathematics-related beliefs, Philipp defines these as:

> […] psychologically held understandings, premises, or propositions about the world that are thought to be true. Beliefs are more cognitive, are felt less intensely, and are harder to change than attitudes. Beliefs might be thought of as lenses that affect one's view of some aspect of the world or as dispositions toward action. Beliefs, unlike knowledge, may be held with various degree of conviction and are not consensual. Beliefs are more cognitive than emotions and attitudes. (Philipp, 2007, p. 259)

According to Green (1971), beliefs are organized in clusters or belief systems, containing both central and peripheral beliefs. People strive for coherence in their beliefs in order to withhold a subjective rationality, which means that a change in central beliefs might affect the entire belief system, thus making central beliefs harder to change (Op't Eynde et al., 2002). Beliefs based on experience or reason can be said to be *evidentially* held. In contrast, *non-evidentially* held beliefs are transferred (from e.g. parents, teachers or society) or derived from other beliefs (Green, 1971). The former tend to be more central, and therefore more powerful and influential. Furthermore, evidentially held beliefs are more likely to be changed with reason; new evidence will make a person reconsider previous experiences that have formed the beliefs and try to adjust the beliefs to fit the new experience/evidence, thereby attaining coherence in the belief system. Teaching methods based on the teacher's transmission of knowledge to the students can endorse non-evidentially held beliefs, which make the beholder reject evidence that does not support them. Thus, for the students to develop critical thinking skills, the teaching must offer the necessary evidence to form the base of their beliefs. However, changing students' beliefs is a complex matter, which, according to Green (1971), can be both difficult and time consuming.

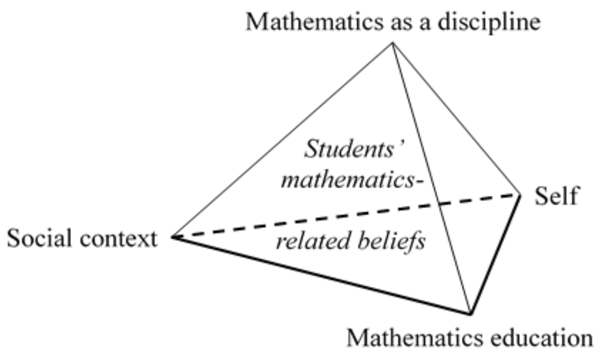

Figure 1. Students' mathematics-related belief system. (Jankvist, 2015a, p. 45)

Op't Eynde, de Corte & Verschaffel (2002) portray students' beliefs about mathematics in three interdependent dimensions: beliefs about *mathematics education,* the *self* (as a learner of mathematics), and the *social context* (typically the class and its norms). Jankvist (2015a) adds a fourth dimension: beliefs about *mathematics as a discipline*. This dimension covers beliefs that are not restricted to a school context, but concerns mathematics in a broader scientific sense as well as the role of mathematics in the world. It includes ”beliefs concerning mathematics as a pure science, an applied science, a system of tools for societal practices as well as the philosophical and epistemological nature of mathematical concepts, theories etc.” (Jankvist, 2015a, p. 45). These beliefs are developed in the context of mathematics education and therefore in interplay with the three other dimensions. Hence, the fourth dimension is placed ”outside” the original triangle, which then forms the ”base” of a tetrahedron (figure 1).

Mathematical overview and judgment

The development of students’ beliefs about mathematics as a discipline is already represented in the Danish curriculum through the notion of *mathematical overview and judgment* (O&J). Mathematics education in Denmark is partly based on a competencies framework described in 2002 in the report ”Competencies and Mathematical Learning” (Niss & Højgaard, 2019). The foundation for mathematical competence is described as two main pillars: eight action-oriented mathematical competenc*ies* (the distinction between competenc*e* and competenc*y* is intentional), and three forms of O&J. While the competencies has been successfully implemented throughout the Danish mathematics programs, the three forms of O&J are a somewhat overlooked part of the curriculum and are primarily intended for students in secondary school. They are characterized as ”insight into essential features of mathematics as a discipline” (Niss & Højgaard, 2019, p. 24), and concern three main issues: 1) the actual application of mathematics within other disciplines and fields of practice, 2) the historical development of mathematics, seen from internal as well as from socio-cultural perspectives, and 3) the nature of mathematics as a scientific discipline (Niss & Højgaard, 2019, pp. 24-25). Hence, these issues are parallel to those included in the fourth dimension of the belief system. As the three forms of O&J are based on not only *knowledge*, but also *beliefs* about mathematics as a discipline (Niss & Højgaard, 2019), they constitute an applicable framework for implementing an increased focus on this part of the students’ belief system. Supported by the curriculum’s inclusion of this framework, it thus represent the dimension of mathematics as a discipline in the intervention in which this case study is situated. Hence, the three types of O&J form the basis of both the didactical principles that are implemented in the teaching as well as the analysis of the students’ beliefs.

In this paper, I address the following research questions: *(1) How can an increased focus on O&J contribute to a change in middle school students’ beliefs about mathematics as a discipline? (2) How do the dimensions of the students’ mathematics-related belief system affect any such change?* These questions are investigated through a case study focusing on the development of one students’ beliefs related to two of the dimensions of his belief system: mathematics as a discipline (represented by the three forms of O&J) and himself as a learner of mathematics.

Method

As beliefs can take time to change (Green, 1971), the intervention proceeds over two years in two 6^{th} grade classes (12-13 years old). Didactical principles have been designed in collaboration with the mathematics teachers. The principles are based on the idea that if the students gain experiences with both the application and the historical development of mathematics as well as mathematical methods and discussions, it can form the basis of reflections about the nature of mathematics and be a stepping-stone towards central and evidentially held beliefs about mathematics as a discipline (c.f. Green, 1971). Although the principles have been adjusted along the way, four main features have remained central: (1) Concrete examples of the application and historical development of mathematics must be included in every teaching module. (2) Every lesson must include room for reflection. (3) The teaching must provide the students with opportunities to apply mathematical methods and participate in dialogue and discussions about scientific and philosophical issues concerning mathematics. (4) The three forms of O&J must be highlighted whenever possible and suitable. Researcher and teachers have worked together on the planning of the lessons, but the actual implementation of the principles have, with a few exceptions, been done by the teachers. Since the principles are quite general, the teachers have had a large amount of autonomy concerning their methodology. However, there have been extensive didactical discussions both in the planning and in the evaluation of the lessons.

The development of the students' beliefs are measured using triangulation of three forms of data: a questionnaire, interviews and classroom observations. The 20-item questionnaire mainly consists of open-ended questions designed by the author of this paper, and it is partly structured from the three forms of O&J, with the addition of questions concerning the students' relation to and beliefs about mathematics as a school subject. 46 students responded to this questionnaire in September 2019. Based on their responses, six students, who represent a diversity in views about and attitudes towards mathematics, were selected for individual, elaborating interviews conducted in November 2019. In October 2020, follow-up interviews were conducted with the same six students. As part of the interviews, the students were asked questions concerning themselves as learners of mathematics. During this year, the teaching have been observed and video recorded once or twice a month.

For analyzing the data of the student in this case (Tom), his statements have been categorized using the four dimensions of the belief system (with O&J representing mathematics as a discipline) along with his learning behavior and attitude towards mathematics. To form an image of his baseline beliefs, statements from the first interview have been combined with answers from the questionnaire. These have then been interpreted using Green's notion of (non-) evidentially held beliefs, as well as the level of consistency, exemplification and justification, as used by Jankvist (2015a) to indicate to which degree students' images of mathematics are *reflected*. Tom's baseline beliefs have then been compared to his statements from the second interview along with relevant episodes from the classroom observations. For the purpose of this study, only results concerning Tom's beliefs about mathematics as a discipline and himself as a learner of mathematics will be presented and related. Tom has been selected for this study as an atypical case. Compared to the five other interviewed students, Tom seems to have contradictory feelings and beliefs about mathematics, and the case exemplifies some of the challenges connected to changing students' beliefs.

The story of Tom

Tom is a student that generally has a negative attitude towards mathematics. According to his mathematics teacher, Tom lacks both motivation and interest in mathematics even though he performs well. Tom confirms this in the questionnaire and in both interviews by repeating that he finds mathematics very boring, very easy, and even a waste of time. Although he is active in classroom discussions and quite competitive, he has a tendency to a maladaptive learning behaviour in his lack of interest and perseverance, which is supported by his own explanations in both interviews, stating that he prefers easy tasks, so that he can get it over with and ”move on to something more useful”.

Tom's baseline beliefs about mathematics as a discipline

In the beginning of the intervention, Tom's statements concerning mathematics as a discipline are often contradictory. In relation to *the application of mathematics*, he states in the questionnaire that mathematics is ”important” and ”good to know”, because it is used for ”pretty much everything”. Yet, he gives no examples of what it might be used for. In the interview, he, on the one hand, appears to perceive mathematics as mostly irrelevant and a waste of time:

Tom: I don't feel that you need it so much. When you know how to give back change in [the supermarket] (…) I don't feel that you need much more than that.

On the other hand, he later exemplifies mathematics as useful:

Tom: I suppose you use it all the time. I mean, what size your clothes are. You just have to remember a number, but it is math. When you have learned that, it's fine.

Interviewer: Can you think of things that math is used for in society?

Tom: Hm. […] If you buy milk. […] How many people can be accommodated in a bus.

Tom's contradictions might be a sign of inconsistent and unstable beliefs. Moreover, the lack of diversity in his examples indicate that the justification of his beliefs are quite limited. The few examples given are restricted to shopping or counting, which could signal that Tom does not have experiences (or at least have not reflected upon his experiences) with the application of mathematics, and thus point to non-evidentially held beliefs. Perhaps this is not uncommon for students his age (Kloosterman, 2002), suggesting that mathematics education in general may not provide sufficient opportunities for students to gain such experiences and/or reflections. This suggestion is supported by Tom's teacher, who expresses that she normally does not focus on relating mathematics to the world outside school.

Also in regards to *the history of mathematics*, Tom seems uncertain. He skips most of the questions in the questionnaire concerning this issue, and in the interview, he is often hesitant and his answers are once again contradictory: ”[Mathematics] comes from that guy, from Egypt or wherever. Or, it has always been there”. It thus seems that Tom has not previously given the origin of mathematics much consideration, and he might not have developed any actual beliefs about this. A similar pattern somewhat applies to Tom's beliefs about *the nature of mathematics,* where his answers in the questionnaire are mostly shallow and evasive. When asked to mark the three most important things in math, he selects ”to come up with your own solution method”, ”to be able to explain what you mean”, and ”to solve problems”. These choices indicate a process/application-oriented view of mathematics, while statements in the interview along with his learning behavior in class suggest a more schema-oriented view with the criteria of success being memorization, speed and correctness.

Tom's beliefs after one year of classroom intervention

A year into the intervention, Tom shows little change in his beliefs. In the second interview, he repeats that he finds mathematics ”boring” and ”way too easy”, and that he still do not see the point in learning mathematics apart from basic arithmetic. Thus, his beliefs about himself as a learner of mathematics are quite consistent. Judging from both interviews and classroom observations, Tom often finds the tasks either too easy or too difficult. He has numerous experiences with finishing quickly and then having to wait for classmates, who need further explanation. In contrast, he struggles with open tasks and tends to give up when the expected procedure is unclear to him. Only once during the first year of the intervention, do I witness Tom deviate from his usual learning behavior. In this specific lesson, the class is introduced to Pythagoras and his theorem, and the teacher shows two examples on the smartboard: one triangle with side lengths 3, 4 and 5, and one with side lengths 6, 8 and 10. The students then work in small groups constructing their own right-angled triangles and checking the validity of the theorem. Tom's group finishes quickly, discovering that in many cases the hypotenuse will not be an integer even though the catheti are. The teacher asks them to search for a Pythagorean triple not already presented in class. Surprisingly, this challenge seems to motivate and engage Tom in a way that makes him turn aside his tendency to finish quickly. As the only one in his group, he pursues a solution for the rest of the lesson, continuing even during the teacher's wrap-up. When I ask him about the situation, Tom seems to reduce his engagement to a matter of competition:

Tom: Well, I needed to find it. Because I had to find it before the others did.

Interviewer: Oh, I see. […] Sometimes […] you like to finish quickly, right? […] But you didn't this time, you actually persisted longer than you had to.

Tom: That's because I have to do something. I cannot just sit there and do nothing.

Instead of perceiving this experience as a situation of motivation and engagement, he accommodates it to his existing beliefs about himself as indifferent, but competitive and eager to finish. His perseverance is excused as passing time even though he actually persisted working beyond the lesson. Hence, Tom's existing beliefs about himself seem to be so central and stable that they are not affected by a single experience of being motivated and engaged.

When it comes to Tom's beliefs about mathematics as a discipline, they have apparently not changed much either. Still, his statements are a bit more consistent, and it seems that he has settled on a certain point of view. He still finds only basic mathematics useful, he thinks of mathematics as having always been there and a schema-orientation is predominant in his statements. Yet, there are small signs of a potential change in his beliefs, all related to the lesson about Pythagoras, to which Tom refers several times. Even though he discarded his enthusiasm as mere competitiveness, he notes that he found this more fun than other themes, and the episode seems to have made an impact on Tom, as the following quotes will show. First concerning the application of mathematics:

Interviewer: So there is nothing you are learning at the moment that you believe you can use?

Tom: Well… If I become a carpenter, I can use Pythagoras.

His otherwise very certain belief that only basic arithmetic is useful has apparently been challenged by this experience in class, causing Tom to make exceptions from his belief. Likewise, his newly attained knowledge about Pythagoras has led to reflections about the historical development of mathematics, making him more certain of its origin and evolvement than he was in the first interview:

Tom: It has evolved. […] People have discovered more stuff. Because, Pythagoras' theorem hasn't always been there. He figured it out.

The concrete example of an ancient mathematician enables Tom to elaborate on his beliefs, using his knowledge as justification. Furthermore, the lesson offers Tom experience with applying mathematical methods (inquiry and systemization) to a problem, and shows a rare example of Tom being motivated. He more or less ”forgets” his beliefs about himself and about the (ir)relevance mathematics. Instead, it initiates a reflection on his existing beliefs about mathematics. This might be a first step of a change in Tom's beliefs, making experiences of this kind a ”way in”.

Discussion

As before mentioned, the ideal beliefs are evidentially held, since they seem to be more stable, more central and more influential, as well as more likely to be changed with reason. Thus, the aim of the intervention was to give the students opportunities to develop evidentially held beliefs based on experiences with the application, history and characteristics of mathematics. Yet, even after a year's intervention, Tom's beliefs have not changed significantly, still being non-evidentially held, lacking exemplifications and justification. Something appears to be hindering this change. The reason might be that his beliefs about himself as clever but bored and indifferent are confirmed in the mathematics lessons by tasks that he can easily solve. These experiences may have a larger impact on him than experiences concerning mathematics as a discipline. Moreover, his beliefs about himself seem to be very central and stable, and thus hinder a change in his beliefs about mathematics as a discipline. If he begins to perceive mathematics as relevant and useful, it will conflict with his indifference, forcing him to make an effort. Similarly, he cannot uphold his emphasis on fast results when applying characteristic mathematical processes like inquiry or modelling. Tom is good at performing a procedure and finishing quickly, and a more process/application-oriented view therefore might mean a change in his beliefs about himself as being skilled.

Nevertheless, the described situation regarding the Pythagorean Theorem seemed to disturb the stability of his beliefs about himself, although he did not admit to this afterwards. Tom was challenged in a way that made him engage in the task, and judging from the second interview, it provided Tom with experiences concerning all three forms of O&J—experiences that caused him to reflect. It is primarily through *reflection* that the increased focus on mathematics as a discipline can cause a change in his beliefs. If these are to change, he needs to reflect on several experiences. The question is, though, how we prevent students' beliefs about themselves from hindering such a development, as seen in the case of Tom. Of course, this study is limited to one student, and the generalizability might be discussed. There might be a number of reasons for Tom's statements and behavior that is impossible to detect through interviews and classroom observations. An example is his age, which is typically connected to the building of identity, and might influence his self-image, his actions in the social context of the classroom and the rigidness of his beliefs. However, we might gain some useful experiences and important insights from this case, leading to new hypotheses to be tested in future research: The four dimensions in students' mathematics-related belief system affect each other. From this case, this also appears to mean that they can counteract the development of one another. Thus, to develop one of the dimensions of the students' belief system, it might not be adequate to address only that dimension alone. However, the interdependence of the dimensions of the belief system does not mean that a focus on one of them automatically will affect the other three. In fact, the case of Tom suggests that if we wish to change one dimension, we have to consider the three other dimensions as well, and be aware that they must support the change in the dimension in focus.

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Bringing Nordic mathematics education into the future – a review of the papers presented at the NORMA20 conference

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The NORMA conferences held triennially gather academics from the Nordic area and beyond. The conferences offer a window into the key topics and outcomes of Nordic mathematics education research. This paper presents an analysis of the 73 regular papers presented at the NORMA20 conference held in Oslo in June 2021 and published in the conference pre- and proceedings. The analysis yields research questions, methods, and key findings for the educational levels spanning kindergarten through to higher education. The majority of the papers present research from primary school-level and pre- and in-service teacher education. Across the educational levels, teaching methods and classroom discourse emerged as the most frequent topics, dominated by empirical studies that applied qualitative methods.

Keywords: Nordic mathematics education, review, educational level, programming, Nordic model

Introduction

Research within the field of mathematics education is a young field within the Nordic community compared to other countries (Grevholm, 2021; Rønning, 2019), and few sources exist that identify what characterises this field. Among the early sources are the edited volumes *Matematikdidaktik i Norden [Mathematics education in the Nordic area*], edited by Björkqvist in 1994, and *Matematikdidaktik–ett nordiskt perspektiv [Mathematics education–a Nordic perspective],* edited by Grevholm in 2001. Both these volumes comprise chapters written in the Nordic languages as well as English, discussing central aspects of Nordic mathematics education in an international perspective. In 2010, the Nordic sourcebook, edited by Sriraman et al. (2010), presented current research from each of the five Nordic countries. According to Sriraman (2010), the aim of this publication was to make Nordic research known and available outside the Nordic community. Since then, a few publications have been published, mainly journal articles, attempting to review the field of Nordic mathematics education research, for instance Grevholm (2021) and Rønning (2019). Rønning (2019) aimed to describe the development of the field of mathematics education as a research domain in the three Scandinavian countries, Denmark, Norway, and Sweden. Linking this to the development of the national school and teacher education systems, Rønning (2019) showed that the research profile and orientation differs across the three countries; however, in all of them, the aspects related to inclusive education (e.g., learning difficulties, misconceptions, and gender issues) and problem solving are central.

In addition to the edited volumes and journal articles, conference proceedings from the Swedish Madif conferences held biannually since 1999 and the NORMA conferences held every third year since 1994 might provide additional insights into research in mathematics education in the Nordic region. The NORMA conferences are mainly attended by researchers from the Nordic countries, and as such, the conference proceedings provide accounts of the current research in the Nordic area (c.f. Grevholm, 2021; Rønning, 2019).

Recently, Grevholm (2021) reviewed the 32 research papers from the Nordic and Baltic countries in the proceedings from NORMA17, revealing that the majority of papers came from Norway and that most papers reported empirical studies applying both qualitative and quantitative methods. Rønning (2019), however, claimed that the majority of mathematics education researchers (and research) in the Nordic region, are from Sweden. Theoretical papers were rare in NORMA17. The empirical papers mostly reported the outcomes of interventions, including teaching approaches and experiments, while some papers reported studies of learning and cognition, including problem solving.

The purpose of the current paper is to add to the insights about research in the Nordic area by reviewing the regular papers presented at the NORMA20 conference that were accepted for publication in the conference pre- and proceedings (Nortvedt et al., 2021; Nortvedt et al., 2022). We argue that these publications might provide a snapshot of the status of Nordic mathematics education research. The conference had two main themes; however, the researchers could relate to Nordic mathematics education in general.

The first theme concerned the Nordic model, as described in Blossing et al. (2014) and Oftedal Telhaug et al. (2006), who claim that the educational systems in the Nordic countries share some characteristics, which is particularly the case at the compulsory education level. For example, the Nordic countries all have a high level of school autonomy, national curricula, inclusive classrooms, and formative grading at the primary school level, in contrast to education systems that have little school autonomy, grading in primary schools, streaming, and tracking. Within the Nordic countries, mathematical competence (e.g., Niss & Højgaard, 2019) is at the core of compulsory-level curricula and lately, all the countries have implemented programming in their national curricula as part of the mathematics subject. Although the Nordic countries have implemented various national-level policies over the past 10 years, and the educational systems are becoming more diverse, a Nordic model is still the focus of discussion (Blossing et al., 2014). In contrast to the similarities at the compulsory school level, higher education in general and teacher education in particular are organised differently across the Nordic countries. In Iceland, Finland, and Norway, teacher education includes a master's degree, whereas in Denmark and Sweden, teacher education lasts for four years and does not include a master's degree. However, it might be argued that even at this level, there are strong similarities, as all countries offer free higher education.

Since the Nordic educational systems have such strong similarities, the emerging issues in educational research might be expected to be similar. The aim of this review is to analyse papers representing different educational levels to identify the key themes, research methods, and outcomes that may highlight the emerging issues within Nordic mathematics education. In this process, our focus is on the whole Nordic area rather than on comparing research representing the individual countries.

Methods

The current study reviews 73 regular papers presented at the NORMA20 conference and included in the pre- and proceedings of the conference (Nortvedt et al., 2021; Nortvedt et al., 2022). The review can be characterised as an overview, with the purpose of describing the characteristics of the research papers presented at NORMA20, following the procedures in Grant and Booth (2009).

Due to the COVID pandemic, NORMA20 was first postponed for one year, and in 2021, it was held as a virtual conference. In 2020, however, the International Programme Committee decided to host a second round of submissions and publish the conference proceedings in two volumes: the preceedings consisted of accepted papers originally submitted in the first round, and the proceedings comprised papers originally submitted in the second round. This process resulted in 36 papers published in the NORMA20 preceedings and 37 papers published in the NORMA20 proceedings. All papers had at least two rounds of peer review.

Data analysis procedures

The data analysis followed a three-step procedure. First, a database was constructed comprising the titles, authors, abstracts, and keywords for each paper. Next, the themes, educational levels, methods, framework(s)/orientations, findings, and implications were categorised and added to the database. All authors of this current review article were involved in the categorisation of the papers. This first overarching classification of the papers shares some aspects with Grevholm (2021), who draws on Niss (2013), for instance in including the research questions, design, and findings. While Grevholm (2021) used Niss's (2013) categories of research study paradigms, the current study focuses on research orientation.

Second, the papers were divided into groups according to educational level, resulting in nine groups of papers (see Table 1). Of these, 31 papers addressed more than one educational level, for instance research on both teachers and students. If one of the educational levels (e.g., students) was the main focus of the paper, the paper would only be included in the analysis of the corresponding group. The categorisation was checked during this phase. Further, the papers belonging to each group were analysed to identify common themes. This analysis was concerned with what characterises the content of the papers at each educational level. Third, the author team discussed the analysis to identify emerging themes across the education-level analysis, which included clarifying the issues related to the categorisation to ensure consistency.

Table 1. Number of papers identified across educational levels

Educational level	Papers (n)
Kindergarten	6
Primary school	27
Lower secondary school	9
Upper secondary school	8
Higher education (mathematics)	5
Pre-service teacher education	15
In-service teacher education	27
Teacher educators	3
Other	6

Results

In total, the analysis included the 73 papers included in the NORMA20 preceedings and proceedings ((Nortvedt et al., 2021; Nortvedt et al., 2022). Of these, 68 papers were empirical and six theoretical, of which two papers had an empirical example. Regarding the research methods applied in the empirical papers, 58 of the accepted papers used a qualitative approach, seven used a quantitative approach, and three used mixed methods. In the next sections, the characteristics of the research papers within each of the educational levels are presented and further discussed.

Kindergarten

Of the accepted papers, six concerned kindergarten,[25] all of which reported some aspect of interaction in mathematics learning, mainly by applying case-study methods or video analysis. For instance, the research questions asked what tasks the teachers might be faced with in leading mathematical discussions in the kindergarten context; how the processes of mathematical inquiry unfold among the collaborating children (and adults) engaged in mathematical activities in kindergarten; and which challenges and opportunities are observed in classroom interactions when the kindergarten class teacher is teaching mathematics. The research foci on the teaching practices and interactions between children and adults may be due to a novel interest among the authors regarding the kindergarten curriculum.

[25] By the term "kindergarten", we refer to the educational level preceding formal education (primary school). We also consistently use the term "students" rather than "pupils" to refer to those enrolled in education at all levels ranging from kindergarten to higher education.

Primary education

Overall, 27 of the papers concerned primary-level education. The majority of these papers reported classroom studies, with a focus on communication and conceptual understanding. Qualitative research methods were mainly applied, including interviews, video analyses, field studies, content analysis, and design research. The authors studied students, teachers, and student teachers in field practice, focusing on a few participants, individuals, or groups of participants. Three of the papers were analyses of textbooks and curricula. Several papers reported numeracy at different grade levels, while others examined beliefs, reasoning, cultural aspects, and societal issues. Further, some papers focused on programming, the use of ICT for testing, and computational thinking by applying different resources. There were also papers about various teaching approaches and their implementation. In the studies on teaching approaches, the emphasis was on communication between teachers and students and the structure of the lessons. In many papers, the researchers worked with the teachers, analysing the teaching and learning from experience with the goal of improving the teaching. Several papers were aimed at developing and trying out research instruments, both regarding data collection and analysis.

Lower-secondary level

Nine papers, all empirical, focused on lower secondary mathematics. Although the qualitative approach dominated the papers—with one offering a mixed study—various analytical frameworks were employed (e.g., Schoenfeld's TRU framework, Bishop's mathematical activities, and semiotic mediation theory). Topically, the papers showed variety, but some commonalities were also visible; for example, several papers investigated dialogue and managing classroom discussions and the process of students' engagement during an activity. Two studies focused on digital tools and how these mediate both teachers' and students' mathematical competencies. The enactment of mathematics through teachers' reflections and particular tasks was the focus of two other papers, one of which focused on exams in Norwegian lower secondary schools, while the other examined North Sámi mathematical records within the context of a particular teaching unit. Despite this topical range, some common themes were found in the conclusions across these studies. These included the importance of "infrastructures" and language register in building students' mathematical (communication) competence, how real-life objects are utilised to support mathematisation and modelling, and the fine line between intended and enacted teaching, which is modulated by teacher beliefs and the particular demands of the activity itself.

Upper-secondary level

Eight out of nine papers involving upper secondary schools were empirical. Based on an instrumental approach, the theoretical paper dealt with evidence for tasks as instruments for developing mathematical competence in calculus. The link to tertiary education and transition from secondary to tertiary education was discussed through the possible construction of such tasks. Tasks and the transition issue were also relevant in an empirical paper focusing on students' metacognition as revealed in integral-area tasks. Two papers directly addressed teaching approaches. One of these, based on data from Chile, focused on the teaching practice linked to a critical approach to statistics. The second paper dealt with inquiry-based mathematics teaching and pointed to mathematical pathologies as a possible starting point for inquiry-based teaching. One paper presented an exploration of the interdisciplinarity between physics and mathematics, concluding that the mathematical emphasis in physics often focuses on the technical use of mathematics rather than its structural use. Vocational students were the focus of attention in two papers, one of which highlighted students' experiences with linear equations, while the other considered how vocational students' goal achievement could be identified in the teachers' descriptions. Digital technology and other potential aspects of mathematical thinking competency that could be offered by vocational experiences were the focus of one of the papers. Lastly, one paper discussed the collaborative learning process among teachers and how this leads to teachers' choosing or rejecting tasks for further development.

Pre-service teacher education

Altogether, 15 papers focused on pre-service teachers. Of these, there was one quantitative study and one theoretical paper, while the others were empirical and qualitative studies. Several methods were adopted in these studies: three studies used video-analysis and observation studies, two studies employed surveys and interviews, while think-aloud protocol and reports, group discussions, reflection papers, and a group-designed survey were used in the other studies.

The themes of all the papers addressing pre-service teacher education varied, but some common threads were found. Most of the studies concentrated on how to incorporate different aspects of mathematics teaching into mathematics classrooms, for instance by surveying preservice teachers' understanding of mathematics teaching. Mathematical modelling with pre-service teachers was highlighted and discussed in three of the papers. These papers concentrated on how to support pre-service teachers in mathematical modelling to recognise students' representation levels and ask appropriate questions at various structural categorical stages, how to help primary teachers in their planning of mathematical modelling activities, and the kind of experiences that preservice teachers and students gain in the context of mathematical modelling. Historical aspects and the use of the history of mathematics in mathematics teaching were dealt with in two of the papers. In addition to modelling and history, learning mathematical concepts, using interactive mathematical maps, integrating programming into mathematics education, Sámi pre-service mathematics teachers traditional body measuring in their teaching, and pathologies as starting points for inquiry-based mathematics education and preservice teachers' fraction representation transformations were studied.

In-service teacher education

Of the 27 in-service teacher education research papers presented at the conference, 25 were empirical papers. One of the remaining two was theoretical, whereas the other was coded as both theoretical and empirical, as it used empirical cases to illustrate theoretical perspectives. Only one study used a quantitative design, and another used a mixed design with both qualitative and quantitative approaches. A total of 25 papers were qualitative studies, in which video observation and interviews were the most prominent methods of data collection. Most papers focused on primary (n = 9) or secondary (n = 8) teacher education. One focused on both primary and secondary education, whereas six explored preschool or kindergarten teacher education. For the rest of the papers, educational level was not relevant or not indicated. In-service teachers' views on programming as part of the mathematics curriculum were explored in two papers, and the collaboration between leaders of professional development (PD) for mathematics in-service teachers and schools was the focus of one of the papers.

The challenges and opportunities related to classroom interactions, with a simultaneous focus on children’s mathematical inquiry, were the main focus across the kindergarten in-service teacher education papers. These challenges and opportunities were mainly related to teachers’ tasks in general and, more specifically, to teaching practices, such as teachers’ noticing, or to classroom interactions. Only one paper focused on toddlers, as the oldest children in kindergarten were the main focus in the few studies from kindergarten in-service teacher education presented at the conference.

In the nine papers reporting on research from primary in-service teacher education, the focus of attention varied to a large extent. Thus, it was difficult to group the papers into categories. Five of them, however, reported research exploring discourse or communication in one way or another. The remaining four papers reported research on primary in-service teacher education exploring socio-mathematical norms, the challenges of translating central words in the mathematics curriculum from Norwegian to Sámi, teachers’ knowledge about what constitutes problem solving, and the interplay between original resources and GeoGebra.

Further, in two out of the eight papers focusing on secondary school in-service teacher education, studies of PD supporting teachers in managing classroom communication were reported. One paper focused on the challenges and opportunities of a dialogic approach to mathematics teaching, and the second one was on how teaching guidelines might support teachers in managing classroom discourse. Two papers explored secondary teachers’ knowledge or competencies, and two focused on the implementation or enactment of tasks and activities. The remaining two papers that explored secondary in-service teacher education focused on how problem solving using whiteboards might contribute to students’ mathematical problem solving process and how teachers relate to their experiences of attainment grouping.

In conclusion, the in-service teacher education research presented at the NORMA20 conference mainly reported qualitative empirical studies from primary and secondary teacher education. Across grade levels, the focus of attention varied to a large extent. However, the challenges and opportunities related to classroom interactions seemed to have received the most attention.

Teacher educators

Three papers presented at the conference focused on teacher educators. All three were empirical with a varied approach, including cross-sectional and longitudinal studies, both quantitative and qualitative. Among the topics studied, two focused on professional development and the idea of a “community of learning”, using a different setup. One study examined the interplay between programme innovators, schools, and school heads while observing dominant discourse practices as the programme unfolded, whereas the other questioned the concept of “project forums” in setting up the playground for a learning community to evolve. The last study in this group took a more didactical approach, examining how definitions are used in mathematics teaching. From the perspective that definitions are there to help students learn and understand the meaning of a new concept, the paper discussed the tensions between the required and preferred features and how practice aligns with the broad recommendations provided within the mathematics education literature.

Higher education

In total, five of the submitted papers presented research conducted in higher education, three of which addressed different mathematical aspects, such as students’ understanding of linear regression. The remaining two papers addressed the role of tasks in developing mathematical competence and the relationship between motivational beliefs and values. The first four papers adopted a qualitative approach, using (task-based) interviews and observation, while the last paper employed a quantitative approach, using a survey.

The red thread: teaching approaches and communication

Across education levels, the majority of the papers concerned different aspects of teaching practices and classroom communication. These are not novel topics; rather, they represent a long-standing research tradition seen not only in the NORMA conferences but also internationally. Many papers addressed tasks and activities related to developing mathematical competence, especially focusing on inquiry-based activities for teaching and learning. This often included collaboration and mathematical discourse, such as researching how students communicate about chance (Julien & Iversen, 2021) and exploring how the signifier 25 × 12 was realised in a discursive classroom when teaching multiplication (Gautam & Bjuland, 2021). As these two examples show, the research that might be grouped into a category such as “discourse” or “communication” varied to a large extent. While some papers focused on students, and some on teachers or pre-service teachers, many addressed both teachers and students. In total, seven of the papers focused on programming, one of the two main themes of the conference, and are therefore explored and used as an in-depth example herein. Few papers used the concept of “the Nordic model”, the second main theme, although they addressed topics under this “umbrella” concept, for instance “mathematics for all” (c.f. Rønning, 2019; UNESCO, 1984).

Programming

Four papers discussed programming as part of mathematics teaching and learning. One of these papers explored teachers’ (n = 20) views on programming and mathematics by applying semi-structured interviews. The researchers found that teachers were convinced that programming would be a potentially powerful tool in learning mathematics, including promoting computational thinking, if only students knew how to use it (Kilhamn et al., 2021). The teachers also claimed that students find programming interesting and engage better with programming tasks than with traditional mathematics tasks. Another of these papers used a survey to collect 445 pre-service and in-service teachers’ views on the usefulness of programming in different areas of mathematics (Kaufmann & Maugesten, 2022). The analysis indicated that the respondents were less convinced of the usefulness of programming in the case of geometry than in other mathematical topic areas. Accordingly, the authors claimed that in- and pre-service teachers might need additional competence and experience in programming generally, both in the context of mathematics and in teaching programming. The remaining two papers discussed programming as part of mathematics teaching. One paper introduced two ways of approaching programming (Bergqvist, 2022). The narrow interpretation manifested activities based on learning to write codes, while the broader interpretation yielded activities in which the focus was on learning how to solve problems by applying programming as a tool. The last paper reported on teachers’ experiences after participating in an intervention study in which the Use-Modify-Create approach to programming was introduced. Participants stated that they used this approach because it has a low threshold for programming, supports mathematical discussion, facilitates problem solving, and strengthens the relationship between programming and mathematics (Maugesten et al., 2022).

Three papers reported classroom studies in which students worked on mathematical problems, two of which focused on the same project, which was introducing Scratch to the students. In this project, the researchers noticed that the students gained power over the language, skills, and practices of using and applying mathematics to different degrees and took ownership of the project (Hauge et al., 2021). Even though the students displayed different levels of mathematical language and understanding of mathematical ideas, they moved towards generalisation and algorithmic thinking. Moreover, in the second paper, which was based on a case study of two students, the authors argued that programming might promote critical learning and productive struggle (Herheim & Johnsen-Høines, 2021). The third paper presented a study on student collaboration while attempting to make a square using a micro:bit. The authors found that most of the student groups struggled in their collaborative problem solving due to a lack of communication while formulating hypotheses in the debugging process (Kaufmann et al., 2022).

A common narrative in the papers focusing on programming is that programming can contribute to mathematical learning, fostering computational thinking and problem solving. However, some obstacles exist, perhaps related to how programming is introduced and scaffolded by the teacher, for instance in supporting collaborative work. Still, the main hurdle might be the teacher's belief that students need to learn to code before they can use programming rather than recognising that the students can change the existing code in their introduction to programming.

The Nordic model

Inclusive education in diverse classrooms is one of the main characteristics of Nordic classrooms and can be seen as one element in a Nordic model of mathematics education (Räsänen et al., 2019). Few papers addressed this specifically; however, many papers addressed whole-class teaching, for instance focusing on discussions (Julien & Iversen, 2021; Manshadi, 2022), exploratory talk and argumentation (Lekaus & Lossius, 2021), or developing teaching design and trying out ideas (Justnes & Mosvold, 2021; Säfström & Sterner, 2021). These and other papers included analyses of student learning that were cognizant of the learning and challenges faced by students at different proficiency levels. As such, the idea of inclusive mathematics education was embedded in most of the NORMA20 research, a few papers addressed students at a specific proficiency levels or non-diverse classrooms. We may argue that identifying what it takes to facilitate learning for all students becomes more challenging when adapted education is not at the forefront. Regarding programming, for instance, students are often asked to write the same code for the same problem, and in this case, little differentiation is included in the task. At the same time, applying problem solving strategies (Buchholtz & Singstad, 2021), and enquiry-based teaching approaches, prominent foci of the NORMA20 papers, offers richer opportunities for differentiating and adapting tasks.

Nordic classrooms are becoming more linguistically and culturally diverse, and within inclusive mathematics education, all students should have the best opportunity to learn mathematics. This includes using contexts that students find familiar and engaging, which has been a main concern in previous research on mathematics teaching in Nordic classrooms (Rønning, 2019). This, too, seemed somewhat hidden in many of the NORMA20 papers. A few exception can be found, for instance papers addressing translation issues related to translating curriculum or mathematics education concepts to North Sámi, which show an awareness of culturally responsive teaching (Norkild et al., 2022; Varjola et al., 2022).

Discussion

Judging by the topics addressed and the research methods applied in NORMA20, a driving force in the NORMA community is the need to improve mathematics education across educational levels. What may not be emergent in the presentation of educational levels, however, is the closeness between schools and research, something that might be inferred also from the analysis performed by Rønning (2019), who discussed the emergence of mathematics education as a research field in the light of the development of the school and teacher education systems in the Scandinavian countries. Regarding the NORMA papers, the authors did not offer an outsider perspective on teaching and learning mathematics; rather, they showed much awareness of the characteristics of the different educational levels. This might not be surprising regarding research on teacher education, as the majority of the NORMA community are teacher educators themselves and thus insiders. However, the same tendencies are seen in studies that reported student learning, in which the research often took a student perspective. However, being a teacher educator involves working with schools, students, and in- and pre-service teachers. This might be a feature that characterises researchers in mathematics education in the Nordic countries, and hence a quality in Nordic research.

Across the levels, many researchers designed and implemented new tools to improve their research while simultaneously improving mathematics education. This represents a willingness to develop as a researcher and to further the research field, and it might be one of the underlying reasons why the majority of studies presented at NORMA are qualitative research. Further, we observed a lack of quantitative-oriented studies, which might allow us to generalise the research findings to a larger population. This is not a novel observation; a similar observation was made by Grevholm (2021), who claimed that the NORMA17 papers were mainly empirical and qualitatively oriented. In order to bring Nordic mathematics education into the future, our research field needs to build on and extend previous research in our field to a larger extent. In addition, when the driving force is to contribute to the improvement of mathematics education, our research needs to reach decision makers such as teachers, school leaders, and teacher educators to provide evidence that they might utilise in their daily work. Moreover, we need to deliver take-home messages to decision makers, for instance by providing evidence that supports our recommendations for mathematics education.

The group of researchers in NORMA20 is diverse, with Ph.D. students and experienced researchers working together or as individuals to investigate different content in mathematics and mathematics education. Some of the papers reported studies that were part of larger projects developed by larger groups of researchers. In the future, we recommend that we, as a community, build on each other's research foci and outcomes to a larger degree while engaging in more mixed methods studies in order to facilitate theoretical discussions that can bring Nordic mathematics education forward by developing theoretical groundings about Nordic mathematics education. Part of this movement could be to continue to grow the collaboration within the NORMA community and within and across institutions and countries.

Compared to the papers reviewed by Grevholm (2021), NORMA20 shows that the NORMA society is growing, based on the number of papers presented. A majority of the papers were Norwegian, while many were Swedish, with only a few contributions from the reminding Nordic countries. This might be seen as a limitation regarding our interest in reviewing the current state of research in Nordic mathematics education.

The tradition of the NORMA conferences has been to bring together researchers in mathematics education from the Nordic and Baltic areas. Interacting, communicating, and sharing ideas has been an important instrument in shaping Nordic research. However, at the last conference, the papers presented and included in the preceedings and proceedings represent fewer countries, and as such, our analysis cannot be extended to the full Nordic area. Hence, bringing Nordic mathematics education research forward involves nurturing the NORMA community, not only with respect to our research methods and foci but also with regard to a diverse representation of the regions and communities across the Nordic and Baltic areas.

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